1 Vector: 3 perspectives - Physics, Math, CS

$$\begin{bmatrix} 7 & 8 & 6 \\ 6 & 8 & 4 \\ 7 & 5 & 4 \end{bmatrix}$$

1.1 physics

- vectors are arrows in space with length and direction
- so vectors can be moved around in space without any issues

1.2 cs

- vectors are ordered lists of numbers

1.3 math

- seeks to generalize both of these views and defines vector operations such as addition and multiplication

1.4 In linear algebra

- graham suggests to view vector as arrow with tail always fixed at origin
- maybe i can imagine a vector as a operation of shifting the origin
- thus when we do vector addition we will start the second vector from the start of first vector, since the origin has been shifted by the first vector
- Note: "but" it could really be shifting of all the points in the coordinate system (as told by graham)

1.5 Scalars

- The numbers that we multiply the vector with to scale the vector in its original direction
- since it's used frequently it just interchangeable with number

2 Linear Combinations, Spans, Basis Vectors

2.1 Vector Coordinates

- vector coordinates are the numbers present in a vector
- Each of the vector coordinates is also a scalar that scales the basis vectors of the coordinate system $\hat{\imath}, \hat{\jmath}$

2.2 Span

Span of $\vec{v}$ and $\vec{w}$ are the set of all of their linear combinations

$$a\vec{v} + b\vec{w}$$

Note: Its common to think a collection of vectors as points, due to clustering/noise

2.3 Linearly dependent Vectors

- If the third vector can be formed by linear combination of the other vectors then the vectors are said to be linearly dependent
- If we cannot get a vector by linear combination of other vectors then then those vectors are called as linearly independent

2.4 Basis

- basis of a vector space is a set of linearly independent vectors that span the full space

3 Linear Transformation and matrices

Transformations are just "functions" just suggests to visualize every function as a movement of coordinates(changing of basis vectors, thus changing of all of coordinate system)

3.1 Linear transformation in graph

- every lines will remain parallel
- and origin will remain at center

3.2 Matrix Transformation

- A 2 dimensional linear transformation of a coordinates system can be entirely described just 4 numbers (a 2d matrix)
- In a matrix each of the column is the final landing basis vector
- thus, matrix $\implies$ transformation data
- $$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \underbrace{x \begin{bmatrix} a \\ c \end{bmatrix} + y \begin{bmatrix} b \\ d \end{bmatrix}}_{\text{intuition}} = \begin{bmatrix} ax + by \\ cx + dy \end{bmatrix}$$

  Note:
- by summary: Matrix $\implies$ set of transformed basis vectors
- thus giving transformation data of the space(coordinate system)

4 Matrix Multiplication as Composition

4.1 Composition matrix

- when a series of transformation occurs the "overall" transformation matrix is called as composition matrix
- this composition matrix is formed by matrix multiplication of each transformation matrix –from right to left
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$$\overset{\overset{f(g(x))}{\text{Read right to left}}}{\xleftarrow{\hspace{10em}}}$$
$$\underbrace{\begin{bmatrix}1 & 1\\0 & 1\end{bmatrix}}_{\text{2nd transformation}} \underbrace{\begin{bmatrix}0 & -1\\1 & 0\end{bmatrix}}_{\text{1st transformation}} = \begin{bmatrix}1 & -1\\1 & 0\end{bmatrix}$$

- thus we can prove the associative nature of matrix multiplication by just intuition alone

$$(AB)C = A(BC)$$

5 Three dimensional linear transformations

Consider a linear transformation with 3d vector as input and output

$$\underset{\text{Input}}{\begin{bmatrix} 2 \\ 6 \\ -1 \end{bmatrix}} \xrightarrow{L(\vec{v})} \underset{\text{Output}}{\begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix}}$$

- This means with the given transformation function L. The vector point (2, 6, -1) is transformed to (3, 2, 0)
- In matrix form L must be a 3 by 3 matrix, since there are 3 basis vectors in 3 dimensions
- Ex:

$$\begin{bmatrix} 0 & 1 & 2 \\ 3 & 4 & 5 \\ 6 & 7 & 8 \end{bmatrix}$$

6 Determinant

- To measure the stretching and squishing of a transformation
- factor by which the area(in 2D) of the system changes
- It is given by "determinant" of a matrix transformation
- generally we compare the area change in basis vectors
- same way in 3d space volume is scaled and given by determinant

6.1 Computing determinant

- For a 2×2 matrix the formula is given by

$$\det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = ad - bc$$

- The intuition for this formula can be obtained by assuming variables c and d to be 0, then a and d are just the multipliers in their axis for basis vectors in their transformation

$$\det\left(\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}\right) = ad - 0 \cdot 0$$

- rigorous proof involves finding area of parallelogram
- For 3×3 Matrix:

$$\begin{aligned} \det\left(\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}\right) &= a \det\left(\begin{bmatrix} e & f \\ h & i \end{bmatrix}\right) \\ &- b \det\left(\begin{bmatrix} d & f \\ g & i \end{bmatrix}\right) \\ &+ c \det\left(\begin{bmatrix} d & e \\ g & h \end{bmatrix}\right) \end{aligned}$$

6.2 Property

- $$\det(M_1 M_2) = \det(M_1)\det(M_2)$$

7 Inverse Matrices, Rank, Column Space

7.1 Usefulness of Matrices

- When we have many linear equations of many variables with degree 1, we can use matrices to solves those equations
- Only things that's happening to those variables are they are scaled by some scalars and added together to get some constant

$$\begin{aligned} 2x + 5y + 3z &= -3 \\ 4x + 0y + 8z &= 0 \\ 1x + 3y + 0z &= 2 \end{aligned}$$

- this is called as a "linear system of equations", this can be written in matrix form as

$$\overbrace{\begin{bmatrix} 2 & 5 & 3 \\ 4 & 0 & 8 \\ 1 & 3 & 0 \end{bmatrix}}^{A} \overbrace{\begin{bmatrix} x \\ y \\ z \end{bmatrix}}^{\vec{x}} = \overbrace{\begin{bmatrix} -3 \\ 0 \\ 2 \end{bmatrix}}^{\vec{v}}$$

$$A\vec{x} = \vec{v}$$

- now the problem of three multi variable linear equations has become a single matrix transformation equation
- Here exactly, a unknown variable vector $\vec{x}$ after applying a transformation A becomes this exact vector $\vec{v}$

7.2 Inverse Matrices

During a transformation, either we can have determinant zero or non-zero, which implies input dimensions are reduced or not reduced —lossy transformation — not sure though

7.2.1 Non-Zero Determinant

- Thus when we consider a transformation with determinant equals zero
- There exists another transformation with the exact opposite/reverse movement of the given transformation
- thus when both of these transformation are applied in series to a input vector will output the same vector

$$\underbrace{A^{-1}A}_{\text{Matrix Multiplication}} = \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{\text{The transformation does nothing}}$$

- Once we have the inverse matrix A^{-1}, we can solve the above equation easily by applying the reverse transformation on the output vector

$$\underbrace{A^{-1}A}_{\text{The "do nothing" matrix}} \qquad \vec{x} = A^{-1}\vec{v}$$

$$\vec{x} = A^{-1}\vec{v}$$

7.2.2 Zero Determinant

- The transformation squishes space into smaller dimension
- No inverse transformation(—matrix) exists, since that would require output of multiple vectors for a single input for the inverse transformation

7.3 Rank for a Transformation

- Means no of dimensions in the output of a transformation
- Rank 1 means line output,
  Rank 2 means plane output,
  Rank 3 means solid output

7.4 Column Space for a Transformation

- Set of all possible output vectors for the given transformation

$$A\vec{x}$$

- the name is Column space is used because, columns in a matrix represent the transformed basis vectors
- and the span of these transformed basis vectors will give us the Column space of the transformation

Note:

- Thus the precise definition of rank will be the no of dimensions in the column space
- when the rank is the highest and equal to no of columns, then we call the matrix "full rank"

7.5 Null Space

- Note: In linear transformations the column space will always include origin
- for a full-rank transformation, only one vector from the input will land on origin, i.e: the zero vector itself
- but for any other lower rank transformations, many input vectors will fall into origin in the output column space
- this set of vectors that land on the origin is called the "Null Space"/ "Kernel" of a matrix-transformation

8 Non Square Matrices

- For a 3×2 matrix,

$$\begin{bmatrix} 3 & 1 \\ 4 & 1 \\ 5 & 9 \end{bmatrix}$$

- this takes in 2D input space and outputs a 3D space

$$\underbrace{\begin{bmatrix} 2 \\ 7 \end{bmatrix}}_{\text{2d Input}} \rightarrow L(\vec{\mathrm{v}}) \rightarrow \underbrace{\begin{bmatrix} 1 \\ 8 \\ 2 \end{bmatrix}}_{\text{3d Output}}$$

- intuitively this can be observed because we know the columns are the transformed basis vectors, and here the columns are given by 3-dimensions and there is only 2 columns which $\implies$ that input space has only 2 basis vectors thus 2 dimensions
- also told as "Column space" of this matrix is 3d

8.1 Linearity

- Here Linearity changes for any transformation
- Eg: line of evenly spaced dots must remain evenly spaced

9 Dot Product and Duality

9.1 Dot product standard introduction

- When given two vectors of same dimensions, dot product of them is sum of product of each pair of basis vector scalars

$$\underbrace{\begin{bmatrix} 2 \\ 7 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 8 \\ 2 \\ 8 \end{bmatrix}}_{\text{Dot product}} = 2 \cdot 8 + 7 \cdot 2 + 1 \cdot 8$$

- It is a "Scalar"
- Geometrically the dot product can be given as the product of length of one vector with projected length of second vector on first vector
- Note: Mechanical work is the dot product of force and displacement vectors. Magnetic flux is the dot product of the magnetic field and the area vectors.

9.2 1×2 Transformation and Dot product

- Consider the transformation of 2d plane to 1d line,

$$\text{Transformation Matrix: } \begin{bmatrix} 2 & 1 \end{bmatrix}$$

- Then when we apply this transformation to a vector, $\vec{v} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$

$$\overbrace{\begin{bmatrix} 1 & -2 \end{bmatrix}}^{\text{Transform}} \underbrace{\begin{bmatrix} 4 \\ 3 \end{bmatrix}}_{\text{Vector}} = 4 \cdot 1 + 3 \cdot -2$$

- Now this just resembles dot product of two vectors
- This suggests a connection between linear transformation of vectors to numbers (1×2 Matrices) and Vectors themselves

9.3 Projection as a Transformation

- when we consider a imaginary line on our 2d space with basis vector ($\vec{u}$)
- Projection of any vector(2d) on this line to get a scalar magnitude(1d) can be considered as a "Projection Transformation" with a 1×2 matrix
- By symmetry, Graham finds the transformation matrix for this projection is the matrix of projections of $\vec{u}$ on x-axis and y-axis

$$\text{Projection Transformation: } \begin{bmatrix} u_x & u_y \end{bmatrix}$$

on the imaginary line can be calculated as

$$\overbrace{\begin{bmatrix} u_x & u_y \end{bmatrix}}^{\text{Transform}} \underbrace{\begin{bmatrix} x \\ y \end{bmatrix}}_{\text{Vector}} = u_x \cdot x + u_y \cdot y$$

- This is exactly similar to dot product of vector with unit vector
- This is why "Taking a dot product with a unit vector" can be interpreted the same as "Projection(length) of the vector on the unit vector"
- For dot product with non-unit vectors:
  - we extend our projection method by
  - first projection the given vector to the other vector's unit vector
  - then scale the projection according to the scale of the other vector

9.4 Duality

- It loosely means natural but surprising correspondence between two different mathematical thing
- Here we can say that the dual of a vector is the linear transformation it encodes
- and the dual of a linear transformation of some space to one dimension is a certain vector in the first space

9.5 Conclusion

- Thus dot products are on surface very useful geometric tools for understanding projections
- and to test whether two vectors are pointing in similar direction (dot product of perpendiculars is zero)
- but on a deeper level we understand that dot product means one vector is matrix transformed onto other vector as number line and scaled by the number line vector subsequently

10 Cross Product and Duality

10.1 Cross Product Standard Introduction - for normal students

- In 2d: The parallelogram area enclosed between two vectors is the cross product magnitude of the two vectors

$$\vec{\mathrm{v}} \times \vec{\mathrm{w}} = \text{Area of Parallelogram}$$

- Trick to remember sign of the cross product:
  - $\hat{\imath} \times \hat{\jmath}$ is positive,
  - so only if the order of vectors is from right to left (In graph, i.e from $\hat{\imath}$ to $\hat{\jmath}$) the cross product is positive
- Resultant vector's direction, given by "right hand rule", is perpendicular to both of the given vectors/parallelogram
- "Notation-Trick" to compute cross product: –this is weird - using $\hat{\imath}, \hat{\jmath}, \ldots$ - symbols as numbers in matrix

$$\begin{aligned}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \times \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} &= \det\left(\begin{bmatrix} \hat{\imath} & v_1 & w_1 \\ \hat{\jmath} & v_2 & w_2 \\ \hat{k} & v_3 & w_3 \end{bmatrix}\right) \\ &= \hat{\imath}(v_2 w_3 - v_3 w_2) + \hat{\jmath}(v_3 w_1 - v_1 w_3) + \hat{k}(v_1 w_2 - v_2 w_1)\end{aligned}$$

10.2 Cross Product as Linear Transformation