1 二维下的情形

定义：

$$DFT : F(u,v) = \sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y)e^(-j2\pi(ux/M+vy/N)), u = 0,1,...,M-1; v = 0,1,...,N-1 \tag{1}$$

$$IDFT : f(x,y) = \frac{1}{MN}\sum_{x=0}^{M-1}\sum_{y=0}^{N-1} F(u,v)e^(-j2\pi(ux/M+vy/N)), x = 0,1,...,M-1; y = 0,1,...,N-1 \tag{2}$$

问题变成：已知矩阵 $x(m,n)$ 长度为 $M_x * N_x$, y(m,n) 长度为 $M_y * N_y$, 通过傅立叶变换计算卷积的过程可描述如下：

$$\begin{gathered} x(m,n) \xrightarrow{M*N\ \ DFT} X(k,l), \quad y(m,n) \xrightarrow{M*N\ \ DFT} Y(k,l) \\ X(k,l)Y(k,l) \xrightarrow{M*N\ \ IDFT} x(m,n) \bigotimes y(m,n) \end{gathered} \tag{3}$$

请问：

$$IDFT(X(k,l)Y(k,l)) = x(m,n) \bigotimes y(m,n) \tag{4}$$

何时成立?

推导分析过程如下：

$$\begin{aligned} z(m,n) &\triangleq \frac{1}{MN}\sum_{k=0}^{M-1}\sum_{l=0}^{N-1} X(k,l)Y(k,L)e^{j2\pi(\frac{km}{M}+\frac{ln}{N})} \\ &= \frac{1}{MN}\sum_{k=0}^{M-1}\sum_{l=0}^{N-1}(\sum_{a=0}^{M-1}\sum_{b=0}^{N-1} x(a,b)e^{-j2\pi(\frac{ka}{M}+\frac{lb}{N})})(\sum_{c=0}^{M-1}\sum_{d=0}^{N-1} y(c,d)e^{-j2\pi(\frac{kc}{M}+\frac{ld}{N})})e^{j2\pi(\frac{km}{M}+\frac{ln}{N})} \\ &= \frac{1}{MN}\sum_{a=0}^{M-1}\sum_{b=0}^{N-1}\sum_{c=0}^{M-1}\sum_{d=0}^{N-1} x(a,b)y(c,d)\sum_{k=0}^{M-1}\sum_{l=0}^{N-1} e^{j2\pi(\frac{k}{M}(m-a-c)+\frac{l}{N}(n-b-d))} \\ &= \sum_{a=0}^{M-1}\sum_{b=0}^{N-1} x(a,b)y(m-a,n-b)_{M,N} \end{aligned} \tag{5}$$

(注：$(x,y)_{M,N} = (s,t)\ if\ x = pM + s\ and\ y = qN + t$, p, q, s, t 为整数, $0 \le s \le M-1, 0 \le t \le N-1$ 上式中，

$$\sum_{k=0}^{M-1}\sum_{l=0}^{N-1} e^{j2\pi(\frac{k}{M}(m-a-c)+\frac{l}{N}(n-b-d))} = \begin{cases} MN,\ if\ m-a-c = pM\ and\ n-b-d = qN,\ \text{p,q 是整数} \\ 0,\ else \end{cases} \tag{6}$$

证 (6)，

$$\sum_{k=0}^{M-1}\sum_{l=0}^{N-1} e^{j2\pi(\frac{k}{M}(m-a-c)+\frac{l}{N}(n-b-d))} = \sum_{k=0}^{M-1}\sum_{l=0}^{N-1} e^{\frac{j2\pi k}{M}(m-a-c)} * \sum_{k=0}^{M-1}\sum_{l=0}^{N-1} e^{\frac{j2\pi l}{N}(n-b-d)} \tag{7}$$

这里仅对前半部分 $\sum_{k=0}^{M-1}\sum_{l=0}^{N-1} e^{\frac{j2\pi k}{M}(m-a-c)}$ 进行讨论，后半部分同理：

- 当 $m-a-c = pM$, p 为整数时：由欧拉公式展开，

$$\begin{aligned} e^{\frac{j2\pi k}{M}(m-a-c)} &= cos(\frac{2\pi k}{M}(m-a-c)) + jsin(\frac{2\pi k}{M}(m-a-c)) \\ &= cos(s\pi kp) + jsin(2\pi kp) \\ &= 1 + 0 \\ &= 1 \end{aligned} \tag{8}$$

因此此时，

$$\sum_{k=0}^{M-1} e^{\frac{2\pi k}{M}(m-a-c)} = M \tag{9}$$

- 当 $m-a-c \neq pM$, p 为整数时：该式为首项为 1，公比为 $e^{\frac{j2\pi}{M}(m-a-c)}$ 的等比数列求和

$$\begin{aligned} \sum_{k=0}^{M-1} e^{\frac{2\pi k}{M}(m-a-c)} &= \frac{1(1-e^{\frac{j2\pi M}{M}(m-a-c)})}{1-e^{\frac{j2\pi}{M}(m-a-c)}} \\ &= \frac{1-e^{j2\pi(m-a-c)}}{1-e^{\frac{j2\pi}{M}(m-a-c)}} \end{aligned} \tag{10}$$

因为 $m-a-c$ 比为整数，且 $m-a-c \neq pM$ 所以有，

$$\begin{aligned} 1-e^{j2\pi(m-a-c)} &= 1 - cos(2\pi(m-a-c)) - sin(2\pi(m-a-c)) \\ &= 1-1-0 \\ &= 0 \end{aligned} \tag{11}$$

$$\begin{aligned} 1-e^{\frac{j2\pi}{M}(m-a-c)} &= 1-cos(\frac{2\pi}{M}(m-a-c))-sin(\frac{2\pi}{M}(m-a-c)) \\ &\neq 0 \end{aligned} \tag{12}$$

所以有 $\sum_{k=0}^{M-1} e^{\frac{2\pi k}{M}(m-a-c)}=0$，证毕

同理,

$$\sum_{k=0}^{M-1}\sum_{l=0}^{N-1} e^{\frac{j2\pi l}{N}(n-b-d)} = \begin{cases} N, if\ n-b-d=qN,\ N\text{是整数} \\ 0, else \end{cases} \tag{13}$$

Fact1: $z(m,n)$ 是横向周期为 M，纵向周期为 N 的序列

Fact2: 当 $M \leq M_x+M_y-1\ and\ N \leq N_x+N_y-1$ 时, $z(m,n)=x(m,n)\bigotimes y(m,n),\ m=0,1,...,M_x+M_y-1,\ n=0,...,N_x+N_y-1$