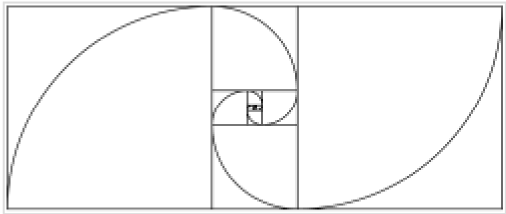
Pell number

In mathematics, the **Pell numbers** are an infinite sequence of integers, known since ancient times, that comprise the denominators of the closest rational approximations to the square root of 2. This sequence of approximations begins $\frac{1}{1}$, $\frac{3}{2}$, $\frac{7}{5}$, $\frac{17}{12}$, and $\frac{41}{29}$, so the sequence of Pell numbers begins with 1, 2, 5, 12, and 29. The numerators of the same sequence of approximations are half the **companion Pell numbers** or **Pell–Lucas numbers**; these numbers form a second infinite sequence that begins with 2, 6, 14, 34, and 82.



The sides of the squares used to construct a silver spiral are the Pell numbers

Both the Pell numbers and the companion Pell numbers may be calculated by means of a recurrence relation similar to that for the Fibonacci numbers, and both sequences of numbers grow exponentially, proportionally to powers of the silver ratio $1 + \sqrt{2}$. As well as being used to approximate the square root of two, Pell numbers can be used to find square triangular numbers, to construct integer approximations to the right isosceles triangle, and to solve certain combinatorial enumeration problems.[1]

As with Pell's equation, the name of the Pell numbers stems from Leonhard Euler's mistaken attribution of the equation and the numbers derived from it to John Pell. The Pell–Lucas numbers are also named after Édouard Lucas, who studied sequences defined by recurrences of this type; the Pell and companion Pell numbers are Lucas sequences.

Pell numbers

The Pell numbers are defined by the recurrence relation:

$$P_n = \begin{cases} 0 & \text{if } n = 0; \\ 1 & \text{if } n = 1; \\ 2P_{n-1} + P_{n-2} & \text{otherwise.} \end{cases}$$

In words, the sequence of Pell numbers starts with 0 and 1, and then each Pell number is the sum of twice the previous Pell number, plus the Pell number before that. The first few terms of the sequence are

> 0, 1, 2, 5, 12, 29, 70, 169, 408, 985, 2378, 5741, 13860, … (sequence A000129 in the OEIS).

Analogously to the Binet formula, the Pell numbers can also be expressed by the closed form formula

$$P_n = \frac{\left(1+\sqrt{2}\right)^n - \left(1-\sqrt{2}\right)^n}{2\sqrt{2}}.$$

For large values of n, the $(1+\sqrt{2})^n$ term dominates this expression, so the Pell numbers are approximately proportional to powers of the silver ratio $1+\sqrt{2}$, analogous to the growth rate of Fibonacci numbers as powers of the golden ratio.

A third definition is possible, from the matrix formula

$$\begin{pmatrix} P_{n+1} & P_n \\ P_n & P_{n-1} \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}^n.$$

Many identities can be derived or proven from these definitions; for instance an identity analogous to Cassini's identity for Fibonacci numbers,

$$P_{n+1}P_{n-1} - P_n^2 = (-1)^n,$$

is an immediate consequence of the matrix formula (found by considering the determinants of the matrices on the left and right sides of the matrix formula).[2]

Approximation to the square root of two

Pell numbers arise historically and most notably in the rational approximation to $\sqrt{2}$. If two large integers x and y form a solution to the Pell equation

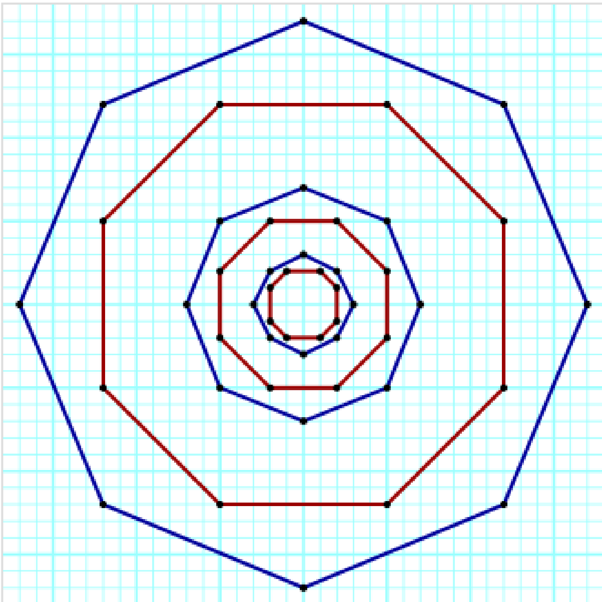
$$x^2 - 2y^2 = \pm 1,$$

then their ratio x/y provides a close approximation to $\sqrt{2}$. The sequence of approximations of this form is

$$\frac{1}{1}, \frac{3}{2}, \frac{7}{5}, \frac{17}{12}, \frac{41}{29}, \frac{99}{70}, \dots$$

where the denominator of each fraction is a Pell number and the numerator is the sum of a Pell number and its predecessor in the sequence. That is, the solutions have the form

$$\frac{P_{n-1} + P_n}{P_n}.$$



Rational approximations to regular octagons, with coordinates derived from the Pell numbers.

The approximation

$$\sqrt{2} \approx \frac{577}{408}$$

of this type was known to Indian mathematicians in the third or fourth century BCE.[3] The Greek mathematicians of the fifth century BCE also knew of this sequence of approximations:[4] Plato refers to the numerators as **rational diameters**.[5] In the second century CE Theon of Smyrna used the term the **side and diameter numbers** to describe the denominators and numerators of this sequence.[6]

These approximations can be derived from the continued fraction expansion of $\sqrt{2}$:

$$\sqrt{2} = [1; 2, 2, 2, \ldots] = 1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \ddots}}}.$$

Truncating this expansion to any number of terms produces one of the Pell-number-based approximations in this sequence; for instance, with seven 2s,

$$[1; 2, 2, 2, 2, 2, 2, 2] = \frac{577}{408}.$$

As Knuth (1994) describes, the fact that Pell numbers approximate $\sqrt{2}$ allows them to be used for accurate rational approximations to a regular octagon with vertex coordinates $(\pm P_i, \pm P_{i+1})$ and $(\pm P_{i+1}, \pm P_i)$. All vertices are equally distant from the origin, and form nearly uniform angles around the origin. Alternatively, the points $(\pm(P_i + P_{i-1}), 0)$, $(0, \pm(P_i + P_{i-1}))$, and $(\pm P_i, \pm P_i)$ form approximate octagons in which the vertices are nearly equally distant from the origin and form uniform angles.

Primes and squares

A **Pell prime** is a Pell number that is prime. The first few Pell primes are

> 2, 5, 29, 5741, 33461, 44560482149, 1746860020068409, 68480406462161287469, ... (sequence A086383 in the OEIS).

The indices of these primes within the sequence of all Pell numbers are

> 2, 3, 5, 11, 13, 29, 41, 53, 59, 89, 97, 101, 167, 181, 191, 523, 929, 1217, 1301, 1361, 2087, 2273, 2393, 8093, ... (sequence A096650 in the OEIS)

These indices are all themselves prime. As with the Fibonacci numbers, a Pell number P_n can only be prime if n itself is prime, because if d is a divisor of n then P_d is a divisor of P_n.

The only Pell numbers that are squares, cubes, or any higher power of an integer are 0, 1, and 169 = 13^2.[7]

However, despite having so few squares or other powers, Pell numbers have a close connection to square triangular numbers.[8] Specifically, these numbers arise from the following identity of Pell numbers:

$$\bigl((P_{k-1} + P_k) \cdot P_k\bigr)^2 = \frac{(P_{k-1} + P_k)^2 \cdot \bigl((P_{k-1} + P_k)^2 - (-1)^k\bigr)}{2}.$$

The left side of this identity describes a square number, while the right side describes a triangular number, so the result is a square triangular number.

Falcón and Díaz-Barrero (2006) proved another identity relating Pell numbers to squares and showing that the sum of the Pell numbers up to P_{4n+1} is always a square:

$$\sum_{i=0}^{4n+1} P_i = \left(\sum_{r=0}^{n} 2^r \binom{2n+1}{2r}\right)^2 = (P_{2n} + P_{2n+1})^2.$$

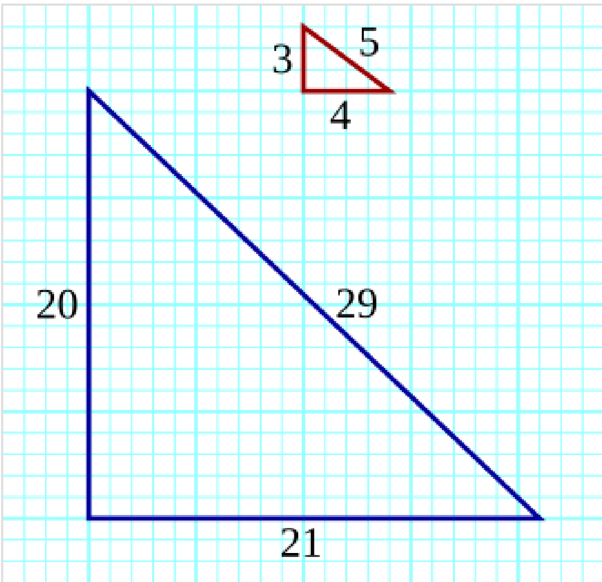
For instance, the sum of the Pell numbers up to P_5, 0 + 1 + 2 + 5 + 12 + 29 = 49, is the square of $P_2 + P_3 = 2 + 5 = 7$. The numbers $P_{2n} + P_{2n+1}$ forming the square roots of these sums,

1, 7, 41, 239, 1393, 8119, 47321, … (sequence A002315 in the OEIS),

are known as the Newman–Shanks–Williams (NSW) numbers.

Pythagorean triples

If a right triangle has integer side lengths *a*, *b*, *c* (necessarily satisfying the Pythagorean theorem $a^2 + b^2 = c^2$), then (*a*,*b*,*c*) is known as a Pythagorean triple. As Martin (1875) describes, the Pell numbers can be used to form Pythagorean triples in which *a* and *b* are one unit apart, corresponding to right triangles that are nearly isosceles. Each such triple has the form



Integer right triangles with nearly equal legs, derived from the Pell numbers.

$$\left(2P_nP_{n+1}, P_{n+1}^2 - P_n^2, P_{n+1}^2 + P_n^2 = P_{2n+1}\right).$$

The sequence of Pythagorean triples formed in this way is

(4,3,5), (20,21,29), (120,119,169), (696,697,985), …

Pell–Lucas numbers

The **companion Pell numbers** or **Pell–Lucas numbers** are defined by the recurrence relation

$$Q_n = \begin{cases} 2 & \text{if } n = 0; \\ 2 & \text{if } n = 1; \\ 2Q_{n-1} + Q_{n-2} & \text{otherwise.} \end{cases}$$

In words: the first two numbers in the sequence are both 2, and each successive number is formed by adding twice the previous Pell–Lucas number to the Pell–Lucas number before that, or equivalently, by adding the next Pell number to the previous Pell number: thus, 82 is the companion to 29, and $82 = 2 \times 34 + 14 = 70 + 12$. The first few terms of the sequence are (sequence A002203 in the OEIS): 2, 2, 6, 14, 34, 82, 198, 478, …

Like the relationship between Fibonacci numbers and Lucas numbers,

$$Q_n = \frac{P_{2n}}{P_n}$$

for all natural numbers n.

The companion Pell numbers can be expressed by the closed form formula

$$Q_n = \left(1+\sqrt{2}\right)^n + \left(1-\sqrt{2}\right)^n.$$

These numbers are all even; each such number is twice the numerator in one of the rational approximations to $\sqrt{2}$ discussed above.

Like the Lucas sequence, if a Pell–Lucas number $\frac{1}{2}Q_n$ is prime, it is necessary that n be either prime or a power of 2. The Pell–Lucas primes are

> 3, 7, 17, 41, 239, 577, … (sequence A086395 in the OEIS).

For these n are

> 2, 3, 4, 5, 7, 8, 16, 19, 29, 47, 59, 163, 257, 421, … (sequence A099088 in the OEIS).

Computations and connections

The following table gives the first few powers of the silver ratio $\delta = \delta_S = 1 + \sqrt{2}$ and its conjugate $\bar{\delta} = 1 - \sqrt{2}$.

n	$(1+\sqrt{2})^n$	$(1-\sqrt{2})^n$
0	$1 + 0\sqrt{2} = 1$	$1 - 0\sqrt{2} = 1$
1	$1 + 1\sqrt{2} = 2.41421\ldots$	$1 - 1\sqrt{2} = -0.41421\ldots$
2	$3 + 2\sqrt{2} = 5.82842\ldots$	$3 - 2\sqrt{2} = 0.17157\ldots$
3	$7 + 5\sqrt{2} = 14.07106\ldots$	$7 - 5\sqrt{2} = -0.07106\ldots$
4	$17 + 12\sqrt{2} = 33.97056\ldots$	$17 - 12\sqrt{2} = 0.02943\ldots$
5	$41 + 29\sqrt{2} = 82.01219\ldots$	$41 - 29\sqrt{2} = -0.01219\ldots$
6	$99 + 70\sqrt{2} = 197.9949\ldots$	$99 - 70\sqrt{2} = 0.0050\ldots$
7	$239 + 169\sqrt{2} = 478.00209\ldots$	$239 - 169\sqrt{2} = -0.00209\ldots$
8	$577 + 408\sqrt{2} = 1153.99913\ldots$	$577 - 408\sqrt{2} = 0.00086\ldots$
9	$1393 + 985\sqrt{2} = 2786.00035\ldots$	$1393 - 985\sqrt{2} = -0.00035\ldots$
10	$3363 + 2378\sqrt{2} = 6725.99985\ldots$	$3363 - 2378\sqrt{2} = 0.00014\ldots$
11	$8119 + 5741\sqrt{2} = 16238.00006\ldots$	$8119 - 5741\sqrt{2} = -0.00006\ldots$
12	$19601 + 13860\sqrt{2} = 39201.99997\ldots$	$19601 - 13860\sqrt{2} = 0.00002\ldots$

The coefficients are the half-companion Pell numbers H_n and the Pell numbers P_n which are the (non-negative) solutions to $H^2 - 2P^2 = \pm 1$. A square triangular number is a number

$$N = \frac{t(t+1)}{2} = s^2,$$

which is both the t-th triangular number and the s-th square number. A *near-isosceles Pythagorean triple* is an integer solution to $a^2 + b^2 = c^2$ where $a + 1 = b$.

The next table shows that splitting the odd number H_n into nearly equal halves gives a square triangular number when n is even and a near isosceles Pythagorean triple when n is odd. All solutions arise in this manner.

n	H_n	P_n	t	t+1	s	a	b	c
0	1	0	0	1	0			
1	1	1				0	1	1
2	3	2	1	2	1			
3	7	5				3	4	5
4	17	12	8	9	6			
5	41	29				20	21	29
6	99	70	49	50	35			
7	239	169				119	120	169
8	577	408	288	289	204			
9	1393	985				696	697	985
10	3363	2378	1681	1682	1189			
11	8119	5741				4059	4060	5741
12	19601	13860	9800	9801	6930			

Definitions

The half-companion Pell numbers H_n and the Pell numbers P_n can be derived in a number of easily equivalent ways.

Raising to powers

$$(1+\sqrt{2})^n = H_n + P_n\sqrt{2}$$

$$(1-\sqrt{2})^n = H_n - P_n\sqrt{2}.$$

From this it follows that there are *closed forms*:

$$H_n = \frac{(1+\sqrt{2})^n + (1-\sqrt{2})^n}{2}.$$

and

$$P_n\sqrt{2} = \frac{(1+\sqrt{2})^n - (1-\sqrt{2})^n}{2}.$$

Paired recurrences

$$H_n = \begin{cases} 1 & \text{if } n = 0; \\ H_{n-1} + 2P_{n-1} & \text{otherwise.} \end{cases}$$

$$P_n = \begin{cases} 0 & \text{if } n = 0; \\ H_{n-1} + P_{n-1} & \text{otherwise.} \end{cases}$$

Reciprocal recurrence formulas

Let n be at least 2.

$$H_n = (3P_n - P_{n-2})/2 = 3P_{n-1} + P_{n-2};$$

$$P_n = (3H_n - H_{n-2})/4 = (3H_{n-1} + H_{n-2})/2.$$

Matrix formulations

$$\begin{pmatrix} H_n \\ P_n \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} H_{n-1} \\ P_{n-1} \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}^n \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

So

$$\begin{pmatrix} H_n & 2P_n \\ P_n & H_n \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}^n.$$

Approximations

The difference between H_n and $P_n\sqrt{2}$ is

$$\left(1 - \sqrt{2}\right)^n \approx (-0.41421)^n,$$

which goes rapidly to zero. So

$$\left(1 + \sqrt{2}\right)^n = H_n + P_n\sqrt{2}$$

is extremely close to $2H_n$.

From this last observation it follows that the integer ratios $\frac{H_n}{P_n}$ rapidly approach $\sqrt{2}$; and $\frac{H_n}{H_{n-1}}$ and $\frac{P_n}{P_{n-1}}$ rapidly approach $1 + \sqrt{2}$.

$H^2 - 2P^2 = \pm 1$

Since $\sqrt{2}$ is irrational, we cannot have $\frac{H}{P} = \sqrt{2}$, i.e.,

$$\frac{H^2}{P^2} = \frac{2P^2}{P^2}.$$

The best we can achieve is either

$$\frac{H^2}{P^2} = \frac{2P^2 - 1}{P^2} \quad \text{or} \quad \frac{H^2}{P^2} = \frac{2P^2 + 1}{P^2}.$$

The (non-negative) solutions to $H^2 - 2P^2 = 1$ are exactly the pairs (H_n, P_n) with n even, and the solutions to $H^2 - 2P^2 = -1$ are exactly the pairs (H_n, P_n) with n odd. To see this, note first that

$$H_{n+1}^2 - 2P_{n+1}^2 = (H_n + 2P_n)^2 - 2(H_n + P_n)^2 = -\left(H_n^2 - 2P_n^2\right),$$

so that these differences, starting with $H_0^2 - 2P_0^2 = 1$, are alternately 1 and −1. Then note that every positive solution comes in this way from a solution with smaller integers since

$$(2P - H)^2 - 2(H - P)^2 = -\left(H^2 - 2P^2\right).$$

The smaller solution also has positive integers, with the one exception: $H = P = 1$ which comes from $H_0 = 1$ and $P_0 = 0$.

Square triangular numbers

The required equation

$$\frac{t(t+1)}{2} = s^2$$

is equivalent to $4t^2 + 4t + 1 = 8s^2 + 1,$ which becomes $H^2 = 2P^2 + 1$ with the substitutions $H = 2t + 1$ and $P = 2s$. Hence the n-th solution is

$$t_n = \frac{H_{2n} - 1}{2} \quad \text{and} \quad s_n = \frac{P_{2n}}{2}.$$

Observe that t and $t + 1$ are relatively prime, so that $\frac{t(t+1)}{2} = s^2$ happens exactly when they are adjacent integers, one a square H^2 and the other twice a square $2P^2$. Since we know all solutions of that equation, we also have

$$t_n = \begin{cases} 2P_n^2 & \text{if } n \text{ is even;} \\ H_n^2 & \text{if } n \text{ is odd.} \end{cases}$$

and $s_n = H_n P_n$.

This alternate expression is seen in the next table.

n	H_n	P_n	t	$t+1$	s	a	b	c
0	1	0						
1	1	1	1	2	1	3	4	5
2	3	2	8	9	6	20	21	29
3	7	5	49	50	35	119	120	169
4	17	12	288	289	204	696	697	985
5	41	29	1681	1682	1189	4059	4060	5741
6	99	70	9800	9801	6930	23660	23661	33461

Pythagorean triples

The equality $c^2 = a^2 + (a + 1)^2 = 2a^2 + 2a + 1$ occurs exactly when $2c^2 = 4a^2 + 4a + 2$ which becomes $2P^2 = H^2 + 1$ with the substitutions $H = 2a + 1$ and $P = c$. Hence the n-th solution is $a_n = \frac{H_{2n+1} - 1}{2}$ and $c_n = P_{2n+1}$.

The table above shows that, in one order or the other, a_n and $b_n = a_n + 1$ are H_nH_{n+1} and $2P_nP_{n+1}$ while $c_n = H_{n+1}P_n + P_{n+1}H_n$.

Notes

1. For instance, Sellers (2002) proves that the number of perfect matchings in the Cartesian product of a path graph and the graph $K_4 - e$ can be calculated as the product of a Pell number with the corresponding Fibonacci number.
2. For the matrix formula and its consequences see Ercolano (1979) and Kilic and Tasci (2005). Additional identities for the Pell numbers are listed by Horadam (1971) and Bicknell (1975).
3. As recorded in the Shulba Sutras; see e.g. Dutka (1986), who cites Thibaut (1875) for this information.
4. See Knorr (1976) for the fifth century date, which matches Proclus' claim that the side and diameter numbers were discovered by the Pythagoreans. For more detailed exploration of later Greek knowledge of these numbers see Thompson (1929), Vedova (1951), Ridenhour (1986), Knorr (1998), and Filep (1999).
5. For instance, as several of the references from the previous note observe, in Plato's Republic there is a reference to the "rational diameter of 5", by which Plato means 7, the numerator of the approximation $\frac{7}{5}$ of which 5 is the denominator.
6. Heath, Thomas Little (1921), *History of Greek Mathematics* (https://archive.org/details/historyofgreekma01heat/page/112/), vol. I, From Thales to Euclid, Oxford University Press, p. 112. Dover reprint, ISBN 9780486240732.
7. Pethő (1992); Cohn (1996). Although the Fibonacci numbers are defined by a very similar recurrence to the Pell numbers, Cohn writes that an analogous result for the Fibonacci numbers seems much more difficult to prove. (However, this was proven in 2006 by Bugeaud et al.)
8. Sesskin (1962). See the square triangular number article for a more detailed derivation.

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External links

- Weisstein, Eric W. "Pell Number" (https://mathworld.wolfram.com/PellNumber.html). *MathWorld*.
- OEIS sequence A001333 (Numerators of continued fraction convergents to sqrt(2)) (https://oeis.org/A001333)—The numerators of the same sequence of approximations