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3D Irregular Packing in an Optimized Cuboid Container

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Abstract. A packing problem for irregular 3D objects approximated by polyhedra is presented. The objects have to be packed into a cuboid of minimum height under continuous rotations, translations and minimum allowable distances between objects. The problem has various applications and arises, e.g. in additive manufacturing. Containment, distance and non-overlapping constraints are described using the phi-function technique. The irregular packing problem is formulated in the form of nonlinear programming problem. A solution algorithm is proposed based on a fast starting point algorithm and efficient local optimization procedure.

*Keywords***:** *operation research; irregular packing; mathematical model; nonlinear optimization; 3D printing.*

1. INTRODUCTION

Packing problems have a wide spectrum of applications, e.g. in materials science, nanotechnology, robotics, medicine, control systems, space engineering, mechanical engineering, aircraft construction. In this paper we consider the problem of packing a collection of irregular objects into a cuboid of minimum height. One of interesting applications of packing problems is in additive manufacturing (AM) (see, e.g., Canellidis et.al., (2010), Gardan, J. (2016), Chen et al., (2015), Araújo et al., (2018) and the references therein)

In Araújo et al., (2018) they review existing general cutting and packing taxonomies and provides a new specification which is more appropriate for classifying the problems encountered in AM. The build volume packing task in AM can be formulated as the open dimension 3D irregular packing problem (Wascher et al. (2007)).

Packing problems are NP-hard and thus solution methodologies generally employ heuristics (see, e.g., Chen et al, (2014), Egeblad et al., (2009), Fasano, (2013), Li et al., (2010), Liu et al., (2015), Smeets et al., (2015), Verkhoturov et al., (2016), and the references therein).

Our approach is based on the analytical descriptions of relations between irregular objects and results in the packing problem formulated as a nonlinear programming problem. We use the phi-function technique (see Stoyan and Romanova, (2013), Stoyan et al., (2015), Stoyan, et al., (2016), Romanova et al., (2018) to describe packing constraints (continuous rotations, non-overlapping and containment of objects into a container, allowable distances between objects). Phi-function approach for a specific 3D packing problem was used earlier in Romanova et al., (2018). In this paper we study another packing problem and propose an efficient starting point algorithm to fasten the overall computations.

2. PROBLEM FORMULATION

We consider here the packing problem in the following setting. Let Ω denote a container, $\Omega=\{(x,y,z)\in R^3 : 0\le x\le l, 0\le y\le w, 0\le z\le h\}$ of variable height h. Let $\{1,2,\dots,N\}=J_N$ and a collection of irregular objects Q_q, $q\in J_N$ be given.

We assume that each object Q_q is approximated by polyhedra with the given accuracy. With each object Q_q we associate its local coordinate system with origin denoted by v_q. The motion of each object Q is defined by a variable vector u of its placement parameters, including a translation vector $v=(x,y,z)$ and a vector θ $=(\theta^1,\theta^2,\theta^3)$ of rotation parameters.

Let $Q(u)=\{p\in R^3 : p=v+M(\theta)\cdot p^0, p^0\in Q^0\}$, where $u=(v,\theta)$, Q^0 denotes the non-translated and non-rotated object Q, $M(\theta)$ is a rotation matrix

$$M(\theta^1,\theta^2,\theta^3)=\begin{pmatrix}\mu_{11} & \mu_{12} & \mu_{13}\\ \mu_{21} & \mu_{22} & \mu_{23}\\ \mu_{31} & \mu_{32} & \mu_{33}\end{pmatrix},$$

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$$\mu_{11} = \cos\theta^1 \cos\theta^3 - \sin\theta^1 \cos\theta^2 \sin\theta^3,$$
$$\mu_{12} = -\cos\theta^1 \sin\theta^3 - \sin\theta^1 \cos\theta^2 \cos\theta^3,$$
$$\mu_{13} = \sin\theta^1 \sin\theta^2,$$
$$\mu_{21} = \sin\theta^1 \cos\theta^3 + \cos\theta^1 \cos\theta^2 \sin\theta^3,$$
$$\mu_{22} = -\sin\theta^1 \sin\theta^3 + \cos\theta^1 \cos\theta^2 \cos\theta^3,$$
$$\mu_{23} = -\cos\theta^1 \sin\theta^2,$$
$$\mu_{31} = \sin\theta^2 \sin\theta^3,\ \mu_{32} = \sin\theta^2 \cos\theta^3,$$
$$\mu_{33} = \cos\theta^2.$$

According to 3D printing technology between each pair of objects Q_q and Q_g, $q < g \in J_N$ as well as, between an object Q_q, $q \in J_N$, and the frontier of container Ω appropriate minimum allowable distances are given. Let $\rho_{qg} > 0$ denote minimum allowable distance between objects Q_q and Q_g and $\rho_q > 0$ denote minimum allowable distance between objects Q_q and $\Omega^* = R^3 \setminus \operatorname{int}\Omega$.

The irregular packing problem may be defined as follows: Pack the set of 3D objects Q_q, $q \in J_N$, within a rectangular container Ω of minimum height, taking into account the given minimum allowable distances.

Assuming that each non-convex object Q_q may be presented as a union of n_q convex objects, we form a set of $n = \sum_{q=1}^{N} n_q$ convex objects K_i, $i \in \{1, 2, \ldots, n\} = I_n$ and vector $\mathbf{a} = (a_1, \ldots, a_n)$, $a_i \in J_N$, where $a_i = q$, if K_i takes part in composition of an object Q_q, $q \in J_N$. Let $I_n = I^1 \cup I^2 \cup \ldots \cup I^N$ be an ordered partition of I_n, where $I^q = \{i \in I_n, a_i = q\}$, $card(I^q) = n_q$, $q \in J_N$. Let each convex object approximated by a convex polyhedron $K_i \subset Q_q$ be defined by its vertices p_s^i, $s = 1, \ldots, m_i$, $i \in I_n$, in the local coordinate system of the object Q_q.

3. MATHEMATICAL MODELING

Let us consider two placement constraints of the irregular packing problem:

$$\operatorname{dist}(Q_q, Q_g) = \min_{a \in Q_q, b \in Q_g} d(a, b) \ge \rho_{qg}, \tag{1}$$

$$\operatorname{dist}(Q_q, \Omega^*) = \min_{a \in Q_q, b \in \Omega^*} d(a, b) \ge \rho_q, \tag{2}$$

where $d(a, b)$ stands for the Euclidean distance between two points $a, b \in R^3$.

To formalize the distance constraints we employ adjusted quasi-phi-functions and adjusted phi-functions. See papers Stoyan and Romanova, (2013) and Stoyan et al., (2016) for definitions of these functions.

Firstly we define the adjusted quasi phi-function to describe distance constraint (1).

The adjusted quasi-phi-function for non-convex objects Q_q and Q_g has the form

$$\widehat{\boldsymbol{\Phi}}'_{qg}(u_q, u_g, u_{qg}) = \min\{\widehat{\Phi}'_{ij}(u_q, u_g, u'_{ij}), i \in I^q, j \in I^g\}, \tag{3}$$

where

$$\widehat{\Phi}'_{ij}(u_q, u_g, u'_{ij} = u_P) = \min\{\Phi^{K_i P}(u_q, u_P), \Phi^{K_j P^*}(u_g, u_P)\} - 0.5\rho_{qg}, \tag{4}$$

is the adjusted quasi-phi-function for convex objects $K_i(u_q)$ and $K_j(u_g)$, $i \in I^q, j \in I^g$, $u_{qg} = (u'_{ij}, i \in I^q, j \in I^g)$, u'_{ij} is a vector of auxiliary variables.

In (4) $\Phi^{K_i P}(u_q, u_P)$ is a phi-function for $K_i(u_q)$ and a half-space

$$P(u_P) = \{(x, y, z): \psi_P = \alpha \cdot x + \beta \cdot y + \gamma \cdot z + \mu_P \le 0\},$$

where $\alpha = \sin\theta_P^1 \sin\theta_P^2$, $\beta = -\cos\theta_P^1 \sin\theta_P^2$, $\gamma = \cos\theta_P^2$, θ_P^1 and θ_P^2 are appropriate variable Euler angles (under $\theta_P^3 = 0$), $u_P = (\theta_P^1, \theta_P^2, \mu_P)$,

$\Phi^{K_j P^*}(u_g, u_P)$ is a phi-function for $K_j(u_g)$ and a half-space $P^*(u_P) = R^3 \setminus \operatorname{int} P(u_P)$,

$$\Phi^{K_i P}(u_q, u_P) = \min_{1 \le k \le m_i} \psi_P(p_k^i),$$

$$\Phi^{K_j P^*}(u_g, u_P) = \min_{1 \le l \le m_j} (-\psi_P(p_l^j)).$$

Based on (3)-(4), we can conclude that

$$\widehat{\boldsymbol{\Phi}}'_{qg}(u_q, u_g, u_{qg}) \ge 0 \Rightarrow \operatorname{dist}(Q_q, Q_g) \ge \rho_{qg}.$$

Then we define the adjusted phi-function to describe distance constraint (2).

An adjusted phi-function for non-convex object $Q_q(u_q)$ and Ω^* can be defined as

$$\widehat{\boldsymbol{\Phi}}_q(u_q) = \min\{\widehat{\Phi}_i(u_q), i \in I^q\}, \tag{5}$$

where

$$\widehat{\Phi}_i(u_q) = \Phi^{K_i \Omega^*}(u_q) - \rho_q, \tag{6}$$

is an adjusted phi-function for objects $K_i(u_q)$ and Ω^*,

$$\Phi^{K_i\Omega^*}(u_q) = \min\{\min_{1\le k\le m_i} \varphi^i_{kj}(u_q),$$
$$j = 1,\ldots,6\},$$
$$\varphi^i_{k1}(u_q) = x_q + p^i_{xk},$$
$$\varphi^i_{k2}(u_q) = -(x_q + p^i_{xk}) + l,$$
$$\varphi^i_{k3}(u_q) = y_q + p^i_{yk},$$
$$\varphi^i_{k4}(u_q) = -(y_q + p^i_{yk}) + w,$$
$$\varphi^i_{k5}(u_q) = z_q + p^i_{zk},$$
$$\varphi^i_{k6}(u_q) = -(z_q + p^i_{zk}) + h.$$

Based on (5)-(6), we can conclude that

$$\widehat{\mathbf{\Phi}}_q(u_q) \ge 0 \Rightarrow \operatorname{dist}(Q_q, \Omega^*) \ge \rho_q.$$

The vector $u \in R^\sigma$ of all variables can be described as follows: $u = (\varsigma, \tau) \in R^\sigma$, where $\varsigma = (h, u_1, u_2, \ldots, u_N)$, h denote the variable height of the cuboid Ω and $u_{a_i} = (v_{a_i}, \theta_{a_i}) = (x_{a_i}, y_{a_i}, z_{a_i}, \theta^1_{a_i}, \theta^2_{a_i}, \theta^3_{a_i})$ is the vector of placement parameters of K_i, $i \in I_n$, an index $a_i \in \{1, 2, \ldots, N\}$ is a component of the vector $\mathbf{a}$, vector $\tau = (u^1_P, \ldots, u^m_P)$ contains auxiliary variables u^s_P for the s-th pair of convex polyhedral objects, $s = 1, \ldots, m$, $m = card(\Xi)$,

$$\Xi = \{(i, j), a_i \ne a_j, i < j = 1, \ldots, n\}. \tag{7}$$

The total number of variables is $\sigma = 1 + 6N + 3m$.
A mathematical model of the irregular packing problem can be stated in the form

$$\min_{u \in W \subset R^\sigma} F(u), \tag{8}$$

$$W = \{u \in R^\sigma : \widehat{\Phi}'_{ij}(u_{a_i}, u_{a_j}) \ge 0, \\ (i, j) \in \Xi, \widehat{\Phi}_i(u_{a_i}) \ge 0, i = 1, 2, \ldots, n\}, \tag{9}$$

where $F(u) = h$, $\widehat{\Phi}'_{ij}(u_{a_i}, u_{a_j})$ is an adjusted quasi-phi-function (4) for objects K_i and K_j, taking into account minimal allowable distance $\rho_{qg} > 0$, $a_i, a_j \in J_N$, under $(i, j) \in \Xi$, Ξ is given by (7), $\widehat{\Phi}_i(u_{a_i})$ is an adjusted phi-function (6) for objects K_i and Ω^*, taking into account minimal allowable distance $\rho_q > 0$.

Each quasi-phi-function inequality in (9) is presented by a system of inequalities with smooth functions. Our model (8)-(9) is a non-convex and continuous nonlinear programming problem.

In order to search for locally optimal packings within a reasonable computational time we propose the following solution algorithm.

4. A SOLUTION ALGORITHM

We employ multi-start strategy that involves the following stages:

Stage 1. Generate a set of vectors of feasible placement parameters of objects placed into the container of height h^0 in the problem (8)-(9).

Stage 2. Search for a local minimum of the objective function $F(u)$ in problem (8)-(9), starting from each point obtained at *Stage* 1.

Stage 3. Choose the best local minimum from those found at *Stage* 2.

The actual search for a local minimum in all optimisation procedures is performed by IPOPT (Wachter and Biegler, (2006)), which is available at an open access noncommercial software depository (https://projects.coin-or.org/Ipopt).

4.1 Starting point algorithm

One of the most popular approaches to get an initial feasible solution consists in a certain arrangement of items placed then one by one in a given order. A position of a new item is based on optimization of a certain merit function. This process terminates when either all items are placed into the container or no space is available for a new item.

Searching for a feasible placement for the next item is a central part of the algorithm. This process is subdivided into a number of cyclically repeated procedures: generation of a candidate for a feasible point and detection of a feasibility of the point. There are several approaches to simplify the procedures, e.g., grid algorithms. However, these approaches are characterized by high consumption of computer resources and require high precision approximations.

We propose a clear and extremely fast starting point algorithm that employs optimization method by groups of variables with respect to the given object sequence. Items and the container are approximated by cuboids of identical small width/height and variable length.

Let us consider a more detailed description of the algorithm.

The key idea of our approach is to approximate both items and containers by elementary cuboids (Δ-cuboids) of identical width Δ and height Δ but different length. Faces of the Δ-cuboids are parallel to coordinate planes of the fixed coordinate system.

Each Δ-cuboid can be entirely characterized by four numbers $\alpha, \beta, \gamma, \tau$:

$$\Delta = \{(x, y, z) \in R^3 : \\ \gamma\Delta x \le x \le (\gamma+1)\Delta x, \tau\Delta z \le z \le (\tau+1)\Delta z, \\ \alpha\Delta y \le y \le \beta\Delta y\} \leftrightarrow [\alpha, \beta, \gamma, \tau].$$

We call such approximation as Δ-approximation. We also differentiate two types of Δ-approximation: an external approximation $P^+ \supset Q$, and an internal approximation $P^- \subset Q$. This approximation guarantees that all placement constraints are satisfied (items become a little bit larger while the container becomes a little bit smaller).

In should be noted that the union, intersection and difference of two Δ-approximated items are also Δ-approximated items.

We take a set of discrete rotations $\theta_m = (\theta_m^1, \theta_m^2, \theta_m^3), m = 1, 2, \ldots, M$ and construct an object $P^- \subset \Omega$ and objects $P_{im}^+ \supset Q_{im}$, $i = 1, \ldots, n.$, $m = 1, 2, \ldots, M$. Here by Q_{im} we denote the object Q_i rotated by angle θ_m and P_{im}^+ is the external Δ-approximation of Q_{im}. Objects $P_i^+ \supset \square_i$, $i = 1, \ldots, n.$ are packed into the object P^{d-} sequentially according to a given sequence $A_j = (a_{j_1}, a_{j_2}, \ldots, a_{j_n})$. We try to pack all objects $P_{j_i m}^+$ on the step i and choose the best variant.

As a result a feasible point $u^0 \in W$ is found.

Generating candidate(s) for a feasible point and detecting feasibility of the point is reduced to a sizes comparison for elementary cuboids.

We nest the first object and construct domain $P_1^- = P^- \setminus P_{j1}^+(u_{j1})$. Then we look for a feasible point for P_{j2}^+ in P_{j1}^-, and construct domain $P_2^- = P_1^- \setminus P_{j2}^+(u_{j2})$. This procedure is repeated until all items are nested within the object P^-. This approach allows us to simplify the procedure of searching for a feasible point to pack the next item.

The algorithm returns a set of vectors of feasible placement parameters $\{u_1^0, u_2^0, \ldots, u_N^0\}$ of objects Q_q, $q \in J_N$, fully placed into the container Ω^0 of height h^0, taking into account distance constraints (1), (2).

4.2 Local optimization algorithm

To search for local optimal solution of problem (8)-(9) we apply algorithm presented in Romanova et al., (2018).

We assume that spheres $S_q^0 \equiv S_q(0)$ of radius r_q and the center point $v_q = (x_q, y_q, z_q)$, circumscribed around each non-translated and non-rotated non-convex polyhedron Q_q, $q = 1, \ldots, N$, as well as, spheres $S_i^0 \equiv S_i(0)$ of radius r_i and the center point $v_{ci} = (x_{ci}, y_{ci}, z_{ci})$ circumscribed around each non-translated and non-rotated convex polyhedron K_i^0, $i = 1, \ldots, n$, are constructed. Let $\varsigma^0 = (h^0, u_1^0, \ldots, u_N^0)$ be the vector of feasible placement parameters of objects $Q_q, q = 1, 2, \ldots, N$, within the container Ω^0.

The algorithm is an iterative procedure and involves the following steps.

Step 1. Assume $k = 1$.

Step 2. Derive the appropriate vector $(v_{c1}^{(k-1)}, \ldots, v_{cn}^{(k-1)})$ of center points of spheres $S_i(u_{a_i}^{(k-1)})$, $i = 1, 2, \ldots, n$.

Step 3. For each sphere $S_i(u_{a_i}^{(k-1)})$ we construct a fixed individual container $\Omega_i^k \supset S_i \supset K_i$ with equal half-sides of length $r_i + \varepsilon$, $i = 1, 2, \ldots, n$, and the center of symmetry point $v_{ci}^{(k-1)}$, assuming $\varepsilon = \sum_{i=1}^{n} r_i / n$.

Step 4. Generate an artificial subset Λ_k^ε described by an inequality system of additional constraints on the translation vector $v_{a_i} = (x_{a_i}, y_{a_i}, z_{a_i})$ of each polyhedron K_i in the form: $\Phi^{S_i \Omega_i^*}(v_{a_i}, v_{a_i}^0) \ge 0$, $i = 1, 2, \ldots, n$, where $\Phi^{S_i \Omega_i^*}(v_{a_i}, v_{a_i}^0)$ is the phi-function for sphere S_i and the object $\Omega_i^* = R^3 \setminus \operatorname{int} \Omega_i$.

Step 5. Form two index sets Ξ_1^k and Ξ_2^k.

The first set Ξ_1^k involves such pairs (i, j) of indexes of convex objects $K_i(u_{a_i}^{(k-1)})$ and $K_j(u_{a_j}^{(k-1)})$ whose individual containers intersect, i.e.

$$\Xi_1^k = \{(i, j) \in \Xi_1^{kS} : \varphi^{\Omega_{ki} \Omega_{kj}}(v_{a_i}, v_{a_j}) < 0\},$$

where $\Xi_1^{kS} = \{(i, j) \in \Xi : \widehat{\Phi}_{ij}(v_{a_i}, v_{a_j}) < 0\}$,

$$\varphi^{\Omega_i \Omega_j}(v_{a_i}, v_{a_j}) = \max\{\varphi_{ij}^s(v_{a_i}, v_{a_j}), s = 1, \ldots, 6\}$$

$$\varphi_{ij}^1(v_{a_i}, v_{a_j}) = (x_i^{(k-1)} - x_j^{(k-1)}) - R_{ij},$$

$$\varphi_{ij}^2(v_{a_i}, v_{a_j}) = (y_i^{(k-1)} - y_j^{(k-1)}) - R_{ij},$$

$$\varphi_{ij}^3(v_{a_i}, v_{a_j}) = (z_i^{(k-1)} - z_j^{(k-1)}) - R_{ij},$$

$$\varphi_{ij}^4(v_{a_i}, v_{a_j}) = -(x_i^{(k-1)} - x_j^{(k-1)}) - R_{ij},$$

$$\varphi_{ij}^5(v_{a_i}, v_{a_j}) = -(y_i^{(k-1)} - y_j^{(k-1)}) - R_{ij},$$

$$\varphi_{ij}^6(v_{a_i}, v_{a_j}) = -(z_i^{(k-1)} - z_j^{(k-1)}) - R_{ij},$$

$$R_{ij} = (r_i + r_j) + \rho_{ij} + 2\varepsilon,$$

$\widehat{\Phi}_{ij}(v_{a_i}, v_{a_j})$ is an adjusted phi-function for spheres S_q and S_g, ($a_i = q, a_j = g$).

The second set Ξ_2^k involves such indexes i of convex objects $K_i(u_{a_i}^{(k-1)})$ whose individual containers intersect the frontier of Ω^*, i.e.

$$\Xi_2^k = \{i \in \Xi_2^{kS} : \hat{\Phi}^{\Omega_i \Omega_\varepsilon^*}(v_{a_i}) < 0\},$$

where $\hat{\Phi}^{\Omega_i \Omega_\varepsilon^*}(v_{a_i})$ is an adjusted phi-function for objects $K_i(u_{a_i}^{(k-1)})$ and $\Omega_\varepsilon^* = R^3 \setminus \operatorname{int} \Omega_\varepsilon$,

$$\Omega_\varepsilon = \{(x, y, z) : \varepsilon \le x \le l^{(k-1)} - \varepsilon,$$
$$\varepsilon \le y \le w^{(k-1)} - \varepsilon, \varepsilon \le z \le h^{(k-1)} - \varepsilon\},$$

$$\Xi_2^{kS} = \{i \in I_n : \hat{\Phi}^{S_{a_i} \Omega_\varepsilon^*}(v_{a_i}) < 0\},$$

$\hat{\Phi}^{S_{a_i} \Omega_\varepsilon^*}(v_{a_i})$ is an adjusted phi-function (13) for a ball S_q, associated with object $Q_q(u_q^{(k-1)}) \supset K_i(u_{a_i}^{(k-1)})$, and the object Ω_ε^*, $a_i = q$,

$$\hat{\Phi}^{\Omega_i \Omega_\varepsilon^*}(v_{a_i}) = \min\{\psi_i^s(v_{a_i}), s = 1, \dots, 6\},$$
$$\psi_i^1(v_{a_i}) = x_i^{(k-1)} - R_i,$$
$$\psi_i^2(v_{a_i}) = y_i^{(k-1)} - R_i,$$
$$\psi_i^3(v_{a_i}) = z_i^{(k-1)} - R_i,$$
$$\psi_i^4(v_{a_i}) = -x_i^{(k-1)} + l - R_i,$$
$$\psi_i^5(v_{a_i}) = -y_i^{(k-1)} + w - R_i,$$
$$\psi_i^6(v_{a_i}) = -z_i^{(k-1)} + h - R_i,$$
$$R_i = r_i + \rho_q + 2\varepsilon.$$

We note that if $(i, j) \notin \Xi_1$, then we do not need to check the distance constraint for the corresponding pair of convex polyhedral objects $K_i(u_{a_i}^{(k-1)})$ and $K_j(u_{a_j}^{(k-1)})$; if $i \notin \Xi_2$, then we do not need to check the distance (or containment) constraint for the a polyhedron $K_i(u_{a_i}^{(k-1)})$ and the object Ω^*.

Step 6. Generate the following k-th subproblem on a subset $W_k = W \cap \Lambda_k^\varepsilon$

$$\min_{u_{w_k} \in W_k \subset R^{\sigma - \sigma_k}} F(u_{w_k}), \qquad (10)$$

where

$$W_k = \{u_{w_k} = (\varsigma, \tau_{w_k}) \in R^{\sigma - \sigma_k} :$$
$$\hat{\Phi}'_{ij}(u_{a_i}, u_{a_j}) \ge 0, (i, j) \in \Xi_1^k,$$
$$\hat{\Phi}_i(u_{a_i}) \ge 0, i \in \Xi_2^k,$$
$$\Phi^{S_i \Omega_{ki}^*}(u_{a_i}) \ge 0, i = 1, \dots, n,$$
$$h \ge h^{(k-1)} - \varepsilon\},$$
$$\sigma_k = 3(m - card(\Xi_1^k)).$$

Step 7. Generate a feasible starting point $u^{(k-1)} = (\varsigma^{(k-1)}, \tau_{w_k}^{(k-1)})$ for the problem (10). Since a vector $\varsigma^{(k-1)}$ is defined above, we find values for the vector of additional variables $\tau_{w_k}^{(k-1)} = (u_P^{(k-1)1}, \dots, u_P^{(k-1)s}, \dots, u_P^{(k-1)m})$ for such $s \in \{1, \dots, m\}$ that $(i, j) \in \Xi_1^k$.

To derive a vector $u_P^{(k-1)s}$ we search for a vector $u_P^{(k-1)s}$ as a vector of feasible parameters of a separating plane for $K_i(u_{a_i}^{(k-1)})$ and $K_j(u_{a_j}^{(k-1)})$, using simple geometrical calculations.

Step 8. Solve the problem (10), starting from the feasible point $u^{(k-1)}$

$$\min_{u_{w_k} \in W_k \subset R^{\sigma - \sigma_k}} F(u_{w_k}),$$

and get a local minimum point $u_{w_k}^* = (\varsigma^{*k}, \tau_{w_k}^{*k})$.

If $u_{w_k}^* \in fr\Lambda_k^\varepsilon$, then we take ς^{*k} as a starting vector ς^k for the next iteration of the algorithm (go to Step 2), otherwise we stop our iterative procedure.

The point $u^* = u^{*k} = (\varsigma^{*k}, \tau^{*k}) \in R^\sigma$ is a point of local minimum of the problem (8)-(9).

Our algorithm typically takes into account only a small subset of all pairs of convex objects since for each convex object only its "ε-neighbors" are involved in consideration. The problem (8)-(9) involves optimization over the set W described by (9) and has a large number of constraints and $O(n^2)$ variables. Our solution approach reduces this problem to a number of optimization subproblems (10) over the set W_k having a smaller number of constraints and only $O(n)$ variables.

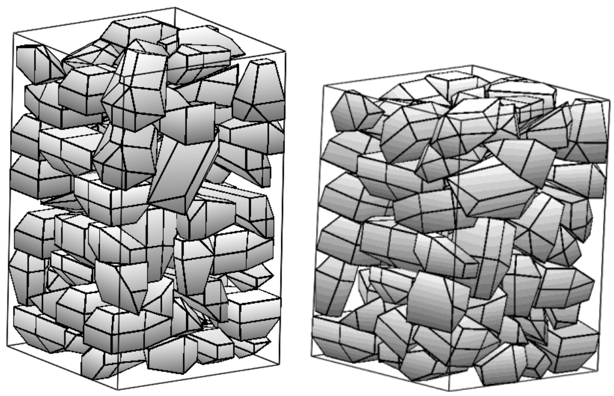
5. COMPUTATIONAL RESULTS

We present two instances to demonstrate the efficiency of our methodology. We have run all experiments on an AMD Athlon 64 X2 5200+ computer, and for the local optimisation we use the IPOPT code (https://projects.coin-or.org/Ipopt).

For each instance we provide: a) a feasible solution h^0 found by our starting point algorithm; b) a local optimal solution h^* found by the decomposition algorithm, starting from the point found by the starting point algorithm.

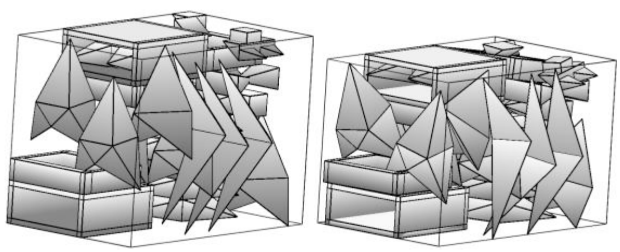
Instance 1. Packing n=80 convex objects.
The container sizes are
a) *l=40, w=40,* ***h⁰= 66.086159*** (we perform 100 *runs,* 2.04 sec per each run, totally 204 *sec.*)
b) *l= 40, w=40,* ***h*= 54.450827*** (2.4 *h.*)

Instance 2. Packing N=30 non-convex objects.
The container sizes are
a) *l=30, w=49,* ***h⁰=35*** (we perform 100 *runs,* 2.52 sec per each run , totally 252 *sec.*)
b) *l=30, w=49,* ***h*=31.564563*** (3.5 *h.*)



a b

Fig.1. Packings of convex objects: a) a feasible solution; b) a local optimal solution.



a b

Fig.2. Packings of non-convex objects: a) a feasible solution; b) a local optimal solution.

6. CONCLUSIONS

From the computational experiment we may conclude that our approach provides a good starting feasible packing very fast, in a few minutes. To improve this solution by a local search much more time is required. So if a real-time reasonable packing decision is required, a starting point heuristic can be applied. Otherwise off-line quality packing can be obtained by combining the starting point heuristic with a local search.

REFERENCES

Araújo, L.J.P. Özcan, E., Atkin, J.A.D., Baumers, M. (2018). Analysis of irregular three-dimensional packing problems in additive manufacturing: a new taxonomy and dataset. *International Journal of Production Research*.

Canellidis, V, J Giannatsis, and V Dedoussis. (2010). Effective nesting of Layer Manufacturing fabricated parts using a Genetic Algorithm and a bottom-left ray casting procedure. In Industrial Engineering and Engineering Management (IEEM), IEEE International Conference on, 547-551. IEEE.

Chen, E.R., Klotsa, D., Engel, M., Damasceno, P.F., Glotzer, S.C. (2014). Complexity in surfaces of densest packings for families of polyhedra. Phys Rev X **4**(1).

Chen, X., Zhang, H., Lin, J., Hu, R., Lu, L., Huang Q.-X., Benes, B., Cohen-Or D., Chen, B. (2015) Dapper: decompose-and-pack for 3D printing. ACM Trans. Graph. **34(**6), 213:1–213:12. 2

Gardan, J. (2016) Additive manufacturing technologies: state of the art and trends. *International Journal of Production Research,* 54 (10): 3118-3132.

Egeblad, J., Nielsen, B.K., Brazil, M. (2009). Translational packing of arbitrary polyhedra. Comp. Geom., **42**(4), 269–288.

Fasano, G. A. (2013). Global Optimisation point of view for non-standard packing problems. J. Glob. Optim., **55**(2), 279–299.

Li, S. X., Zhao, J., Lu, P., Xie, Y. (2010). Maximum packing densities of basic 3D objects. Chin. Scien. Bull., **55**(2), 114–119.

Liu, X., Liu, J., Cao, A., Yao, Z. (2015). HAPE3D – a new constructive algorithm for the 3D irregular packing problem. Frontiers of Information Technology & Electronic Engineering, **16(**5), 380–390.

Romanova T., Bennell J., Stoyan Y., Pankratov, A. (2018). Packing of non-convex polyhedra with continuous rotations using nonlinear optimization. *European J. of Operational Research*, 268, 37–53.

Smeets, B., Odenthal, T., Vanmaercke, S., Ramon, H. (2015). Polygon-based contact description for modeling arbitrary polyhedra in the Discrete Element Method. Computer Methods in Applied Mechanics and Engineering, **290**, 277-289.

Stoyan, Y., Pankratov, A., Romanova, T. (2016). Quasi-phi-functions and optimal packing of ellipses. J. of Glob. Optim., **65** (2), 283–307.

Stoyan, Yu., Pankratov, A., Romanova, T., Chugay, A. (2015). Optimized object packings using quasi-phi-functions. Chapter in contributed book "*Optimized Packings and Their Applications*", Editors/ G. Fasano and J.Pintér/ Springer Optimization and its Applications. – New York, 105, 265-291.

Stoyan, Yu., Romanova, T. (2013). Mathematical models of placement optimization: two- and three-dimensional problems and applications. Chapter in contributed book "*Modeling and Optimization in Space Engineering*", Editors/ G. Fasano and J.Pintér/ Springer Optimization and its Applications.73, 363-388.

Verkhoturov, M., Petunin, A., Verkhoturova, G., Danilov, K., Kurennov, D. (2016). The 3D Object Packing Problem into a Parallelepiped Container Based on Discrete-LogicalRepresentation. IFAC-PapersOnLine **49** (12): 1-5..

Wachter, A., Biegler, L. T. (2006). On the implementation of an interior-point filter line-search algorithm for large-scale nonlinear programming. Math. Program., **106** (1), 25–57.

Wascher, G., Hauner, H., Schumann, H. (2007). An improved typology of cutting and packing problems. Eur. J. Oper. Res., **183**(3), 1109–1130.