Computational Methods for Astrophysical Applications

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Lesson 3: Temporal Discretizations

- **Explicit methods:**
  stability and von Neumann analysis, stencil, Lax-Friedrichs scheme, CFL condition, leapfrog and Lax-Wendroff method
- **Runge-Kutta methods:**
  predictor-corrector, fourth-order four-step
- **Implicit methods:**
  backward Euler, Crank-Nicolson, trapezoidal method
- **Examples**

Explicit methods

- from time-independent BVP to **mixed Boundary Value-Initial Value problem**: spatio-temporal discretizations needed
- already 1D: quantities depend on **x** and t

  $\Rightarrow$ from ODE+BCs to PDE
- consider prototype **hyperbolic equation**

$$\frac{\partial u}{\partial t} = -v\,\frac{\partial u}{\partial x}$$

  $\Rightarrow$ given constant 'advection speed' v
- arbitrary initial shape $u_0(x, t = 0)$, exact solution

$$u(x,t) = u_0(x - vt)$$

  $\Rightarrow$ errors can be quantified precisely

- discretize both space and time, discrete time levels $t^n = n\Delta t$

  $\Rightarrow$ initial condition specifies spatial variation $u_0(x, t^0 = 0)$

- **explicit time integration:** values at t^{n+1} computed from available information on time level t^n

  $\Rightarrow$ e.g. use second-order central difference for spatial derivative, forward difference for $\partial/\partial t$

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} = -v\,\frac{u_{i+1}^n - u_{i-1}^n}{2\,\Delta x}$$

$\Rightarrow$ rearrange to

$$u_i^{n+1} = u_i^n - v\,\frac{\Delta t}{\Delta x}\,\frac{u_{i+1}^n - u_{i-1}^n}{2}$$

$\Rightarrow$ Euler's **Forward Time Central Space** scheme: **useless**!

- **useless?** Method is **consistent** since LTE vanish in limit $\Delta x \to 0$ and $\Delta t \to 0$
  - $\Rightarrow$ accuracy: 2nd order space, 1st order time $\to$ overall 1st
  - $\Rightarrow$ failure is related to numerical stability
- round-off errors should not grow during time progression
  - $\Rightarrow$ evaluate by **von Neumann method**
  - $\Rightarrow$ numerical solution = exact $+$ round-off error $\epsilon(x,t)$
  - $\Rightarrow$ represent $\epsilon(x,t)$ in Fourier series, analyse Fourier term

$$\epsilon_k(x,t) = \hat{\epsilon}_k \mathrm{e}^{\lambda t} \mathrm{e}^{\mathrm{i}kx}$$

- numerically stable scheme: for all spatial wavenumbers k

$$\left| \frac{\epsilon_k^{n+1}}{\epsilon_k^n} \right| = \left| \mathrm{e}^{\lambda \Delta t} \right| \leq 1$$

- FTCS scheme and von Neumann analysis yields

$$e^{\lambda \Delta t} = 1 - \frac{v \Delta t}{\Delta x} \frac{e^{ik\Delta x} - e^{-ik\Delta x}}{2} = 1 - i \frac{v \Delta t}{\Delta x} \sin(k \Delta x)$$

$\Rightarrow$ scheme is **unconditionally unstable** since for all k

$$\left| \frac{\epsilon_k^{n+1}}{\epsilon_k^n} \right| > 1$$

- **three cures** to save stability
  - $\Rightarrow$ add 'numerical diffusion' to damp nonphysical instability
  - $\Rightarrow$ impose same space-time symmetry as original PDE
  - $\Rightarrow$ use implicit scheme

- adding diffusion: advection-diffusion equation has form

$$\frac{\partial u}{\partial t} = -v \frac{\partial u}{\partial x} + \mathcal{D} \frac{\partial^2 u}{\partial x^2}$$

 $\Rightarrow$ diffusion coefficient $\mathcal{D}$

- replace u_i^n by spatial average between x_{i-1} and x_{i+1}, arrive at

$$\boxed{u_i^{n+1} = \frac{u_{i+1}^n + u_{i-1}^n}{2} - v \frac{\Delta t}{\Delta x} \frac{u_{i+1}^n - u_{i-1}^n}{2}}$$

 $\Rightarrow$ **Lax–Friedrichs scheme** (or Lax scheme)

 $\Rightarrow$ rearrange to form

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} = -v \frac{u_{i+1}^n - u_{i-1}^n}{2\Delta x} + \frac{(\Delta x)^2}{2\Delta t} \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{(\Delta x)^2}$$

 $\Rightarrow$ numerical dissipation with $\mathcal{D} \equiv \frac{(\Delta x)^2}{2\Delta t}$

CFL condition

- perform von Neumann stability analysis for Lax–Friedrichs

$$e^{\lambda \Delta t} = \cos(k\Delta x) - \mathrm{i}\,\frac{v\Delta t}{\Delta x}\,\sin(k\Delta x)$$

$\Rightarrow$ **conditional stability** requiring Courant number C

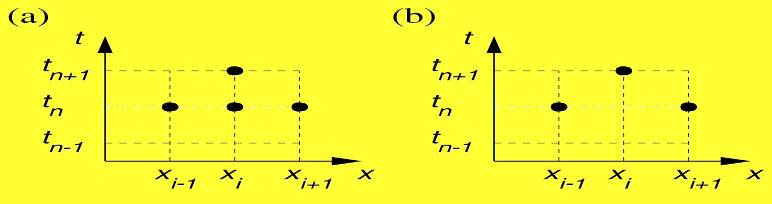
$$C \equiv \frac{|v|\Delta t}{\Delta x} \leq 1$$

$\Rightarrow$ limitation of the time step Δt for a given resolution Δx

$\Rightarrow$ **Courant–Friedrichs–Lewy** condition (1928)

$\Rightarrow$ necessary (not sufficient) condition for stability!

- in (x, t) space, we identify **stencil** of a method
  - $\Rightarrow$ stencils visualizes discrete dependence of method



  - $\Rightarrow$ stencil for FTCS (a) versus Lax-Friedrichs (b)
- **hyperbolic** PDE and **physical characteristics**
  - $\Rightarrow$ the advection equation is hyperbolic as

$$\frac{\partial u}{\partial t} + \frac{\partial \left(\underbrace{vu}_{F} \right)}{\partial x} = 0$$

and Flux Jacobian $\frac{\partial F}{\partial u} = v$ is real number, ‘characteristic speed’
  - $\Rightarrow$ $x - t$ curves given by $\xi = x - vt$ yield $d\,u = 0$ or u constant

CFL and domain of dependence (DOD)

- for second order wave equation

$$\frac{\partial^2 u}{\partial t^2} - v^2 \frac{\partial^2 u}{\partial x^2} = 0$$

$\Rightarrow$ factorizes to

$$\left(\frac{\partial}{\partial t} - v\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} + v\frac{\partial}{\partial x}\right) u = 0$$

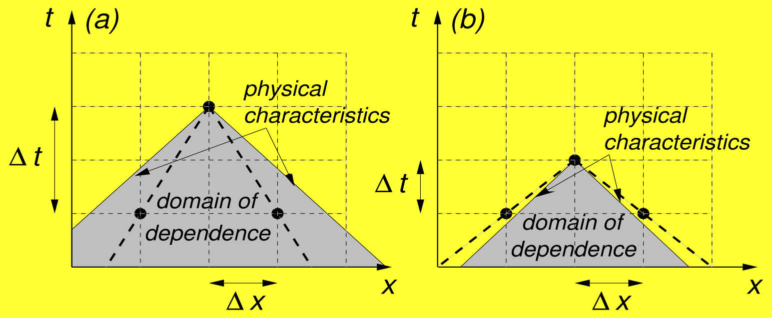
$\Rightarrow$ general solution has left and right going wave with

$$u = f(x - vt) + g(x + vt)$$

$\Rightarrow$ initial shapes $f(x)$, $g(x)$ combine

$\Rightarrow$ 2 characteristics $\frac{dx}{dt} = \pm v$

- illustrate CFL for second order wave equation:
the domain of dependence of the differential equation should be contained in the DOD of the discretised equations



$\Rightarrow$ stability means physical DOD contained in stencil bounds (numerical DOD), hence Δt small enough (right case)

- note: linear advection $+$ wave equation: DOD only involves 1 or 2 points from $t = 0$ $\leftrightarrow$ HD: DOD bounds set by $v \pm c_s$ with c_s sound speed, delimits $t = 0$ interval

- Second cure: maintain space-time symmetry of the PDE
  - $\Rightarrow$ use central discretisation for both x and t
  - $\Rightarrow$ obtain **leapfrog** scheme

$$u_i^{n+1} = u_i^{n-1} - \frac{\Delta t}{\Delta x}\left(F_{i+1}^n - F_{i-1}^n\right)$$

  - $\Rightarrow$ numerical flux function for advection is $F_i^n \equiv v u_i^n$
  - $\Rightarrow$ conditionally stable and second-order accurate
  - $\Rightarrow$ multiple time levels involved: $n-1$, n, $n+1$
  - $\Rightarrow$ potential problem: even/oneven time levels may 'decouple'

Lax-Wendroff scheme

- other 2nd order accurate scheme, exploit Taylor series

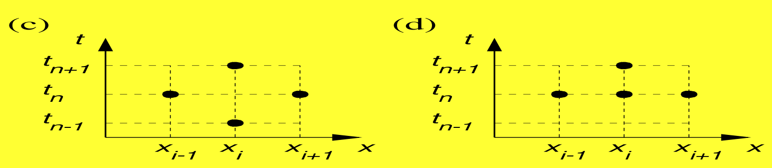
$$u(x, t + \Delta t) \approx u(x, t) + \Delta t \, \frac{\partial u}{\partial t}(x, t) + \tfrac{1}{2}(\Delta t)^2 \, \frac{\partial^2 u}{\partial t^2}(x, t)$$

$\Rightarrow$ now use equation $\frac{\partial u}{\partial t} = -v \frac{\partial u}{\partial x}$ to replace time derivatives
$\Rightarrow$ finally obtain Lax–Wendroff scheme

$$u_i^{n+1} = u_i^n - \tfrac{1}{2} \frac{\Delta t}{\Delta x} \, v \, (u_{i+1}^n - u_{i-1}^n) + \tfrac{1}{2} \frac{(\Delta t)^2}{(\Delta x)^2} \, v^2 \, (u_{i+1}^n - 2u_i^n + u_{i-1}^n)$$

$\Rightarrow$ uses first cure idea: numerical dissipation
$\Rightarrow$ explicit (one-step) two-level scheme, conditionally stable

- stencils for leapfrog and Lax–Wendroff



Runge-Kutta methods

- semi-discretization for spatio-temporal PDE
  - $\Rightarrow$ method of lines: first discretize space, obtain ODEs in time
  - $\Rightarrow$ obtain initial value problem with ODE

$$\frac{du}{dt} = f(t, u)$$

augmented with initial condition $u(t^0) = u^0$

- Runge–Kutta methods use weighted averages of $f(t, u)$ in different points in time interval $[t^n, t^{n+1}]$
  - $\Rightarrow$ weights to achieve certain order of accuracy (in time here)

- **predictor-corrector or two-step** method takes

$$u^{n+1} = u^n + \Delta t\, k^{n2} + \mathcal{O}(\Delta t)^3$$

where we have

$$k^{n1} = f(t^n, u^n), \qquad k^{n2} = f(t^n + \tfrac{1}{2}\Delta t, u^n + \tfrac{1}{2}\Delta t k^{n1})$$

$\Rightarrow$ k^{n2} corresponds to evaluating f at time $t^{n+\frac{1}{2}}$

- classic **fourth-order four-step Runge–Kutta** method uses

$$u^{n+1} = u^n + \Delta t \, \tfrac{1}{6}(k^{n1} + 2k^{n2} + 2k^{n3} + k^{n4}) + \mathcal{O}(\Delta t)^5$$

where

$$k^{n1} = f(t^n, u^n), \qquad k^{n2} = f(t^n + \tfrac{1}{2}\Delta t, u^n + \tfrac{1}{2}\Delta t k^{n1}),$$

$$k^{n3} = f(t^n + \tfrac{1}{2}\Delta t, u^n + \tfrac{1}{2}\Delta t k^{n2}), \qquad k^{n4} = f(t^n + \Delta t, u^n + \Delta t k^{n3})$$

$\Rightarrow$ fourth-order accurate (for time derivative) in step size Δt

- **same ideas possible for space discretization**

  $\Rightarrow$ will be **topic of first assignment!**
- adaptive RK schemes: adjust step size (time/space)

Temporal discretizations: implicit methods

- third cure to instability of Euler FTCS scheme: evaluate spatial derivative at t^{n+1}

$$u_i^{n+1} = u_i^n - v\,\frac{\Delta t}{\Delta x}\,\frac{u_{i+1}^{n+1} - u_{i-1}^{n+1}}{2}$$

$\Rightarrow$ **Backwards in Time, Central in Space Euler** scheme

$\Rightarrow$ u_i^{n+1} not expressed in terms of values at time t^n: **implicit**

- von Neumann stability analysis for BTCS scheme

$$\left|e^{\lambda\Delta t}\right| = \frac{1}{\left|1 + i\frac{v\Delta t}{\Delta x}\sin\left(k\Delta x\right)\right|} < 1 \qquad \text{for all } k$$

$\Rightarrow$ **unconditionally stable**, any (large) time step Δt allowed

- note: **stability does not imply accuracy**

$\Rightarrow$ large Δt affects accuracy, defines time resolution:
behavior may involve physical timescale that needs to be resolved!

- implicit backward Euler: first order in time

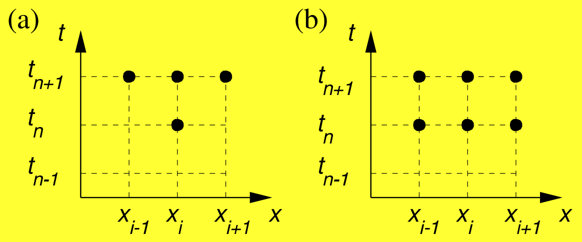
- spatial differences as average of n-th and $(n+1)$-th time step

$$u_i^{n+1} = u_i^n - \frac{1}{4} v \frac{\Delta t}{\Delta x} \left(u_{i+1}^{n+1} + u_{i+1}^n - u_{i-1}^{n+1} - u_{i-1}^n \right)$$

$\Rightarrow$ second order **Crank–Nicolson method**

$\Rightarrow$ **Exercise**: show that this scheme is unconditionally stable, 2nd order accurate

- stencils for BTCS (left) and Crank-Nicolson (right)



Semi-discretisation and implicit methods

- many practical implementations use ‘method of lines’

  $\Rightarrow$ vector $\mathbf{u}$ of unknowns after first spatial discretization

  $\Rightarrow$ obtain ODE system

$$\frac{d\mathbf{u}}{dt} = \mathbf{f}(\mathbf{u})$$

  $\Rightarrow$ RHS vector function $\mathbf{f}$ could even be nonlinear in $\mathbf{u}$

- discretize ODE in time using parameter α in

$$\mathbf{u}^{n+1} = \mathbf{u}^n + \Delta t \left[\alpha \mathbf{f}(\mathbf{u}^{n+1}) + (1-\alpha)\mathbf{f}(\mathbf{u}^n) \right]$$

  $\Rightarrow$ note case $\alpha = 0$: explicit (unstable) forward Euler method

$$\mathbf{u}^{n+1} = \mathbf{u}^n + \Delta t \left[\alpha \mathbf{f}(\mathbf{u}^{n+1}) + (1-\alpha)\mathbf{f}(\mathbf{u}^n)\right]$$

- $\alpha = 1$ is **implicit backward Euler**
- $\alpha = 1/2$ gives second-order accuracy, **trapezoidal method**
  $\Rightarrow$ Crank-Nicolson for central discretization of flux in $\mathbf{f}$

- when $\mathbf{f}$ nonlinear: linearize using

$$\mathbf{f}(\mathbf{u}^{n+1}) \approx \mathbf{f}(\mathbf{u}^n) + \frac{\partial \mathbf{f}^n}{\partial \mathbf{u}} (\mathbf{u}^{n+1} - \mathbf{u}^n)$$

$\Rightarrow$ introduces matrix $\frac{\partial \mathbf{f}^n}{\partial \mathbf{u}}$ called ***"Jacobian matrix"*** of $\mathbf{f}$

- using this linearization, rewrite scheme to

$$
\begin{aligned}
\mathbf{u}^{n+1} &= \mathbf{u}^n + \Delta t \alpha \frac{\partial \mathbf{f}^n}{\partial \mathbf{u}} \left(\mathbf{u}^{n+1} - \mathbf{u}^n\right) + \Delta t \mathbf{f}(\mathbf{u}^n) \\
\Rightarrow \qquad & \left[I - \Delta t \alpha \frac{\partial \mathbf{f}^n}{\partial \mathbf{u}} \right] \delta \mathbf{u} = \Delta t \mathbf{f}(\mathbf{u}^n)
\end{aligned}
$$

$\Rightarrow$ linear system for the unknowns $\delta \mathbf{u} \equiv \mathbf{u}^{n+1} - \mathbf{u}^n$

$\Rightarrow$ represents **first step of Newton iteration**, may be expanded to full iteration for nonlinear problems

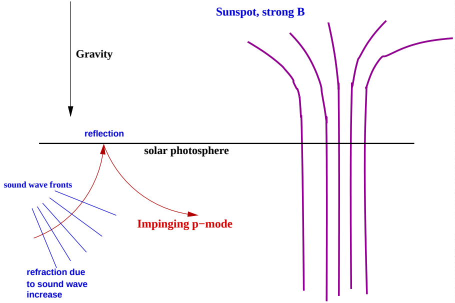
Examples

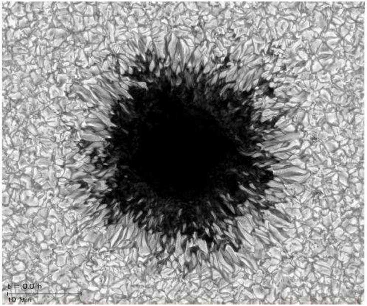
- **numerically solving linearized MHD equations**
- 1st assignment: linear HD equations for stratified plane-parallel stellar atmosphere, govern wave dynamics: pressure gradient and gravity as restoring force
  - $\Rightarrow$ gravito-acoustic *p*-, *g*-modes
  - $\Rightarrow$ also have *f*-mode, surface character
- in MHD magneto-gravito-acoustic waves

- First example: study of *p*-mode interactions with sunspots
- Second example: study of solar coronal loop oscillations
  - $\Rightarrow$ both represent **magneto-seismology**
  - $\Rightarrow$ use different spatio-temporal discretizations

Sunspot seismology

- solve **linear MHD equations for gravitationally stratified atmosphere**, containing ‘sunspot’ as slab of vertical **B**



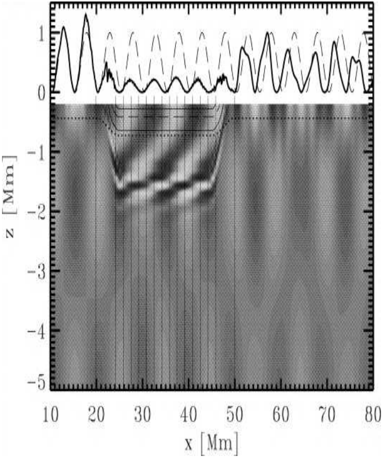


- region exterior to spot: vertical hydrostatic equilibrium
  - $\Rightarrow$ pressure increases downwards
  - $\Rightarrow$ upwards pressure gradient balances downwards gravity

- numerical strategy:
  - $\Rightarrow$ **explicit Lax–Wendroff type scheme, finite difference**
  - $\Rightarrow$ IVP-BVP in 2 dimensions: driving *p*-mode at left boundary
- **study how impinging *f*- or *p*-modes are partially converted to slow magneto-atmospheric gravity waves within the stratified magnetic slab**
- conversion where 2 natural speeds of system near-equal
  - $\Rightarrow$ in sunspot: sound speed varies with depth
  - $\Rightarrow$ magnetic Alfvén speed $v_A = B/\sqrt{\rho}$
  - $\Rightarrow$ mode coupling/conversion at $\beta \approx 1$ region
  - $\Rightarrow$ where $\beta = 2p/B^2$ ratio thermal/magnetic pressure

Active region seismology

- Interaction p-modes with sunspots:
  - decompose in in- & outgoing waves,
  - absorb up to 50 % of impingent acoustic power!
- Candidate linear MHD processes:
  - driving frequencies in Alfvén range causing local resonant Ohmic dissipation
  - stratification: **mode conversion** to downward propagating magneto-acoustic wavemodes at $\beta \approx 1$ layers.



Cally & Bogdan, ApJL **486**, L67 (1997)

http://www.hao.ucar.edu/

Coronal seismology

- TRACE spacecraft (NASA launch ’98):
  - http://trace.lmsal.com/
  - 3D field transition region/corona,
  - unprecedented views coronal loops:

TRACE movie of erupting filament

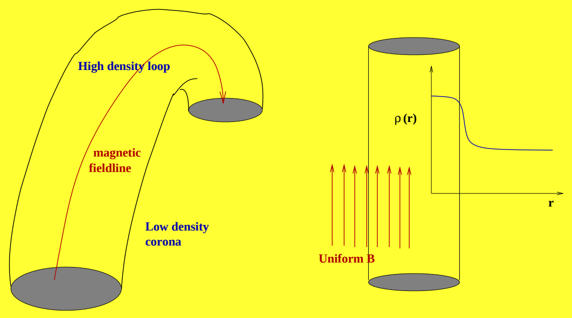
TRACE movie of oscillating filament

- Studies of MHD waves in loops:
  - coronal seismology.
  - invert for loop parameters

Damped loop oscillations

- observed loop displacements:
  $\Rightarrow$ **oscillation amplitudes, periods, damping time**
- damp within few periods
  $\Rightarrow$ MHD mode conversion/dissipation

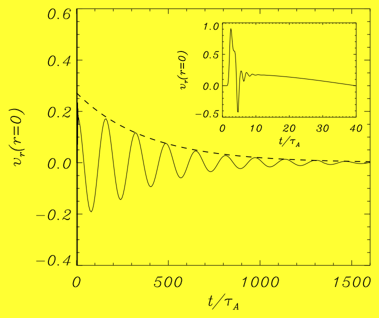
- cylindrical coronal loop model, zero β (no slow waves)
  $\Rightarrow$ no gravity, line-tied loop, density variation $\rho(r)$ only



- **Spatial discretization**:
  - $\Rightarrow$ Fourier handling ignorable θ, z directions
  - $\Rightarrow$ finite element treatment of radial direction
  - $\Rightarrow$ semi-discretize linear MHD equations
- focus on single Fourier mode
  - $\Rightarrow$ isolate $m = 1$ **kink** displacements of a line-tied loop
- **temporal**: 2nd order Crank–Nicolson (implicit)

Damped loop oscillations

- temporal behavior of radial velocity at loop axis



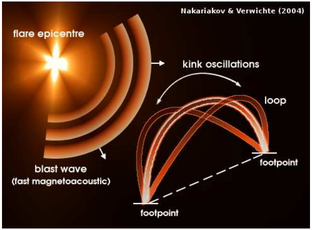
$v_r(r,t)$ **and** $v_\theta(r,t)$ **movies** Terradas *et al.*, ApJL **642**, 533 (2006)

$\Rightarrow$ identifies various phases of loop dynamics

- identifies **various phases of loop dynamics**
  - $\Rightarrow$ **transient** with attenuated short-period oscillations
  - $\Rightarrow$ leaky loop eigenmode, wave energy propagates away
  - $\Rightarrow$ **damped, longer period oscillation** dominates later
  - $\Rightarrow$ global kink eigenoscillation,

coupled to Alfvén in thin layer due to $\rho(r)$ variation

  - $\Rightarrow$ localized (Ohmic) resistive dissipation damps



- **succesfully explains (quantitative!) damping behavior**

Gravito-acoustic waves and Rayleigh-Taylor instability

- **Intro:** derivation of governing second order ODE
  - $\Rightarrow$ analytic solutions for exponentially stratified medium
  - $\Rightarrow$ govern ***p*- and *g*-modes**
- **Discussion of numerical assignment:**
  - $\Rightarrow$ use of **shooting method** for obtaining eigenoscillations
  - $\Rightarrow$ analyse special case of **incompressible plane slab**

[**material adapted from Goedbloed & Poedts, Principles of MHD, CUP 2004, and lectures by Hans Goedbloed**]

Introduction: gravito-acoustic waves

- Use **hydrodynamic equations governing stratified slab**
  - ⇒ **force balance** for hydrostatic equilibrium
  - ⇒ illustrate process of **linearization about equilibrium**
- specify to **gravito-acoustic waves in plane slab**
  - ⇒ take gravity $\mathbf{g} = -g\hat{e}_x$
  - ⇒ obtain second order ODE for displacement x-component
  - ⇒ complement with rigid boundary conditions

Hydrodynamic equations

- **Euler equations for gas dynamics**, with (external) gravity **g**, in terms of density ρ, velocity vector **v** and pressure p

  $\Rightarrow$ express conservation of mass

$$\boxed{\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0}$$

  $\Rightarrow$ momentum equation (Newton's law)

$$\boxed{\underbrace{\rho \left(\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} \right)}_{\text{mass} \times \text{acceleration}} = \underbrace{-\nabla p + \rho \mathbf{g}}_{\text{force}}}$$

  $\Rightarrow$ pressure evolution

$$\boxed{\frac{\partial p}{\partial t} + \mathbf{v} \cdot \nabla p + \gamma p \nabla \cdot \mathbf{v} = 0}$$

  with constant ratio of specific heats γ, ideal gas value $\frac{5}{3}$

- **express conservation of mass, momentum, energy**
- look for **hydrostatic equilibrium**, time-independent $\frac{\partial}{\partial t} = 0$
  - $\Rightarrow$ static equilibrium sets $\mathbf{v} = \mathbf{0}$, write $p \equiv p_0(\mathbf{r})$ and $\rho \equiv \rho_0(\mathbf{r})$
  - $\Rightarrow$ only **force balance** remains

$$\mathbf{0} = -\nabla p_0 + \rho_0 \mathbf{g}$$

  - $\Rightarrow$ 1D: gravity and pressure isolevels mutually $\perp$
  - $\Rightarrow$ define x-direction, equilibrium has $p_0(x)$, $\rho_0(x)$ related

- **linearize HD equations about static equilibrium**
  - $\Rightarrow$ $\mathbf{v} = \mathbf{v}_1(\mathbf{r}, t)$, $\rho = \rho_0(x) + \rho_1(\mathbf{r}, t)$ and $p = p_0(x) + p_1(\mathbf{r}, t)$
  - $\Rightarrow$ insert, split off equilibrium
  - $\Rightarrow$ keep linear terms in small quantities $\mathbf{v}_1$, ρ_1, and p_1
- governs **small perturbations about static equilibrium**

HD wave equation

- Equilibrium of plane slab with constant external gravity field $\mathbf{g} = (-g, 0, 0)$

$$\nabla p_0 = \rho_0 \mathbf{g} \quad \Rightarrow \quad p_0'(x) = -\rho_0(x) g$$

- Linearized HD equations:

$$\begin{aligned}
\frac{\partial \rho_1}{\partial t} + \mathbf{v}_1 \cdot \nabla \rho_0 + \rho_0 \nabla \cdot \mathbf{v}_1 &= 0 \\
\rho_0 \frac{\partial \mathbf{v}_1}{\partial t} + \nabla p_1 - \rho_1 \mathbf{g} &= 0 \\
\frac{\partial p_1}{\partial t} + \mathbf{v}_1 \cdot \nabla p_0 + \gamma p_0 \nabla \cdot \mathbf{v}_1 &= 0
\end{aligned}$$

- With $\mathbf{v}_1 = \partial \boldsymbol{\xi} / \partial t \;\Rightarrow$ **Wave equation for gravito-acoustic waves in plane slab:**

$$\boxed{\rho_0 \frac{\partial^2 \boldsymbol{\xi}}{\partial t^2} - \nabla(\gamma p_0 \nabla \cdot \boldsymbol{\xi}) - \rho_0 \nabla(\mathbf{g} \cdot \boldsymbol{\xi}) + \rho_0 \mathbf{g} \nabla \cdot \boldsymbol{\xi} = 0}$$

- equilibrium 1D (x-direction from gravity)
  - $\Rightarrow$ displacement vector $\boldsymbol{\xi}(\mathbf{r}, t) \equiv \boldsymbol{\xi}(x, y, z, t)$ in general
  - $\Rightarrow$ ignorable coordinates y, z: use Fourier mode $\mathrm{e}^{i(k_y y + k_z z)}$
  - $\Rightarrow$ **normal mode** analysis, time dependence $\mathrm{e}^{-i\omega t}$
  - $\Rightarrow$ together, change to **eigenvalue problem** where

$$\boxed{\boldsymbol{\xi}(\mathbf{r}, t) = \hat{\boldsymbol{\xi}}(x)\, \mathrm{e}^{i(k_y y + k_z z - \omega t)}}$$

- for given equilibrium, and given mode numbers k_y, k_z
  $\Rightarrow$ obtain eigenfrequencies ω with corresponding eigenfunctions $\hat{\boldsymbol{\xi}}(x)$

- choose coordinate system such that $k_z = 0$, **rename $k_y \equiv k$**
  - $\Rightarrow$ no physical direction other than gravity (x)
- governing vector equation for $\hat{\xi}(x)$
  - $\Rightarrow$ **express as second order ODE for $\hat{\xi}_x \equiv \xi$**

$$\frac{d}{dx}\left(\frac{\gamma p_0\,\rho_0\omega^2}{\rho_0\omega^2 - k^2\gamma p_0}\,\frac{d\xi}{dx}\right) + \left[\rho_0\omega^2 - \frac{k^2\rho_0^2 g^2}{\rho_0\omega^2 - k^2\gamma p_0} - \left(\frac{k^2\gamma p_0\,\rho_0 g}{\rho_0\omega^2 - k^2\gamma p_0}\right)'\right]\xi = 0$$

  - $\Rightarrow$ 2 BCs, rigid walls at $x = 0$ and $x = 1$ require

$$\xi(x=0) = \xi(x=1) = 0$$

  - $\Rightarrow$ note: γ, k, g given constants
  - $\Rightarrow$ ω is (constant) eigenfrequency to be determined
  - $\Rightarrow$ known $\rho_0(x)$ and $p_0(x)$ functions related by $p_0' = -\rho_0 g$

Gravito-acoustic waves

- *Exponentially stratified medium with constant sound speed*

$$\rho_0 = \mathrm{e}^{-\alpha x}\,, \quad p_0 = \mathrm{e}^{-\alpha x} \quad \Rightarrow \; c^2 = \frac{\gamma p_0}{\rho_0} = \gamma = \mathrm{const}$$

$$p_0' = -\alpha p_0 = -\rho_0 g \quad \Rightarrow \; \alpha = \frac{\rho_0 g}{p_0} = g = \mathrm{const}$$

Spectral equation reduces to

$$\frac{c^2\omega^2}{\omega^2 - k^2c^2}\,\frac{d}{dx}\left(\mathrm{e}^{-\alpha x}\,\frac{d\xi}{dx}\right) + \left(\omega^2 - \frac{k^2g^2}{\omega^2 - k^2c^2} + \alpha\,\frac{k^2c^2g}{\omega^2 - k^2c^2}\right)\mathrm{e}^{-\alpha x}\,\xi = 0$$

- introduce Brunt–Väisäläa frequency $N^2 = (\gamma - 1)\frac{g^2}{c^2}$ find

$$\frac{d^2\xi}{dx^2} - \alpha\,\frac{d\xi}{dx} + \frac{\omega^4 - k^2c^2\,\omega^2 + k^2c^2N^2}{c^2\,\omega^2}\,\xi = 0$$

$\Rightarrow$ 2nd order differential equation with constant coefficients

- trivial solutions
$$\xi = C e^{(\frac{1}{2}\alpha \pm \mathrm{i}q)x}\,, \qquad q \equiv \sqrt{-\frac{1}{4}\alpha^2 + \frac{\omega^4 - k^2 c^2\,\omega^2 + k^2 c^2 N^2}{c^2\,\omega^2}}$$
Expression under root > 0 for oscillatory solutions, satisfy BCs with quantized $q = n\pi \quad (n = 1, 2, \ldots)$
- *Dispersion equation of gravito-acoustic waves* from

$$\omega^4 - (k^2 + q^2 + \frac{1}{4}\alpha^2)c^2\omega^2 + k^2 c^2 N^2 = 0$$

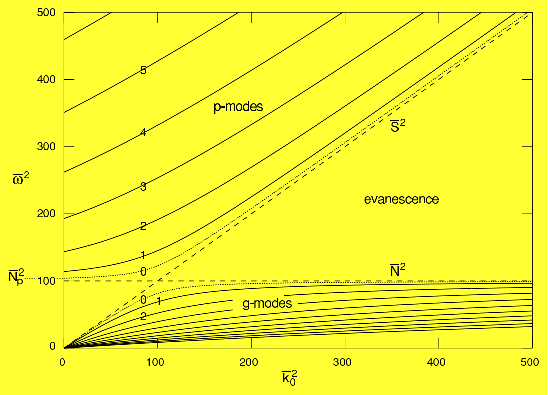
with solutions

$$\boxed{\omega^2_{p,g} = \frac{1}{2}k^2_{\mathrm{eff}}c^2 \left[1 \pm \sqrt{1 - \frac{4k^2 N^2}{k^4_{\mathrm{eff}} c^2}} \right]}\,, \quad k^2_{\mathrm{eff}} \equiv k^2 + q^2 + \frac{1}{4}\alpha^2$$

where k_{eff} is effective total 'wave number'

  - Branch with $+$ sign: *acoustic waves* or *p-modes*
  - Branch with $-$ sign: *gravity waves* or *g-modes*

Dispersion diagram p- and g-modes



- Frequencies *p*-modes *increase* monotonically, cluster at ∞
- Frequencies *g*-modes *decrease* monotonically, cluster at 0

- thus far: general ODE, solved analytically for special case

  $\Rightarrow$ when (numerical) solution strategy for governing ODE known, use special case to test/verify/quantify the errors!

- **Actual assignment**: consider simpler **incompressible limit**

  $\Rightarrow$ formally move sound speed to infinity

$$c^2 \equiv \frac{\gamma p_0}{\rho_0} \rightarrow \infty$$

  $\Rightarrow$ incompressible modes $\nabla \cdot \boldsymbol{\xi} = 0$, with $c^2 \nabla \cdot \boldsymbol{\xi}$ finite!

- ODE for **waves in incompressible gravitating plasma slab**

$$\frac{d}{dx}\left[\rho_0 \omega^2 \frac{d\xi}{dx}\right] - k^2 \left[\rho_0 \omega^2 + \rho_0' g\right] \xi = 0$$

  $\Rightarrow$ with BCs $\xi(0) = \xi(1) = 0$.

Assignment: summary

- ODE for **waves in incompressible gravitating slab**

$$\frac{d}{dx}\left[\rho_0\omega^2\frac{d\xi}{dx}\right] - k^2\left[\rho_0\omega^2 + \rho_0' g\right]\xi = 0$$

$\Rightarrow$ with BCs $\xi(0) = \xi(1) = 0$.

$\Rightarrow$ analyse for linear density profile $\rho_0(x) = 1 + \sigma x$

$\Rightarrow$ study waves $\omega^2 > 0$ or instabilities $\omega^2 < 0$ driven by gravity

$\Rightarrow$ pressure profile no longer in wave description

- **study modes for varying k^2, σ, g, i.e. solve for eigenmodes $\omega^2(k^2, \sigma, g)$, with corresponding eigenfunctions $\xi(x)$**

$\Rightarrow$ interpret physically: **instabilities are known as Rayleigh-Taylor instabilities: heavy fluid atop a lighter one represents a gravitationally unstable situation!**

- solve ODE numerically, set up a programme that would allow for easy generalization to different equilibrium density profiles

Assignment: extra input

- core problem: solve 2nd order ODE of generic form

$$\frac{d}{dx}\left[P(x;\omega^2)\,\frac{d\xi}{dx}\right] - Q(x;\omega^2)\,\xi = 0\,, \qquad \xi(0) = \xi(1) = 0$$

  $\Rightarrow$ appearance of squared quantity ω^2

  $\Rightarrow$ rescaled problem, implicit choice units length, mass, time

  $\Rightarrow$ corresponding rescaled function ξ

- introduce auxiliary variable $\psi \equiv P\frac{d\xi}{dx}$

  $\Rightarrow$ alternative formulation as two 1st order ODEs

$$\begin{aligned}\frac{d\xi}{dx} &= \psi/P \\ \frac{d\psi}{dx} &= Q\xi\end{aligned}$$

  $\Rightarrow$ turned BVP into IVP, can exploit **shooting method**

Shooting method

- system of 1st order ODEs for P non-zero on domain $[0, 1]$

$$\begin{aligned} \frac{d\xi}{dx} &= \psi/P \\ \frac{d\psi}{dx} &= Q\xi \end{aligned}$$

- take starting guess for ω^2, insert it in $P(x;\omega^2)$ and $Q(x;\omega^2)$, integrate the ODE system (e.g. use Runge-Kutta) from $x = 0$ to $x = 1$, with initial values

$$\begin{aligned} \xi(0) &= 0 \\ \frac{d\xi}{dx}(0) &\equiv \psi(0)/P(0,\omega^2) = 1 \end{aligned}$$

$\Rightarrow$ solution $\xi(x)$ may not satisfy RHS BC $\xi(1) = 0$

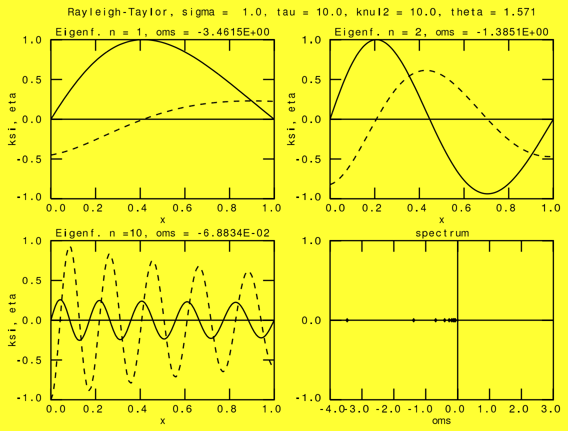
$\Rightarrow$ guess new value for ω^2 that 'brings us closer to' satisfying both BCs, repeat till RHS BC satisfied within prechosen accuracy

- how change ω^2 in iteration?

  $\Rightarrow$ rely on fact that the **number of zero values for $\xi(x)$ on $[0,1]$ is monotonic in parameter ω^2** (oscillation theorem, known for MHD by Goedbloed & Sakanaka, 1974)

  $\Rightarrow$ **(in)**stability: start with large **(negative)** guess

  $\Rightarrow$ decreasing $|\omega^2|$ moves zeros into domain



Evaluation

- first assignment asks to implement your own solver.
  - $\Rightarrow$ Use language/software package you're most familiar with!
- **when using** `Maple, Matlab, Mathematica, ...`, it is likely you resort to pre-implemented library routines for solving ODE systems. That is fine, BUT we then expect a written account of the numerical methodology used in those, and also expect a somewhat deeper physical analysis, parameter study, result interpretation/presentation, ...
- **when use programming language** you already master (`Fortran, C, ...`), or learn now, we expect the emphasis on development/design of full code, input/output control and strategies, analysis of (preliminary) resulting data (basic plots for 1D functions, ...).

- **Both are ok**, in the spirit of this course we would recommend/prefer the second approach, but in any case will use the above distinction in evaluating. This assignment amounts to **8 out of the 20 points** of this course

  $\Rightarrow$ division 8/6/6 for 1st, 2nd assignment, oral

- **Hand-in on 14 november**

  $\Rightarrow$ **Required**: *short* $\leq$ *5 page description* of approach and/or implementation, and result verifications and validation, error quantification (includes figures, reference list, ...).

  $\Rightarrow$ in addition to these 5 pages: *a full printout of actual code*

- use PC/laptop available to you, or ask use PCs at IvS/CmPA