Numerical study on 2D Riemann problems using state of the art solvers

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Outline

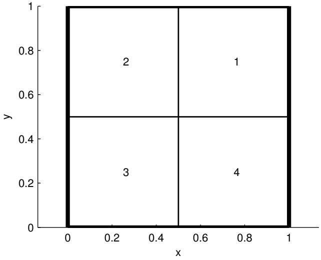
Introduction

Riemann problem in 2D

A Riemann problem consists of a system of conservation equation together with a piecewise constant initial condition.

$$U_t + F(U)_x + G(U)_y = 0$$

$$U_0(x,y) = \begin{cases} u_1 & (x,y) \in [0.5,1]^2 \\ u_2 & (x,y) \in [0,0.5] \times [0.5,1] \\ u_3 & (x,y) \in [0,0.5]^2 \\ u_4 & (x,y) \in [0.5,1] \times [0,0.5] \end{cases}$$



The equations

Euler equation for compressible fluids

$$U_t + F(U)_x + G(U)_y = 0$$

with

$$U = \begin{pmatrix} \rho \\ \rho u \\ \rho v \\ e \end{pmatrix} \quad F = \begin{pmatrix} \rho u \\ \rho u^2 + p \\ \rho u v \\ u(e+p) \end{pmatrix} \quad G = \begin{pmatrix} \rho v \\ \rho u v \\ \rho u^2 + p \\ v(e+p) \end{pmatrix}$$

Equation describing the dynamics of a compressible fluid. Conserved quantities are: ρ density, ρu momentum in x , ρv momentum in y and the total energy e.

Numerical methods

5 different numerical schemes tested:

- TVDLF
- HLL
- HLLC
- TVD-MUSCL
- FD

TVDLF - Total Variation Diminishing (TVD) concept

Total variation of numerical approximation of u:

$$TV(u^n) = \sum_{i=0}^{N} |u_{i+1}^n - u_i^n|$$

Scheme is total variation diminishing in time if

$$TV(u^{n+1}) \leq TV(u^n) \quad \forall n$$

TVDLF

First order Lax-Friedrichs scheme:

$$U_j^{n+1} = U_j^n - \frac{\Delta t}{\Delta x}\left(F_{j+1/2} - F_{j-1/2}\right) + \frac{1}{2}\left(\Phi_{j+1/2} - \Phi_{j-1/2}\right)$$

where

$$F_{j+1/2} = \frac{F_j + F_{j+1}}{2}$$

$$\Phi_{j+1/2} = U_{j+1} - U_j$$

Scheme is TVD $\Leftrightarrow$ CFL condition satisfied
Scheme is first order accurate
Reduce numerical diffusion (with local Courant number, maximal speed):

$$\Phi_{j+1/2} = \frac{\Delta t}{\Delta x} c_{j+1/2}^{max}\left(U_{j+1} - U_j\right)$$

TVDLF - second order spatial accuracy

Linear approximation of U and fluxes at boundary interfaces
U at $x_{j+1/2}$, linear interpolation:

$$U^L_{j+1/2} = U^n_j + \frac{1}{2}\bar{\Delta} U^n_j$$

$$U^R_{j+1/2} = U^n_{j+1} - \frac{1}{2}\bar{\Delta} U^n_{j+1}$$

Limited slopes $\bar{\Delta} U$ will be defined later

$$F_{j+1/2} = \frac{F(U^L_{j+1/2}) + F(U^R_{j+1/2})}{2}$$

$$\Phi_{j+1/2} = \frac{\Delta t}{\Delta x} c^{max}_{j+1/2} \left(U^R_{j+1/2} - U^L_{j+1/2} \right)$$

TVDLF - slope limiter

Required to ensure TVD property
For example *minmod* limiter

$$\bar{\Delta U_j} = minmod(\Delta U_{j-1/2}, \Delta U_{j+1/2})$$

with

$$\Delta U_{j+1/2} = U_{j+1} - U_j$$

and

$$minmod(w_1, w_2, \ldots, w_n) =$$

$$sgn(w_1) max[0, min(|w_1|, sgn(w1)w_2, \ldots, sgn(w_1)w_n)]$$

The *minmod* function takes the argument with the smallest modulus when all arguments have the same signs and otherwise it is zero.

TVDLF - Temporally second order accuracy

Use Hancock's predictor step:

$$U_j^{n+1/2} = U_j^n - \frac{1}{2}\frac{\Delta t}{\Delta x}\left[F(U_j^n + \frac{1}{2}\bar{\Delta}U_j^n) - F(U_j^n - \frac{1}{2}\bar{\Delta}U_j^n)\right]$$

used for calculating linear extrapolations:

$$U_{j+1/2}^L = U_j^{n+1/2} + \frac{1}{2}\bar{\Delta}U_j^n$$

$$U_{j+1/2}^R = U_{j+1}^{n+1/2} - \frac{1}{2}\bar{\Delta}U_{j+1}^n$$

TVD-MUSCL

MUSCL = Monotonic Upstream Scheme for Conservation Laws

- same Hancock predictor step and upwinding as TVDLF
- upwinding is applied for characteristic variables rather than conservative variables

Characteristic variables $\vec{r}^k$

- linear combinations of conservative variables
- right eigenvectors of $\partial\vec{F}/\partial\vec{U}$

$$\frac{\partial\vec{F}}{\partial\vec{U}}\vec{r}^k = c^k\vec{r}^k$$

- eigenvalues c^k are real
- eigenvectors form a complete orthogonal basis
- left eigenvectors $\vec{l}^k$:

$$\vec{l}^k \cdot \vec{r}^k = \delta_{k,m}$$

TVD-MUSCL

Modify $\vec{\Phi}$:

$$\vec{\Phi} = \frac{\Delta t}{\Delta x} \sum_k \vec{r}^k |c^k| \vec{l}^k \cdot (\vec{U}^R - \vec{U}^L)$$

where $\vec{r}^k$, c^k and $\vec{l}^k$ are calculated for $U_{j+1/2}$

- $\vec{l}^k \cdot (\vec{U}^R - \vec{U}^L)$ determines jump in k-th characteristic variable
- multiplication by $\vec{r}^k$ transforms result back to conservation variables

Comparison to TVDLF

- Advantage: use of eigenvalue c^k instead of largest eigenvalue c^{max} $\Rightarrow$ upwinding is accurate for each characteristic variable $\Rightarrow$ less numerical diffusion
- Disadvantage: left and right eigenvectors must be calculated for each cell interface (can be very expensive)

FD

Finite Differences
The flow variables are given as point-wise values U_j at locations x_j as:

$$U_j(t) = U(x_j, t)$$

Difference formulae of a given order of accuracy can be derived from Taylor expansion around the grid points

FD

Example, 1D, grid spacing Δx:
$\rightarrow$ a first order spatial derivative can be approximated by the centered finite difference formula:

$$\frac{U_{j+1} - U_{j-1}}{2\Delta x}$$

$$= \frac{1}{2\Delta x}\left[U_j + \Delta x \partial_x U + \frac{(\Delta x)^2}{2!}\partial_{xx} U + \dots\right]$$

$$-\frac{1}{2\Delta x}\left[U_j - \Delta x \partial_x U + \frac{(\Delta x)^2}{2!}\partial_{xx} U + \dots\right]$$

$$= \partial_x U + O((\Delta x)^2)$$

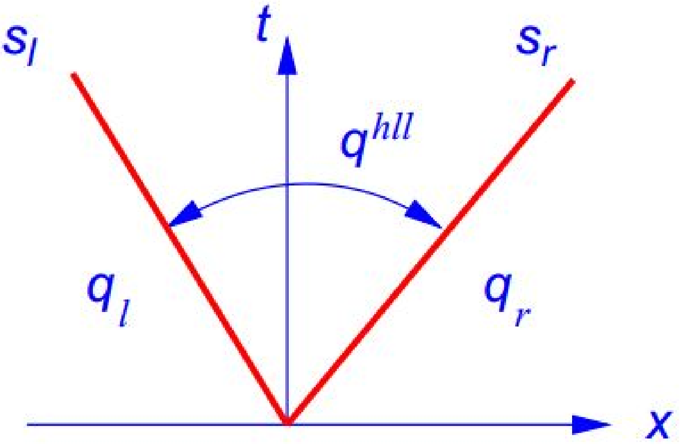
HLL

Riemann solver of Harten, Lax and van Leer
System of one-dimensional conservation laws:

$$q_t + f(q)_x = 0$$

with

$$q(x,0) = \begin{cases} q_l & \text{if } x < 0 \\ q_r & \text{if } x > 0 \end{cases}$$


s_l
t
s_r
q^{hll}
q_l
q_r
x

HLL

Integrating over control volume:

$$q^{hll} = \frac{s_r q_r - s_l q_l + f_l - f_r}{s_r - s_l}$$

Following approximation is proposed:

$$\tilde{q}(x,t) = \begin{cases} q_l & \text{if } \frac{x}{t} \leq s_l \\ q^{hll} & \text{if } s_l \leq \frac{x}{t} \leq s_r \\ q_r & \text{if } \frac{x}{t} \geq s_r \end{cases}$$

Corresponding flux:

$$f^{hll}_{i+1/2} = \begin{cases} f_l & \text{if } 0 \leq s_l \\ \frac{s_r f_l - s_l f_r + s_l s_r (q_r - q_l)}{s_r - s_l} & \text{if } s_l \leq 0 \leq s_r \\ f_r & \text{if } 0 \geq s_r \end{cases}$$

HLL

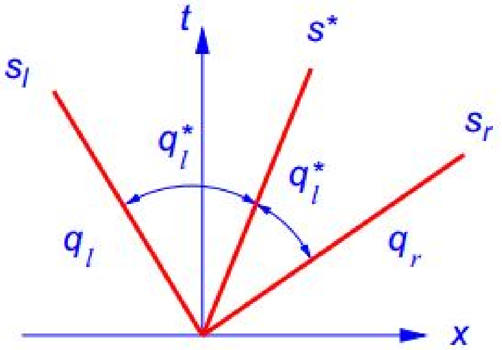
These formulas can be used in the explicit conservative formula:

$$q_i^{n+1} = q_i^n + \frac{\Delta t}{\Delta x}\left[f_{i-1/2} - f_{i+1/2}\right]$$

Wave speeds s_l and s_r must be estimated.

HLLC

Modification of HLL scheme in which the missing contact and shear waves are restored.



HLLC

$$\tilde{q}(x,t) = \begin{cases} q_l & \text{if } \frac{x}{t} \le s_l \\ q_l^* & \text{if } s_l \le \frac{x}{t} \le s^* \\ q_r^* & \text{if } s^* \le \frac{x}{t} \le s_r \\ q_r & \text{if } \frac{x}{t} \ge s_r \end{cases}$$

$$f_{i+1/2}^{hllc} = \begin{cases} f_l & \text{if } 0 \le s_l \\ f_l + s_l(q_l^* - q_l) & \text{if } s_l \le 0 \le s^* \\ f_r + s_r(q_r^* - q_r) & \text{if } s^* \le 0 \le s_r \\ f_r & \text{if } 0 \ge s_r \end{cases}$$

Intermediate states q_l^* and q_r^* can be determined.
Wave speeds s_l, s_r and s^* need to be estimated.

MPI-AMRVAC and HPC

Maar heel even aanhalen... Iedereen kent dit toch al.

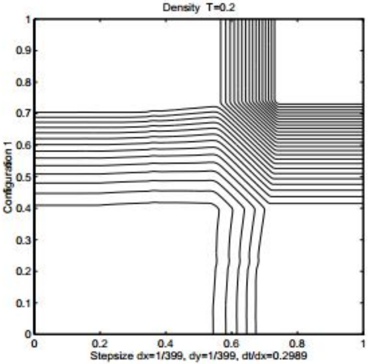
Personal working method

Talk about the scrip we wrote to set up the project and run everything.
Tell them we ran the code succsefully for small problem on the cw network on 5 computers. =¿ network was not stable or something else went wrong.

Configuration 1 - Results from paper

CONFIGURATION 1. $\overrightarrow{R_{32}}$ $\overset{\overrightarrow{R_{21}}}{\underset{\overrightarrow{R_{34}}}{}}$ $\overrightarrow{R_{41}}$: the initial data are

$p_2 = 0.4$ $u_2 = -0.7259$	$\rho_2 = 0.5197$ $v_2 = 0$	$p_1 = 1$ $u_1 = 0$	$\rho_1 = 1$ $v_1 = 0$
$p_3 = 0.0439$ $u_3 = -0.7259$	$\rho_3 = 0.1072$ $v_3 = -1.4045$	$p_4 = 0.15$ $u_4 = 0$	$\rho_4 = 0.2579$ $v_4 = -1.4045$



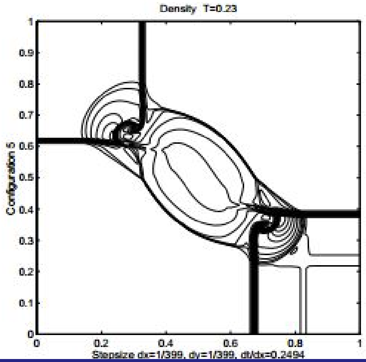
Configuration 1 - Own results

movie

Configuration 5 - Results from paper

CONFIGURATION 5. $\begin{matrix} & J_{21}^- & \\ J_{32}^- & & J_{41}^- \\ & J_{34}^- & \end{matrix}$: the initial data are

$p_2 = 1$	$\rho_2 = 2$	$p_1 = 1$	$\rho_1 = 1$
$u_2 = -0.75$	$v_2 = 0.5$	$u_1 = -0.75$	$v_1 = -0.5$
$p_3 = 1$	$\rho_3 = 1$	$p_4 = 1$	$\rho_4 = 3$
$u_3 = 0.75$	$v_3 = 0.5$	$u_4 = 0.75$	$v_4 = -0.5$



Configuration 5 - Own results

movie

Configuration 5 - Own results

High Resolution - HLLC