Binary Trees

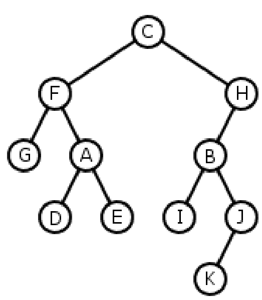
Why Care?

A binary tree is just a k-ary tree with k = 2. Because k is only 2,

- Binary trees are a bit simpler and easier to understand than trees with a large or unbounded number of children
- There are special traversals besides the usual breadth-first and depth-first traversals
- It is fun (or at least a valuable brain exercise) to generate the formula for the number of distinct binary tree shapes for a given number of nodes
- Binary tree nodes have an elegant linked representation with "left" and "right" subtrees
- Binary trees form the basis for efficient representations of sets, dictionaries, and priority queues

Special Traversals

Like all oriented trees, breadth-first and depth-first traversals exist for binary trees, but there are others:



- **Preorder**: C F G A D E H B I J K
- **Inorder**: G F D A E C I B K J H
- **Postorder**: G D E A F I K J B H C

Binary Trees as Strings

Three simple rules:

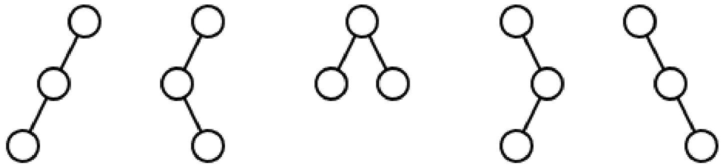
- Write the empty tree as: `()`
- Write a non empty tree as: `(leftSubtree root rightSubtree)`
- An empty subtree can be, and usually is, omitted.

So for example:

- The tree with one element A is: `(() A ())`, or just `(A)`
- The tree with A as the root and B as the left child is: `((() B ()) A ())`, or just `((B) A)`
- The tree with A as the root and B as the right child is: `(() A (() B ()))`, or just `(A (B))`
- The tree in the previous section is: `(((G) F (D (A) E)) C (((I) B ((K) J)) H))`

How Many Binary Trees Are There?

There are five distinct shapes of binary trees with three nodes:



But how many are there for n nodes?

Let C(n) be the number of distinct binary trees with n nodes. This is equal to the number of trees that have a root, a left subtree with j nodes, and a right subtree of (n-1)-j nodes, for each j. That is,

```
C(n) = C(0)C(n-1) + C(1)C(n-2) + ... + C(n-1)C(0)
```

which is

$$C_0 = 1 \qquad \text{and} \qquad C_n = \sum_{i=0}^{n-1} C_i C_{n-1-i} \quad \text{for } n \geq 1.$$

The first few terms:

```
C(0) = 1
C(1) = C(0)C(0) = 1
C(2) = C(0)C(1) + C(1)C(0) = 2
C(3) = C(0)C(2) + C(1)C(1) + C(2)C(0) = 5
C(4) = C(0)C(3) + C(1)C(2) + C(2)C(1) + C(3)C(0) = 14
```

You can prove

$$C_n = \frac{1}{n+1}\binom{2n}{n} \qquad \text{for } n \geq 0.$$

Here's the number of 8-node binary trees:

```
         1    16!       16×15×14×13×12×11×10
C(8) = - × ---- = -------------------- = 13×11×10 = 1430
         9    8!8!         8×7×6×5×4×3×2×1
```

Also see [Wikipedia's article on the Catalan Numbers](https://en.wikipedia.org/wiki/Catalan_number).

An Implementation of Binary Trees

As is common with trees (though quite unlike lists), node classes are visible to clients. Here is a simple binary tree node interface, again using the visitor pattern for traversals.

BinaryTreeNode.java

```java
package edu.lmu.cs.collections;

/**
 * A simple interface for binary trees.  An empty binary tree is
 * represented with the value null; a non-empty tree by its root
 * node.
 */
 public interface BinaryTreeNode<E> {

    /**
     * Returns the data stored in this node.
     */
    E getData();

    /**
     * Modifies the data stored in this node.
     */
    void setData(E data);

    /**
     * Returns the parent of this node, or null if this node is a root.
     */
    BinaryTreeNode<E> getParent();

    /**
     * Returns the left child of this node, or null if it does
     * not have one.
     */
    BinaryTreeNode<E> getLeft();

    /**
     * Removes child from its current parent and inserts it as the
     * left child of this node.  If this node already has a left
     * child it is removed.
     *
     * @exception IllegalArgumentException if the child is
     * an ancestor of this node, since that would make
     * a cycle in the tree.
     */
    void setLeft(BinaryTreeNode<E> child);

    /**
     * Returns the right child of this node, or null if it does
     * not have one.
     */
    BinaryTreeNode<E> getRight();

    /**
     * Removes child from its current parent and inserts it as the
     * right child of this node.  If this node already has a right
     * child it is removed.
     * @exception IllegalArgumentException if the child is
     * an ancestor of this node, since that would make
     * a cycle in the tree.
     */
    void setRight(BinaryTreeNode<E> child);

    /**
     * Removes this node, and all its descendants, from whatever
     * tree it is in.  Does nothing if this node is a root.
     */
```

```
    void removeFromParent();

    /**
     * Visits the nodes in this tree in preorder.
     */
    void traversePreorder(Visitor visitor);

    /**
     * Visits the nodes in this tree in postorder.
     */
    void traversePostorder(Visitor visitor);

    /**
     * Visits the nodes in this tree in inorder.
     */
    void traverseInorder(Visitor visitor);

    /**
     * Simple visitor interface.
     */
    public interface Visitor {
        <E> void visit(BinaryTreeNode<E> node);
    }
}
```

We can implement this interface with links

LinkedBinaryTreeNode.java

```
package edu.lmu.cs.collections;

/**
 * An implementation of the BinaryTreeNode interface in which
 * each node stores direct links to its left child, its right
 * child, and its parent.
 *
 * <p>LinkedBinaryTreeNode objects are pretty mean: if one tries
 * to mix them up with different kinds of binary tree nodes,
 * and exception may be thrown.</p>
 */
public class LinkedBinaryTreeNode<E> implements BinaryTreeNode<E> {
    protected E data;
    protected LinkedBinaryTreeNode<E> parent;
    protected LinkedBinaryTreeNode<E> left;
    protected LinkedBinaryTreeNode<E> right;

    /**
     * Constructs a node as the root of its own one-element tree.
     * This is the only public constructor.  The only trees that
     * clients can make directly are simple one-element trees.
     */
    public LinkedBinaryTreeNode(E data) {
        this.data = data;
    }

    /**
     * Returns the data stored in this node.
     */
    public E getData() {
        return data;
    }
```

```
    /**
     * Modifies the data stored in this node.
     */
    public void setData(E data) {
        this.data = data;
    }

    /**
     * Returns the parent of this node, or null if this node is a root.
     */
    public BinaryTreeNode<E> getParent() {
      return parent;
    }

    /**
     * Returns the left child of this node, or null if it does
     * not have one.
     */
    public BinaryTreeNode<E> getLeft() {
      return left;
    }

    /**
     * Removes child from its current parent and inserts it as the
     * left child of this node.  If this node already has a left
     * child it is removed.
     * @exception IllegalArgumentException if the child is
     * an ancestor of this node, since that would make
     * a cycle in the tree.
     */
    public void setLeft(BinaryTreeNode<E> child) {
        // Ensure the child is not an ancestor.
        for (LinkedBinaryTreeNode<E> n = this; n != null; n = n.parent) {
            if (n == child) {
                throw new IllegalArgumentException();
            }
        }

        // Ensure that the child is an instance of LinkedBinaryTreeNode.
        LinkedBinaryTreeNode<E> childNode = (LinkedBinaryTreeNode<E>)child;

        // Break old links, then reconnect properly.
        if (this.left != null) {
            left.parent = null;
        }
        if (childNode != null) {
            childNode.removeFromParent();
            childNode.parent = this;
        }
        this.left = childNode;
    }

    /**
     * Returns the right child of this node, or null if it does
     * not have one.
     */
    public BinaryTreeNode<E> getRight() {
      return right;
    }

    /**
     * Removes child from its current parent and inserts it as the
```

```
     * right child of this node.  If this node already has a right
     * child it is removed.
     * @exception IllegalArgumentException if the child is
     * an ancestor of this node, since that would make
     * a cycle in the tree.
     */
    public void setRight(BinaryTreeNode<E> child) {
        // Ensure the child is not an ancestor.
        for (LinkedBinaryTreeNode<E> n = this; n != null; n = n.parent) {
            if (n == child) {
                throw new IllegalArgumentException();
            }
        }

        // Ensure that the child is an instance of LinkedBinaryTreeNode.
        LinkedBinaryTreeNode<E> childNode = (LinkedBinaryTreeNode<E>)child;

        // Break old links, then reconnect properly.
        if (right != null) {
            right.parent = null;
        }
        if (childNode != null) {
            childNode.removeFromParent();
            childNode.parent = this;
        }
        this.right = childNode;
    }

    /**
     * Removes this node, and all its descendants, from whatever
     * tree it is in.  Does nothing if this node is a root.
     */
    public void removeFromParent() {
        if (parent != null) {
            if (parent.left == this) {
                parent.left = null;
            } else if (parent.right == this) {
                parent.right = null;
            }
            this.parent = null;
        }
    }

    /**
     * Visits the nodes in this tree in preorder.
     */
    public void traversePreorder(BinaryTreeNode.Visitor visitor) {
        visitor.visit(this);
        if (left != null) left.traversePreorder(visitor);
        if (right != null) right.traversePreorder(visitor);
    }

    /**
     * Visits the nodes in this tree in postorder.
     */
    public void traversePostorder(Visitor visitor) {
        if (left != null) left.traversePostorder(visitor);
        if (right != null) right.traversePostorder(visitor);
        visitor.visit(this);
    }

    /**
```

```
     * Visits the nodes in this tree in inorder.
     */
    public void traverseInorder(Visitor visitor) {
        if (left != null) left.traverseInorder(visitor);
        visitor.visit(this);
        if (right != null) right.traverseInorder(visitor);
    }
}
```

For fun, here is a panel class that can draw a binary tree.

BinaryTreePanel.java

```
package edu.lmu.cs.collections;

import java.awt.Color;
import java.awt.FontMetrics;
import java.awt.Graphics;
import java.awt.Point;
import java.awt.Rectangle;
import java.lang.reflect.Field;
import java.util.HashMap;
import java.util.Map;

import javax.swing.JPanel;

/**
 * A panel that maintains a picture of a binary tree.
 */
public class BinaryTreePanel extends JPanel {
    private BinaryTreeNode<?> tree;
    private int gridwidth;
    private int gridheight;

    /**
     * Stores the pixel values for each node in the tree.
     */
    private Map<BinaryTreeNode<?>, Point> coordinates =
        new HashMap<BinaryTreeNode<?>, Point>();

    /**
     * Constructs a panel, saving the tree and drawing parameters.
     */
    public BinaryTreePanel(BinaryTreeNode<?> tree, int gridwidth, int gridheight) {
        this.tree = tree;
        this.gridwidth = gridwidth;
        this.gridheight = gridheight;
    }

    /**
     * Changes the tree rendered by this panel.
     */
    public void setTree(BinaryTreeNode<?> root) {
        tree = root;
        repaint();
    }

    /**
     * Draws the tree in the panel.  First it computes the coordinates
     * of all the nodes with an inorder traversal, then draws them
     * with a postorder traversal.
     */
```

```
    public void paintComponent(final Graphics g) {
        super.paintComponent(g);

        if (tree == null) {
            return;
        }

        tree.traverseInorder(new BinaryTreeNode.Visitor() {
            private int x = gridwidth;
            public void visit(BinaryTreeNode node) {
                coordinates.put(node, new Point(x, gridheight * (depth(node)+1)));
                x += gridwidth;
            }
        });

        tree.traversePostorder(new BinaryTreeNode.Visitor() {
            public void visit(BinaryTreeNode node) {
                String data = node.getData().toString();
                Point center = (Point)coordinates.get(node);
                if (node.getParent() != null) {
                    Point parentPoint = (Point)coordinates.get(node.getParent());
                    g.setColor(Color.black);
                    g.drawLine(center.x, center.y, parentPoint.x, parentPoint.y);
                }
                FontMetrics fm = g.getFontMetrics();
                Rectangle r = fm.getStringBounds(data, g).getBounds();
                r.setLocation(center.x - r.width/2, center.y - r.height/2);
                Color color = getNodeColor(node);
                Color textColor =
                    (color.getRed() + color.getBlue() + color.getGreen() < 382)
                    ? Color.white
                    : Color.black;
                g.setColor(color);
                g.fillRect(r.x - 2 , r.y - 2, r.width + 4, r.height + 4);
                g.setColor(textColor);
                g.drawString(data, r.x, r.y + r.height);
            }
        });
    }

    /**
     * Returns a color for the node.  If the node is of a class with a
     * field called "color", and that field currently contains a
     * non-null value, then that value is returned.  Otherwise
     * a default color of yellow is returned.
     */
    Color getNodeColor(BinaryTreeNode<?> node) {
        try {
            Field field = node.getClass().getDeclaredField("color");
            return (Color)field.get(node);
        } catch (Exception e) {
            return Color.yellow;
        }
    }

    private int depth(BinaryTreeNode<?> node) {
        return (node.getParent() == null) ? 0 : 1 + depth(node.getParent());
    }
}
```