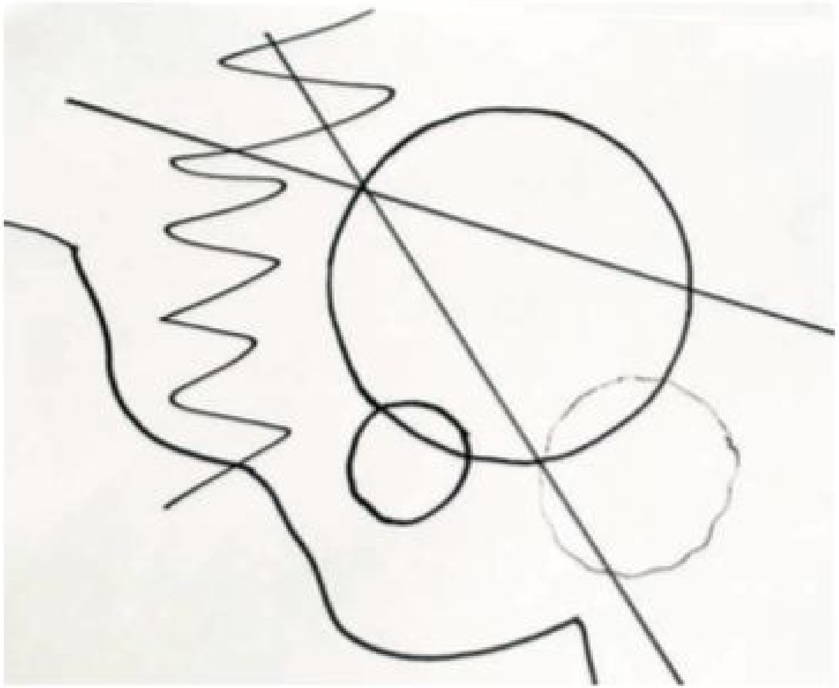
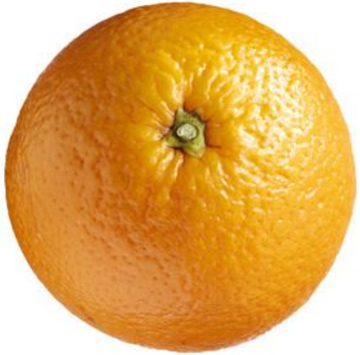
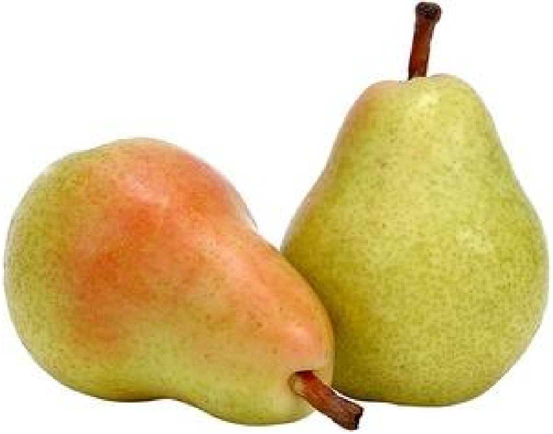
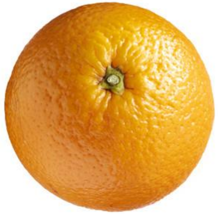
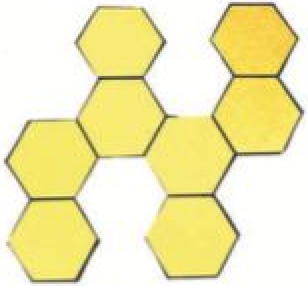
Crocheting Hyperbolic Surfaces

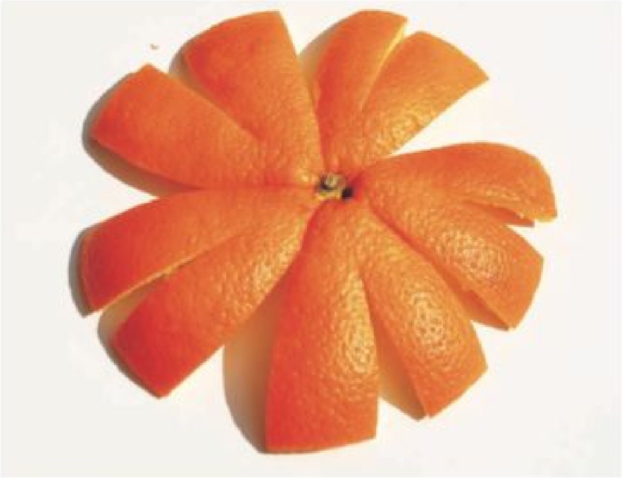



Positive curvature

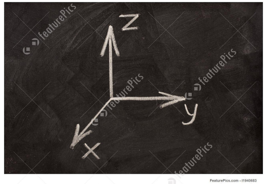






Negative curvature

Euclidean space: Real coordinate space with an imposed Euclidean structure



Let **R** denote a field of real numbers. An element $\mathbf{R}^n$ is written

$$\mathbf{x} = (x_1, x_2, \ldots, x_n),$$

Where each x_i is a real number. The vector space operators on $\mathbf{R}^n$ are defined by

$$\mathbf{x} + \mathbf{y} = (x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n),$$
$$a\,\mathbf{x} = (ax_1, ax_2, \ldots, ax_n).$$

The vector space $\mathbf{R}^n$ comes with a standard basis:

$$\begin{aligned}\mathbf{e}_1 &= (1, 0, \ldots, 0),\\ \mathbf{e}_2 &= (0, 1, \ldots, 0),\\ &\vdots\\ \mathbf{e}_n &= (0, 0, \ldots, 1).\end{aligned}$$

An arbitrary vector in $\mathbf{R}^n$ can then be written as:

$$\mathbf{x} = \sum_{i=1}^{n} x_i \mathbf{e}_i.$$

The inner product of any two vectors **x** and **y** is defined by

$$\mathbf{x} \cdot \mathbf{y} = \sum_{i=1}^{n} x_i y_i = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n.$$

The “length” of a vector is described by the inner product of x with itself

$$\|\mathbf{x}\| = \sqrt{\mathbf{x} \cdot \mathbf{x}} = \sqrt{\sum_{i=1}^{n} (x_i)^2}.$$

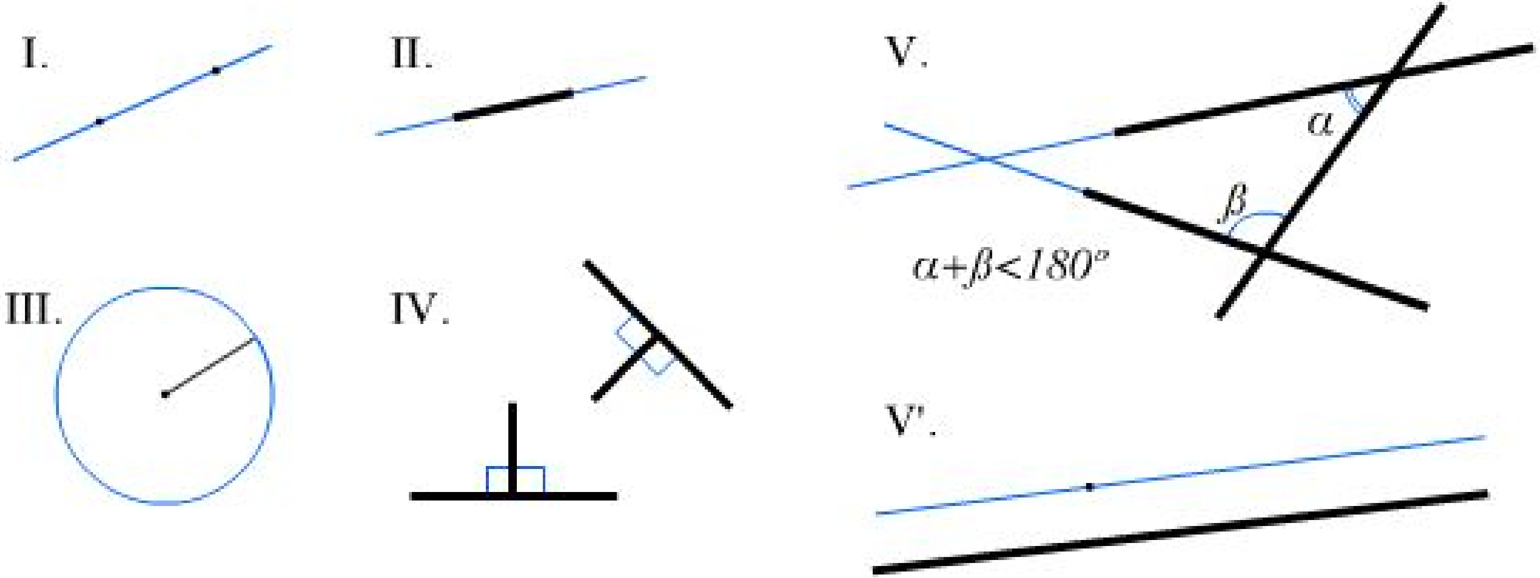
The **(non-obtuse) angle** θ (0° ≤ θ ≤ 180°) between **x** and **y** is then given by

$$\theta = \cos^{-1}\left(\frac{\mathbf{x} \cdot \mathbf{y}}{\|\mathbf{x}\| \|\mathbf{y}\|}\right)$$

The norm is used to define the Euclidean metric, a distance function, on $\mathbf{R}^n$ by

$$d(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} - \mathbf{y}\| = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}.$$

Euclid's 5 axioms that define his space



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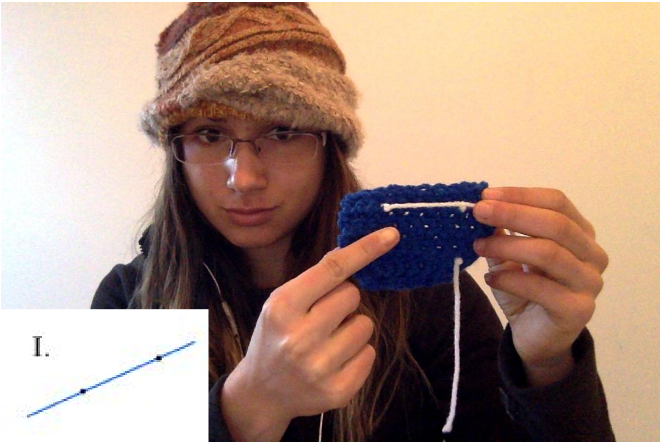
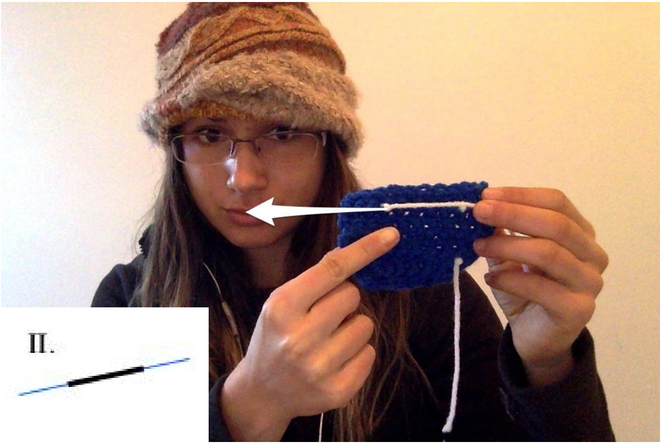
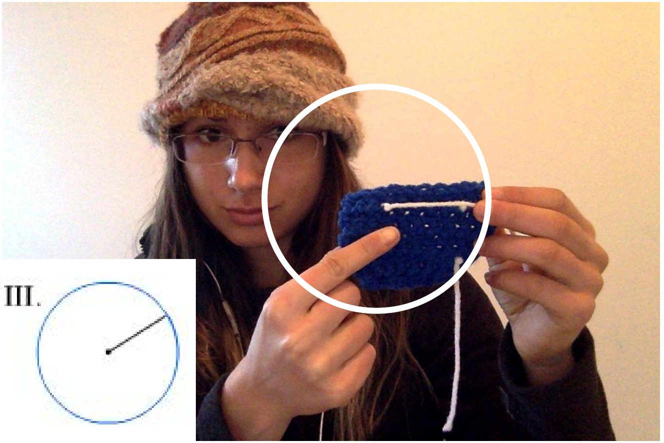
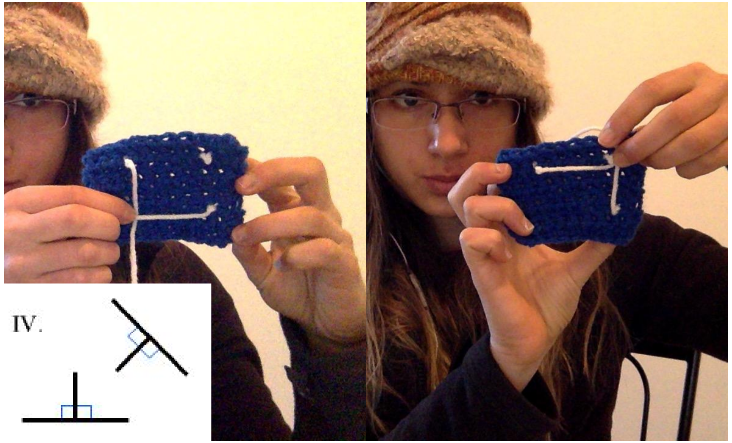
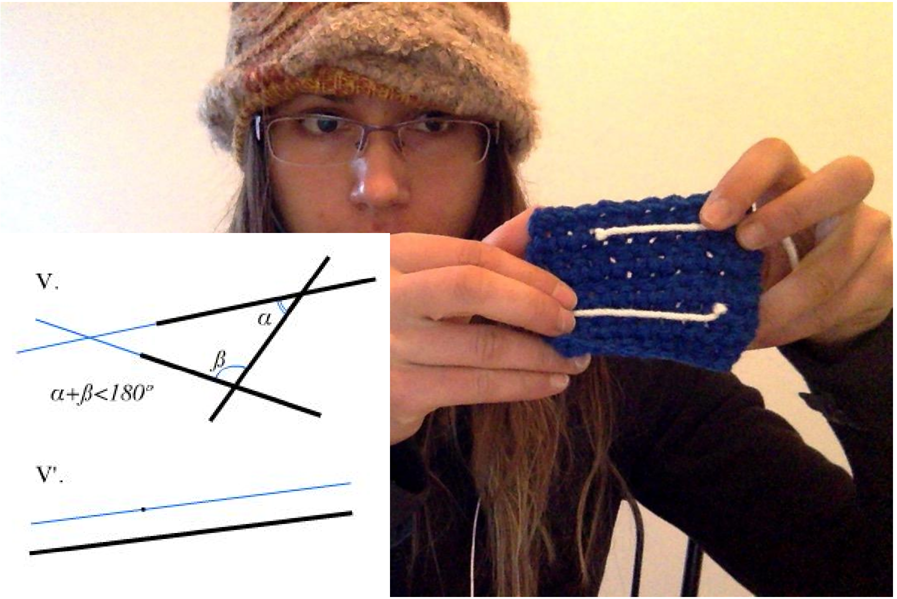








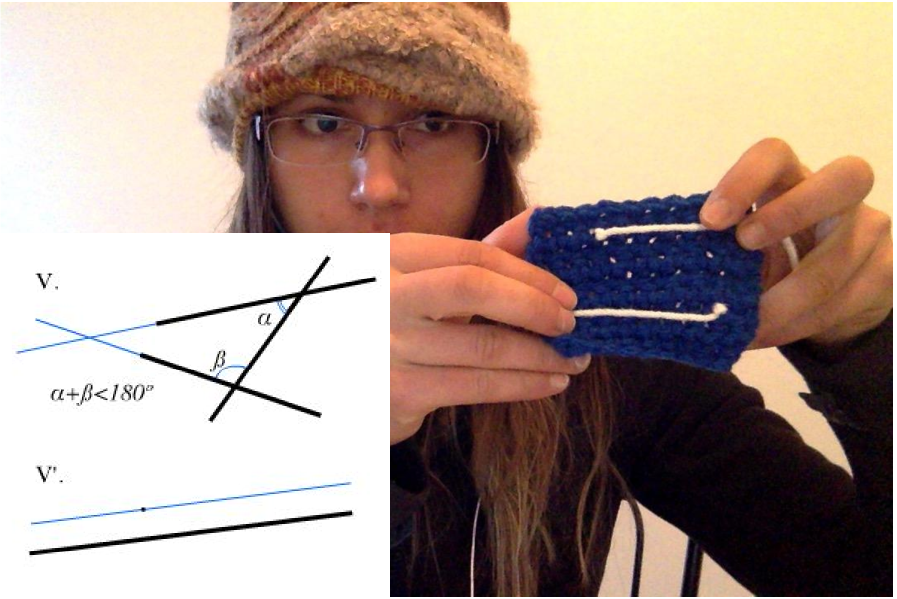
The fifth postulate



Aka: "Given a line A and a point P not on A, there is no more than <u>one</u> line through P that is parallel to A."

Non-euclidean spaces break the 5th postulate!

The fifth postulate



Aka: "Given a line A and a point P not on A, there is no more than <u>one</u> line through P that is parallel to A."

Non-euclidean spaces break the 5th postulate!

Reimann

Hyperbolic

Define our coordinate system on the hyperbolic plane

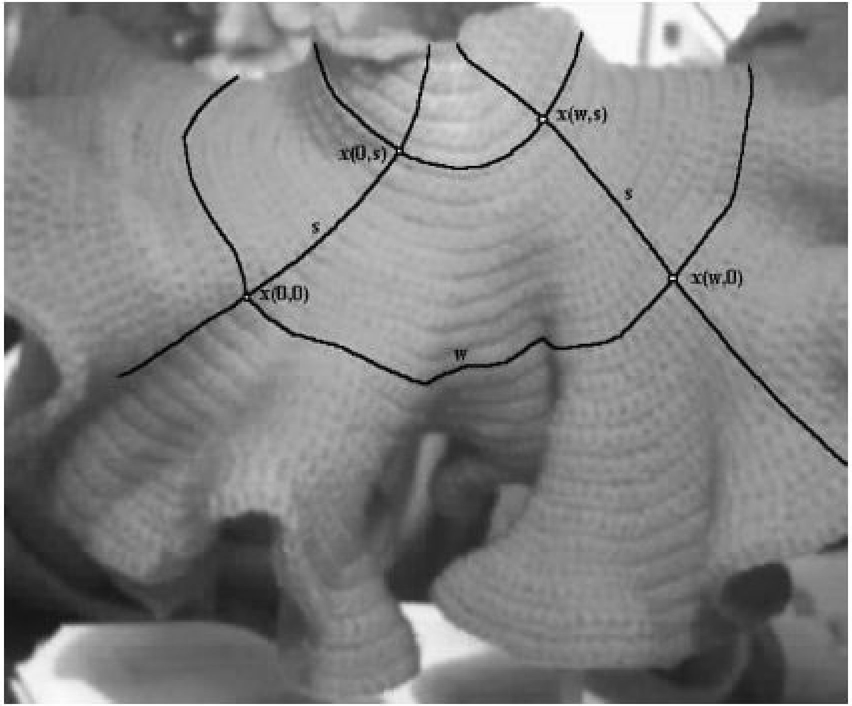
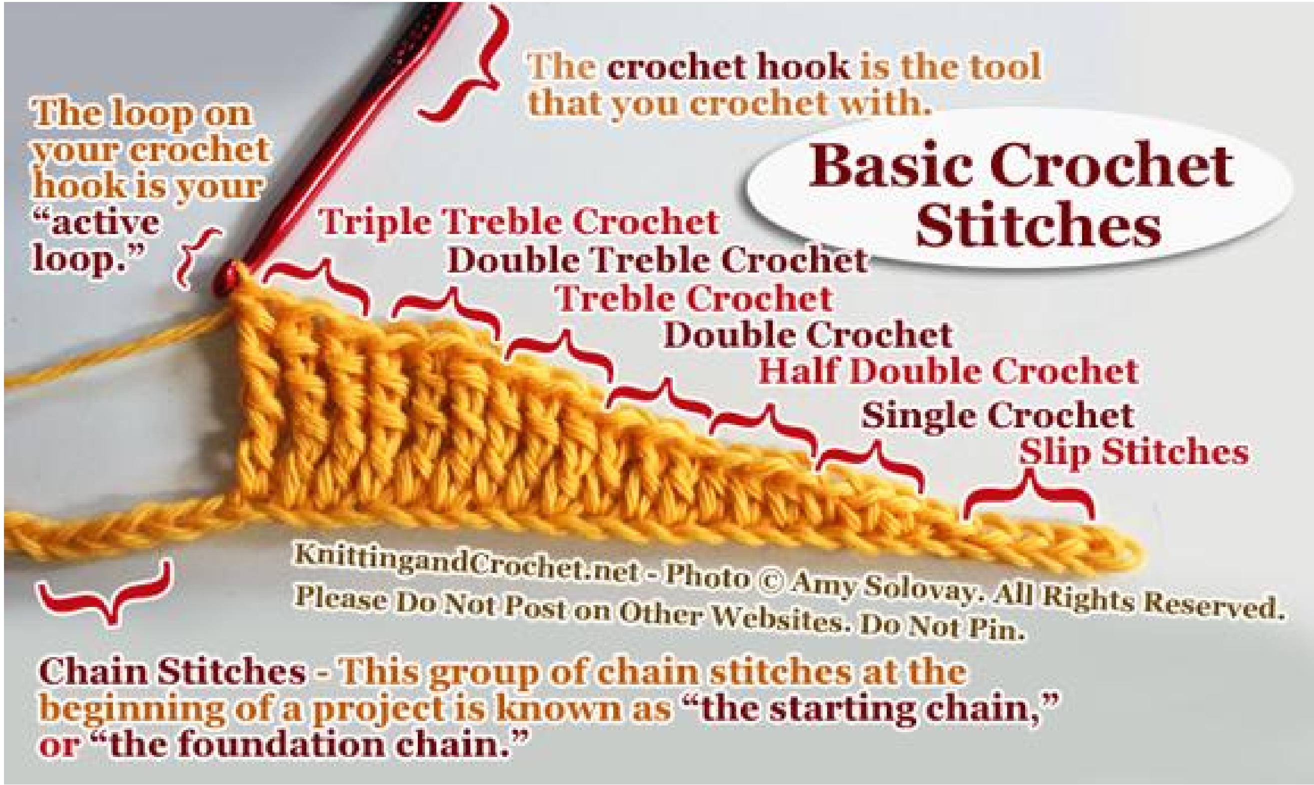


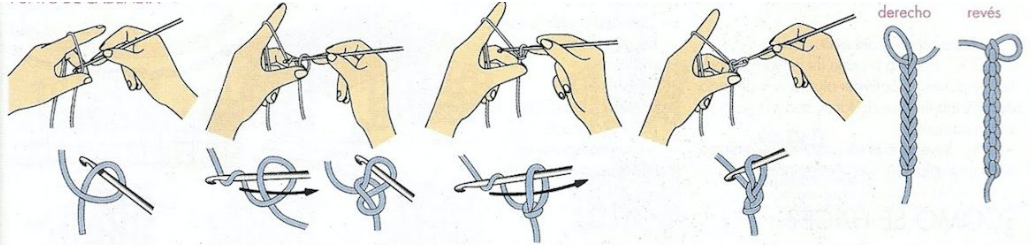
Figure 8. **Geodesic rectangular coordinates on annular hyperbolic plane.**

How do we crochet a hyperbolic surface?

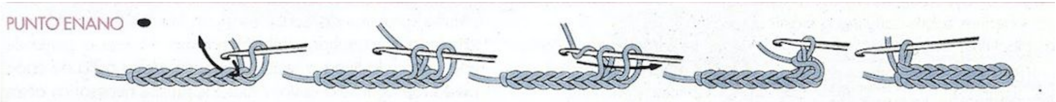
Quick intro to crochet lingo



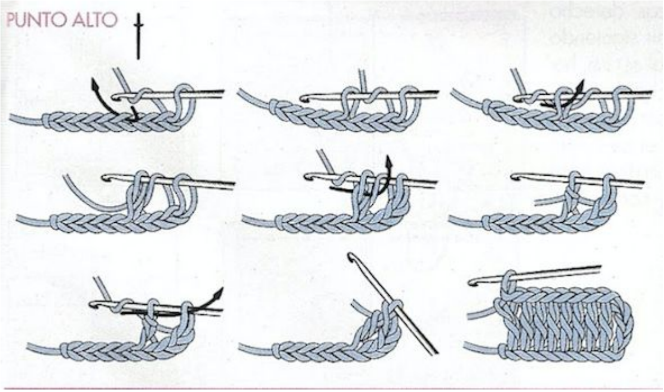
Chaining



Single Stitch



Double Stitch



Crocheting a hyperbolic plane

$$C = 2\pi R \cdot \sinh(r/R),$$

$$C = \pi R \cdot (e^{r/R} - e^{-r/R}),$$

$$C(n) = \pi R \cdot (e^{n \cdot h/R} - e^{-n \cdot h/R}),$$

Where:

- C = intrinsic circumference
- R = radius of hyperbolic plane
- r = intrinsic radius of circle

and,
r = n*h

Where:

- n = number of rows
- h = height of a stitch

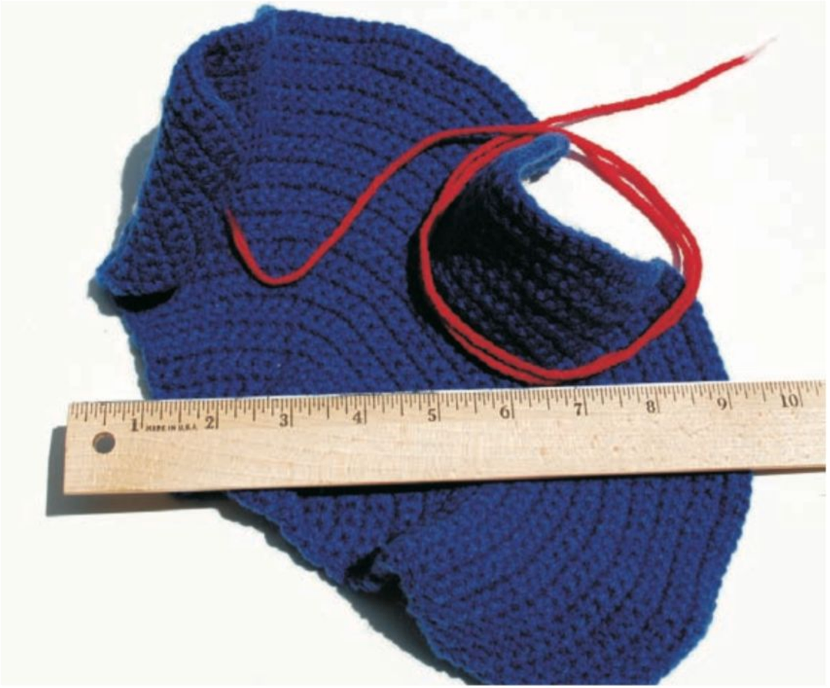
Determine when to increase the stitches/row using the ratio:

$$C(n)/C(n-1),$$

Where this ratio is of the form of a fraction: $(k+1)/k.$

n	$C(n)$	$C(n)/C(n-1)$	Nearby fraction	Increase ratio
1	3.5			
2	7.0	2.0	2/1	1 to 2
3	10.8	1.5	3/2	2 to 3
4	14.8	1.4		
5	19.1	1.3	4/3	3 to 4
6	23.9	1.25	5/4	4 to 5
7	29.4	1.23		
8	35.5	1.21		
9	42.6	1.20	6/5	5 to 6
10	50.6	1.19		
11	60.0	1.19		
12	70.8	1.18		
13	83.4	1.18		
14	98.0	1.18		
15	115.1	1.17	7/6	6 to 7
16	135.0	1.17		
17	158.2	1.17		
18	185.4	1.17		
19	217.2	1.17		

By Diana Taimina

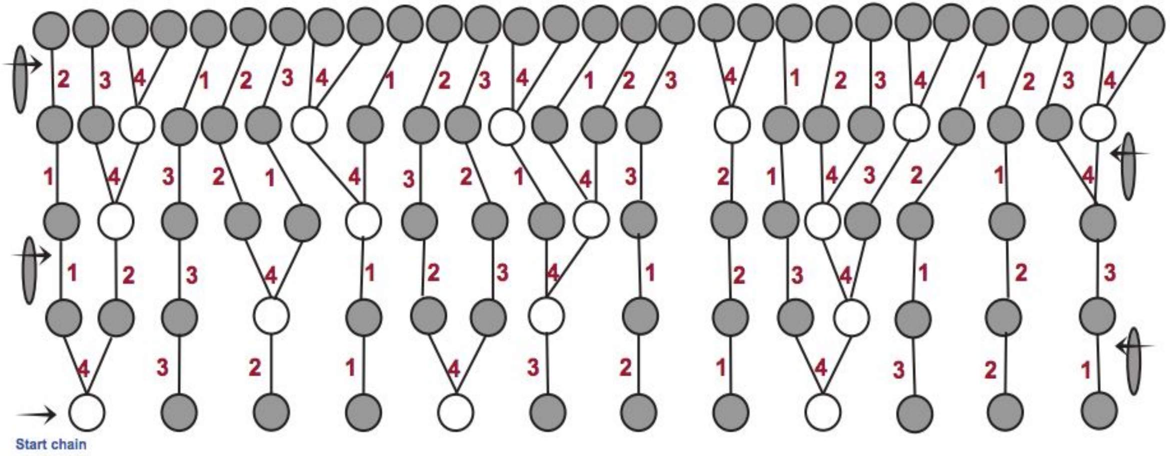


How to determine the radius of the hyperbolic plane:
here, the radius is about 2 inches.

My method

Double every Nth stitch.
Example: N=4

row		total stitches
5		29
4		23
3		18
2		15
1		12



More generally this is expressed as

r_n = a * ((1 + 1/N) ^ (n-1))

where,

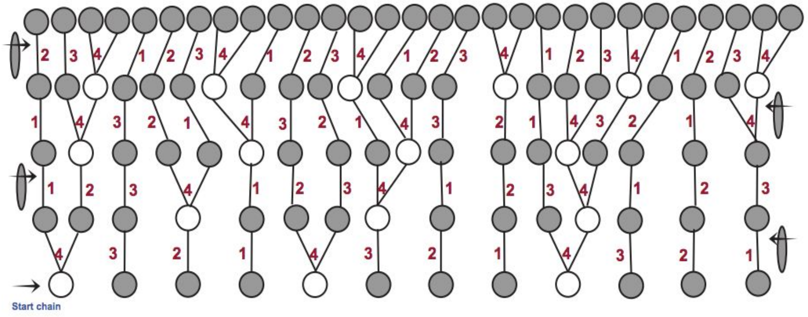
- n = row

- r_n = number of stitches in the nth row

- a = number of stitches in the annulus, or the first row

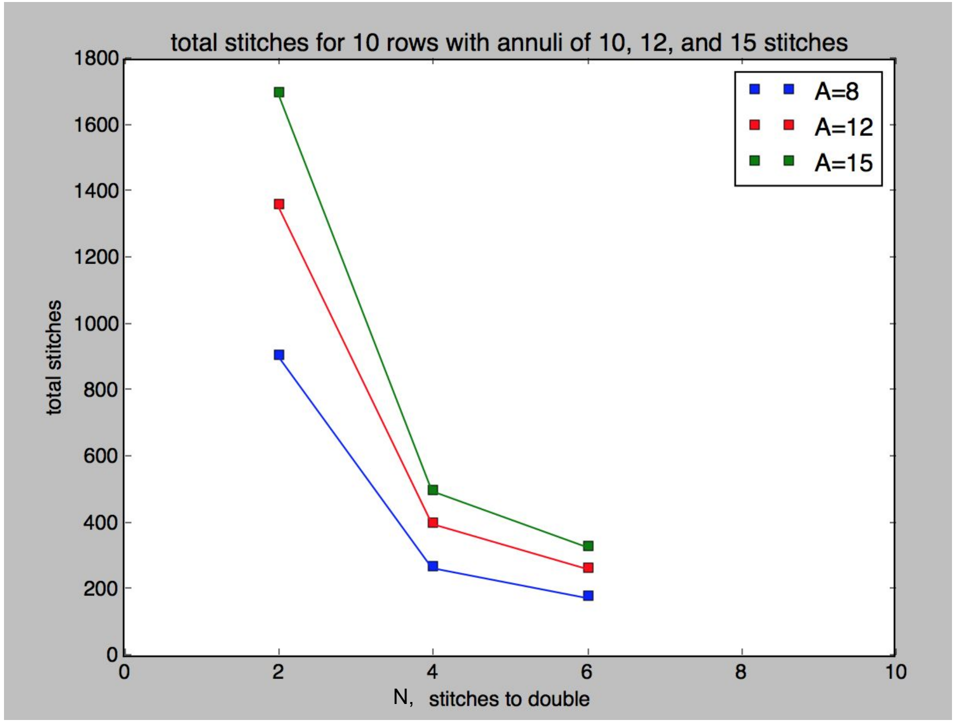
- N = where double stitch occurs. For example, if N=4, I count to the fourth stitch, add two single stitches on the fourth stitch, and repeat.

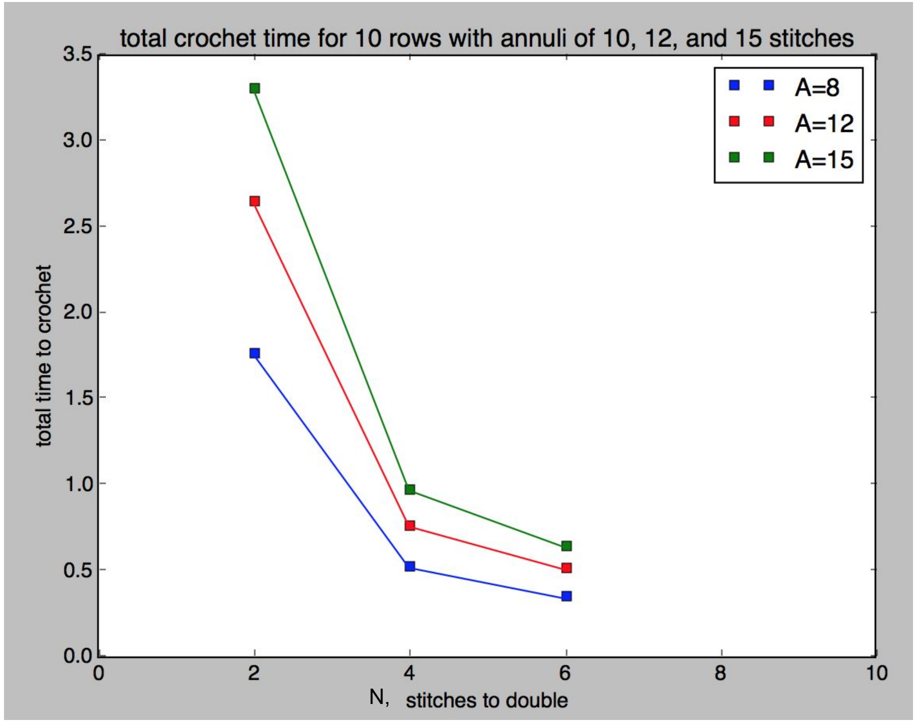
row label	# stitches/row counted from crochet diagram	# stitches/row predicted by formula r_n = 12 * ((1 + 1/4) ^ (n-1))	stitch difference
r_1	12	12	0
r_2	15	15	0
r_3	18	18	0
r_4	23	23	0
r_5	29	29	0



Fun tests!

Parameters
n = 10 (total rows)
a = 8, 12, or 15 (stitches in annulus)
N = 2, 4, 6 (stitches to double)





Fun tests

If:
- crochet rate = 1 stitch/7 sec
- n_final = **10** (n = row)

	a = 8 stitches	approximate time to crochet	a = 12 stitches	approximate time to crochet	a = 15 stitches	approximate time to crochet
N = 2	906	1.76 hours	1359	2.64 hours	1699	3.30 hours
N = 4	266	0.52 hours	399	0.76 hours	498	0.97 hours
N = 6	176	0.342 hours	264	0.51 hours	330	0.64 hours

If:
- crochet rate = 1 stitch/7 sec
- n_final = **100** (n = row)

	a = 8 stitches	time to crochet
N = 2	6.5049 * 10^18	1.44 * 10^12 years = 1.4 trillion years!
N = 4	2.6181 * 10^10	5811 years
N = 6	2.3764 * 10^8	52 years

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HYPERBOLIC CROCHET BASICS

Here we present a taxonomy of basic hyperbolic crochet models. To crochet a hyperbolic structure one simply increases stitches at a regular rate in every row. The more often you increase, the more quickly the model will ruffle up.

Model by Daina Taimina

Hyperbolic Plane: Step 1. To crochet a basic hyperbolic plane, begin with a line of chain stitches. (We recommend 15 or 20 stitches for your first try.) **Step 2.** After the line of chains, begin the first row by crocheting 5 stitches then increasing in the sixth stitch. (You may use single, half double, or double crochet as you choose.) Keep on repeating this pattern - crochet 5 stitches, increase 1; crochet 5 stitches, increase one - until the end of the row. **Step 3.** Turn around and repeat the pattern in the next row and all subsequent rows.

More Curly Hyperbolic Plane: In our first model the rate of increase is one in every 6 stitches. To make a more ruffled model, increase more rapidly. In our next model the rate of increase is one in every 4 stitches. **Step 1.** Begin with a line of chain stitches. **Step 2.** After the line of chains, begin the first row by crocheting 3 stitches then increasing in the fourth stitch. Keep repeating the pattern: Crochet 3 stitches, increase 1; crochet 3 stitches, increase one.

Crafters are encouraged to try out different rates of increase. Different types of yarn behave in different ways. To make a structurally rigid model like a coral, use synthetic yarn and a small hook. For a floppy kelp-like model, use soft wools and a larger hook.

Pseudosphere: In this model one does hyperbolic crochet round a circle. **Step 1.** Begin with a line of chains. **Step 2.** After a dozen stitches, you need to turn the line into a circle. To do this, crochet three stitches into the last chain and then join this group of stitches into a tiny cone. **Step 3.** Begin to crochet around the edge of the cone increasing at a regular rate. Here the rate of increase is one in every 3 stitches.

Another pseudosphere: Here the rate of increase is one in every 2 stitches, so the model ruffles up faster. If you increase at a regular rate you always achieve a mathematically perfect shape. This form is the hyperbolic equivalent of a cone - its tip extends to infinity.

Making a perfect pseudosphere is not necessary if you want to crochet corals. Instead, you may start with a circle of chains and do hyperbolic crochet round this loop. **Step 1.** Crochet 4 chains. **Step 2.** Join chains into a circle. **Step 3.** Begin to crochet around the loop increasing at a regular rate as you spiral out. Here we increase in every stitch.

Double Hyperbolic Plane: Here one does hyperbolic crochet around both *sides* of a line, working in a racetrack pattern. **Step 1.** Begin with a line of chains. **Step 2.** Crochet along one side of the chain increasing at a regular rate. (In this model we increase one in every 2 stitches.) **Step 3.** At the end of the row, increase 5 stitches in the last chain, then turn around and come back along the other side, continuing to increase at the same rate. **Step 4.** Continue hyperbolic crochet around the racetrack in all following rows. This kelp-like form is two hyperbolic planes joined together.

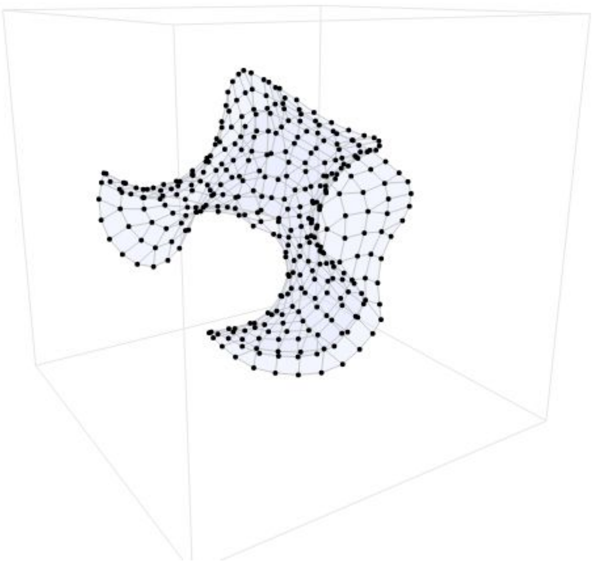
Seed-Pod Model: Step 1. To get this pretty form begin with a line of 15 chains. **Step 2.** In the first row crochet around the chain on both sides, increasing in every stitch. **Step 3.** In the second row, increase in two out of three stitches. **Step 4.** In the third row increase in every second stitch. **Step 5.** In the fourth row, increase in every third stitch. And so on. There are seed-pods with this structure.

Model by Spring Pace

Living organisms are always irregular. In order to achieve natural looking corals, you need to vary the rate of increase within the model. We encourage crafters to experiment for themselves.

All models by the Institute For Figuring, unless stated.

But how do you determine the 3D conformation of a 2D hyperbolic plane?



Crochet simulator

(https://github.com/timhutton/crochet-simulator)

For this I used the formula from Daina Taimina's excellent book: each row has n stitches, where:

```
n = 2*pi*R*sinh(h*i/R)/w
```

where R=2.0cm is the radius of curvature (controls the frilliness), h=0.55cm is the stitch height, w=0.55cm is the stitch width, i is the row number (1,2,3,...) and sinh() is the hyperbolic sine function.

References

- “Euclidean space” http://www.cs.mcgill.ca/~rwest/link-suggestion/wpcd_2008-09_augmented/wp/e/Euclidean_space.htm
- “How to crochet hyperbolic corals” http://www.stewardschool.org/file/bil/IFF-CrochetReef-HowToHandout-1-2.pdf
- “Crocheting Adventures with Hyperbolic Planes” Diana Taimina