Assignment on Calibration & Hedging

Delta Hedging

Thu Hang, DO

class

QFRM II

Jan 2019

List of Figures

Nomenclature

Π	Profit/Loss of hedging position
σ	Volatility of underlying asset
A	Portion of stocks in hedging portfolio
B	Portion of cash in hedging portfolio
K	Strike price
N	Number of discretization
Nmc	Number of simulation
P	Hedging portfolio
r	Interest rate
S_t	Price of underlying asset at time t
T	Time to maturity
V	Option price

1 Introduction

Option value is exposed to price movement of the underlying asset so it is often hedged by other instruments to offset this risk. The most common scheme is to delta hedge the option with the underlying asset; the hedging portfolio consists of the underlying asset and cash, or banking account.

$$P_t = A_t S_t + B_t$$

To delta hedge is to establish a delta neutral position, in which the delta of the hedging portfolio and that of the hedged option cancel out each other.

$$\frac{\partial \Pi}{\partial S} = \frac{\partial P}{\partial S} - \frac{\partial V}{\partial S} = 0 \iff A = \frac{\partial V}{\partial S}$$

In this report, we consider a hedging portfolio on an European Call Option with the following parameters:

$$S_0 = 1 \qquad r = 0.05 \qquad T = 5 \qquad K = 1.5 \qquad \Delta t = 0.01$$

We will analyze the hedging portfolio under different assumption of the volatility σ where it can be constant or stochastic. The portion of stock in the hedging portfolio is:

$$A_t = \frac{\partial V}{\partial S}(S_t) = N(d1(t, S_t))$$

2 Analysis Result

Hedging in discrete time

Standard Black-Scholes model assumes continuous hedging, which is almost impossible in practice. Figure 1 shows the evolution of option price (V) and the hedging portfolio (P_{act}) in discrete time. The hedging portfolio is revised at every time step $\Delta t = 0.01$. We can see a small gap between V and P_{act}. This is due to discrete hedging.

Figure 1: Evolution of option price and hedging portfolio

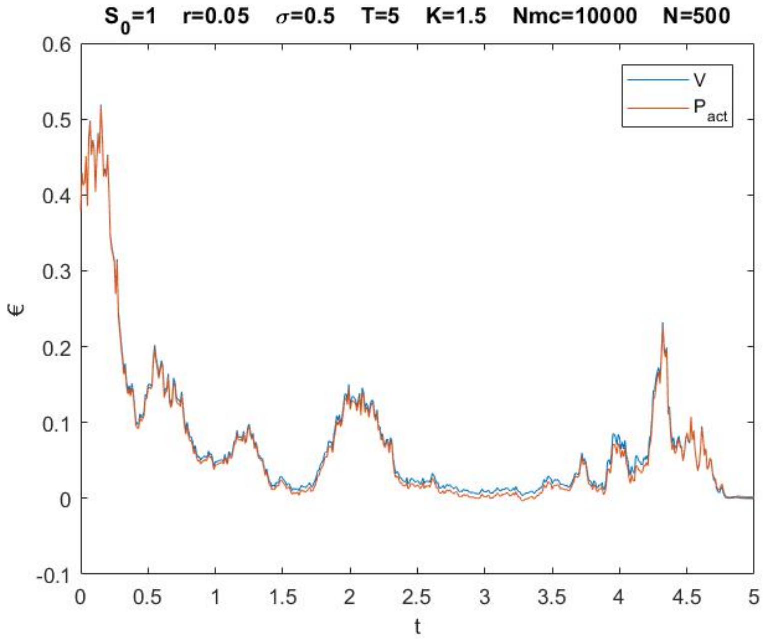
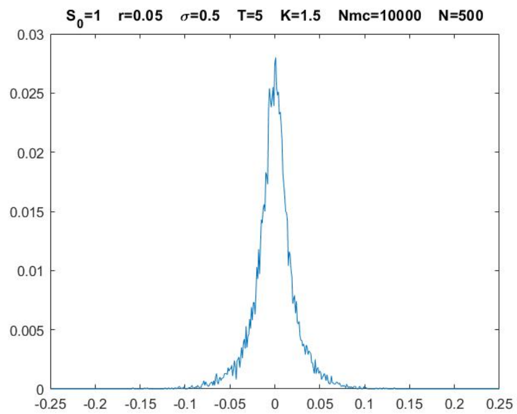


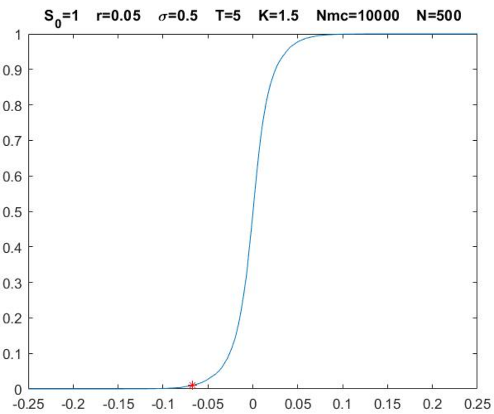
Figure 2 illustrates the P&L distribution of the hedging portfolio. The density distribution is symmetric around 0 but it is not normal distribution. The peak of density function is a lot more pointed than that of the normal distribution. The CDF graph implies that we have a risk of 1% to lose at least 0.05.

Figure 2: P&L distribution

(a) PDF



(b) CDF

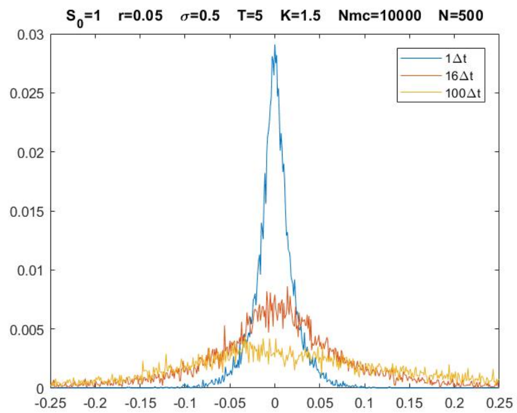


The effect of infrequent hedging

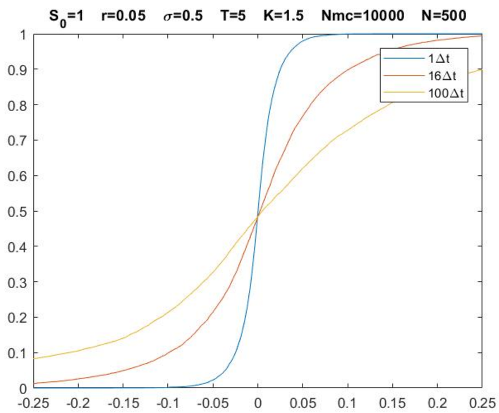
Figure 3 compares the profit and loss distribution with different hedging periods. The blue line shows the P&L distribution of a portfolio that is revised every time step, while red and yellow lines shows that of a portfolio that is updated every $16\Delta t$ and $100\Delta t$ respectively. It is visible in Figure $3a$ that red and yellow lines have much fatter tails than blue lines. This indicates that infrequent hedging affects negatively to P&L in the sense that P&L varies much wider than frequent hedging. It is equivalent to say that VaR of longer hedging period will be much lower than that of shorter hedging period under same risk level.

Figure 3: P&L distribution at different hedging periods

(a)

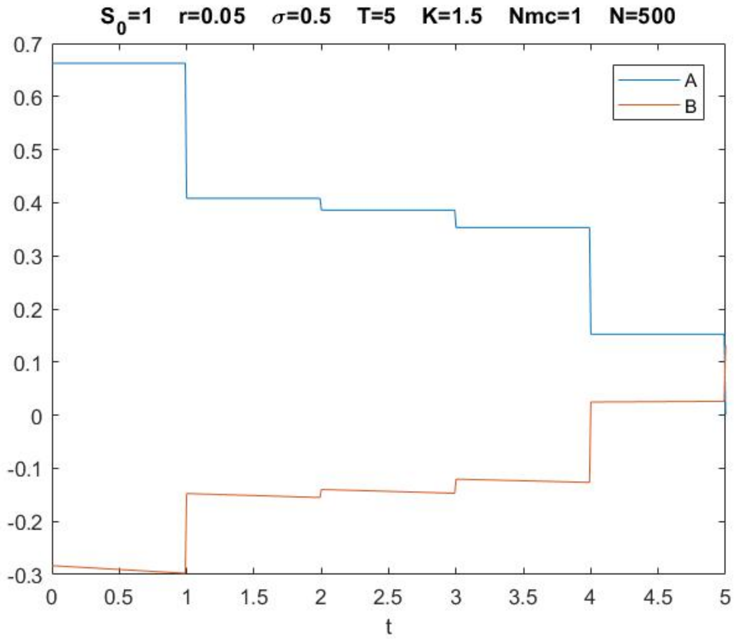


(b)



We further analyze the portion of stock and cash in the hedging portfolio that is rebalanced every $100\Delta t$. While the portion of stock remains unchanged during the interval, the cash portion decreases slightly. This is due to interest rate factor ($r = 0.05$) and to the fact that cash portion is negative at the beginning $t = 0$. If the cash balance were positive at $t = 0$, it would increase during the interval.

Figure 4: Proportion of stock and cash on hedging portfolio



The impact of stochastic volatility

We first consider a simple stochastic volatility model: volatility remains at 0.5 and it has 5% to drop to 0.3.

$$\sigma_t = \begin{cases} 0.5 \text{ if } p < 0.95 \text{ where } p \sim \mathcal{U}[0,1] \\ 0.3 \text{ otherwise} \end{cases}$$

Figure 5a depicts the evolution of volatility over time and Figure 5b shows the evolution of option price under this assumption about volatility and hedging portfolio. We can see in Figure 5b the wide gap between the option price and hedging portfolio at every drop of volatility accordingly to Figure 5a.

Figure 5: Evolution of option price and portfolio - Stochastic volatility Model 1

(a) (b)

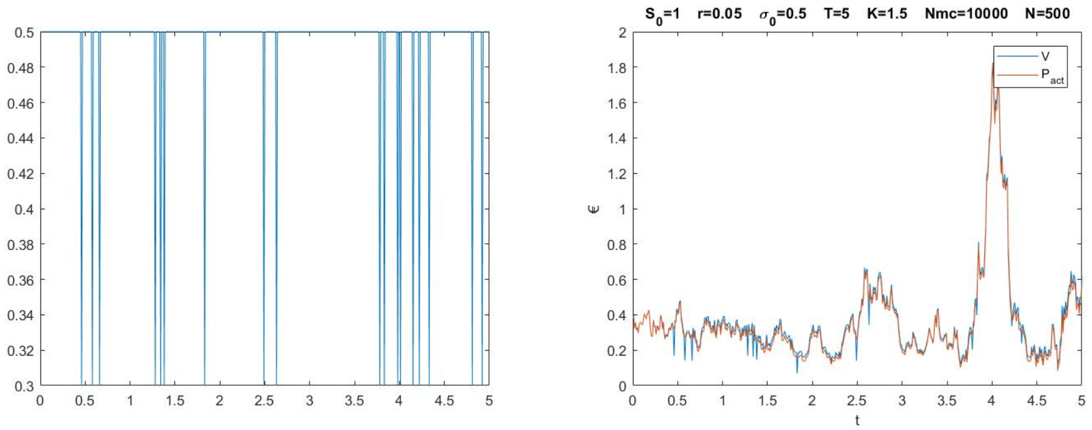
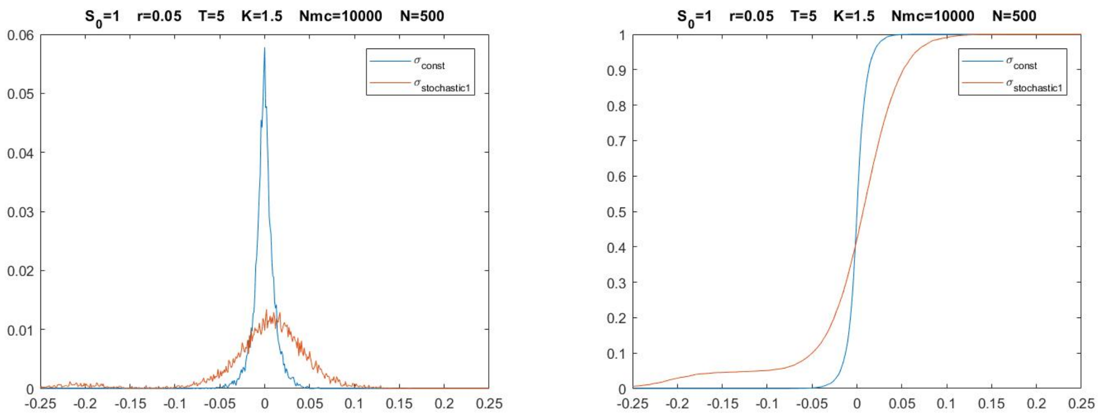


Figure 6 shows that the P&L density distribution is slightly skewed to the left, which indicates that there is higher chance to earn profit from this hedging position. However, a much fatter tail implies that the risk of losing money is much greater than constant volatility model.

Figure 6: P&L distribution - Volatility constant vs. stochastic 1

(a) (b)



We consider a second stochastic volatility model, in which the volatility has 5%

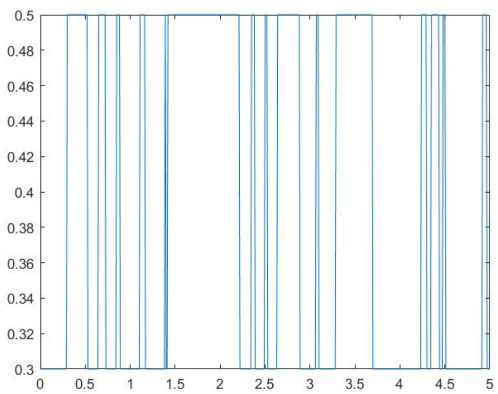
to opt between 0.5 and 0.3 and $\sigma_0 = 0.5$.

$$\sigma_{t+\Delta t} = \begin{cases} 0.5 \text{ if } p < 0.05 \text{ and } \sigma_t = 0.3 \text{ where } p \sim \mathcal{U}[0,1] \\ 0.3 \text{ if } p < 0.05 \text{ and } \sigma_t = 0.5 \\ \sigma_t \text{ otherwise} \end{cases}$$

Again, we can see a wide gap between the option price and hedging portfolio. It is evident that stochastic volatility is more difficult to hedge than constant volatility.

Figure 7: Evolution of option price and hedging portfolio - Stochastic volatility Model 2

(a)



(b)

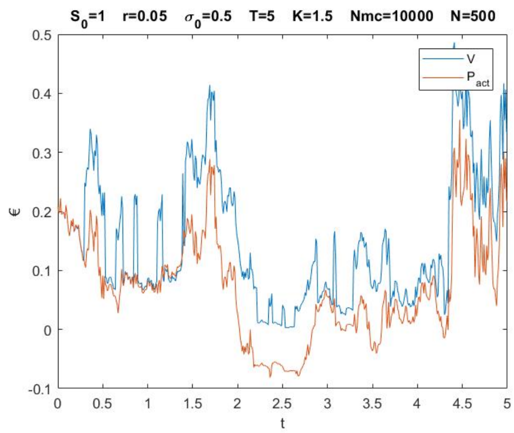
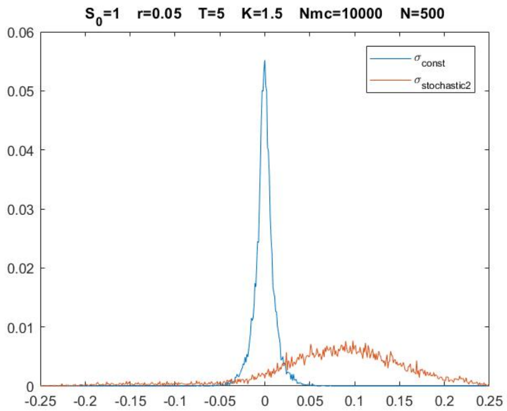


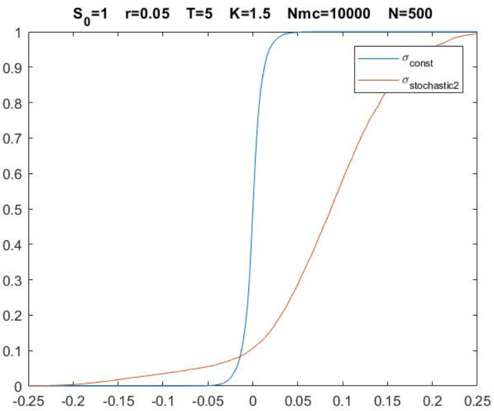
Figure 8 again depicts the similar pattern as Model 1 as the profit and loss distribution is skewed to the left and has much wider tail than constant volatility model. Model 2 has even fatter tail than Model 1 because the volatility is more volatile than in Model 1.

Figure 8: P&L distribution - Stochastic volatility Model 2

(a)



(b)



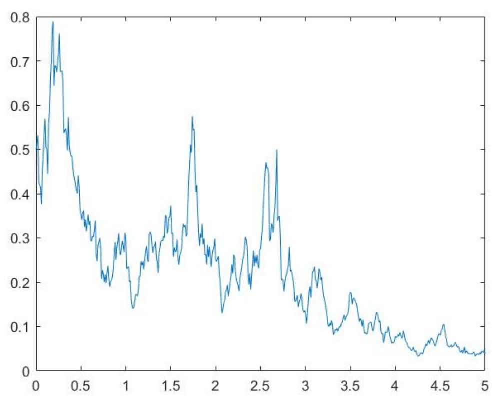
Finally, we consider the stochastic volatility model where

$$\sigma_t = \int_0^t \sigma_s dW_s$$

Figure 9 illustrates a huge difference between the hedging portfolio and option price. This is due to a strong volatility of volatility. The profit and loss distribution also show the fattest tail out of the 3 stochastic volatility model.

Figure 9: Evolution of option price and hedging portfolio - Stochastic volatility Model 3

(a)



(b)

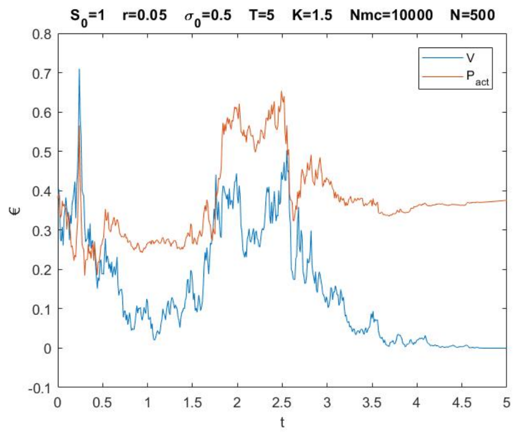
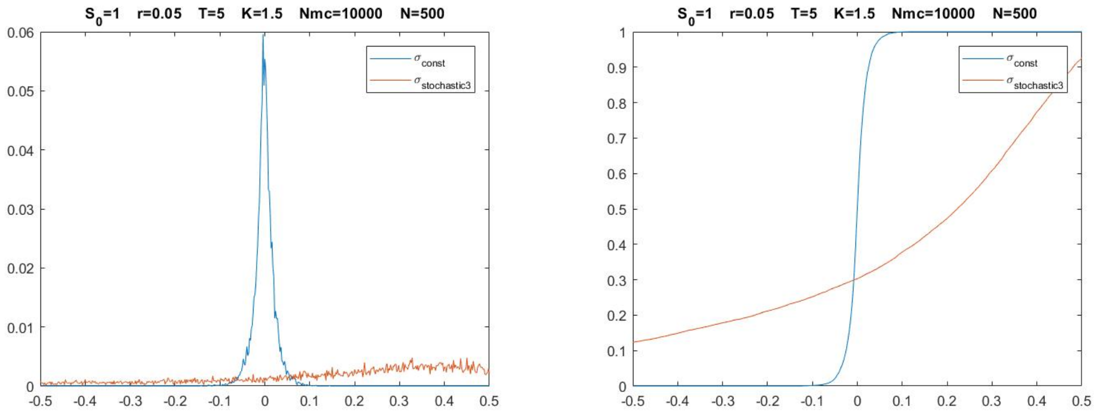


Figure 10: P&L distribution - Stochastic volatility Model 3

(a) (b)



3 Conclusion

From this analysis, we can conclude that:

- Infrequent hedging leads to higher chance of losing money as the tail of P&L density distribution is fatter.
- Stochastic volatility is more difficult to hedge than constant volatility; the more volatile the volatility is, the less corresponding the hedging portfolio is to the option price.
- Delta hedge requires frequent rebalancing; hence, it is costly in practice.