Faster Optimistic Online Mirror Descent for Extensive-form Games

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A Proof of Theorem 1

Lemma 1. *(Lemma 26 in [30]) Let $\mathcal{X} \in \mathbb{R}^n$ be a close convex set. Let $r : \mathcal{X} \to \mathbb{R}$ be a convex differentiable function. Assume that $\boldsymbol{x}^+ := \operatorname{argmin}_{\boldsymbol{x} \in \mathcal{X}} \langle \boldsymbol{l}, \boldsymbol{x} \rangle + r(\boldsymbol{x})$ for any $\boldsymbol{l} \in \mathbb{R}^n$ and $\boldsymbol{x}' \in \mathcal{X}$. Then, for any $\boldsymbol{x} \in \mathcal{X}$,*

$$\langle \boldsymbol{l}, \boldsymbol{x} - \boldsymbol{x}^+ \rangle + r(\boldsymbol{x}) - r(\boldsymbol{x}^+) \geq \mathcal{B}_r(\boldsymbol{x} \| \boldsymbol{x}^+). \tag{1}$$

Lemma 2. *(Lemma 27 in [30]) Let $\mathcal{X} \in \mathbb{R}^n$ be a close convex set. Let $r : \mathcal{X} \to \mathbb{R}$ be a convex differentiable function and let $q : \mathcal{X} \to \mathbb{R}$ be a proper and differentiable function. Assume that $\boldsymbol{x}^+ := \operatorname{argmin}_{\boldsymbol{x} \in \mathcal{X}} \langle \boldsymbol{l}, \boldsymbol{x} \rangle + q(\boldsymbol{x}) + \mathcal{B}_r(\boldsymbol{x} \| \boldsymbol{x}')$ for any $\boldsymbol{l} \in \mathbb{R}^n$ and $\boldsymbol{x}' \in \mathcal{X}$. Then, for any $\boldsymbol{x} \in \mathcal{X}$,*

$$\langle \boldsymbol{l}, \boldsymbol{x} - \boldsymbol{x}^+ \rangle + q(\boldsymbol{x}) - q(\boldsymbol{x}^+) + \mathcal{B}_r(\boldsymbol{x} \| \boldsymbol{x}') - \mathcal{B}_r(\boldsymbol{x}^+ \| \boldsymbol{x}') \geq \mathcal{B}_{r+q}(\boldsymbol{x} \| \boldsymbol{x}^+). \tag{2}$$

Proof. **Ada-OOMD**: Firstly,

$$\sum_{t=1}^{T} \langle \boldsymbol{l}^t, \boldsymbol{x}^t - \boldsymbol{x}' \rangle = \underbrace{\sum_{t=1}^{T} \langle \boldsymbol{m}^t, \boldsymbol{x}^t - \boldsymbol{z}^{t+1} \rangle}_{\mathbf{Term}_1} + \underbrace{\sum_{t=1}^{T} \langle \boldsymbol{l}^t, \boldsymbol{z}^{t+1} - \boldsymbol{x}' \rangle}_{\mathbf{Term}_2} + \underbrace{\sum_{t=1}^{T} \langle \boldsymbol{l}^t - \boldsymbol{m}^t, \boldsymbol{x}^t - \boldsymbol{z}^{t+1} \rangle}_{\mathbf{Term}_3}. \tag{3}$$

For $\mathbf{Term}_1$, according to Lemma 2, let $\boldsymbol{l} \leftarrow \boldsymbol{m}^t$, $\boldsymbol{x} \leftarrow \boldsymbol{z}^{t+1}$, $\boldsymbol{x}^+ \leftarrow \boldsymbol{x}^t$, $\boldsymbol{x}' \leftarrow \boldsymbol{z}^t$, $q \leftarrow q^t$, and $r \leftarrow q^{0:t-1}$, then,

$$\langle \boldsymbol{m}^t, \boldsymbol{x}^t - \boldsymbol{z}^{t+1} \rangle \leq q^t(\boldsymbol{z}^{t+1}) - q^t(\boldsymbol{x}^t) + \mathcal{B}_{q^{0:t-1}}(\boldsymbol{z}^{t+1} \| \boldsymbol{z}^t) - \mathcal{B}_{q^{0:t-1}}(\boldsymbol{x}^t \| \boldsymbol{z}^t) - \mathcal{B}_{q^{0:t}}(\boldsymbol{z}^{t+1} \| \boldsymbol{x}^t). \tag{4}$$

For $\mathbf{Term}_2$, according to Theorem 3 in [30], we have

$$\sum_{t=1}^{T} \langle \boldsymbol{l}^t, \boldsymbol{z}^{t+1} - \boldsymbol{x}' \rangle \leq q^{0:T}(\boldsymbol{x}') - \sum_{t=0}^{T} q^t(\boldsymbol{z}^{t+1}) - \sum_{t=1}^{T} \mathcal{B}_{q^{0:t-1}}(\boldsymbol{z}^{t+1} \| \boldsymbol{z}^t). \tag{5}$$

So, $\mathbf{Term}_1 + \mathbf{Term}_2$ satisfies

$$\begin{aligned} \mathbf{Term}_1 + \mathbf{Term}_2 \leq & q^{0:T}(\boldsymbol{x}') - q^0(\boldsymbol{x}^0) + \sum_{t=1}^{T} \left(q^t(\boldsymbol{z}^{t+1}) - q^t(\boldsymbol{z}^{t+1}) - q^t(\boldsymbol{x}^t) \right) \\ & - \sum_{t=1}^{T} \left(\mathcal{B}_{q^{0:t-1}}(\boldsymbol{x}^t \| \boldsymbol{z}^t) + \mathcal{B}_{q^{0:t}}(\boldsymbol{z}^{t+1} \| \boldsymbol{x}^t) \right). \end{aligned} \tag{6}$$

Combine that above equation with $\mathbf{Term}_3$ gives Theorem 1.

B Proof of Theorem 2

Lemma 3. *[27] For any $\boldsymbol{x}, \boldsymbol{x}' \in \mathcal{X}$ and loss $\boldsymbol{l} \in \mathbb{R}^{\sum_{j\in\mathcal{J}} n_j}$, let $\hat{\boldsymbol{r}}_j$ be the instantaneous regret at decision point j under strategy $\boldsymbol{x}$ and loss $\boldsymbol{l}$, then, $\langle \boldsymbol{l}, \boldsymbol{x}\rangle - \langle \boldsymbol{l}, \boldsymbol{x}'\rangle = \sum_{j\in\mathcal{J}} \boldsymbol{x}'[p_j]\langle \hat{\boldsymbol{r}}_j, \hat{\boldsymbol{x}}'_j\rangle$.*

Proof. According to Theorem 1, we have

$$\begin{aligned}\sum_{t=1}^{T}\langle \boldsymbol{l}^t, \boldsymbol{x}^t - \boldsymbol{x}'\rangle \leq & \underbrace{q^{0:T}(\boldsymbol{x}') - \sum_{t=0}^{T} q^t(\boldsymbol{x}^t)}_{\mathbf{Term}_1} + \underbrace{\sum_{t=1}^{T}\langle \boldsymbol{l}^t - \boldsymbol{m}^t, \boldsymbol{x}^t - \boldsymbol{z}^{t+1}\rangle}_{\mathbf{Term}_2} \\ & \underbrace{- \sum_{t=1}^{T}\left(\mathcal{B}_{q^{0:t-1}}(\boldsymbol{x}^t \| \boldsymbol{z}^t) + \mathcal{B}_{q^{0:t}}(\boldsymbol{z}^{t+1} \| \boldsymbol{x}^t)\right)}_{\mathbf{Term}_3}.\end{aligned} \tag{7}$$

for $q^{0:T}(\boldsymbol{x}')$ in $\mathbf{Term}_1$, we have

$$q^{0:T}(\boldsymbol{x}') = \sum_{j\in\mathcal{J}} x'_{p_j}\beta_j^T \frac{1}{2\eta}\|\hat{\boldsymbol{x}}'_j\|_2^2 \leq \frac{1}{2\eta}\sum_{j\in\mathcal{J}} x'_{p_j}\beta_j^T. \tag{8}$$

For $-q^t(\boldsymbol{x}^t)$ in $\mathbf{Term}_1$, we have

$$-q^t(\boldsymbol{x}^t) = q^{0:t-1}(\boldsymbol{x}^t) - q^{0:t}(\boldsymbol{x}^t) = \sum_{j\in\mathcal{J}} x^t_{p_j}\left(\frac{1}{2\eta}\beta_j^{t-1}\|\hat{\boldsymbol{x}}^t_j\|_2^2 - \frac{1}{2\eta}\beta_j^t\|\hat{\boldsymbol{x}}^t_j\|_2^2\right) \leq 0. \tag{9}$$

Since $q^0(\boldsymbol{x}^0) > 0$, we have $\mathbf{Term}_1 \leq \frac{1}{2\eta}\sum_{j\in\mathcal{J}} x'_{p_j}\beta_j^T$.

For $\mathbf{Term}_2$, According to Lemma 3, we have

$$\begin{aligned}\langle \boldsymbol{l}^t - \boldsymbol{m}^t, \boldsymbol{x}^t - \boldsymbol{z}^{t+1}\rangle &= \sum_{j\in\mathcal{J}} z^{t+1}_{p_j}\langle \hat{\boldsymbol{r}}^t_j - \hat{\boldsymbol{r}}'^t_j, \hat{\boldsymbol{z}}^{t+1}_j\rangle \\ &= \sum_{j\in\mathcal{J}} z^{t+1}_{p_j}\langle \hat{\boldsymbol{r}}^t_j - \hat{\boldsymbol{r}}'^t_j, \hat{\boldsymbol{z}}^{t+1}_j - \hat{\boldsymbol{x}}^t_j\rangle.\end{aligned} \tag{10}$$

where $\hat{\boldsymbol{r}}'^t_j$ is the instantaneous under $\boldsymbol{m}^t$ and $\boldsymbol{x}^t$. The last equality is according to the definition of instantaneous regret. For $\mathbf{Term}_2$, we have

$$\mathcal{B}_{q^{0:t-1}}(\boldsymbol{x}^t \| \boldsymbol{z}^t) + \mathcal{B}_{q^{0:t}}(\boldsymbol{z}^{t+1} \| \boldsymbol{x}^t) \geq \mathcal{B}_{q^{0:t}}(\boldsymbol{z}^{t+1} \| \boldsymbol{x}^t) = \sum_{j\in\mathcal{J}} z^{t+1}_{p_j}\frac{1}{2\eta}\beta_j^t\|\hat{\boldsymbol{z}}^{t+1}_j - \hat{\boldsymbol{x}}^t_j\|_2^2. \tag{11}$$

So, for $\mathbf{Term}_2 + \mathbf{Term}_3$, we have

$$
\begin{aligned}
&\mathbf{Term}_2 + \mathbf{Term}_3\\
=&\sum_{t=1}^{T}\sum_{j\in\mathcal{J}} z_{p_j}^{t+1}\left(\langle \hat{\boldsymbol{r}}_j^t - \hat{\boldsymbol{r}}_j'^t, \hat{\boldsymbol{z}}_j^{t+1} - \hat{\boldsymbol{x}}_j^t\rangle - \frac{1}{2\eta}\beta_j^t\|\hat{\boldsymbol{z}}_j^{t+1} - \hat{\boldsymbol{x}}_j^t\|_2^2\right)\\
\leq&\sum_{t=1}^{T}\sum_{j\in\mathcal{J}} z_{p_j}^{t+1}\left(\frac{\eta\|\hat{\boldsymbol{r}}_j^t - \hat{\boldsymbol{r}}_j'^t\|_2^2}{2\beta_j^t} + \frac{1}{2\eta}\beta_j^t\|\hat{\boldsymbol{z}}_j^{t+1} - \hat{\boldsymbol{x}}_j^t\|_2^2 - \frac{1}{2\eta}\beta_j^t\|\hat{\boldsymbol{z}}_j^{t+1} - \hat{\boldsymbol{x}}_j^t\|_2^2\right)\\
=&\frac{1}{2}\sum_{t=1}^{T}\sum_{j\in\mathcal{J}} z_{p_j}^{t+1}\frac{\eta\|\hat{\boldsymbol{r}}_j^t - \hat{\boldsymbol{r}}_j'^t\|_2^2}{\beta_j^t}.
\end{aligned} \tag{12}
$$

The first inequality is according to the Fenchel-Young inequality. Combine all the above equations, then,

$$
\begin{aligned}
\sum_{t=1}^{T}\langle \boldsymbol{l}^t, \boldsymbol{x}^t - \boldsymbol{x}'\rangle \leq&\frac{1}{2\eta}\sum_{j\in\mathcal{J}} x'_{p_j}\beta_j^T + \frac{1}{2}\sum_{t=1}^{T}\sum_{j\in\mathcal{J}} z_{p_j}^{t+1}\frac{\eta\|\hat{\boldsymbol{r}}_j^t - \hat{\boldsymbol{r}}_j'^t\|_2^2}{\beta_j^t}\\
\leq&\frac{1}{2}\sum_{j\in\mathcal{J}}\left(\frac{\beta_j^T}{\eta} + \sum_{t=1}^{T}\frac{\eta\|\hat{\boldsymbol{r}}_j^t - \hat{\boldsymbol{r}}_j'^t\|_2^2}{\beta_j^t}\right).
\end{aligned} \tag{13}
$$

C Proof of Corollary 1

We first present a lemma from [32]. For completeness, the proof is also quoted.

Lemma 4. *[32] Let $a_0 \geq 0$ and $f : [0, +\infty) \to [0, +\infty)$ a non-increasing function. Then*

$$\sum_{t=1}^{T} a_t f\left(a_0 + \sum_{k=1}^{t} a_k\right) \leq \int_{a_0}^{\sum_{t=0}^{T} a_t} f(x)dx. \tag{14}$$

Proof (32). Denote by $s_t = \sum_{k=0}^{t} a_k$.

$$a_t f\left(a_0 + \sum_{k=1}^{t} a_k\right) = a_t f(s_t) \leq \int_{s_{t-1}}^{s_t} f(x)dx. \tag{15}$$

Summing over $t = 1, \ldots, T$, we have the stated bound.

Proof. According to Lemma 4, we have

$$
\begin{aligned}
\sum_{t=1}^{T}\frac{\eta\|\hat{\boldsymbol{r}}_j^t - \hat{\boldsymbol{r}}_j'^t\|_2^2}{\beta_j^t} =&\sum_{t=1}^{T}\frac{\eta\|\hat{\boldsymbol{r}}_j^t - \hat{\boldsymbol{r}}_j'^t\|_2^2}{\sqrt{\sum_{k=1}^{t}\|\hat{\boldsymbol{r}}_j^k - \hat{\boldsymbol{r}}_j'^k\|_2^2}}\\
\leq&2\eta\sqrt{\sum_{t=1}^{T}\|\hat{\boldsymbol{r}}_j^t - \hat{\boldsymbol{r}}_j'^t\|_2^2}.
\end{aligned} \tag{16}
$$

So,

$$\begin{aligned}\sum_{t=1}^{T}\langle \boldsymbol{l}^t, \boldsymbol{x}^t - \boldsymbol{x}'\rangle \leq& \frac{1}{2}\sum_{j\in\mathcal{J}}\left(\frac{\beta_j^T}{\eta} + \sum_{t=1}^{T}\frac{\eta\|\hat{\boldsymbol{r}}_j^t - \hat{\boldsymbol{r}}_j'^t\|_2^2}{\beta_j^t}\right)\\ \leq& \frac{1}{2}\left(2\eta + \frac{1}{\eta}\right)\sum_{j\in\mathcal{J}}\sqrt{\sum_{t=1}^{T}\|\hat{\boldsymbol{r}}_j^t - \hat{\boldsymbol{r}}_j'^t\|_2^2}.\end{aligned} \tag{17}$$