Physicist's Grimoire
Volume I: Mechanics

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1 Introduction

"Physics is not a collection of facts; it is a collection of stories."
-Brian Greene

2 Scalars, Vectors, and Fields

"Physics is mathematical not because we know so much about the physical world, but because we know so little; it is only its mathematical properties that we can discover."
-Bertrand Russell

Before beginning our study of physics, we must first acquaint ourselves with a tool that will prove to be instrumental in our work, vectors.
Some quantities, such as mass, time, and density can be described completely by a single number and a unit. However, many other important quantities in physics have a direction associated with them. We designate any quantity that is described only with a number as a **scalar**. which are written plainly as symbols: A. In contrast, a quantity described with both a **direction** and a **magnitude** (a description of the "size" or "largeness") is called a **vector**, which is written as a symbol with an arrow overhead: $\vec{A}$. While scalars can be operated on using ordinary arithmetic like numbers, vector calculations require a special set of rules. We can draw a vector as a line with an arrowhead at its tip, where the length of the line shows the magnitude of the vector, and the relative direction of the line shows the relative direction of the vector.

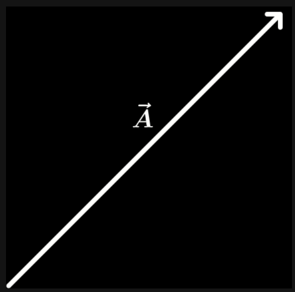

Fig. 2.1: The vector $\vec{A}$ in the plane.

The negative $-\vec{A}$ of any vector $\vec{A}$ will have the same magnitude but opposite direction of $\tilde{\mathbf{A}}$. We represent the magnitude of a vector $\vec{A}$ as A or alternatively $\left|\vec{A}\right|$.

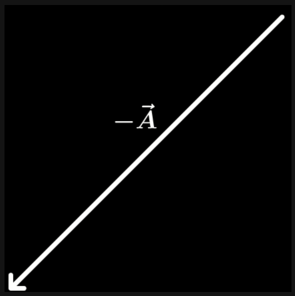

Fig. 2.2: The vector $-\vec{A}$ in the plane.

Note that while scalars may be positive or negative, the magnitude of a vector must always be positive or zero.

2.1 Vector Addition and Subtraction

The **vector sum** or **resultant** $\vec{C}$ of two vectors $\vec{A}$ and $\vec{B}$ is expressed as

$$\vec{C} = \vec{A} + \vec{B}$$

We bold the plus sign to indicate that adding two vector quantities is a different process than adding two scalar quantities. Adding vectors can be seen as a geometric process. In adding any two vectors $\vec{A}$ and $\vec{B}$ we

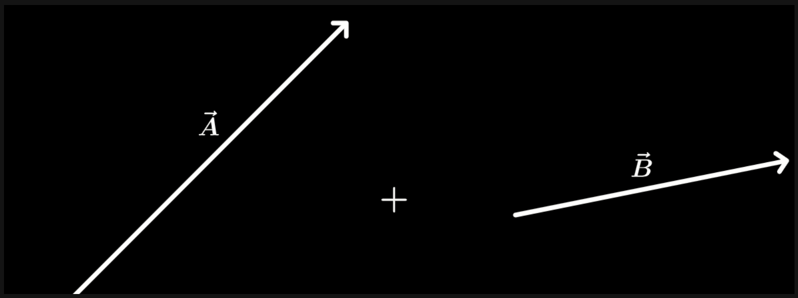

Fig. 2.3: Adding $\vec{A}$ and $\vec{B}$.

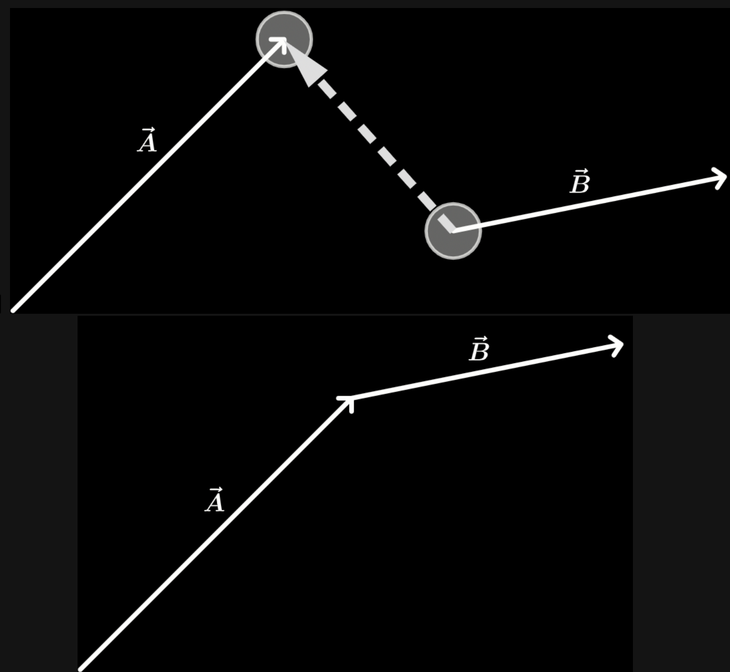

Fig. 2.4: Placing the tail of $\vec{B}$ at the tip of $\vec{A}$.

1. Place the tail of $\vec{B}$ at the at the tip of $\vec{A}$.

2. Draw the resultant vector $\vec{C}$ from the tail of $\vec{A}$ to the tip of $\vec{B}$

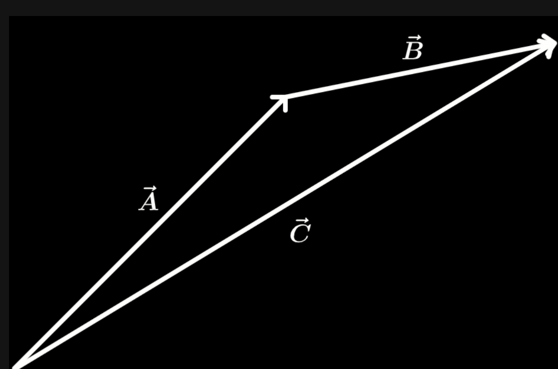

Fig. 2.5: $\vec{A}+\vec{B}=\vec{C}$

If we reverse the order that we add the vectors and place the tail of $\vec{A}$ at the tip of $\vec{B}$ instead, we find that the resultant vector is the same.

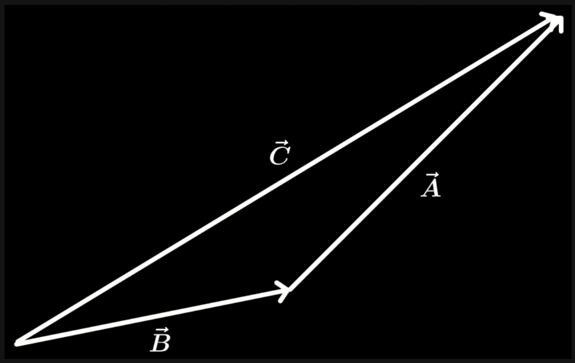

Fig. 2.6: $\vec{B}+\vec{A}=\vec{C}$

This shows that vector addition obeys the the commutative law, meaning that

$$\boxed{\vec{A}+\vec{B}=\vec{B}+\vec{A}} \tag{2.1}$$

Vector subtraction is a similar process, we can see that to subtract $\vec{B}$ from $\vec{A}$,

$$\boxed{\vec{A}-\vec{B}=\vec{A}+(-\vec{B})} \tag{2.2}$$

Geometrically, subtracting $\vec{B}$ from $\vec{A}$ entails reversing the direction of $\vec{B}$ and adding it to $\vec{A}$.

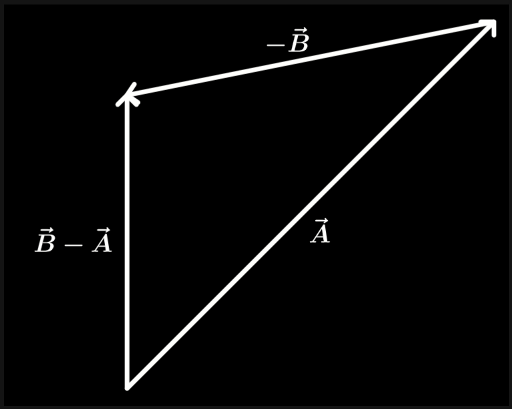

Fig. 2.7: $\vec{A}-\vec{B}$

Vectors can also be multiplied by scalars. When a vector $\vec{A}$ is multiplied by a scalar c, the result $c\vec{A}$ has magnitude $|c|A$. If c is positive, $c\vec{A}$ is in the same direction as $\vec{A}$. If c is negative, $c\vec{A}$ is in the direction opposite to $\vec{A}$.

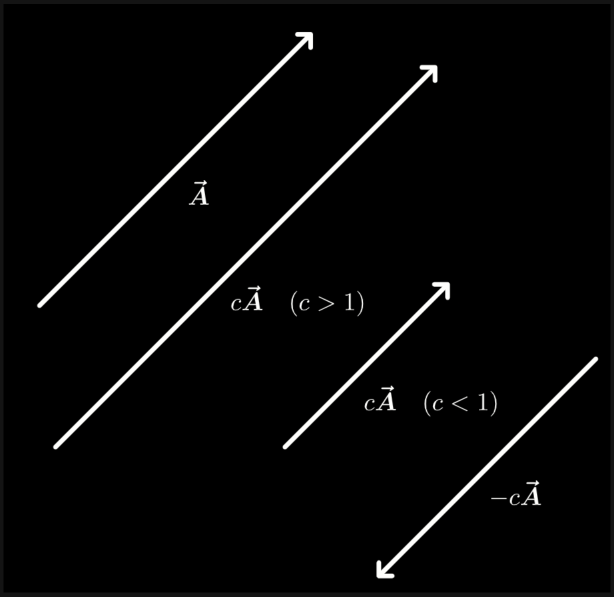

Fig. 2.8: Scalar multiplication of $\vec{A}$ by various values of c.

2.2 Components of Vectors

Suppose we have a vector $\vec{A}$ lying in the coordinate plane with its tail at the origin. We can represent any vector in the plane as the sum of a vector parallel to the x-axis and a vector parallel to the y-axis. We label the two vectors $\vec{A}_x$ and $\vec{A}_y$ and call them the **component vectors** of $\vec{A}$.

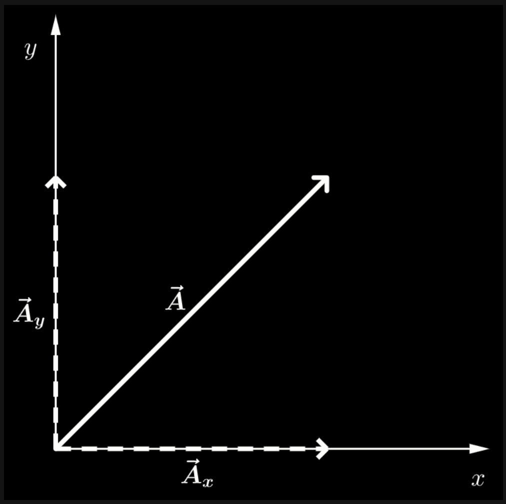

Fig. 2.9: $\vec{A}$ and its component vectors

The vector sum of $\vec{A}_x$ and $\vec{A}_y$ is $\vec{A}$

$$\vec{A}_x + \vec{A}_y = \vec{A}$$

The magnitudes of the component vectors $\vec{A}_x$ and $\vec{A}_y$ are the numbers A_x and A_y. We call A_x and A_y the **components** of A.

We can calculate the components of $\vec{A}$ if we know its magnitude and direction. The direction of the vector is described as its angle (θ) relative to some reference direction (the positive x-axis for now). From the definition of trigonometric functions,

$$\boxed{\frac{A_x}{A} = \cos(\theta) \quad \frac{A_y}{A} = \sin(\theta) \quad \tan\theta = \frac{A_y}{A_x}} \tag{2.3}$$

Note that these equations only hold when θ is measured from the positive x-axis.

2.3 Vector Calculations with Components

These are some useful ways components can be used in working with vectors. From the Pythagorean Theorem,

$$\boxed{A = \sqrt{{A_x}^2 + {A_y}^2}} \tag{2.4}$$

If the vector $\vec{D} = c\vec{A}$, each component of $\vec{D}$ is the is the product of c and the corresponding component of $\vec{A}$:

$$D_x = cA_x \quad D_y = cA_y$$

If $\vec{R} = \vec{A} + \vec{B}$, then

$$R_x = A_x + B_x \quad R_y = A_y + B_y$$

We have only discussed vectors in the two dimensional plane, however the components method holds for vectors in planes of any number of dimensions.

2.4 Unit Vectors

A **unit vector** is a vector with a magnitude of 1, with no units. The purpose of the unit vector is only to point to a direction in space. It offers a convenient notation for expressions involving components. We can define a unit vector $\hat{\imath}$ that points in the direction of the positive x-axis, a unit vector $\hat{\jmath}$ that points in the direction of the positive y-axis, and a unit vector $\hat{k}$ that points in the direction of the positive z-axis. We can then re-express the relationship between component vectors and components as

$$\boxed{\vec{A}_x = A_x\hat{\imath} \quad \vec{A}_y = A_y\hat{\jmath} \quad \vec{A}_z = A_z\hat{z}} \tag{2.5}$$

and a vector can be expressed as

$$\boxed{\vec{A} = A_x\hat{\imath} + A_y\hat{\jmath} + A_z\hat{k}} \tag{2.6}$$

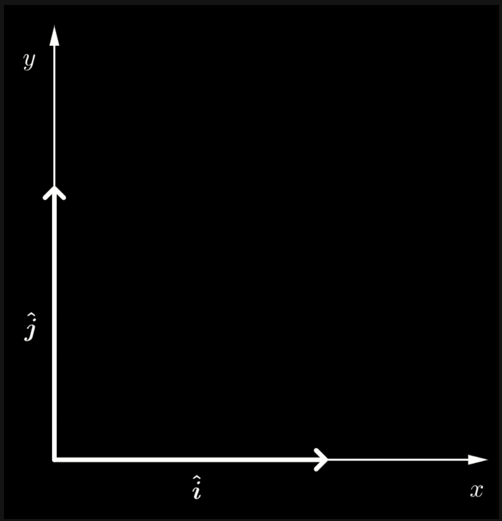

Fig. 2.10: The unit vectors $\vec{\imath}$ and $\vec{\jmath}$ in the plane.

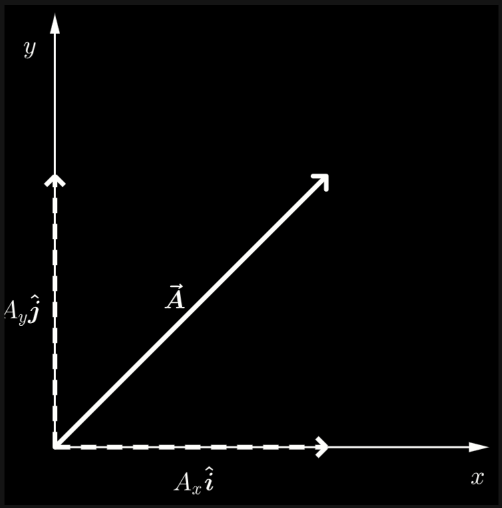

Fig. 2.11: $\vec{A}$ and its component vectors as scaled unit vectors.

If $\vec{R} = \vec{A} + \vec{B}$, then

$$\begin{aligned}\vec{R} &= \vec{A} + \vec{B} \\ &= A_x\hat{\imath} + A_y\hat{\jmath} + A_z\hat{k} + B_x\hat{\imath} + B_y\hat{\jmath} + B_z\hat{k} \\ &= (A_x + B_x)\hat{\imath} + (A_y + B_y)\hat{\jmath} + (A_z + B_z)\hat{k} \\ &= R_x\hat{\imath} + R_y\hat{\jmath} + R_z\hat{k}\end{aligned}$$

2.5 Products of Vectors

2.5.1 Scalar/Dot Product

The **scalar product** or **dot product** of two vectors $\vec{A}$ and $\vec{B}$ is denoted by $\vec{A} \cdot \vec{B}$. The quantity $\vec{A} \cdot \vec{B}$ is a scalar. The scalar product can be thought of the magnitude of $\vec{A}$ multiplied by the magnitude of $\vec{B}$ in the direction of $\vec{A}$. If the angle between the directions of the two vectors is ϕ, then

$$\boxed{\vec{A} \cdot \vec{B} = AB\cos(\phi)} \tag{2.7}$$

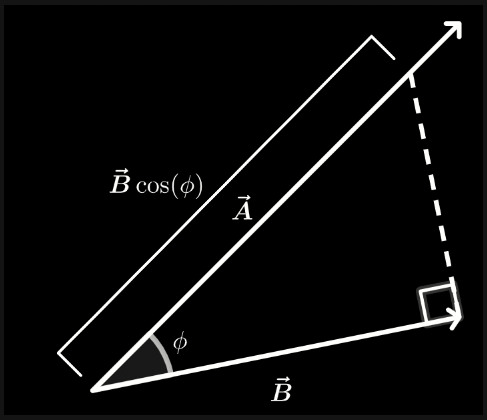

Fig. 2.12: The component of B in the direction of A

The dot product, being a scalar, may be positive, negative, or zero. The dot product obeys the commutative law of multiplication, meaning that

$$\vec{A} \cdot \vec{B} = \vec{B} \cdot \vec{A} \tag{2.8}$$

$$A(B\cos(\phi)) = B(A\cos(\phi)) \tag{2.9}$$

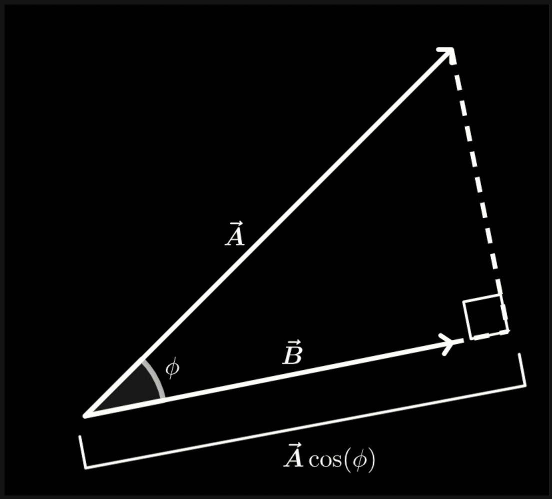

Fig. 2.13: The component of A in the direction of B

An interesting result can be found from expanding the multiplied vectors,

$$\begin{aligned}
\vec{A}\cdot\vec{B} &= (A_x\hat{\imath} + A_y\hat{\jmath} + A_z\hat{k}) \cdot (B_x\hat{\imath} + B_y\hat{\jmath} + B_z\hat{k}) \\
&= A_x\hat{\imath}\cdot B_x\hat{\imath} + A_x\hat{\imath}\cdot B_y\hat{\jmath} + A_x\hat{\imath}\cdot B_z\hat{k} \\
&+ A_y\hat{\jmath}\cdot B_x\hat{\imath} + A_y\hat{\jmath}\cdot B_y\hat{\jmath} + A_y\hat{\jmath}\cdot B_z\hat{k} \\
&+ A_z\hat{k}\cdot B_x\hat{\imath} + A_z\hat{k}\cdot B_y\hat{\jmath} + A_z\hat{k}\cdot B_z\hat{k} \\
&= A_xB_x\hat{\imath}\cdot\hat{\imath} + A_xB_y\hat{\imath}\cdot\hat{\jmath} + A_xB_z\hat{\imath}\cdot\hat{\jmath} \\
&+ A_yB_x\hat{\jmath}\cdot\hat{\imath} + A_yB_y\hat{\jmath}\cdot\hat{\jmath} + A_yB_z\hat{\jmath}\cdot\hat{k} \\
&+ A_zB_x\hat{k}\cdot\hat{\imath} + A_zB_y\hat{k}\cdot\hat{\jmath} + A_zB_z\hat{k}\cdot\hat{k}
\end{aligned}$$

It can be seen that

$$\begin{aligned}
\hat{\imath}\cdot\hat{\imath} = \hat{\jmath}\cdot\hat{\jmath} = \hat{k}\cdot\hat{k} = (1)(\cos(0)) = 1 \\
\hat{\imath}\cdot\hat{\jmath} = \hat{\imath}\cdot\hat{k} = \hat{\jmath}\cdot\hat{k} = (1)(1)\cos\left(\frac{\pi}{2}\right) = 0
\end{aligned}$$

Since six of the terms in our summation are zero, we simply have

$$\boxed{\vec{A}\cdot\vec{B} = A_xB_x + A_yB_y + A_zB_z} \tag{2.10}$$

Thus, the scalar product of two vectors is the sum of the products of their respective components.

2.5.2 Vector/Cross Product

The **vector product** or **cross product** of two vectors $\vec{A}$ and $\vec{B}$ is denoted by $\vec{A}\times\vec{B}$. The quantity $\vec{A}\times\vec{B}$ is a vector. The scalar product can be thought of the magnitude of $\vec{A}$ multiplied by the magnitude of $\vec{B}$ in the direction perpendicular to $\vec{A}$. If the angle between the directions of the two vectors is ϕ, then

$$\boxed{\vec{A}\times\vec{B} = AB\sin(\phi)} \tag{2.11}$$

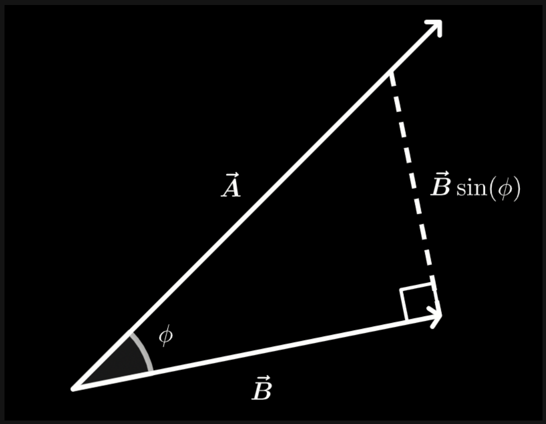

Fig. 2.14: The componen of B perpendicular to A.t

The cross product, being a vector, may be positive or negative but not zero. The direction of the cross product is perpendicular to the two vectors. To figure out which of the two perpendicular direction the cross product points, we use the right hand rule:

1. Point the fingers of your right hand in the direction of the first vector
2. Point the palm of your right hand in the direction of the second vector
3. Curl your ring and little fingers.
4. Stick your thumb out.

The direction of the cross product will be the direction your thumb points in.

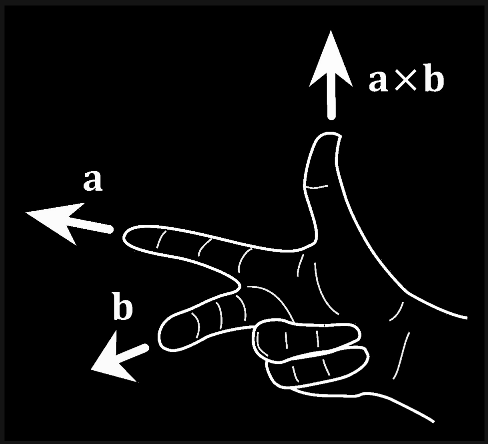

Fig. 2.15: The right hand rule.

Note that if you reverse the order that you multiply the vectors, the direction of the vector product will be reversed. Mathematically,

$$\vec{A} \times \vec{B} = -\vec{B} \times \vec{A} \tag{2.12}$$

$$A(B\sin(\phi)) = B(A\sin(\phi)) \tag{2.13}$$

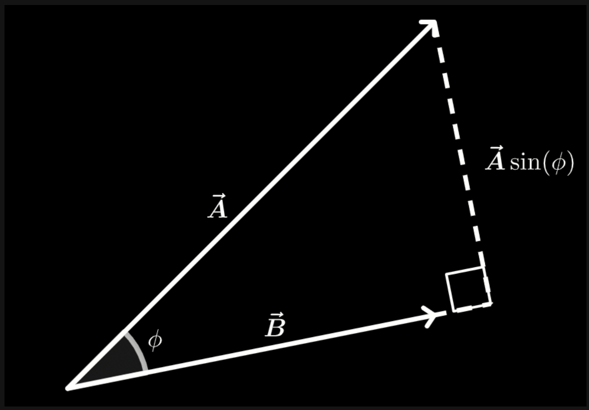

Fig. 2.16: The component of $\vec{A}$ perpendicular to $\vec{B}$.

This is to say that the cross product is not commutative, but rather anti-commutative.

As with the dot product, we can expand the vectors to yield a useful result

$$\begin{aligned}
\vec{A} \times \vec{B} &= (A_x\hat{\imath} + A_y\hat{\jmath} + A_z\hat{k}) \times (B_x\hat{\imath} + B_y\hat{\jmath} + B_z\hat{k}) \\
&= A_x\hat{\imath} \times B_x\hat{\imath} + A_x\hat{\imath} \times B_y\hat{\jmath} + A_x\hat{\imath} \times B_z\hat{k} \\
&+ A_y\hat{\jmath} \times B_x\hat{\imath} + A_y\hat{\jmath} \times B_y\hat{\jmath} + A_y\hat{\jmath} \times B_z\hat{k} \\
&+ A_z\hat{k} \times B_x\hat{\imath} + A_z\hat{k} \times B_y\hat{\jmath} + A_z\hat{k} \times B_z\hat{k} \\
&= A_xB_x\hat{\imath} \times \hat{\imath} + A_xB_y\hat{\imath} \times \hat{\jmath} + A_xB_z\hat{\imath} \times \hat{\jmath} \\
&+ A_yB_x\hat{\jmath} \times \hat{\imath} + A_yB_y\hat{\jmath} \times \hat{\jmath} + A_yB_z\hat{\jmath} \times \hat{k} \\
&+ A_zB_x\hat{k} \times \hat{\imath} + A_zB_y\hat{k} \times \hat{\jmath} + A_zB_z\hat{k} \times \hat{k}
\end{aligned}$$

It can be seen that

$$\hat{\imath} \times \hat{\imath} = \hat{\jmath} \times \hat{\jmath} = \hat{k} \times \hat{k} = (1)(\sin(0)) = \mathbf{0}$$

The zero has been bolded to indicate that the products result in a **zero vector**: a vector with all components equal to zero and an undefined direction. Using the right hand rule, we see that

$$
\begin{aligned}
\hat{i} \times \hat{j} = -\hat{j} \times \hat{i} = \hat{k} \\
\hat{j} \times \hat{k} = -\hat{k} \times \hat{j} = \hat{i} \\
\hat{i} \times \hat{k} = -\hat{k} \times \hat{i} = \hat{j}
\end{aligned}
$$

Rewriting, we are left with

$$\boxed{\vec{A} \times \vec{B} = (A_y B_z - A_z B_y)\hat{i} + (A_z B_x - A_x B_z)\hat{j} + (A_x B_y - A_y B_x)\hat{k}} \quad (2.14)$$

An axis system where $\hat{i}+\hat{j} = \hat{k}$ is called a **right-handed coordinate system**. An axis system where $\hat{i} + \hat{j} = -\hat{k}$ is called a **left-handed coordinate system**. We should only ever use right handed systems.

2.6 Fields

Fields will prove to be another important tool for our work. Suppose we have a vector-valued function such as

$$f(x, y) = x + y$$

A **vector field** is a quantity that has assigns a vector to each point in space and time.

3 Introduction to Dynamics

We begin our study of mechanics with dynamics. For years, the study of dynamics was synonymous with the study of mechanics, though mechanics has come to encompass a larger collection of branches. Dynamics is the branch of mechanics concerned with describing the motion of objects influenced by forces. We are left then, to define forces.

A brief note on convention: For now, we will ignore the size, shape, and structure of objects - though it will become important in later chapters. Instead, we will pick one infinitesimally small point on the object (a particle) and represent the entire object as that point.

3.0.1 Force

Few other physical quantities are as central and wide reaching to physics as force. Force can be described as a "push" or "pull" between objects. For example, every time you pull a door open, you are exerting a force on the door handle. Forces do not necessitate physical contact either and can be mediated through empty space: the Earth pulls objects towards it with a gravitational force, whether they are in contact with it or not.

Suppose a force is exerted on an object. The force acting on the object can be visualized as follows:

Forces, being vector quantities, can be summed to a **net force**:

$$\sum \vec{F} = \vec{F}_1 + \vec{F}_2 + ... + \vec{F}_n$$

3.1 Newton's First Law

3.1.1 Development of Newton's First Law[H]

The Western physical tradition is often traced back to Aristotle. Aristotle first hypothesized two types of motion: "natural" - downwards motion - and "violent" - sideways motion - which alludes to the tendency for objects to fall directly downwards. According to Aristotelian physics, violent motion is caused by some immediate action, whereupon the actions fails to act, the object assumes natural motion, which is directed by the air. This is the foundation of force. About 900 years later, a Greek thinker, Philoponus, argues that violent motion imparts an "impetus" into the object, which is carried by the body itself. This idea will prove to serve as the foundation for the quantity "momentum", which will be discussed in a later chapter.

In 1638, Galileo Galilei published his *Two New Sciences*, where he stated:

> Imagine any particle projected along a horizontal plane without friction; then we know, from what has been more fully explained in the preceding pages, that this particle will move along this same plane with a motion which is uniform and perpetual, provided the plane has no limits.

Galileo, however, mistakenly thought that such a particle would follow the curvature of the Earth. Descartes corrected this misconception in his *Principles of Philosophy*

> First Law of Nature: Each thing when left to itself continues in the same state; so any moving body goes on moving until something stops it.
> Second Law of Nature: Each moving thing if left to itself moves in a straight line; so any body moving in a circle always tends to move away from the centre of the circle.

Newton then wrote these laws in the common condensed form, for whom it is named.

3.1.2 Newton's First Law

In conventional English, we now state Newton's First Law:

> Every body continues in its state of rest, or of uniform motion in a straight line, unless it is compelled to change that state by forces impressed on it.

It follows from this definition that once a body has been set in motion it is no longer necessary to exert a force on it to maintain it in motion.

This is may seem contrary to everyday experience. If one were to move a book resting on a table, they may exert a force on it, but it would shortly come to rest after the influence of the force is removed. It appears as though Newton's First Law has been violated - the book has not maintained it's uniform motion!; however, the cessation of the motion of the book is due to the influence of a frictional force applied by the table.

This definition lacks precision. What exactly is meant by "uniform motion" is unclear. Like thinkers of the past, we will come back to this definition to refine it at a later time.

4 Kinematics

"The primary and most beautiful of Nature's qualities is motion, which agitates her at all times." - Marquis de Sade

We begin our study of mechanics with kinematics, which is the branch of mechanics concerned with describing motion. For now, we will ignore the size, shape, and structure of objects. Instead, we will pick one infinitesimally small point on the object (a particle) and represent the entire object as that point.

4.1 Kinematics Quantities

4.1.1 Displacement

We can define the **position** ($\vec{r}$) of a particle in three dimensions by

$$\boxed{\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}} \tag{4.1}$$

If a particle were to move from position $\vec{r_1}$ at time t_1 to position $\vec{r_2}$ at time t_2, we define the change in position of the particle as its displacement ($\Delta\vec{r}$).

$$\boxed{\Delta\vec{r} = \vec{r_1} - \vec{r_2}} \tag{4.2}$$

Notice that the displacement differs from the total distance travelled by the object, it only depends on the length of a straight line between two objects, and is independent of the actual path taken between, which we will refer to as distance.

Example 4.1. *Suppose a car travels in a full circle of radius r in time Δt. Find both the displacement and distance traveled by the car.*

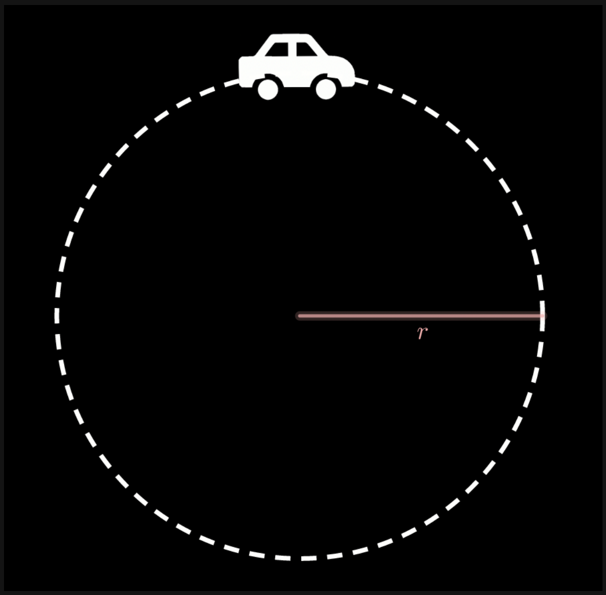

Fig. 4.1: Example 3.1

Solution 1. *To find the displacement, we will use 4.2:*

$$\Delta\vec{r} = \vec{r_1} - \vec{r_2} = \boxed{0\text{m}}$$

The distance is the total length of the path travelled by the car. This length is equal to the circumference of the circle.

$$\textit{Distance travelled} = \textit{circumference of path} = \boxed{\pi r^2}$$

As displacement and distance are descriptions of length, the SI units of both quantities is meters [m].

4.2 Velocity

4.2.1 Average Velocity

We define average velocity ($\vec{v}_{av}$) to be the displacement of an object divided by the time interval in which the displacement happens,

$$\boxed{\vec{v}_{av} = \frac{\Delta r}{\Delta t} = \frac{\vec{r_2} - \vec{r_1}}{t_2 - t_1}} \tag{4.3}$$

Graphically, the average velocity can be interpreted as the slope of a position vs. time graph. Let's just look at the x-components of the displacement and average velocity: $\Delta\vec{x}$ and $\vec{v}_{av,x}$

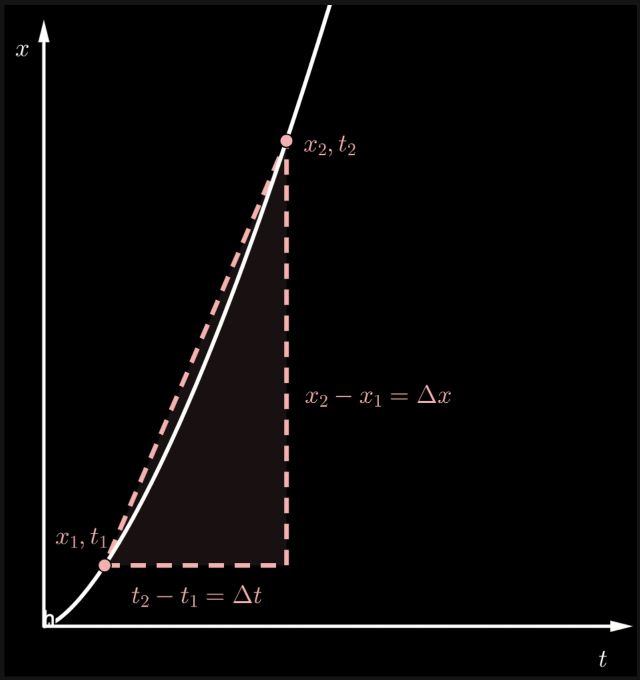

Fig. 4.2: $v_{av,x} =$ the slope of the $x vs. t$ graph

The scalar analogue of velocity is speed.

$$\text{av speed} = \frac{\text{total distance}}{\Delta t} \tag{4.4}$$

Example 4.2. *Find both the average velocity and average speed of the car in 4.1.*

Solution 2. *We can find the average velocity with 4.3:*

$$\begin{aligned} \vec{v}_{av} = \frac{\Delta r}{\Delta t} &= \frac{r_2 - r_1}{\Delta t} \\ &= \frac{0 - 0}{\Delta t} \\ &= \boxed{0\text{m/s}} \end{aligned}$$

To find the average speed, we divide the total distance travelled that we found in 4.1 by the time interval:

$$\textit{av speed} = \frac{\textit{total distance}}{\Delta t} = \frac{\pi r^2}{\Delta t}$$

The SI units of both velocity and speed is meters/second [m/s]

4.2.2 Instantaneous Velocity (C)

Often, it is more useful to find the velocity at a point rather than over an interval. This leads us to the methods of differential calculus. To begin, let's rewrite the point (x_2, t_2) as $(x_1 + \Delta x, t_1 + \Delta t)$.

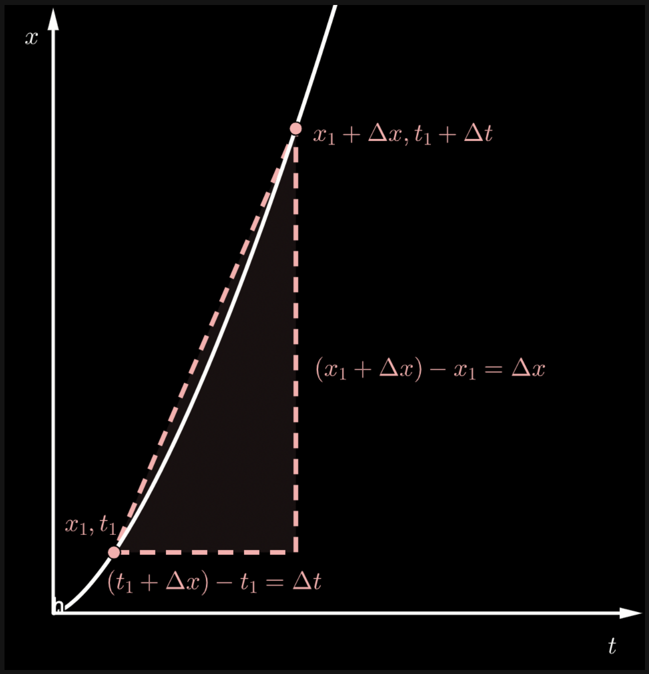

Fig. 4.3: Rewriting (x_2, t_2) as $(x_1 + \Delta x, t_1 + \Delta t)$

Then, let Δx and Δt get smaller and smaller until they approach 0.

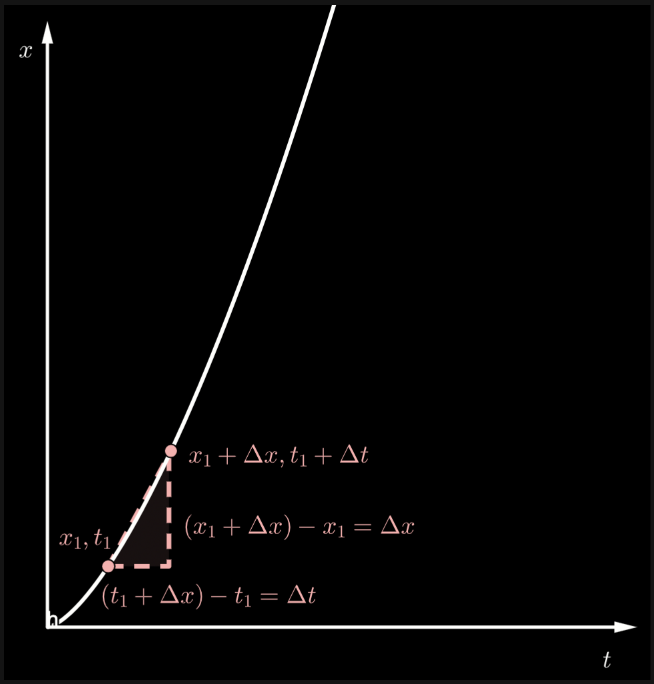

Fig. 4.4: Δx and Δt approach 0.

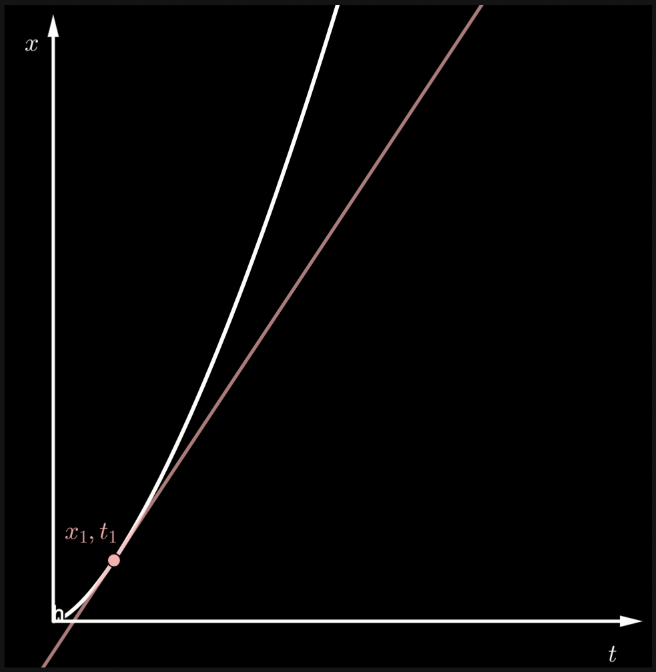

Fig. 4.5: The velocity at (x_1, t_1) is the slope of the tangent line at that point.

We thus define the instantaneous velocity ($\vec{v}$) as

$$\boxed{\vec{v} = \lim_{\Delta t \to 0} \frac{\Delta \vec{r}}{\Delta t} = \frac{d\vec{r}}{dt}} \tag{4.5}$$

We note that the direction of the velocity is indicated by the direction of the displacement. In other words, velocity always points in the same direction as displacement.

4.3 Acceleration

4.3.1 Average Acceleration

Similar to the relationship between displacement and velocity, we define average acceleration ($\vec{a}_{av}$) as

$$\boxed{\vec{a}_{av} = \frac{\vec{v}_2 - \vec{v}_1}{t_2 - t_1} = \frac{\Delta v}{\Delta t}} \tag{4.6}$$

4.3.2 Instantaneous Acceleration (C)

and make the link to instantaneous acceleration as follows ($\vec{a}$)

$$\boxed{\vec{a} = \lim_{\Delta t \to 0} \frac{\Delta \vec{v}}{\Delta t} = \frac{d\vec{v}}{dt} = \frac{d^2\vec{r}}{dt^2}} \tag{4.7}$$

Unlike $\vec{v}$ and $\vec{r}$, $\vec{v}$ and $\vec{a}$ do not have to point in the same direction and can point at any angle relative to one another. In fact, any time an object is not moving in a straight line, there is a component of $\vec{a}$ that points perpendicular to $\vec{v}$.

Average acceleration can also be interpreted as the slope of a velocity vs time graph.

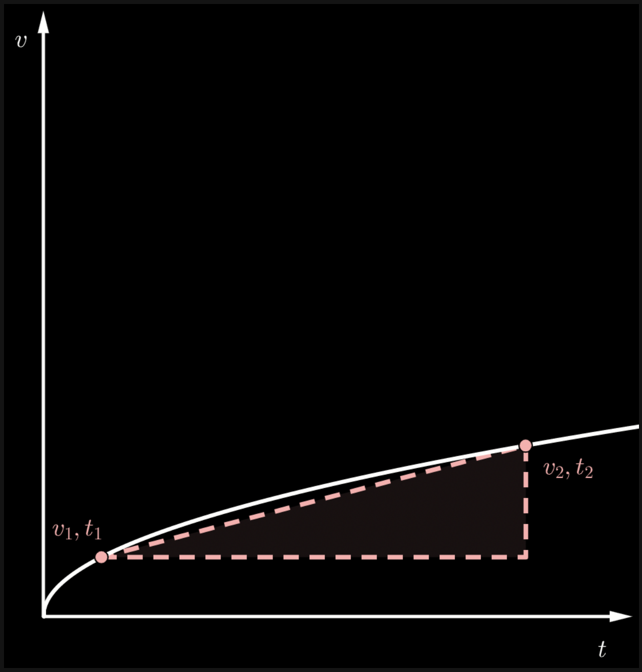

Fig. 4.6: Average Acceleration

Similarly, instantaneous acceleration at a point is the slope of the tangent line of a velocity vs time graph at that point.

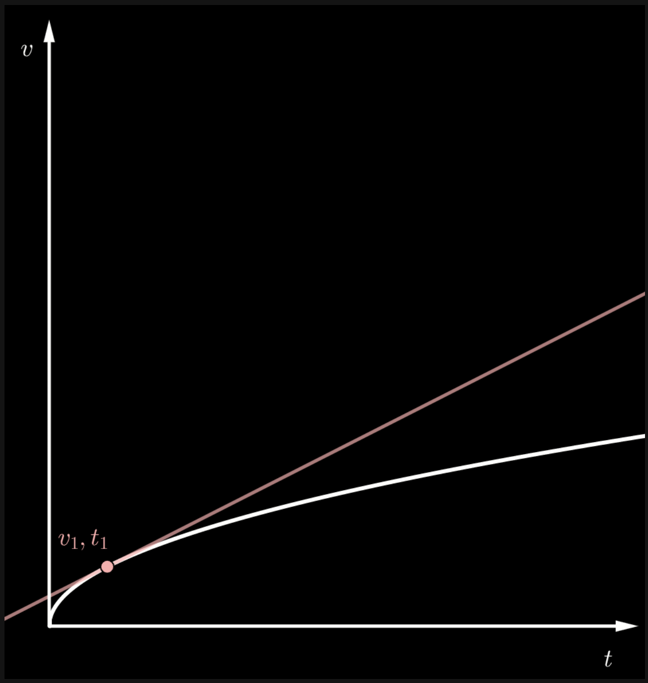

Fig. 4.7: Instantaneous Acceleration

4.4 Constant Acceleration Kinematics

Often, we will face situations where the acceleration is constant. We can derive relationships between the 5 kinematics quantities (Δt, Δr, $\vec{v_0}$, $\vec{v_1}$, $\vec{a}$ we've studied thus far. We call these relationships the constant acceleration equations. They are only valid when the acceleration is constant.

4.5 Free Fall

4.5.1 Projectile Motion

4.6 Circular Motion