PLURIDISCIPLINE REPORT

on the subject of

A study on application of Monte Carlo method in evaluating risks

by

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class

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Nomenclature

Π_t	Debt value of the portfolio
σ	Stock's annual volatility
B	Threshold under which the company is considered bankrupted
N	Number of stocks in a portfolio
N_{mc}	Number of simulations
R	Recovery rate
r	Annual interest rate
S_t	Stock price at time t
T	Time horizon in which the default event is examined
W_t	Geometric Brownian Motion follows normal distribution $\mathcal{N}(0, t)$
X	Loss at the end of the given time horizon

1 Introduction

The global financial market, in the past decades, has suffered from several unprecedented crises: Dot-com bubble (2000-2002), Global Financial Crisis (2007-2009), Greek government-debt crisis (2009-now), etc. Despite the complexity of their causes, almost all of those crises shared a similar driver which is the inability of governments, financial institutions and investors to assess and control credit risk.

What exactly is credit risk? In general, it is the risk that a borrower fails to meet his/her obligated payment for the debt. The reason can be due to either a fraud or the borrowers facing financial hardship or disruption in cash flow, etc.

Effective methods to quantify credit risk are becoming more and more vital for financial institutions. Value-at-Risk (VaR) is perhaps the most common risk measure in banking industry ever since it was integrated into the Basel Accord as a requirement to assess capital adequacy. VaR basically means the potential loss within a predefined confidence interval over a time horizon. Despite its popularity, this technique suffers from two major drawbacks such as it is not always sub-additive and it cannot measure the loss beyond VaR. Thus, CVaR (Conditional Value-at-Risk, Average Value-at-Risk or Expected shortfall) was introduced as an extension of VaR to measure extreme loss (loss that goes beyond VaR).

In this report, we put an effort into understanding the practice application of the two aforementioned measures: VaR and CVaR. In addition, we attempt to study two aspects of expected loss that were suggested in Basel II, the chance that borrowers not paying (Probability of default - PD) and the loss amount in case of default (Loss given default - LGD).

Given the scope of this report, all data are presumed. We then apply Monte Carlo method alongside with some of its techniques to simulate probabilistic events and study their effects on the preceding measurements of our presupposed portfolios.

2 Analysis Result

2.1 Risk of a single stock

2.1.1 A stock's probability of default

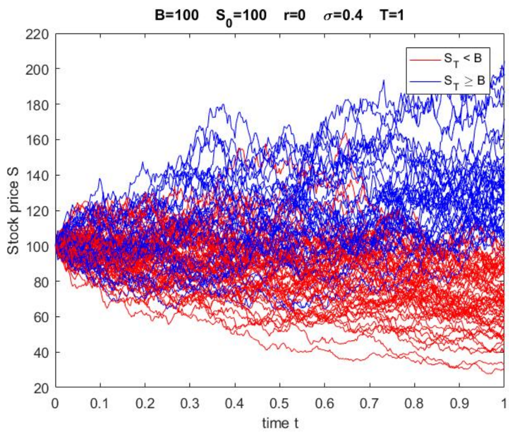
In this section, we study the odds that a single company defaults. Merton model (1974) considers bankruptcy as the event that a company's total asset is lower than its total debt at a specific date. A way to assess its total asset value is to evaluate its stock price which is assumed to follow a geometric Brownian motion:

$$S_{t+\Delta t} = S_t \exp\left(\sigma W_{\Delta t} + \left(r - \frac{\sigma^2}{2}\right)\Delta t\right)$$

Figure 1 displays the stock evolution based on 100 simulations over the given period $T = 1$ year under different threshold value $B = 100, 50, 36$ respectively.

Figure 1: 1 Stock - Price Evolution

(a)



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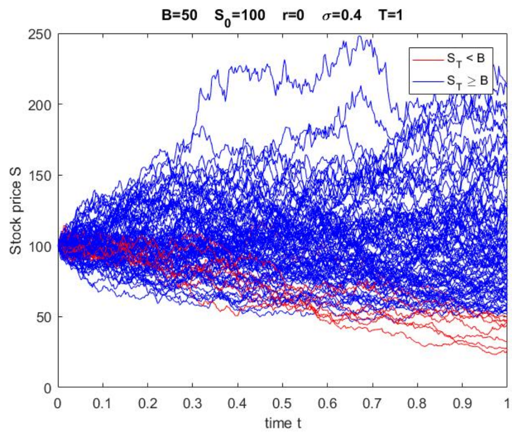

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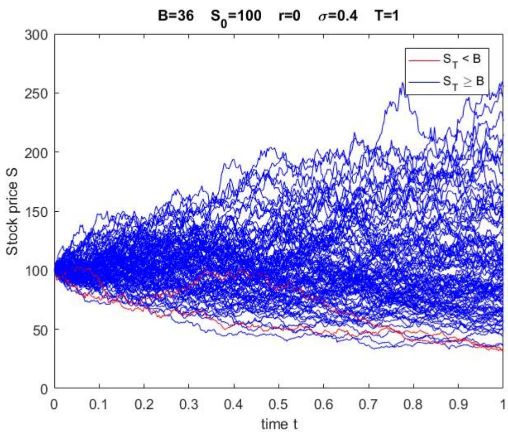


The red lines indicate the paths that the year-end stock price is lower than the given threshold ($S_T < B$) while the blue lines indicates the other case. It is easy to notice that the lower the threshold, the fewer the red lines. This goes in line with the outcome in Table 1 which presents the probability of default under a threshold B. This estimation is the result of 10,000 Monte Carlo simulation.

$$P(S_T < B) = \frac{1}{N_{mc}} \sum_{i=1}^{N_{mc}} \mathbb{1}_{S_T<B}$$

Table 1: 1 Stock - Threshold B and Probability of Default

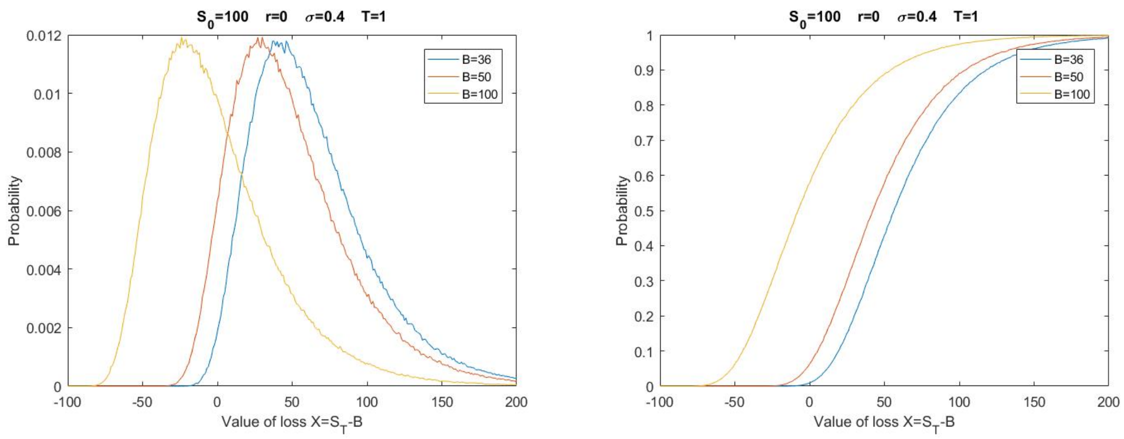
B	PD
100	57.65%
50	6.12%
36	1.08%

2.1.2 Loss from a single stock

We define the loss on this stock as $X = S_T - B$. Figure 2 shows that the loss follows log-normal distribution. This is a direct result from the fact that S_T is log-normal distributed and B is a constant parameter. Moreover, threshold B only affects the mean value but not the shape of the distribution. Figure 2a clarifies this statement as the density functions are identical in shape but each peaks differently.

Figure 2: 1 Stock - Loss distribution given B

(a) Density distribution (b) Cumulative distribution



2.1.3 Value at Risk and Conditional Value at Risk

We take a further look at each case and estimate its VaR at different α by 3 methods: (i) analytical method, (ii) sorting method and (iii) Robbins-Monro stochastic approximation. After that, CVaR will be calculated using the following formular:

$$CVaR_\alpha = \mathbb{E}\Big[X|X \leq VaR_\alpha\Big]$$

Analytical method

For a single stock, this method is rather simple to calculate and easy to understand as we know the loss distribution is log-normal. However, for a portfolio of stocks or options, this method is difficult due to complicated distribution function. Hence, we only run this method for a single stock.

$$\begin{aligned}
& & P(S_T - B \leq VaR) = \alpha \\
\iff & & P\Bigg(W_T \leq \frac{1}{\sigma}\Big(\ln\frac{VaR + B}{S_0} - \big(r - \frac{\sigma^2}{2}\big)T\Big)\Bigg) = \alpha \\
\iff & & \frac{1}{\sigma\sqrt{T}}\Big(\ln\frac{VaR + B}{S_0} - \big(r - \frac{\sigma^2}{2}\big)T\Big) = \mathcal{N}^{-1}(\alpha) \\
\iff & & S_0\exp(\sigma\sqrt{T}\mathcal{N}^{-1}(\alpha) + (\big(r - \frac{\sigma^2}{2}\big)T)) - B = VaR
\end{aligned}$$

Table 2: 1 Stock - VaR estimation by Analytical method

	T=365 days			T=10 days		
α	B=100	B=50	B=36	B=100	B=50	B=36
10%	-44.71	5.29	19.29	-8.34	41.66	55.66
1%	-63.60	-13.60	0.40	-14.46	35.54	49.54
0.1%	-73.18	-23.18	-9.18	-18.68	31.32	45.32

Sorting method

Sorting method allows us to estimate VaR as the α^{th}-quantile of an ascending ordered sequence of the simulated loss X.

$$VaR_\alpha(X) = X_{\lfloor \alpha N_{mc} \rfloor}$$

This method is much simpler than analytical method and works for any distribution. One thing to consider regarding this method is its precision when we want to estimate VaR at extremely low α.

Figure 3: 1 Stock - VaR estimation by Sorting method

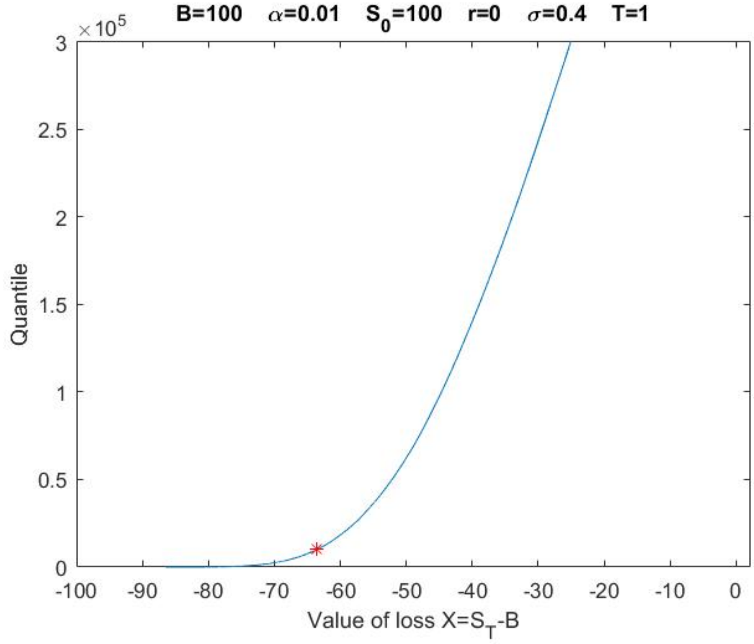


Table 3: 1 Stock - VaR estimation by Sorting method

	T=365 days			T=10 days		
α	B=100	B=50	B=36	B=100	B=50	B=36
10%	-44.93	5.55	19.26	-8.35	41.61	55.66
1%	-63.54	-13.61	0.47	-14.55	35.42	49.59
0.1%	-73.37	-22.85	-8.53	-18.68	31.30	45.37

Table 4: 1 Stock - CVaR estimation by Sorting method

	T=365 days			T=10 days		
α	B=100	B=50	B=36	B=100	B=50	B=36
10%	-53.64	-3.65	10.32	-11.15	38.86	52.86
1%	-68.10	-17.95	-3.94	-16.32	33.68	47.65
0.1%	-75.85	-26.19	-11.62	-20.14	29.84	43.76

Robbins-Monro stochastic approximation

Estimating VaR by Robbins-Monro algorithm is the most challenging method among the three as we need to choose a "good step" (β) and a starting point (z_0) that is rationally near the true value for the approximating process. The theory tells us to transform the probability in form of expectation of indicator function:

$$P(X \leq VaR) = \alpha \iff \mathbb{E}[\mathbb{1}_{X \leq VaR} - \alpha] = 0$$

Then, the sequence converges in probability to VaR.

$$VaR_{\alpha}(X) = \lim_{n\to\infty} z_n - \gamma_{n+1}(\mathbb{1}_{X\leq z_n} - \alpha)$$

where

$$\gamma_n = \frac{\beta}{(n+1)^{\lambda}} \text{ we take } \lambda = 0.9$$

As for β, we decide to take automate guess as $1/\alpha, 10/\alpha, 100/\alpha$. Figure 4 demonstrates our test at different β and it turns out that this method produces the similar result with two other methods.

Figure 4: 1 Stock - VaR estimation by Robbins-Monro method

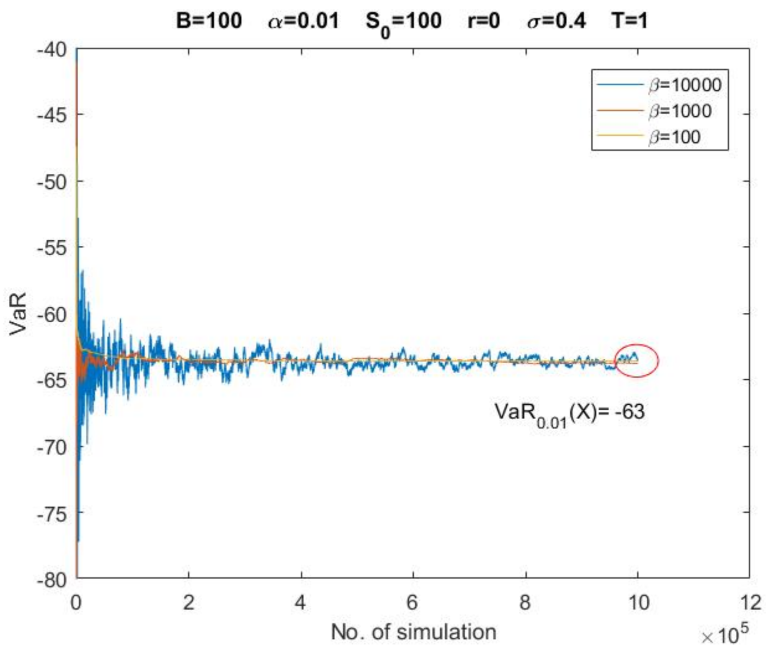


Table 5: 1 Stock - VaR estimation by Robbins-Monro method $\beta = 10/\alpha$

	T=365 days			T=10 days		
α	B=100	B=50	B=36	B=100	B=50	B=36
10%	-44.70	5.16	19.29	-8.34	41.68	55.64
1%	-63.64	-13.82	0.60	-14.45	35.46	49.50
0.1%	-72.97	-23.24	-9.20	-18.45	31.23	45.24

Table 6: 1 Stock - CVaR estimation by Robbins-Monro method

	T=365 days			T=10 days		
α	B=100	B=50	B=36	B=100	B=50	B=36
10%	-53.63	-3.79	10.35	-11.13	38.86	52.84
1%	-68.00	-18.10	-3.92	-16.37	33.54	47.60
0.1%	-75.64	-25.90	-11.87	-21.24	29.86	43.80

A major trouble of classic Monte Carlo simulation method is that it fails to tell us the information of the tail of a distribution, which is severely small. A way to tackle this issue is to adopt Important Sampling method to get a sufficient amount of samples from another distribution.

In this report, we choose to sample from a normal distribution that is similar to our original distribution of the Brownian motion W_T with shifted mean value by a term θT. Equation 1 justifies this idea.

$$\begin{aligned}
\mathbb{E}\Big[\phi(W_T)\Big] &= \int_R \phi(x)\frac{1}{\sqrt{2\pi T}}e^{-x^2/2T}dx \\
&= \int_R \phi(x)\frac{1}{\sqrt{2\pi T}}e^{-x^2/2T}e^{-(x-\theta T)^2/2T}e^{(x+\theta T)^2/2T}dx \\
&= \int_R \phi(x)e^{-\theta Tx+\frac{\theta^2 T}{2}}\frac{1}{\sqrt{2\pi T}}e^{-(x-\theta T)^2/2T}dx \\
&= \mathbb{E}_Q\Big[\phi(\bar{W}_T)e^{-\theta\bar{W}_T+\frac{\theta^2 T}{2}}\Big]
\end{aligned} \tag{1}$$

where

$$W_T \sim \mathcal{N}(0,T) \text{ and } \bar{W}_T \sim \mathcal{N}(\theta T, T)$$

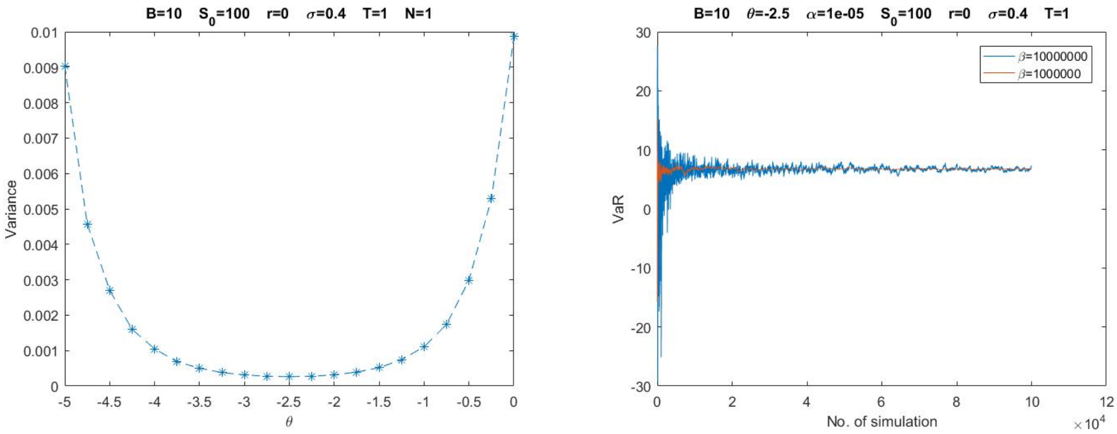
Choosing the right θ required a lot of effort so as the variance is minimized. In order to have a rational choice, we decide to analyze the relationship between θ and variance of a targeted function $\mathbb{E}_Q\Big[\phi(\bar{W}_T)e^{-\theta\bar{W}_T+\frac{\theta^2 T}{2}}\Big]$ and choose the θ that gives the lowest estimate for variance.

$$\begin{aligned}
&Var\Big[\phi(W_T+\theta T)e^{-\theta W_T-\frac{\theta^2 T}{2}}\Big] \approx \\
&\frac{1}{N_{mc}}\sum_{i=1}^{N_{mc}}\Big(\phi(W_T+\theta T)e^{-\theta W_T-\frac{\theta^2 T}{2}}\Big)^2 - \Big(\frac{1}{N_{mc}}\sum_{i=1}^{N_{mc}}\phi(W_T+\theta T)e^{-\theta W_T-\frac{\theta^2 T}{2}}\Big)^2
\end{aligned} \tag{2}$$

We set the goal to estimate the possibility of default at threshold as low as 10 $P(S_T < 10)$ and VaR at α as small as 0.001%. By sorting method, we are aware that $VaR_{0.01}(X) = 26.43$, we predict that $VaR_{10^{-5}}(X)$ will be much lower. We set our objective function $\phi(W_T) = \mathbb{1}_{X\leq 26.43}$ and find that $\theta = -2.5$ satisfies our condition (Figure 5a). With the help of Importance Sampling, we can deduce $PD \approx 1.3 \times 10^{-8}$ and $VaR \approx 6.81$ (Figure 5b).

Figure 5: 1 Stock - VaR estimate by Importance sampling method

(a) θ-variance relationship (b) VaR estimate



2.2 Risk of a portfolio of stocks

2.2.1 Loss from a portfolio of stocks

It is a general consensus that you never put all eggs in one basket. Thus, investors are more interested in a portfolio of stocks than a single stocks because they can benefit from diversified risk. In this section, we tackle a portfolio of N=125 stocks, each shares the same parameters $S_0 = 100$ and the interest rate $r = 0$.

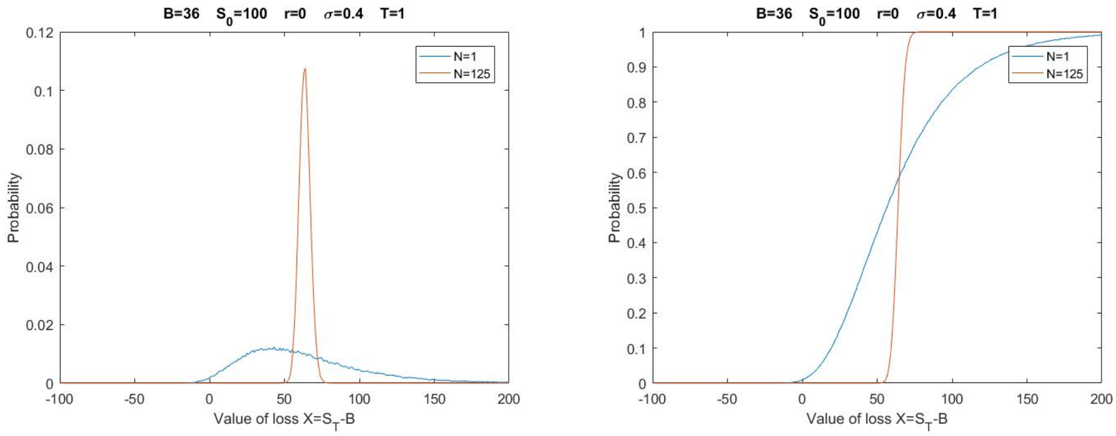
First, we study a homogeneous portfolio where all stocks share same volatility $\sigma = 0.4$ in $T = 1$ year, each stock is independent of another and has the same recovery rate $R = 0.3$. Recovery rate is the percentage of loss that we can "recover" in case of default. Hence, its complement $1 - R$ is an estimate for Loss Given Default (LGD). The loss of the portfolio is the average loss of stocks.

$$X = \frac{1}{N} \sum_{i=1}^{N} S_T^i - B$$

Figure 6 compares the loss distribution between a single stock and a portfolio of stocks. It depicts that the loss distribution of a single stock has fatter tail than that of the portfolio. This implies that portfolio's variance, or its risk, is less than a single stock's.

Figure 6: N Stock - Loss distribution of N independent homogeneous stocks

(a) Density distribution (b) Cumulative distribution



2.2.2 Value at Risk and Conditional Value at Risk

Earlier, we have incorporated Importance Sampling method to estimate the risk for one single stock, yet for a portfolio of stocks, it will not be as simple as the value of the portfolio now depends on not one but many independent stocks. The Radon-Nikodym now will be:

$$\mathbb{E}\Big[\phi(W_T^{(1)}, W_T^{(2)}, ..., W_T^{(N)})\Big] = \mathbb{E}_Q\Big[\phi(\bar{W}_T^{(1)}, \bar{W}_T^{(2)}, ..., \bar{W}_T^{(N)}) \exp\Big(-\langle\theta, \bar{W}_T\rangle + \frac{||\theta||^2 T}{2}\Big)\Big]$$

Our method of guessing θ is the same as in Section 2.1 and is available in Appendix. Table 7 shows our estimation of VaR and CVaR for $B = 100$, the result for $B = 50$ and $B = 36$ are available in the Appendix (Table 11 and Table 12). Compared to the VaR and CVaR of a single stock, we can tell that the VaR and CVaR of the portfolio is much lower (which indicates the fact that a portfolio of N companies is less risky than a single one). The normal Robbins-Monro method once again could not deal with extreme level of α:

Table 7: N Independent Stock - VaR-CVaR at B=100

	Sorting		Robbins-Monro		Robbins-Monro IS	
α	VaR	CVaR	VaR	CVaR	VaR	CVaR
1%	-8.35	-9.52	-8.36	-9.49	-8.33	9.39
0.1%	-10.93	-11.79	-10.88	-11.79	-10.97	-11.84
0.01%	-12.80	-13.54	-	-	-12.88	-13.65
0.001%	-14.85	-15.41	-	-	-14.66	-15.33

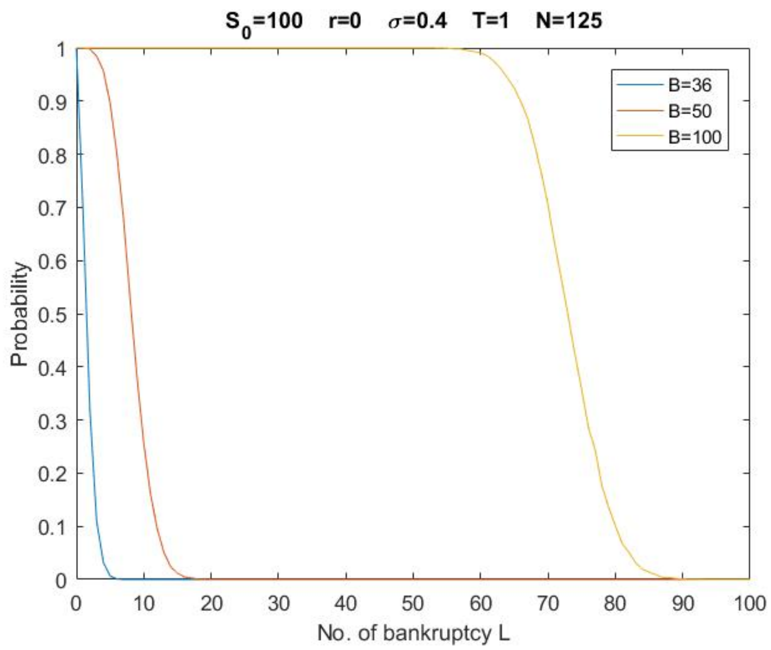
2.2.3 The number of bankruptcies

Other than the loss distribution of the portfolio, the number of bankruptcy L in the portfolio is another aspect that draws our attention:

$$L = \sum_{i=1}^{N} \mathbb{1}_{S_T^i < B}$$

Figure 7 sketches the right-tail of the distribution of the number of bankruptcy $P(L \geq K)$ for 3 different thresholds B. A common feature in this graph is the downward sloping of the distribution. It is rational in the way that it is less likely to have many defaults in the portfolio. The graph also suggests that lower B leads to a lower probability to have at least K bankruptcies. For example, $P(L \geq 10) \approx 0\%$ for $B = 36$, it goes up to 30% for $B = 50$ and it is almost surely that $L \geq 10$ when $B = 100$.

Figure 7: N Stock - Distribution of number of bankruptcies



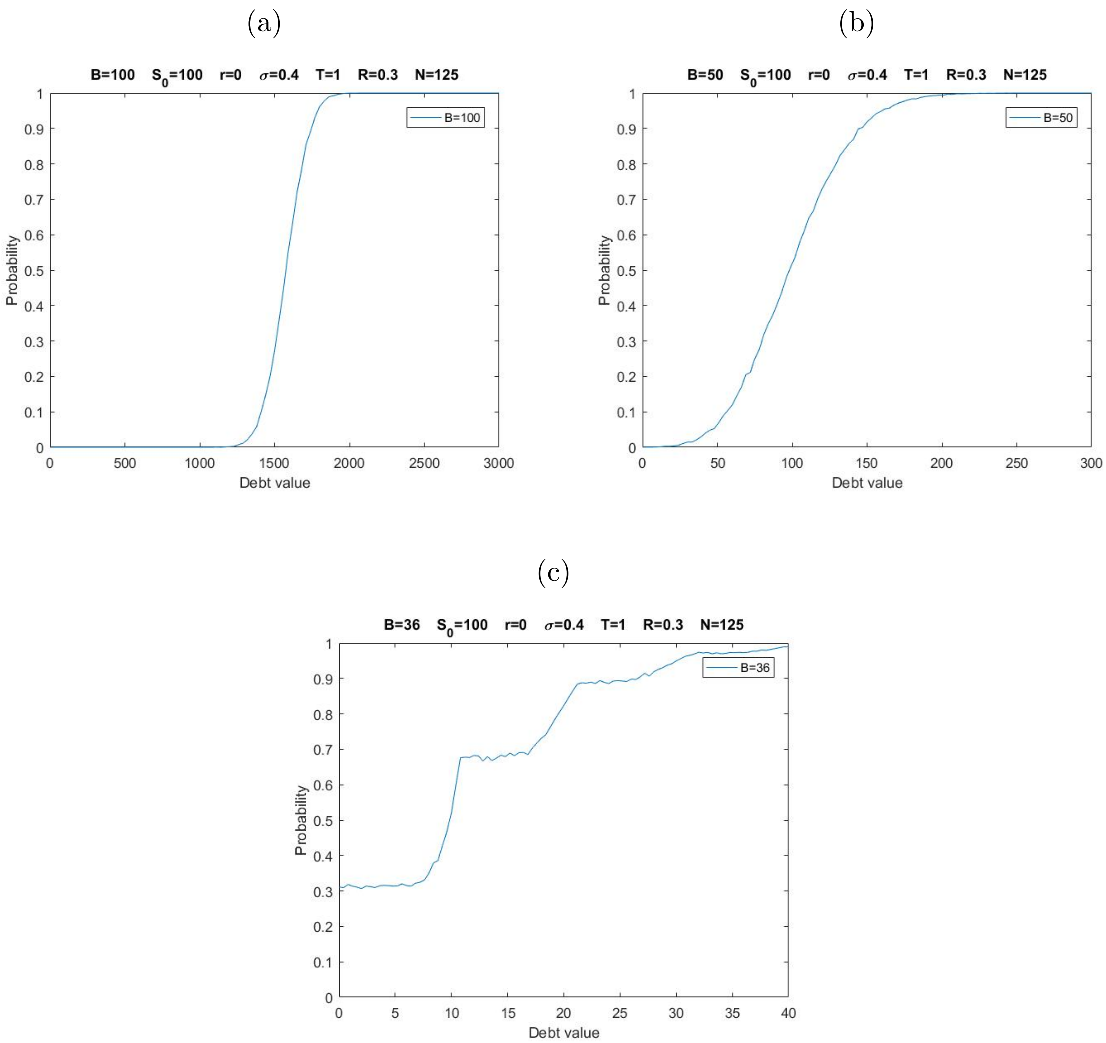
2.2.4 Recovery of the loss

Afore, we have only pondered the "loss" facet of our portfolio: the PD and the number of default. Nonetheless, when a company goes bankrupt, it can liquidate its remaining assets and pay back a part of its debt. The remaining debt Π_T that can be "recovered" is define as:

$$\Pi_T = \sum_{i \in D_T} R_i S_T^i \qquad \text{where } D_T := \{1 \leq i \leq N | S_T^i - B \leq 0\}$$

Figure 8 exhibits the cumulative distribution of Π_T for different threshold $B = 100, 50, 36$, i.e. $P(\Pi_T \leq x | S_T < B)$.

Figure 8: N Stock - Debt distribution of N independent homogeneous stocks

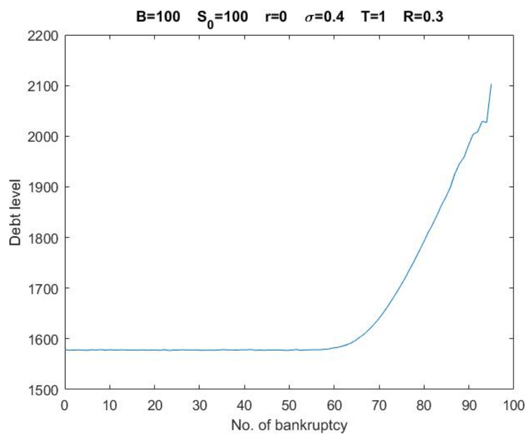


From Figure 8, we can see that the higher B is, the higher the value of debt that may occur. For example, with $B = 100$, the value of debt are in the range from 1000 to 2000 with probability of almost 100% but for $B = 50$, the range is only from 0 to 200. For $B = 36$, the figure looks quite different compared to the other value of B as we can see the big step at the value of 10, 20, etc. This can be explained as for $B = 36$, the number of bankrupted company mostly ranges from 0 to 6, so for every additional bankrupted company, it causes a jump of 30% of its remaining value ($R = 30\%$) to the debt value.

Next, we study the conditional expectation of the remaining debt value Π_T given the number of bankrupted company L: $E[\Pi_T|L > K]$.

Figure 9: N stock - Conditional Expectation of Debt of N independent companies

(a)



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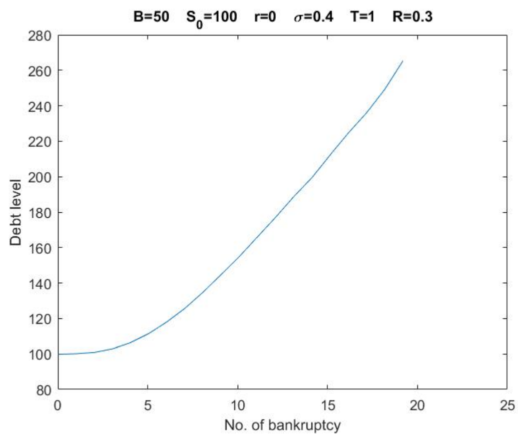

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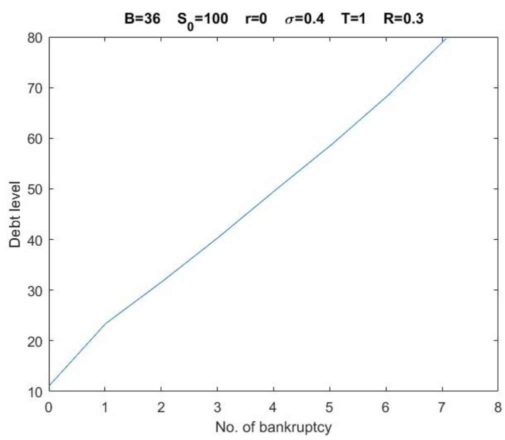


Figure 9 illustrates 3 different lines for the conditional expectation of debt for 3 different threshold $B = 36, 50, 100$ knowing the number of bankruptcies. We can see that the conditional expectation is increasing. This is obviously true as the higher the number of bankruptcy, the higher the value of the debt. However, when $B = 100$, the expectation of debt is much higher compared to the two other cases. $E[\Pi_T|L > 0]$ when $B = 100$ is almost 1600, which means if there is any case of bankruptcy, you are expected to have the recovered debt of 1600, but for $B = 50$ and $B = 36$, there are only about 100 and 10 respectively.

2.2.5 Brownian motion's correlation effect to stocks portfolio

Stocks may moves somewhat in relation to each other. Thus, it is crucial to study the effect of correlation to our hypothetical portfolio. Now supposing our N dimensional Brownian motion W_T is positively correlated with correlation coefficient ρ. We apply Cholesky method to decompose this vector to an uncorrelated Brownian motion B_T:

$$W_T^i = \sum_{j=1}^{N} M_{ij} B_T^i$$

Then, the stock price is given as

$$S_t^{(i)} = S_0^{(i)} \exp\Big((r - \frac{\sigma_i^2}{2})t + \sigma_i \sum_{j=1}^{N} M_{ij} B_t^j \Big) \ i \leq N$$

where M_{ij} is our Cholesky lower matrix

$$MM^T = Cov = \begin{pmatrix} 1 & \rho & \dots & \dots & \rho \\ \rho & 1 & \rho & \dots & \rho \\ \dots & \dots & \dots & \dots & \dots \\ \rho & \dots & \dots & \rho & 1 \end{pmatrix}$$

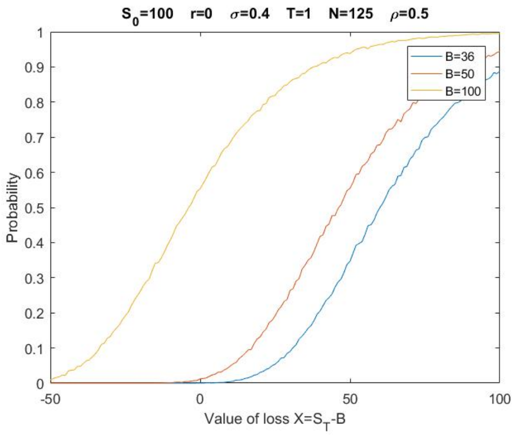
So in order to calculate the risk distribution and indicator, we only need to replace W_T with $\sum\limits_{j=1}^{N} M_{ij} B_T^i$ and all of the functions now will be based on the new independent Brownian Motion B_T.

Studying the effects of correlation to the portfolio, we applied the correlation coefficient $\rho = 0.5$. Two almost identical portfolios, one of N independent stocks and one of N correlated stock, are investigated for the purpose of comparison.

Figure 10 presents the loss distribution of correlated portfolio and compare its density with that of a non-correlated one. A glance of the result agrees with our expectation that correlated portfolio derives greater risk because its loss distribution has much fatter tail than the non-correlated.

Figure 10: N Correlated Stock - Loss distribution of correlated portfolio

(a) CDF at 3 thresholds



(b) PDF of non- vs. correlated portfolio

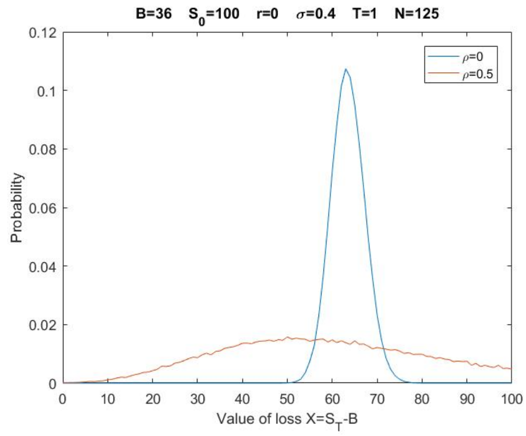
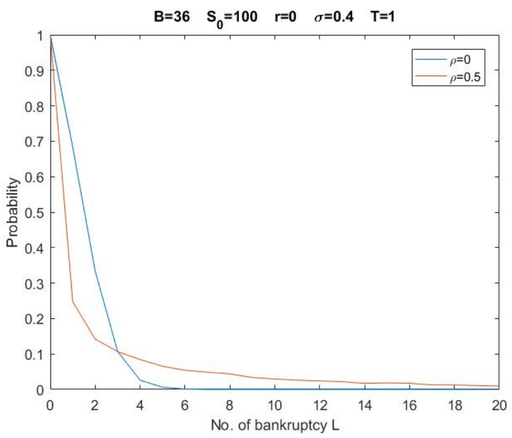


Figure 11 compares the distribution of number of bankruptcies at different level of threshold for two portfolios represented with blue and orange lines, respectively. From the figure, we can say that a correlated portfolio is riskier than a non-correlated one because the orange lines go much further than the blue lines. In particular, when the threshold B is 50, a correlated portfolio may experience as many as 20 bankruptcies with a probability of 12.5% while for a non-correlated portfolio this incident only occurs with 2×10^{-5}% of probability.

Figure 11: N Correlated Stock - Distribution of number of bankruptcies

(a)



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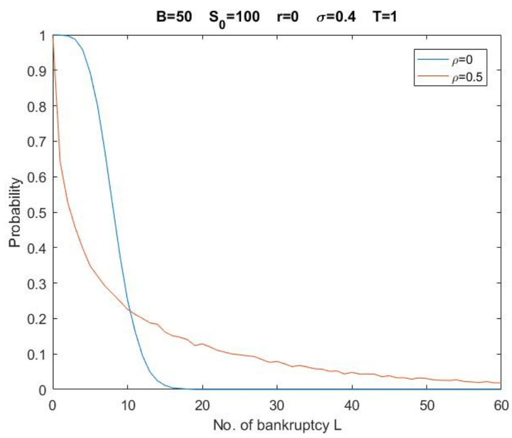


Figure 12: N Correlated Stock - Debt distribution of non- vs. correlated portfolio

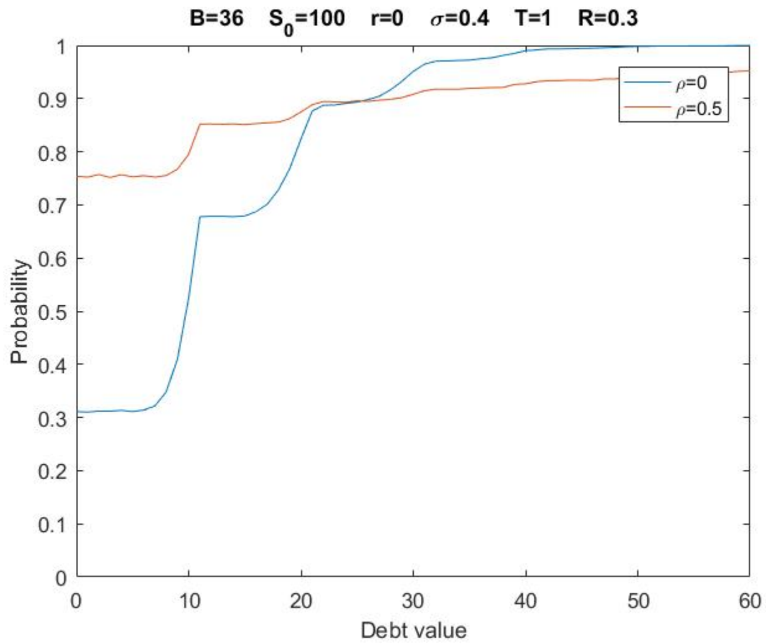
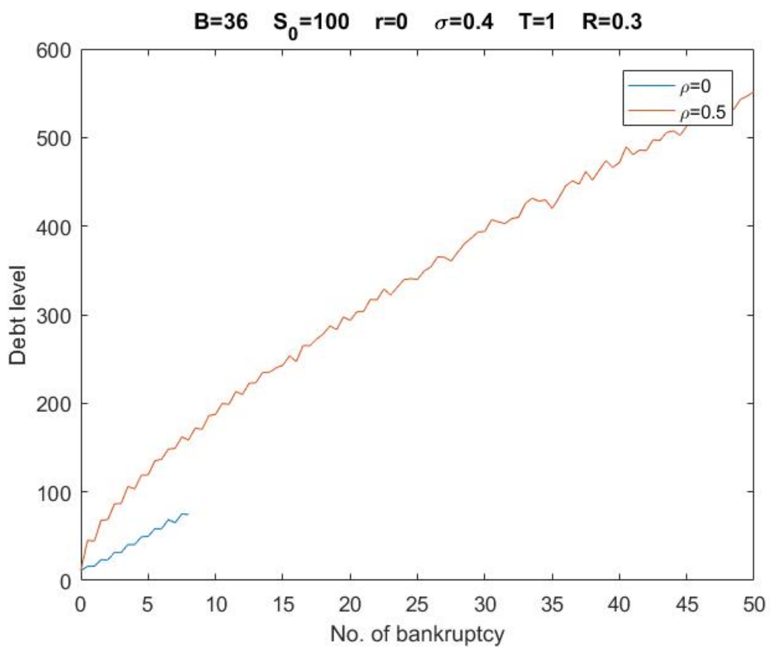


Figure 13 is intriguing because the conditional expectation of debt of a non-correlated portfolio cuts off at roughly 7 bankruptcies and that of a correlated one goes till 50 bankruptcies. This can be explained by the trait that we observe in Figure 11: when $P(L > K) \approx 0$, we can no longer estimate our $\mathbb{E}[\Pi_T | L > K]$.

Figure 13: N Correlated Stock - Conditional Expectation of Debt of non- vs. correlated portfolio



Using Cholesky decomposition, the method of calculating VaR and CVaR using sorting and general Robbins-Monro technique stay almost the same. We only have to replace W_T with $\sum_{j=1}^{N} M_{ij} B_T^i$ and everything will work. However, if we wants to apply Importance Sampling, things will get more complicated as it is not easy to evaluate the effect of changing the probability space to $W_T + \theta T$ when W_T is correlated. So instead of changing the probability space of W_T, we tried to change the probability space regarding to the independent Brownian Motion B_T to $\bar{B}_T (= B_T + \theta T)$:

$$\begin{aligned} E[\phi(W_T^{(1)}, W_T^{(2)}, ...W_T^{(N)})] &= E\left[\phi(\sum_{j=1}^{N} M_{1,j} B_T^{(j)}, \sum_{j=1}^{N} M_{2,j} B_T^{(j)}, ... \sum_{j=1}^{N} M_{3,j} B_T^{(j)})\right] \\ &= E[\hat{\phi}(B_T^{(1)}, B_T^{(2)}, ...B_T^{(N)})] \\ &= E_Q\left[\hat{\phi}(\bar{B}_T^{(1)}, \bar{B}_T^{(2)}, ...\bar{B}_T^{(N)}) \exp\Big(- \langle \theta, \bar{B} \rangle + \frac{||\theta||^2 T}{2}\Big)\right] \end{aligned}$$

Using this method of changing probability space, we could achieve similar results compared to the sorting technique and general Robbins-Monro technique:

Table 8: N Correlated Stock - VaR-CVaR with B=100

	Sorting method		Robbins-Monro		Robbins-Monro IS	
α	VaR	CVaR	VaR	CVaR	VaR	CVaR
1%	-50.31	-54.79	-51.01	-55.25	-50.29	-54.77
0.1%	-60.45	-63.72	-60.48	-63.36	-60.52	-62.54
0.01%	-65.15	-67.39	-	-	-66.83	-68.74
0.001%	-72.02	-73.43	-	-	-71.17	-73.28

2.2.6 Inhomogeneous portfolio of stocks

Until now, we have examined a portfolio of stocks with identical parameters; nevertheless, in practice this assumption is unrealistic. This necessitates an analysis of a portfolio of stocks with different parameters. Here, we assume stocks are divided into 5 groups of volatility:

$$\sigma_i = 0.2 \text{ for } 1 \leq i \leq 25$$

$$\sigma_i = 0.25 \text{ for } 26 \leq i \leq 50$$

$$\sigma_i = 0.3 \text{ for } 51 \leq i \leq 45$$

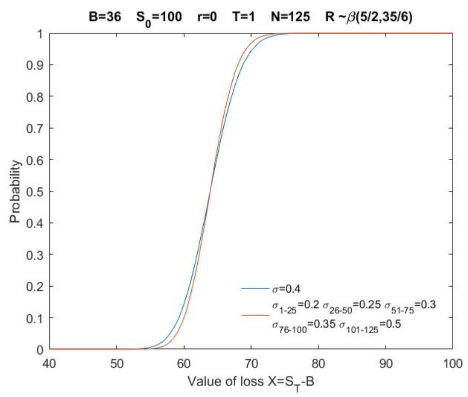
$$\sigma_i = 0.35 \text{ for } 76 \leq i \leq 100$$

$$\sigma_i = 0.5 \text{ for } 101 \leq i \leq 125$$

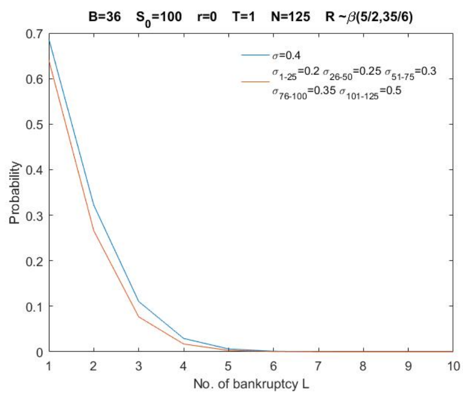
We further assume that recovery rate follows Beta distribution with mean 0.3 and standard deviation 0.15, that is to say $R \sim \beta(\frac{5}{2}, \frac{35}{6})$, and generate random recovery rate by Acceptance-Rejection method (Appendix D). Figure 14 compares loss distribution, number of bankruptcies distribution and conditional expectation of debt level between the homogeneous portfolio and a inhomogeneous one. The sole figure that shows a distinctive feature of the latter is Figure 14*c* where it presents a much smoother distribution for the latter compared to the "step" distribution of the former. A randomly generated recovery rate accounts for this difference.

Figure 14: N Stock - Comparision between homogeneous and inhomogeneous portfolio

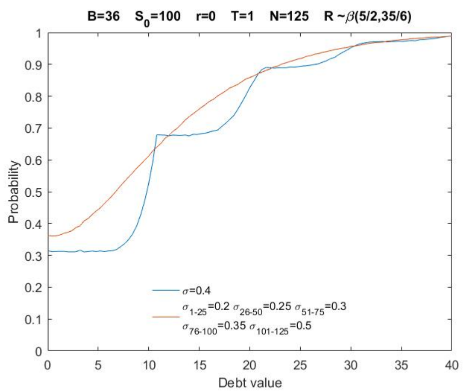
(a) $P(Average(S_T) < B)$



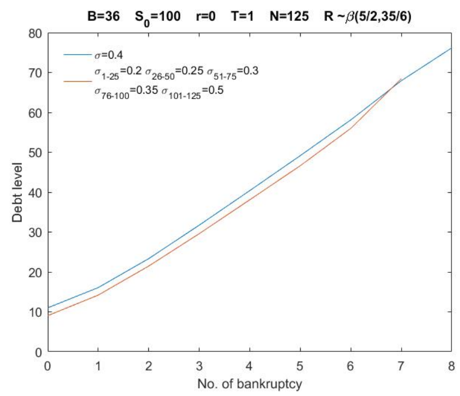
(b) $P(L \geq K)$



(c) $P(\Pi_T \leq x | S_T < B)$



(d) $E[\Pi_T \leq x | L \geq K]$



2.3 A portfolio of options

Throughout the report, we have gradually studied the portfolio of a single stocks and N number of stocks. However, in reality, stocks is not the only financial instrument that contains credit risk. The change of value in bonds, foreign currencies and a wide range of other options and derivatives can be the reason leading to the loss of an investor.

2.3.1 Portfolio of Independent Options

In this part of the report, three different portfolios consisted of put and call options will be taken into account:

$$P_1 = \sum_{i=1}^{10} 10Call_i(t, S_t^{(i)}) + 5Put_i(t, S_t^{(i)})$$

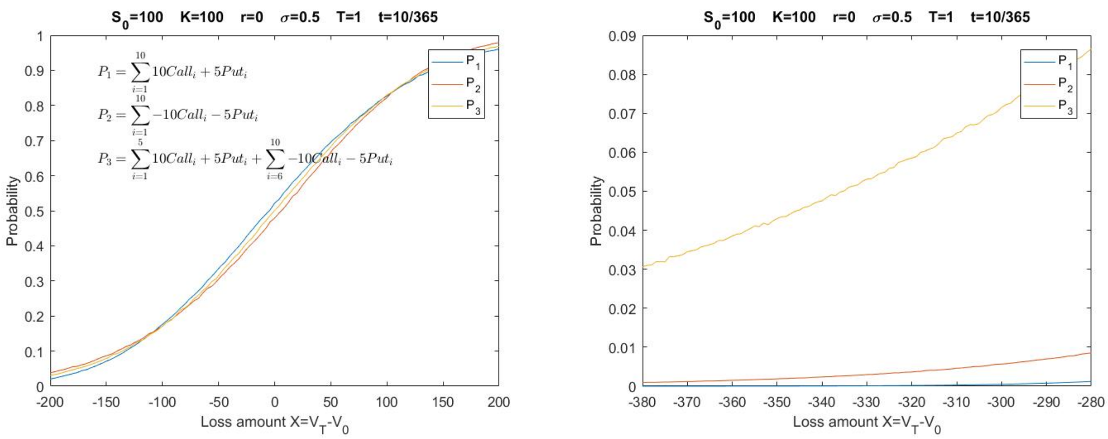
$$P_2 = \sum_{i=1}^{10} -10Call_i(t, S_t^{(i)}) - 5Put_i(t, S_t^{(i)})$$

$$P_3 = \sum_{i=1}^{5} 10Call_i(t, S_t^{(i)}) + 5Put_i(t, S_t^{(i)}) + \sum_{i=6}^{10} -10Call_i(t, S_t^{(i)}) - 5Put_i(t, S_t^{(i)})$$

Figure 15a visualizes the loss distribution of the three portfolios. Despite the minimal difference among the three distribution, when we magnify the left tail of the loss distribution by Importance Sampling method, it is visible that P_3 is the most likely among the three to produce a given loss amount.

Figure 15: Loss distribution of three Put Call portfolios

(a) (b)



From this point onward, we take into account P_1 as the prime portfolio to investigate. Table 9 summarises VaR of 10 days at different α levels. For small α of 0.01% and 0.001%, once again the normal Robbins-Monro did not work. Importance Sampling method was used with $\theta = -6$ which gave the smallest variance (Figure 16a).

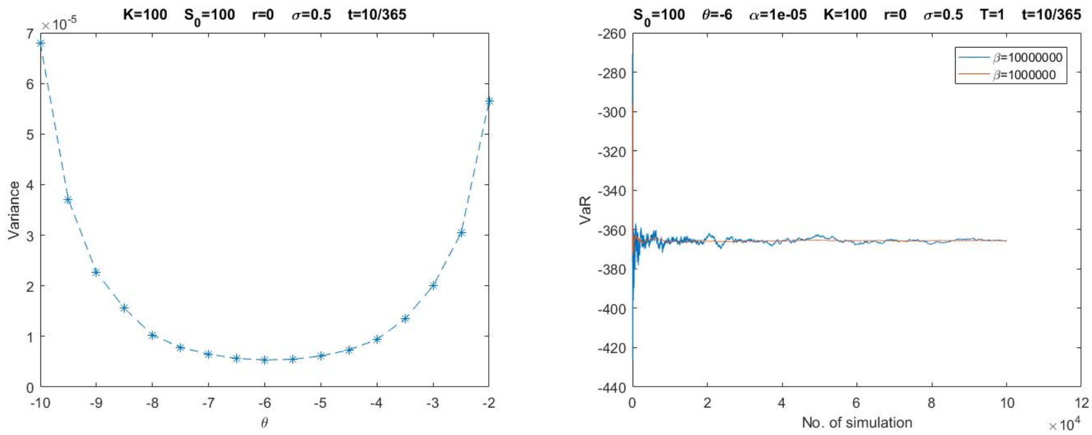
Table 9: Independent Option Portfolio - VaR-CVaR with a=10, b=5

	Sorting method		Robbins-Monro		Robbins-Monro IS	
α	VaR	CVaR	VaR	CVaR	VaR	CVaR
10%	-133.09	-175.40	-135.89	-177.69	-134.75	-175.21
1%	-223.68	-250.34	-224.05	-251.93	-223.27	-249.83
0.1%	-281.76	-301.19	-283.73	-304.52	-284.35	-305.36
0.01%	-332.77	-348.91	-	-	-329.68	-345.45
0.001%	-356.23	-368.34	-	-	-365.66	-379.09

Figure 16: Independent Option Portfolio - VaR estimation by Robbins-Monro method

(a) θ-Variance analysis

(b) VaR estimation

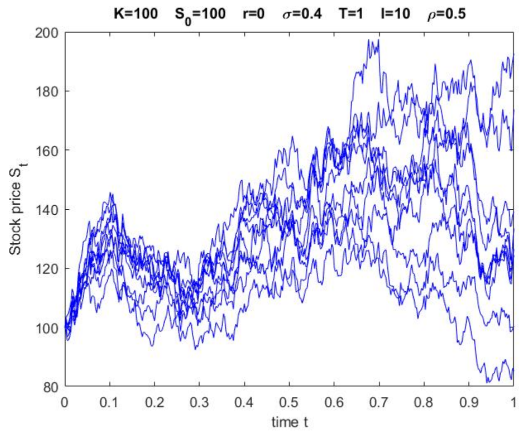


2.3.2 Portfolio of Correlated Options

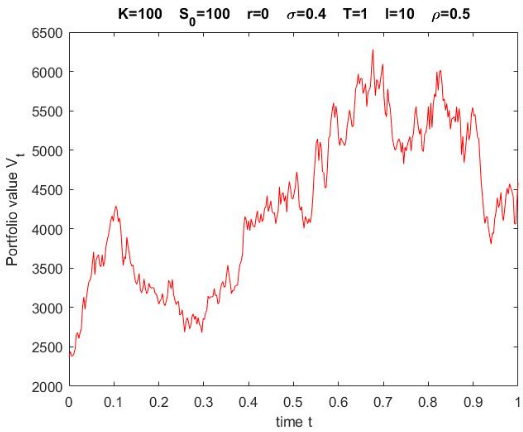
In the previous part, we have just studied the portfolio without the element of correlation. With the inclusion of correlation, the risk nature of our portfolio may change dramatically. Figure 17a shows the evolution of 10 correlated stocks. As we used the correlation coefficient $\rho = 0.5$, most of the stocks share the same pattern and if we take a look at Figure 17b, we can see that our portfolio of Put and Call based on those stocks also moved accordingly.

Figure 17: Evolution of option portfolio and its underlying correlated stocks

(a) Correlated stocks evolution



(b) Portfolio evolution



Using Cholesky decomposition and the method of Important Sampling for correlated Brownian Motion, we calculated the value of VaR and CVaR for the correlated portfolio in Table 10. Compared to the independent counterpart, it is easy to recognize that the VaR and CVaR of correlated portfolio is much lower than the independent one.

Table 10: Correlated Option Portfolio - VaR-CVaR with a=10, b=5

	Sorting method		Robbins-Monro		Robbins-Monro IS	
α	VaR	CVaR	VaR	CVaR	VaR	CVaR
1%	-444.61	-480.82	-446.02	-481.18	-444.77	-480.13
0.1%	-522.58	-544.71	-522.63	-541.94	-523.55	-544.63
0.01%	-567.60	-578.68	-	-	-566.62	-579.49
0.001%	-599.81	-603.01	-	-	-594.78	-601.99

2.4 An example of θ estimation by Robbins-Monro method

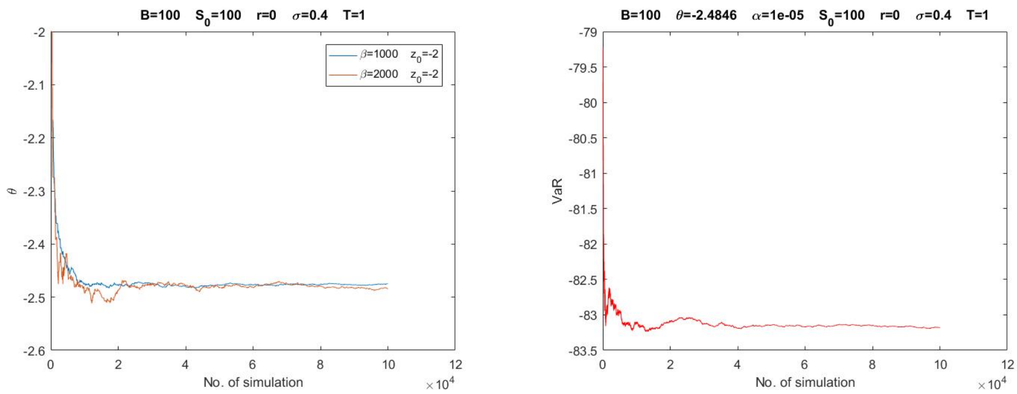
Beforehand, we have studied the link between θ and variance by the "smile". Another method proposed by Lelong (2007) [?] is to endorse Robbins-Monro algorithm to minimize the variance of a targeted function, or to find the point where the gradient is zero.

$$\mathbb{E}\Big[(\theta T - W_T)\phi(W_T)^2 e^{-\theta W_T + \frac{\theta^2 T}{2}}\Big] = 0$$

This goes back to the original Robbins-Monro problem: finding a fixed point where its expected function is zero. Our objective here is to estimate VaR at $\alpha = 0.001\%$. First, we need to estimate the parameter θ to change the probability measure. As we are aware that $VaR_{0.01} = -63$ (Table 2,4,6), we predict that $VaR_{10^{-5}}(X)$ will be much lower. We set $\phi(W_T) = \mathbb{1}_{X\leq -63}$ and find θ so as to reduce the variance of this function. Figure 21a tells us that this condition is satisfied at $\theta \approx -2.5$. We then use this θ to estimate VaR also by Robbins-Monro method and it gives us $VaR_{10^{-5}}(X) \approx -83$. We further analyze the variance of $\mathbb{E}[\mathbb{1}_{X\leq -83}]$ and it brings about a variance of around 4×10^{-09}.

Figure 18: Robbins-Monro method application in estimating θ

(a) θ approximation (b) $VaR_{10^{-5}}(X)$ approximation



In Appendix C we further use this method to evaluate θ of portfolios.

3 Conclusion

Having reckoned sizable extent of risk measures, we jump to the following conclusion from this report:

(i) A portfolio of numeral stocks profits from diversified risk.

(ii) Correlation effect in a portfolio may amplify its risk.

(iii) Sorting method is straightforward and applicable for most portfolio composition; however, it is incapable of evaluating VaR/CVaR at utterly rare occasions. In this case, Robbins-Monro method comes in handy as it can be coupled with Importance Sampling technique and deliver reasonably accurate estimation.

Appendix

A Tables of VaR/CVaR results

A.1 VaR/CVaR for N non-correlated stocks

Table 11: N Independent Stock - VaR-CVaR at B=36

	Sorting		Robbins-Monro		Robbins-Monro IS	
α	VaR	CVaR	VaR	CVaR	VaR	CVaR
1%	55.66	54.58	55.63	54.51	55.50	54.37
0.1%	53.07	52.06	53.13	52.22	52.97	51.84
0.01%	51.03	50.31	-	-	51.02	50.26
0.001%	49.07	48.47	-	-	49.33	48.66

Table 12: N Independent Stock - VaR-CVaR at B=50

	Sorting		Robbins-Monro		Robbins-Monro IS	
α	VaR	CVaR	VaR	CVaR	VaR	CVaR
1%	41.65	40.51	41.65	40.51	41.66	40.53
0.1%	39.07	38.18	39.05	38.15	39.09	38.21
0.01%	37.01	36.19	-	-	37.05	36.28
0.001%	35.12	34.65	-	-	35.18	34.52

A.2 VaR/CVaR for N correlated stocks

Table 13: N Correlated Stock - VaR-CVaR at B=50

	Sorting method		Robbins-Monro		Robbins-Monro IS	
α	VaR	CVaR	VaR	CVaR	VaR	CVaR
1%	-0.25	-4.55	-0.13	-4.57	-0.18	-4.9
0.1%	-10.01	-12.43	-10.31	-13.11	-9.60	-12.63
0.01%	-16.08	-19.07	-	-	-15.99	-18.78
0.001%	-21.51	-23.34	-	-	-21.56	-22.78

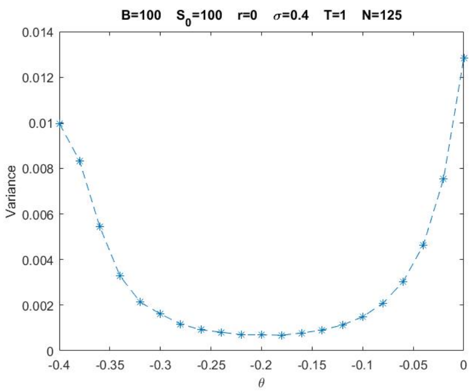
Table 14: N Correlated Stock - VaR-CVaR at B=36

	Sorting method		Robbins-Monro		Robbins-Monro IS	
α	VaR	CVaR	VaR	CVaR	VaR	CVaR
1%	13.49	8.95	13.25	8.87	13.35	8.85
0.1%	3.98	1.11	3.84	0.75	3.8331	0.88
0.01%	-2.63	-4.38	-	-	-2.63	-4.53
0.001%	-7.04	-8.75	-	-	-7.59	-8.62

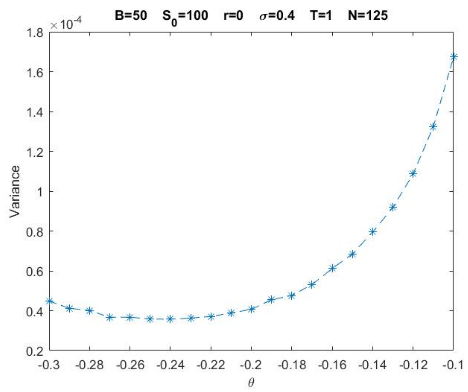
B θ-Variance analysis

Figure 19: Variance analysis - Estimating θ and VaR of N-stock portfolio

(a)



(b)



(c)

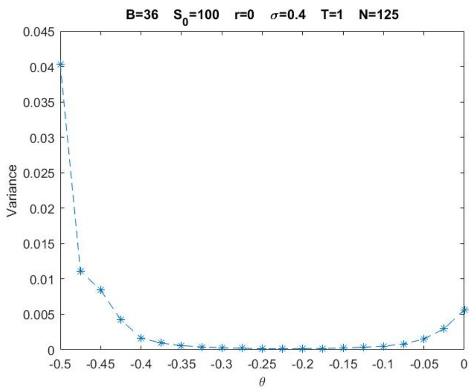
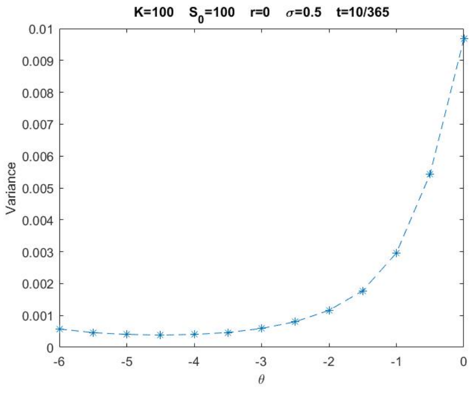
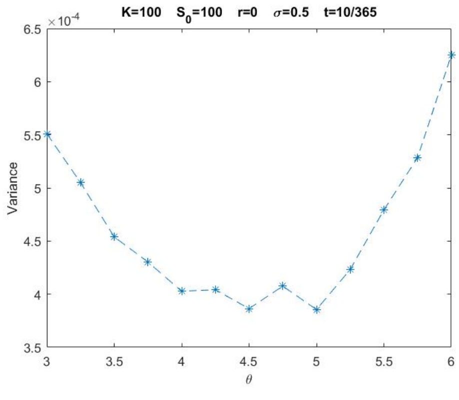


Figure 20: Variance analysis - Estimating θ options portfolio

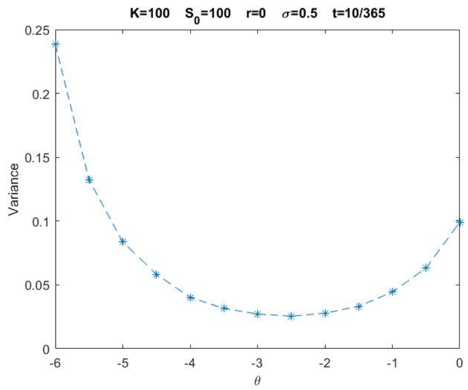
(a) P_1



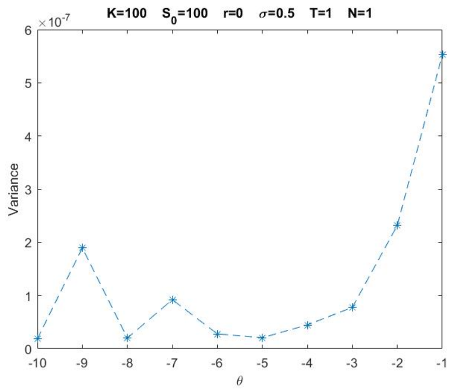
(b) P_2



(c) P_3



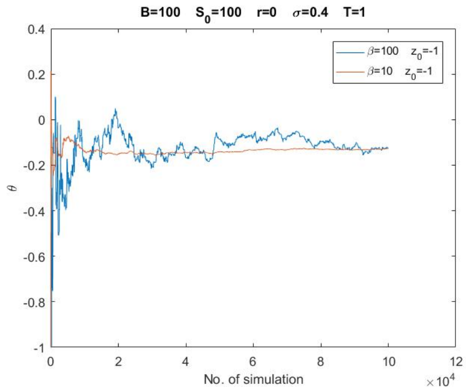
(d) P_1-correlated



C Lelong's method in estimating θ

Figure 21: LeLong's method - Estimating θ and VaR of N-stock portfolio

(a) θ approximation



(b) $VaR_{10^{-5}}(X)$ approximation

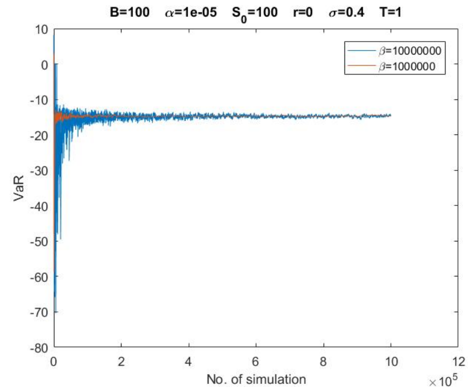
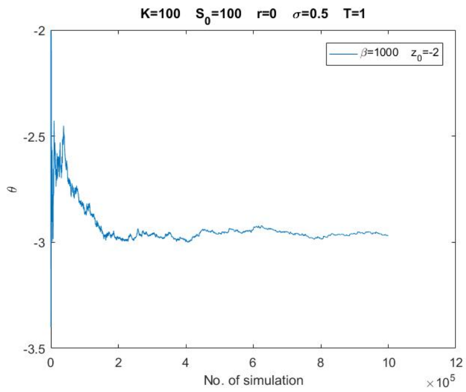
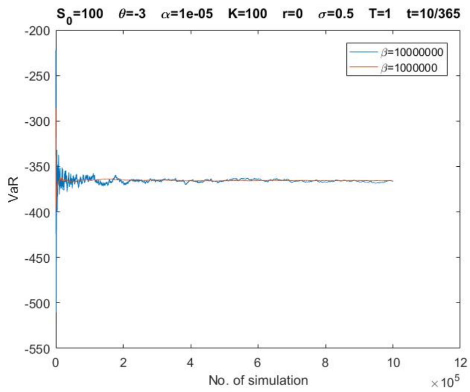


Figure 22: LeLong's method - Estimating θ and VaR of option portfolio

(a) θ approximation



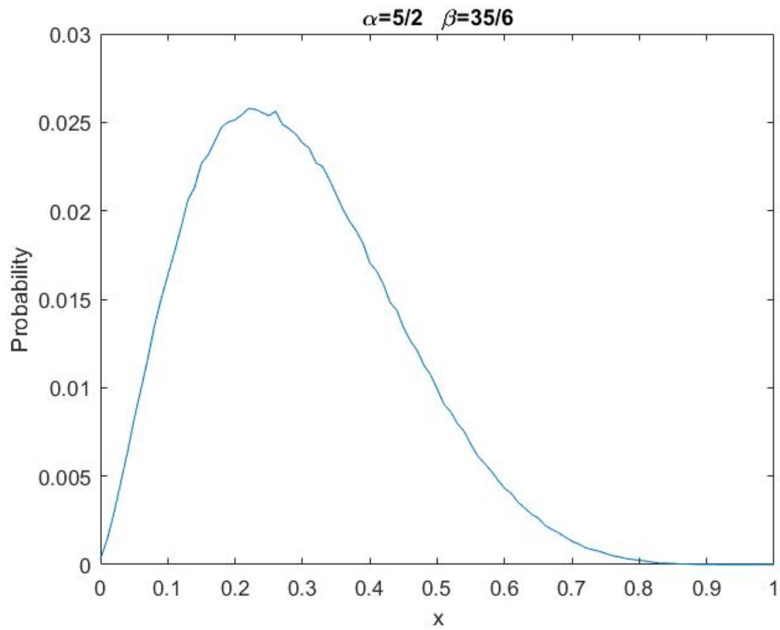
(b) $VaR_{10^{-5}}(X)$ approximation



D Beta distribution

The idea of this method is to generate samples from another distribution that covers the density function of the random variable and keep the portion in the region under the graph of its density; thus, the density function must be known. As for this report, we choose the bound function to be the uniform density function $\mathcal{U}([0,1])$ multiplied by a constant c which is the sup of Beta density.

Figure 23: Beta density distribution by Acceptance-Rejection method



E Default date as a stopping time

In this section, we define the insolvency as the event at the first date τ at which the stock value drops below the threshold.

$$\tau = \inf\{\ t_k \quad | \quad S_{t_k}^{(i)} \leq B_i\}$$

Figure 24: Stopping time - 125 stocks evolution cut-off at date of default τ

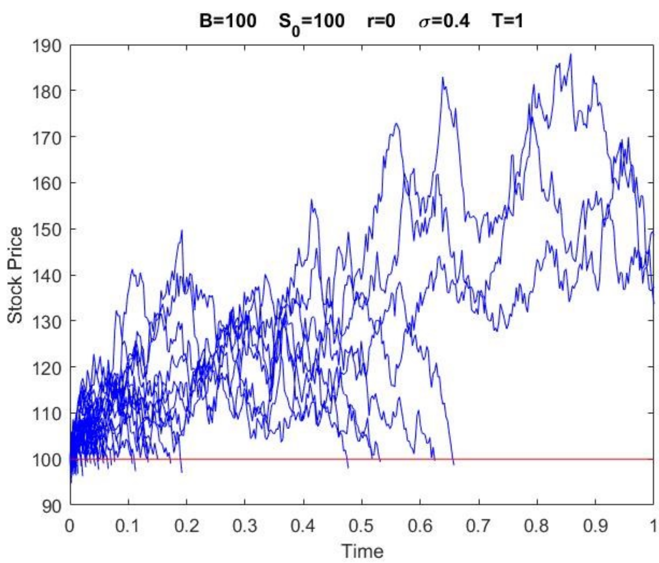


Figure 25, 26, 27, 28 demonstrate our analysis result for the same aspects that we discussed in Section 2: P_1 notes the portfolio in Section 2.1 and P_2 is the portfolio with new default definition. It is noteworthy that while it is more likely for P_2 to experience default event, the its debt distribution and conditional expectation of debt is lower than those of P_1. This is due to the fact that the loss is given at time of default and not accumulated for other future loss.

Figure 25: Stopping time - Loss distribution for different thresholds

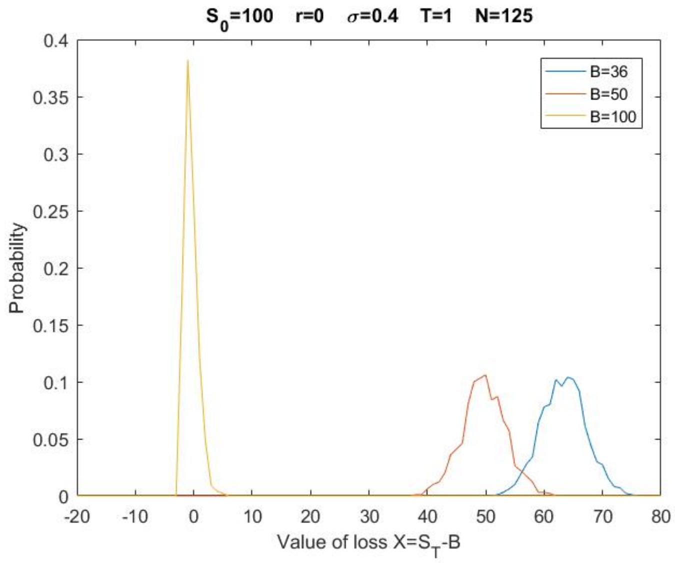
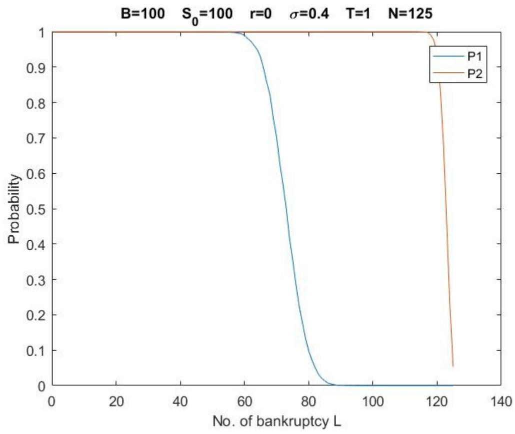
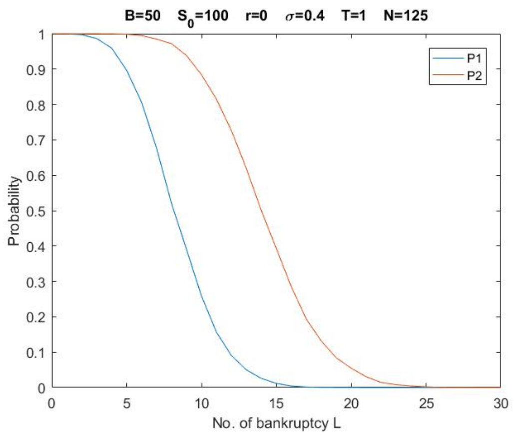


Figure 26: Stopping time - Distribution of number of bankruptcies

(a)



(b)



(c)

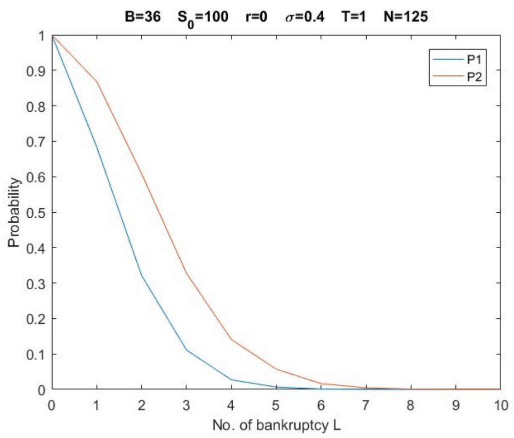
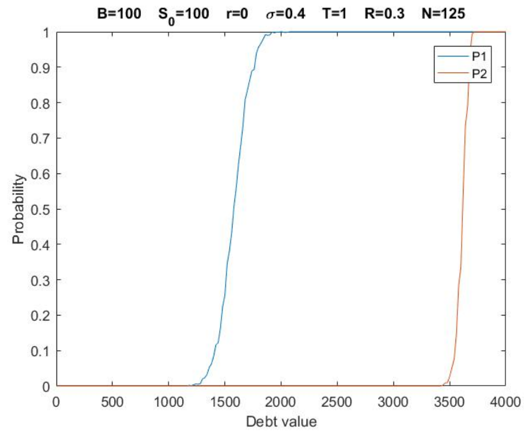
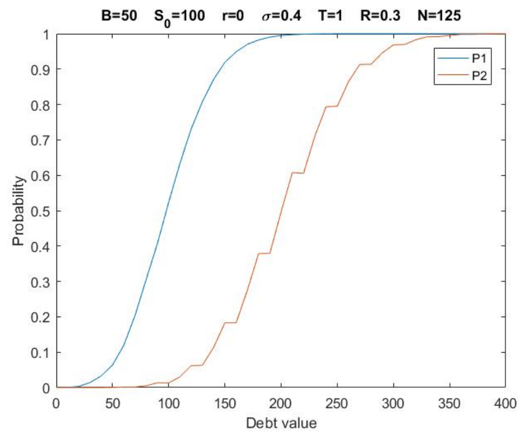


Figure 27: Stopping time - Debt distribution

(a)



(b)



(c)

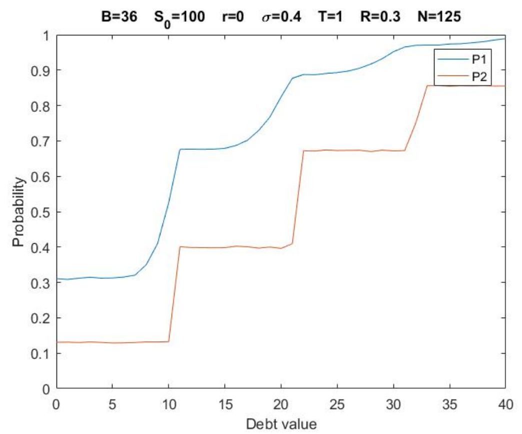
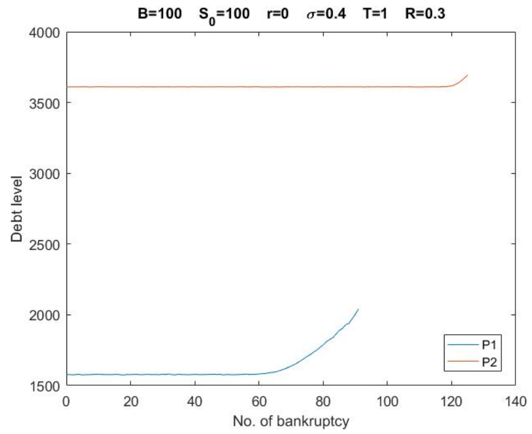
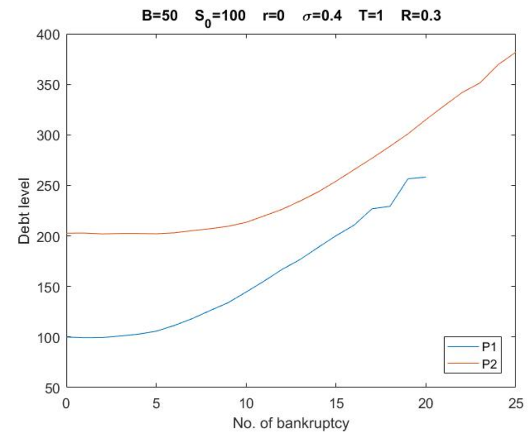


Figure 28: Stopping time - Conditional Expectation of Debt

(a)



(b)



(c)