# Numerical study on 2D Riemann problems using state of the art solvers

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January , 2015

# Outline

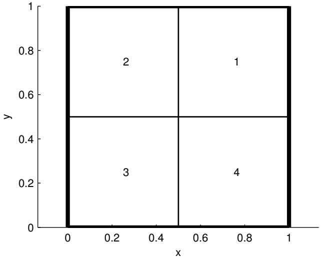
# Introduction

## Riemann problem in 2D

A Riemann problem consists of a system of conservation equation together with a piecewise constant initial condition.

$$U_t + F(U)_x + G(U)_y = 0$$

$$U_0(x,y) = \begin{cases} u_1 & (x,y) \in [0.5,1]^2 \\ u_2 & (x,y) \in [0,0.5] \times [0.5,1] \\ u_3 & (x,y) \in [0,0.5]^2 \\ u_4 & (x,y) \in [0.5,1] \times [0,0.5] \end{cases}$$



# The equations

## Euler equation for compressible fluids

$$U_t + F(U)_x + G(U)_y = 0$$

with

$$U = \begin{pmatrix} \rho \\ \rho u \\ \rho v \\ e \end{pmatrix} \quad F = \begin{pmatrix} \rho u \\ \rho u^2 + p \\ \rho u v \\ u(e+p) \end{pmatrix} \quad G = \begin{pmatrix} \rho v \\ \rho u v \\ \rho u^2 + p \\ v(e+p) \end{pmatrix}$$

Equation describing the dynamics of a compressible fluid. Conserved quantities are: $\rho$ density, $\rho u$ momentum in $x$ , $\rho v$ momentum in $y$ and the total energy $e$.

# Numerical methods

5 different numerical schemes tested:

- TVDLF
- HLL
- HLLC
- TVD-MUSCL
- FD

# TVDLF - Total Variation Diminishing (TVD) concept

Total variation of numerical approximation of u:

$$TV(u^n) = \sum_{i=0}^{N} |u_{i+1}^n - u_i^n|$$

Scheme is total variation diminishing in time if

$$TV(u^{n+1}) \leq TV(u^n) \quad \forall n$$

# TVDLF

First order Lax-Friedrichs scheme:

$$U_j^{n+1} = U_j^n - \frac{\Delta t}{\Delta x}\left(F_{j+1/2} - F_{j-1/2}\right) + \frac{1}{2}\left(\Phi_{j+1/2} - \Phi_{j-1/2}\right)$$

where

$$F_{j+1/2} = \frac{F_j + F_{j+1}}{2}$$

$$\Phi_{j+1/2} = U_{j+1} - U_j$$

Scheme is TVD $\Leftrightarrow$ CFL condition satisfied

Scheme is first order accurate

Reduce numerical diffusion (with local Courant number, maximal speed):

$$\Phi_{j+1/2} = \frac{\Delta t}{\Delta x} c_{j+1/2}^{max}\left(U_{j+1} - U_j\right)$$

# TVDLF - second order spatial accuracy

Linear approximation of U and fluxes at boundary interfaces
U at $x_{j+1/2}$, linear interpolation:

$$U^L_{j+1/2} = U^n_j + \frac{1}{2}\bar{\Delta} U^n_j$$

$$U^R_{j+1/2} = U^n_{j+1} - \frac{1}{2}\bar{\Delta} U^n_{j+1}$$

Limited slopes $\bar{\Delta} U$ will be defined later

$$F_{j+1/2} = \frac{F(U^L_{j+1/2}) + F(U^R_{j+1/2})}{2}$$

$$\Phi_{j+1/2} = \frac{\Delta t}{\Delta x} c^{max}_{j+1/2}\left(U^R_{j+1/2} - U^L_{j+1/2}\right)$$

# TVDLF - slope limiter

Required to ensure TVD property
For example *minmod* limiter

$$\bar{\Delta} U_j = minmod(\Delta U_{j-1/2}, \Delta U_{j+1/2})$$

with

$$\Delta U_{j+1/2} = U_{j+1} - U_j$$

and

$$minmod(w_1, w_2, \ldots, w_n) =$$

$$sgn(w_1) max[0, min(|w_1|, sgn(w1)w_2, \ldots, sgn(w_1)w_n)]$$

The *minmod* function takes the argument with the smallest modulus when all arguments have the same signs and otherwise it is zero.

# TVDLF - Temporally second order accuracy

Use Hancock's predictor step:

$$U_j^{n+1/2} = U_j^n - \frac{1}{2}\frac{\Delta t}{\Delta x}\left[F(U_j^n + \frac{1}{2}\bar{\Delta}U_j^n) - F(U_j^n - \frac{1}{2}\bar{\Delta}U_j^n)\right]$$

used for calculating linear extrapolations:

$$U_{j+1/2}^L = U_j^{n+1/2} + \frac{1}{2}\bar{\Delta}U_j^n$$

$$U_{j+1/2}^R = U_{j+1}^{n+1/2} - \frac{1}{2}\bar{\Delta}U_{j+1}^n$$

# TVD-MUSCL

MUSCL = Monotonic Upstream Scheme for Conservation Laws

- same Hancock predictor step and upwinding as TVDLF
- upwinding is applied for characteristic variables rather than conservative variables

Characteristic variables $\vec{r}^k$

- linear combinations of conservative variables
- right eigenvectors of $\partial\vec{F}/\partial\vec{U}$

$$\frac{\partial\vec{F}}{\partial\vec{U}}\vec{r}^k = c^k\vec{r}^k$$

- eigenvalues $c^k$ are real
- eigenvectors form a complete orthogonal basis
- left eigenvectors $\vec{l}^k$:

$$\vec{l}^k \cdot \vec{r}^k = \delta_{k,m}$$

# TVD-MUSCL

Modify $\vec{\Phi}$:

$$\vec{\Phi} = \frac{\Delta t}{\Delta x} \sum_k \vec{r}^k |c^k| \vec{l}^k \cdot (\vec{U^R} - \vec{U^L})$$

where $\vec{r}^k$, $c^k$ and $\vec{l}^k$ are calculated for $U_{j+1/2}$

- $\vec{l}^k \cdot (\vec{U^R} - \vec{U^L})$ determines jump in k-th characteristic variable
- multiplication by $\vec{r}^k$ transforms result back to conservation variables

Comparison to TVDLF

- Advantage: use of eigenvalue $c^k$ instead of largest eigenvalue $c^{max}$ $\Rightarrow$ upwinding is accurate for each characteristic variable $\Rightarrow$ less numerical diffusion
- Disadvantage: left and right eigenvectors must be calculated for each cell interface (can be very expensive)

# FD

Finite Differences

The flow variables are given as point-wise values $U_j$ at locations $x_j$ as:

$$U_j(t) = U(x_j, t)$$

Difference formulae of a given order of accuracy can be derived from Taylor expansion around the grid points

# FD

Example, 1D, grid spacing $\Delta x$:
$\rightarrow$ a first order spatial derivative can be approximated by the centered finite difference formula:

$$\frac{U_{j+1} - U_{j-1}}{2\Delta x}$$

$$= \frac{1}{2\Delta x}\left[U_j + \Delta x \partial_x U + \frac{(\Delta x)^2}{2!}\partial_{xx} U + \dots\right]$$

$$-\frac{1}{2\Delta x}\left[U_j - \Delta x \partial_x U + \frac{(\Delta x)^2}{2!}\partial_{xx} U + \dots\right]$$

$$= \partial_x U + O((\Delta x)^2)$$

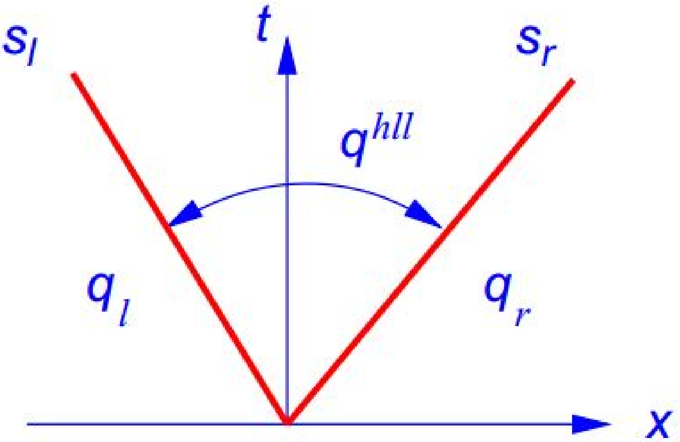
# HLL

Riemann solver of Harten, Lax and van Leer
System of one-dimensional conservation laws:

$$q_t + f(q)_x = 0$$

with

$$q(x,0) = \begin{cases} q_l & if\ x < 0 \\ q_r & if\ x > 0 \end{cases}$$


$s_l$
$t$
$s_r$
$q^{hll}$
$q_l$
$q_r$
$x$

# HLL

Integrating over control volume:

$$q^{hll} = \frac{s_r q_r - s_l q_l + f_l - f_r}{s_r - s_l}$$

Following approximation is proposed:

$$\tilde{q}(x,t) = \begin{cases} q_l & if \ \frac{x}{t} \leq s_l \\ q^{hll} & if \ s_l \leq \frac{x}{t} \leq s_r \\ q_r & if \ \frac{x}{t} \geq s_r \end{cases}$$

Corresponding flux:

$$f^{hll}_{i+1/2} = \begin{cases} f_l & if \ 0 \leq s_l \\ \frac{s_r f_l - s_l f_r + s_l s_r (q_r - q_l)}{s_r - s_l} & if \ s_l \leq 0 \leq s_r \\ f_r & if \ 0 \geq s_r \end{cases}$$

# HLL

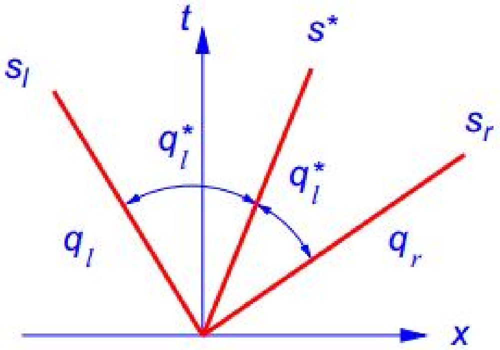
These formulas can be used in the explicit conservative formula:

$$q_i^{n+1} = q_i^n + \frac{\Delta t}{\Delta x}\left[f_{i-1/2} - f_{i+1/2}\right]$$

Wave speeds $s_l$ and $s_r$ must be estimated.

# HLLC

Modification of HLL scheme in which the missing contact and shear waves are restored.



# HLLC

$$\tilde{q}(x,t) = \begin{cases} q_l & \text{if } \frac{x}{t} \le s_l \\ q_l^* & \text{if } s_l \le \frac{x}{t} \le s^* \\ q_r^* & \text{if } s^* \le \frac{x}{t} \le s_r \\ q_r & \text{if } \frac{x}{t} \ge s_r \end{cases}$$

$$f_{i+1/2}^{hllc} = \begin{cases} f_l & \text{if } 0 \le s_l \\ f_l + s_l(q_l^* - q_l) & \text{if } s_l \le 0 \le s^* \\ f_r + s_r(q_r^* - q_r) & \text{if } s^* \le 0 \le s_r \\ f_r & \text{if } 0 \ge s_r \end{cases}$$

Intermediate states $q_l^*$ and $q_r^*$ can be determined.
Wave speeds $s_l$, $s_r$ and $s^*$ need to be estimated.

# MPI-AMRVAC and the HPC

# Server interaction

A script can do this better than we do!

- No need to ssh
- All changes can be made locally
- Bulk submitting
- No need to wait on the compilation
- Compresses and downloads all automatic

Les then 10 minutes to compare 5 methods on a new problem.

# Some problems

- Compiler error: segmentation problem
- Some of the problems ran only on one node
- The visualisation software crashed a lot

# Configuration 1 - Results from paper

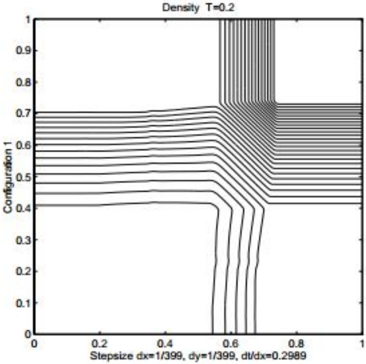
CONFIGURATION 1. $\overrightarrow{R_{32}}$ $\overrightarrow{R_{21}}$ $\overrightarrow{R_{34}}$ $\overrightarrow{R_{41}}$ : the initial data are

$p_2 = 0.4$ $\quad \rho_2 = 0.5197$ $\quad p_1 = 1$ $\quad \rho_1 = 1$

$u_2 = -0.7259$ $\quad v_2 = 0$ $\quad u_1 = 0$ $\quad v_1 = 0$

$p_3 = 0.0439$ $\quad \rho_3 = 0.1072$ $\quad p_4 = 0.15$ $\quad \rho_4 = 0.2579$

$u_3 = -0.7259$ $\quad v_3 = -1.4045$ $\quad u_4 = 0$ $\quad v_4 = -1.4045$



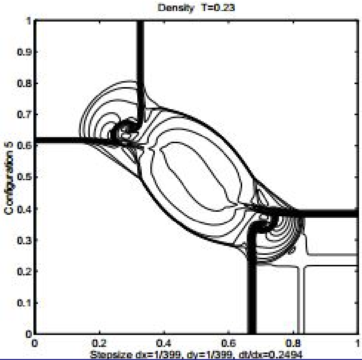
# Configuration 1 - Own results

movie

# Configuration 5 - Results from paper

CONFIGURATION 5. $J_{32}^-$ $\overset{J_{21}^-}{\underset{J_{34}^-}{}}$ $J_{41}^-$ : the initial data are

| | | | |
|---|---|---|---|
| $p_2 = 1$ | $\rho_2 = 2$ | $p_1 = 1$ | $\rho_1 = 1$ |
| $u_2 = -0.75$ | $v_2 = 0.5$ | $u_1 = -0.75$ | $v_1 = -0.5$ |
| $p_3 = 1$ | $\rho_3 = 1$ | $p_4 = 1$ | $\rho_4 = 3$ |
| $u_3 = 0.75$ | $v_3 = 0.5$ | $u_4 = 0.75$ | $v_4 = -0.5$ |

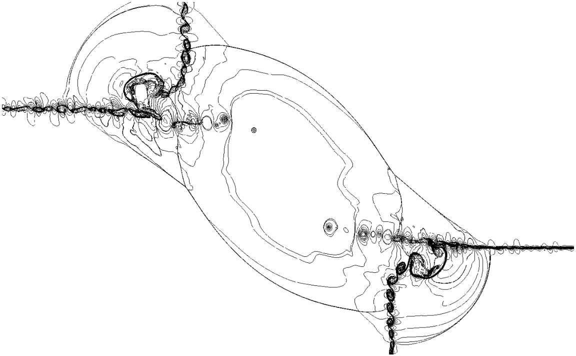


# Configuration 5 - Own results

movie

# Configuration 5 - Own results

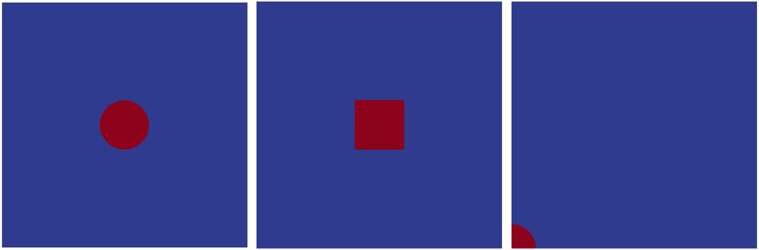
High Resolution - HLLC



# Setup for the explosion

We tried several parameters

- Size of the explosion centre
- Pressure difference
- Location of the explosion



# Comparing two explosions

Movie

# 3d rayleigh taylor