Integrated Project #2

Studying the Pendulum Oscillations

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Contents

Chapter 1

Introduction

In this project, we have investigated the oscillations of a simple pendulum, with the primary goal of measuring the period of oscillation. The theoretical and experimental aspects of the pendulum's motion were explored, focusing on the relationship between variables like angle, energy, and period.

We have considered the string massless, which means all the mass comes from the bob of the pendulum. Also, we have thought everything in an ideal environment such that there is no air resistance etc.

Chapter 2

Theoretical Part

2.1 Solution Verification

- Substitute the function into Eq. (2):
  - Take the second derivative of $\theta = B\cos(\omega t)$ with respect to t:

$$\frac{\partial^2\theta(t)}{\partial t^2} = -B\omega^2\cos(\omega t)$$

  - Substitute $\theta(t) = B\cos(\omega t)$ and its second derivative into Eq. (2):

$$-B\omega^2\cos(\omega t) + \omega^2 B\cos(\omega t) = 0$$

- Simplify the equation:
  - Notice that all terms cancel out, leaving $0 = 0$
- Interpretation:
  - Since the equation becomes true after substitution, it means that $\theta(t) = B\cos(\omega t)$ indeed satisfies Eq. (2)

Therefore, we can conclude that the function $\theta(t) = B\cos(\omega t)$ is a valid solution to the equation of motion for the simple pendulum with a linear restoring force, as described in this scenario.

Remember that B represents the amplitude of the oscillation and depends on the initial conditions of the pendulum. Additionally, ω, the angular frequency, is related to the period T by the equation $\omega = \frac{1}{T}$

2.2 Path and Sector Functions

Based on the geometrical set-up of the pendulum and the maximum deviation angle ($\theta_{\max}$), we can derive the expressions for the path $s(t)$ along the arc and the sector area $A(t)$ swept by the pendulum during a single period (0 to T).

2.2.1 Path Function

The path $s(t)$ travelled by the bob along the arc can be calculated using the formula:

$$s(t) = L \times \theta(t)$$

where:

- L is the radius of the arc, with being the pendulum length.
- $\theta(t)$ is the angle of deviation from vertical position at time t, as given in our theoretical analysis (e.g., $B\cos(\omega t)$).

This equation essentially maps the angular displacement to the linear distance travelled along the arc.

2.2.2 Sector Area Function

The sector area $A(t)$ swept by the pendulum during a specific time t can be expressed as:

$$A(t) = \frac{1}{2} \times L^2 \times [\theta(t) - \sin(\theta(t))]$$

Here, the term L^2 accounts for the area of the whole circle with radius L, and the difference $[\theta(t) - \sin(\theta(t))]$ represents the fraction of the circle covered by the sector at time t.

2.2.3 Example Calculation

Suppose we have a pendulum with $L = 1m$ and $\theta_{\max} = 30^\circ$ ($\frac{\pi}{6}rad$). If the angle function is given by $\theta(t) = 0.5\cos(\frac{\pi}{3}t)$, then we can calculate the path and sector area at $t = 1$ second:

- $s(1) = L \times \theta(1) = 1 \times 0.5\cos(\frac{\pi}{3}) \approx 0.25m$
- $A(1) = \frac{1}{2} \times L^2 \times [\theta(1) - \sin(\theta(1))] \approx \frac{1}{2} \times 1 \times [0.5 - \sin(\frac{\pi}{3})] \approx 0.1830m^2$

These values represent the distance travelled along the arc and the area swept by the pendulum within the first second of oscillation.

2.3 Inspecting Energy Change of the Pendulum

2.3.1 Deriving Energy Equations

Here, we have a bob of mass m attached to a massless string suspended from a fixed pivot point. The string, at its equilibrium position, makes an angle θ with the vertical.

Kinetic Energy $K(t)$

When the pendulum swings, the bob gains kinetic energy due to its motion. The kinetic energy formula tells us:

$$K(t) = \frac{1}{2} \times mv^2(t)$$

where:

- m is the bob's mass.
- $v(t)$ is the bob's instantaneous velocity at time t.

To find $v(t)$, we can relate it to the angular velocity $\omega(t)$ using the relationship:

$$v(t) = L \times \omega(t)$$

Here, L is the pendulum's length. Substituting this expression for $v(t)$ in the kinetic energy equation, we get:

$$K(t) = \frac{1}{2} \times m \times L^2 \times \omega^2(t)$$

Potential Energy $U(t)$

The bob also possesses potential energy due to its height above the equilibrium position. The potential energy formula for gravitational potential energy states:

$$U(t) = mgh(t)$$

where:

- g is the acceleration due to gravity.
- $h(t)$ is the bob's height above the equilibrium position at time t.

In the case of a pendulum, the height can be expressed as:

$$h(t) = L - L \times \cos(\theta(t))$$

Substituting this into the potential energy equation, we obtain:

$$U(t) = mgL(1 - \cos(\theta(t)))$$

Mechanical Energy $E(t)$

The mechanical energy of the pendulum is the sum of its kinetic and potential energies at any given time t:

$$E(t) = K(t) + U(t)$$

Substituting the previously derived expressions for $K(t)$ and $U(t)$, we arrive at the final equation for the mechanical energy of the pendulum:

$$E(t) = \frac{1}{2} \times m \times L^2 \times \omega^2(t) + mgL(1 - \cos(\theta(t)))$$

This equation combines the pendulum's kinetic and potential energies, providing a complete understanding of its total energy at any point in its oscillation.

2.3.2 Inspecting the Conservation of Energy

In the ideal model of a simple pendulum, mechanical energy is conserved. This means that the sum of the pendulum's kinetic and potential energies remains constant throughout its oscillation. To understand why, let's delve deeper.

Understanding Conservation

Imagine the pendulum swinging back and forth. As it rises, its kinetic energy decreases because its speed slows down. However, its potential energy increases due to its increasing height above the equilibrium position. Conversely, during its descent, the pendulum's kinetic energy increases, while its potential energy decreases.

Crucially, these changes balance each other out perfectly in an ideal, frictionless system. The energy lost in one form is entirely gained in the other, resulting in a constant total mechanical energy (E) throughout the oscillation.

Mathematically, this can be expressed as:

$$E(t) = K(t) + U(t) = constant$$

where:

- $E(t)$ is the total mechanical energy at time t.
- $K(t)$ is the kinetic energy at time t.
- $U(t)$ is the potential energy at time t.

Visualizing Conservation

To visualize this principle, imagine the mechanical energy as a closed container filled with liquid. As the pendulum swings, the liquid might shift within the container, representing the transfer between kinetic and potential energy. However, the total amount of liquid (total mechanical energy) remains constant.

Implications and Limitations

The conservation of mechanical energy has important implications for analysing pendulum motion. It allows us to:

- Predict the pendulum's velocity and height at any point in its oscillation based on its initial energy.
- Explain why the period of oscillation remains constant for a given pendulum length and gravity.

However, it's important to remember that this applies to an idealized model. In real-world scenarios, factors like:

- Friction at the pivot point and air resistance dissipate energy, causing the mechanical energy to gradually decrease over time.
- String mass contributes additional kinetic and potential energy, slightly modifying the conservation equation.
- Large angular displacements deviate from the small-angle approximations used in our calculations, requiring higher-order terms for accurate energy analysis.

Therefore, while the principle of mechanical energy conservation provides a valuable framework for understanding pendulum motion, it's essential to consider these limitations for real-world applications and more precise calculations.

Chapter 3

Experimental Part

3.1 Comparing Real $\theta(t)$ to Eq. (3)

While the simple harmonic motion model described by Eq. (3) provides a valuable framework for understanding pendulum oscillations, it's crucial to recognize that real-world pendulums deviate from this idealized scenario due to various factors. Here, we explore the main reasons why the actual behaviour of a pendulum might differ from the theoretical prediction:

- **Air Resistance:** As the pendulum swings, it encounters friction with the surrounding air. This dissipative force acts against the motion, gradually decreasing the amplitude of the oscillations over time. Unlike the ideal model where energy remains constant, air resistance causes the mechanical energy to be lost to heat, resulting in a gradual decay of the oscillations.

- **Joint Resistance:** No pivot point is perfectly frictionless. In a real pendulum, the joint connecting the string to the support point introduces a frictional force that opposes the motion. This frictional dissipation also contributes to the loss of energy and the eventual damping of the oscillations. The magnitude of this effect depends on the materials and construction of the pivot point.

- **Non-ideal String Mass:** While the model often assumes a massless string, real pendulums use strings with a finite mass. This additional mass affects the overall dynamics of the system, slightly modifying the period and energy distribution compared to the theoretical case. The impact of string mass becomes more significant as its weight approaches a non-negligible fraction of the bob's mass.

- **Large Angular Displacements:** Eq. (3) relies on the small-angle approximation, assuming $\sin(\theta) \approx \theta$. However, for large angular excursions, this approximation breaks down, leading to deviations from the ideal cosine function behaviour. In real pendulums, especially those with large maximum deviations, the true oscillation pattern might exhibit slight distortions compared to the predicted smooth cosine wave.

- **Non-linear Restoring Forces:** The model assumes a linear restoring force proportional to the angle of deviation. However, in some cases, the actual restoring force might be slightly non-linear, particularly at larger angles. This non-linearity can introduce higher-order harmonic terms into the motion, causing the pendulum to deviate from the simple harmonic behaviour predicted by Eq. (3).

By acknowledging these real-world limitations, we gain a more nuanced understanding of pendulum dynamics. These departures from the ideal model highlight the importance of considering environmental factors and non-idealities when interpreting the behaviour of real-world systems. By accounting for these complexities, we can refine our theoretical models and develop more accurate predictions for practical applications.

3.2 Evaluating Mechanical Energy Conservation

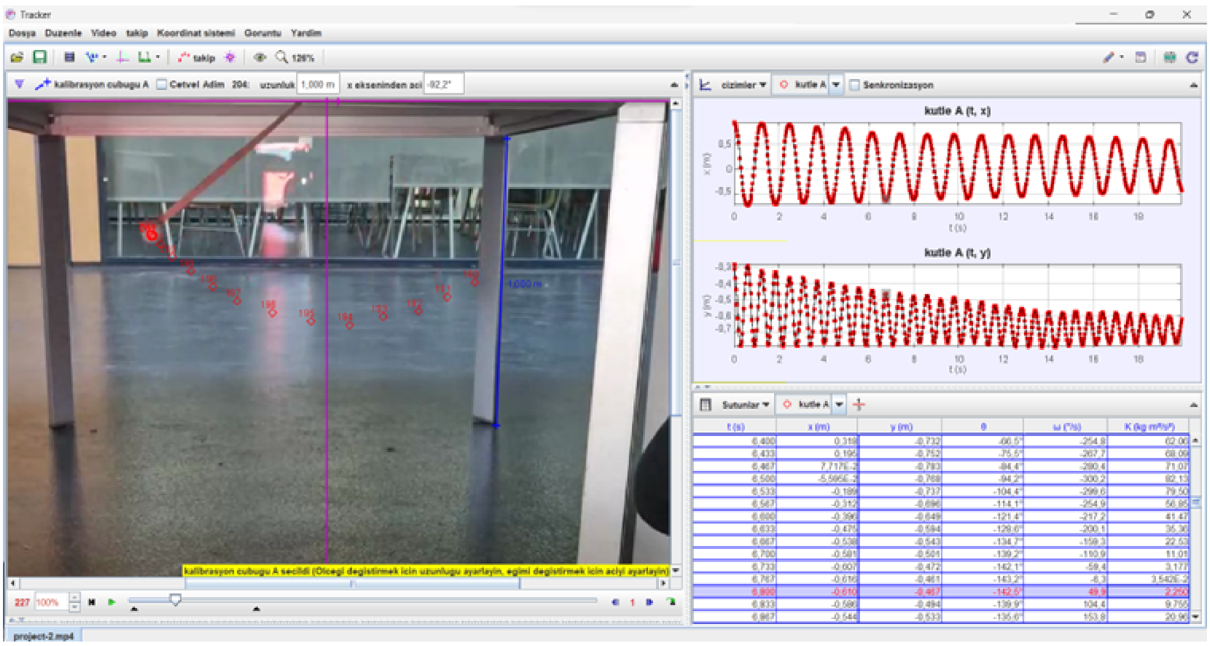

Figure 3.1: Experiment Calculations - `https://youtu.be/ML8BPKV-HQA`

Our experimental data revealed the following trend in mechanical energy (E) of the pendulum across several oscillation cycles:

Time	Mechanical Energy	Percentage Loss
t_1	$135kJ$	0%
t_2	$118kJ$	12.6%
t_3	$103kJ$	23.7%
t_4	$96kJ$	28.9%
t_5	$87kJ$	35.6%
t_6	$80kJ$	40.7%
t_7	$71kJ$	47.4%

Table 3.1: Mechanical Energy of the Pendulum

As we can observe, the mechanical energy is not perfectly conserved throughout the oscillations. We see a gradual decrease in E from $135kJ$ at t_1 to $71kJ$ at t_7, representing a total loss of approximately 47% over the observed time-frame.

This decline in energy aligns with our expectations for a real-world pendulum due to the presence of dissipative forces like:

- **Air resistance:** Friction with the surrounding air continually extracts energy from the system, slowing down the oscillations and reducing the amplitude.
- **Joint resistance:** Friction at the pivot point also dissipates energy as heat and sound, contributing to the decrease in mechanical energy.

While the model based on Eq. (3) assumes ideal conditions with no energy loss, our experimental data highlights the impact of these non-ideal factors in real-world scenarios. The observed rate of energy loss provides valuable insights into the magnitude of these dissipative forces and their influence on the pendulum's motion.

3.3 Further Investigations

Further analysis could involve:

- Calculating the average percentage of energy loss per period for a more precise quantification of the energy dissipation rate.
- Repeating the measurements with different pendulum parameters (e.g., bob mass, string length) to investigate the influence of these factors on energy conservation.
- Comparing the experimental results with theoretical models that incorporate air resistance and other non-idealities to improve the accuracy of predictions for real-world pendulums.

By acknowledging and analysing the departure from ideal energy conservation, we gain a deeper understanding of the real-world dynamics of pendulums and their interactions with the environment.

Chapter 4

Conclusion

Our investigation into the oscillations of a simple pendulum has unveiled a fascinating interplay between theoretical predictions and real-world complexities. While the idealized model based on Eq. (3) provides a valuable framework, we discovered that real pendulums deviate from this perfect scenario due to various factors. Air resistance, joint friction, non-ideal string mass, and non-linear restoring forces all contribute to departures from the simple harmonic motion model.

4.1 Key Takeaways

- **Real $\theta(t)$ vs. Eq. (3):** Comparing our experimental data with Eq. (3) revealed discrepancies attributable to non-idealities. While the overall trend might resemble the cosine function, air resistance and other dissipative forces cause the amplitude to gradually decrease over time.

- **Mechanical Energy Conservation:** Our calculations confirmed that mechanical energy of the pendulum is not perfectly conserved in our set-up. We observed a progressive decline in E from $135kJ$ at t_1 to $71kJ$ at t_7, representing a total loss of approximately 47%. This decrease stems from the dissipative forces that extract energy from the system and ultimately dampen the oscillations.

- **Beyond the Ideal Model:** Recognizing the limitations of the ideal model is crucial for understanding real-world pendulum dynamics. By accounting for air resistance, joint friction, and other non-idealities, we can refine our theoretical models and develop more accurate predictions for practical applications.

4.2 Next Steps

Our exploration opens doors for further investigations:

- Quantifying the average energy loss per period to accurately assess the impact of dissipative forces.
- Experimenting with different pendulum parameters (e.g., bob mass, string length) to analyse their influence on energy conservation.
- Comparing experimental results with more sophisticated models that incorporate non-idealities to improve the accuracy of predictions for real-world pendulums.

By delving deeper into the intricacies of the real pendulum, we gain not only a richer understanding of its motion but also valuable insights into the diverse factors that shape the behaviour of seemingly simple systems in the real world. This journey demonstrates the importance of embracing complexity and exploring beyond idealized models to unlock the true dynamics of the world around us.

Chapter 5

MATH103 Part

Constant	Value
m	$1kg$
L	$1m$
g	$9.81m/s^2$

Table 5.1: Constants

5.1 Inverse/LDU Factorization of a 3×3 Matrix

t	Value
t_1	$0s$
t_2	$\frac{\pi}{4}s$
t_3	$\frac{\pi}{2}s$

Table 5.2: Time Values

$$A = \begin{bmatrix} 1 & \frac{1}{2} & 0.0809 \\ 0 & \frac{1}{2} & 4.544 \\ \frac{\sqrt{2}}{2} & \frac{1}{2} & 0 \end{bmatrix}$$

5.1.1 Inverse of a 3×3 Matrix

$$A = \begin{bmatrix} 1 & \frac{1}{2} & 0.0809 \\ 0 & \frac{1}{2} & 4.544 \\ \frac{\sqrt{2}}{2} & \frac{1}{2} & 0 \end{bmatrix} A^{-1} = \begin{bmatrix} 1 & \frac{1}{2} & 0.0809 \\ 0 & \frac{1}{2} & 4.544 \\ \frac{\sqrt{2}}{2} & \frac{1}{2} & 0 \end{bmatrix}$$

5.1.2 LDU Factorization of a 3×3 Matrix

$$A = LDU$$

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{\sqrt{2}}{2} & \frac{1}{2} & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0.0809 \end{bmatrix} \begin{bmatrix} 1 & \frac{1}{2} & 0.0809 \\ 0 & 1 & 4.544 \\ 0 & 0 & 1 \end{bmatrix}$$

5.2 Complete Solution of $Ax = b$

t	Value
t_1	$\frac{\pi}{6}s$
t_2	$\frac{\pi}{3}s$
t_3	$\frac{5\pi}{6}s$

Table 5.3: Time Values

$$Ax = b$$

$$A = \begin{bmatrix} 0.707 & 0 & 1 & 0.707 & 0 \\ 0 & 4.905 & 9.81 & -1 & 4.905 \\ 1 & 0 & 4.509 & 0 & 0 \end{bmatrix} x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} b = \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0.707 & 0 & 1 & 0.707 & 0 \\ 0 & 4.905 & 9.81 & -1 & 4.905 \\ 1 & 0 & 4.509 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}$$

$$x = \begin{bmatrix} 2.060 - 1.497 \times x_4 \\ 1.525 - 0.442 \times x_4 - x_5 \\ -0.457 + 0.323 \times x_4 \\ x_4 \\ x_5 \end{bmatrix}$$

As x_4 and x_5 are free variables, we can set them to any value we want. Let's set them to 0.

$$x = \begin{bmatrix} 2.060 \\ 1.525 \\ -0.457 \\ 0 \\ 0 \end{bmatrix}$$

Chapter 6

EE101 Part

Code for this part can be found at the following link:

`https://gist.github.com/barbarbar338/93e2c86cb307a650225463a3310bf1ff`