

Parameters

$$a > 0 \quad (1)$$

$$b > 0 \quad (2)$$

$$K \geq 0 \quad (3)$$

$$D_0 \geq 0 \quad (4)$$

$$\tau \geq 0 \quad (5)$$

Potential

$$V(x) = \frac{a}{4}x^4 - \frac{b}{2}x^2 + \frac{b^2}{4a} = \frac{a}{4}x^2 \left(x - \sqrt{\frac{2b}{a}} \right) \left(x + \sqrt{\frac{2b}{a}} \right) + \frac{b^2}{4a} \quad (6)$$

Force

$$F(x) = -\frac{\partial V}{\partial x} = -ax^3 + bx \quad (7)$$

We have 2 Hypervolumes

$$\mathcal{V}_t \subset \mathbb{R}^{n_t} \quad (8)$$

$$\mathcal{V}_\tau \subset \mathbb{R}^{n_\tau} \quad (9)$$

$$\mathcal{V} = \mathcal{V}_t \times \mathcal{V}_\tau \quad (10)$$

$$n_t \geq n_\tau \quad (11)$$

$$\dot{X}_t = F(X_t, X_\tau) dt + \sqrt{2D_0} dB_t \quad (12)$$

$$= -aX_t^3 dt + bX_t dt + K(X_t - X_\tau) dt + \sqrt{2D_0} dB_t \quad (13)$$

$$F(x, x_\tau) = F_1(x) + F_2(x_\tau) = -ax^3 + (b + K)x - Kx_\tau \quad (14)$$

Fokker-Planck Equation

From the Langevin equation

$$\dot{X} = -aX^3(t) + bX(t) + K(X(t) - X(t - \tau)) + \sqrt{2D_0}\xi(t) \quad (15)$$

With the uncorrelated Gaussian noise ξ

$$\langle \xi(t) \rangle = 0 \quad (16)$$

$$\langle \xi(t)\xi(t') \rangle = \delta(t - t') \quad (17)$$

The infinite hierarchy

$$\partial_t \rho_1(x, t) = -\partial_x \int_{\mathcal{V}_\tau} dx_\tau F(x, x_\tau) \rho_2(x, t; x_\tau, t - \tau) + D_0 \partial_x^2 \rho_1(x, t) \quad (18)$$

$$\partial_t \rho_2(x, t; x_\tau, t - \tau) = -\partial_{x_\tau} \int_{\mathcal{V}_\tau} dx_{2\tau} F(x_\tau, x_{2\tau}) \rho_3(x, t; x_\tau, t - \tau; x_{2\tau}, t - 2\tau) + \dots \quad (19)$$

$$\vdots \quad (20)$$

Conditional probability densities

$$\rho_1(x, t) = \langle \delta[X(t) - x] \rangle \quad (21)$$

$$\rho_2(x, t; x_\tau, t - \tau) = \langle \delta[X(t) - x] \delta[X(t - \tau) - x_\tau] \rangle \quad (22)$$

Novikov's theorem

$$\partial_t \rho_1(x, t) = -\partial_x \int_{\mathcal{V}_\tau} dx_\tau F(x, x_\tau) \rho_2(x, t; x_\tau, t - \tau) + D_0 \partial_x^2 \rho_1(x, t) \quad (23)$$

$$\partial_t \rho_2(x, t; x_\tau, t - \tau) = -\partial_x F(x, x_\tau) \rho_2(x, t; x_\tau, t - \tau) - \partial_{x_\tau} \int_{\mathcal{V}_\tau} dx_{2\tau} F(x_\tau, x_{2\tau}) \rho_3(x, t; x_\tau, t - \tau; x_{2\tau}, t - 2\tau) \quad (24)$$

$$+ D_0 \left(\partial_x^2 + \partial_{x_\tau}^2 + \partial_x \partial_{x_\tau} \left(\frac{\delta X(t)}{\delta \xi(t - \tau)} \Big|_{X(t)=x}^{X(t-\tau)=x_\tau} \right) \right) \rho_2(x, t; x_\tau, t - \tau) \quad (25)$$

Crazy cross diffusion term for a test functional Λ of the noise process

$$\langle \Lambda[\xi] \xi(t) \rangle = \left\langle \frac{\delta \Lambda}{\delta \xi(t)} \right\rangle \quad (26)$$

Auxiliary variables

$$j = 1, 2, \dots, N \quad (27)$$

$$X(t) \rightarrow (X_0, X_1, X_2, \dots, X_N), \quad \mathbb{R} \rightarrow \mathbb{R}^{N+1} \quad (28)$$

Slice the interval

$$\Delta t = \frac{\tau}{N}, \quad X_j(t) = X(t - j\Delta t) \quad (29)$$

$$\Rightarrow X_0(t) = X(t), \quad X_N(t) = X(t - \tau) \quad (30)$$

Dynamics

$$\dot{X}_j = \lim_{\Delta t \rightarrow 0} \frac{X_j(t + \Delta t) - X_j(t)}{\Delta t} \approx \frac{N}{\tau} (X_{j-1}(t) - X_j(t)) \quad (31)$$

Langevin equation and Markovian system

$$\dot{X}_0 = F(X_0(t), X_N(t)) + \sqrt{2D_0} \xi(t) \quad (32)$$

$$\dot{X}_1 = \frac{N}{\tau} (X_0(t) - X_1(t)) \quad (33)$$

$$\vdots \quad (34)$$

$$\dot{X}_N = \frac{N}{\tau} (X_{N-1}(t) - X_N(t)) \quad (35)$$

Projection onto dynamics of X_0

$$X_j(t) \Big|_{j>0} = \int_{-\infty}^t dt' \frac{N}{\tau} \exp\left(-\frac{N}{\tau}(t-t')\right) X_{j-1}(t) \quad (36)$$

Iterate

$$X_{j+1}(t) = \int_{-\infty}^t dt' \left(\frac{N}{\tau}\right)^2 (t-t') \exp\left(-\frac{N}{\tau}(t-t')\right) X_{j-1}(t) \quad (37)$$

$$X_{j+2}(t) = \int_{-\infty}^t dt' \frac{1}{2} \left(\frac{N}{\tau}\right)^3 (t-t')^2 \exp\left(-\frac{N}{\tau}(t-t')\right) X_{j-1}(t) \quad (38)$$

$$X_{j+N-1}(t) = \int_{-\infty}^t dt' \frac{1}{(N-1)!} \left(\frac{N}{\tau}\right)^N (t-t')^{N-1} \exp\left(-\frac{N}{\tau}(t-t')\right) X_{j-1}(t) \quad (39)$$

For $j = 1$

$$X_N(t) = \int_{-\infty}^t dt' \frac{1}{(N-1)!} \left(\frac{N}{\tau}\right)^N (t-t')^{N-1} \exp\left(-\frac{N}{\tau}(t-t')\right) X_0(t) \quad (40)$$

Keep N not only countable, but finite. Let $j \in \{1, 2, \dots, N\}$

$$\dot{X}_0 = F(X_0(t), X_N(t)) + \sqrt{2D_0} \xi(t) \quad (41)$$

$$\dot{X}_j = \frac{N}{\tau} (X_{j-1}(t) - X_j(t)) \quad (42)$$

Fokker-Planck equation for the Markovian system

$$\rho_{N+1}(x_0, \dots, x_N; t) = \langle \delta(x_0 - X_0(t)) \dots \delta(x_N - X_N(t)) \rangle \quad (43)$$

$$\partial_t \rho_{N+1}(x_0, \dots, x_N; t) = \left(- \sum_{j=0}^N \partial_{x_j} f_j(x_j, x_{j-1}) + D_0 \partial_{x_0}^2 \right) \rho_{N+1}(x_0, \dots, x_N; t) \quad (44)$$

$$f_j(x_j, x_{j-1}) = \begin{cases} F(x_0, x_N), & j = 0 \\ \frac{N}{\tau} (x_{j-1} - x_j), & j > 0 \end{cases} \quad (45)$$

Natural boundary condition

$$\lim_{x_j \rightarrow \pm\infty} \rho_{N+1}(x_0, \dots, x_N; t) = 0 \quad (46)$$

Integrating the Fokker-Planck equation over all auxiliary variables

$$\int_{\mathbb{R}} dx_1 \cdots \int_{\mathbb{R}} dx_N \partial_t \rho_{N+1}(x_0, \dots, x_N; t) = \int_{\mathbb{R}} dx_1 \cdots \int_{\mathbb{R}} dx_N \left(- \sum_{j=0}^N \partial_{x_j} f_j(x_j, x_{j-1}) + D_0 \partial_{x_0}^2 \right) \rho_{N+1}(x_0, \dots, x_N; t) \quad (47)$$

$$\int_{\mathbb{R}} dx_1 \cdots \int_{\mathbb{R}} dx_N \rho_{N+1}(x_0, \dots, x_N; t) = \rho_1(x_0; t) \quad (48)$$

$$\int_{\mathbb{R}} dx_1 \cdots \int_{\mathbb{R}} dx_N \sum_{j=0}^N \partial_{x_j} f_j(x_j, x_{j-1}) \rho_{N+1}(x_0, \dots, x_N; t) = \partial_{x_0} \int_{\mathbb{R}} dx_N F(x_0, x_N) \rho_2(x_0, x_N; t) \quad (49)$$

$$\partial_t \rho_1(x_0, t) = -\partial_{x_0} \int_{\mathbb{R}} dx_N F(x_0, x_N) \rho_2(x_0, x_N; t) + D_0 \partial_{x_0}^2 \rho_1(x_0, t) \quad (50)$$

We take the limit $N \rightarrow \infty$ again

$$X_0(t) = X(t), \quad X_N(t) = X(t - \tau) \quad (51)$$

This leads to a coordinate transformation

$$\{x_0, t\} \rightarrow \{x, t\}, \quad \{x_N, t\} \rightarrow \{x_\tau, t - \tau\} \quad (52)$$

And therefore to the first Fokker-Planck equation

$$\partial_t \rho_1(x, t) = -\partial_x \int_{\mathbb{R}} dx_\tau F(x_0, x_\tau) \rho_2(x, t; x_\tau, t - \tau) + D_0 \partial_x^2 \rho_1(x, t) \quad (53)$$

Integrating the Fokker-Planck equation over all auxiliary variables but not x_N

$$\int_{\mathbb{R}} dx_1 \cdots \int_{\mathbb{R}} dx_{N-1} \partial_t \rho_{N+1}(x_0, \dots, x_N; t) = \int_{\mathbb{R}} dx_1 \cdots \int_{\mathbb{R}} dx_{N-1} \left(- \sum_{j=0}^N \partial_{x_j} f_j(x_j, x_{j-1}) + D_0 \partial_{x_0}^2 \right) \rho_{N+1}(x_0, \dots, x_N; t) \quad (54)$$

$$\int_{\mathbb{R}} dx_1 \cdots \int_{\mathbb{R}} dx_{N-1} \rho_{N+1}(x_0, \dots, x_N; t) = \rho_2(x_0, x_N; t) \quad (55)$$

$$\int_{\mathbb{R}} dx_1 \cdots \int_{\mathbb{R}} dx_{N-1} \sum_{j=0}^N \partial_{x_j} f_j(x_j, x_{j-1}) \rho_{N+1}(x_0, \dots, x_N; t) = \partial_{x_0} F(x_0, x_N) \rho_2(x_0, x_N; t) - \frac{N}{\tau} \partial_{x_N} \int_{\mathbb{R}} dx_{N-1} (x_N - x_{N-1}) \rho_3(x_0, x_{N-1}, x_N; t) \quad (56)$$

$$\partial_t \rho_2(x_0, x_N, t) = -\partial_{x_0} F(x_0, x_N) \rho_2(x_0, x_N; t) + \frac{N}{\tau} \partial_{x_N} \int_{\mathbb{R}} dx_{N-1} (x_N - x_{N-1}) \rho_3(x_0, x_{N-1}, x_N; t) + D_0 \partial_{x_0}^2 \rho_2(x_0, x_N, t) \quad (57)$$

Taking again the limit $N \rightarrow \infty$ and interpret the auxiliary variables under the coordinate transformation

$$\left\{ \frac{N}{\tau} (x_{N-1} - x_N), t \right\} \rightarrow \{\dot{x}, t - \tau\} \quad (58)$$

This forms an alternative hierarchy

$$\partial_t \rho_2(x, t; x_\tau, t - \tau) = -\partial_x F(x, x_\tau) \rho_2(x, t; x_\tau, t - \tau) + \partial_{x_\tau} \int_{\mathbb{R}} d\dot{x}_\tau \dot{x}_\tau \rho_3(x, t; \dot{x}_\tau, t - \tau; x_\tau, t - \tau) + D_0 \partial_x^2 \rho_2(x, t; x_\tau, t - \tau) \quad (59)$$

Closure via mean field approximation

$$\rho_3(x, t; \dot{x}_\tau, t - \tau; x_\tau, t - \tau) \sim \rho_2(x, t; x_\tau, t - \tau) \rho_1(\dot{x}_\tau, t - \tau) \quad (60)$$

We arrive at a system of 2 Fokker-Planck equations, that are delay partial integro-differential equations.

$$\partial_t \rho_1(x, t) = -\partial_x \int_{\mathbb{R}} dx_\tau F(x_0, x_\tau) \rho_2(x, t; x_\tau, t - \tau) + D_0 \partial_x^2 \rho_1(x, t) \quad (61)$$

$$\partial_t \rho_2(x, t; x_\tau, t - \tau) = -\partial_x F(x, x_\tau) \rho_2(x, t; x_\tau, t - \tau) + \partial_{x_\tau} \int_{\mathbb{R}} d\dot{x}_\tau \dot{x}_\tau \rho_2(x, t; x_\tau, t - \tau) \rho_1(\dot{x}_\tau, t - \tau) + D_0 \partial_x^2 \rho_2(x, t; x_\tau, t - \tau) \quad (62)$$

$$\frac{\partial \rho_1}{\partial t} = -\frac{\partial F_1}{\partial x} \rho_1(x, t) - F_1(x) \frac{\partial \rho_1}{\partial x} - \frac{\partial}{\partial x} \int_{\mathcal{V}_\tau} dx_\tau F_2(x_\tau) \rho_2(x, t; x_\tau, t - \tau) + D_0 \frac{\partial^2 \rho_1}{\partial x^2} \quad (63)$$

$$\frac{\partial \rho_2}{\partial t} = -\frac{\partial F_1}{\partial x} \rho_2(x, t; x_\tau, t - \tau) - F_1(x) \frac{\partial \rho_2}{\partial x} - F_2(x_\tau) \frac{\partial \rho_2}{\partial x} + \frac{\partial}{\partial x_\tau} \int_{\mathcal{V}_\tau} d\dot{x}_\tau \dot{x}_\tau \rho_2(x, t; x_\tau, t - \tau) \rho_1(\dot{x}_\tau, t - \tau) + D_0 \frac{\partial^2 \rho_1}{\partial x^2} \quad (64)$$

$$\frac{\partial \rho_1}{\partial t} = (3ax^2 - b - K) \rho_1(x, t) + (ax^3 - xb - xK) \frac{\partial \rho_1}{\partial x} + K \frac{\partial}{\partial x} \int_{\mathcal{V}_\tau} dx_\tau x_\tau \rho_2(x, t; x_\tau, t - \tau) + D_0 \frac{\partial^2 \rho_1}{\partial x^2} \quad (65)$$

$$\frac{\partial \rho_2}{\partial t} = (3ax^2 - b - K) \rho_2(x, t; x_\tau, t - \tau) + (ax^3 - xb - xK) \frac{\partial \rho_2}{\partial x} + K x_\tau \frac{\partial \rho_2}{\partial x} + \frac{\partial \rho_2}{\partial x_\tau} \left(\int_{\mathcal{V}_\tau} d\dot{x}_\tau \dot{x}_\tau \rho_1(\dot{x}_\tau, t - \tau) \right) + D_0 \frac{\partial^2 \rho_2}{\partial x^2} \quad (66)$$