Computer Vision

Assignment 1

Tongji University
School of Software Engineering

ID:	1754060
Name:	Zhe Zhang
Adviser:	Prof. Lin Zhang
Time:	Mon. 2-4 [1-17]
Email:	dbzdbz@tongji.edu.cn

Assignment 1

Contents

1 Problem 1(Math)

In our lectures, we mentioned that matrices that can represent isometries can form a group. Specifically, in 3D space, the set comprising matrices $\{\mathbf{M}_i\}$ is actually a group, where $\mathbf{M}_i = \begin{bmatrix} \mathbf{R}_i & \mathbf{t}_i \\ \mathbf{0} & 1 \end{bmatrix} \in \mathbb{R}^{4\times4}, \mathbf{R}_i \in \mathbb{R}^{3\times3}$ is an orthonormal matrix and $\mathbf{t}_i \in \mathbb{R}^{3\times1}$ is a vector.

Please prove that the set $\{\mathbf{M}_i\}$ forms a group.

Hint: You need to prove that $\{\mathbf{M}_i\}$ satisfies the four properties of a group, i.e., closure, associativity, existence of identity element, and existence of inverse element for each group element.

Solution:

Suppose that the element of group is M_i, and the operation on this group is **Matrix Multiply**.

- **Closure**

  $\forall M_i, M_j \in \{M_i\}$

$$M_i \otimes M_j = \begin{bmatrix} R_i & t_i \\ 0 & 1 \end{bmatrix} \begin{bmatrix} R_j & t_j \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} R_iR_j & R_it_j + t_i \\ 0 & 1 \end{bmatrix} \tag{1}$$

  $R_i, R_j \in \mathbb{R}^{3\times3} \Rightarrow R_iR_j \in \mathbb{R}^{3\times3}$
  $t_i, t_j \in \mathbb{R}^{3\times1} \Rightarrow R_it_j + t_i \in \mathbb{R}^{3\times1}$

  Hence, $M_i \otimes M_j \in \{M_i\}$, the operation of this group meet the requirement of closure.

- **Associativity**

$$\begin{aligned} M_i \otimes (M_j \otimes M_k) &= \begin{bmatrix} R_i & t_i \\ 0 & 1 \end{bmatrix} \begin{bmatrix} R_j & t_j \\ 0 & 1 \end{bmatrix} \begin{bmatrix} R_k & t_k \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} R_i & t_i \\ 0 & 1 \end{bmatrix} \begin{bmatrix} R_jR_k & R_jt_k + t_j \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} R_iR_jR_k & R_i(R_jt_k + t_j) + t_i \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} R_iR_jR_k & R_iR_jt_k + R_it_j + t_i \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} R_iR_j & R_it_j + t_i \\ 0 & 1 \end{bmatrix} \begin{bmatrix} R_k & t_k \\ 0 & 1 \end{bmatrix} \\ &= (M_i \otimes M_j) \otimes M_k \end{aligned} \tag{2}$$

  Hence, $M_i \otimes (M_j \otimes M_k) = (M_i \otimes M_j) \otimes M_k$, the operation of this group meet the requirement of associativity.

- **Existence of identity element**

  $\forall M_i \in \{M_i\}$

  $\exists g = \begin{bmatrix} A & B \\ 0 & 1 \end{bmatrix}$, where $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

  $$\begin{bmatrix} A & B \\ 0 & 1 \end{bmatrix} \begin{bmatrix} R_i & t_i \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} R_i & t_i \\ 0 & 1 \end{bmatrix} \begin{bmatrix} A & B \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} R_i & t_i \\ 0 & 1 \end{bmatrix} \tag{3}$$

  Hence, $gM_i = M_i g = M_i$, the operation of this group meet the requirement of existence of identity element for each group element, and the identity element is g.

- **Existence of inverse element**

  $\forall M_i \in \{M_i\}$

  $\exists A = \begin{bmatrix} R_i^{-1} & -R_i^{-1}t_i \\ 0 & 1 \end{bmatrix}$

  $$\begin{bmatrix} R_i^{-1} & -R_i^{-1}t_i \\ 0 & 1 \end{bmatrix} \begin{bmatrix} R_i & t_i \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} R_i & t_i \\ 0 & 1 \end{bmatrix} \begin{bmatrix} R_i^{-1} & -R_i^{-1}t_i \\ 0 & 1 \end{bmatrix} = E_{4\times 4} \tag{4}$$

  Because R_i is an orthonormal matrix, R_i^{-1} must exist, and $R_iR_i^{-1} = R_i^{-1}R_i = E$

  Hence, $AM_i = M_iA = E$, the operation of this group meet the requirement of existence of inverse element for each group element, and the inverse element for M_i is $\begin{bmatrix} R_i^{-1} & -R_i^{-1}t_i \\ 0 & 1 \end{bmatrix}$.

2 Problem 2(Math)

Gaussian function is

$$G(x,y;\sigma)=\frac{1}{2\pi\sigma^2}\exp\left(-\frac{x^2+y^2}{2\sigma^2}\right)$$

The scale-normalized Laplacian of Gaussian (LOG) is

$$LoG=\sigma^2\nabla^2 G$$

Please verify that Difference of Gaussian (DOG)

$$DoG=G(x,y;k\sigma)-G(x,y;\sigma)$$

can be a good approximation of LoG.

Solution:

First, calculate the LoG and DoG

- **LoG:**

$$LoG=\sigma^2\bigtriangledown^2 G=\frac{x^2+y^2-2\sigma^2}{2\pi\sigma^4}exp(-\frac{x^2+y^2}{2\sigma^2}) \tag{5}$$

some of the part in Formula(5) are as follow:

$$\begin{aligned}\frac{\partial G}{\partial x}&=-\frac{x}{2\pi\sigma^4}exp(-\frac{x^2+y^2}{2\sigma^2})\\ \frac{\partial G}{\partial y}&=-\frac{y}{2\pi\sigma^4}exp(-\frac{x^2+y^2}{2\sigma^2})\end{aligned}$$

$$\begin{aligned}\frac{\partial^2 G}{\partial x^2}&=\frac{\partial\left(\frac{\partial G}{\partial x}\right)}{\partial x}+\frac{\partial\left(\frac{\partial G}{\partial y}\right)}{\partial x}\\ &=-\frac{1}{2\pi\sigma^4}\left[exp\left(-\frac{x^2+y^2}{2\sigma^2}\right)+xexp\left(-\frac{x^2+y^2}{2\sigma^2}\right)\left(-\frac{2x}{2\sigma^2}\right)\right]\\ &=-\frac{1}{2\pi\sigma^4}\left[\left(1-\frac{x^2}{\sigma^2}\right)exp\left(-\frac{x^2+y^2}{2\sigma^2}\right)\right]\\ &=\frac{x^2-\sigma^2}{2\pi\sigma^6}exp\left(-\frac{x^2+y^2}{2\sigma^2}\right)\\ \frac{\partial^2 G}{\partial y^2}&=\frac{y^2-\sigma^2}{2\pi\sigma^6}exp\left(-\frac{x^2+y^2}{2\sigma^2}\right)\end{aligned} \tag{6}$$

$$\bigtriangledown^2 G=\frac{\partial^2 G}{\partial x^2}+\frac{\partial^2 G}{\partial y^2}=\frac{x^2+y^2-2\sigma^2}{2\pi\sigma^6}exp(-\frac{x^2+y^2}{2\sigma^2})$$

- **DoG:**

$$\begin{aligned} DoG &= G(x,y;k\sigma) - G(x,y;\sigma) \\ &= \frac{1}{2\pi(k\sigma)^2}exp(-\frac{x^2+y^2}{2(k\sigma)^2}) - \frac{1}{2\pi\sigma^2}exp(-\frac{x^2+y^2}{2\sigma^2}) \end{aligned} \tag{7}$$

Notice that the first-order partial derivative of the two-dimensional Gaussian function with respect to σ is:

$$\begin{aligned} \frac{\partial G}{\partial \sigma} &= -\frac{1}{\pi\sigma^3}exp(-\frac{x^2+y^2}{2\sigma^2}) + \frac{1}{2\pi\sigma^5}exp(-\frac{x^2+y^2}{2\sigma^2})(x^2+y^2) \\ &= \frac{x^2+y^2-2\sigma^2}{2\pi\sigma^5}exp(-\frac{x^2+y^2}{2\sigma^2}) \end{aligned} \tag{8}$$

Hence, we can find that

$$\frac{\partial G}{\partial \sigma} = \sigma \bigtriangledown^2 G \tag{9}$$

Represent $\frac{\partial G}{\partial \sigma}$ in another way:

$$\begin{aligned} \frac{\partial G}{\partial \sigma} &= \lim_{\Delta\sigma \to 0} \frac{G(x,y,\sigma+\Delta\sigma - G(x,y,\sigma)}{\Delta\sigma} \\ &\xlongequal{\text{let t} = \sigma+\Delta\sigma} \lim_{t \to \sigma} \frac{G(x,y,t) - G(x,y,\sigma)}{t-\sigma} \\ &\xlongequal{\text{let t} = \text{k}\sigma} \frac{G(x,y,k\sigma) - G(x,y,\sigma)}{k\sigma - \sigma} \end{aligned} \tag{10}$$

From Formula(10) and Formula(9), we get

$$\begin{aligned} (k-1)\sigma\frac{\partial G}{\partial \sigma} &= G(x,y,k\sigma) - G(x,y,\sigma) \\ (k-1)\sigma\sigma \bigtriangledown^2 G &= DoG \\ (k-1)LoG &= DoG \end{aligned} \tag{11}$$

(Here, we suppose that $k\sigma = \sigma + \Delta\sigma, k = \frac{\Delta\sigma}{\sigma} + 1$)

From what has been discussed above $DoG = (k-1)LoG$, it means that DoG hold the same shape trend with LoG, the diffenence is amplitude and something small. Hence, DoG can be a good approximation of LoG.

3 Problem 3(Math)

In the lecture, we talked about the least square method to solve an over-determined linear system $A\mathbf{x} = b, A \in \mathbb{R}^{m\times n}, \mathbf{x} \in \mathbb{R}^{n\times 1}, m > n, \text{rank}(A) = n$. The closedform solution is $\mathbf{x} = \left(A^T A\right)^{-1} A^T b$. Try to prove that $A^T A$ is non-singular (or in other words, it is invertible).

Solution:

From the assumption we know that $A \in \mathbb{R}^{m\times n}$ and $A^T \in \mathbb{R}^{n\times m}$, so we got that $A^T A \in \mathbb{R}^{n\times n}$, it is a square matrix.

Because $rank(A) = A$, we know that A has full column rank, and we can also know that $rank(A^T) = n$, A^T has full row rank.(according to property(3))

According to properties of matrix ranking,

$$\begin{gathered} r(A) + r(A^T) - n \leq r(AA^T) \leq min\{n, m\} \\ \Rightarrow r(AA^T) = n \end{gathered} \tag{12}$$

Hence, AA^T has both full row rank and full column rank. So $A^T A = (AA^T)^T$ also has both full row rank and full column rank.

From what has been discussed above $A^T A$ is non-sigular.

"Transpose does change the rank of matrix"

Proof: According to the matrix ranking property we know that column rank is equal to the row rank of a matrix, so

$$rank(A) = rrank(A) = crank(A) \tag{13}$$

And we also have $rrank(A) = crank(A^T)$, $crank(A) = rrank(A^T)$. Hence, $rank(A) = rank(A^T)$