Constraints and Lagrangian Multipliers

constraint $g(q,t) = 0$

<u>Ex.</u> Surface of a Sphere

$$g = x^2 + y^2 + z^2 - R^2 = 0$$

$$J[q_1, q_2] = \int L[q_1, q_2, \dot{q}_1, \dot{q}_2, t]\,dt$$
$$g(q_1, q_2) = 0$$

consider:

$$S[q_1, q_2, \lambda] = \int \left[L[\ldots] + \lambda(t)\, g(q_1, q_2) \right] dt$$

and q_1, q_2, λ as independant

$$\frac{d}{dt}\frac{\partial S}{\partial \dot{q}_1} - \frac{\partial S}{\partial q_1} = \frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q_1} - \lambda \frac{\partial g}{\partial q_1} = 0$$

$$\frac{d}{dt}\frac{\partial S}{\partial \dot{q}_2} - \frac{\partial S}{\partial q_2} = \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_2} - \frac{\partial L}{\partial q_2} - \lambda \frac{\partial g}{\partial q_2} = 0$$

$$\frac{d}{dt}\frac{\partial S}{\partial \dot{\lambda}} - \frac{\partial S}{\partial \lambda} = \underbrace{-g = 0}_{\text{constraint eqn}}$$

In general, n variables, m constraints

$$\boxed{\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} - \sum_{j=1}^{m} \lambda_j \frac{\partial g}{\partial q_i} = 0}$$

$q_j = (q_1, \ldots, q_n, t)$

$i = 1, \ldots, n$

$j = 1, \ldots, m$

Several Variables

$$\mathbf{q} = (q_1, \dots, q_n) \xrightarrow{\text{functional}} J[\mathbf{q}] = \int_{t_1}^{t_2} L(\mathbf{q}, \dot{\mathbf{q}}, t)\, dt$$

path variations:

$$\boldsymbol{\eta} = (\eta_1, \dots, \eta_n) \qquad \mathbf{q}(t, \varepsilon) = \mathbf{q}(t, 0) + \varepsilon \boldsymbol{\eta}(t)$$

$$\boldsymbol{\eta}(t_1) = \boldsymbol{\eta}(t_2) = 0$$

$$\delta J = \int_{t_1}^{t_2} \sum_{k=1}^{n} \left(\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_k} - \frac{\partial L}{\partial q_k} \right) \eta_k \, dt \overset{!}{=} 0$$

$\delta J = 0$ for any $\boldsymbol{\eta}$, in particular $\boldsymbol{\eta}(\eta, 0, 0, \dots, 0)$

$$\boxed{\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_k} - \frac{\partial L}{\partial q_k} = 0} \qquad k = 1, \dots, n \qquad L = L(q_1, \dots, q_n, \dot{q}_1, \dots, \dot{q}_n, t)$$

<u>Ex.</u> Geodesic in the Plane

$$J = \int ds = \int \sqrt{dx^2 + dy^2} = \int_{t_1}^{t_2} \underbrace{\sqrt{\dot{x}^2 + \dot{y}^2}}_{L[\dot{x}(t), \dot{y}(t)]} \, dt$$

$$\frac{\partial L}{\partial \dot{x}} = \frac{\dot{x}}{\sqrt{\dot{x}^2 + \dot{y}^2}} = c_x \qquad \frac{\dot{x}}{c_x} = \frac{\dot{y}}{c_y}$$

$$\frac{\partial L}{\partial \dot{y}} = \frac{\dot{y}}{\sqrt{\dot{x}^2 + \dot{y}^2}} = c_y$$

$$\int_{t_1}^{t_2} \longmapsto \frac{x - x_1}{c_x} = \frac{y - y_1}{c_y} \longrightarrow y = Ax + B$$

$$\int_{t_1}^{t_2} \longmapsto \frac{x_2 - x_1}{c_x} = \frac{y_2 - y_1}{c}$$

divide

line

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} \longrightarrow \boxed{y = y_1 + \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)}$$

Invariance of Euler-Lagrange

under coordinate changes

$(q(t), t) \longmapsto (y(x), x)$

$$J[q] = \int_{t_1}^{t_2} L[q, \dot{q}, t]\, dt \longmapsto J[y] = \int_{x_1}^{x_2} F[y, y', x]\, dx$$

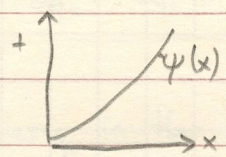
$$\frac{\partial}{\partial t}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = 0 \quad \longleftarrow ? \longrightarrow \quad \frac{d}{dx}\frac{\partial F}{\partial y'} - \frac{\partial F}{\partial y} = 0$$

$$\left(\frac{d}{dx}\frac{\partial F}{\partial y'} - \frac{\partial F}{\partial y}\right) = \left(\frac{d}{dt}\frac{dL}{d\dot{q}} - \frac{dL}{dq}\right)\frac{\partial(t,q)}{\partial(x,y)}$$

Ex.

$t = \psi(x)$

$q = y$



$\frac{\partial \psi}{\partial x} = \psi' > 0$

$$\frac{\partial(t,q)}{\partial(x,y)} = \det\begin{pmatrix} \frac{\partial t}{\partial x} & \frac{\partial q}{\partial x} \\ \frac{\partial t}{\partial y} & \frac{\partial q}{\partial y} \end{pmatrix} = \begin{vmatrix} \psi' & 0 \\ 0 & 1 \end{vmatrix} = \psi'$$

$$dt = \psi'(x)\, dx \longmapsto \frac{d}{dt} = \frac{1}{\psi'}\frac{d}{dx}$$

$$\dot{q} = \frac{dq}{dt} = \frac{dy}{dt} = \frac{1}{\psi'}\frac{dy}{dx} = \frac{y'(x)}{\psi'(x)}$$

$$J = \int L[q, \dot{q}, t]\, dt = \int \underbrace{L\left[y, \frac{y'}{\psi'}, \psi(x)\right]\psi'}_{F[y, y', x]}\, dx$$

$$\frac{\partial F}{\partial y} = \left[\frac{\partial L}{\partial q}\frac{\partial y}{\partial y} + \frac{\partial L}{\partial \dot{q}}\underbrace{\frac{\partial}{\partial y}\left(\frac{y'}{\psi'(x)}\right)}_{0} + \frac{\partial L}{\partial t}\underbrace{\frac{\partial \psi}{\partial y}}_{0}\right]\psi' + L\underbrace{\frac{\partial \psi'}{\partial y}}_{0} = \frac{\partial L}{\partial q}\psi'$$

$$\frac{\partial F}{\partial y'} = \left[\frac{\partial L}{\partial \dot{q}}\frac{\partial}{\partial y'}\left[\frac{y'}{\psi'}\right]\right]\psi' = \frac{\partial L}{\partial \dot{q}}$$

$$\frac{d}{dx}\frac{\partial F}{\partial y'} = \psi'\frac{d}{dt}\frac{\partial F}{\partial y'} = \psi'\frac{d}{dt}\frac{\partial L}{\partial \dot{q}}$$

$$\left(\frac{d}{dx}\frac{\partial F}{\partial y'} - \frac{\partial F}{\partial y}\right) = \left(\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q}\right)\psi'$$

Intro to Lagrangian Mechanics

A) Generalized coordinates and configuration space

- N particles → 3N coordinates $\quad \mathbb{r}_i \in \mathbb{R}^3 \quad i=1,\dots,N$
- Rigid bodies → M angles θ_i
- K constraints of form $g(\mathbb{r}, \theta, t) = 0$

↳ degrees of freedom: $\quad n = (3N + M) - K$

→ can introduce generalized coordinates $q = (q_1, \dots, q_N)$

→ n-linearly independant coordinates q_i

→ q_i are axes of n-dim configuration space

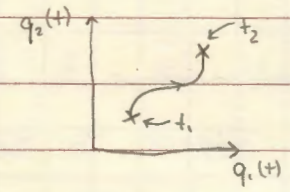
→ There is a mapping:

$$\vec{\mathbb{r}}_1 = \vec{\mathbb{r}}_1(q_1, \dots, q_n; t) \qquad \theta_1 = \theta_1(q_1, \dots, q_n; t)$$
$$\vdots \qquad\qquad\qquad \vdots$$
$$\vec{\mathbb{r}}_N = \vec{\mathbb{r}}_N(q_1, \dots, q_n; t) \qquad \theta_M = \theta_M(q_1, \dots, q_n; t)$$

→ in configuration space

point = state of mech system

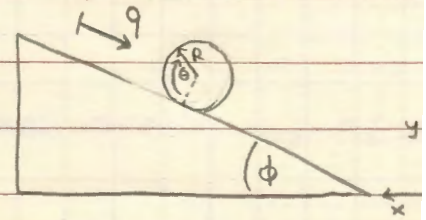
path = evolution of a system



Ex. Barrel Rolling Down an Inclined Plane

Coordinates: • center-of-motion: x, y, z

• rotation about CM: θ

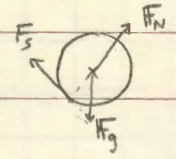


constraints: 1) $z=0$ $\quad$ 2) $y = x\tan\phi$ $\quad$ 3) $v_{cm} = R\dot{\theta}$

DoF: $4-3=1$. Pick q as generalized coordinate

$$\left.\begin{aligned} x &= x_0 - q\cos\phi & z &= 0 \\ y &= y_0 - q\sin\phi & \theta &= \frac{q}{R} \end{aligned}\right\} \quad q \to x, y, z, \theta$$

Solve Using Newton



CM: $\quad m\ddot{q} = mg\sin\phi - F_s \qquad F_s = mR\ddot{\theta} = m\ddot{q}$

rotation: $\quad (mR^2)\ddot{\theta} = RF_s$

↳ $2m\ddot{q} = mg\sin\phi \implies \boxed{\ddot{q} = \frac{g}{2}\sin\phi} \qquad q(t) = \frac{g}{4}\sin\phi\, t^2$

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B) Hamilton's principle

Assumptions

1) All forces that are not forces of constraint are conservative, i.e. derivable from a potential U that depends on q only.

$$U = U(q) \quad [\mathbb{F} = -\nabla U]$$

Generalizable to forces derivable from $U = U(q, \dot{q}, t)$. (includes friction)

2) Constraints are <u>holonomic</u>, i.e.

$$g(\mathbf{r}, t) = 0$$

eq. of constraint only depends on coord.

<u>Theorem</u>: Hamilton's Principle

The motion of the system from time t_1 to time t_2 is such that the line integral [the <u>action</u>]

$$J = \int_{t_1}^{t_2} L[q_1, \ldots, q_n, \dot{q}_1, \ldots, \dot{q}_n; t]\, dt$$

has a stationary value for the actual path of motion. Here L is the <u>Lagrangian</u>, which is the difference of kinetic and potential energy:

$$L[q, \dot{q}, t] = T[q, \dot{q}, t] - U(q)$$

<u>Corollary</u>:

Given you work with n-linearly independant variables $q_1, \ldots, q_n$ (n = DoF)

$$\boxed{\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0} \qquad i = 1, \ldots, n$$

Ex. (same as before)

$$T = \frac{1}{2}m\dot{q}^2 + \frac{1}{2}mR^2\dot{\theta}^2 = m\dot{q}^2 \qquad U = mgy = mgy_0 - mgq\sin\phi$$

$$L = m\dot{q}^2 - mgy_0 + mgq\sin\phi$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = 2m\ddot{q} - mg\sin\phi \rightarrow \boxed{\ddot{q} = \frac{g}{2}\sin\phi}$$

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Hamilton's Principle

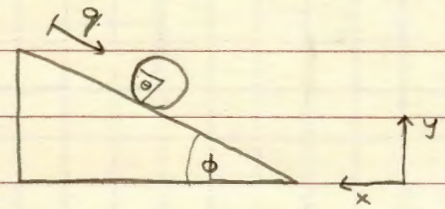
$$J = \int_{t_1}^{t_2} L[q(t), \dot{q}(t), t]\,dt \qquad L = T - U$$

Method 1 Generalized Coordinates

$$q = (q_1, q_2, q_3, \ldots, q_n) \qquad \text{linearly independant}$$

$$\frac{\partial}{\partial t}\frac{\partial L}{\partial \dot{q}_k} - \frac{\partial L}{\partial q_k} = 0 \qquad k = 1, 2, \ldots, n$$

Ex. Barrel on Incline (with Method 1)



- constraints: $y = x\tan\phi$; $\dot{q} = R\dot{\theta} \rightarrow q = R\theta$
- gen. coordinate: q
- mapping $q \rightarrow x, y, \theta$

$$\left.\begin{aligned} x &= x_0 - q\cos\phi \\ y &= y_0 - q\sin\phi \end{aligned}\right\} \dot{x}^2 + \dot{y}^2 = \dot{q}^2$$

$$R\theta = q \rightarrow R^2\dot{\theta}^2 = \dot{q}^2$$

- energies: $L = T - U$

$$= \frac{m}{2}(\dot{x}^2 + \dot{y}^2) + \frac{mR^2}{2}\dot{\theta}^2 - mgy$$

$$= m\dot{q}^2 - mgy_0 + mgq\sin\phi$$

$$0 = \frac{d}{dt}(2m\dot{q}) + mg\sin\phi = 2m\ddot{q} - mg\sin\phi \longmapsto \ddot{q} = \frac{g}{2}\sin\phi$$

Method 2 Undetermined Multiplier

- picked coordinate: $q_1, \ldots, q_m$
- left k holonomic constraints $\quad g_i(q_1, \ldots, q_n) = 0$

$$g(q,t) \rightarrow \frac{\partial g}{\partial t} = \sum_{i=1}^{m} \frac{\partial g}{\partial q_i}\frac{\partial q_i}{\partial t} + \frac{\partial g}{\partial t}$$

$$= \sum_i A(q,t)\dot{q}_i + B(q,t)$$

Ex. $g = q - R\theta$

$$\frac{\partial g}{\partial t} = \underset{A_1}{1} \cdot \underset{q_1}{q} + \underset{A_2}{(-R)}\underset{q_2}{\dot{\theta}}$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} - \sum_{j=1}^{k} \lambda_j \frac{\partial g}{\partial q_i} = 0$$

$$g_j(q_1, \ldots, q_n) = 0 \quad i = 1 \ldots m \quad j = 1 \ldots k$$

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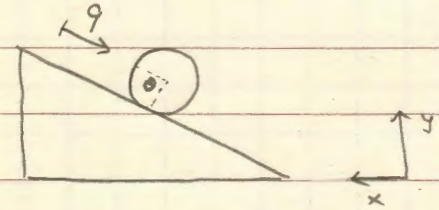
Ex. Barrel on Incline (with Method 2)

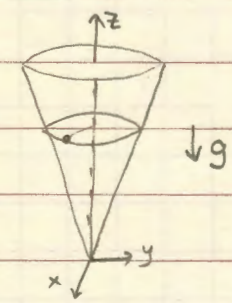


- coord.: $q_1 \Leftrightarrow q$; $q_2 \Leftrightarrow \theta$
- constraint: $g = q - R\theta = 0$

$$L = T - U$$
$$= \left(\frac{m}{2}\dot{q}^2 + \frac{mR^2}{2}\dot{\theta}\right) - (mgy_0 - mgq\sin\phi)$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} - \lambda\frac{\partial g}{\partial q} = m\ddot{q} - mg\sin\phi - \lambda\cdot 1 = 0$$
$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} - \lambda\frac{\partial g}{\partial \theta} = mR^2\ddot{\theta} - \lambda(-R) = 0$$
$$q - R\theta = 0$$
$$m\ddot{q} - mg\sin\phi + m\ddot{q} = 0$$
$$\ddot{q} = \frac{g}{2}\sin\phi$$

Ex Bead in Cone



- symmetry → cyl. coor. ρ, ϕ, z
- constraint: $\frac{\rho}{z} = \tan\alpha \mapsto \dot{z} = \dot{\rho}\frac{\cos\alpha}{\sin\alpha}$
- gen. coord.: $q_1 \Leftrightarrow \rho$, $q_2 \Leftrightarrow \phi$

$$T = \frac{m}{2}v^2 = \frac{m}{2}\left(\dot{\rho}^2 + \rho^2\dot{\phi}^2 + \dot{z}^2\right)$$
$$= \frac{m}{2}\left(\dot{\rho}^2 + \rho^2\dot{\phi}^2 + \dot{\rho}^2\frac{\cos^2\alpha}{\sin^2\alpha}\right)$$
$$= \frac{m}{2}\left(\frac{\dot{\rho}}{\sin^2\alpha} + \rho^2\dot{\phi}^2\right)$$

$$U = mgz = mg\rho\frac{\cos\alpha}{\sin\alpha}$$

$$L = T - U$$
$$= \frac{m}{\sin^2\alpha}\left(\frac{\dot{\rho}^2}{2} + \frac{\rho^2\dot{\phi}^2}{2}\sin^2\alpha\right) - mg\rho\frac{\cos\alpha}{\sin\alpha}$$

$$0 = \frac{d}{dt}\frac{\partial L}{\partial \dot{\rho}} - \frac{\partial L}{\partial \rho} = \frac{d}{dt}\left(\frac{m}{\sin^2\alpha}\dot{\rho}\right) - \left(\frac{m}{\sin^2\alpha}\rho\dot{\phi}\sin^2\alpha - mg\frac{\cos\alpha\sin\alpha}{\sin^2\alpha}\right)$$
$$= \frac{m}{\sin^2\alpha}\left(\ddot{\rho} - \rho\dot{\phi}^2\sin^2\alpha + g\cos\alpha\sin\alpha\right) \qquad (1)$$

$$0 = \frac{d}{dt}\frac{\partial L}{\partial \dot{\phi}} - \frac{\partial L}{\partial \phi}, \quad \text{constant} = \frac{\partial L}{\partial \dot{\phi}} = m\rho^2\dot{\phi} \qquad (2)$$

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(2) Angular Momentum

$$\ell = \overset{\text{rho}}{\rho} \cdot \overset{\text{"Pee"}}{p} = \overset{\text{rho}}{\dot{\rho}} \cdot m\rho\dot{\phi} = m\rho^2\dot{\phi} = \text{const.}$$

$$\hookrightarrow \dot{\phi}^2 = \frac{\ell^2}{m^2\rho^4}$$

(1) $$0 = \ddot{\rho} - \frac{\ell^2 \sin^2\alpha}{m^2\rho^3} + g\cos\alpha\sin\alpha$$

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Conservative Holonomic Systems

① Constraints? (Holonomic?)

② Coordinate Choice

- Symmetry (spherical / cylindrical)
- can you eliminate constraints and generalized coordinate q?
- can you find q such that L depends on $\dot{q}_i$ but not q_i.
- Pick q s.t. $q = 0$ at an equilibrium

③ Energies

- Kinetic: write T in cartesian coordinates const.

$$T = \frac{1}{2} m \sum_{i=1}^{n} (\dot{x}_i^2 + \dot{y}_i^2 + \dot{z}_i)$$

$$T = \frac{1}{2} I \dot{\theta}_i^2$$

- Use $q \mapsto T$ map

a) Static Trans $\quad x = x(q), \quad y = y(q), \ldots$

$$\dot{x} = \frac{\partial x}{\partial t} = \sum_{k=1}^{n} \frac{\partial x}{\partial q_k} \dot{q}_k$$

$$T = T(q, \dot{q}) = \sum_{j,k} A_{jk}(q) \dot{q}_j \dot{q}_k$$

quadratic velocities

b) Time dep Trans

$$x = x(q, t)$$

$$\dot{x} = \sum_{k=1}^{n} \frac{\partial x}{\partial q_k} \dot{q}_k + \frac{\partial x}{\partial t} \longrightarrow T = T(q, \dot{q}; t)$$

④ $L = T - U$

- Simplify: const $\longrightarrow C$, $F(t) \longrightarrow$, $\frac{d}{dt} F(q,t) \longrightarrow$ differential

$$J = \int_{t_1}^{t_2} \underbrace{\left[L + C + f(t) + \frac{d}{dt} F(q, t)\right]}_{\tilde{L}} dt \qquad \dot{\phi} = f$$

$$= \underbrace{\left[\int_{t_1}^{t_2} L\, dt\right]}_{\delta J = \delta \dot{J}} + C(t_2 - t_1) + \phi(t_2) - \phi(t_1) + F(q, t)\Big|_{q_1 t_1}^{q_2 t_2}$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = \frac{\partial}{\partial t}\frac{\partial \tilde{L}}{\partial \dot{q}} - \frac{\partial \tilde{L}}{\partial q}$$

⑤ Euler-Lagrange

a) cylindrical: $\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0$

b) multipliers: $\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} - \sum_j \lambda_j \frac{\partial g_j}{\partial q_i} = 0 \qquad g_j(q, t) = 0$

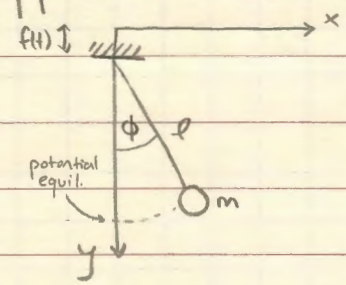
- $L = L[q_1, \dot{q}_1, \dot{q}_2]$ ← no t dep

no q_2 dep and q_2 is lin indep. $\rightarrow \frac{\partial L}{\partial \dot{q}_2} = \text{const.}$ "generalized momentum p_2"

q_2 = cyclical coor $\rightarrow p_2$ = const

EXAMPLE

Ex Pendulum with Vertically Oscillating Support



constraints: $x^2 + (y + f(t))^2 = \ell^2$

symmetries: 2D plane, circular sym.

↳ gen coor. $q = \phi$

map: $\phi \to (x, y)$

$$x = \ell \sin\phi \quad , \quad y = f(t) + \ell\cos\phi$$

energies:
$$T = \frac{m}{2}(\dot{x}^2 + \dot{y}^2)$$
$$= \frac{m}{2}\left[(\ell\dot{\phi}\cos\phi)^2 + (\dot{f} - \ell\dot{\phi}\sin\phi)^2\right]$$
$$= \frac{m}{2}\ell^2\dot{\phi}^2 + \frac{m}{2}\dot{f}^2 - m\ell\dot{f}\dot{\phi}\sin\phi$$

$$U = -mgy = -mgf - mg\ell\cos\phi$$

$$L = T - U = \frac{m}{2}\ell^2\dot{\phi}^2 + \frac{m}{2}\dot{f}^2 - m\ell\dot{f}\dot{\phi}\sin\phi + mgf + mg\ell\cos\phi$$

$$F_1(t) = \frac{m}{2}\dot{f}(t)^2 + mgf(t)$$

$$F_2(\phi, t) = m\ell\dot{f}\cos\phi$$

$$\frac{d}{dt}F_2 = -m\ell\dot{f}\dot{\phi}\sin\phi + m\ell\ddot{f}\cos\phi$$

$$-m\ell\dot{f}\dot{\phi}\sin\phi = m\ell\ddot{f}\cos\phi + \frac{d}{dt}F_2$$ ← can be omitted

$$\boxed{L = \frac{m}{2}\ell^2\dot{\phi}^2 + m\ell\ddot{f}\cos\phi + mg\ell\cos\phi}$$

$$\frac{d}{dt}\frac{\partial L}{\partial\dot{\phi}} - \frac{\partial L}{\partial\phi} = 0$$

$$m\ell^2\ddot{\phi} + m\ell\ddot{f}\sin\phi + mg\ell\sin\phi = 0$$

$$\ddot{\phi} + \left(\frac{g}{\ell} + \frac{\ddot{f}}{\ell}\right)\sin\phi = 0 \qquad \omega_0^2 = \frac{g}{\ell}$$

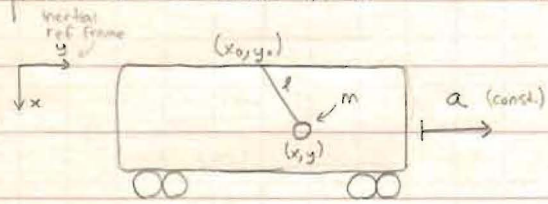
Assume $f = A\cos(\omega_0 t)$, ϕ = small

$$\ddot{\phi} + \left(\omega_0^2 + \frac{A}{\ell}\omega_0^2\cos(\omega_0 t)\right)\phi = 0$$

Mathieu Eqn.

EXAMPLE

Ex. Determine the Amplitude Oscillation Freq. of a Pendulum in Accel. Train



constraint: $(x-x_0)^2+(y-y_0)^2=l^2$

support: $x_s=0$, $y_s=\frac{a}{2}t^2+v_0t$

coord. map: $\theta \to x, y$; $x=l\cos\theta$

$$y=l\sin\theta+\frac{a}{2}t^2+v_0t$$

energies: $T=\frac{m}{2}(\dot{x}^2+\dot{y}^2)$

time dep term can be dropped

$$=\frac{m}{2}\left(l^2\dot{\theta}^2\sin^2\theta+l^2\dot{\theta}^2\cos^2\theta+(at+v_0)^2+2(at+v_0)l\dot{\theta}\cos\theta\right)$$

$$=\frac{m}{2}\left(l^2\dot{\theta}^2+\frac{d}{dt}\left(2(at+v_0)l\sin\theta\right)-2al\sin\theta\right)$$

$$U=-mgx=-mgl\cos\theta$$

$$L=T-U=\frac{m}{2}l^2\dot{\theta}^2-mal\sin\theta+mgl\cos\theta$$

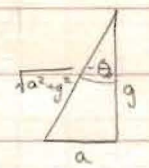
Euler-Lagrange: $0=\frac{d}{dt}\frac{\partial L}{\partial\dot{\theta}}-\frac{\partial L}{\partial\theta}=ml^2\ddot{\theta}-\left[-mal\cos\theta-mgl\sin\theta\right]$

$$\ddot{\theta}=-\frac{a}{l}\cos\theta-\frac{g}{l}\theta$$

equilibrium angle

Find equilibrium: $\ddot{\theta}_e=\dot{\theta}_e=0$

$$0=\frac{a}{l}\cos\theta_e+\frac{g}{l}\sin\theta_e \longrightarrow \frac{a}{g}=\tan(-\theta_e)$$



taylor expand: $\theta=\theta_e+\phi$ (ϕ is small)

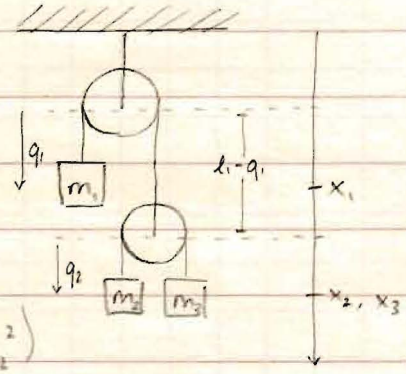
$$\ddot{\theta}=\ddot{\phi}=-\frac{a}{l}\cos(\theta_e+\phi)-\frac{g}{l}\sin(\theta_e+\phi)$$

$$=\frac{1}{l}\underbrace{\left(-a\cos\theta_e-g\sin\theta_e\right)}_{0 \text{ at equilibrium}}+\frac{\phi}{l}\left(a\sin\theta_e-g\cos\theta_e\right)+\ldots$$

$$=\frac{\phi}{l}\left(-\sqrt{a^2+g^2}\right)$$

$$\boxed{\ddot{\phi}=-\frac{\sqrt{a^2+g^2}}{l}\phi} \qquad \omega=\sqrt{\frac{\sqrt{a^2+g^2}}{l}}$$

Ex Double Atwood Machine



map: $q_1, q_2 \longrightarrow x_1, x_2, x_3$

$$x_1 = q_1 \;,\quad x_2 = (l_1 - q_1) + q_2 \;,\quad x_3 = (l_1 - q_1) + (l_2 - q_2)$$

$$\dot{x}_1 = \dot{q}_1 \;,\quad x_2 = -\dot{q}_1 + \dot{q}_2 \;,\quad x_3 = -\dot{q}_1 - \dot{q}_2$$

energies: $T = \frac{m_1}{2}\dot{x}_1^2 + \frac{m_2}{2}\dot{x}_2^2 + \frac{m_3}{2}\dot{x}_3^2$

$$= \frac{m_1}{2}\dot{q}_1^2 + \frac{m}{2}(\dot{q}_2^2 - \dot{q}_1^2) + \frac{m_3}{2}(\dot{q}_1^2 + 2\dot{q}_1\dot{q}_2 + \dot{q}_2^2)$$

$$= \frac{1}{2}(m_1 + m_2 + m_3)\dot{q}_1^2 + \frac{1}{2}(m_2 + m_3)\dot{q}_2^2 + (-m_2 + m_3)\dot{q}_1\dot{q}_2$$

$U = -m_1 g x_1 - m_2 g x_2 - m_3 g x_3$

$$= -m_1 g q_1 - m_2 g(l - q_1 + q_2) - m_3 g(l_1 + l_2 - q_1 - q_2)$$

$$= -g(m_1 - m_2 - m_3)q_1 - g(m_2 - m_3)q_2 + \text{const.}$$

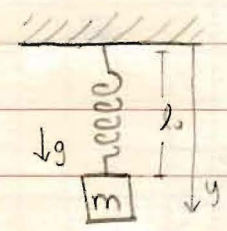
$L = T - U$

$$= (m_1 + m_2 + m_3)\frac{\dot{q}_1^2}{2} + (m_2 + m_3)\frac{\dot{q}_2^2}{2} + (m_3 - m_2)\dot{q}_1\dot{q}_2 + (m_1 - m_2 - m_3)g q_1 + (m_2 - m_3)g q_2$$

$$0 = \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_1} - \frac{\partial L}{\partial q_1} = (m_1 + m_2 + m_3)\ddot{q}_1 + (m_3 - m_2)\ddot{q}_2 - (m_1 - m_2 - m_3)g$$

$$0 = \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_2} - \frac{\partial L}{\partial q_2} = (m_2 + m_3)\ddot{q}_2 + (m_3 - m_2)\ddot{q}_1 - (m_2 - m_3)g$$

Ex Mass on Spring



$$T = \frac{1}{2}m\dot{y}^2$$

$$U_s = \frac{k}{2}(y - l_0)^2 \qquad U_g = -mgy$$

$$L = \frac{m}{2}\dot{y}^2 - \frac{k}{2}y^2 + k l_0 y - \cancel{\frac{k}{2}l_0^2}^{\text{const}} + mgy$$

$$\frac{\partial L}{\partial y} - \frac{d}{dt}\frac{\partial L}{\partial \dot{y}} = m\ddot{y} + ky - k l_0 - mg = 0$$

$$\ddot{y} + \frac{k}{m}y = \left(\frac{k}{m}l_0 + g\right)$$

equilibrium $\quad \ddot{y} = 0 \qquad y_0 = \frac{m}{k}\left(\frac{k}{m}l_0 + g\right)$

$$y = y_0 + x \qquad \ddot{x} + \frac{k}{m}x = 0 \longrightarrow \ddot{x} = -\omega_0^2 x$$

Gold Fibre

Conserved Quantities

constants of motion $\left(= \text{mathematical constraints}\right) \Leftrightarrow$ (not essential force) but conserved P/L/E

geometric (holonomic) constraints $\Leftrightarrow$ constraint forces

Structure of Lagrangian $\longrightarrow$ const. motion

Def: $\boxed{\text{generalized momentum}: \quad p_i = \frac{\partial L}{\partial \dot{q}_i}}$

Def: $\boxed{\text{Hamiltonian}: \quad H = \sum_{i=1}^{n} \left(\dot{q}_i \frac{\partial L}{\partial \dot{q}_i} \right) - L}$

Ex. $L = L(\dot{q}_1, \dot{q}_2, q)$ does not depend on time t $\longrightarrow$ const. Hamiltonian

does not depend on q_1 $\longrightarrow$ q_1 is a cyclic variable

$p_1 = \frac{\partial L}{\partial \dot{q}_1} = \text{const.}$

$0 = \frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q}$

proof:

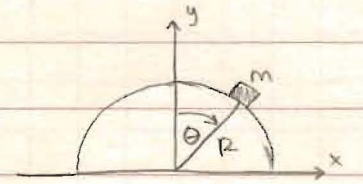
$$\frac{d}{dt} H = \frac{d}{dt}\left(\sum_{i=1}^{n} \left\{ \frac{\partial L[q, \dot{q}, t]}{\partial \dot{q}_i} q_i \right\} - L[q, \dot{q}, t] \right)$$

$$= \sum_{i=1}^{n} \left(\cancel{\ddot{q}_i \frac{\partial L}{\partial \dot{q}_i}} + \dot{q}_i \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} \right) - \left(\sum_{i=1}^{n} \left\{ \frac{\partial L}{\partial q_i} \dot{q}_i + \cancel{\frac{\partial L}{\partial \dot{q}_i} \ddot{q}_i} \right\} + \frac{\partial L}{\partial t} \right)$$

$$= \sum_{i}^{n} \left\{ \dot{q}_i \overbrace{\left(\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} \right)}^{0} \right\} - \frac{\partial L}{\partial t}$$

$\blacksquare$

Ex. Mass on Cylinder



Slides down from rest

1) EOM

2) At what θ does m leave the cylinder

1) generalized coordinates

- constraint $g = \rho - R = 0$
- cyl. coord. $x = \rho \sin\theta = R \sin\theta$

  $y = \rho \cos\theta = R\cos\theta$
- generalized coordinate θ

$L = T - U = \frac{mR^2}{2}\dot{\theta}^2 - mgR\cos\theta$

$L(\theta, \dot{\theta}) = \frac{mR^2\dot{\theta}^2}{2} - mgR\cos\theta$

conserved Hamiltonian

$$H = \dot{\theta}\frac{\partial L}{\partial \dot{\theta}} - L$$
$$= \dot{\theta}(mR^2\dot{\theta}) - (T - U)$$
$$= 2T - (T - U)$$
$$= T + U = E = \text{const.}$$

$E_0 = \frac{mR^2\dot{\theta}^2}{2} + mgR\cos\theta$

$\rightarrow \dot{\theta}^2 = -\frac{2g}{R}\cos\theta + \frac{2E_0}{mR^2}$

EOM

$$0 = \frac{d}{dt}\frac{\partial L}{\partial \dot\theta} - \frac{\partial L}{\partial \theta} = mR^2\ddot\theta - mgR\sin\theta$$

$$\ddot\theta = \frac{g}{R}\sin\theta$$

Progress possible because $E = \text{const}$

$$\dot\theta^2 = -\frac{2g}{R}\cos\theta + \frac{2E_0}{mR^2}$$

$$\cancel{2}\dot\theta\ddot\theta = \frac{\cancel{2}g}{R}\dot\theta\sin\theta$$

$$t = \int dt = \int \frac{d\theta}{\sqrt{\frac{2E_0}{mR^2} - \frac{2g}{R}\cos\theta}} \longmapsto \text{elliptic function}$$

GoldFibra

Forces

No Constraints

$q_i = x_i \longrightarrow L = T - U = \sum \frac{m}{2}\dot{x}^2 - U(x)$

$0 = \frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} \longrightarrow 0 = m\ddot{x}_i - \underbrace{\left(-\frac{\partial U}{\partial x_i}\right)}_{F_i} \longrightarrow F_i = m\ddot{x}_i$

Lagrange = Newton

conservative force $\longleftarrow F = -\nabla U$

Note: $p_i = m\dot{x}_i = \frac{\partial L}{\partial \dot{x}_i}$ $\qquad F_i = -\frac{\partial U}{\partial x}$

With Constraints

$q = q_1, \ldots, q_n \qquad n = \text{DoF} \qquad x = x_i(q, t)$

$p_j = \frac{\partial L}{\partial \dot{q}_j}$ (generalized momentum)

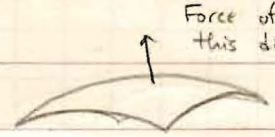
$Q_j^{cons} = -\frac{\partial U[x(q,t)]}{\partial q_j} = \sum -\frac{\partial U}{\partial x_i}\frac{\partial x_i}{\partial q_j} = \sum F_i^{cons}\frac{\partial x_i}{\partial q_j}$ (generalized cons. force)

Forces of Constraint

$q = q_1 \ldots q_m \qquad m > n$ DOF

not generalized coord. but can be curvilinear

constraint: $g(q) = 0$ — defines surface $\longrightarrow$



normal direction: Force of constraint has this direction

$\nabla g = \sum_{j=1}^{m} \frac{\partial g}{\partial q_j}\hat{e}_{q_j}$

$\rightarrow$ generalized force of constraint in a direction of q_j has force $\qquad Q_j^{const} = \lambda \frac{\partial g}{\partial q_j}$

In method of undetermined multiplicatiers: $\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} - \frac{\partial L}{\partial q_j} - \underbrace{\sum_{\nu=1}^{k} \lambda \frac{\partial g_\nu}{\partial q_j}}_{j^{th}\text{ component of net generalized force of constraint}} = 0$

k constraints

FORCES II

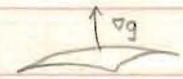
Finding Forces of Constraint

coordinates $q = q_1 \dots q_m$ $\qquad m > \text{DoF}$

constraints $g_j(q, t) = 0$ $\qquad j = 1, \dots k$

Euler-Lagrange eq. w/ undetermined multipliers

$$\left.\begin{aligned} \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} - \sum_{j=1} \lambda_j \frac{\partial g_j}{\partial q_i} &= 0 \\ g_j(q_i, t) &= 0 \end{aligned}\right\} \begin{array}{l} m+k \text{ eqn. of motion} \\ m+k \text{ unknowns} \\ \uparrow\ \uparrow \\ q_i(t)\ \lambda_j(t) \end{array}$$

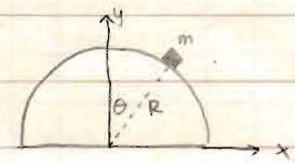


Def Generalized force of constraint $\qquad Q_i^{\text{const.}} = \sum_{j=1}^{k} \lambda_j \frac{\partial g_j}{\partial q_i}$

$|Q_i^{\text{const.}}|$ is the magnitude of the net force of constraint in the q_i direction

Ex. m on Cylinder

At what θ does m leave the surface?



coord. $q_1 = \rho \qquad q_2 = \theta$

$$x = \rho\sin\theta \qquad \dot{x} = \dot{\rho}\sin\theta + \rho\dot{\theta}\cos\theta$$

$$y = \rho\cos\theta \qquad \dot{y} = \dot{\rho}\cos\theta - \rho\dot{\theta}\sin\theta$$

$$L = \frac{m}{2}(\dot{\rho}^2 + \rho^2\dot{\theta}^2) - mg\rho\cos\theta \qquad \text{constraint: } g = \rho - R = 0$$

① $0 = \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} - \lambda\frac{\partial g}{\partial \theta} = m\rho^2\ddot{\theta} + 2m\rho\dot{\rho}\dot{\theta} - mg\rho\sin\theta - \lambda = 0$

② $0 = \frac{d}{dt}\frac{\partial L}{\partial \dot{\rho}} - \frac{\partial L}{\partial \rho} - \lambda\frac{\partial g}{\partial \rho} = m\ddot{\rho} - mg\dot{\theta}^2 + mg\cos\theta - \lambda = 0$

③ $\rho = R \rightarrow \ddot{\rho} = \dot{\rho} = 0$

Recall $H = E = \text{const} \longrightarrow E_0 = E \longrightarrow mgR\cos\theta_0 = \frac{m}{2}R^2\dot{\theta}^2 + mgR\cos\theta$

Solve ③ for λ $\qquad R\dot{\theta}^2 = -g(2\cos\theta - 2\cos\theta_0)$

$$\lambda = mg(3\cos\theta - 2\cos\theta_0)$$

$$|F_N| = |Q_i^{\text{const.}}| = \left|\lambda\frac{\partial g}{\partial \rho}\right| = |\lambda| \longrightarrow \boxed{\cos\theta_{crit} = \frac{2}{3}\cos\theta_0}$$

NOETHER'S THEOREM

Symmetry = Conservation Laws

symmetry = Invariance of a physical law under a continuous (differential) coordinate transformation

① $q_k' = q_k + \Delta q_k$: translational invariance of Lagrangian w.r.t. gen. coord. q_k

↳ L does not depend on q_k

↳ $p_k = \frac{\partial L}{\partial \dot{q}_k} = \text{const}$ (conservation of gen. momentum)

- Assume $q_k = q_1 = x$, then $p_x = \text{const.}$ (along x)

  translational invariance ↔ conservation of linear momentum

- $q_1 = \theta \xrightarrow[\text{Lagrangian}]{} L = \frac{I}{2}\dot{\theta}^2 + \frac{m}{2}\dot{q}_2 - U(q_2)$

  $\text{const.} = \frac{\partial L}{\partial \dot{\theta}} = I\dot{\theta} = L$ ← angular momentum

  rotational invariance ↔ conservation of angular momentum

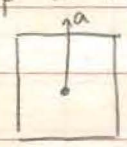
② $t' = t + \Delta t$: invariance of L w.r.t time

translation ↔ conservation of Hamiltonian

- Kinetic Energy T has form $T = \sum_{jk} a_{jk}\dot{q}_j\dot{q}_k$

  ↳ then true if coord. map is time independant $x_i = x_i(q_l)$

  counter example: <u>Elevator</u>



$m\ddot{\theta} + \frac{m}{\ell}(g+a)\sin\theta = 0$

  Hamiltonian conserved; <u>not</u> Energy

$\dot{\theta} = \ddot{\theta} = 0 \rightarrow E = \frac{m}{2}(at)^2 + mg\left(\frac{at^2}{2} - \ell\right) + \text{const.}$

- $U = U(q_l)$

HAMILTONIAN MECHANICS

Want equations of motion in terms of generalized momenta p_k and H

$\mathbf{q} = q_1 \dots q_n$ $\qquad$ $n = \text{DoF}$

$\dot{\mathbf{q}} = \dot{q}_1 \dots \dot{q}_n$

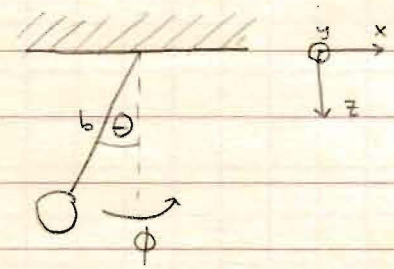
$\mathbf{p} = p_1 \dots p_n$

Procedure to find $H(\mathbf{q}, \mathbf{p}, t)$

- find Lagrangian $L(\mathbf{q}, \dot{\mathbf{q}}, t)$
- solve $p_k = \frac{\partial L(\mathbf{q}, \dot{\mathbf{q}}, t)}{\partial \dot{q}_k}$ that for $\dot{q}_k = \dot{q}_k(\mathbf{q}, \mathbf{p}, t)$
- use $\dot{q}_k(\mathbf{q}, \mathbf{p}, t)$ in def of Hamiltonian

$$H(\mathbf{q}, \mathbf{p}, t) = \sum_{k=1}^{n} p_k \dot{q}_k - L(\mathbf{q}, \dot{\mathbf{q}}, t)$$

<u>Ex</u> Spherical Pendulum



spherical coordinates: ρ, θ, ϕ

<u>constraint</u>: $\rho = b$

<u>coor. map</u>: $x = b\cos\phi \sin\theta$

$y = b\sin\phi \sin\theta$

$z = b\cos\theta$

$\dot{x} = -b\dot{\phi}\sin\phi\sin\theta + b\dot{\theta}\cos\theta\cos\phi$

$\dot{y} = b\dot{\phi}\cos\phi\sin\theta + b\dot{\theta}\sin\phi\cos\theta$

$\dot{z} = -b\dot{\theta}\sin\theta$

$T = \frac{m}{2}\left(b^2\dot{\phi}^2\sin^2\theta + b\dot{\theta}^2\right)$

$U = -mgb\cos\theta$

$L = \frac{mb^2}{2}\dot{\phi}^2\sin^2\theta + \frac{mb^2}{2}\dot{\theta}^2 + mbg\cos\theta$

<u>generalized momentum</u>

$\frac{\partial L}{\partial \dot{\theta}} = p_\theta = mb^2\dot{\theta}$ $\qquad$ $\frac{\partial L}{\partial \dot{\phi}} = p_\phi = mb^2\sin^2\theta\,\dot{\phi}$

$\dot{\theta} = \frac{p_\theta}{mb^2}$ $\qquad$ $\dot{\phi} = \frac{p_\phi}{mb^2\sin^2\theta}$

<u>Hamiltonian</u>: $H = p_\theta\dot{\theta} + p_\phi\dot{\phi} - \left(\frac{1}{2}(mb^2\sin^2\theta\,\dot{\phi})\dot{\phi} + \frac{1}{2}(mb^2\dot{\theta})\dot{\theta} + mbg\cos\theta\right)$

$$\boxed{H = \frac{p_\theta^2}{2mb^2} + \frac{p_\phi^2}{2mb^2\sin^2\theta} - mgb\cos\theta}$$

or $H = T + U$

Hamiltonian Equations of Motion

(Canonical Eq.)

I) $H = H(q, p, t)$

$$dH = \sum_{k=1}^{n} \left(\frac{\partial H}{\partial q_k} dq_k + \frac{\partial H}{\partial p_k} dp_k \right) + \frac{\partial H}{\partial t} dt$$

II) $H = \sum_{k=1}^{n} p_k \dot{q}_k - L(q, \dot{q}, t)$

$$dH = \sum_{k=1}^{n} \left(\dot{q}_k dp_k + p_k d\dot{q}_k \right) - \left(\sum_{k=1}^{n} \left(\overbrace{\frac{\partial L}{\partial q_k}}^{\dot{p}_k} dq_k + \overbrace{\frac{\partial L}{\partial \dot{q}_k}}^{p_k} d\dot{q}_k \right) \right) - \overbrace{\frac{\partial L}{\partial t}}^{\frac{\partial H}{\partial t}} dt$$

Know: $\frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t}$ and $\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_k} - \frac{\partial L}{\partial q_k} = 0 \rightarrow \dot{p}_k = \frac{\partial L}{\partial \dot{q}_k}$

$$\frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t} = \frac{\partial H}{\partial t}$$

If H does not depend on t explicitely then $H = \text{const.}$ with respect to t

$$\boxed{\begin{aligned} \dot{q}_k &= \frac{\partial H}{\partial p_k} \\ -\dot{p}_k &= \frac{\partial H}{\partial q_k} \end{aligned}}$$

Ex. (cont'd)

$$\dot{\phi} = \frac{\partial H}{\partial p_\phi} = \frac{p_\phi}{mb^2 \sin^2\theta} \rightarrow \text{given } \theta(t)\text{, can integrate to get } \phi(t)$$

$$\dot{p}_\phi = -\frac{\partial H}{\partial \phi} = 0 \rightarrow p_\phi = \text{const.} = l_z$$

$$\dot{\theta} = \frac{\partial H}{\partial p_z} = \frac{p_\theta}{mb^2}$$

$$\dot{p}_\theta = -\frac{\partial H}{\partial \theta} = -\frac{p_\phi^2 \cos\theta}{mb^2 \sin^3\theta} - mgb\sin\theta$$

special case: $p_\phi = l_z = 0$

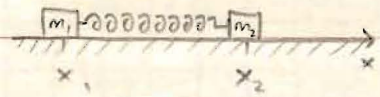
$$\dot{\theta} = \frac{p_\theta}{mb^2} \longmapsto \ddot{\theta} = \frac{\dot{p}_\theta}{mb^2} = -\frac{g}{b}\sin\theta$$

$$\dot{p}_\theta = -mgb\sin\theta$$ simple pendulum

HAMILTONIAN DYNAMICS

Spaces

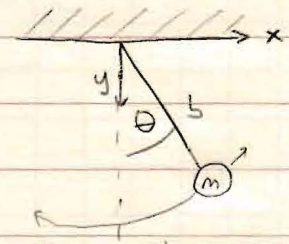
consider 1D motion of 2 masses:



	Newtonian	Lagrangian	Hamiltonian
What do you need to specify to know motion in future?	$x_1(0)$ $\dot{x}_1(0)$ $x_2(0)$ $\dot{x}_2(0)$	$q_1(0) = x_{cm}$ $\dot{q}_1(0)$ $q_2(0) = x_2 - x_1$ $\dot{q}_2(0)$	$q_1(0)$ $p_1(0)$ $q_2(0)$ $p_2(0)$
In what space do you picture the motion?	motion of 2 points in physical space have 1D space $\longrightarrow x$	motion of 1 point in configuration space have 2D space q_2, q_1	motion of 1 point in phase space have 4D space " p_1, p_2, q_1, q_2 "

Ex. Simple Pendulum

coordinates x, y ($\dot{x}$ and $\dot{y}$ implied)



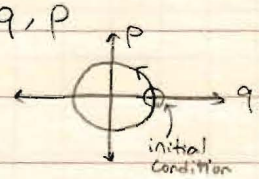
Lagrangian

$q = \theta$

($\dot{q}$ is implicit)

Hamiltonian

q, p

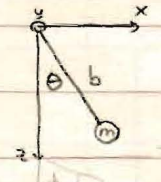


Each point in phase space uniquely specifies state of system (= initial conditions)

→ associated trajectory

→ trajectories cannot cross

Ex. Spherical Pendulum



$$T = \frac{m}{2} \mathbf{v} \cdot \mathbf{v} = \frac{m}{2}\left(b^2\dot{\theta}^2 + b^2 \sin^2\theta\, \dot{\phi}^2\right)$$

$$U = -mgz = -mgb\cos\theta$$

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = \frac{\partial T}{\partial \dot{\theta}} = mb^2\dot{\theta} \qquad p_\phi = \frac{\partial L}{\partial \dot{\phi}} = \frac{\partial T}{\partial \dot{\phi}} = mb^2\sin^2\theta\,\dot{\phi}$$

$$H = E = T + U = \frac{p_\theta^2}{2mb^2} + \frac{p_\phi^2}{2mb^2\sin^2\theta} - mgb\cos\theta$$

$\theta = F^{-1}(t; \ell, E, c_1)$

to satisfy $\theta(0)$

to satisfy $p_\theta(0)$

to satisfy $p_\phi(0)$

4D problem: $p_\theta\ \theta\ p_\phi\ \phi \longrightarrow H = H(p_\theta, p_\phi, \theta)$

- no ϕ dependence (cyc → p_ϕ const., ϕ ign
- no t dependence → $E = H =$ const.

canonical eqns:

$$\dot{\phi} = \frac{\partial H}{\partial p_\phi} = \frac{p_\phi}{mb^2\sin^2\theta} \qquad \dot{p}_\phi = -\frac{\partial H}{\partial \phi} = 0 \rightarrow p_\phi = \ell = \text{const.}$$

$$\dot{\theta} = \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{mb^2} \qquad \dot{p}_\theta = \frac{\ell^2\cos\theta}{mb^2\sin^3\theta} - mgb\sin\theta \longrightarrow \text{effective 2D problem}$$

- ϕ dyn. decouples from p_θ-θ dyn
- for known $\theta(t)$ can integrate to get $\phi(t)$

$$E = \frac{1}{2}mb^2\dot{\theta}^2 + \frac{\ell^2}{2mb^2\sin^2\theta} - mgb\cos\theta \rightarrow \frac{d\theta}{dt} = \sqrt{\frac{2E}{mb^2} - \frac{\ell^2}{m^2b^4\sin^2\theta} + \frac{2g\cos\theta}{b}} \rightarrow$$

integrate to $t = F(\theta; \ell,$ and invert $\theta = F^{-1}(t;$

Perturbation Solution (Spherical Pend. cont'd)

special solution : $\dot{\Theta}_{eq} = 0 \rightarrow$ Find Θ_{eq} and solutions closely

Ansatz : $\Theta = \Theta_{eq} + \varepsilon\tilde{\Theta}$ $(\varepsilon \ll 1)$ and $p_\Theta = p_{\Theta_{eq}} + \varepsilon\tilde{p}_\Theta$

$$\dot{\Theta} = \varepsilon\dot{\tilde{\Theta}}_{eq} = \frac{p_{\Theta_{eq}} + \varepsilon\tilde{p}_\Theta}{mb^2}$$

$$\left(mb^2\sin^3(\Theta_{eq} + \varepsilon\tilde{\Theta})\right)\left(\dot{p}_{\Theta_{eq}} + \varepsilon\dot{\tilde{p}}_\Theta\right) = l^2\cos(\Theta_{eq} + \varepsilon\tilde{\Theta}) - m^2b^3g\sin^4(\Theta_{eq} + \varepsilon\tilde{\Theta})$$

$$\left(mb^2\sin^3(\Theta_{eq}) + \underbrace{3mb^2\sin^2(\Theta_{eq})\cos(\Theta_{eq})\,\varepsilon\tilde{\Theta}}_{\Theta(\varepsilon)}\right)\left(\overset{0}{\dot{p}_{\Theta_{eq}}} + \varepsilon\dot{\tilde{p}}_\Theta\right) =$$

$$(l^2\cos\Theta_{eq} - l^2\sin\Theta_{eq}\cdot\varepsilon\tilde{\Theta}) - m^2b^3g\sin^4(\Theta_{eq}) + 4m^2b^3g\sin^3\Theta_{eq}\cos\Theta_{eq}$$

<u>order ε^0:</u> $0 = \frac{p_{\Theta_{eq}}}{mb^2} \rightarrow p_{\Theta_{eq}} = 0$

$$0 = l^2\cos\Theta_{eq} - m^2b^3g\sin^4\Theta_{eq} \longrightarrow \Theta_{eq}$$

<u>order ε :</u> $\dot{\tilde{\Theta}} = \frac{\tilde{p}_\Theta}{mb^2} \rightarrow \ddot{\tilde{\Theta}} = \frac{\dot{\tilde{p}}_\Theta}{mb^2}$

$$mb^2\sin^3\Theta_{eq}\,\dot{\tilde{p}}_\Theta = -(l^2\sin\Theta_{eq} + 4m^2b^3g\sin^3\Theta_{eq}\cos\Theta_{eq})\tilde{\Theta}$$

$$\ddot{\tilde{\Theta}} = -\underbrace{\left(\frac{l^2\sin\Theta_{eq} + 4m^2b^3\sin^3\Theta_{eq}\cos\Theta_{eq}}{m^2b^4\sin^3\Theta_{eq}}\right)}_{\omega_0^2}\tilde{\Theta}$$

$$\Theta = \Theta_{eq} + A\cos(\omega_0 t + B)$$

Modern Hamiltonian Mechanics

A) $H = H(p, q, t)$

B) When is $H = E$? When is $H = \text{const}$?

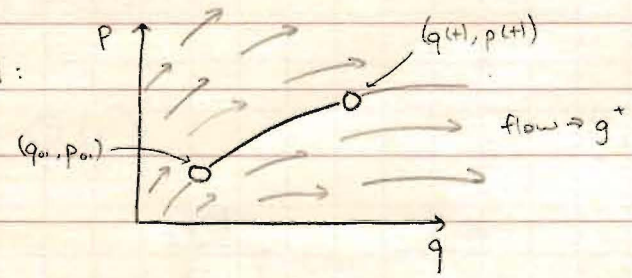
- $T = a\dot{q}_1\dot{q}_2$ $\quad U = -kq_1$ $\quad H = T + U = E$ $\quad \frac{\partial H}{\partial t} = 0 = \frac{dH}{dt}$, $H = \text{const.}$
- $T = a\dot{q}_1\dot{q}_2 + b\dot{q}_1\dot{q}_2 + cq_1$ $\quad U = kq_1$ $\quad H \neq T + U$, $H = \text{const.}$
- $T = e^{-t}q_1\dot{q}_1\dot{q}_2$ $\quad U = e^{-t}kq_1$ $\quad H = T + U = E$, $H \neq \text{const.}$

$$T = \sum_{ij} a_{ij}(q, t)\,\dot{q}_i\dot{q}_j + \overbrace{\sum_{ij} b_i(q, t)\,\dot{q}_i + c(q, t)}^{\text{If } b \text{ and } c \text{ are zero then } H = T + U}$$

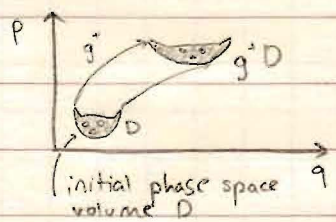
Liouville's Theorem

Phase space volume is preserved:

$$\underset{g^t D}{\text{volume of}} = \underset{D}{\text{volume of}}.$$



$$\begin{bmatrix} q(t) \\ p(t) \end{bmatrix} = g^t \begin{bmatrix} q_0 \\ p_0 \end{bmatrix}$$



Ex. Flow g^t for Harmonic Oscillator

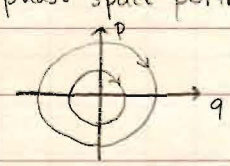
$$H = \frac{p^2}{2} + \frac{q^2}{2}$$

$$\left.\begin{aligned} \dot{q} &= \frac{\partial H}{\partial p} = p \\ \dot{p} &= -\frac{\partial H}{\partial q} = -q \end{aligned}\right\} \ddot{q} = -q$$

$$\left.\begin{aligned} q(t) &= q_0 \cos t + p_0 \sin t \\ p(t) &= -q \sin t + p_0 \cos t \end{aligned}\right\} \text{defines } g^t$$

$$\begin{bmatrix} q(t) \\ p(t) \end{bmatrix} = \overbrace{\begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix}}^{g^t \text{ is rotation around origin}} \begin{bmatrix} q_0 \\ p_0 \end{bmatrix}$$

phase space portrait



Modern Hamiltonian Dynamics

Liouville's Thm

initial volume: $V_0 = \int dp_0\, dq_0$

at time t $\quad V(t) = \int dp(t)\, dq(t)$

$$\overset{\text{change of variables}}{=} \int \frac{\partial(p(t), q(t))}{\partial(p_0, q_0)}$$

$$\frac{\partial(p(t), q(t))}{\partial(p_0, q_0)} = \det\begin{pmatrix} \frac{\partial p(t)}{\partial p_0} & \frac{\partial p(t)}{\partial q_0} \\ \frac{\partial q(t)}{\partial p_0} & \frac{\partial q(t)}{\partial q_0} \end{pmatrix} = \det\begin{pmatrix} 1 - \frac{\partial^2 H}{\partial q_0 \partial p_0} t + \mathcal{O}(t^2) & -\frac{\partial^2 H}{\partial q_0 \partial p_0} t + \mathcal{O}(t^2) \\ \frac{\partial^2 H}{\partial p_0 \partial q_0} t + \mathcal{O}(t^2) & 1 + \frac{\partial^2 H}{\partial q_0 \partial p_0} t + \mathcal{O}(t^2) \end{pmatrix}$$

$$= 1 + \mathcal{O}(t^2)$$

Need to know $p(t), q(t)$ for $t \to 0$

$$\frac{dp}{dt} = -\frac{\partial H}{\partial q} \to p(t) = p_0 - \frac{\partial H}{\partial q}\Big|_{\substack{p=p_0 \\ q=q_0 \\ t=0}} (t) + \mathcal{O}(t^2)$$

$$\lim_{t\to 0} = \frac{p(t) - p_0}{t} \qquad \frac{\partial H}{\partial q_0}$$

$$\frac{dq}{dt} = \frac{\partial H}{\partial q} \to q(t) = q_0 + \frac{\partial H}{\partial p_0} t + \mathcal{O}(t^2)$$

$$V(t) = \int (1 + \mathcal{O}(t^2))\, dp_0\, dq_0$$

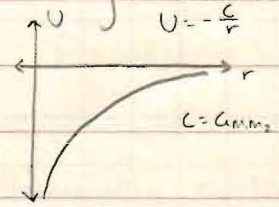
$$\frac{dV(t)}{dt} = \lim_{t\to 0} \frac{V(t) - V_0}{t} = \lim_{t\to 0} \frac{\int \mathcal{O}(t^2)\, dp_0\, dq_0}{t} = 0$$

$$\boxed{V = \text{const.}}$$

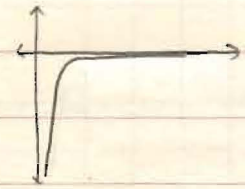
CENTRAL FORCE MOTION I

Potentials $U(r)$

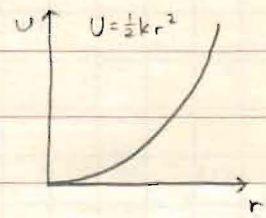
① Gravity / Coulomb $U = -\frac{c}{r}$



② Yukawa Potential $U = -\frac{k}{r} e^{-r/a}$



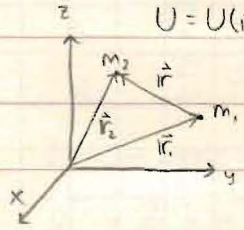
③ 3D Harmonic Oscillator



④ "Dipole like" Potential $U = \frac{k}{r^2}$

Two-Body Central Potential Motion

$U = U(\vec{r})$ $\quad r = |\vec{r}|$



DoF: 6

PS dgm: 12
Phase space

Phase Space
- for cyclic variable q_i (-2) (one is const. of prop. eg. p_i)
- other const of motion (eg. E) (-1)

Translational Symmetry

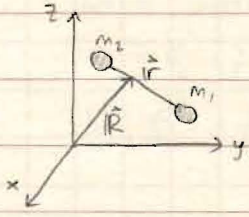
- motion of two mass total system is invariant

$$\vec{r}_1 \to \vec{r}_1 + c \qquad \vec{r}_2 \to \vec{r}_2 + c$$

→ the 3 momenta are conserved

→ maybe there are 3 cyclic variables

CM of coordinates



CM: $\vec{R} = \frac{m_1 \vec{r}_1 + m_2 \vec{r}_2}{m_1 + m_2}$

coor dif: $\vec{r} = \vec{r}_1 - \vec{r}_2$

total mass: $M = m_1 + m_2$

reduced mass: $\mu = \frac{1}{m_1} + \frac{1}{m_2} = \frac{m_1 m_2}{m_1 + m_2}$

$$\vec{r}_1 = \vec{R} + \frac{\mu}{m_1}\vec{r}$$

$$|\dot{\vec{r}}_1|^2 = \left(\dot{\vec{R}} + \frac{\mu}{m_1}\dot{\vec{r}}\right)\cdot\left(\dot{\vec{R}} + \frac{\mu}{m_1}\dot{\vec{r}}\right)$$

$$= |\dot{\vec{R}}|^2 + 2\frac{\mu}{m_1}(\dot{\vec{r}}\cdot\dot{\vec{R}}) + \left(\frac{\mu}{m_1}\right)^2|\dot{\vec{r}}|^2$$

$$L = \frac{m}{2}|\dot{\vec{r}}_1|^2 + \frac{m}{2}|\dot{\vec{r}}_2|^2 - U(r)$$

$$= \frac{1}{2}M|\dot{\vec{R}}|^2 + \frac{1}{2}\mu|\dot{\vec{r}}|^2 - U(r)$$

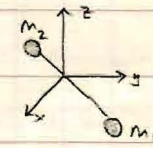
CENTRAL FORCE MOTION I (cont'd)

Translational Symmetry

$\mathbb{R} = (X, Y, Z)$ are cyclic,

$\mathbb{P} = \text{const.}$ $\quad (P_x = M\dot{X} = \text{const.})$

can pick co-moving coordinate where $\mathbb{R} = 0$ $\longrightarrow$



Remaining system:

$$L = \frac{\mu}{2}|\dot{\mathbb{r}}|^2 - U(r)$$

DoF: 3
PSdgm: 6

$\rightarrow$ equivalent 1 body motion, "particle of mass μ"

Rotational symmetry / conservation of angular momentum

$$L = \frac{\mu}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - U(r)$$

$$p_x = \frac{\partial L}{\partial \dot{x}} = \mu\dot{x} \qquad p_y = \mu\dot{y} \qquad p_z = \mu\dot{z} \qquad \longrightarrow \quad \mathbb{p} = \mu\dot{\mathbb{r}}$$

$$H = \frac{p_x^2}{2\mu} + \frac{p_y^2}{2\mu} + \frac{p_z^2}{2\mu} + U(r)$$

$$\dot{p}_x = -\frac{\partial H}{\partial x} = -\frac{\partial U}{\partial r}\frac{\partial r}{\partial x} = -\frac{\partial U}{\partial r}\frac{x}{r} \qquad \frac{\partial r}{\partial x} = \frac{\partial}{\partial x}\sqrt{x^2+y^2+z^2} = \frac{x}{r}$$

$$\dot{p}_y = -\frac{\partial U}{\partial r}\frac{y}{r} \qquad \dot{p}_z = -\frac{\partial U}{\partial r}\frac{z}{r} \qquad \longrightarrow \qquad \dot{\mathbb{p}} = -\frac{\partial U}{\partial r}\frac{\vec{\mathbb{r}}}{r}$$

$$\mathbb{L} = \mathbb{r} \times \mathbb{p}$$

$$\begin{aligned}\dot{\mathbb{L}} &= \dot{\mathbb{r}} \times \mathbb{p} + \mathbb{r} \times \dot{\mathbb{p}} \\ &= \frac{1}{\mu}\underbrace{(\mathbb{p} \times \mathbb{p})}_{0} + \underbrace{(\mathbb{r} \times \mathbb{r})}_{0}\left(-\frac{\partial U}{\partial r}\frac{1}{r}\right) \\ &= 0\end{aligned}$$

$\mathbb{L} = \text{const.} \longrightarrow$ 3 const. of motion

- pick coordinate such that $\mathbb{L} = L\hat{e}_z$
  - $\hookrightarrow$ planar motion (-2PS, direction of $\mathbb{L}$ is const.)

$$L = \frac{\mu}{2}(\dot{x}^2 + \dot{y}^2) - U(\sqrt{x^2+y^2})$$

- pick polar coordinate

$$L = \frac{\mu}{2}(\dot{r}^2 + r^2\dot{\phi}^2) - U(r)$$

mag of $\mathbb{L}$ is const.

$$H = \frac{p_r^2}{2\mu} + \frac{p_\phi^2}{2\mu} + U(r) \longrightarrow p_\phi = \mu r^2\dot{\phi} = \ell = \text{const.}$$

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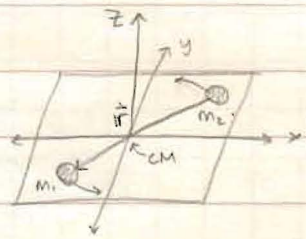
CENTRAL FORCE MOTION II

$\vec{r} = \vec{r}_1 - \vec{r}_2$

$\mu = \dfrac{m_1 m_2}{m_1 + m_2}$

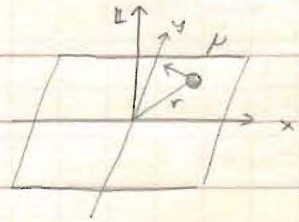


$$L = \frac{m_1}{2}|\dot{\vec{r}}_1|^2 + \frac{m_2}{2}|\dot{\vec{r}}_2|^2 - U(r)$$

- trans. symmetry origin is at CM
- rotational symmetry

$\longrightarrow$ angular momentum $\mathbb{L} = \text{const.}$

$\longrightarrow$ pick z-axis along $\mathbb{L}$



$$L = \frac{\mu}{2}(\dot{r}^2 + r^2\dot{\phi}^2) - U(r)$$

$$p_r = \mu\dot{r} \qquad p_\phi = \mu r^2 \dot{\phi}$$

$$H = \frac{p_r^2}{2\mu} + \frac{p_\phi^2}{2\mu r^2} + U(r) \qquad H = T + U = E$$

Canonical Equations

$$\dot{p}_\phi = -\frac{\partial H}{\partial \phi} = 0 \rightarrow p_\phi = \mu r^2 \dot{\phi} = \text{const} = \ell \qquad \ell = |\mathbb{L}|$$

$$\dot{\phi} = \frac{\partial H}{\partial p_\phi} = \frac{\ell}{\mu r^2}$$

$$\dot{p}_r = -\frac{\partial H}{\partial r} = -\frac{\partial}{\partial r}\left(\frac{\ell^2}{2\mu r^2} + U(r)\right)$$

$$\dot{r} = \frac{\partial H}{\partial p_r} = \frac{p_r}{\mu}$$

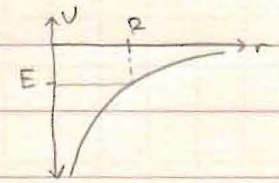
- time translational invariance $\qquad \dfrac{dH}{dt} = \dfrac{\partial H}{\partial t} = 0 \rightarrow H = \text{const.} = E$

$\longrightarrow$ Energy conservation

$$E = \frac{\mu}{2}\dot{r}^2 + \frac{\ell^2}{2\mu r^2} + U(r) \longrightarrow \dot{r} = \pm\sqrt{\frac{2}{\mu}\left(E - U - \frac{\ell^2}{2\mu r^2}\right)}$$

$\longrightarrow$ integrable

Simple Case : Radial Infall



$\ell = 0 \longrightarrow \dot{\phi}_0 = 0 \qquad r_0 = R \qquad \dot{r}_0 = 0$

"dipole-like" $\quad U = -\dfrac{k}{r^2}$

$$E = 0 + 0 + U(R) = -\frac{k}{R^2}$$

infall

$$\frac{dr}{dt} = -\sqrt{\frac{2k}{\mu}\left(-\frac{1}{R^2} + \frac{1}{r^2}\right)} \longrightarrow t = \int_R^r \frac{-r'\,dr'}{\sqrt{\frac{2k}{\mu}}\sqrt{1 - \frac{r'^2}{R^2}}}$$

$$y = 1 - \left(\frac{r'}{R}\right)^2 \qquad dy = -\frac{2r'\,dr'}{R^2}$$

$$-r'\,dr' = \frac{R^2}{2}\,dy$$

$$t = \int_0^{1-(\frac{r}{R})^2} \sqrt{\frac{\mu}{2k}}\, R^2 \frac{dy}{2\sqrt{y}}$$

$$= \sqrt{\frac{\mu}{2k}}\, R^2 \sqrt{1 - \frac{r^2}{R^2}}$$

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CENTRAL FORCE MOTION II (cont'd)

Two questions

1) to get $r(t)$, $\phi(t)$

$$t = \int \frac{dr}{\pm\sqrt{\frac{2}{\mu}\left(E - U(r) - \frac{\ell^2}{2\mu r^2}\right)}}$$

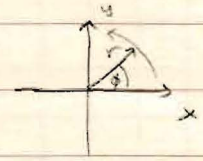
$\longrightarrow$ integrate $t(r)$

$\longrightarrow$ invert $r(t)$

$$\phi = \int \frac{\ell}{\mu (r(t))^2}\, dt$$

$\longrightarrow$ integrate $\phi(t)$

2) to get orbit $\phi(r)$ [or $r(\phi)$]



$$\frac{d\phi}{dt} = \frac{\ell}{\mu r^2} \qquad \frac{d\phi}{dr} = \frac{d\phi}{dt}\frac{dt}{dr} = \frac{\ell}{\mu r^2}\frac{1}{\dot{r}}$$

$$\phi = \int \frac{\frac{\ell}{r^2}\, dr}{\pm\mu\sqrt{\frac{2}{\mu}\left(E - U(r) - \frac{\ell^2}{2\mu r^2}\right)}}$$

Effective Potential

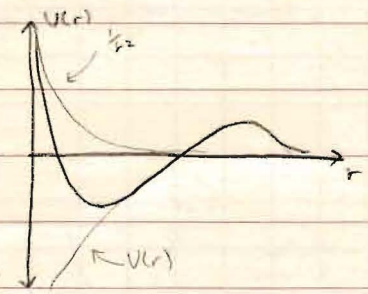
$$E = \underbrace{\frac{\mu}{2}\dot{r}^2}_{T_{eff}} + \underbrace{\overbrace{\frac{\ell^2}{2\mu r^2}}^{U_c} + U(r)}_{U_{eff}}$$

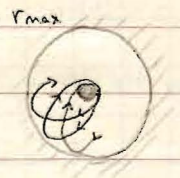
$$U_c = \frac{\ell^2}{2\mu r^2}$$

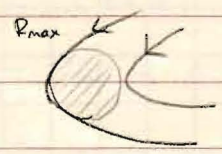
$$F_c = -\frac{\partial U_c}{\partial r} = \frac{\ell^2}{\mu r^3} = \mu r \dot{\phi}^2$$

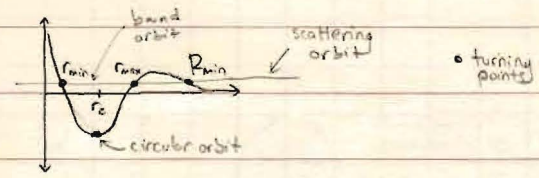
"centrifugal force" (fictitious force)

centripital acceleration









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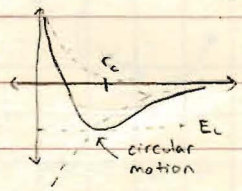
CENTRAL FORCE MOTION III

Central Force

$$H = \frac{p_r^2}{2\mu} + \underbrace{\frac{\ell^2}{2\mu r^2} + U(r)}_{U_{eff}}$$



canonical eqns of motion

$$p_\phi = \ell = \text{const.}$$

$$\dot\phi = \frac{\ell}{\mu r^2} \longrightarrow \ell = \mu r^2 \dot\phi$$

$$\left.\begin{aligned} \mu\ddot r &= -\frac{\partial U_{eff}}{\partial r} \\ \dot p_r &= -\frac{\partial}{\partial r}\left(U(r)\right) \\ \dot r &= \frac{p_r}{\mu} \end{aligned}\right\}$$

effective potential force $= F_{eff}$

Circular Motion

$$r(t) = r_c \longrightarrow \dot r = \ddot r = 0$$

$$F_{eff}(r_c) = -\left.\frac{\partial U_{eff}}{\partial r}\right|_{r=r_c} = 0 \longrightarrow \text{gives you } r_c$$

Nearly Circular Motion

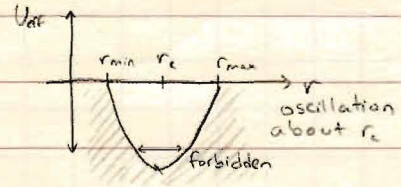
consider perturbation: $r = r_c + \eta(t)$

$$p_r = \tilde p_r \qquad \text{(note: } p_\phi \text{ no longer constant)}$$

pick $\dot\phi_0 \rightarrow \dot\phi_0 + \dot{\tilde\phi}_0$

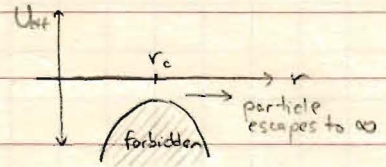
Stability of circular motion:

- Δ stable motion - U_{eff} has local minimum $\frac{\partial^2 U_{eff}}{\partial r^2} > 0$



$$r(t) = r_c + A\cos(\omega_r t + B)$$

- Δ unstable motion - U_{eff} is a local maximum $\frac{\partial^2 U_{eff}}{\partial r^2} < 0$



$$r(t) = r_c + Ae^{\alpha t} + Be^{-\alpha t}$$

Ex

$$U = -\frac{k}{(n-1)r^{n-1}} \qquad F = -\frac{k}{r^n}$$

$$U_{eff} = \frac{\ell^2}{2\mu r^2} - \frac{k}{(n-1)r^{n-1}}$$

$$\boxed{\begin{array}{lll} n<3 & \frac{\partial^2 U_{eff}}{\partial r^2} > 0 & \omega_r^2 \rightarrow r(t) = r_c + A\cos(\omega_r t + B) \\ n>3 & \frac{\partial^2 U_{eff}}{\partial r^2} < 0 & -\alpha^2 \rightarrow r(t) = r_c + Ae^{\alpha t} + Be^{-\alpha t} \end{array}}$$

$$\frac{\partial U_{eff}}{\partial r} = -\frac{\ell^2}{\mu r^3} + \frac{k}{r^n}$$

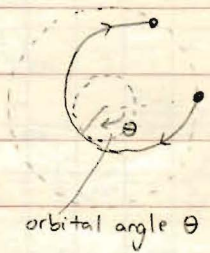
$$\frac{\partial^2 U_{eff}}{\partial r^2} = \frac{3\ell^2}{\mu r^4} - \frac{nk}{r^{n+1}} = \frac{\ell^2}{\mu r_c^4}\left(3 - n\underbrace{\frac{k\mu}{\ell^2 r_c^{n-3}}}_{1 \text{ in some cases}}\right)$$

circular orbit: $-\frac{\partial U_{eff}}{\partial r_c} = F_{eff} = \frac{\ell^2}{\mu r_c^3} - \frac{k}{r_c^n} = 0 \longrightarrow r_c^{n-3} = \frac{\mu k}{\ell^2}$

stability: $\mu\ddot r = F_{eff}(r)$, $r = r_c + \eta(t)$; $\ddot\eta = (3-n)\frac{\ell^2}{\mu^2 r_c^2}$

CENTRAL FORCE MOTION III (cont'd)

Closed vs. Open Orbits



closed orbit – for n radial oscillations with

$$T_r = \frac{2\pi}{\omega_r}$$

you can go around exactly n t

m & $n \in 1, 2, 3, \ldots$

$$m T_r = n T_\phi \rightarrow \frac{T_r}{T_\phi} = \frac{\omega_r}{\omega_\phi} = \frac{n}{m}$$

$$\Theta = \Theta\left(\frac{T_r}{2}\right) - \Theta(0) = \omega_\phi \frac{T_r}{2} = \frac{2\pi}{T_\phi}\frac{T_r}{2} = \pi \frac{\omega_\phi}{\omega_r} = \frac{n}{m}\pi$$

not-closed orbit – $\frac{\omega_\phi}{\omega_r}$ is irrational

Bertrand's Theorem

The only closed, stable orbits are for:

1) $U(r) = -\frac{k}{r}$

2) $U(r) = \frac{k}{2} r^2$

CENTRAL FORCE MOTION IV

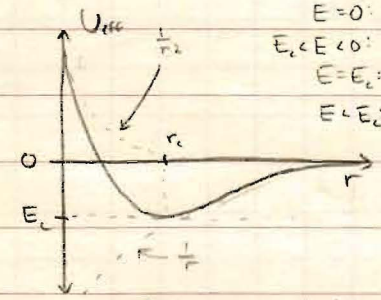
Gravitational Potential

$$U = -\frac{k}{r} \qquad k = Gm_1m_2$$

$$U_{eff} = \frac{\ell^2}{2\mu r^2} - \frac{k}{r} \quad ; \quad \ell = \mu r^2 \dot{\phi}$$

$$F_{eff} = -\frac{\partial U_{eff}}{\partial r} = \frac{\ell^2}{\mu r^3} - \frac{k}{r^2}$$

$$E = \frac{1}{2}\mu \dot{r}^2 + \frac{\ell^2}{2\mu r^2} - \frac{k}{r}$$



$E > 0$: scattering
$E = 0$: marginally bound
$E_c < E < 0$: bound orbit
$E = E_c$: circular orbit
$E < E_c$: physically not allowed

circular motion: $F_{eff} = 0 \longrightarrow r_c = \frac{\ell^2}{\mu k} \longrightarrow E_c = \frac{\ell^2}{\mu k}\left(\frac{\mu k}{\ell^2}\right)^2 - k\left(\frac{\mu k}{\ell^2}\right) = -\frac{\mu k^2}{2\ell^2}$

Method 1: find $\phi(r) \leftrightarrow r(\phi)$ by reducing problem to known integral

$$\frac{d\phi}{dr} = \frac{d\phi}{dt}\frac{dt}{dr} = \left(\frac{\ell}{\mu r^2}\right)\frac{1}{\sqrt{\frac{2}{\mu}\left(E - \frac{\ell^2}{2\mu r^2} + \frac{k}{r}\right)}}$$

non-dimensionalize: $\sqrt{\left|\frac{E_c}{E_c}\right|}$, $|E_c| = \frac{\mu k^2}{2\ell^2}$ $\hookrightarrow$ $\phi = \int \frac{\frac{\ell}{r^2}\,dr}{\sqrt{2\mu\left(E - \frac{\ell^2}{2\mu r^2} + \frac{k}{r}\right)}}$

$$\phi = \int \frac{\frac{\ell}{r^2}\,dr}{\sqrt{2\mu\left(\frac{\mu k^2}{2\ell^2}\right)\left[E\left(\frac{2\ell^2}{\mu k^2}\right) - \frac{\ell^2}{2\mu r^2}\left(\frac{2\ell^2}{\mu k^2}\right) + \frac{k}{r}\left(\frac{2\ell^2}{\mu k^2}\right)\right]}}$$

$$= \int \frac{\frac{\ell^2}{\mu k r^2}\,dr}{\sqrt{\frac{2E\ell^2}{\mu k^2} - \left(\frac{\ell^2}{\mu k r}\right)^2 - 2\left(\frac{\ell^2}{\mu k r}\right)}}$$

$$\rightarrow u = \frac{\ell^2}{\mu k r} = \frac{r_c}{r} \qquad du = -\frac{\ell^2}{\mu k r^2}\,dr$$

$$\phi = \int \frac{-du}{\sqrt{\frac{2E\ell^2}{\mu k^2} - (u^2 - 2u)}} = \int \frac{-du}{\sqrt{\left(\frac{2E\ell^2}{\mu k^2} + 1\right) - (u-1)^2}}$$

$$\rightarrow z = u - 1 \qquad dz = du \quad ; \quad \varepsilon = \sqrt{\frac{2E\ell^2}{\mu k^2} + 1} = \sqrt{1 - \frac{E}{E_c}}$$

$$\phi = \int \frac{-dz}{\sqrt{\varepsilon^2 - z^2}}$$

$$\longrightarrow \quad z = \varepsilon\cos\psi \qquad dz = -\varepsilon\sin\psi\,d\psi = -\varepsilon\sqrt{1 - \frac{z^2}{\varepsilon^2}}\,d\psi \qquad -dz = \sqrt{\varepsilon^2 - z^2}\,d\psi$$

$$\phi = \int d\psi$$

$$\phi = \psi + \phi_0 \rightarrow \cos(\phi - \phi_0) = \cos\psi = \frac{z}{\varepsilon} = \frac{u-1}{\varepsilon} = \frac{\frac{r_c}{r} - 1}{\varepsilon}$$

$$r = \frac{r_c}{1 + \varepsilon\cos(\phi - \phi_0)}$$ can set $\phi_0 = 0$

$$\boxed{r = \frac{r_c}{1 + \varepsilon\cos\phi}}$$

<u>Method II</u>: find $r(\phi)$ by reducing problem to known 2nd order eq.

$$\mu \ddot{r} = F_{eff} = \frac{\ell^2}{\mu r^3} - \frac{k}{r^2}$$

$$u = \frac{1}{r} \qquad \frac{du}{d\phi} = \frac{du}{dr}\frac{dr}{d\phi} = \left(-\frac{1}{r^2}\right)\frac{\dot{r}}{\dot{\phi}}$$

$$= -\frac{1}{r^2}\frac{\mu r^2}{\ell}\dot{r} = -\frac{\mu}{\ell}\dot{r}$$

$$\frac{d^2u}{d\phi^2} = -\frac{\mu}{\ell}\frac{d\dot{r}}{d\phi} = -\frac{\mu}{\ell}\frac{\ddot{r}}{\dot{\phi}}$$

$$= -\frac{\mu}{\ell}\frac{\mu r^2}{\ell}\ddot{r}$$

$$= -\frac{\mu}{\ell^2}\frac{1}{u^2}(\mu\ddot{r})$$

$$= -\frac{\mu}{\ell^2}\frac{1}{u^2}\left(\frac{\ell^2}{\mu}u^3 - ku^2\right)$$

$$= -u + \frac{k\mu}{\ell^2}$$

$$\boxed{r = \frac{r_c}{1+\varepsilon\cos\phi}}$$

$$u'' + u = \frac{1}{r_c}$$

$$u = \frac{1}{r} = \frac{1}{r_c} + A\cos(\phi + \beta)$$

can set to zero

$$= \frac{1+\varepsilon\cos\phi}{r_c}$$

Central Force Motion V

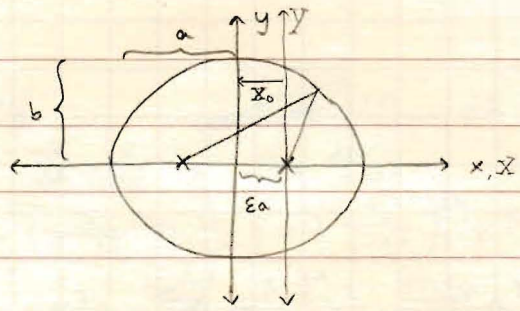
Gravitational Potential

$$\ell = \mu r^2 \dot{\phi}^2 \qquad r_c = \frac{\ell^2}{\mu k}$$

$$U(r) = -\frac{k}{r} \qquad E_c = -\frac{\mu k^2}{2\ell^2}$$

$$U_{eff} = \frac{\ell^2}{2\mu r^2} - \frac{k}{r} \qquad \varepsilon = \sqrt{1 + \frac{2\ell^2}{\mu k^2} E} = \sqrt{1 + \frac{E}{|E_c|}}$$

$$\boxed{r = \frac{r_c}{1 + \varepsilon \cos\phi}}$$



$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$(x, y) = (a\cos\theta, b\sin\theta)$$

$$X_{cm} = r\cos\phi$$

$$Y_{cm} = r\sin\phi$$

Ellipse

$$r_c = r + \varepsilon r\cos\phi \longrightarrow r = r_c - \varepsilon r\cos\phi \rightarrow r^2 = r_c^2 - 2\varepsilon r r\cos\phi + \varepsilon$$

$$X^2 + Y^2 = r_c^2 - 2\varepsilon r_c X + \varepsilon^2 X$$

$$X^2(1-\varepsilon^2) + 2\varepsilon r_c X + Y^2 = r_c^2$$

$$(1-\varepsilon^2)\left(X^2 + \frac{2\varepsilon r_c}{(1-\varepsilon^2)} X + \frac{\varepsilon^2 r_c^2}{(1-\varepsilon^2)^2}\right) + Y^2 = r_c^2 + \frac{\varepsilon^2 r_c^2}{(1-\varepsilon^2)^2}$$

$$(1-\varepsilon^2)(X - X_0)^2 + Y^2 = \frac{r_c^2}{1-\varepsilon^2} \qquad X_0 = -\frac{\varepsilon r_c}{1-\varepsilon^2}$$

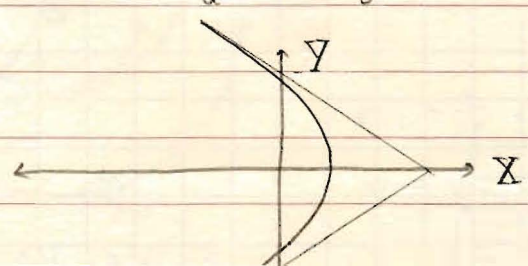
$$\frac{(X - X_0)^2}{\frac{r_c^2}{(1-\varepsilon^2)^2}} + \frac{Y^2}{\frac{r_c^2}{1-\varepsilon^2}} = 1 \qquad a = \frac{r_c}{1-\varepsilon^2}$$

$$b = \frac{r_c}{\sqrt{1-\varepsilon^2}}$$

Hyperbola

$$1 < \varepsilon \longrightarrow \frac{(X - X_0)^2}{a^2} - \frac{Y^2}{b^2} = 1 \qquad a = \frac{r_c}{\varepsilon^2 - 1} \qquad b = \frac{r_c}{\sqrt{\varepsilon^2 - 1}}$$

$$X_0 = +\varepsilon a$$

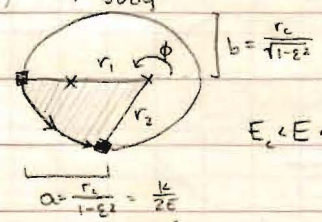


DATE 11/18/13
TOPIC

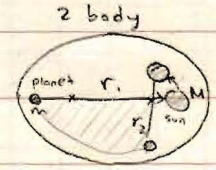
Central Force Motion VI

Kepler's Laws

(I) 1 body $\xleftrightarrow[\text{transfer}]{\text{CM coordinate}}$ 2 body



$E_c < E < 0$



$M_{sun} \gg m_{planet}$

$\mu \approx m_{planet}$

$r_{planet} \approx r$

$$r_c = \frac{\ell^2}{\mu k} \qquad \varepsilon = \sqrt{1 - \frac{2\ell^2|E|}{\mu k^2}} \qquad E_c = -\frac{\mu k^2}{2\ell^2}$$

$$\ell = \mu r^2 \dot{\phi}$$

$$dA = \frac{r^2 d\phi}{2} \qquad \frac{dA}{dt} = \frac{r^2}{2}\dot{\phi} = \frac{\ell}{2\mu} = \text{const.}$$

(II) $\frac{dA}{dt} = \frac{\ell}{2\mu} \longrightarrow dt = \frac{2\mu}{\ell} dA$ Period $\tau = \frac{2\mu}{\ell} A_{ellipse} = \frac{2\mu\pi}{\ell} ab$

(III) $ab = \left(\frac{r_c}{\sqrt{1-\varepsilon^2}}\right)\left(\frac{r_c}{1-\varepsilon^2}\right) = \left(\frac{r_c}{1-\varepsilon^2}\right)^{3/2} r_c^{1/2} = a^{3/2}\sqrt{r_c} = a^{3/2}\frac{\ell}{\sqrt{\mu k}}$

$$\tau^2 = \frac{4\mu^2\pi^2}{\ell^2} a^3 \frac{\ell^2}{\mu k} = \frac{4\pi^2\mu}{k} a^3 \longrightarrow \tau^2 \propto a^3$$

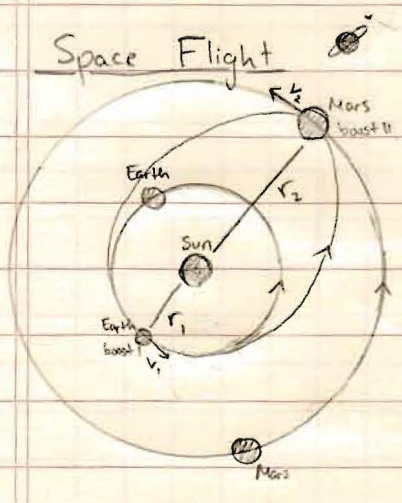
Kepler

I) Planets move on elliptical orbits (w/ the sun at one focus).

II) Area swept out by radius per time is a constant

III) $[\text{period}]^2 \propto \left[\begin{smallmatrix}\text{semi-major}\\ \text{axis}\end{smallmatrix}\right]^3$

Space Flight



Hochmann Transfer

- low energy orbit
- 2 boosts only

$$E_1 = -\frac{k}{2r_1} = \frac{1}{2}mv_1^2 - \frac{k}{r}$$

$$v_1 = \sqrt{\frac{k}{mr_1}} \qquad E_t = -\frac{k}{2}\frac{1}{\frac{r_1+r_2}{2}} = \frac{1}{2}mv_t^2 - \frac{k}{r_1}$$

$$v_t = \sqrt{\frac{2k}{mr_1}\frac{r_2}{r_1+r_2}}$$

$$\Delta V_{boost1} = V_{t1} - V_1$$

$$v_2 = \sqrt{\frac{k}{mr^2}} \qquad V_{t2} = \sqrt{\frac{k}{mr_c}\frac{2r_1}{(r_1+r_2)}}$$

$$\Delta V_{boost2} = V_2 - V_{t2}$$

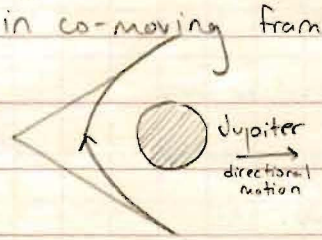
Gold Fibre

DATE 11/18/13
TOPIC

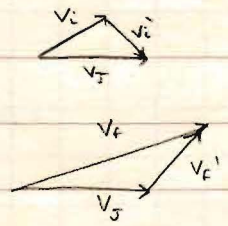
CENTRAL FORCE MOTION VI

Sling-Shot

in co-moving frame



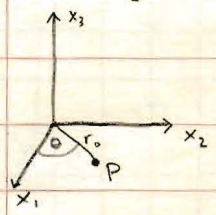
in frame of sun



→ get a boost by stealing momentum from Jupiter

Non-Inertial Reference Frames

Rotating Object



fixed coordinate system Σ_F

unit vectors $\{\hat{e}_{1F}, \hat{e}_{2F}, \hat{e}_{3F}\}$

Rotating point P in x-y plane

Ⓐ Vector Notation

$$\mathbf{r}(t) = [r_0 \cos\theta]\,\hat{e}_{1F} + [r_0 \sin\theta]\,\hat{e}_{2F}$$

Ⓑ Matrix Algebra

$$\mathbf{r} = \mathbb{R}^T \mathbf{r}_0 \Leftrightarrow \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}_F = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} r_0 \\ 0 \\ 0 \end{pmatrix}$$

rotation matrix

T = transpose

In general, for rotation matrices:

$$\mathbb{R} = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Ⓒ Index Notation (for components)

$$x_i(t) = \sum_j R^T_{ij}\, x_j(0)$$

The Derivative

(of the rotating object above)

Ⓐ Vector Notation

$\dot{\theta} = \omega$

$$\frac{d\mathbf{r}}{dt} = [-\dot{\theta} r_0 \sin\theta]\,\hat{e}_{1F} + [\dot{\theta} r_0 \cos\theta]\,\hat{e}_{2F}$$

pick: $\omega\,\hat{e}_{3F} \Leftrightarrow \begin{pmatrix} 0 \\ 0 \\ \omega \end{pmatrix}$ $\quad \begin{pmatrix} 0 \\ 0 \\ \omega \end{pmatrix} \times \begin{pmatrix} r_0 \cos\theta \\ r_0 \sin\theta \\ 0 \end{pmatrix}_F = \begin{pmatrix} -\omega r_0 \sin\theta \\ \omega r_0 \cos\theta \\ 0 \end{pmatrix}_F$

$$\left[\frac{d\mathbf{r}}{dt}\right]_{\text{fixed}} = \boldsymbol{\omega} \times \mathbf{r}$$

Ⓑ Matrix Algebra

$$\frac{d\mathbf{r}}{dt} = \frac{d\mathbb{R}^T}{dt}\mathbf{r}_0 = \frac{d\mathbb{R}^T}{dt}\mathbb{R}\mathbf{r}$$

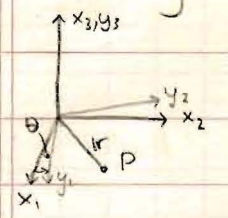
$$\frac{d\mathbb{R}^T}{dt}\mathbb{R}\mathbf{r}(t) = \boldsymbol{\omega} \times \mathbf{r}$$

Ⓒ Index Notation

$$\sum_{ij}{}' \frac{dR^T_{ki}}{dt} R_{ij} r_j = \sum_{ij}{}' \varepsilon_{kij}\, \omega_i r_j$$

Non-Inertial Reference Frames

Rotating Coordinates



object fixed in Σ'_F

fixed coor. Σ_F rotating coor. Σ_r

$\{\hat{e}_{1F}, \hat{e}_{2F}, \hat{e}_{3F}\}$ $\{\hat{e}_{1r}, \hat{e}_{2r}, \hat{e}_{3r}\}$

As an abstract vector, there is no single vector $\mathbb{r}$, but $\mathbb{r}$ has two component forms:

Ⓐ Vector Notation

$$\mathbb{r} = \mathbb{x} = \sum_{i=1}^{3} x_i \hat{e}_{iF} \qquad \mathbb{r} = \mathbb{y} = \sum_{j=1}^{3} y_j \hat{e}_{jr}$$

Ⓑ Matrix Algebra

$$\mathbb{y} = \mathbb{R}\mathbb{x} \qquad \mathbb{x} = \mathbb{R}^T \mathbb{y} \qquad \mathbb{R} = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The Derivative

P fixed in $\Sigma_F \rightarrow x_i = \text{const.}$

$$\left[\frac{d\mathbb{r}}{dt}\right]_{fixed} = \sum_i \frac{dx_i}{dt} \hat{e}_{iF} = 0$$

P in rotating Σ_F

given

$$\left[\frac{d\mathbb{r}}{dt}\right]_{rot} = -\omega \times \mathbb{r}$$

Check matrix algebra

$$\dot{\mathbb{y}} = \dot{\mathbb{R}}\mathbb{x} + \mathbb{R}\dot{\mathbb{x}} = \dot{\mathbb{R}}\mathbb{x}$$

$$= \dot{\mathbb{R}}\mathbb{R}^T\mathbb{y}$$

$$\dot{\mathbb{R}}\mathbb{R}^T = \begin{pmatrix} -\omega\sin\theta & \omega\cos\theta & 0 \\ -\omega\cos\theta & -\omega\sin\theta & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \omega\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\dot{\mathbb{R}}\mathbb{R}^T\mathbb{y} = \omega\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} \omega y_2 \\ -\omega y_1 \\ 0 \end{pmatrix} = -\begin{pmatrix} 0 \\ 0 \\ \omega \end{pmatrix} \times \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

Last Time

Rotating object $\mathbf{r}$ in a fixed frame Σ_F $\left[\frac{d\mathbf{r}}{dt}\right]_{fixed} = \vec{\omega} \times \vec{\mathbf{r}}$

Rotating frame Σ_r in fixed ref. frame Σ_F + object fixed in Σ_F

$$\left[\frac{d\vec{\mathbf{r}}}{dt}\right]_{rot} = -\omega \times \mathbf{r}$$

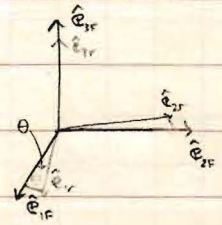
$$\sum_{ij} \frac{dR_{kj}}{dt} R^T_{ij} y_j = -\sum_{ij} \varepsilon_{kij} \omega_i y_j$$

A general vector $\mathbf{q}$ in a rotating coordinate system

Ex. $\mathbf{q} = \omega, \mathbf{r}, \mathbf{F}, \ldots$

How are $\left[\frac{d\mathbf{q}}{dt}\right]_{fixed}$; $\left[\frac{d\mathbf{q}}{dt}\right]_{rot}$ related?

Consider unit vectors:



$$\hat{e}_{1r} = \cos\theta\, \hat{e}_{1F} + \sin\theta\, \hat{e}_{2F}$$

$$\hat{e}_{2r} = -\sin\theta\, \hat{e}_{1F} + \cos\theta\, \hat{e}_{2F}$$

$$\hat{e}_{3r} = \hat{e}_{3F}$$

$$e_{jr} = \sum_{i=1}^{3} R_{ij} \hat{e}_{iF} \qquad \hat{e}_{iF} = \sum_{j=1}^{3} R^T_{ij} \hat{e}_{jr}$$

Vector $\mathbf{q}$ has two representations: Σ_F and Σ_r

$$\mathbf{q} = \sum_i x_i \hat{e}_{iF} \qquad \mathbf{q} = \sum_j y_j \hat{e}_{jr}$$

$$\sum_{ij} x_i R^T_{ij} \hat{e}_{jr} = \sum_{ij} R_{ji} x_i \hat{e}_{jr}$$

$$x_i = \sum_j R^T_{ij} y_j \qquad y_j = \sum R_{ji} x_i$$

$$\left[\frac{d\mathbf{q}}{dt}\right]_{rot} = \sum_j \frac{dy_j}{dt} \hat{e}_{jr} = \sum_{ji} \frac{d}{dt}\left(R_{ji} x_i\right) \hat{e}_{jr}$$

$$= \sum_{ji} x_i \frac{dR_{ji}}{dt} \hat{e}_{jr} + \sum_{ji} R_{ji} \frac{dx_i}{dt} \hat{e}_{jr}$$

$$= \sum_{jik} \frac{dR_{ji}}{dt} R^T_{ik} y_k \hat{e}_{jr} + \sum_i \frac{dx_i}{dt} \sum_j R^T_{ij} \hat{e}_{jr} \qquad \hat{e}_{iF}$$

$$= -\sum_j \underbrace{\sum_{ik} \varepsilon_{jik} \omega_i y_k \hat{e}_{jr}}_{[\omega \times \mathbf{q}]_j} + \underbrace{\sum_i \frac{dx_i}{dt} \hat{e}_{iF}}_{\left[\frac{d\mathbf{q}}{dt}\right]_{fixed}}$$

$$\boxed{\left[\frac{d\mathbf{q}}{dt}\right]_{fixed} = \left[\frac{d\mathbf{q}}{dt}\right]_{rot} + \omega \times \mathbf{q}}$$

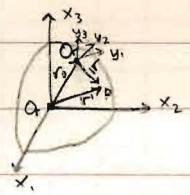
Examples

$q = \mathbf{r}$ — $\mathbf{r}$ is corotating Σ_r (fixed in Σ_r) — $\left[\frac{d\mathbf{r}}{dt}\right]_{fixed} = \boldsymbol{\omega} \times \mathbf{r}$

$q = \mathbf{r}$ — $\mathbf{r}$ is fixed in Σ_f — $0 = \left[\frac{d\mathbf{r}}{dt}\right]_{rot} + \boldsymbol{\omega} \times \mathbf{r}$

$q = \boldsymbol{\omega}$ — $\left[\frac{d\boldsymbol{\omega}}{dt}\right]_{fixed} = \left[\frac{d\boldsymbol{\omega}}{dt}\right]_{rot} + \underbrace{\boldsymbol{\omega} \times \boldsymbol{\omega}}_{0} \equiv \dot{\boldsymbol{\omega}}$

Example



- fixed Σ_f with origin O_F at center of Earth
- Σ'_r has origin (O_r) on surface of Earth and it co-rotates
- $\mathbf{r}_0$ in vector from O_F to O_r
- $\mathbf{r}$ is in Σ_r
- $\mathbf{r}'$ is in Σ_F

$$\mathbf{r}' = \mathbf{r}_0 + \mathbf{r}$$

$$\left[\frac{d\mathbf{r}'}{dt}\right]_{fixed} = \left[\frac{d\mathbf{r}_0}{dt}\right]_{fixed} + \left[\frac{d\mathbf{r}}{dt}\right]_{fixed}$$

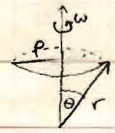
$$= \underbrace{\left[\frac{d\mathbf{r}_0}{dt}\right]_{fixed}}_{\boldsymbol{\omega} \times \mathbf{r}} + \left[\frac{d\mathbf{r}}{dt}\right]_{rot} + \boldsymbol{\omega} \times \mathbf{r}$$

$\mathbf{r}_0$ is corotating

Call $\left[\frac{d\mathbf{r}'}{dt}\right]_{fixed} \equiv \mathbf{v}_F$ $\qquad \left[\frac{d\mathbf{r}}{dt}\right]_{rot} \equiv \mathbf{v}_r$

$$\mathbf{v}_F = \mathbf{v}_r + \boldsymbol{\omega} \times (\mathbf{r}_0 + \mathbf{r})$$

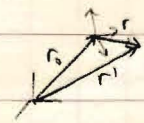
$$\boxed{\mathbf{v}_F = \mathbf{v}_r + \boldsymbol{\omega} \times (\mathbf{r}_0 + \mathbf{r})}$$

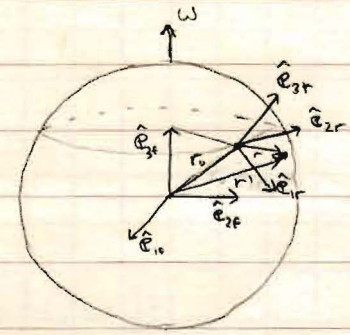


$$|\mathbf{v}| = \rho\omega = r\sin\theta\,\omega = |\boldsymbol{\omega}\times\mathbf{r}| \qquad \mathbf{v} = \boldsymbol{\omega}\times\mathbf{r}$$

Arbitrary Vector in Rotating Coordinates

$$\left[\frac{d\mathbf{q}}{dt}\right]_{fixed} = \left[\frac{d\mathbf{q}}{dt}\right]_{rot} + \boldsymbol{\omega}\times\mathbf{q}$$





$$\mathbf{r}' = \mathbf{r}_0 + \mathbf{r}$$

$$\underbrace{\left[\frac{d\mathbf{r}'}{dt}\right]_{fixed}}_{\mathbf{v}_f} = \left[\frac{d\mathbf{r}_0}{dt}\right]_{fixed} + \underbrace{\left[\frac{d\mathbf{r}}{dt}\right]_{rot}}_{\mathbf{v}_r} + \boldsymbol{\omega}\times\mathbf{r}$$

Ex. Σ_r is corotating with Earth

$$\rightarrow \left[\frac{d\mathbf{r}_0}{dt}\right]_{fixed} = \boldsymbol{\omega}\times\mathbf{r}_0$$

$$\mathbf{v}_f = \mathbf{v}_r + \boldsymbol{\omega}\times(\mathbf{r}_0 + \mathbf{r})$$

$$\underbrace{\left[\frac{d^2}{dt^2}\mathbf{r}'\right]_{fixed}}_{\mathbf{a}_f} = \underbrace{\left[\frac{d^2}{dt^2}\mathbf{r}_0\right]_{fixed}}_{[\ddot{\mathbf{r}}_0]_f} + \underbrace{\left[\frac{d}{dt}\left[\frac{d\mathbf{r}}{dt}\right]_{rot}\right]_{fixed}}_{\underbrace{\left[\frac{d^2\mathbf{r}}{dt^2}\right]_{rot}}_{\mathbf{a}_r} + \underbrace{\boldsymbol{\omega}\times\left[\frac{d\mathbf{r}}{dt}\right]_{rot}}_{\mathbf{v}_r}} + \underbrace{\left[\frac{d\boldsymbol{\omega}}{dt}\right]_{fixed}\times\mathbf{r}}_{\dot{\boldsymbol{\omega}}\times\mathbf{r}} + \underbrace{\boldsymbol{\omega}\times\left[\frac{d\mathbf{r}}{dt}\right]_{fixed}}_{\underbrace{\boldsymbol{\omega}\times\left[\frac{d\mathbf{r}}{dt}\right]_{rot}}_{\mathbf{v}_r} + \boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r})}$$

acceleration:

$$\boxed{\mathbf{a}_f = \mathbf{a}_r + [\ddot{\mathbf{r}}_0]_f + \dot{\boldsymbol{\omega}}\times\mathbf{r} + \boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r}) + 2\boldsymbol{\omega}\times\mathbf{v}_r}$$

In inertial fixed frame Σ_f:

$$m\mathbf{a}_f = \mathbf{F}$$

In non-inertial frame Σ_r:

If we presume (erroneously) that Newton's 2nd Law should have the form $m\mathbf{a}_r = \mathbf{F}_{eff}$ then there are additional noninertial terms (fictitious forces).

flip page

$$m\mathbf{a}_r = \mathbf{F}_{eff} = \mathbf{F} - m[\ddot{\mathbf{r}}_o]_f - m\dot{\boldsymbol{\omega}}\times\mathbf{r} - m\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r}) - 2m\boldsymbol{\omega}\times\mathbf{v}_r$$

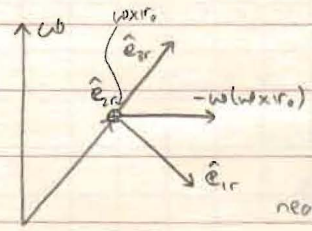
acceleration of origin — angular acceleration term — centrifugal force — Coriolis force

Ex. Σ_r fixed to surface of Earth

$$[\ddot{\mathbf{r}}_o]_f = \boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r}_o)$$

Ex. Earth $\omega \approx$ constant

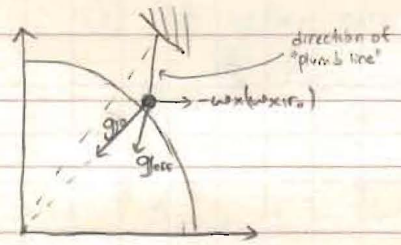
$$\dot{\boldsymbol{\omega}}\times\mathbf{r} = 0$$



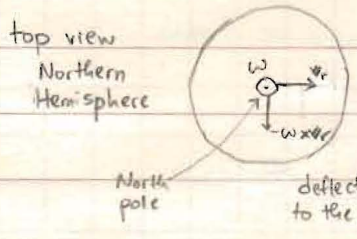
On Earth

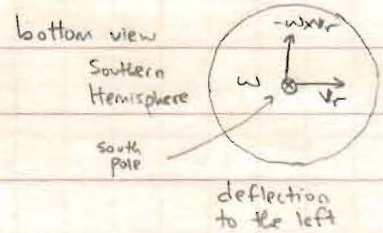
$$F_{CF} = -m\boldsymbol{\omega}\times(\boldsymbol{\omega}\times[\mathbf{r}_o + \mathbf{r}])$$

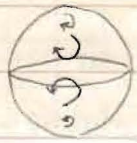
near surface: $F_{CF} \approx -m\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r}_o)$

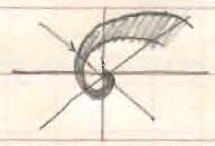


Coriolis Force





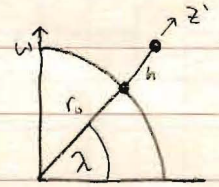


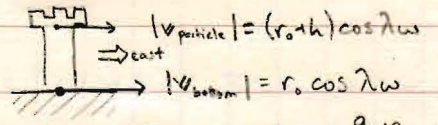


Free Fall

We drop a mass from the top of an elevated tower ($h = 100\,m$).
Where, with respect to the plumb line, does the mass hit the bottom?

(A) Back-of-the-envelope estimate (in Σ_p)

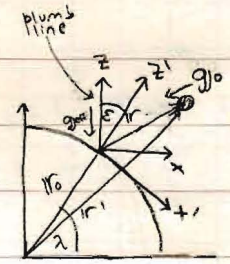




time to fall T: $\quad z' = -\frac{g}{2}t^2 + h \longrightarrow T = \sqrt{\frac{2h}{g}}$

$$d_{east} = v_{particle}T - v_{bottom}T$$
$$= h\omega\cos\lambda\sqrt{\frac{2h}{g}}$$

over estimation
should have 2/3



(B) A better soln (in Σ'_r)

$$a_r = \frac{F}{m} - \underbrace{[\ddot{r}_0]_f}_{\omega\times(\omega\times r_0)} - \underbrace{\dot{\omega}\times r}_{0} - \overset{\text{coriolis}}{2\omega\times v_f} - \overset{\text{centrifugal}}{\omega\times(\omega\times r)}$$

Approx I.
$$\frac{F}{m} = -\frac{GM}{(r')^3}r' = \Big(\underbrace{-\overset{\frac{GM}{r_0^2}}{g}\hat{e}_{z'}}_{g_0}\Big) + \underbrace{\Big[g\hat{e}_z - \frac{GM}{(r_0+r)^3}(r_0+r)\Big]}_{\text{the difference in magnitude and direction of force of gravity at particle locations and bottom}}$$

$\frac{F}{m} \approx g_0$ $\qquad$ $\left[\text{size} < \frac{h}{r_0} \ll 1\right]$

Work in x-y-z coord. (z along plumb line)

$$\omega \Rightarrow \begin{pmatrix}\omega_x\\ \omega_y\\ \omega_z\end{pmatrix} = \begin{pmatrix}-\omega\cos(\lambda+\varepsilon)\\ 0\\ \omega\sin(\lambda+\varepsilon)\end{pmatrix}$$

small quantity

$$\omega = |\omega| = \frac{2\pi}{24\,hrs} \approx 7.3\times10^{-5}\,\frac{rad}{sec}$$

$$a_r = \underbrace{g_0 - \omega\times(\omega\times r_0)}_{g_{eff}} - 2\omega\times v_r - \omega\times(\omega\times r)$$

$$\psi = \lambda + \varepsilon \approx \lambda$$

$$\hat{x} = 0 + 2\omega\sin(\psi)\dot{y} + \omega^2\sin(\psi)[\sin(\psi)x + \cos(\psi)z]$$
$$\hat{y} = 0 - 2\omega\sin(\psi)\dot{x} - 2\omega\cos(\psi)\dot{z} + \omega^2 y$$
$$\hat{z} = -g_{eff} + 2\omega\sin(\psi)\dot{y} + \omega^2\sin(\psi)[\sin(\psi)x + \cos(\psi)y]$$

Approx II Neglect terms of second order in ω

Perturbation Solution

$x = x^{(0)} + x^{(1)}$ $\qquad x^{(0)} \propto \omega^0$

$y = y^{(0)} + y^{(1)}$ $\qquad x^{1} \propto \omega^1$

$z = z^{(0)} + z^{(1)}$

zeroth order

$\ddot{x}^{(0)} = 0 \rightarrow x^{(0)} = 0$

$\ddot{y}^{(0)} = 0 \rightarrow y^{(0)} = 0$

$\ddot{z}^{(0)} = 0 \rightarrow z^{(0)} = h - \frac{g_{eff}}{2} t^2$

initial conditions: $x = y = 0$; $z = h$; $\dot{x} = \dot{y} = \dot{z} = 0$

first order

$\ddot{x}^{(1)} = 0 \rightarrow x^{(1)} = 0 \rightarrow x = 0$

$\ddot{y}^{(1)} = (-2\omega\cos\varphi)(-g_{eff}t) \rightarrow y^{(1)} = g_{eff}\,\omega\cos(\varphi)\frac{t^3}{3} \rightarrow y = g_{eff}\,\omega\cos(\varphi)\frac{t^3}{3}$

$\ddot{z}^{(1)} = 0 \rightarrow z^{(1)} = 0 \rightarrow z = h - \frac{g_{eff}}{2} t^2$

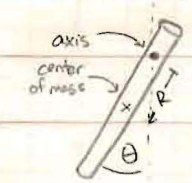
fall time $T = \sqrt{\frac{2h}{g_{eff}}}$

$$d_{east} = y(T) = \frac{2}{3} h\omega\cos(\lambda + \varepsilon)\sqrt{\frac{2h}{g_{eff}}}$$

$$\boxed{d_{east} = \frac{2}{3} h\omega\cos(\lambda + \varepsilon)\sqrt{\frac{2h}{g_{eff}}}}$$

Rigid Body Motion I

(A) Review: 2D Rotation



uniform rod
- mass m
- length l

moment of inertia

$$T = \frac{1}{2} I \omega^2 = \frac{1}{2} I \dot{\theta}^2$$

look like energies for a point mass

$$U = mgh_{cm} = mgR(1-\cos\theta)$$

$$L = T - U = \frac{1}{2} I \dot{\theta}^2 + mgR\cos\theta - \cancel{mgR}$$

$$0 = \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} = I\ddot{\theta} + mgR\sin\theta \xrightarrow{\text{small angle}} \ddot{\theta} = -\frac{mgR}{I}\theta$$

$$\omega = \sqrt{\frac{g}{\beta R}}$$

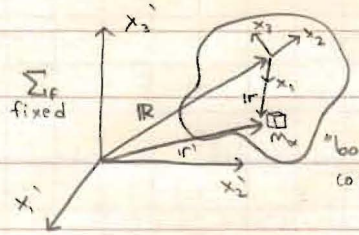
$$I = I_{cm} + mR^2$$

$$= \frac{1}{12} m l^2 + mR^2 = mR^2 \overbrace{\left(\frac{l^2}{12R^2} + 1\right)}^{\beta}$$

dimensionless

(B) 3D Rotation

$I \to \mathbb{I}$ inertia tensor ; components I_{ij}



"body coordinate" corotating

$$T = \frac{1}{2}\sum_\alpha m_\alpha (\mathbb{v}_{\alpha,f})^2$$

$$\mathbb{v}_{\alpha,f} = \mathbb{V} + \mathbb{v}_{\alpha,r} + \omega \times \mathbb{r}_\alpha$$

$\left(\frac{d\mathbb{R}}{dt}\right)_f$ motion of origin

0 for rigid body

$$T = \frac{1}{2}\sum_\alpha m_\alpha (\mathbb{V} + \omega\times\mathbb{r}_\alpha)\cdot(\mathbb{V} + \omega\times\mathbb{r}_\alpha)$$

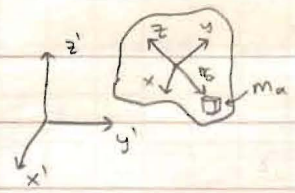
$\sum_\alpha m_\alpha = M$

$$= \underbrace{\frac{1}{2}\sum_\alpha m_\alpha V^2}_{T_{trans} = \frac{1}{2}MV^2} + \underbrace{\frac{1}{2}\sum_\alpha m_\alpha (\omega\times\mathbb{r}_\alpha)\cdot(\omega\times\mathbb{r}_\alpha)}_{T_{rot} = \frac{1}{2}\sum_{i,j=1}^{3} I_{ij}\omega_i\omega_j} + \sum_\alpha m_\alpha \mathbb{V}\cdot(\omega\times\mathbb{r}_\alpha)$$

$$\mathbb{V}\cdot\Big(\omega\times\underbrace{\Big(\sum_\alpha m_\alpha \mathbb{r}_\alpha\Big)}_{\mathbb{r}_{cm}M}\Big) = 0$$

if we pick origin at cm. b/c then $\mathbb{r}_{cm} = 0$

Rigid Body Motion II



Found: $T_{rot} = \frac{1}{2}\sum_{\alpha=1}^{N} m_\alpha (\omega \times r_\alpha)\cdot(\omega \times r_\alpha)$

$= \frac{1}{2}\sum_{ij=1}^{3} I_{ij}\,\omega_i \omega_j$

$= \frac{1}{2}\,\omega \cdot \mathbb{I} \cdot \omega$

use: $[\omega \times r_\alpha]_\nu = \sum_{ij} \varepsilon_{ij\nu}\,\omega_i\, x_{\alpha j}$

$\sum_\nu \varepsilon_{ij}\,\varepsilon_{\ell k \nu} = \delta_{i\ell}\,\delta_{jk} - \delta_{ik}\,\delta_{j\ell}$ $\qquad\qquad$ $\sum_{\ell=1}^{3} \delta_{j\ell}\,\omega_\ell = \omega_j$

$$
\begin{aligned}
T_{rot} &= \frac{1}{2}\sum_\alpha m_\alpha \sum_\nu \sum_{ij} \sum_{\ell k} \varepsilon_{ij\nu}\,\omega_i x_{\alpha,j}\;\varepsilon_{\ell k \nu}\,\omega_\ell x_{\alpha,k} \\
&= \frac{1}{2}\sum_\alpha m_\alpha \sum_{ijk\ell=1}^{3} (\delta_{i\ell}\delta_{jk} - \delta_{ik}\delta_{j\ell})\,\omega_i \omega_\ell x_{\alpha j} x_{\alpha k} \\
&= \frac{1}{2}\sum_\alpha m_\alpha \Big(\underbrace{\sum_i \omega_i^2}_{\sum_i\sum_j \omega_i\omega_j\delta_{ij}} \sum x_{\alpha,k}^2 - \underbrace{\sum_{ij} \omega_i \omega_j}_{\nu}\, x_{\alpha j} x_{\alpha i}\Big) \\
&= \frac{1}{2}\sum_{ij} \omega_i \omega_j \Big[\sum_\alpha m_\alpha \Big(\delta_{ij}\sum_k^3 x_{\alpha k}^2 - x_{\alpha j} x_{\alpha i}\Big)\Big]
\end{aligned}
$$

$$I_{ij} = \sum_{\alpha=1}^{N} m_\alpha \Big(\delta_{ij}\sum_{k=1}^{3} x_{\alpha k}^2 - x_{\alpha i} x_{\alpha j}\Big)$$

continuum: define mass density at r : $\rho(r)$

$$= \iiint_V \rho(r)\Big(\delta_{ij}\sum_{k=1}^{3} x_k^2 - x_i x_j\Big)\,dx_1\,dx_2\,dx_3$$

$x_1 \leftrightarrow x \qquad x_2 \leftrightarrow y \qquad x_3 \leftrightarrow z \qquad \sum_{k=1}^{3} x_k^2 = x^2 + y^2 + z^2 = r^2$

$$
\begin{pmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{pmatrix}
=
\begin{pmatrix} \sum_\alpha m_\alpha (r_\alpha^2 - x_\alpha^2) & -\sum_\alpha m_\alpha x_\alpha y_\alpha & -\sum_\alpha m_\alpha x_\alpha z_\alpha \\ \cdots & \sum_\alpha m_\alpha (r_\alpha^2 - y_\alpha^2) & -\sum m_\alpha y_\alpha z_\alpha \\ \cdots & \cdots & \sum_\alpha m_\alpha (r_\alpha^2 - z_\alpha^2) \end{pmatrix}
$$

$\mathbb{I}$ is real and symmetric $\rightarrow$ diagonalizable

$\rightarrow$ can pick orthogonal, real basis s.t.

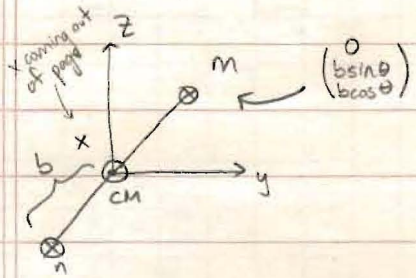
$$\mathbb{I} = \begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{pmatrix}$$

principal moments of inertia $\leftrightarrow$ eigenvalues

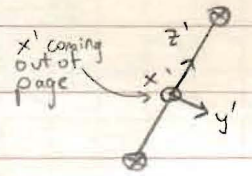
corresponding basis vector $\qquad$ principal axes of inertia $\leftrightarrow$ eigenvectors

Rigid Body Motion II

Ex 2 point masses connected by massless rod



$$\mathbb{I} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1-\sin^2\theta & -\sin\theta\cos\theta \\ 0 & -\sin\theta\cos\theta & 1-\cos^2\theta \end{pmatrix}$$



$$\mathbb{I} = \begin{pmatrix} 2mb^2 & 0 & 0 \\ 0 & 2mb^2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Shown:

$$T_{rot} = \frac{1}{2}\omega \cdot \mathbb{I} \cdot \omega = \frac{1}{2}\sum_{i,j=1}^{3} I_{ij}\omega_i\omega_j$$
$$= \frac{1}{2}\left(I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2\right)$$

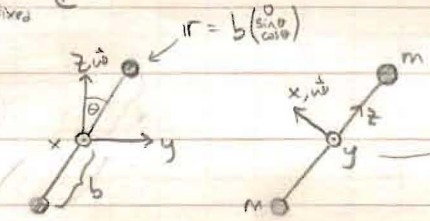
Angular Momentum

$$\mathbb{L} = \overleftrightarrow{\mathbb{I}} \cdot \omega \qquad L_i = \sum_j I_{ij}\omega_j$$

Newton II

$$\left[\frac{d\mathbb{L}}{dt}\right]_{fixed} = \tau \quad \text{torque}$$

$$\omega = \begin{pmatrix} 0 \\ 0 \\ \omega \end{pmatrix}$$

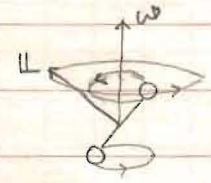


$$\omega = \begin{pmatrix} \omega \\ 0 \\ 0 \end{pmatrix}$$

$$\mathbb{I} = \begin{pmatrix} 2mb^2 & 0 & 0 \\ 0 & 2mb^2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \mathbb{L} = \mathbb{I} \cdot \omega = \begin{pmatrix} 2mb^2\omega \\ 0 \\ 0 \end{pmatrix}$$

$$\mathbb{I} = 2mb^2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos^2\theta & -\sin\theta\cos\theta \\ 0 & \sin\theta\cos\theta & \sin^2\theta \end{pmatrix}$$

$$\mathbb{L} = \mathbb{I} \cdot \omega = 2mb^2 \begin{pmatrix} 0 \\ -\sin\theta\cos\theta\,\omega \\ \omega\sin^2\theta \end{pmatrix}$$
$$= \underbrace{2mb^2\omega\sin\theta}_{|\mathbb{L}|} \begin{pmatrix} 0 \\ -\cos\theta \\ \sin\theta \end{pmatrix}$$



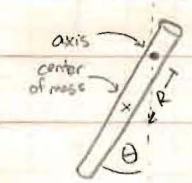
If ω is not along a principal axes, then $\mathbb{L} \nparallel \omega$.

1) If we want ω = const. need torque because $\mathbb{L}$ changes directions

2) What if $\tau = 0$

Rigid Body Motion I

(A) Review: 2D Rotation



uniform rod
- mass m
- length l

moment of inertia

$$T = \frac{1}{2} I \omega^2 = \frac{1}{2} I \dot{\theta}^2$$

look like energies for a point mass

$$U = mgh_{cm} = mgR(1-\cos\theta)$$

$$L = T - U = \frac{1}{2} I \dot{\theta}^2 + mgR\cos\theta - \cancel{mgR}$$

$$0 = \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} = I\ddot{\theta} + mgR\sin\theta \xrightarrow{\text{small angle}} \ddot{\theta} = -\frac{mgR}{I}\theta$$

$$\omega = \sqrt{\frac{g}{\beta R}}$$

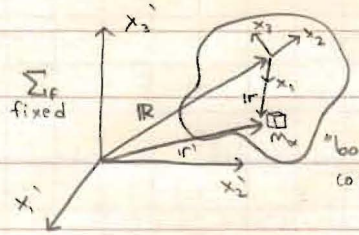
$$I = I_{cm} + mR^2$$

$$= \frac{1}{12} m l^2 + mR^2 = mR^2 \overbrace{\left(\frac{l^2}{12R^2} + 1\right)}^{\beta}$$

dimensionless

(B) 3D Rotation

$I \to \mathbb{I}$ inertia tensor ; components I_{ij}



"body coordinate" corotating

$$T = \frac{1}{2}\sum_\alpha m_\alpha (\mathbb{v}_{\alpha,f})^2$$

$$\mathbb{v}_{\alpha,f} = \mathbb{V} + \mathbb{v}_{\alpha,r} + \omega \times \mathbb{r}_\alpha$$

$\left(\frac{d\mathbb{R}}{dt}\right)_f$ motion of origin

0 for rigid body

$$T = \frac{1}{2}\sum_\alpha m_\alpha (\mathbb{V} + \omega\times\mathbb{r}_\alpha)\cdot(\mathbb{V} + \omega\times\mathbb{r}_\alpha)$$

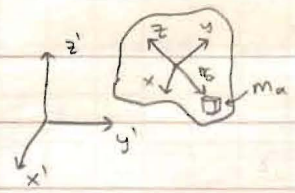
$\sum_\alpha m_\alpha = M$

$$= \underbrace{\frac{1}{2}\sum_\alpha m_\alpha V^2}_{T_{trans} = \frac{1}{2}MV^2} + \underbrace{\frac{1}{2}\sum_\alpha m_\alpha (\omega\times\mathbb{r}_\alpha)\cdot(\omega\times\mathbb{r}_\alpha)}_{T_{rot} = \frac{1}{2}\sum_{i,j=1}^{3} I_{ij}\omega_i\omega_j} + \sum_\alpha m_\alpha \mathbb{V}\cdot(\omega\times\mathbb{r}_\alpha)$$

$$\mathbb{V}\cdot\Big(\omega\times\underbrace{\Big(\sum_\alpha m_\alpha \mathbb{r}_\alpha\Big)}_{\mathbb{r}_{cm}M}\Big) = 0$$

if we pick origin at cm. b/c then $\mathbb{r}_{cm} = 0$

Rigid Body Motion II



Found: $T_{rot} = \frac{1}{2}\sum_{\alpha=1}^{N} m_\alpha (\omega \times r_\alpha)\cdot(\omega \times r_\alpha)$

$= \frac{1}{2}\sum_{ij=1}^{3} I_{ij}\,\omega_i \omega_j$

$= \frac{1}{2}\,\omega \cdot \mathbb{I} \cdot \omega$

use: $[\omega \times r_\alpha]_\nu = \sum_{ij} \varepsilon_{ij\nu}\,\omega_i\, x_{\alpha j}$

$\sum_\nu \varepsilon_{ij}\,\varepsilon_{\ell k \nu} = \delta_{i\ell}\,\delta_{jk} - \delta_{ik}\,\delta_{j\ell}$ $\qquad\qquad$ $\sum_{\ell=1}^{3} \delta_{j\ell}\,\omega_\ell = \omega_j$

$$
\begin{aligned}
T_{rot} &= \frac{1}{2}\sum_\alpha m_\alpha \sum_\nu \sum_{ij} \sum_{\ell k} \varepsilon_{ij\nu}\,\omega_i x_{\alpha,j}\;\varepsilon_{\ell k \nu}\,\omega_\ell x_{\alpha,k} \\
&= \frac{1}{2}\sum_\alpha m_\alpha \sum_{ijk\ell=1}^{3} (\delta_{i\ell}\delta_{jk} - \delta_{ik}\delta_{j\ell})\,\omega_i \omega_\ell x_{\alpha j} x_{\alpha k} \\
&= \frac{1}{2}\sum_\alpha m_\alpha \Big(\underbrace{\sum_i \omega_i^2}_{\sum_i\sum_j \omega_i\omega_j\delta_{ij}} \sum x_{\alpha,k}^2 - \underbrace{\sum_{ij} \omega_i \omega_j}_{\nu}\, x_{\alpha j} x_{\alpha i}\Big) \\
&= \frac{1}{2}\sum_{ij} \omega_i \omega_j \Big[\sum_\alpha m_\alpha \Big(\delta_{ij}\sum_k^3 x_{\alpha k}^2 - x_{\alpha j} x_{\alpha i}\Big)\Big]
\end{aligned}
$$

$$I_{ij} = \sum_{\alpha=1}^{N} m_\alpha \Big(\delta_{ij}\sum_{k=1}^{3} x_{\alpha k}^2 - x_{\alpha i} x_{\alpha j}\Big)$$

continuum: define mass density at r : $\rho(r)$

$$= \iiint_V \rho(r)\Big(\delta_{ij}\sum_{k=1}^{3} x_k^2 - x_i x_j\Big)\,dx_1\,dx_2\,dx_3$$

$x_1 \leftrightarrow x \qquad x_2 \leftrightarrow y \qquad x_3 \leftrightarrow z \qquad \sum_{k=1}^{3} x_k^2 = x^2 + y^2 + z^2 = r^2$

$$
\begin{pmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{pmatrix}
=
\begin{pmatrix} \sum_\alpha m_\alpha (r_\alpha^2 - x_\alpha^2) & -\sum_\alpha m_\alpha x_\alpha y_\alpha & -\sum_\alpha m_\alpha x_\alpha z_\alpha \\ \cdots & \sum_\alpha m_\alpha (r_\alpha^2 - y_\alpha^2) & -\sum m_\alpha y_\alpha z_\alpha \\ \cdots & \cdots & \sum_\alpha m_\alpha (r_\alpha^2 - z_\alpha^2) \end{pmatrix}
$$

$\mathbb{I}$ is real and symmetric $\rightarrow$ diagonalizable

$\rightarrow$ can pick orthogonal, real basis s.t.

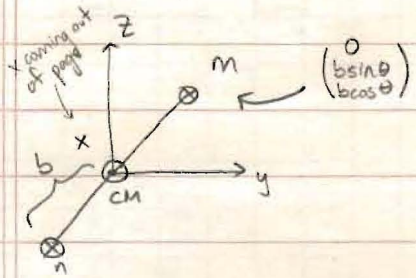
$$\mathbb{I} = \begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{pmatrix}$$

principal moments of inertia $\leftrightarrow$ eigenvalues

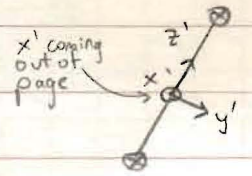
corresponding basis vector $\qquad$ principal axes of inertia $\leftrightarrow$ eigenvectors

Rigid Body Motion II

Ex 2 point masses connected by massless rod



$$\mathbb{I} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1-\sin^2\theta & -\sin\theta\cos\theta \\ 0 & -\sin\theta\cos\theta & 1-\cos^2\theta \end{pmatrix}$$



$$\mathbb{I} = \begin{pmatrix} 2mb^2 & 0 & 0 \\ 0 & 2mb^2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Shown:

$$T_{rot} = \frac{1}{2}\omega \cdot \mathbb{I} \cdot \omega = \frac{1}{2}\sum_{i,j=1}^{3} I_{ij}\omega_i\omega_j$$
$$= \frac{1}{2}\left(I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2\right)$$

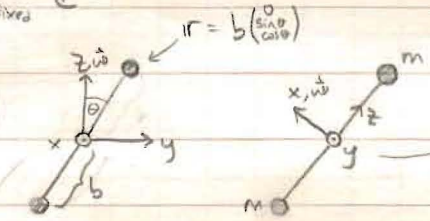
Angular Momentum

$$\mathbb{L} = \overleftrightarrow{\mathbb{I}} \cdot \omega \qquad L_i = \sum_j I_{ij}\omega_j$$

Newton II

$$\left[\frac{d\mathbb{L}}{dt}\right]_{fixed} = \tau \quad \text{torque}$$

$$\omega = \begin{pmatrix} 0 \\ 0 \\ \omega \end{pmatrix}$$

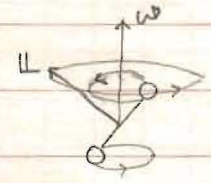


$$\omega = \begin{pmatrix} \omega \\ 0 \\ 0 \end{pmatrix}$$

$$\mathbb{I} = \begin{pmatrix} 2mb^2 & 0 & 0 \\ 0 & 2mb^2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \mathbb{L} = \mathbb{I} \cdot \omega = \begin{pmatrix} 2mb^2\omega \\ 0 \\ 0 \end{pmatrix}$$

$$\mathbb{I} = 2mb^2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos^2\theta & -\sin\theta\cos\theta \\ 0 & \sin\theta\cos\theta & \sin^2\theta \end{pmatrix}$$

$$\mathbb{L} = \mathbb{I} \cdot \omega = 2mb^2 \begin{pmatrix} 0 \\ -\sin\theta\cos\theta\,\omega \\ \omega\sin^2\theta \end{pmatrix}$$
$$= \underbrace{2mb^2\omega\sin\theta}_{|\mathbb{L}|} \begin{pmatrix} 0 \\ -\cos\theta \\ \sin\theta \end{pmatrix}$$



If ω is not along a principal axes, then $\mathbb{L} \nparallel \omega$.

1) If we want ω = const. need torque because $\mathbb{L}$ changes directions

2) What if $\tau = 0$

Rigid Body Motion IV

Euler's Equations

Def. $\mathbb{L} = \mathbb{I} \cdot \omega$ (general)

In body frame and principal axes

$$L_i = I_i \omega_i(t) \qquad i = 1, 2, 3$$

Newton's 2nd Law: $\left[\frac{d\mathbb{L}}{dt}\right]_{fixed} = \mathbb{T}$

$$\left[\frac{d\mathbb{L}}{dt}\right]_{fixed} = \left[\frac{d\mathbb{L}}{dt}\right]_{rot} + \omega \times \mathbb{L}$$

$$= \left[\frac{d}{dt}(\mathbb{I} \cdot \omega)\right]_{rot} + \omega \times \mathbb{L}$$

$$= \mathbb{I} \cdot \dot{\omega} + \omega \times (\mathbb{I} \cdot \omega)$$

Newton II: $\mathbb{I} \cdot \dot{\omega} + \omega \times (\mathbb{I} \cdot \omega) = \mathbb{T}$ (body frame)

Euler's eqn: $I_k \dot{\omega}_k + \sum_{ij}' \varepsilon_{ijk} \omega_i I_j \omega_j = \tau_k$ (body frame, principal axes)

Recall:

$$\varepsilon_{123} = \varepsilon_{312} = \varepsilon_{231} = 1$$

$$\varepsilon_{132} = \varepsilon_{213} = \varepsilon_{321} = -1$$

$$\varepsilon = 0 \text{ otherwise}$$

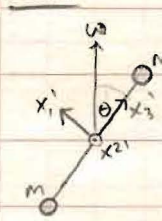
Euler's Eqn

$$I_1 \dot{\omega}_1 - (I_2 - I_3)\omega_3 \omega_2 = \tau_1$$

$$I_2 \dot{\omega}_2 - (I_3 - I_1)\omega_1 \omega_3 = \tau_2$$

$$I_3 \dot{\omega}_3 - (I_1 - I_2)\omega_2 \omega_1 = \tau_3$$

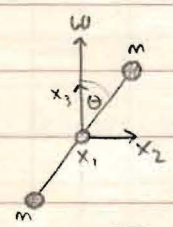
Ex.



body frame

$\Sigma_b : x_1', x_2', x_3'$

$\omega = \omega \begin{pmatrix} \sin\theta \\ 0 \\ \cos\theta \end{pmatrix} \qquad \dot{\omega} = 0$



fixed frame

$\Sigma_f : x_1, x_2, x_3$

In Σ_b: $\mathbb{I} = 2mb^2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Euler's Eqns

$$0 = \tau_1$$

$$2mb^2\omega^2 \sin\theta\cos\theta\, \tau_2$$

$$0 = \tau_3$$

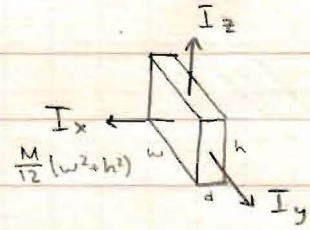
In Σ_f: $\hat{e}_2'$ rotates in $x_1 - x_2$ plane with angular speed ω

$$\mathbb{T} = 2mb^2\omega^2 \sin\theta\cos\theta\,(\cos(\omega t)\hat{e}_1 + \sin(\omega t)\hat{e}_2)$$

Rigid Body Motion V

No Torque

$$I_1\dot{\omega}_1 - (I_2 - I_3)\omega_2\omega_3 = 0$$
$$I_2\dot{\omega}_2 - (I_3 - I_1)\omega_1\omega_3 = 0$$
$$I_3\dot{\omega}_3 - (I_1 - I_2)\omega_1\omega_2 = 0$$



$I_x > I_y > I_z$

Throw block so that it spins around one principal axis, I_1:

- perfect throw: $\omega_0 = \begin{pmatrix}\Omega\\0\\0\end{pmatrix}$

  soln: $\omega(t) = \begin{pmatrix}\Omega\\0\\0\end{pmatrix}$

- imperfect throw: $\omega_0 = \begin{pmatrix}\Omega\\\tilde{\omega}_{20}\\\tilde{\omega}_{30}\end{pmatrix}$ ← tiny rotations about other axes

  will $\omega(t)$ stay close to perfect throw soln?

  Perturbation Analysis

  Ansatz $\omega = \begin{pmatrix}\Omega\\0\\0\end{pmatrix} + \begin{pmatrix}\tilde{\omega}_1\\\tilde{\omega}_2\\\tilde{\omega}_3\end{pmatrix}$ ← small

  to first order in $\tilde{\omega}$ the Euler eqn's are:

$$I_1\dot{\tilde{\omega}}_1 - \underbrace{\cancel{(I_1 - I_3)\tilde{\omega}_2\tilde{\omega}_3}}_{2^{nd}\text{ order}} = 0 \rightarrow \tilde{\omega}_1 = 0$$

$$\left.\begin{aligned} I_2\dot{\tilde{\omega}}_2 - (I_3 - I_2)\Omega\tilde{\omega}_3 &= 0 \\ I_3\dot{\tilde{\omega}}_3 - (I_1 - I_2)\Omega\tilde{\omega}_2 &= 0 \end{aligned}\right\} \quad \ddot{\tilde{\omega}}_2 - \underbrace{\left(\frac{I_3 - I_1}{I_2}\Omega\right)\left(\frac{I_2 - I_1}{I_3}\Omega\right)}_{\substack{>0\ \tilde{\omega}_2\text{ is oscillatory} \\ =0\ \tilde{\omega}_2\text{ is linear in } t \\ <0\ \tilde{\omega}_2\text{ exponential}}}\tilde{\omega}_2 = 0$$

① Stable - pert. soln. stays close to perfect throw soln.

if $I_3 > I_1$ and $I_2 > I_1$ → I_1 is a minimum principal moment

or $I_3 < I_1$ and $I_2 < I_1$ → I_1 is a maximum

② Unstable - pert. soln. grows, deviates from perfect throw soln

$I_2 \leq I_1 \leq I_3$

$I_3 \leq I_1 \leq I_2$