



Perfect number

In number theory, a **perfect number** is a positive integer that is equal to the sum of its positive proper divisors, that is, divisors excluding the number itself.^[1] For instance, 6 has proper divisors 1, 2, and 3, and $1 + 2 + 3 = 6$, so 6 is a perfect number. The next perfect number is 28, because $1 + 2 + 4 + 7 + 14 = 28$.

The first seven perfect numbers are 6, 28, 496, 8128, 33550336, 8589869056, and 137438691328.^[2]

The sum of proper divisors of a number is called its aliquot sum, so a perfect number is one that is equal to its aliquot sum. Equivalently, a perfect number is a number that is half the sum of all of its positive divisors; in symbols, $\sigma_1(n) = 2n$ where σ_1 is the sum-of-divisors function.

This definition is ancient, appearing as early as Euclid's *Elements* (VII.22) where it is called τέλειος ἀριθμός (*perfect*, *ideal*, or *complete number*). Euclid also proved a formation rule (IX.36) whereby $\frac{q(q+1)}{2}$ is an even perfect number whenever q is a prime of the form $2^p - 1$ for positive integer p —what is now called a Mersenne prime. Two millennia later, Leonhard Euler proved that all even perfect numbers are of this form.^[3] This is known as the Euclid–Euler theorem.

It is not known whether there are any odd perfect numbers, nor whether infinitely many perfect numbers exist.

History

In about 300 BC Euclid showed that if $2^p - 1$ is prime then $2^{p-1}(2^p - 1)$ is perfect. The first four perfect numbers were the only ones known to early Greek mathematics, and the mathematician Nicomachus noted 8128 as early as around AD 100.^[4] In modern language, Nicomachus states without proof that every perfect number is of the form $2^{n-1}(2^n - 1)$ where $2^n - 1$ is prime.^{[5][6]} He seems to be unaware that n itself has to be prime. He also says (wrongly) that the perfect numbers end in 6 or 8 alternately. (The first 5 perfect numbers end with digits 6, 8, 6, 8, 6; but the sixth also ends in 6.) Philo of Alexandria in his first-century book "On the creation" mentions perfect numbers, claiming that the world was created in 6 days and the moon orbits in 28 days because 6 and 28 are perfect. Philo is followed by Origen,^[7] and by Didymus the Blind, who adds the observation that there are only four perfect numbers that are less than 10,000. (Commentary on Genesis 1. 14–19).^[8] Augustine of Hippo defines perfect numbers in *The City of God* (Book XI, Chapter 30) in the early 5th century AD, repeating the claim that God created the world in 6 days because 6 is the smallest perfect number. The Egyptian mathematician Ismail ibn Fallūs (1194–1252) mentioned the next three perfect numbers (33,550,336; 8,589,869,056; and 137,438,691,328) and listed a few more which are now known to be incorrect.^[9] The first known

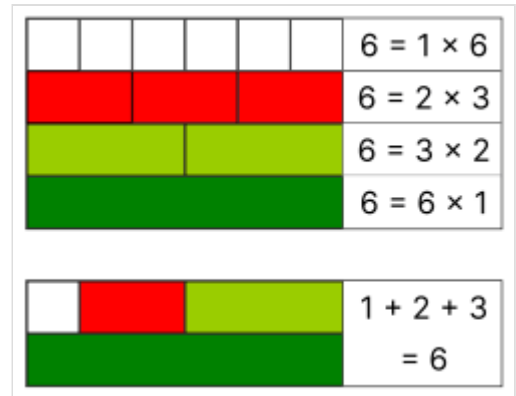


Illustration of the perfect number status of the number 6

European mention of the fifth perfect number is a manuscript written between 1456 and 1461 by an unknown mathematician.^[10] In 1588, the Italian mathematician Pietro Cataldi identified the sixth (8,589,869,056) and the seventh (137,438,691,328) perfect numbers, and also proved that every perfect number obtained from Euclid's rule ends with a 6 or an 8.^{[11][12][13]}

Even perfect numbers

Unsolved problem in mathematics



Are there infinitely many perfect numbers?

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Euclid proved that $2^{p-1}(2^p - 1)$ is an even perfect number whenever $2^p - 1$ is prime (*Elements*, Prop. IX.36).

For example, the first four perfect numbers are generated by the formula $2^{p-1}(2^p - 1)$, with p a prime number, as follows:

$$p = 2 : 2^1(2^2 - 1) = 2 \times 3 = 6$$

$$p = 3 : 2^2(2^3 - 1) = 4 \times 7 = 28$$

$$p = 5 : 2^4(2^5 - 1) = 16 \times 31 = 496$$

$$p = 7 : 2^6(2^7 - 1) = 64 \times 127 = 8128.$$

Prime numbers of the form $2^p - 1$ are known as Mersenne primes, after the seventeenth-century monk Marin Mersenne, who studied number theory and perfect numbers. For $2^p - 1$ to be prime, it is necessary that p itself be prime. However, not all numbers of the form $2^p - 1$ with a prime p are prime; for example, $2^{11} - 1 = 2047 = 23 \times 89$ is not a prime number.^[a] In fact, Mersenne primes are very rare: of the approximately 4 million primes p up to 68,874,199, $2^p - 1$ is prime for only 48 of them.^[14]

While Nicomachus had stated (without proof) that *all* perfect numbers were of the form $2^{n-1}(2^n - 1)$ where $2^n - 1$ is prime (though he stated this somewhat differently), Ibn al-Haytham (Alhazen) circa AD 1000 was unwilling to go that far, declaring instead (also without proof) that the formula yielded only every even perfect number.^[15] It was not until the 18th century that Leonhard Euler proved that the formula $2^{p-1}(2^p - 1)$ indeed yields all the even perfect numbers. Thus, there is a one-to-one correspondence between even perfect numbers and Mersenne primes; each Mersenne prime generates one even perfect number, and vice versa. This result is often referred to as the Euclid–Euler theorem.

An exhaustive search by the GIMPS distributed computing project has shown that the first 50 even perfect numbers are $2^{p-1}(2^p - 1)$ for

$p = 2, 3, 5, 7, 13, 17, 19, 31, 61, 89, 107, 127, 521, 607, 1279, 2203, 2281, 3217, 4253, 4423, 9689, 9941, 11213, 19937, 21701, 23209, 44497, 86243, 110503, 132049, 216091, 756839, 859433, 1257787, 1398269, 2976221, 3021377, 6972593, 13466917, 20996011, 24036583, 25964951, 30402457, 32582657, 37156667, 42643801, 43112609, 57885161, 74207281, 77232917$ [OEIS: A000043](#).^[14]

Two higher perfect numbers have also been discovered, namely those for which $p = 82589933$ and 136279841 . Although it is still possible there may be others within this range, initial but exhaustive tests by GIMPS have revealed no other perfect numbers for p below 138277717 . As of October 2024, 52 Mersenne primes are known,^[16] and therefore 52 even perfect numbers (the largest of which is $2^{136279840} \times (2^{136279841} - 1)$ with 82,048,640 digits). It is not known whether there are infinitely many perfect numbers, nor whether there are infinitely many Mersenne primes.

As well as having the form $2^{p-1}(2^p - 1)$, each even perfect number is the $(2^p - 1)$ -th triangular number (and hence equal to the sum of the integers from 1 to $2^p - 1$) and the 2^{p-1} -th hexagonal number. Furthermore, each even perfect number except for 6 is the $\frac{2^p+1}{3}$ -th centered nonagonal number and is equal to the sum of the first $2^{\frac{p-1}{2}}$ odd cubes (odd cubes up to the cube of $2^{\frac{p+1}{2}} - 1$):

$$6 = 2^1(2^2 - 1) = 1 + 2 + 3,$$

$$\begin{aligned} 28 &= 2^2(2^3 - 1) = 1 + 2 + 3 + 4 + 5 + 6 + 7 \\ &= 1^3 + 3^3 \end{aligned}$$

$$\begin{aligned} 496 &= 2^4(2^5 - 1) = 1 + 2 + 3 + \dots + 29 + 30 + 31 \\ &= 1^3 + 3^3 + 5^3 + 7^3 \end{aligned}$$

$$\begin{aligned} 8128 &= 2^6(2^7 - 1) = 1 + 2 + 3 + \dots + 125 + 126 + 127 \\ &= 1^3 + 3^3 + 5^3 + 7^3 + 9^3 + 11^3 + 13^3 + 15^3 \end{aligned}$$

$$\begin{aligned} 33550336 &= 2^{12}(2^{13} - 1) = 1 + 2 + 3 + \dots + 8189 + 8190 + 8191 \\ &= 1^3 + 3^3 + 5^3 + \dots + 123^3 + 125^3 + 127^3 \end{aligned}$$

Even perfect numbers (except 6) are of the form

$$T_{2^p-1} = 1 + \frac{(2^p - 2) \times (2^p + 1)}{2} = 1 + 9 \times T_{(2^p-2)/3}$$

with each resulting triangular number $T_7 = 28$, $T_{31} = 496$, $T_{127} = 8128$ (after subtracting 1 from the perfect number and dividing the result by 9) ending in 3 or 5, the sequence starting with $T_2 = 3$, $T_{10} = 55$, $T_{42} = 903$, $T_{2730} = 3727815$, ...^[17] It follows that by adding the digits of any even perfect number (except 6), then adding the digits of the resulting number, and repeating this process until a single digit (called the digital root) is obtained, always produces the number 1. For example, the digital root of 8128 is 1, because $8 + 1 + 2 + 8 = 19$, $1 + 9 = 10$, and $1 + 0 = 1$. This works with all perfect numbers $2^{p-1}(2^p - 1)$ with odd prime p and, in fact, with *all* numbers of the form $2^{m-1}(2^m - 1)$ for odd integer (not necessarily prime) m .

Owing to their form, $2^{p-1}(2^p - 1)$, every even perfect number is represented in binary form as p ones followed by $p - 1$ zeros; for example:

$$\begin{aligned}
6_{10} &= 2^2 + 2^1 = 110_2 \\
28_{10} &= 2^4 + 2^3 + 2^2 = 11100_2 \\
496_{10} &= 2^8 + 2^7 + 2^6 + 2^5 + 2^4 = 111110000_2 \\
8128_{10} &= 2^{12} + 2^{11} + 2^{10} + 2^9 + 2^8 + 2^7 + 2^6 = 1111111000000_2
\end{aligned}$$

Thus every even perfect number is a pernicious number.

Every even perfect number is also a practical number (cf. Related concepts).

Odd perfect numbers

Unsolved problem in mathematics

? *Are there any odd perfect numbers?*

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It is unknown whether any odd perfect numbers exist, though various results have been obtained. In 1496, Jacques Lefèvre stated that Euclid's rule gives all perfect numbers,^[18] thus implying that no odd perfect number exists, but Euler himself stated: "Whether ... there are any odd perfect numbers is a most difficult question".^[19] More recently, Carl Pomerance has presented a heuristic argument suggesting that indeed no odd perfect number should exist.^[20] All perfect numbers are also harmonic divisor numbers, and it has been conjectured as well that there are no odd harmonic divisor numbers other than 1.

Any odd perfect number N must satisfy the following conditions:

- $N > 10^{1500}$.^[21]
- N is not divisible by 105.^[22]
- N is of the form $N \equiv 1 \pmod{12}$ or $N \equiv 117 \pmod{468}$ or $N \equiv 81 \pmod{324}$.^[23]
- The largest prime power p^a that divides N is greater than 10^{62} .^[21]
- The largest prime factor of N is greater than 10^8 ,^[24] and less than $\sqrt[3]{3N}$.^[25]
- The second largest prime factor is greater than 10^4 ,^[26] and is less than $\sqrt[5]{2N}$.^[27]
- The third largest prime factor is greater than 100,^[28] and less than $\sqrt[6]{2N}$.^[29]
- N has at least 101 prime factors and at least 10 distinct prime factors.^{[21][30]} If 3 does not divide N , then N has at least 12 distinct prime factors.^[31]
- N is of the form

$$N = q^\alpha p_1^{2e_1} \cdots p_k^{2e_k},$$

where:

- q, p_1, \dots, p_k are distinct odd primes (Euler).
- $q \equiv \alpha \equiv 1 \pmod{4}$ (Euler).
- The smallest prime factor of N is at most $\frac{k-1}{2}$.^[32]
- $N < 2^{(4^{k+1}-2^{k+1})}$.^{[33][34]}
- $\alpha + 2e_1 + 2e_2 + 2e_3 + \cdots + 2e_k \geq \frac{99k-224}{37}$.^{[32][35][36]}

- $qp_1p_2p_3 \cdots p_k < 2N^{\frac{17}{26}}$.^[37]
- $\frac{1}{q} + \frac{1}{p_1} + \frac{1}{p_2} + \cdots + \frac{1}{p_k} < \ln 2$.^{[38][39]}

Furthermore, several minor results are known about the exponents e_1, \dots, e_k .

- Not all $e_i \equiv 1 \pmod{3}$.^[40]
- Not all $e_i \equiv 2 \pmod{5}$.^[41]
- If all $e_i \equiv 1 \pmod{3}$ or $2 \pmod{5}$, then the smallest prime factor of N must lie between 10^8 and 10^{1000} .^[41]
- More generally, if all $2e_i+1$ have a prime factor in a given finite set S , then the smallest prime factor of N must be smaller than an effectively computable constant depending only on S .^[41]
- If $(e_1, \dots, e_k) = (1, \dots, 1, 2, \dots, 2)$ with t ones and u twos, then $\frac{t-1}{4} \leq u \leq 2t + \sqrt{\alpha}$.^[42]
- $(e_1, \dots, e_k) \neq (1, \dots, 1, 3)$,^[43] $(1, \dots, 1, 5)$, $(1, \dots, 1, 6)$.^[44]
- If $e_1 = \dots = e_k = e$, then
 - e cannot be 3,^[45] 5, 24,^[46] 6, 8, 11, 14 or 18.^[44]
 - $k \leq 2e^2 + 8e + 2$.^[47]

In 1888, Sylvester stated:^[48]

... a prolonged meditation on the subject has satisfied me that the existence of any one such [odd perfect number]—its escape, so to say, from the complex web of conditions which hem it in on all sides—would be little short of a miracle.

On the other hand, several odd integers come close to being perfect. René Descartes observed that the number $D = 3^2 \cdot 7^2 \cdot 11^2 \cdot 13^2 \cdot 22021 = (3 \cdot 1001)^2 \cdot (22 \cdot 1001 - 1) = 198585576189$ would be an odd perfect number if only 22021 ($= 19^2 \cdot 61$) were a prime number. The odd numbers with this property (they would be perfect if one of their composite factors were prime) are the Descartes numbers. Many of the properties proven about odd perfect numbers also apply to Descartes numbers, and Pace Nielsen has suggested that sufficient study of these numbers may lead to a proof that no odd perfect numbers exist.^[49]

Minor results

All even perfect numbers have a very precise form; odd perfect numbers either do not exist or are rare. There are a number of results on perfect numbers that are actually quite easy to prove but nevertheless superficially impressive; some of them also come under Richard Guy's strong law of small numbers:

- The only even perfect number of the form $n^3 + 1$ is 28 (Makowski 1962).^[50]
- 28 is also the only even perfect number that is a sum of two positive cubes of integers (Gallardo 2010).^[51]
- The reciprocals of the divisors of a perfect number N must add up to 2 (to get this, take the definition of a perfect number, $\sigma_1(n) = 2n$, and divide both sides by n):

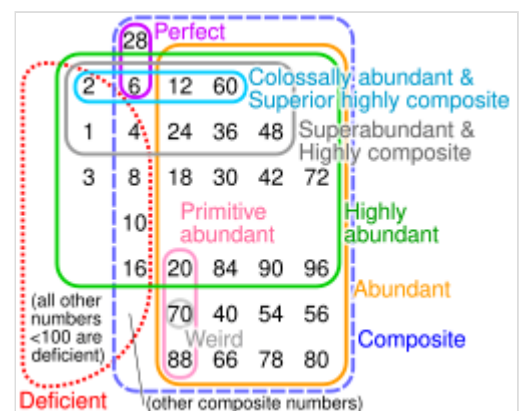
- For 6, we have $\frac{1}{6} + \frac{1}{3} + \frac{1}{2} + \frac{1}{1} = \frac{1}{6} + \frac{2}{6} + \frac{3}{6} + \frac{6}{6} = \frac{1+2+3+6}{6} = \frac{2 \cdot 6}{6} = 2$;
- For 28, we have $\frac{1}{28} + \frac{1}{14} + \frac{1}{7} + \frac{1}{4} + \frac{1}{2} + \frac{1}{1} = 2$, etc.
- The number of divisors of a perfect number (whether even or odd) must be even, because N cannot be a perfect square.^[52]
 - From these two results it follows that every perfect number is an Ore's harmonic number.
- The even perfect numbers are not trapezoidal numbers; that is, they cannot be represented as the difference of two positive non-consecutive triangular numbers. There are only three types of non-trapezoidal numbers: even perfect numbers, powers of two, and the numbers of the form $2^{n-1}(2^n + 1)$ formed as the product of a Fermat prime $2^n + 1$ with a power of two in a similar way to the construction of even perfect numbers from Mersenne primes.^[53]
- The number of perfect numbers less than n is less than $c\sqrt{n}$, where $c > 0$ is a constant.^[54] In fact it is $o(\sqrt{n})$, using little-o notation.^[55]
- Every even perfect number ends in 6 or 28 in base ten and, with the only exception of 6, ends in 1 in base 9.^{[56][57]} Therefore, in particular the digital root of every even perfect number other than 6 is 1.
- The only square-free perfect number is 6.^[58]

Related concepts

The sum of proper divisors gives various other kinds of numbers. Numbers where the sum is less than the number itself are called deficient, and where it is greater than the number, abundant. These terms, together with *perfect* itself, come from Greek numerology. A pair of numbers which are the sum of each other's proper divisors are called amicable, and larger cycles of numbers are called sociable. A positive integer such that every smaller positive integer is a sum of distinct divisors of it is a practical number.

By definition, a perfect number is a fixed point of the restricted divisor function $s(n) = \sigma(n) - n$, and the aliquot sequence associated with a perfect number is a constant sequence. All perfect numbers are also **\mathcal{S}** -perfect numbers, or Granville numbers.

A semiperfect number is a natural number that is equal to the sum of all or some of its proper divisors. A semiperfect number that is equal to the sum of all its proper divisors is a perfect number. Most abundant numbers are also semiperfect; abundant numbers which are not semiperfect are called weird numbers.



Euler diagram of numbers under 100:

- Abundant
- Primitive abundant
- Highly abundant
- Superabundant and highly composite
- Colossally abundant and superior highly composite
- Weird
- Perfect
- Composite
- Deficient

See also

- [Harmonic divisor number](#)
- [Hyperperfect number](#)
- [Leinster group](#)
- [List of Mersenne primes and perfect numbers](#)
- [Multiply perfect number](#)
- [Superperfect numbers](#)
- [Unitary perfect number](#)

Notes

- a. All factors of $2^p - 1$ are congruent to 1 mod $2p$. For example, $2^{11} - 1 = 2047 = 23 \times 89$, and both 23 and 89 yield a remainder of 1 when divided by 22. Furthermore, whenever p is a [Sophie Germain prime](#)—that is, $2p + 1$ is also prime—and $2p + 1$ is congruent to 1 or 7 mod 8, then $2p + 1$ will be a factor of $2^p - 1$, which is the case for $p = 11, 23, 83, 131, 179, 191, 239, 251, \dots$ [OEIS: A002515](#).

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