Computer Vision

Assignment 2

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1 Problem 1(Math)

In the augmented Euclidean plane, there is a line $x-3y+4=0$, what is the homogeneous coordinate of the infinity point of this line?

Solution: According to the theory of "On the project plane the intersection of two line l, l' is $x = l \times l'$"

- Equation of the infinite line: $0x + 0y + z = 0$
- Equation of this specific line: $\frac{x}{z} - 3\frac{y}{z} + 4 = 0 \Rightarrow x - 3y + 4z = 0$

The simultaneous equations solve the intersection coordinates:

$$\begin{cases} 0x + 0y + z & = 0 \\ x - 3y + 4z & = 0 \end{cases} \tag{1}$$

The line coordinates of two lines are $(0, 0, 1)$ and $(1, -3, 4)$. The cross point is

$$(0,0,1)^T \times (1,-3,4)^T = \begin{vmatrix} i & j & k \\ 0 & 0 & 1 \\ 1 & -3 & 4 \end{vmatrix} = - \begin{vmatrix} i & j \\ 1 & -3 \end{vmatrix} = 3i + j \tag{2}$$

Take $i = 1$, the homogeneous corrdinate of the infinity point of this line is $(3, 1, 0)$

2 Problem 2(Math)

A,B,C and D are four points in 3D Euclidean space, their coordinates are $(x_i, y_i, z_i), i = 1, 2, 3, 4$, respectively. Please prove that:

$$\text{These four points are coplanar} \Leftrightarrow \begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix} = 0$$

Solution:

let

$$\begin{cases} \vec{a} = B - A = (x_2 - x_1, y_2 - y_1, z_2 - z_1) \\ \vec{b} = C - A = (x_3 - x_1, y_3 - y_1, z_3 - z_1) \\ \vec{c} = D - A = (x_4 - x_1, y_4 - y_1, z_4 - z_1) \end{cases}$$

- From the left side:

$$\begin{aligned} &\text{These four points are coplanar} \\ &\Leftrightarrow \vec{a}, \vec{b}, \vec{c} \text{ are coplanar} \\ &\Leftrightarrow (\vec{a}, \vec{b}, \vec{c}) = 0 \\ &\Leftrightarrow (\vec{a} \times \vec{b}) \cdot \vec{c} = 0 \\ &\Leftrightarrow \begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \\ x_4 - x_1 & y_2 - y_1 & z_4 - z_1 \end{vmatrix} = 0 \end{aligned} \tag{3}$$

- From the right side:

$$\begin{aligned} &\begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix} = 0 \xlongequal{r_4 - r_1 \ \ r_3 - r_1 \ \ r_2 - r_1} \begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 & 0 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 & 0 \\ x_4 - x_1 & y_4 - y_1 & z_4 - z_1 & 0 \end{vmatrix} = 0 \\ &\xlongequal{expandalongthefirstrow} \\ &x_1 \begin{vmatrix} y_2 - y_1 & z_2 - z_1 & 0 \\ y_3 - y_1 & z_3 - z_1 & 0 \\ y_4 - y_1 & z_4 - z_1 & 0 \end{vmatrix} - y_1 \begin{vmatrix} x_2 - x_1 & z_2 - z_1 & 0 \\ x_3 - x_1 & z_3 - z_1 & 0 \\ x_4 - x_1 & z_4 - z_1 & 0 \end{vmatrix} \\ &+ z_1 \begin{vmatrix} x_2 - x_1 & y_2 - y_1 & 0 \\ x_3 - x_1 & y_3 - y_1 & 0 \\ x_4 - x_1 & y_4 - y_1 & 0 \end{vmatrix} - \begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \\ x_4 - x_1 & y_4 - y_1 & z_4 - z_1 \end{vmatrix} = 0 \end{aligned} \tag{4}$$

Because the first 3 parts all equal to 0, so we can also get

$$\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \\ x_4 - x_1 & y_2 - y_1 & z_4 - z_1 \end{vmatrix} = 0$$

Hence, the left side is equivalent to the right side.

3 Problem 3(Math)

On the normalized retinal plane, suppose that p_n is an ideal point of projection without considering distortion. If distortion is considered, $p_n = (x, y)^T$ is mapped to $p_d = (x_d, y_d)^T$ which is also on the normalized retinal plane. Their relationship is,

$$\begin{cases} x_d &= x(1 + k_1 r^2 + k_2 r^4 + 2\rho_1 xy + \rho_2(r^2 + 2x^2) + xk_3 r^6 \\ y_d &= y(1 + k_1 r^2 + k_2 r^4 + 2\rho_2 xy + \rho_2(r^2 + 2y^2) + yk_3 r^6 \end{cases}$$

where $r^2 = x^2 + y^2$ For performing nonlinear optimization in the pipeline of camera calibration, we need to compute the Jacobian matrix of p_d w.r.t p_n, i.e., $\frac{dp_n}{dp_n^T}$ It should be noted that in this question p_d is the function of p_n and all the other parameters can be regarded as constants.

Solution:

Substitute $r^2 = x^2 + y^2$ into the system

$$\begin{aligned} x_d &= x + xk_1(x^2 + y^2) + xk_2(x^4 + 2x^2y^2 + y^4) + 2\rho_1 xy + \rho_2(3x^2 + y^2) + xk_3(x^6 + 3x^4y^2 + 3x^2y^4 + y^6) \\ &= x + k_1(x^3 + xy^2) + k_2(x^5 + 2x^3y^2 + xy^4) + 2\rho_1 xy + \rho_2(3x^2 + y^2) + k_3(x^7 + 3x^5y^2 + 3x^3y^4 + xy^6) \end{aligned}$$

And symmetriclly, we can expand y_d

$$y_d = y + k_1(yx^3 + y^3) + k_2(x^4y + 2x^2y^3 + y^5) + 2\rho_2 xy + \rho_1(x^2 + 3y^2) + k_3(x^6y + 3x^4y^3 + 3x^2y^5 + y^7)$$

What we want to calculate is

$$\frac{dp_d}{dp_n^T} = \begin{bmatrix} \frac{\partial x_d}{\partial x_n} & \frac{\partial x_d}{\partial y_n} \\ \frac{\partial y_d}{\partial x_n} & \frac{\partial y_d}{\partial y_n} \end{bmatrix} \tag{5}$$

$$\begin{aligned} \frac{\partial x_d}{\partial x_n} &= 1 + k_1(3x^2 + y^2) + k_2(5x^4 + 6y^2x^2 + y^4) + 2\rho_1 y + 6\rho_2 x + k_3(7x^6 + 15y^2x^4 + 9y^4x^2 + y^6) \\ &= 7k_3x^6 + (5k_2 + 15k_3y^2)x^4 + (3k_1 + 6k_2y^2 + 9k_3y^4)x^2 + 6\rho_2 x + (k_3y^6 + k_2y^4 + k_1y^2 + 2\rho_1 y + 1) \end{aligned}$$

$$\begin{aligned} \frac{\partial x_d}{\partial y_n} &= 2k_1xy + k_2(4x^3y + 4xy^3) + 2\rho_1 x + 2\rho_2 y + k_3(6x^5y + 12x^3y^3 + 6xy^5) \\ &= 6k_3xy^5 + (4k_2x + 12k_3x^3)y^3 + (2k_1x + 2\rho_2 + 6k_3x^5 + 4k_2x^3)y + 2\rho_1 x \end{aligned}$$

And symmeticlly, exchange $x \leftrightarrow y$ and $\rho_1 \leftrightarrow \rho_2$

$$\frac{\partial y_d}{\partial x_n} = 6k_3yx^5 + (4k_2y + 12k_3y^3)x^3 + (2k_1y + 2\rho_1 + 6k_3y^5 + 4k_2y^3)x + 2\rho_2 y$$

$$\frac{\partial y_d}{\partial y_n} = 7k_3y^6 + (5k_2 + 15k_3x^2)y^4 + (3k_1 + 6k_2x^2 + 9k_3x^4)y^2 + 6\rho_1 y + (k_3x^6 + k_2x^4 + k_1x^2 + 2\rho_2 x + 1)$$

Considering the complexity of calculation, I use **Symbolic Calculation** in MATLAB for validation. The MATLAB code:

```
syms x y k1 k2 rho1 rho2 k3
r = (x^2 + y^2)^(1/2);

xd(x,y) = x*(1+k1*r^2+k2*r^4) + 2*rho1*x*y + rho2*(r^2+2*x^2) + x*k3*r^6;
Dxd_Dx = diff(xd(x,y),x)
expand(Dxd_Dx)
Dxd_Dy = diff(xd(x,y),y);
expand(Dxd_Dy)

yd(x,y) = y*(1+k1*r^2+k2*r^4) + 2*rho2*x*y + rho1*(r^2+2*y^2) + y*k3*r^6;
Dyd_Dx = diff(yd(x,y),x);
expand(Dyd_Dx)
Dyd_Dy = diff(yd(x,y),y);
expand(Dyd_Dy)
```

4 Problem 4(Math)

In our lecture, we mebtioned that for performing nonlinear optimization in the pipeline of camera calibration, we need to compute the Jacobian of the rotation matrix(represented in a vector) w.r.t its axis-angle representation. In this question, your task is to derive the concrete formula of this Jacobian matrix.

Suppose that $r = \theta n \in \mathbb{R}^{3\times 1}$, where $n = \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix}$ is a 3D unit vector and θ is a real number denoting the rotation angle.

With **Rodrigues formula**, r can be converted to its rotation matrix form, $R = cos\theta I + (1 - cos\theta)nn^T + sin\theta n^\wedge$ and obviously $R \triangleq \begin{bmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{bmatrix}$ is a 3×3 matrix.

Denote u by the vectorized form of R, i.e., $u \triangleq (R_{11}, R_{12}, R_{13}, R_{21}, R_{22}, R_{23}, R_{31}, R_{32}, R_{33})^T$

Please give the concrete form of Jacobian matrix of U w.r.t r, i.e., $\frac{du}{dr^T} \in \mathbb{R}^{9\times 3}$.

In order to make it easy to check your result, please follow the following notation requirements, $\alpha \triangleq sin\theta, \beta \triangleq cos\theta, \gamma \triangleq 1 - cos\theta$

In other words, the ingredients appearing in your fornula are restricted to $\alpha, \beta, \gamma, \theta, n_1, n_2, n_3$.

Solution:

According to **Rodrigues formula**

$$\begin{aligned} R &= \beta I + \gamma nn^T + \alpha n^\wedge \\ n &= \begin{bmatrix} n_1 & n_2 & n_3 \end{bmatrix}^T \\ n^\wedge &= \begin{bmatrix} 0 & -n_3 & n_2 \\ n_3 & 0 & -n_1 \\ -n_2 & n_1 & 0 \end{bmatrix} \end{aligned} \tag{6}$$

$$\begin{aligned} R &= \begin{bmatrix} \beta & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \beta \end{bmatrix} + \gamma \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} \begin{bmatrix} n_1 n_2 n_3 \end{bmatrix} + \alpha \begin{bmatrix} 0 & -n_3 & n_2 \\ n_3 & 0 & -n_1 \\ -n_2 & n_1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} \beta + \gamma n_1^2 & \gamma n_1 n_2 - \alpha n_3 & \gamma n_1 n_3 + \alpha n_2 \\ \gamma n_1 n_2 + \alpha n_3 & \beta + \gamma n_2^2 & \gamma n_2 n_3 - \alpha n_1 \\ \gamma n_1 n_3 - \alpha n_2 & \gamma n_2 n_3 + \alpha n_1 & \beta + \gamma n_3^3 \end{bmatrix} \end{aligned} \tag{7}$$

$$\frac{du}{dr^T} = \begin{bmatrix} \frac{dR_{11}}{dr^T} \\ \frac{dR_{12}}{dr^T} \\ \frac{dR_{13}}{dr^T} \\ \frac{dR_{21}}{dr^T} \\ \frac{dR_{22}}{dr^T} \\ \frac{dR_{23}}{dr^T} \\ \frac{dR_{31}}{dr^T} \\ \frac{dR_{32}}{dr^T} \\ \frac{dR_{33}}{dr^T} \end{bmatrix}$$

$$\frac{dR_{11}}{dr^T} = \frac{d(\beta + \gamma n_1^2)}{d(\theta n)^T} = \frac{1}{\theta} \begin{bmatrix} 2\gamma n_1 & 0 & 0 \end{bmatrix}$$

$$\frac{dR_{12}}{dr^T} = \frac{d(\gamma n_1 n_2 - \alpha n_3)}{d(\theta n)^T} = \frac{1}{\theta} \begin{bmatrix} \gamma n_2 & \gamma n_1 & -\alpha \end{bmatrix}$$

$$\frac{dR_{13}}{dr^T} = \frac{d(\gamma n_1 n_3 + \alpha n_2)}{d(\theta n)^T} = \frac{1}{\theta} \begin{bmatrix} \gamma n_3 & \alpha & \gamma n_1 \end{bmatrix}$$

$$\frac{dR_{21}}{dr^T} = \frac{d(\gamma n_1 n_2 + \alpha n_3)}{d(\theta n)^T} = \frac{1}{\theta} \begin{bmatrix} \gamma n_2 & \gamma n_1 & \alpha \end{bmatrix}$$

$$\frac{dR_{22}}{dr^T} = \frac{d(\beta + \gamma n_2^2)}{d(\theta n)^T} = \frac{1}{\theta} \begin{bmatrix} 0 & 2\gamma n_2 & 0 \end{bmatrix}$$

$$\frac{dR_{23}}{dr^T} = \frac{d(\gamma n_2 n_3 - \alpha n_1)}{d(\theta n)^T} = \frac{1}{\theta} \begin{bmatrix} -\alpha & \gamma n_3 & \gamma n_2 \end{bmatrix}$$

$$\frac{dR_{31}}{dr^T} = \frac{d(\gamma n_1 n_3 - \alpha n_2)}{d(\theta n)^T} = \frac{1}{\theta} \begin{bmatrix} \gamma n_3 & -\alpha & \gamma n_1 \end{bmatrix}$$

$$\frac{dR_{32}}{dr^T} = \frac{d(\gamma n_2 n_3 + \alpha n_1)}{d(\theta n)^T} = \frac{1}{\theta} \begin{bmatrix} \alpha & \gamma n_3 & \gamma n_2 \end{bmatrix}$$

$$\frac{dR_{33}}{dr^T} = \frac{d(\beta + \gamma n_3^3)}{d(\theta n)^T} = \frac{1}{\theta} \begin{bmatrix} 0 & 0 & 2\gamma n_3 \end{bmatrix}$$

Hence, the final result is

$$\frac{du}{dr^T} = \frac{1}{\theta} \begin{bmatrix} 2\gamma n_1 & 0 & 0 \\ \gamma n_2 & \gamma n_1 & -\alpha \\ \gamma n_3 & \alpha & \gamma n_1 \\ \gamma n_2 & \gamma n_1 & \alpha \\ 0 & 2\gamma n_2 & 0 \\ -\alpha & \gamma n_3 & \gamma n_2 \\ \gamma n_3 & -\alpha & \gamma n_1 \\ \alpha & \gamma n_3 & \gamma n_2 \\ 0 & 0 & 2\gamma n_3 \end{bmatrix} \tag{8}$$