Parameters

$$a > 0 \tag{1}$$
$$b > 0 \tag{2}$$
$$K \geq 0 \tag{3}$$
$$D_0 \geq 0 \tag{4}$$
$$\tau \geq 0 \tag{5}$$

Potential

$$V(x) = \frac{a}{4}x^4 - \frac{b}{2}x^2 + \frac{b^2}{4a} = \frac{a}{4}x^2\left(x - \sqrt{\frac{2b}{a}}\right)\left(x + \sqrt{\frac{2b}{a}}\right) + \frac{b^2}{4a} \tag{6}$$

Force

$$F(x) = -\frac{\partial V}{\partial x} = -ax^3 + bx \tag{7}$$

We have 2 Hypervolumes

$$\mathcal{V}_t \subset \mathbb{R}^{n_t} \tag{8}$$
$$\mathcal{V}_\tau \subset \mathbb{R}^{n_\tau} \tag{9}$$
$$\mathcal{V} = \mathcal{V}_t \times \mathcal{V}_\tau \tag{10}$$
$$n_t \geq n_\tau \tag{11}$$

$$\begin{aligned}\dot{X}_t &= F(X_t, X_\tau)\,\mathrm{d}t + \sqrt{2D_0}\,\mathrm{d}B_t &(12)\\ &= -aX_t^3\,\mathrm{d}t + bX_t\,\mathrm{d}t + K(X_t - X_\tau)\,\mathrm{d}t + \sqrt{2D_0}\,\mathrm{d}B_t &(13)\end{aligned}$$

$$F(x, x_\tau) = F_1(x) + F_2(x_\tau) = -ax^3 + (b+K)x - Kx_\tau \tag{14}$$

Fokker-Planck Equation

From the Langevin equation

$$\dot{X} = -aX^3(t) + bX(t) + K(X(t) - X(t-\tau)) + \sqrt{2D_0}\xi(t) \tag{15}$$

With the uncorrelated Gaussian noise ξ

$$\langle \xi(t) \rangle = 0 \tag{16}$$
$$\langle \xi(t)\xi(t') \rangle = \delta(t - t') \tag{17}$$

The infinite hierarchy

$$\partial_t \rho_1(x,t) = -\partial_x \int\limits_{\mathcal{V}_\tau} \mathrm{d}x_\tau\, F(x, x_\tau)\rho_2(x,t;x_\tau,t-\tau) + D_0\partial_x^2\rho_1(x,t) \tag{18}$$

$$\partial_t \rho_2(x,t;x_\tau,t-\tau) = -\partial_{x_\tau} \int\limits_{\mathcal{V}_\tau} \mathrm{d}x_{2\tau}\, F(x_\tau, x_{2\tau})\rho_3(x,t;x_\tau,t-\tau;x_{2\tau},t-2\tau) + \dots \tag{19}$$

$$\vdots \tag{20}$$

Conditional probability densities

$$\rho_1(x,t) = \langle \delta[X(t) - x] \rangle \tag{21}$$
$$\rho_2(x,t;x_\tau,t-\tau) = \langle \delta[X(t) - x]\delta[X(t-\tau) - x_\tau] \rangle \tag{22}$$

Novikov's theorem

$$\partial_t \rho_1(x,t) = -\partial_x \int\limits_{\mathcal{V}_\tau} \mathrm{d}x_\tau\, F(x, x_\tau)\rho_2(x,t;x_\tau,t-\tau) + D_0\partial_x^2\rho_1(x,t) \tag{23}$$

$$\begin{aligned}\partial_t \rho_2(x,t;x_\tau,t-\tau) &= -\partial_x F(x,x_\tau)\rho_2(x,t;x_\tau,t-\tau) - \partial_{x_\tau}\int\limits_{\mathcal{V}_\tau} \mathrm{d}x_{2\tau}\, F(x_\tau, x_{2\tau})\rho_3(x,t;x_\tau,t-\tau;x_{2\tau},t-2\tau) &(24)\\ &\quad + D_0\left(\partial_x^2 + \partial_{x_\tau}^2 + \partial_x\partial_{x_\tau}\left(\frac{\delta X(t)}{\delta\xi(t-\tau)}\bigg|_{X(t)=x}^{X(t-\tau)=x_\tau}\right)\right)\rho_2(x,t;x_\tau,t-\tau) &(25)\end{aligned}$$

Crazy cross diffusion term for a test functional Λ of the noise process

$$\langle \Lambda[\xi]\xi(t)\rangle = \left\langle \frac{\delta\Lambda}{\delta\xi(t)} \right\rangle \tag{26}$$

Auxiliary variables

$$j = 1, 2, \dots, N \tag{27}$$
$$X(t) \to (X_0, X_1, X_2, \dots, X_N), \quad \mathbb{R} \to \mathbb{R}^{N+1} \tag{28}$$

Slice the interval

$$\Delta t = \frac{\tau}{N}, \quad X_j(t) = X(t - j\Delta t) \tag{29}$$
$$\Rightarrow X_0(t) = X(t), \quad X_N(t) = X(t-\tau) \tag{30}$$

Dynamics

$$\dot{X}_j = \lim_{\Delta t \to 0} \frac{X_j(t+\Delta t) - X_j(t)}{\Delta t} \approx \frac{N}{\tau}(X_{j-1}(t) - X_j(t)) \tag{31}$$

Langevin equation and Markovian system

$$\dot{X}_0 = F(X_0(t), X_N(t)) + \sqrt{2D_0}\xi(t) \tag{32}$$
$$\dot{X}_1 = \frac{N}{\tau}(X_0(t) - X_1(t)) \tag{33}$$
$$\vdots \tag{34}$$
$$\dot{X}_N = \frac{N}{\tau}(X_{N-1}(t) - X_N(t)) \tag{35}$$

Projection onto dynamics of X_0

$$X_j(t)\Big|_{j>0} = \int_{-\infty}^{t} \mathrm{d}t' \, \frac{N}{\tau} \exp\left(-\frac{N}{\tau}(t-t')\right) X_{j-1}(t) \tag{36}$$

Iterate

$$X_{j+1}(t) = \int_{-\infty}^{t} \mathrm{d}t' \left(\frac{N}{\tau}\right)^2 (t-t') \exp\left(-\frac{N}{\tau}(t-t')\right) X_{j-1}(t) \tag{37}$$
$$X_{j+2}(t) = \int_{-\infty}^{t} \mathrm{d}t' \, \frac{1}{2}\left(\frac{N}{\tau}\right)^3 (t-t')^2 \exp\left(-\frac{N}{\tau}(t-t')\right) X_{j-1}(t) \tag{38}$$
$$X_{j+N-1}(t) = \int_{-\infty}^{t} \mathrm{d}t' \, \frac{1}{(N-1)!}\left(\frac{N}{\tau}\right)^N (t-t')^{N-1} \exp\left(-\frac{N}{\tau}(t-t')\right) X_{j-1}(t) \tag{39}$$

For $j = 1$

$$X_N(t) = \int_{-\infty}^{t} \mathrm{d}t' \, \frac{1}{(N-1)!}\left(\frac{N}{\tau}\right)^N (t-t')^{N-1} \exp\left(-\frac{N}{\tau}(t-t')\right) X_0(t) \tag{40}$$

Keep N not only countable, but finite. Let $j \in \{1, 2, \dots, N\}$

$$\dot{X}_0 = F(X_0(t), X_N(t)) + \sqrt{2D_0}\xi(t) \tag{41}$$
$$\dot{X}_j = \frac{N}{\tau}(X_{j-1}(t) - X_j(t)) \tag{42}$$

Fokker-Planck equation for the Markovian system

$$\rho_{N+1}(x_0, \dots, x_N; t) = \langle \delta(x_0 - X_0(t)) \dots \delta(x_N - X_N(t)) \rangle \tag{43}$$

$$\partial_t \rho_{N+1}(x_0, \dots, x_N; t) = \left(-\sum_{j=0}^{N} \partial_{x_j} f_j(x_j, x_{j-1}) + D_0 \partial_{x_0}^2\right) \rho_{N+1}(x_0, \dots, x_N; t) \tag{44}$$
$$f_j(x_j, x_{j-1}) = \begin{cases} F(x_0, x_N), & j = 0 \\ \frac{N}{\tau}(x_{j-1} - x_j), & j > 0 \end{cases} \tag{45}$$

Natural boundary condition

$$\lim_{x_j \to \pm\infty} \rho_{N+1}(x_0, \dots, x_N; t) = 0 \tag{46}$$

Integrating the Fokker-Planck equation over all auxiliary variables

$$\int_{\mathbb{R}} \mathrm{d}x_1 \cdots \int_{\mathbb{R}} \mathrm{d}x_N \, \partial_t \rho_{N+1}(x_0, \dots, x_N; t) = \int_{\mathbb{R}} \mathrm{d}x_1 \cdots \int_{\mathbb{R}} \mathrm{d}x_N \left(-\sum_{j=0}^{N} \partial_{x_j} f_j(x_j, x_{j-1}) + D_0 \partial_{x_0}^2 \right) \rho_{N+1}(x_0, \dots, x_N; t) \tag{47}$$

$$\int_{\mathbb{R}} \mathrm{d}x_1 \cdots \int_{\mathbb{R}} \mathrm{d}x_N \, \rho_{N+1}(x_0, \dots, x_N; t) = \rho_1(x_0; t) \tag{48}$$

$$\int_{\mathbb{R}} \mathrm{d}x_1 \cdots \int_{\mathbb{R}} \mathrm{d}x_N \sum_{j=0}^{N} \partial_{x_j} f_j(x_j, x_{j-1}) \rho_{N+1}(x_0, \dots, x_N; t) = \partial_{x_0} \int_{\mathbb{R}} \mathrm{d}x_N \, F(x_0, x_N) \rho_2(x_0, x_N; t) \tag{49}$$

$$\partial_t \rho_1(x_0, t) = -\partial_{x_0} \int_{\mathbb{R}} \mathrm{d}x_N \, F(x_0, x_N) \rho_2(x_0, x_N; t) + D_0 \partial_{x_0}^2 \rho_1(x_0, t) \tag{50}$$

We take the limit $N \to \infty$ again

$$X_0(t) = X(t), \quad X_N(t) = X(t - \tau) \tag{51}$$

This leads to a coordinate transformation

$$\{x_0, t\} \to \{x, t\}, \quad \{x_N, t\} \to \{x_\tau, t - \tau\} \tag{52}$$

And therefore to the first Fokker-Planck equation

$$\partial_t \rho_1(x, t) = -\partial_x \int_{\mathbb{R}} \mathrm{d}x_\tau \, F(x_0, x_\tau) \rho_2(x, t; x_\tau, t - \tau) + D_0 \partial_x^2 \rho_1(x, t) \tag{53}$$

Integrating the Fokker-Planck equation over all auxiliary variables but not x_N

$$\int_{\mathbb{R}} \mathrm{d}x_1 \cdots \int_{\mathbb{R}} \mathrm{d}x_{N-1} \, \partial_t \rho_{N+1}(x_0, \dots, x_N; t) = \int_{\mathbb{R}} \mathrm{d}x_1 \cdots \int_{\mathbb{R}} \mathrm{d}x_{N-1} \left(-\sum_{j=0}^{N} \partial_{x_j} f_j(x_j, x_{j-1}) + D_0 \partial_{x_0}^2 \right) \rho_{N+1}(x_0, \dots, x_N; t) \tag{54}$$

$$\int_{\mathbb{R}} \mathrm{d}x_1 \cdots \int_{\mathbb{R}} \mathrm{d}x_{N-1} \, \rho_{N+1}(x_0, \dots, x_N; t) = \rho_2(x_0, x_N; t) \tag{55}$$

$$\int_{\mathbb{R}} \mathrm{d}x_1 \cdots \int_{\mathbb{R}} \mathrm{d}x_{N-1} \sum_{j=0}^{N} \partial_{x_j} f_j(x_j, x_{j-1}) \rho_{N+1}(x_0, \dots, x_N; t) = \partial_{x_0} F(x_0, x_N) \rho_2(x_0, x_N; t) - \frac{N}{\tau} \partial_{x_N} \int_{\mathbb{R}} \mathrm{d}x_{N-1} \, (x_N - x_{N-1}) \rho_3(x_0, x_{N-1}, x_N; t) \tag{56}$$

$$\partial_t \rho_2(x_0, x_N, t) = -\partial_{x_0} F(x_0, x_N) \rho_2(x_0, x_N; t) + \frac{N}{\tau} \partial_{x_N} \int_{\mathbb{R}} \mathrm{d}x_{N-1} \, (x_N - x_{N-1}) \rho_3(x_0, x_{N-1}, x_N; t) + D_0 \partial_{x_0}^2 \rho_2(x_0, x_N, t) \tag{57}$$

Taking again the limit $N \to \infty$ and interpret the auxiliary variables under the coordinate transformation

$$\left\{ \tfrac{N}{\tau}(x_{N-1} - x_N), t \right\} \to \{\dot{x}, t - \tau\} \tag{58}$$

This forms an alternative hierarchy

$$\partial_t \rho_2(x, t; x_\tau, t - \tau) = -\partial_x F(x, x_\tau) \rho_2(x, t; x_\tau, t - \tau) + \partial_{x_\tau} \int_{\mathbb{R}} \mathrm{d}\dot{x}_\tau \, \dot{x}_\tau \rho_3(x, t; \dot{x}_\tau, t - \tau; x_\tau, t - \tau) + D_0 \partial_x^2 \rho_2(x, t; x_\tau, t - \tau) \tag{59}$$

Closure via mean field approximation

$$\rho_3(x, t; \dot{x}_\tau, t - \tau; x_\tau, t - \tau) \sim \rho_2(x, t; x_\tau, t - \tau) \rho_1(\dot{x}_\tau, t - \tau) \tag{60}$$

We arrive at a system of 2 Fokker-Planck equations, that are delay partial integro-differential equations.

$$\partial_t \rho_1(x, t) = -\partial_x \int_{\mathbb{R}} \mathrm{d}x_\tau \, F(x_0, x_\tau) \rho_2(x, t; x_\tau, t - \tau) + D_0 \partial_x^2 \rho_1(x, t) \tag{61}$$

$$\partial_t \rho_2(x, t; x_\tau, t - \tau) = -\partial_x F(x, x_\tau) \rho_2(x, t; x_\tau, t - \tau) + \partial_{x_\tau} \int_{\mathbb{R}} \mathrm{d}\dot{x}_\tau \, \dot{x}_\tau \rho_2(x, t; x_\tau, t - \tau) \rho_1(\dot{x}_\tau, t - \tau) + D_0 \partial_x^2 \rho_2(x, t; x_\tau, t - \tau) \tag{62}$$

1D System

$$\frac{\partial \rho_1}{\partial t} = -\frac{\partial F_1}{\partial x}\rho_1(x,t) - F_1(x)\frac{\partial \rho_1}{\partial x} - \frac{\partial}{\partial x}\int\limits_{\mathcal{V}_\tau} \mathrm{d}x_\tau \, F_2(x_\tau)\rho_2(x,t;x_\tau,t-\tau) + D_0\frac{\partial^2 \rho_1}{\partial x^2} \tag{63}$$

$$\frac{\partial \rho_2}{\partial t} = -\frac{\partial F_1}{\partial x}\rho_2(x,t;x_\tau,t-\tau) - F_1(x)\frac{\partial \rho_2}{\partial x} - F_2(x_\tau)\frac{\partial \rho_2}{\partial x} + \frac{\partial}{\partial x_\tau}\int\limits_{\mathcal{V}_\tau} \mathrm{d}\dot{x}_\tau \, \dot{x}_\tau\rho_2(x,t;x_\tau,t-\tau)\rho_1(\dot{x}_\tau,t-\tau) + D_0\frac{\partial^2 \rho_1}{\partial x^2} \tag{64}$$

$$\frac{\partial \rho_1}{\partial t} = \big(3ax^2 - b - K\big)\rho_1(x,t) + \big(ax^3 - xb - xK\big)\frac{\partial \rho_1}{\partial x} + K\frac{\partial}{\partial x}\int\limits_{\mathcal{V}_\tau} \mathrm{d}x_\tau \, x_\tau\rho_2(x,t;x_\tau,t-\tau) + D_0\frac{\partial^2 \rho_1}{\partial x^2} \tag{65}$$

$$\frac{\partial \rho_2}{\partial t} = \big(3ax^2 - b - K\big)\rho_2(x,t;x_\tau,t-\tau) + \big(ax^3 - xb - xK\big)\frac{\partial \rho_2}{\partial x} + Kx_\tau\frac{\partial \rho_2}{\partial x} + \frac{\partial \rho_2}{\partial x_\tau}\left(\int\limits_{\mathcal{V}_\tau} \mathrm{d}\dot{x}_\tau \, \dot{x}_\tau\rho_1(\dot{x}_\tau,t-\tau)\right) + D_0\frac{\partial^2 \rho_2}{\partial x^2} \tag{66}$$

test