1 Vector: 3 perspectives - Physics, Math, CS

$$\begin{bmatrix} 7 & 8 & 6 \\ 6 & 8 & 4 \\ 7 & 5 & 4 \end{bmatrix}$$

1.1 physics

- vectors are arrows in space with length and direction
- so vectors can be moved around in space without any issues

1.2 cs

- vectors are ordered lists of numbers

1.3 math

- seeks to generalize both of these views and defines vector operations such as addition and multiplication

1.4 In linear algebra

- graham suggests to view vector as arrow with tail always fixed at origin
- maybe i can imagine a vector as a operation of shifting the origin
- thus when we do vector addition we will start the second vector from the start of first vector, since the origin has been shifted by the first vector
- Note: "but" it could really be shifting of all the points in the coordinate system (as told by graham)

1.5 Scalars

- The numbers that we multiply the vector with to scale the vector in its original direction
- since it's used frequently it just interchangeable with number

2 Linear Combinations, Spans, Basis Vectors

2.1 Vector Coordinates

- vector coordinates are the number numbers present in a vector
- Each of the vector coordinates is also a scalar that scales the basis vectors of the coordinate system $\hat{i}, \hat{j}$

2.2 Span

Span of $\vec{v}$ and $\vec{w}$ are the set of all of their linear combinations

$$a\vec{v} + b\vec{w}$$

Note: Its common to think a collection of vectors as points, due to clustering/noise

2.3 Linearly dependent Vectors

- If the third vector can be formed by linear combination of the other vectors then the vectors are said to be linearly dependent
- If we cannot get a vector by linear combination of other vectors then then those vectors are called as linearly independent

2.4 Basis

- basis of a vector space is a set of linearly independent vectors that span the full space

3 Linear Transformation and matrices

Transformations are just "functions" just suggests to visualize every function as a movement of coordinates(changing of basis vectors, thus changing of all of coordinate system)

3.1 Linear transformation in graph

- every lines will remain parallel
- and origin will remain at center

3.2 Matrix Transformation

- A 2 dimensional linear transformation of a coordinates system can be entirely described just 4 numbers (a 2d matrix)
- In a matrix each of the column is the final landing basis vector
- thus, matrix $\implies$ transformation data
- $$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \underbrace{x \begin{bmatrix} a \\ c \end{bmatrix} + y \begin{bmatrix} b \\ d \end{bmatrix}}_{\text{intuition}} = \begin{bmatrix} ax + by \\ cx + dy \end{bmatrix}$$

Note:

- by summary: Matrix $\implies$ set of transformed basis vectors
- thus giving transformation data of the space(coordinate system)

4 Matrix Multiplication as Composition

4.1 Composition matrix

- when a series of transformation occurs the "overall" transformation matrix is called as composition matrix
- this composition matrix is formed by matrix multiplication of each transformation matrix –from right to left
- $$\overset{\text{Read right to left}}{\overset{f(g(x))}{\xleftarrow{\hspace{8em}}}}$$
  $$\underbrace{\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}}_{\text{2nd transformation}} \underbrace{\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}}_{\text{1st transformation}} = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}$$
- thus we can prove the associative nature of matrix multiplication by just intuition alone
  $$(AB)C = A(BC)$$

5 Three dimensional linear transformations

Consider a linear transformation with 3d vector as input and output

$$\underset{\text{Input}}{\begin{bmatrix} 2 \\ 6 \\ -1 \end{bmatrix}} \xrightarrow{L(\vec{v})} \underset{\text{Output}}{\begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix}}$$

- This means with the given transformation function L. The vector point (2, 6, -1) is transformed to (3, 2, 0)
- In matrix form L must be a 3 by 3 matrix, since there are 3 basis vectors in 3 dimensions
- Ex:

$$\begin{bmatrix} 0 & 1 & 2 \\ 3 & 4 & 5 \\ 6 & 7 & 8 \end{bmatrix}$$

6 Determinant

- To measure the stretching and squishing of a transformation
- factor by which the area(in 2D) of the system changes
- It is given by "determinant" of a matrix transformation
- generally we compare the area change in basis vectors
- same way in 3d space volume is scaled and given by determinant

6.1 Computing determinant

- For a 2×2 matrix the formula is given by

$$det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = ad - bc$$

- The intuition for this formula can be obtained by assuming variables c and d to be 0, then a and d are just the multipliers in their axis for basis vectors in their transformation

$$det\left(\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}\right) = ad - 0 \cdot 0$$

- rigorous proof involves finding area of parallelogram

6.2 Property

- $$det(M_1 M_2) = det(M_1)det(M_2)$$

7 Usefulness of Matrices

- When we have many linear equations of many variables with degree 1, we can use matrices to solves those equations
- Only things that's happening to those variables are they are scaled by some scalars and added together to get some constant

$$\begin{aligned} 2x + 5y + 3z &= -3 \\ 4x + 0y + 8z &= 0 \\ 1x + 3y + 0z &= 2 \end{aligned}$$

- this is called as a "linear system of equations", this can be written in matrix form as

$$\overbrace{\begin{bmatrix} 2 & 5 & 3 \\ 4 & 0 & 8 \\ 1 & 3 & 0 \end{bmatrix}}^{A} \overbrace{\begin{bmatrix} x \\ y \\ z \end{bmatrix}}^{\vec{x}} = \overbrace{\begin{bmatrix} -3 \\ 0 \\ 2 \end{bmatrix}}^{\vec{v}}$$

$$A\vec{x} = \vec{v}$$

- now the problem of three multi variable linear equations has become a single matrix transformation equation
- Here exactly, a unknown variable vector $\vec{x}$ after applying a transformation A becomes this exact vector $\vec{v}$

7.1 Inverse Matrices

During a transformation, either we can have determinant zero or non-zero, which implies input dimensions are reduced or not reduced —lossy transformation — not sure though

7.1.1 Non-Zero Determinant

- Thus when we consider a transformation with determinant equals zero
- There exists another transformation with the exact opposite/reverse movement of the given transformation
- thus when both of these transformation are applied in series to a input vector will output the same vector

$$\underbrace{A^{-1}A}_{\text{Matrix Multiplication}} = \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{\text{The transformation does nothing}}$$

- Once we have the inverse matrix A^{-1}, we can solve the above equation easily by applying the reverse transformation on the output vector

$$\underbrace{A^{-1}A}_{\text{The "do nothing" matrix}} \qquad \vec{x} = A^{-1}\vec{v}$$

$$\vec{x} = A^{-1}\vec{v}$$

7.1.2 Zero Determinant

- The transformation squishes space into smaller dimension
- No inverse transformation(—matrix) exists, since that would require output of multiple vectors for a single input for the inverse transformation

7.2 Rank for a Transformation

- Means no of dimensions in the output of a transformation
- Rank 1 means line output,
  Rank 2 means plane output,
  Rank 3 means solid output

7.3 Column Space for a Transformation

- Set of all possible output vectors for the given transformation

$$A\vec{x}$$

- the name is Column space is used because, columns in a matrix represent the transformed basis vectors
- and the span of these transformed basis vectors will give us the Column space of the transformation

Note:

- Thus the precise definition of rank will be the no of dimensions in the column space
- when the rank is the highest and equal to no of columns, then we call the matrix "full rank"

7.4 Null Space

- Note: In linear transformations the column space will always include origin
- for a full-rank transformation, only one vector from the input will land on origin, i.e: the zero vector itself
- but for any other lower rank transformations, many input vectors will fall into origin in the output column space
- this set of vectors that land on the origin is called the "Null Space"/ "Kernel" of a matrix-transformation