Parameters

$$a > 0 \tag{1}$$
$$b > 0 \tag{2}$$
$$K \geq 0 \tag{3}$$
$$D_0 \geq 0 \tag{4}$$
$$\tau \geq 0 \tag{5}$$

Potential

$$V(x) = \frac{a}{4}x^4 - \frac{b}{2}x^2 = \frac{a}{4}x^2\left(x - \sqrt{\frac{2b}{a}}\right)\left(x + \sqrt{\frac{2b}{a}}\right) \tag{6}$$

Force

$$F(x) = -\frac{\partial V}{\partial x} = -ax^3 + bx \tag{7}$$

We have 2 Hypervolumes

$$\mathcal{V}_t \subset \mathbb{R}^{n_t} \tag{8}$$
$$\mathcal{V}_\tau \subset \mathbb{R}^{n_\tau} \tag{9}$$
$$\mathcal{V} = \mathcal{V}_t \times \mathcal{V}_\tau \tag{10}$$
$$n_t \geq n_\tau \tag{11}$$

$$\begin{aligned} \dot{X}_t &= F(X, X_\tau)\,\mathrm{d}t + \sqrt{2D_0}\,\mathrm{d}B_t & (12) \\ &= -aX_t^3\,\mathrm{d}t + bX_t\,\mathrm{d}t + K(X_t - X_\tau)\,\mathrm{d}t + \sqrt{2D_0}\,\mathrm{d}B_t & (13) \end{aligned}$$

$$F(x, x_\tau) = F_1(x) + F_2(x_\tau) = -ax^3 + (b+K)x - Kx_\tau \tag{14}$$

$$\begin{aligned} \frac{\partial}{\partial t}\rho_1 &= -\frac{\partial}{\partial x}\int\limits_{\mathcal{V}_\tau} \mathrm{d}x_\tau\, F(x, x_\tau)\rho_2(x, t; x_\tau, t-\tau) + D_0\frac{\partial^2}{\partial x^2}\rho_1(x,t) & (15) \\ &= -\frac{\partial}{\partial x}F_1(x)\int\limits_{\mathcal{V}_\tau} \mathrm{d}x_\tau\, \rho_2(x, t; x_\tau, t-\tau) - \frac{\partial}{\partial x}\int\limits_{\mathcal{V}_\tau} \mathrm{d}x_\tau\, F_2(x_\tau)\rho_2(x, t; x_\tau, t-\tau) + D_0\frac{\partial^2}{\partial x^2}\rho_1(x,t) & (16) \\ &= -\frac{\partial}{\partial x}F_1(x)\rho_1(x,t) - \frac{\partial}{\partial x}\int\limits_{\mathcal{V}_\tau} \mathrm{d}x_\tau\, F_2(x_\tau)\rho_2(x, t; x_\tau, t-\tau) + D_0\frac{\partial^2}{\partial x^2}\rho_1(x,t) & (17) \end{aligned}$$

$$\frac{\partial}{\partial t}\rho_1 = -\frac{\partial F_1}{\partial x}\rho_1(x,t) - F_1(x)\frac{\partial}{\partial x}\rho_1(x,t) - \frac{\partial}{\partial x}\int\limits_{\mathcal{V}_\tau} \mathrm{d}x_\tau\, F_2(x_\tau)\rho_2(x, t; x_\tau, t-\tau) + D_0\frac{\partial^2}{\partial x^2}\rho_1(x,t) \tag{18}$$

$$\begin{aligned} \frac{\partial}{\partial t}\rho_2 &= -\frac{\partial}{\partial x}F(x, x_\tau)\rho_2(x, t; x_\tau, t-\tau) + D_0\frac{\partial^2}{\partial x^2}\rho_2(x, t; x_\tau, t-\tau) + \frac{\partial}{\partial x_\tau}\rho_2(x, t; x_\tau, t-\tau)\int\limits_{\mathcal{V}_t} \mathrm{d}\dot{x}_\tau\, \rho_1(\dot{x}_\tau, t-\tau)\dot{x}_\tau & (19) \\ &= -\frac{\partial}{\partial x}F(x, x_\tau)\rho_2(x, t; x_\tau, t-\tau) + D_0\frac{\partial^2}{\partial x^2}\rho_2(x, t; x_\tau, t-\tau) + \frac{\partial}{\partial x_\tau}\rho_2(x, t; x_\tau, t-\tau)\left(\int\limits_{\mathcal{V}_t} \mathrm{d}\dot{x}_\tau\, \rho_1(\dot{x}_\tau, t-\tau)\dot{x}_\tau\right) & (20) \\ &= -\frac{\partial}{\partial x}F_1(x)\rho_2(x, t; x_\tau, t-\tau) - F_2(x_\tau)\frac{\partial}{\partial x}\rho_2(x, t; x_\tau, t-\tau) + D_0\frac{\partial^2}{\partial x^2}\rho_2(x, t; x_\tau, t-\tau) + \frac{\partial}{\partial x_\tau}\rho_2(x, t; x_\tau, t-\tau)\left(\int\limits_{\mathcal{V}_t} \mathrm{d}\dot{x}_\tau\, \rho_1(\dot{x}_\tau, t-\tau)\dot{x}_\tau\right) & (21) \end{aligned}$$

$$\begin{aligned} \frac{\partial f}{\partial x} &\sim \frac{f(x_{i+1}) - f(x_{i-1})}{2\Delta x} & (22) \\ \frac{\partial^2 f}{\partial x^2} &\sim \frac{2f(x_i) - f(x_{i-1}) - f(x_{i+1})}{\Delta x^2} & (23) \end{aligned}$$

$$\int\limits_{\mathcal{V}} \mathrm{d}x\, f(x) = \Delta x \sum_{i=1}^{N_{\mathcal{V}}} f(x_i) \tag{24}$$