2-D Discontinuous Galerkin Analysis of EM Wave Propagation in Time Domain

Supervised by ...

..., ..., ...

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Overview

Introduction

A small overview of the Task is:

- Waveguide with PEC boundary.
- Cavity acting as a "filter".
- TD-DG-FEM Approach in 2D.
- S11 Analysis of the Waveguide, using a Gaussian pulse.

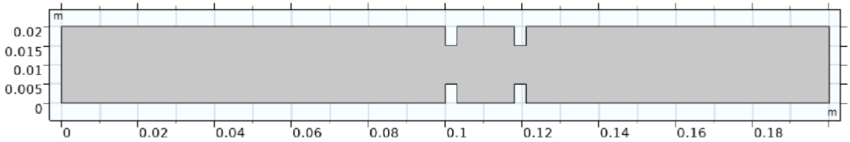

Figure: Geometry of the Waveguide, $\mu_r = \varepsilon_r = 1$

Weak Form Derivation

We start by using Faraday's (1) and Ampere's law (2) in free space:

$$\nabla \times \vec{E} + \mu \frac{\partial \vec{H}}{\partial t} = 0 \tag{1}$$

$$\nabla \times \vec{H} - \varepsilon \frac{\partial \vec{E}}{\partial t} = 0 \tag{2}$$

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Apply the weighted residual method locally over a single element:

$$\iiint_{(\Omega^e)} \vec{w}_i \cdot \left(\mu \frac{\partial \vec{H}}{\partial t} + \nabla \times \vec{E} \right) dV = 0 \qquad \forall \vec{w}_i \in (H^1(\Omega^e))^3 \tag{3}$$

$$\iiint_{(\Omega^e)} \vec{w}_i \cdot \left(\varepsilon \frac{\partial \vec{E}}{\partial t} - \nabla \times \vec{H} \right) dV = 0 \qquad \forall \vec{w}_i \in (H^1(\Omega^e))^3 \tag{4}$$

Weak Form Derivation

Split into two integrals:

$$\iiint_{(\Omega^e)} \mu \vec{w}_i \cdot \frac{\partial \vec{H}}{\partial t}\, dV + \iiint_{(\Omega^e)} \vec{w}_i \cdot (\nabla \times \vec{E})\, dV = 0 \qquad \forall \vec{w}_i \in (H^1(\Omega^e))^3 \tag{3.1}$$

$$\iiint_{(\Omega^e)} \varepsilon \vec{w}_i \cdot \frac{\partial \vec{E}}{\partial t}\, dV - \iiint_{(\Omega^e)} \vec{w}_i \cdot (\nabla \times \vec{H})\, dV = 0 \qquad \forall \vec{w}_i \in (H^1(\Omega^e))^3 \tag{4.1}$$

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$$\iiint_{(\Omega^e)} \varepsilon \vec{w}_i \cdot \frac{\partial \vec{E}}{\partial t}\, dV - \iiint_{(\Omega^e)} \vec{w}_i \cdot (\nabla \times \vec{H})\, dV = 0 \qquad \forall \vec{w}_i \in (H^1(\Omega^e))^3 \tag{4.1}$$

Use following vector identity to move the curl off of our trial functions:

$$\nabla \cdot (\vec{a} \times \vec{b}) = \vec{b} \cdot (\nabla \times \vec{a}) - \vec{a} \cdot (\nabla \times \vec{b}) \tag{5}$$

Weak Form Derivation

$$\iiint_{(\Omega^e)} \mu \vec{w}_i \cdot \frac{\partial \vec{H}}{\partial t}\, dV + \iiint_{(\Omega^e)} \vec{E} \cdot (\nabla \times \vec{w}_i)\, dV + \iiint_{(\Omega^e)} \nabla \cdot (\vec{E} \times \vec{w}_i)\, dV = 0 \qquad (3.2)$$

$$\iiint_{(\Omega^e)} \varepsilon \vec{w}_i \cdot \frac{\partial \vec{E}}{\partial t}\, dV - \iiint_{(\Omega^e)} \vec{H} \cdot (\nabla \times \vec{w}_i)\, dV - \iiint_{(\Omega^e)} \nabla \cdot (\vec{H} \times \vec{w}_i)\, dV = 0 \qquad (4.2)$$

Weak Form Derivation

$$\iiint_{(\Omega^e)} \mu \vec{w}_i \cdot \frac{\partial \vec{H}}{\partial t}\, dV + \iiint_{(\Omega^e)} \vec{E} \cdot (\nabla \times \vec{w}_i)\, dV + \iiint_{(\Omega^e)} \nabla \cdot (\vec{E} \times \vec{w}_i)\, dV = 0 \tag{3.2}$$

$$\iiint_{(\Omega^e)} \varepsilon \vec{w}_i \cdot \frac{\partial \vec{E}}{\partial t}\, dV - \iiint_{(\Omega^e)} \vec{H} \cdot (\nabla \times \vec{w}_i)\, dV - \iiint_{(\Omega^e)} \nabla \cdot (\vec{H} \times \vec{w}_i)\, dV = 0 \tag{4.2}$$

Use Gauss' law on the divergent term, which results in a boundary integral:

$$\iiint_{(\Omega^e)} \mu \vec{w}_i \cdot \frac{\partial \vec{H}}{\partial t}\, dV + \iiint_{(\Omega^e)} \vec{E} \cdot (\nabla \times \vec{w}_i)\, dV = \oiint_{(\partial\Omega^e)} \vec{w}_i \cdot (\vec{E} \times \vec{n})\, dS \tag{3.3}$$

$$\iiint_{(\Omega^e)} \varepsilon \vec{w}_i \cdot \frac{\partial \vec{E}}{\partial t}\, dV - \iiint_{(\Omega^e)} \vec{H} \cdot (\nabla \times \vec{w}_i)\, dV = \oiint_{(\partial\Omega^e)} \vec{w}_i \cdot (\vec{n} \times \vec{H})\, dS \tag{4.3}$$

Weak Form Derivation

Define a central flux over the boundary faces:

$$\vec{E} \approx \frac{1}{2}(\vec{E}^{\text{int}} + \vec{E}^{\text{ext}}) =: \{\vec{E}\} \qquad \text{on } \partial\Omega^e \tag{6}$$

$$\oiint_{(\partial\Omega^e)} \vec{w}_i \cdot (\vec{E} \times \vec{n})\, dS \approx \frac{1}{2} \oiint_{(\partial\Omega^e)} \vec{w}_i \cdot (\vec{E}^{\text{int}} \times \vec{n} + \vec{E}^{\text{ext}} \times \vec{n})\, dS \tag{7}$$

$$\oiint_{(\partial\Omega^e)} \vec{w}_i \cdot (\vec{n} \times \vec{H})\, dS \approx \frac{1}{2} \oiint_{(\partial\Omega^e)} \vec{w}_i \cdot (\vec{n} \times \vec{H}^{\text{int}} + \vec{n} \times \vec{H}^{\text{ext}})\, dS \tag{8}$$

Weak Form Derivation

Final weak formulation with central flux in three dimensions, which allows for discontinuities over boundaries:

$$\iiint_{(\Omega^e)} \mu \vec{w}_i \cdot \frac{\partial \vec{H}}{\partial t} \, dV + \iiint_{(\Omega^e)} \vec{E} \cdot (\nabla \times \vec{w}_i) \, dV = \oiint_{(\partial\Omega^e)} \vec{w}_i \cdot \left(\{\vec{E}\} \times \vec{n} \right) \, dS \tag{3.4}$$

$$\iiint_{(\Omega^e)} \varepsilon \vec{w}_i \cdot \frac{\partial \vec{E}}{\partial t} \, dV - \iiint_{(\Omega^e)} \vec{H} \cdot (\nabla \times \vec{w}_i) \, dV = \oiint_{(\partial\Omega^e)} \vec{w}_i \cdot \left(\vec{n} \times \{\vec{H}\} \right) \, dS \tag{4.4}$$

Application to TE Mode

In our case we reduce the problem to a two dimensional one, where our vectors are defined as follows:

$$\vec{E} = \begin{pmatrix} 0 \\ 0 \\ E_z \end{pmatrix}, \quad \vec{H} = \begin{pmatrix} H_x \\ H_y \\ 0 \end{pmatrix}, \quad \vec{w_i} = \begin{pmatrix} u_i \\ v_i \\ w_i \end{pmatrix}, \quad \vec{n} = \begin{pmatrix} n_x \\ n_y \\ 0 \end{pmatrix}$$

$$E_z \in H^1(\Omega^e) \ , \quad H_x \text{ and } H_y \in H^1(\Omega^e) \ , \quad \vec{w_i} \in (H^1(\Omega^e))^3$$

Application to TE Mode

We can decouple our equation (3.4) into two new ones with respect to u_i and v_i

$$\iint_{(\Omega^e)} \mu u_i \frac{\partial H_x}{\partial t}\, dS - \iint_{(\Omega^e)} E_z \frac{\partial u_i}{\partial y}\, dS = -\oint_{(\partial\Omega^e)} u_i n_y \{E_z\}\, dl \tag{9}$$

$$\iint_{(\Omega^e)} \mu v_i \frac{\partial H_y}{\partial t}\, dS + \iint_{(\Omega^e)} E_z \frac{\partial v_i}{\partial x}\, dS = \oint_{(\partial\Omega^e)} v_i n_x \{E_z\}\, dl \tag{10}$$

$$\iint_{(\Omega^e)} \varepsilon w_i \frac{\partial E_z}{\partial t}\, dS + \iint_{(\Omega^e)} H_y \frac{\partial w_i}{\partial x} - H_x \frac{\partial w_i}{\partial y}\, dS = \oint_{(\partial\Omega^e)} w_i \left(n_x\{H_y\} - n_y\{H_x\}\right) dl \tag{11}$$

Boundary Conditions: ABC

Assume that our signal is a superposition of planar, electromagnetic waves, each with following description:

$$\vec{H}(\vec{x},t) = \vec{H}_0 e^{i(\omega t - \vec{k}\cdot\vec{x})}, \qquad \vec{E}(\vec{x},t) = \vec{E}_0 e^{i(\omega t - \vec{k}\cdot\vec{x})} \tag{12}$$

Since Faraday's law is linear we can solve for each frequency separately:

$$\nabla \times E = -\mu \frac{\partial \vec{H}}{\partial t} = -i\mu\omega\vec{H} \tag{13}$$

$$\nabla \times \vec{E} = -i\vec{k} \times \vec{E} \tag{14}$$

Boundary Condition: ABC at Outlet

In the case of the outlet we need tangential continuity for Faraday's law and we assume that locally, the plane wave propagates normal to the boundary, so $\vec{k} = k\vec{n}$:

$$\vec{n} \times (\nabla \times \vec{E}) = -\mu \vec{n} \times \frac{\partial \vec{H}}{\partial t} \tag{15}$$

$$-ik\vec{n} \times (\vec{n} \times \vec{E}) = -i\mu\omega(\vec{n} \times \vec{H}) \tag{16}$$

We define the wave impedance $Z := \sqrt{\mu\varepsilon^{-1}}$. Since $\omega k^{-1} = c$ is constant, we can superimpose the solutions and get following boundary condition for the total field:

$$\vec{n} \times (\vec{n} \times \vec{E}) = Z(\vec{n} \times \vec{H}) \text{ on } \partial\Omega_{\text{outlet}} \tag{17}$$

Boundary Conditions: ABC and Source at Inlet

The same equation as for the outlet can be applied to the scattered field at the inlet with $\vec{E}_{scat} = \vec{E} - \vec{E}_{inc}$

$$\vec{n} \times (\vec{n} \times \vec{E}_{scat}) = Z(\vec{n} \times \vec{H}_{scat}) \tag{18}$$

$$\vec{n} \times (\vec{n} \times (\vec{E} - \vec{E}_{inc})) = Z(\vec{n} \times \vec{H}) - Z(\vec{n} \times \vec{H}_{inc}) \tag{19}$$

The incoming field is again locally assumed to have $\vec{k} = -k\vec{n}$:

$$\vec{n} \times (\vec{n} \times (\vec{E} - \vec{E}_{inc})) = Z(\vec{n} \times \vec{H}) + \vec{n} \times (\vec{n} \times \vec{E}_{inc}) \tag{20}$$

$$\vec{n} \times (\vec{n} \times (\vec{E} - 2\vec{E}_{inc})) = Z(\vec{n} \times \vec{H}) \text{ on } \partial\Omega_{\text{inlet}} \tag{21}$$

Boundary Conditions: PEC

For a PEC the tangential components of the electric field must vanish:

$$\vec{n} \times \vec{E} = 0 \text{ on } \partial\Omega_{\mathrm{wall}} := \partial\Omega \setminus (\partial\Omega_{\mathrm{inlet}} \cup \partial\Omega_{\mathrm{outlet}}) \tag{22}$$

From Faraday's and Gauss' law for the magnetic flux we can then derive following restrictions on the magnetic field.

$$\vec{n} \cdot \vec{H} = 0 \text{ on } \partial\Omega_{wall} \tag{23}$$

$$\vec{n} \times \vec{H} = \vec{J}_s \text{ on } \partial\Omega_{wall} \tag{24}$$

$\vec{J}_s$ can be nonzero, so when simulating with "ghost cells" we impose artificial boundary conditions onto the tangential field, such that its value is not halfed by the central flux:

$$H^{\mathrm{int}}_{tangential} = 2\{H_{tangential}\} \text{ on } \partial\Omega_{wall} \tag{25}$$

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Application of BC to TE Mode

If we apply the derived conditions on our special 2-D DG-FEM case, we end up with following boundary conditions:

$$E_z = 0 \text{ on } \partial\Omega_{\mathrm{wall}} \tag{26}$$

$$n_x H_x + n_y H_y = 0 \text{ on } \partial\Omega_{\mathrm{wall}} \tag{27}$$

$$n_x H_y - n_y H_x = 2\{n_x H_y - n_y H_x\} \text{ on } \partial\Omega_{\mathrm{wall}} \tag{28}$$

$$E_z + Z H_y = 0 \text{ on } \partial\Omega_{\mathrm{outlet}} \tag{29}$$

$$E_z - Z H_y = 2E_{z,inc} \text{ on } \partial\Omega_{\mathrm{inlet}} \tag{30}$$

Final Equations: Incorporating the BC

All of our boundary terms enter via the Flux terms. Each Element's Boundary consists of a sum of different boundary types:

$$\oint_{(\partial\Omega^e)} (\cdot) dl = \int_{\Gamma_{\text{interior}}} (\cdot) dl + \int_{\Gamma_{\text{walls}}} (\cdot) dl + \int_{\Gamma_{\text{outlet}}} (\cdot) dl + \int_{\Gamma_{\text{inlet}}} (\cdot) dl \tag{31}$$

And for different integrals, the corresponding boundary integrals are changed accordingly:

Final Equations: Incorporating the BC

$$\iint_{(\Omega^e)} \mu u_i \frac{\partial H_x}{\partial t}\, dS - \iint_{(\Omega^e)} E_z \frac{\partial u_i}{\partial y}\, dS = -\oint_{(\partial\Omega^e)} u_i n_y \{\vec{E}_z\}\, dl \tag{9}$$

$$\iint_{(\Omega^e)} \mu v_i \frac{\partial H_y}{\partial t}\, dS + \iint_{(\Omega^e)} E_z \frac{\partial v_i}{\partial x}\, dS = \oint_{(\partial\Omega^e)} v_i n_x \{\vec{E}_z\}\, dl \tag{10}$$

PEC: $E_z = 0$ on $\partial\Omega_{\text{wall}}$

$$\int_{\Gamma_{\text{wall}}} u_i n_y \{\vec{E}_z\}\, dl = \int_{\Gamma_{\text{wall}}} v_i n_x \{\vec{E}_z\}\, dl = 0 \tag{32}$$

Final Equations: Incorporating the BC

$$\iint_{(\Omega^e)} \varepsilon w_i \frac{\partial E_z}{\partial t}\, dS + \iint_{(\Omega^e)} H_y \frac{\partial w_i}{\partial x} - H_x \frac{\partial w_i}{\partial y}\, dS = \oint_{(\partial\Omega^e)} w_i \left(n_x\{H_y\} - n_y\{H_x\}\right)\, dl \quad (11)$$

PEC: $n_x H_x + n_y H_y = 0$ on $\partial\Omega_{\mathrm{wall}}$

Automatically enforced, because if H_x or H_y is normal, correspondingly $n_y = 0$ or $n_x = 0$.

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Inlet: $E_z - ZH_y = 2E_{\text{inc}}$ on $\partial\Omega_{\text{inlet}}$, $n_x = -1, n_y = 0$

$$\int_{\Gamma_{\text{inlet}}} w_i n_x \{H_y\} = -\int_{\Gamma_{\text{inlet}}} w_i Z^{-1}\{E_z - 2E_{inc}\} dl \quad (33)$$

$$= -\int_{\Gamma_{\text{inlet}}} w_i Z^{-1}(E_z - E_{inc}) dl \quad (34)$$

E_{inc} is known to us, we can therefore regard this as a simple linear form.

Spatial Discretization: Matrix Formulation

We Replace our Sobolev spaces with a discrete space, thus, using linearity of the Integral and the basis of the discrete space, we can write:

$$\mathbf{M}\left\{\frac{\partial}{\partial t}\vec{U}\right\} - (\mathbf{K}+\mathbf{F})\vec{U} = \varphi\vec{(t)} \qquad \mathbf{U} = \begin{bmatrix} H_x \\ H_y \\ E_z \end{bmatrix}, \quad \vec{\varphi}(t) = \begin{bmatrix} 0 \\ \varphi^v(t) \\ \varphi^w(t) \end{bmatrix} \tag{35}$$

$$\frac{\partial}{\partial t}\vec{U} = \mathbf{M^{-1}}\left[(\mathbf{K}+\mathbf{F})\vec{U} + \vec{\varphi}(t)\right] \tag{36}$$

Spatial Discretization: Matrix Formulation

$$\mathbf{M} = \begin{bmatrix} \mu\mathbf{m} & 0 & 0 \\ 0 & \mu\mathbf{m} & 0 \\ 0 & 0 & \varepsilon\mathbf{m} \end{bmatrix}, \qquad (\mathbf{m})_{i,j} = \iint_{(\Omega^e)} u_j \; v_i \; dS \tag{37}$$

$$\mathbf{K} = \begin{bmatrix} 0 & 0 & -\mathbf{k}^y \\ 0 & 0 & \mathbf{k}^x \\ -\mathbf{k}^y & \mathbf{k}^x & 0 \end{bmatrix}, \qquad (\mathbf{k}^{(\cdot)})_{i,j} = \iint_{(\Omega^e)} u_j \frac{\partial}{\partial(\cdot)} v_i dS \tag{38}$$

Spatial Discretization: Matrix Formulation

$$\mathbf{F} = \begin{bmatrix} 0 & 0 & \mathbf{f}^y_{\text{int}} \\ 0 & -Z\mathbf{b}_{\text{abc}} & -\mathbf{f}^x_{\text{int}} \\ \mathbf{f}^y_{\text{int}} + 2\mathbf{f}^y_{\text{wall}} & -\mathbf{f}^x_{\text{int}} - 2\mathbf{f}^x_{\text{wall}} & -Z^{-1}\mathbf{b}_{\text{abc}} \end{bmatrix}, \tag{39}$$

$$(\mathbf{f}^{(\cdot)}_{(\dots)})_{i,j} = \int_{\Gamma_{\dots}} u_j n_{(\cdot)} \{v_i\} dl, \quad (\mathbf{b}_{(\dots)})_{i,j} = \int_{\Gamma_{\dots}} u_j v_i dl \tag{40}$$

Spatial Discretization: RHS

The incoming wave that is prescribed via the flux over the inlet, enters the Linear System of Equation via a Right-Hand-Side Term, or put differently a linear form. It is given by:

$$\vec{\varphi^{v}}(t)_j = h(t)\int_{\partial\Omega} f v_j dl, \quad \vec{\varphi^{w}}(t)_j = h(t)\int_{\partial\Omega} f u_j dl, \tag{41}$$

$$f(y) = \sin\left(\frac{\pi y}{h}\right), \quad h(t) = \sin(2\pi f t)\cdot\exp\left(-[(t-t_0)f]^2\right) \tag{42}$$

Temporal Discretization: Timestepping Scheme and Stability

We now have an equation of the form $\dot{u} = f(u,t)$, that we can integrate with the explicit RK4 scheme, characterized by it's butcher scheme:

$$\begin{array}{c|c} \mathfrak{c} & \mathfrak{A} \\ \hline & \mathfrak{b} \end{array} \quad := \quad \begin{array}{c|cccc} 0 & & & & \\ 1/2 & 1/2 & & & \\ 1/2 & 0 & 1/2 & & \\ 1 & 0 & 0 & 1 & \\ \hline & 1/6 & 1/3 & 1/3 & 1/6 \end{array} \tag{43}$$

From [Hesthaven and Warburton, 2002] we know our CFL condition for above RK-scheme:

$$\Delta t \leq \frac{4\sqrt{2}h_{\text{min}}}{C \cdot c \cdot (p+1)^2} \approx C' \frac{h_{\text{min}}}{c(p+1)^2} \tag{44}$$

Mesh creation using GMSH

The Mesh was created using GMSH, or to be more specific, the gmsh library for C++. For the mesh generation we used the frontal Delauney algorithm. The mesh looks as follows:

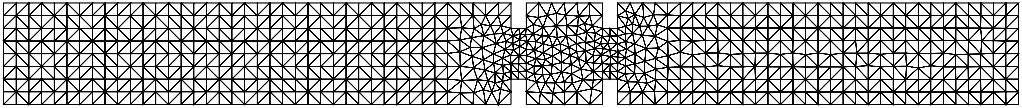

Figure: Mesh used for the S11 Analysis. $h_{\text{min}} = 0.00125$

For this Mesh and polynomial of order 2:

$$\Delta t \leq C' \frac{h_{\text{min}}}{c(p+1)^2} \approx 10^{-12}\text{s}$$

MFEM: Implementation of Custom Class

```cpp
class DGBlockInverse {
    DenseTensor inv_mass_blocks;
    const FiniteElementSpace &fes;

public:
    //Constructor: Calculate each Inverse Matrix locally, avoid assembly of big Matrix
    DGBlockInverse(FiniteElementSpace &fes_, double constant_val) : fes(fes_) {
        const int ne = fes.GetNE();
        const int dof = fes.GetFE(0)->GetDof();
        inv_mass_blocks.SetSize(dof, dof, ne);

        ConstantCoefficient coeff(constant_val);
        MassIntegrator mass_integ(coeff);
        DenseMatrix M_elem;

        for (int i = 0; i < ne; i++) {
            mass_integ.AssembleElementMatrix(*fes.GetFE(i),
                                             *fes.GetElementTransformation(i), M_elem);
            inv_mass_blocks(i) = M_elem;
            inv_mass_blocks(i).Invert();
        }
    }
```

MFEM: Implementation of Custom Class

```
    //Mult method called by Time-Stepper, use pre computed Mass Inverse Matrices
    void Mult(const Vector &x, Vector &y) {
        y.SetSize(x.Size());
        const int ne = fes.GetNE();
        Array<int> dofs;
        Vector x_loc, y_loc;

        for (int i = 0; i < ne; i++) {
            fes.GetElementVDofs(i, dofs);
            x.GetSubVector(dofs, x_loc);
            y_loc.SetSize(dofs.Size());
            inv_mass_blocks(i).Mult(x_loc, y_loc);
            y.SetSubVector(dofs, y_loc);
        }
    }
};
```

Videos Created with Paraview

S11 Analysis

The idea of the S11 Analysis is to find the resonance frequency of our waveguide. We do so by Analysing how much of our incidence Wave is reflected, e.g.

$$S11(f) = 20\log\left(\left|\frac{E_{\mathsf{ref}}(f)}{E_{\mathsf{inc}}(f)}\right|\right) \tag{45}$$

To solve for all frequencies, we solve in the time domain and send in a gaussian puls (multiple frequencies) and use Fourier transformation to obtain S11 directly for a wide range of frequencies.

$$S11(f) = 20\log\left(\left|\frac{\mathcal{F}\{E_{\mathsf{ref}}(t)\}}{\mathcal{F}\{E_{\mathsf{inc}}(t)\}}\right|\right) \tag{46}$$

S11 Analysis

To visualize and extract information for the S11 analysis, we used the open source application Paraview.

Figure: Probe Location for S11 analysis

To calculate the Fourier transform of our extracted data, we used a Python script, specifically Numpy.

S11 Analysis

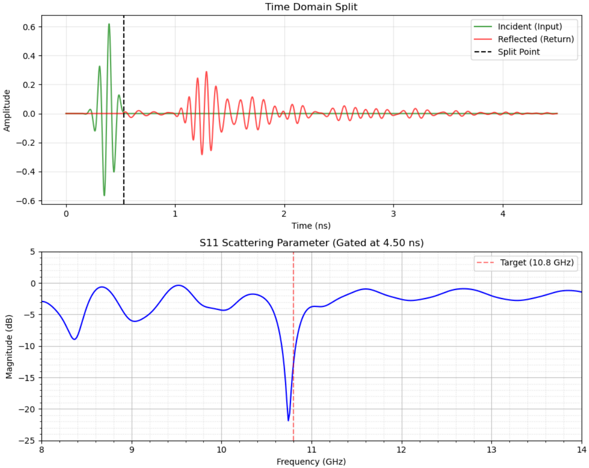

Figure: Upper plot: value of E_z at the previously mentioned probe location. Lower plot: S11 of the Frequencies after the split point

S11 Analysis: Discrepancy

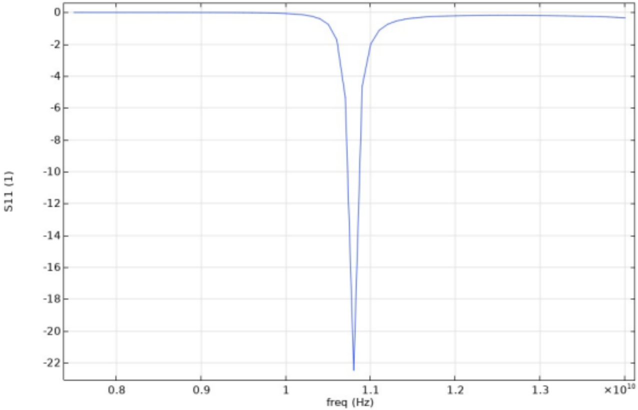

Figure: The plot shows the S11 analysis we should have expected

S11 Analysis: Discussion

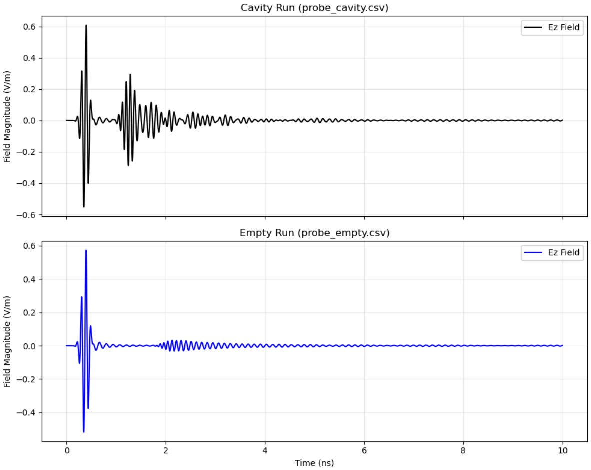

Figure: E_z over time for the Waveguide with cavity (upper) and without (lower)

S11 Analysis: Discussion

Possible causes for the small errors include:

- Cut of the signal after the simulation, producing small frequencies.
- FFT errors.
- Numerical Error arising from the simulation

Faced Challenges

- Lack of Documentation for MFEM, specifically the DG-FEM aspect of it.
- Wrong or partially incorrect Weak Formulation.
- Mesh generation and singularities in the corners, partly due to a too large timestep.

References

Anderson, R., Andrej, J., Barker, A., Bramwell, J., Camier, J.-S., Cerveny, J., Dobrev, V., Dudouit, Y., Fisher, A., Kolev, T., Pazner, W., Stowell, M., Tomov, V., Akkerman, I., Dahm, J., Medina, D., and Zampini, S. (2021).
MFEM: A modular finite element methods library.
Computers & Mathematics with Applications, 81:42–74.

Hesthaven, J. and Warburton, T. (2002).
Nodal high-order methods on unstructured grids: I. time-domain solution of maxwell's equations.
Journal of Computational Physics, 181(1):186–221.

Final Equations: Incorporating the BC

PEC: $n_x H_y - n_y H_x = 2\{n_x H_y - n_y H_x\}$ on $\partial\Omega_{\text{wall}}$

$$\int_{\Gamma_{\text{wall}}} w_i \left(n_x\{H_y\} - n_y\{H_x\}\right)\, dl = \int_{\Gamma_{\text{wall}}} 2w_i \left(n_x\{H_y\} - n_y\{H_x\}\right)\, dl \tag{47}$$

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PEC: $n_x H_x + n_y H_y = 0$ on $\partial\Omega_{\text{wall}}$

Automatically enforced, because if H_x or H_y is normal, correspondingly $n_y = 0$ or $n_x = 0$.

Final Equations: Incorporating the BC

Outlet: $E_z + ZH_y = 0$ on $\partial\Omega_{\text{outlet}}, n_x = 1, n_y = 0$

$$\int_{\Gamma_{\text{outlet}}} v_i n_x \{E_z\}\, dl = -Z \int_{\Gamma_{\text{outlet}}} v_i n_x \{H_y\} dl \tag{48}$$

$$= -Z \int_{\Gamma_{\text{outlet}}} v_i n_x H_y dl \tag{49}$$

Final Equations: Incorporating the BC

Outlet: $E_z + ZH_y = 0$ on $\partial\Omega_{\text{outlet}}, n_x = 1, n_y = 0$

$$\begin{aligned}
\int_{\Gamma_{\text{outlet}}} v_i n_x \{E_z\}\, dl &= -Z \int_{\Gamma_{\text{outlet}}} v_i n_x \{H_y\} dl && (48)\\
&= -Z \int_{\Gamma_{\text{outlet}}} v_i n_x H_y dl && (49)
\end{aligned}$$

$$\begin{aligned}
\int_{\Gamma_{\text{outlet}}} w_i n_x \{H_y\}\, dl &= -Z^{-1} \int_{\Gamma_{\text{outlet}}} w_i n_x \{\vec{E}_z\}\, dl && (50)\\
&= -Z^{-1} \int_{\Gamma_{\text{outlet}}} w_i n_x E_z\, dl && (51)
\end{aligned}$$

Final Equations: Incorporating the BC

Inlet: $E_z - ZH_y = 2E_{\mathsf{inc}}$ on $\partial\Omega_{\mathsf{inlet}}$, $n_x = -1, n_y = 0$

$$\int_{\Gamma_{\text{inlet}}} u_i n_y \{\vec{E_z}\}\, dl = \int_{\Gamma_{\text{inlet}}} w_i n_y \{H_x\}\, dl = 0 \tag{52}$$

Final Equations: Incorporating the BC

Inlet: $E_z - ZH_y = 2E_{\text{inc}}$ on $\partial\Omega_{\text{inlet}}$, $n_x = -1, n_y = 0$

$$\int_{\Gamma_{\text{inlet}}} u_i n_y \{\vec{E}_z\}\, dl = \int_{\Gamma_{\text{inlet}}} w_i n_y \{H_x\}\, dl = 0 \tag{52}$$

$$\int_{\Gamma_{\text{inlet}}} v_i n_x \{\vec{E}_z\}\, dl = -\int_{\Gamma_{\text{inlet}}} v_i \{Z\vec{H}_y + 2E_{z,inc}\}\, dl \tag{53}$$

$$= -\int_{\Gamma_{\text{inlet}}} v_i (Z\vec{H}_y + E_{z,inc}) dl \tag{54}$$

E_{inc} is only defined on the outside, as soon as it enters the domain it is taken into account for in E_z

Final Equations: Incorporating the BC

Inlet: $E_z - ZH_y = 2E_{\text{inc}}$ on $\partial\Omega_{\text{inlet}}$, $n_x = -1, n_y = 0$

$$\int_{\Gamma_{\text{inlet}}} w_i n_x \{H_y\} = -\int_{\Gamma_{\text{inlet}}} w_i Z^{-1} \{E_z - 2E_{inc}\} dl \tag{55}$$

$$= -\int_{\Gamma_{\text{inlet}}} w_i Z^{-1} (E_z - E_{inc}) dl \tag{56}$$

E_{inc} is known to us, we can therefore regard this as a simple linear form.

Temporal Discretization: Stability

The chosen RK-Scheme has a stability Function of:

$$\dot{u} = \lambda u : \quad S(z) = 1 + z + \frac{z^2}{2} + \frac{z^3}{6} + \frac{z^4}{24}, \quad z = \Delta t \lambda \tag{57}$$

Rewriting our Equation to the form of $\dot{u} = \lambda u$, yields us an estimate for the largest eigenvalue, found in [Hesthaven and Warburton, 2002].

$$|\lambda_{\mathsf{max}}| \leq C \frac{c}{h_{\mathsf{min}}} \frac{(p+1)^2}{2} \tag{58}$$

Assuming our Eigenvalues are all imaginary, as central flux is energy conserving, we can intercept our stability function with the imaginary axis, yielding:

$$\Delta t \leq \frac{4\sqrt{2} h_{\mathsf{min}}}{C \cdot c \cdot (p+1)^2} \approx C' \frac{h_{\mathsf{min}}}{c(p+1)^2} \approx 10^{-12} s \tag{59}$$

MFEM

MFEM [Anderson et al., 2021] is a free, lightweight, scalable C++ library for Finite Element Methods. It is developed and maintained by Center of Applied Scientific Computing of the USA.

It provides some helpful tools such as Args Parser, Visualisation, Timesteppers and Bilinear Form Integrators, which we use heavily in our code.

Additional things were added by hand, such as the `DGBlockInverse`.