Assignment on Calibration & Hedging

Calibration of Vasicek model

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class

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1 Introduction

Vasicek model is a well-known mathematical model that depicts the evolution of interest rates. It is characterized by 3 parameters η, γ and σ as in the following SDE:

$$dr_t = (\eta - \gamma r_t)dt + \sigma dW_t \tag{1}$$

To calibrate Vasicek model is to best estimate these 3 parameters, the idea is to minimise the error term between the calibrated model and the market data. In this report, we consider approaches in calibrating Vasicek model:

(i) Using the zero-coupon rates

(ii) Using historical data

2 Main idea

2.1 Use of Zero-Coupon Bond

Let $P(t,T)$ denote the price of a zero-coupon bond at time t having the maturity date T. In theory we know that:

$$P(t,T) = e^{Y(t,T)(T-t)} \iff Y(t,T) = -\frac{\ln P(t,T)}{T-t}$$

where we can estimate r_t from $Y(t,T)$ as

$$r_t = \lim_{t \to T} Y(t,T)$$

On the other hand, the discounted bond price $e^{-\int_t^T r_s ds} P(t,T)$ is a martingale under the risk-neutral measure Q, or equivalently:

$$P(t,T) = E_Q\left[e^{-\int_t^T r_s ds} \middle| \mathcal{F}_t\right]$$

By solving this equation, we obtain the PDE system:

$$\begin{cases} -rP + \frac{\partial P}{\partial t} + (\eta - \gamma r)\frac{\partial B}{\partial r} + \frac{1}{2}\sigma^2 \frac{\partial^2 B}{\partial r^2} = 0 \\ P(t = T, r) = 1 \end{cases}$$

$$\iff P(t,r) = e^{A(t,T) - rB(t,T)} \text{ where}$$

$$\begin{cases} A(t,T) = (B - (T-t))(\frac{\eta\gamma - \sigma^2/2}{\gamma^2}) - \frac{\sigma^2 B^2}{4\gamma} \\ B(t,T) = \frac{-e^{-\gamma(T-t)}}{\gamma} \end{cases}$$

Now, we can have a direct connection between $Y(t,T)$ and (η,γ,σ)

$$Y(t,T) = -\frac{A(t,T) - r_0 B(t,T)}{T-t}$$

Finally, we use the Levernberg-Marquart Alforithm (LMA) to find the triplet (η,γ,σ) in order to best fit the following set of data

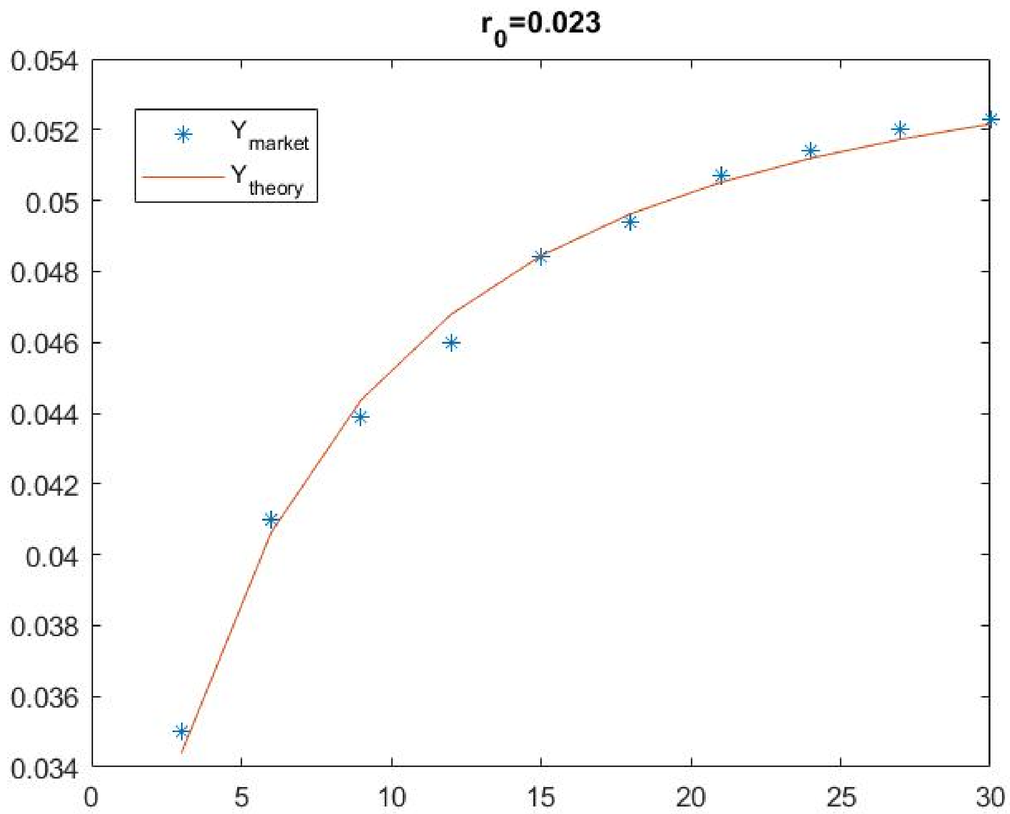
Table 1: Yield curve data $Y(t=0,T)$ at different maturities

T_i	3	6	9	12	15	18	21	24	27	30
Y^i_{market}	0.035	0.041	0.0439	0.046	0.0484	0.0494	0.0507	0.0514	0.052	0.0523

Figure 1 suggests a "good" approximate of the parameters because the theoretical values (in red lines) are close to the market data (in blue asterisks).

$$\eta = 0.0154 \qquad \gamma = 0.2154 \qquad \sigma = 0.0377$$

Figure 1: Calibration of Vasicek Model by ZC-Bond



Using the calibrated parameters, we try to find the interest rate for any given spontaneous short rate r_0 and any maturity date ^{T}, this is depicted in Figure 2.

Figure 2: Interest rates by Vasicek model for given (η, γ, σ)

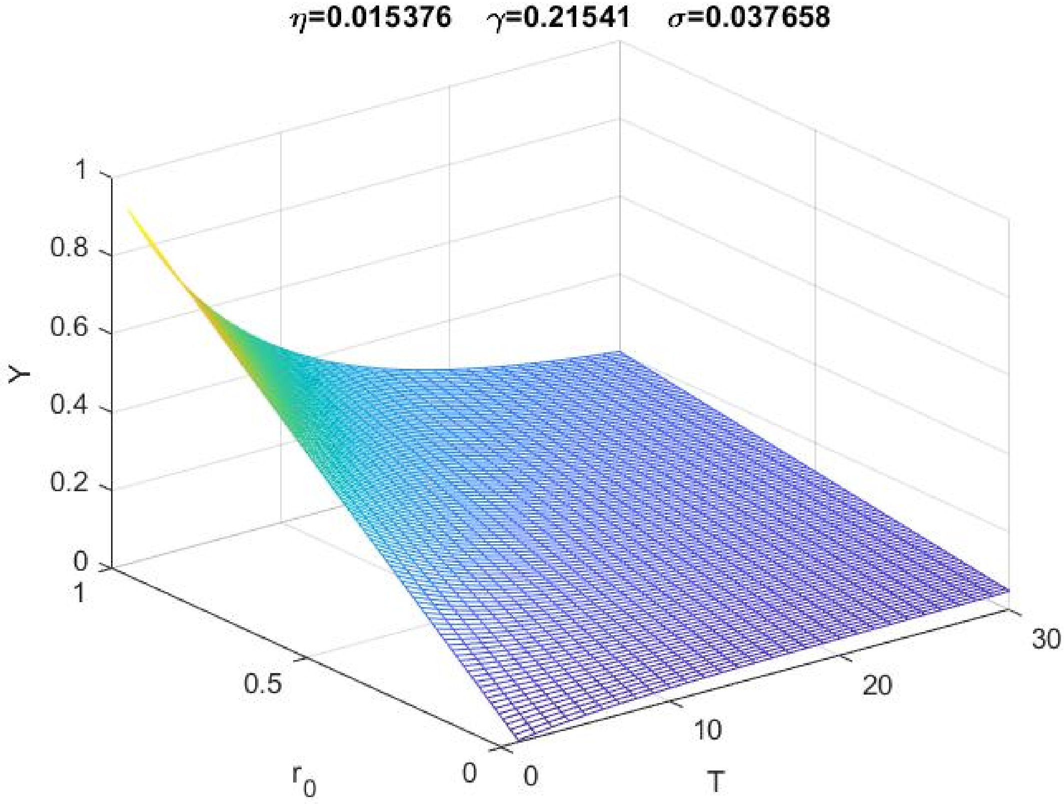


Figure 3: Analysis of interest with fix η

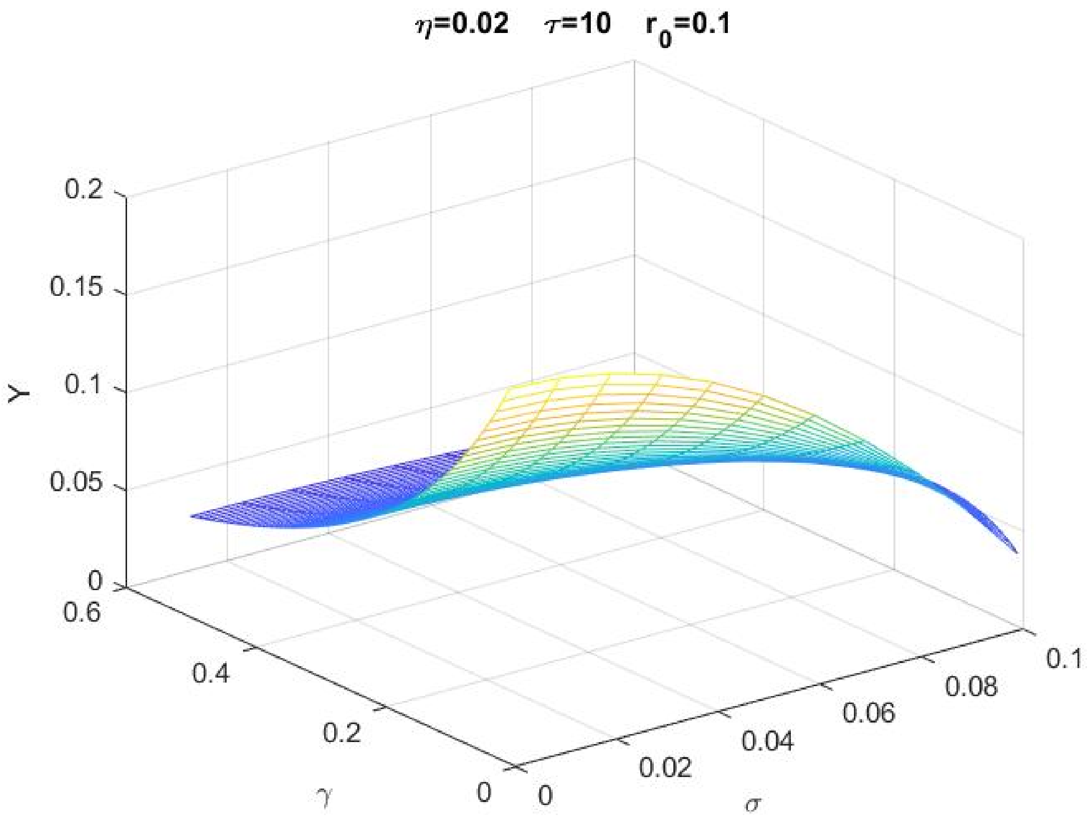
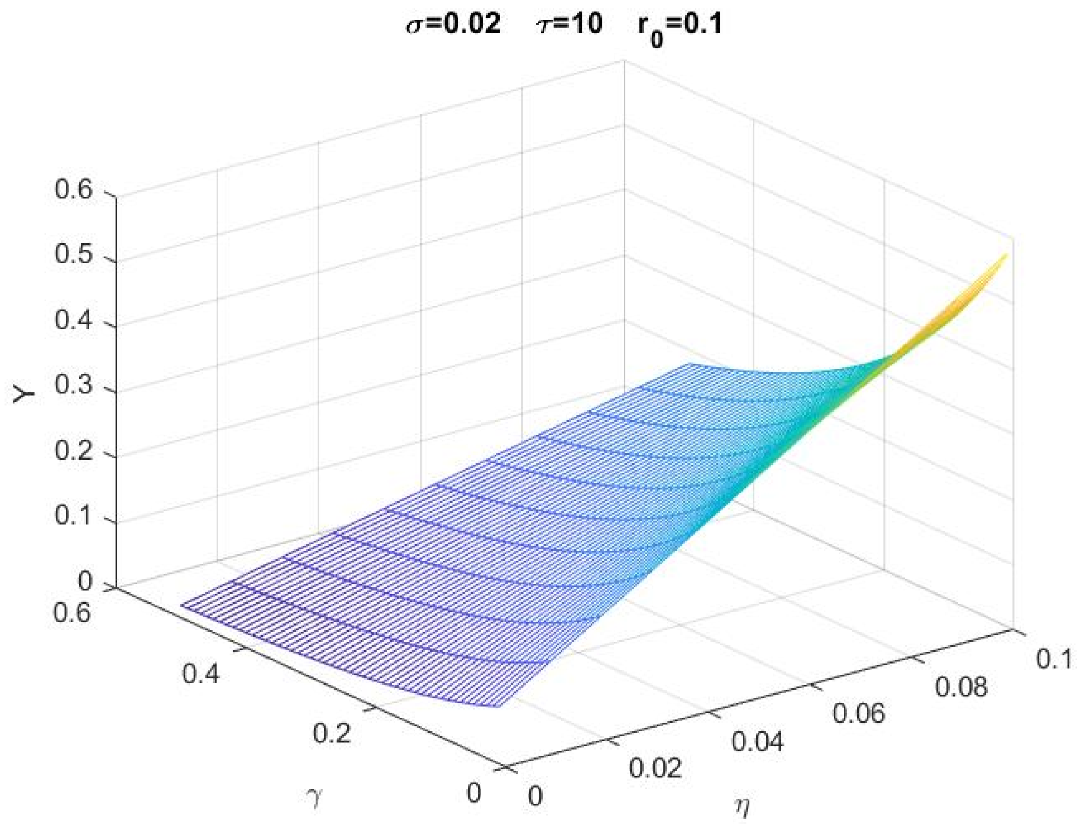


Figure 4: Analysis of interest with fix σ



2.2 Use of historical data

The solution for equation (1) is

$$r_t = r_0 e^{-\gamma t} + \frac{\eta}{\gamma}(1 - e^{-\gamma t}) + \sigma \int_0^t e^{-\gamma(t-s)} dW_s$$

So we can conclude that

$$E[r_t] = r_0 e^{-\gamma t} + \frac{\eta}{\gamma}(1 - e^{-\gamma t})$$

$$Var[r_t] = \frac{\sigma^2}{2\gamma}(1 - e^{-2\gamma t})$$

Note that if we approximate r_t by a sequence of r_{t_i} with $t_i = it/_n$

$$
\begin{aligned}
r_{t_{i+1}} &= r_0 e^{-\gamma t_{i+1}} + \frac{\eta}{\gamma}(1 - e^{-\gamma t_{i+1}}) + \sigma \int_0^{t_{i+1}} e^{-\gamma(t_{i+1}-s)} dW_s \\
&= r_0 e^{-\gamma t_i + \Delta t} + \frac{\eta}{\gamma}(1 - e^{-\gamma t_i + \Delta t}) + \sigma \int_0^{t_i + \Delta t} e^{-\gamma(t_i + \Delta t - s)} dW_s \\
&= e^{-\gamma \Delta t}\left(r_0 e^{-\gamma t_i} + \frac{\eta}{\gamma}(1 - e^{-\gamma t_i}) + \sigma \int_0^{t_i} e^{-\gamma(t_i - s)} dW_s\right) \\
&\quad + \frac{\eta}{\gamma}(1 - e^{-\gamma \Delta t}) + \sigma \int_{t_i}^{t_{i+1}} e^{-\gamma(t_{i+1}-s)} dW_s \\
&= e^{-\gamma \Delta t} r_{t_i} + \frac{\eta}{\gamma}(1 - e^{-\gamma \Delta t}) + \sigma \int_{t_i}^{t_{i+1}} e^{-\gamma(t_{i+1}-s)} dW_s
\end{aligned}
$$

and the last term follows $\sigma\sqrt{\frac{1-e^{-2\gamma\Delta t}}{2\gamma}}\mathcal{N}(0,1)$. Now the calibration problem comes down to a simple optimization problem: to minimize the errors of an affine function $y = ax + b$ where

$$
\begin{cases} a = e^{-\gamma \Delta t} \\ b = \frac{\eta}{\gamma}(1 - e^{-\gamma \Delta t}) \end{cases} \iff \begin{cases} \gamma = -\frac{\ln a}{\Delta t} \\ \eta = \frac{b\gamma}{\Delta t} \end{cases}
$$

Finally, the total variation (D) shall be an approximate of the variance of $r_{\Delta t}$ so

$$
D = \sigma \sqrt{\frac{1 - e^{2\gamma \Delta t}}{2\gamma}} \iff \sigma = D \sqrt{\frac{-2 \ln a}{\Delta t (1 - a^2)}}
$$

We first generate a sample of 1000 r_t where $\eta = 0.6, \gamma = 0.4, \sigma = 0.08$ and use this method to calibrate the Vasicek model. Figure 6 illustrates the fitted affine function (in red line) with the sample data (in blue dots). The result is relatively close to the true value that we used to generate the sample

$$
\eta = 0.4659 \qquad \gamma = 0.3208 \qquad \sigma = 0.0791
$$

Figure 5: Calibration of interest rate by historical data

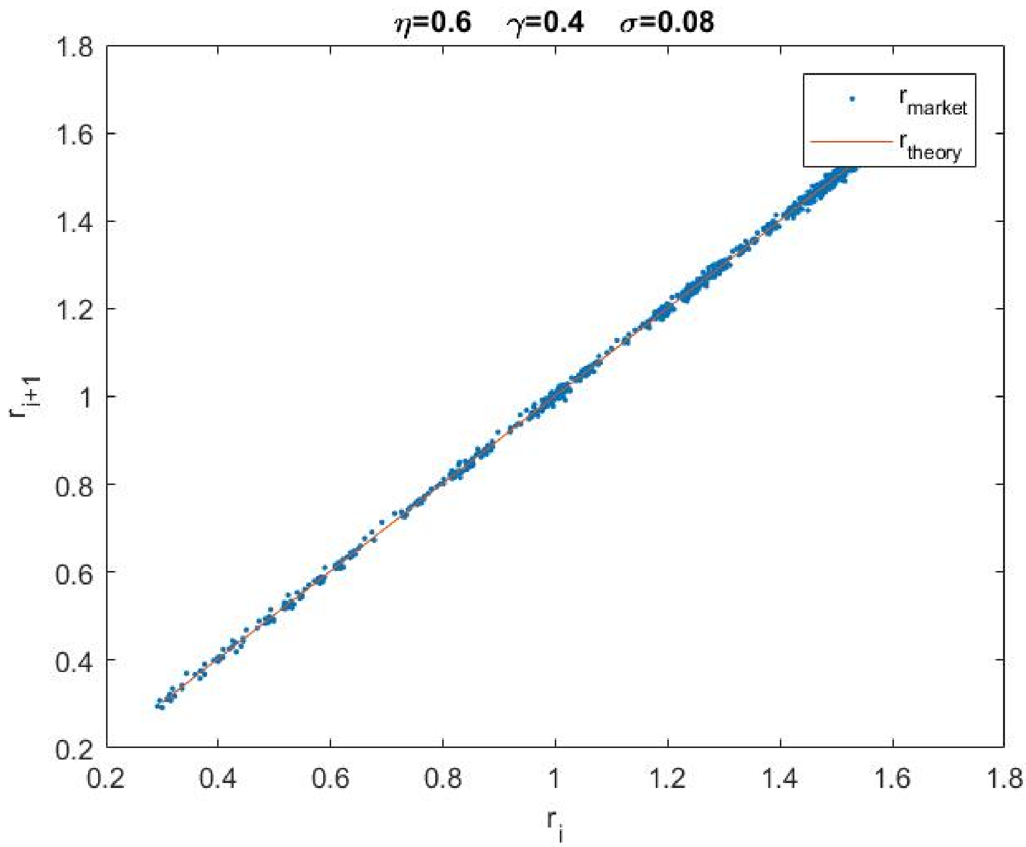


Figure 5 shows the evolution of the interest rate that we generated by the given parameters (in blue lines) and compares it with the fitted curve that we calibrated using the sample data (in red lines). The graph shows a relatively close result between the sample data and the theoretically generated data.

Figure 6: Interest rates evolution