2
Relativity II

Chapter Outline

In this chapter we extend the theory of special relativity to classical mechanics, that is, we give relativistically correct expressions for momentum, Newton's second law, and the famous equivalence of mass and energy. The final section, on general relativity, deals with the physics of accelerating reference frames and Einstein's theory of gravitation.

2.1 RELATIVISTIC MOMENTUM AND THE RELATIVISTIC FORM OF NEWTON'S LAWS

The conservation of linear momentum states that when two bodies collide, the total momentum remains constant, assuming the bodies are isolated (that is, they interact only with each other). Suppose the collision is described in a reference frame S in which momentum is conserved. If the velocities of the colliding bodies are calculated in a second inertial frame S′ using the Lorentz transformation, and the classical definition of momentum $\mathbf{p} = m\mathbf{u}$ applied, one finds that momentum *is not* conserved in the second reference frame. However, because the laws of physics are the same in all inertial frames, momentum must be conserved in all frames if it is conserved in any one. This application of the principle of relativity demands that we modify the classical definition of momentum.

To see how the classical form $\mathbf{p} = m\mathbf{u}$ fails and to determine the correct relativistic definition of $\mathbf{p}$, consider the case of an inelastic collision

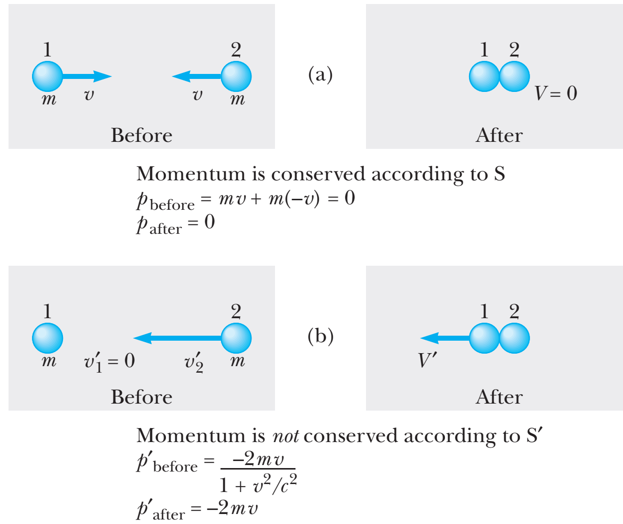

Figure 2.1 (a) An inelastic collision between two equal clay lumps as seen by an observer in frame S. (b) The same collision viewed from a frame S′ that is moving to the right with speed v with respect to S.

between two particles of equal mass. Figure 2.1a shows such a collision for two identical particles approaching each other at speed v as observed in an inertial reference frame S. Using the classical form for momentum, $\mathbf{p} = m\mathbf{u}$ (we use the symbol $\mathbf{u}$ for particle velocity rather than $\mathbf{v}$, which is reserved for the relative velocity of two reference frames), the observer in S finds momentum is conserved as shown in Figure 2.1a. Suppose we now view things from an inertial frame S′ moving to the right with speed v relative to S. In S′ the new speeds are v_1', v_2' and V' (see Fig. 2.1b). If we use the Lorentz velocity transformation

$$u_x' = \frac{u_x - v}{1 - (u_x v/c^2)}$$

to find v_1', v_2' and V', and the classical form for momentum, $\mathbf{p} = m\mathbf{u}$, will momentum be conserved according to the observer in S′? To answer this question we first calculate the velocities of the particles in S′ in terms of the given velocities in S.

$$v_1' = \frac{v_1 - v}{1 - (v_1 v/c^2)} = \frac{v - v}{1 - (v^2/c^2)} = 0$$

$$v_2' = \frac{v_2 - v}{1 - (v_2 v/c^2)} = \frac{-v - v}{1 - [(-v)(v)/c^2]} = \frac{-2v}{1 + (v^2/c^2)}$$

$$V' = \frac{V - v}{1 - (Vv/c^2)} = \frac{0 - v}{1 - [(0)v/c^2]} = -v$$

Checking for momentum conservation in S′, we have

$$p'_{\text{before}} = mv'_1 + mv'_2 = m(0) + m\left[\frac{-2v}{1 + (v^2/c^2)}\right] = \frac{-2mv}{1 + (v^2/c^2)}$$

$$p'_{\text{after}} = 2mV' = -2mv$$

Thus, *in the frame S′*, the momentum before the collision is not equal to the momentum after the collision, and *momentum is not conserved.*

It can be shown (see Example 2.6) that momentum is conserved in both S and S′, (and indeed in all inertial frames), if we redefine momentum as

$$\mathbf{p} \equiv \frac{m\mathbf{u}}{\sqrt{1 - (u^2/c^2)}} \tag{2.1}$$

Definition of relativistic momentum

where **u** is the velocity of the particle and *m is the proper mass,* that is, the mass measured by an observer at rest with respect to the mass.[1] Note that when u is much less than c, the denominator of Equation 2.1 approaches unity and **p** approaches $m\mathbf{u}$. Therefore, the relativistic equation for **p** reduces to the classical expression when u is small compared with c. Because it is a simpler expression, Equation 2.1 is often written $\mathbf{p} = \gamma m\mathbf{u}$, where $\gamma = 1/\sqrt{1 - (u^2/c^2)}$. Note that this γ has the same functional form as the γ in the Lorentz transformation, but here γ contains u, the particle speed, while in the Lorentz transformation, γ contains v, the relative speed of the two frames.

The **relativistic form of Newton's second law** is given by the expression

$$\mathbf{F} = \frac{d\mathbf{p}}{dt} = \frac{d}{dt}(\gamma m\mathbf{u}) \tag{2.2}$$

where **p** is given by Equation 2.1. This expression is reasonable because it preserves classical mechanics in the limit of low velocities and requires **the momentum of an isolated system (F = 0) to be conserved relativistically as well as classically.** It is left as a problem (Problem 3) to show that the relativistic acceleration a of a particle *decreases* under the action of a constant force applied in the direction of **u**, as

$$a = \frac{F}{m}(1 - u^2/c^2)^{3/2}$$

From this formula we see that as the velocity approaches c, the acceleration caused by any finite force approaches zero. Hence, it is impossible to accelerate a particle from rest to a speed equal to or greater than c.

[1] In this book we shall always take m to be constant with respect to speed, and we call m the speed invariant mass, or proper mass. Some physicists refer to the mass in Equation 2.1 as the rest mass, m_0, and call the term $m_0/\sqrt{1 - (u^2/c^2)}$ the relativistic mass. Using this description, the relativistic mass is imagined to increase with increasing speed. We exclusively use the invariant mass m because we think it is a clearer concept and that the introduction of relativistic mass leads to no deeper physical understanding.

EXAMPLE 2.1 Momentum of an Electron

An electron, which has a mass of 9.11×10^{-31} kg, moves with a speed of $0.750c$. Find its relativistic momentum and compare this with the momentum calculated from the classical expression.

Solution Using Equation 2.1 with $u = 0.750c$, we have

$$\begin{aligned} p &= \frac{mu}{\sqrt{1 - (u^2/c^2)}} \\ &= \frac{(9.11 \times 10^{-31}\ \text{kg})(0.750 \times 3.00 \times 10^8\ \text{m/s})}{\sqrt{1 - [(0.750c)^2/c^2]}} \\ &= 3.10 \times 10^{-22}\ \text{kg}\cdot\text{m/s} \end{aligned}$$

The incorrect classical expression would give

$$\text{momentum} = mu = 2.05 \times 10^{-22}\ \text{kg}\cdot\text{m/s}$$

Hence, for this case the correct relativistic result is 50% greater than the classical result!

EXAMPLE 2.2 An Application of the Relativistic Form of $\mathbf{F} = d\mathbf{p}/dt$: The Measurement of the Momentum of a High-Speed Charged Particle

Suppose a particle of mass m and charge q is injected with a relativistic velocity $\mathbf{u}$ into a region containing a magnetic field $\mathbf{B}$. The magnetic force $\mathbf{F}$ on the particle is given by $\mathbf{F} = q\mathbf{u} \times \mathbf{B}$. If $\mathbf{u}$ is perpendicular to $\mathbf{B}$, the force is radially inward, and the particle moves in a circle of radius R with $|\mathbf{u}|$ constant. From Equation 2.2 we have

$$\mathbf{F} = \frac{d\mathbf{p}}{dt} = \frac{d}{dt}(\gamma m\mathbf{u})$$

Solution Because the magnetic force is always perpendicular to the velocity, it does no work on the particle, and hence the speed, u, and γ are both constant with time. Thus, the magnitude of the force on the particle is

$$F = \gamma m \left| \frac{d\mathbf{u}}{dt} \right| \tag{2.3}$$

Substituting $F = quB$ and $|d\mathbf{u}/dt| = u^2/R$ (the usual definition of centripetal acceleration) into Equation 2.3, we can solve for $p = \gamma mu$. We find

$$p = \gamma mu = qBR \tag{2.4}$$

Equation 2.4 shows that the momentum of a relativistic particle of known charge q may be determined by measuring its radius of curvature R in a known magnetic field, $\mathbf{B}$. This technique is routinely used to determine the momentum of subatomic particles from photographs of their tracks in space.

2.2 RELATIVISTIC ENERGY

We have seen that the definition of momentum and the laws of motion required generalization to make them compatible with the principle of relativity. This implies that the relativistic form of the kinetic energy must also be modified.

To derive the relativistic form of the work–energy theorem, let us start with the definition of work done by a force F and make use of the definition of relativistic force, Equation 2.2. That is,

$$W = \int_{x_1}^{x_2} F\,dx = \int_{x_1}^{x_2} \frac{dp}{dt}\,dx \tag{2.5}$$

where we have assumed that the force and motion are along the x-axis. To perform this integration and find the work done on a particle or the relativistic kinetic energy as a function of the particle velocity u, we first evaluate dp/dt:

$$\frac{dp}{dt} = \frac{d}{dt}\frac{mu}{\sqrt{1 - (u^2/c^2)}} = \frac{m\left(\dfrac{du}{dt}\right)}{[1 - (u^2/c^2)]^{3/2}} \tag{2.6}$$

Substituting this expression for dp/dt and $dx = u\,dt$ into Equation 2.5 gives

$$W = \int_{x_1}^{x_2} \frac{m\left(\dfrac{du}{dt}\right)u\,dt}{[1 - (u^2/c^2)]^{3/2}} = m\int_0^u \frac{u\,du}{[1 - (u^2/c^2)]^{3/2}}$$

where we have assumed that the particle is accelerated from rest to some final velocity u. Evaluating the integral, we find that

$$W = \frac{mc^2}{\sqrt{1 - (u^2/c^2)}} - mc^2 \qquad (2.7)$$

Recall that the work–energy theorem states that the work done by all forces acting on a particle equals the change in kinetic energy of the particle. Because the initial kinetic energy is zero, we conclude that the work W in Eq. 2.7 is equal to the relativistic kinetic energy K, that is,

$$K = \frac{mc^2}{\sqrt{1 - (u^2/c^2)}} - mc^2 \qquad (2.8)$$

Relativistic kinetic energy

At low speeds, where $u/c \ll 1$, Equation 2.8 should reduce to the classical expression $K = \frac{1}{2}mu^2$. We can check this by using the binomial expansion $(1 - x^2)^{-1/2} \approx 1 + \frac{1}{2}x^2 + \cdots$, for $x \ll 1$, where the higher-order powers of x are ignored in the expansion. In our case, $x = u/c$, so that

$$\frac{1}{\sqrt{1 - (u^2/c^2)}} = \left(1 - \frac{u^2}{c^2}\right)^{-1/2} \approx 1 + \frac{1}{2}\frac{u^2}{c^2} + \cdots$$

Substituting this into Equation 2.8 gives

$$K \approx mc^2\left(1 + \frac{1}{2}\frac{u^2}{c^2} + \cdots\right) - mc^2 = \frac{1}{2}mu^2$$

which agrees with the classical result. A graph comparing the relativistic and nonrelativistic expressions for u as a function of K is given in Figure 2.2. Note that in the relativistic case, the particle speed never exceeds c, regard-

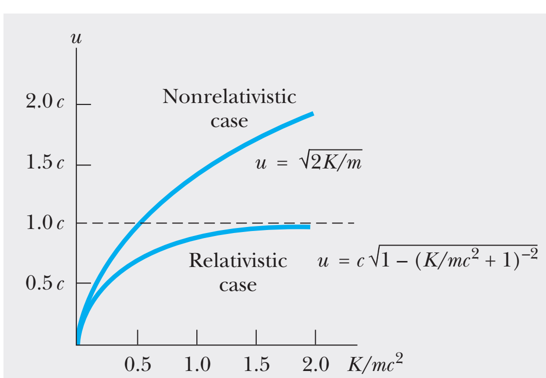

Figure 2.2 A graph comparing the relativistic and nonrelativistic expressions for speed as a function of kinetic energy. In the relativistic case, u is always *less* than c.

less of the kinetic energy, as is routinely confirmed in very high energy particle accelerator experiments. The two curves are in good agreement when $u \ll c$.

It is instructive to write the relativistic kinetic energy in the form

$$K = \gamma mc^2 - mc^2 \tag{2.9}$$

where

$$\gamma = \frac{1}{\sqrt{1 - u^2/c^2}}$$

The constant term mc^2, which is independent of the speed, is called the **rest energy** of the particle. The term γmc^2, which depends on the particle speed, is therefore the sum of the kinetic and rest energies. We define γmc^2 to be the **total energy** E, that is,

Definition of total energy

$$E = \gamma mc^2 = K + mc^2 \tag{2.10}$$

Mass–energy equivalence

The expression $E = \gamma mc^2$ is Einstein's famous mass–energy equivalence equation, which shows that mass is a measure of the total energy in all forms. Although we have been considering single particles for simplicity, Equation 2.10 applies to macroscopic objects as well. In this case it has the remarkable implication that any kind of energy added to a "brick" of matter—electric, magnetic, elastic, thermal, gravitational, chemical—actually increases the mass! Several end-of-chapter questions and problems explore this idea more fully. Another implication of Equation 2.10 is that a small mass corresponds to an enormous amount of energy because c^2 is a very large number. This concept has revolutionized the field of nuclear physics and is treated in detail in the next section.

In many situations, the momentum or energy of a particle is measured rather than its speed. It is therefore useful to have an expression relating the total energy E to the relativistic momentum p. This is accomplished using $E = \gamma mc^2$ and $p = \gamma mu$. By squaring these equations and subtracting, we can eliminate u (Problem 7). The result, after some algebra, is

Energy–momentum relation

$$E^2 = p^2c^2 + (mc^2)^2 \tag{2.11}$$

When the particle is at rest, $p = 0$, and so we see that $E = mc^2$. That is, the total energy equals the rest energy. For the case of particles that have zero mass, such as photons (massless, chargeless particles of light), we set $m = 0$ in Equation 2.11, and find

$$E = pc \tag{2.12}$$

This equation is an *exact* expression relating energy and momentum for photons, which always travel at the speed of light.

Finally, note that because the mass m of a particle is independent of its motion, m must have the same value in all reference frames. On the other hand, the total energy and momentum of a particle depend on the reference frame in which they are measured, because they both depend on velocity. Because m is a constant, then according to Equation 2.11 the quantity $E^2 - p^2c^2$ must have the same value in all reference frames. That is, $E^2 - p^2c^2$ is *invariant* under a Lorentz transformation.

When dealing with electrons or other subatomic particles, it is convenient to express their energy in **electron volts (eV)**, since the particles are usually given this energy by acceleration through a potential difference. The conversion factor is

$$1 \text{ eV} = 1.60 \times 10^{-19} \text{ J}$$

For example, the mass of an electron is 9.11×10^{-31} kg. Hence, the rest energy of the electron is

$$m_e c^2 = (9.11 \times 10^{-31} \text{ kg})(3.00 \times 10^8 \text{ m/s})^2 = 8.20 \times 10^{-14} \text{ J}$$

Converting this to electron volts, we have

$$m_e c^2 = (8.20 \times 10^{-14} \text{ J})(1 \text{ eV}/1.60 \times 10^{-19} \text{ J}) = 0.511 \text{ MeV}$$

where 1 MeV $= 10^6$ eV. Finally, note that because $m_e c^2 = 0.511$ MeV, the mass of the electron may be written $m_e = 0.511 \text{ MeV}/c^2$, accounting for the practice of measuring particle masses in units of MeV$/c^2$.

EXAMPLE 2.3 The Energy of a Speedy Electron

An electron has a speed $u = 0.850c$. Find its total energy and kinetic energy in electron volts.

Solution Using the fact that the rest energy of the electron is 0.511 MeV together with $E = \gamma mc^2$ gives

$$E = \frac{m_e c^2}{\sqrt{1 - (u^2/c^2)}} = \frac{0.511 \text{ MeV}}{\sqrt{1 - [(0.85c)^2/c^2]}}$$
$$= 1.90(0.511 \text{ MeV}) = 0.970 \text{ MeV}$$

The kinetic energy is obtained by subtracting the rest energy from the total energy:

$$K = E - m_e c^2 = 0.970 \text{ MeV} - 0.511 \text{ MeV} = 0.459 \text{ MeV}$$

EXAMPLE 2.4 The Energy of a Speedy Proton

The total energy of a proton is three times its rest energy.

(a) Find the proton's rest energy in electron volts.

Solution

$$\begin{aligned}\text{rest energy} &= m_p c^2\\ &= (1.67 \times 10^{-27} \text{ kg})(3.00 \times 10^8 \text{ m/s})^2\\ &= (1.50 \times 10^{-10} \text{ J})(1 \text{ eV}/1.60 \times 10^{-19} \text{ J})\\ &= 938 \text{ MeV}\end{aligned}$$

(b) With what speed is the proton moving?

Solution Because the total energy E is three times the rest energy, $E = \gamma mc^2$ gives

$$E = 3m_p c^2 = \frac{m_p c^2}{\sqrt{1 - (u^2/c^2)}}$$

$$3 = \frac{1}{\sqrt{1 - (u^2/c^2)}}$$

Solving for u gives

$$\left(1 - \frac{u^2}{c^2}\right) = \frac{1}{9} \qquad \text{or} \qquad \frac{u^2}{c^2} = \frac{8}{9}$$

$$u = \frac{\sqrt{8}}{3}c = 2.83 \times 10^8 \text{ m/s}$$

(c) Determine the kinetic energy of the proton in electron volts.

Solution

$$K = E - m_p c^2 = 3m_p c^2 - m_p c^2 = 2m_p c^2$$

Because $m_p c^2 = 938$ MeV, $K = 1876$ MeV.

(d) What is the proton's momentum?

Solution We can use Equation 2.11 to calculate the momentum with $E = 3m_p c^2$:

$$E^2 = p^2c^2 + (m_p c^2)^2 = (3m_p c^2)^2$$
$$p^2c^2 = 9(m_p c^2)^2 - (m_p c^2)^2 = 8(m_p c^2)^2$$
$$p = \sqrt{8}\,\frac{m_p c^2}{c} = \sqrt{8}\,\frac{(938 \text{ MeV})}{c} = 2650\,\frac{\text{MeV}}{c}$$

Note that the unit of momentum is left as MeV$/c$ for convenience.

2.3 MASS AS A MEASURE OF ENERGY

The equation $E = \gamma mc^2$ as applied to a particle suggests that even when a particle is at rest ($\gamma = 1$) it still possesses enormous energy through its mass. The clearest experimental proof of the equivalence of mass and energy occurs in nuclear and elementary particle interactions in which both the conversion of mass into energy and the conversion of energy into mass take place. Because of this convertibility from the currency of mass into the currency of energy, we can no longer accept the separate classical laws of the conservation of mass and the conservation of energy; we must instead speak of a single unified law, **the conservation of mass–energy.** Simply put, this law requires that **the sum of the mass–energy of a system of particles before interaction must equal the sum of the mass–energy of the system after interaction where the mass–energy of the ith particle is defined as the total relativistic energy**

Conservation of mass–energy

$$E_i = \frac{m_i c^2}{\sqrt{1 - (u_i^2/c^2)}}$$

To understand the conservation of mass–energy and to see how the relativistic laws possess more symmetry and wider scope than the classical laws of momentum and energy conservation, we consider the simple inelastic collision treated earlier.

As one can see in Figure 2.1a, *classically* momentum is conserved but kinetic energy is not because the total kinetic energy before collision equals mu^2 and the total kinetic energy after is zero (we have replaced the v shown in Figure 2.1 with u). Now consider the same two colliding clay lumps using the relativistic mass–energy conservation law. If the mass of each lump is m, and the mass of the composite object is M, we must have

$$E_{\text{before}} = E_{\text{after}}$$

$$\frac{mc^2}{\sqrt{1 - (u^2/c^2)}} + \frac{mc^2}{\sqrt{1 - (u^2/c^2)}} = Mc^2$$

or

$$M = \frac{2m}{\sqrt{1 - (u^2/c^2)}} \tag{2.13}$$

Because $\sqrt{1 - (u^2/c^2)} < 1$, the composite mass M is greater than the sum of the two individual masses! What's more, it is easy to show that the mass increase of the composite lump, $\Delta M = M - 2m$, is equal to the sum of the incident kinetic energies of the colliding lumps ($2K$) divided by c^2:

$$\Delta M = \frac{2K}{c^2} = \frac{2}{c^2}\left(\frac{mc^2}{\sqrt{1 - (u^2/c^2)}} - mc^2\right) \tag{2.14}$$

Thus, we have an example of the conversion of kinetic energy to mass, and the satisfying result that in relativistic mechanics, kinetic energy is not lost in an inelastic collision but shows up as an increase in the mass of the final composite object. In fact, the deeper symmetry of relativity theory shows that *both relativistic mass–energy and momentum are always conserved in a collision,* whereas classical methods show that momentum is conserved but kinetic energy is not unless the collision is perfectly elastic. Indeed, as we show in Example 2.6, relativistic momentum and energy are inextricably linked because momentum conservation only holds in all inertial frames if mass–energy conservation also holds.

EXAMPLE 2.5

(a) Calculate the mass increase for a completely inelastic head-on collision of two 5.0-kg balls each moving toward the other at 1000 mi/h (the speed of a fast jet plane). (b) Explain why measurements on macroscopic objects reinforce the relativistically *incorrect* beliefs that mass is conserved ($M = 2m$) and that kinetic energy is lost in an inelastic collision.

Solution (a) $u = 1000$ mi/h $= 450$ m/s, so

$$\frac{u}{c} = \frac{4.5 \times 10^2 \text{ m/s}}{3.0 \times 10^8 \text{ m/s}} = 1.5 \times 10^{-6}$$

Because $u^2/c^2 << 1$, substituting

$$\frac{1}{\sqrt{1 - (u^2/c^2)}} \approx 1 + \frac{1}{2}\frac{u^2}{c^2}$$

in Equation 2.14 gives

$$\begin{aligned} \Delta M &= 2m\left(\frac{1}{\sqrt{1 - (u^2/c^2)}} - 1\right) \\ &\approx 2m\left(1 + \frac{1}{2}\frac{u^2}{c^2} - 1\right) \approx \frac{mu^2}{c^2} \\ &= (5.0 \text{ kg})(1.5 \times 10^{-6})^2 = 1.1 \times 10^{-11} \text{ kg} \end{aligned}$$

(b) Because the mass increase of 1.1×10^{-11} kg is an unmeasurably minute fraction of $2m$ (10 kg), it is quite natural to believe that the mass remains constant when macroscopic objects suffer an inelastic collision. On the other hand, the change in kinetic energy from mu^2 to 0 is so large (10^6 J) that it is readily measured to be lost in an inelastic collision of macroscopic objects.

Exercise 1 Prove that $\Delta M = 2\Delta K/c^2$ for a completely inelastic collision, as stated.

EXAMPLE 2.6

Show that use of the relativistic definition of momentum

$$p = \frac{mu}{\sqrt{1 - (u^2/c^2)}}$$

leads to momentum conservation in both S and S′ for the inelastic collision shown in Figure 2.1.

Solution In frame S:

$$p_{\text{before}} = \gamma mv + \gamma m(-v) = 0$$

$$p_{\text{after}} = \gamma MV = (\gamma M)(0) = 0$$

Hence, momentum is conserved in S. Note that we have used M as the mass of the two combined masses after the collision and allowed for the possibility in relativity that M is not necessarily equal to $2m$.

In frame S′:

$$\begin{aligned} p'_{\text{before}} &= \gamma m v'_1 + \gamma m v'_2 = \frac{(m)(0)}{\sqrt{1 - (0)^2/c^2}} \\ &+ \frac{m}{\{\sqrt{1 - [-2v/1 + (v^2/c^2)]^2}\}(1/c^2)} \times \left(\frac{-2v}{1 + v^2/c^2}\right) \end{aligned}$$

After some algebra, we find

$$\frac{m}{\{\sqrt{1 - [2v/1 + (v^2/c^2)]^2}\}(1/c^2)} = \frac{m(1 + v^2/c^2)}{(1 - v^2/c^2)}$$

and we obtain

$$p'_{\text{before}} = \frac{m(1 + v^2/c^2)}{(1 - v^2/c^2)}\left(\frac{-2v}{1 + v^2/c^2}\right) = \frac{-2mv}{(1 - v^2/c^2)}$$

$$p'_{\text{after}} = \gamma MV' = \frac{M(-v)}{\sqrt{1 - [(-v)^2/c^2]}} = \frac{-Mv}{\sqrt{1 - v^2/c^2}}$$

To show that momentum is conserved in S′, we use the fact that M is not simply equal to $2m$ in relativity. As shown, the combined mass, M, formed from the collision of two particles, each of mass m moving toward each other with speed v, is greater than $2m$. This occurs because of the equivalence of mass and energy, that is, the kinetic energy of the incident particles shows up in relativity theory as a tiny increase in mass, which can actually be measured as thermal energy. Thus, from Equation 2.13, which results from imposing the conservation of mass–energy, we have

$$M = \frac{2m}{\sqrt{1 - (v^2/c^2)}}$$

Substituting this result for M into p'_{after}, we obtain

$$\begin{aligned} p'_{\text{after}} &= \frac{2m}{\sqrt{1 - (v^2/c^2)}} \frac{-v}{\sqrt{1 - (v^2/c^2)}} \\ &= \frac{-2mv}{1 - (v^2/c^2)} = p'_{\text{before}} \end{aligned}$$

Hence, momentum is conserved in both S and S′, provided that we use the correct relativistic definition of momentum, $p = \gamma mu$, and assume the conservation of mass–energy.

The absence of observable mass changes in inelastic collisions of macroscopic objects impels us to look for other areas to test this law, where particle velocities are higher, masses are more precisely known, and forces are stronger than electrical or mechanical forces. This leads us to consider nuclear reactions, because nuclear masses can be measured very precisely with a mass spectrometer, nuclear forces are much stronger than electrical forces, and decay products are often produced with extremely high velocities.

Perhaps the most direct confirmation of the conservation of mass–energy occurs in the decay of a heavy radioactive nucleus at rest into several lighter particles emitted with large kinetic energies. For such a nucleus of mass M undergoing *fission* into particles with masses M_1, M_2, and M_3 and having speeds u_1, u_2, and u_3, conservation of total relativistic energy requires

Fission

$$Mc^2 = \frac{M_1c^2}{\sqrt{1-(u_1^2/c^2)}} + \frac{M_2c^2}{\sqrt{1-(u_2^2/c^2)}} + \frac{M_3c^2}{\sqrt{1-(u_3^2/c^2)}} \tag{2.15}$$

Because the square roots are all less than 1, $M > M_1 + M_2 + M_3$ and the loss of mass, $M - (M_1 + M_2 + M_3)$, appears as energy of motion of the products. This **disintegration energy** released per fission is often denoted by the symbol Q and can be written for our case as

$$Q = [M - (M_1 + M_2 + M_3)]c^2 = \Delta mc^2 \tag{2.16}$$

EXAMPLE 2.7 A Fission Reaction

An excited $^{236}_{92}\text{U}$ nucleus decays at rest into $^{90}_{37}\text{Rb}$, $^{143}_{55}\text{Cs}$, and several neutrons, $^{1}_{0}\text{n}$. (a) By conserving charge and the total number of protons and neutrons, write a balanced reaction equation and determine the number of neutrons produced. (b) Calculate by how much the combined "offspring" mass is less than the "parent" mass. (c) Calculate the energy released per fission. (d) Calculate the energy released in kilowatt hours when 1 kg of uranium undergoes fission in a power plant that is 40% efficient.

Solution (a) In general, an element is represented by the symbol $^{A}_{Z}\text{X}$, where X is the symbol for the element, A is the number of neutrons plus protons in the nucleus (mass number), and Z is the number of protons in the nucleus (atomic number). Conserving charge and number of nucleons gives

$$^{236}_{92}\text{U} \longrightarrow {}^{90}_{37}\text{Rb} + {}^{143}_{55}\text{Cs} + 3\,^{1}_{0}\text{n}$$

So three neutrons are produced per fission.

(b) The masses of the decay particles are given in Appendix B in terms of atomic mass units, u, where $1\text{ u} = 1.660 \times 10^{-27}\text{ kg} = 931.5\text{ MeV}/c^2$.

$$\begin{aligned}\Delta m = M_\text{U} - (M_\text{Rb} + M_\text{Cs} + 3m_\text{n}) &= 236.045563\text{ u}\\ &-(89.914811\text{ u} + 142.927220\text{ u}\\ &+ (3)(1.008665)\text{ u})\\ &= 0.177537\text{ u} = 2.9471 \times 10^{-28}\text{ kg}\end{aligned}$$

Therefore, the reaction products have a combined mass that is about 3.0×10^{-28} kg less than the initial uranium mass.

(c) The energy given off per fission event is just Δmc^2. This is most easily calculated if Δm is first converted to mass units of MeV/c^2. Because $1\text{ u} = 931.5\text{ MeV}/c^2$,

$$\begin{aligned}\Delta m &= (0.177537\text{ u})(931.5\text{ MeV}/c^2)\\ &= 165.4\text{ MeV}/c^2\end{aligned}$$

$$\begin{aligned}Q = \Delta mc^2 &= 165.4\,\frac{\text{MeV}}{c^2}\,c^2 = 165.4\text{ MeV}\\ &= -165.4\text{ MeV}\end{aligned}$$

(d) To find the energy released by the fission of 1 kg of uranium we need to calculate the number of nuclei, N, contained in 1 kg of ^{236}U.

$$N = \frac{(6.02 \times 10^{23}\text{ nuclei/mol})}{(236\text{ g/mol})}(1000\text{ g})$$
$$= 2.55 \times 10^{24}\text{ nuclei}$$

The total energy produced, E, is

$$E = (\text{efficiency})NQ$$
$$= (0.40)(2.55 \times 10^{24}\text{ nuclei})(165\text{ MeV/nucleus})$$
$$= 1.68 \times 10^{26}\text{ MeV}$$
$$= (1.68 \times 10^{26}\text{ MeV})(4.45 \times 10^{-20}\text{ kWh/MeV})$$
$$= 7.48 \times 10^{6}\text{ kWh}$$

Exercise 2 How long will this amount of energy keep a 100-W lightbulb burning?

Answer $\approx$ 8500 years.

We have considered the simplest case showing the conversion of mass to energy and the release of this nuclear energy: the decay of a heavy unstable element into several lighter elements. However, the most common case is the one in which the mass of a composite particle is *less than* the sum of the particle masses composing it. By examining Appendix B, you can see that the mass of any nucleus is less than the sum of its component neutrons and protons by an amount Δm. This occurs because the nuclei are stable, *bound* systems of neutrons and protons (bound by strong attractive nuclear forces), and in order to disassociate them into separate nucleons an amount of energy Δmc^2 must be supplied to the nucleus. This energy or work required to pull a bound system apart, leaving its component parts free of attractive forces and at rest, is called the binding energy, BE. Thus, we describe the mass and energy of a bound system by the equation

$$Mc^2 + BE = \sum_{i=1}^{n} m_i c^2 \qquad (2.17)$$

where M is the bound system mass, the m_i's are the free component particle masses, and n is the number of component particles. Two general comments are in order about Equation 2.17. First, it applies quite generally to any type of system bound by attractive forces, whether gravitational, electrical (chemical), or nuclear. For example, the mass of a water molecule is less than the combined mass of two free hydrogen atoms and a free oxygen atom, although the mass difference cannot be directly measured in this case. (The mass difference can be measured in the nuclear case because the forces and the binding energy are so much greater.) Second, Equation 2.17 shows the possibility of liberating huge quantities of energy, BE, if one reads the equation from right to left; that is, one collides nuclear particles with a small but sufficient amount of kinetic energy to overcome proton repulsion and fuse the particles into new elements with less mass. Such a process is called *fusion,* one example of which is a reaction in which two deuterium nuclei combine to form a helium nucleus, releasing 23.9 MeV per fusion. (See Chapter 14 for more on fusion processes.) We can write this reaction schematically as follows:

Fusion

$$^{2}_{1}\text{H} + ^{2}_{1}\text{H} \longrightarrow ^{4}_{2}\text{He} + 23.9\text{ MeV}$$

EXAMPLE 2.8

(a) How much lighter is a molecule of water than two hydrogen atoms and an oxygen atom? The binding energy of water is about 3 eV. (b) Find the fractional loss of mass per gram of water formed. (c) Find the total energy released (mainly as heat and light) when 1 gram of water is formed.

Solution (a) Equation 2.17 may be solved for the mass difference as follows:

$$\Delta m = (m_{\mathrm{H}} + m_{\mathrm{H}} + m_{\mathrm{O}}) - M_{\mathrm{H_2O}} = \frac{BE}{c^2} = \frac{3\text{ eV}}{c^2}$$

$$= \frac{(3.0\text{ eV})(1.6 \times 10^{-19}\text{ J/eV})}{(3.0 \times 10^8\text{ m/s})^2} = 5.3 \times 10^{-36}\text{ kg}$$

(b) To find the fractional loss of mass per molecule we divide Δm by the mass of a water molecule, $M_{\mathrm{H_2O}} = 18\mathrm{u} = 3.0 \times 10^{-26}$ kg:

$$\frac{\Delta m}{M_{\mathrm{H_2O}}} = \frac{5.3 \times 10^{-36}\text{ kg}}{3.0 \times 10^{-26}\text{ kg}} = 1.8 \times 10^{-10}$$

Because the fractional loss of mass per molecule is the same as the fractional loss per gram of water formed, 1.8×10^{-10} g of mass would be lost for each gram of water formed. This is much too small a mass to be measured directly, and this calculation shows that nonconservation of mass does not generally show up as a measurable effect in chemical reactions.

(c) The energy released when 1 gram of H_2O is formed is simply the change in mass when 1 gram of water is formed times c^2:

$$E = \Delta mc^2 = (1.8 \times 10^{-13}\text{ kg})(3.0 \times 10^8\text{ m/s})^2 \approx 16\text{ kJ}$$

This energy change, as opposed to the decrease in mass, is easily measured, providing another case similar to Example 2.5 in which mass changes are minute but energy changes, amplified by a factor of c^2, are easily measured.

2.4 CONSERVATION OF RELATIVISTIC MOMENTUM AND ENERGY

So far we have considered only cases of the conservation of mass–energy. By far, however, the most common and strongest confirmation of relativity theory comes from the daily application of relativistic momentum and energy conservation to elementary particle interactions. Often the measurement of momentum (from the path curvature in a magnetic field—see Example 2.2) and kinetic energy (from the distance a particle travels in a known substance before coming to rest) are enough when combined with conservation of momentum and mass–energy to determine fundamental particle properties of mass, charge, and mean lifetime.

EXAMPLE 2.9 Measuring the Mass of the π^+ Meson

The π^+ meson (also called the *pion*) is a subatomic particle responsible for the strong nuclear force between protons and neutrons. It is observed to decay at rest into a μ^+ meson (muon) and a neutrino,[2] denoted v. Because the neutrino has no charge and little mass (talk about elusive!), it leaves no track in a bubble chamber. (A bubble chamber is a large chamber filled with liquid hydrogen that shows the tracks of charged particles as a series of tiny bubbles.) However, the track of the charged muon is visible as it loses kinetic energy and comes to rest (Fig. 2.3). If the mass of the muon is known to be 106 MeV/c^2, and the kinetic energy, K, of the muon is measured to be 4.6 MeV from its track length, find the mass of the π^+.

Solution The decay equation is $\pi^+ \rightarrow \mu^+ + v$. Conserving energy gives

$$E_\pi = E_u + E_v$$

[2]Neutrino, from the Italian, means "little tiny neutral one." Following this practice, neutron should probably be neutrone (pronounced noo-trōn-eh) or "great big neutral one."

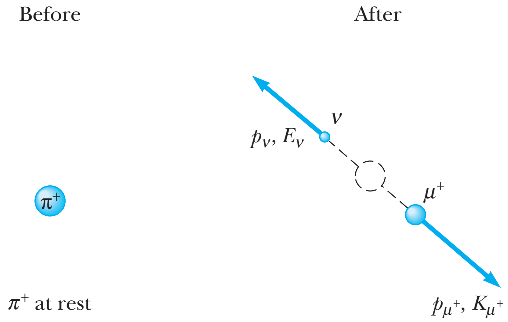

Figure 2.3 (Example 2.9) Decay of the pion at rest into a neutrino and a muon.

Because the pion is at rest when it decays, and the neutrino has negligible mass,

$$m_\pi c^2 = \sqrt{(m_\mu c^2)^2 + (p_\mu^2 c^2)} + p_v c \qquad (2.18)$$

Conserving momentum in the decay yields $p_\mu = p_v$. Substituting the muon momentum for the neutrino momentum in Equation 2.18 gives the following expression for the rest energy of the pion in terms of the muon's mass and momentum:

$$m_\pi c^2 = \sqrt{(m_\mu c^2)^2 + (p_\mu^2 c^2)} + p_\mu c \qquad (2.19)$$

Finally, to obtain p_μ from the measured value of the muon's kinetic energy, K_μ, we start with Equation 2.11, $E_\mu^2 = p_\mu^2 c^2 + (m_\mu c^2)^2$, and solve it for $p_\mu^2 c^2$:

$$\begin{aligned} p_\mu^2 c^2 &= E_\mu^2 - (m_\mu c^2)^2 = (K_\mu + m_\mu c^2)^2 - (m_\mu c^2)^2 \\ &= K_\mu^2 + 2K_\mu m_\mu c^2 \end{aligned}$$

Substituting this expression for $p_\mu^2 c^2$ into Equation 2.19 yields the desired expression for the pion mass in terms of the muon's mass and kinetic energy:

$$\begin{aligned} m_\pi c^2 = \sqrt{(m_\mu^2 c^4 + K_\mu^2 + 2K_\mu m_\mu c^2} \\ + \sqrt{K_\mu^2 + 2K_\mu m_\mu c^2} \end{aligned} \qquad (2.20)$$

Finally, substituting $m_\mu c^2 = 106$ MeV and $K_\mu = 4.6$ MeV into Equation 2.20 gives

$$m_\pi c^2 = 111 \text{ MeV} + 31 \text{ MeV} \approx 1.4 \times 10^2 \text{ MeV}$$

Thus, the mass of the pion is

$$m_\pi = 140 \text{ MeV}/c^2$$

This result shows why this particle is called a meson; it has an intermediate mass (from the Greek word *mesos* meaning "middle") between the light electron ($0.511 \text{ MeV}/c^2$) and the heavy proton ($938 \text{ MeV}/c^2$).

2.5 GENERAL RELATIVITY

Up to this point, we have sidestepped a curious puzzle. Mass has two seemingly different properties: a *gravitational attraction* for other masses and an *inertial* property that represents a resistance to acceleration. To designate these two attributes, we use the subscripts g and i and write

$$\text{Gravitational property:} \qquad F_g = G\frac{m_g m_g'}{r^2}$$

$$\text{Inertial property:} \qquad \sum F = m_i a$$

The value for the gravitational constant G was chosen to make the magnitudes of m_g and m_i numerically equal. Regardless of how G is chosen, however, the strict proportionality of m_g and m_i has been established experimentally to an extremely high degree: a few parts in 10^{12}. Thus, it appears that gravitational mass and inertial mass may indeed be exactly proportional.

But why? They seem to involve two entirely different concepts: a force of mutual gravitational attraction between two masses, and the resistance of a single mass to being accelerated. This question, which puzzled Newton and many other physicists over the years, was answered by Einstein in 1916 when he published his theory of gravitation, known as the *general theory of relativity*. Because it is a mathematically complex theory, we offer merely a hint of its elegance and insight.

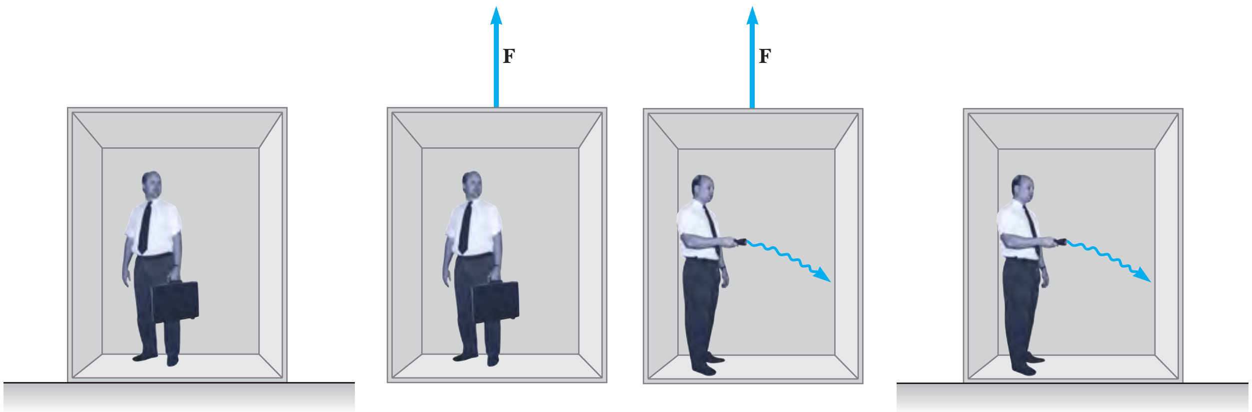

Figure 2.4 (a) The observer is at rest in a uniform gravitational field **g**, directed downward. (b) The observer is in a region where gravity is negligible, but the frame is accelerated by an external force **F** that produces an acceleration **g** directed upward. According to Einstein, the frames of reference in parts (a) and (b) are equivalent in every way. No local experiment can distinguish any difference between the two frames. (c) In the accelerating frame, a ray of light would appear to bend downward due to the acceleration of the elevator. (d) If parts (a) and (b) are truly equivalent, as Einstein proposed, then part (c) suggests that a ray of light would bend downward in a gravitational field.

In Einstein's view, the dual behavior of mass was evidence of a very intimate and basic connection between the two behaviors. He pointed out that no mechanical experiment (such as dropping an object) could distinguish between the two situations illustrated in Figures 2.4a and 2.4b. In Figure 2.4a, a person is standing in an elevator on the surface of a planet and feels pressed into the floor, due to the gravitational force. In Figure 2.4b, the person is in an elevator in empty space accelerating upward with $a = g$. The person feels pressed into the floor with the same force as in Figure 2.4a. In each case, an object released by the observer undergoes a downward acceleration of magnitude g relative to the floor. In Figure 2.4a, the person is in an inertial frame in a gravitational field. In Figure 2.4b, the person is in a noninertial frame accelerating in gravity-free space. Einstein's claim is that these two situations are *completely* equivalent. Because the two reference frames in relative acceleration can no longer be distinguished from one another, this extends the idea of complete physical equivalence to reference frames *accelerating translationally* with respect to each other. This solved another philosophical issue raised by Einstein, namely the artificiality of confining the principle of relativity to nonaccelerating frames.

Einstein carried his idea further and proposed that *no* experiment, mechanical or otherwise, could distinguish between the two cases. This extension to include all phenomena (not just mechanical ones) has interesting consequences. For example, suppose that a light pulse is sent horizontally across an elevator that is accelerating upward in empty space, as in Figure 2.4c. From the point of view of an observer in an inertial frame outside of the elevator, the light travels in a straight line while the floor of the elevator accelerates upward. According to the observer on the elevator, however, the trajectory of the light pulse bends downward as the floor of the elevator (and the observer) accelerates upward. Therefore, based on the equality of parts (a) and (b) of the figure for all phenomena, Einstein proposed that **a beam of light should also be defelected downward or fall in a gravitational field,** as in Figure 2.4d. Experiments have verified the effect, although the bending is small. A laser aimed at the horizon falls less than 1 cm after traveling 6000 km.

The two postulates of Einstein's **general theory of relativity** are

Postulates of general relativity

- The laws of nature have the same form for observers in any frame of reference, whether accelerated or not.
- In the vicinity of any point, a gravitational field is equivalent to an accelerated frame of reference in the absence of gravitational effects. (This is the **principle of equivalence.**)

An interesting effect predicted by the general theory is that time is altered by gravity. A clock in the presence of gravity runs slower than one located where gravity is negligible. Consequently, the frequencies of radiation emitted by atoms in the presence of a strong gravitational field are *redshifted* to lower frequencies when compared with the same emissions in the presence of a weak field. This gravitational redshift has been detected in spectral lines emitted by atoms in massive stars. It has also been verified on the Earth by comparing the frequencies of gamma rays (a high-energy form of electromagnetic radiation) emitted from nuclei separated vertically by about 20 m (see Section 3.7).

The second postulate suggests that a gravitational field may be "transformed away" at any point if we choose an appropriate accelerated frame of reference—a freely falling one. Einstein developed an ingenious method of describing the acceleration necessary to make the gravitational field "disappear." He specified a concept, the *curvature of spacetime,* that describes the gravitational effect at every point. In fact, the curvature of spacetime completely replaces Newton's gravitational theory. According to Einstein, there is no such thing as a gravitational force. Rather, the presence of a mass causes a curvature of spacetime in the vicinity of the mass, and this curvature dictates the spacetime path that all freely moving objects must follow. In 1979, John Wheeler (b. 1911, American theoretical physicist) summarized Einstein's general theory of relativity in a single sentence: "Space tells matter how to move and matter tells space how to curve."

As an example of the effects of curved spacetime, imagine two travelers moving on parallel paths a few meters apart on the surface of the Earth and maintaining an exact northward heading along two longitude lines. As they observe each other near the equator, they will claim that their paths are exactly parallel. As they approach the North Pole, however, they notice that they are moving closer together, and they will actually meet at the North Pole. Thus, they will claim that they moved along parallel paths, but moved toward each other, *as if there were an attractive force between them.* They will make this conclusion based on their everyday experience of moving on flat surfaces. From our perspective, however, we realize that they are walking on

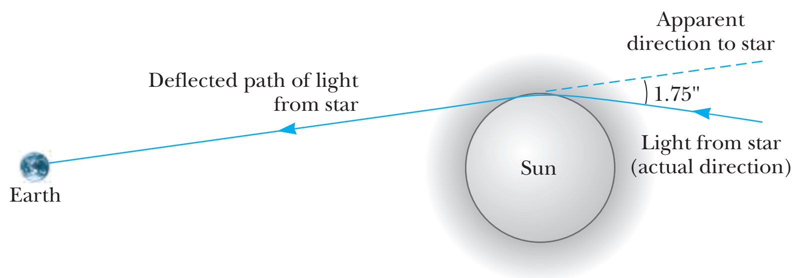

Figure 2.5 Deflection of starlight passing near the Sun. Because of this effect, the Sun or some other remote object can act as a *gravitational lens.* In his general theory of relativity, Einstein calculated that starlight just grazing the Sun's surface should be deflected by an angle of 1.75 s of arc.

a curved surface, and it is the geometry of the curved surface that causes them to converge, rather than an attractive force. In a similar way, general relativity replaces the notion of forces with the movement of objects through curved spacetime.

An important prediction of the general theory of relativity is that a light ray passing near the Sun should be deflected in the curved spacetime created by the Sun's mass. This prediction was confirmed when astronomers detected the bending of starlight near the Sun during a total solar eclipse that occurred shortly after World War I (Fig. 2.5). When this discovery was announced, Einstein became an international celebrity. (See the web essay by Clifford Will for other important tests and ramifications of general relativity at http://info.brookscole.com/mp3e.)

If the concentration of mass becomes very great, as is believed to occur when a large star exhausts its nuclear fuel and collapses to a very small volume, a **black hole** may form. Here, the curvature of spacetime is so extreme that, within a certain distance from the center of the black hole, all matter and light become trapped, as discussed in Section 3.7.

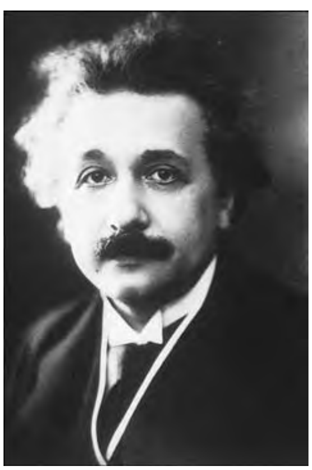

Figure 2.6 Albert Einstein. Gravity imaging was another triumph for Einstein since he pointed out that it might occur in 1936. (*Courtesy of AIP/Niels Bohr Library*).

Gravitational Radiation, or A Good Wave Is Hard to Find

Gravitational radiation is a subject almost as old as general relativity. By 1916, Einstein had succeeded in showing that the field equations of general relativity admitted wavelike solutions analogous to those of electromagnetic theory. For example, a dumbbell rotating about an axis passing at right angles through its handle will emit gravitational waves that travel at the speed of light. Gravitational waves also carry energy away from the dumbbell, just as electromagnetic waves carry energy away from a light source. Also, like electromagnetic (em) waves, gravity waves are believed to have a dual particle and wave nature. The gravitational particle, the graviton, is believed to have a mass of zero, to travel at the speed c, and to obey the relativistic equation $E = pc$.

In 1968, Joseph Weber initiated a program of gravitational-wave detection using as detectors massive aluminum bars, suspended in vacuum and isolated from outside forces. Gravity waves are notoriously more difficult to detect than

Figure 2.7 Joseph Weber working on a bar detector at the University of Maryland in the early 1970's. The fundamental frequency of the bar was 1660 Hz. Piezoelectric crystals around the center of the bar convert tiny mechanical vibrations to electrical signals. (*Courtesy of AIP Emilio Segre Visual Archives*)

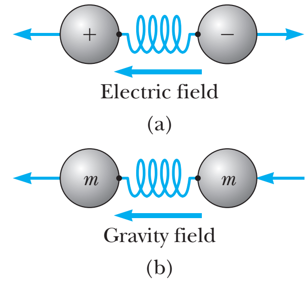

Figure 2.8 Simple models of em and gravity wave detectors. The detectors are shown as two "charges" with a spring sandwiched in between, the idea being that the waves exert forces on the charges and set the spring vibrating in proportion to the wave intensity. The detector will be particularly sensitive when the wave frequency matches the natural frequency of the spring–mass system. (a) Equal and opposite electric charges move in opposite directions when subjected to an em wave and easily excite the spring. (b) A metal bar gravity wave detector can be modeled by a spring connecting two equal masses; however, a wave encountering both masses in phase will not cause the spring to vibrate.

em waves not only because gravitational forces are much weaker than electric forces but also because gravitational "charge" or mass only comes in one variety, positive. Figure 2.8a shows why a dipolar em wave detector is much more sensitive than a gravitational bar detector shown in Figure 2.8b. Nevertheless, as shown in Figure 2.9, if the distance between detecting masses is of the same order of magnitude as the wavelength of the gravity wave, passing gravitational waves exert a weak net oscillating force that alternately compresses and extends the bar lengthwise.

Tiny vibrations of the bar are detected by crystals attached to the bar that convert the vibrations to electrical signals. Currently, a dozen laboratories around the world are engaged in building and improving the basic "Weber bar" detector, striving to reduce noise from thermal, electrical, and environmental sources in order to detect the very weak oscillations produced by a gravitational wave. For a bar of 1 meter in length, the challenge is to detect a variation in length smaller than 10^{-20} m, or 10^{-5} of the radius of a proton. This sensitivity is predicated on a massive nearby *catastrophic* source of gravitational waves, such as the gravitational collapse of a star to form a black hole at the center of our galaxy. Thus, gravity waves are not only hard to detect but also hard to generate with great intensity. It is interesting that collapsing star models predict a collapse to take about a millisecond, with production of gravity waves of frequency around 1 kHz and wavelengths of several hundred km.

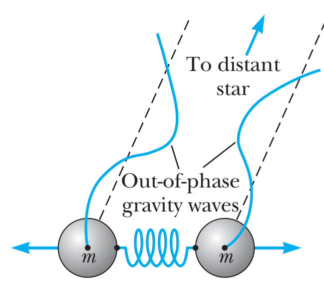

Figure 2.9 If the gravity wave detector is of the same size as the wavelength of the radiation detected, the waves arrive out-of-phase at the two masses and the system starts to vibrate.

(a)

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Figure 2.10 (a) Prototype LIGO apparatus with 40 m arms.
a: Tony Tyson/Lucent Technologies/Bell Labs Innovations;

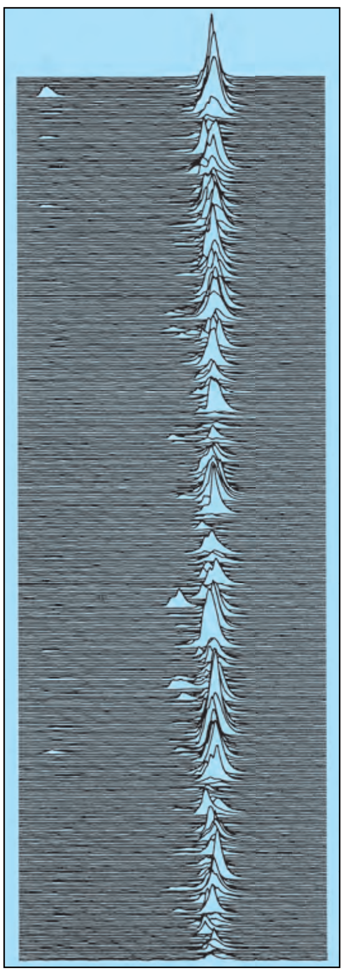
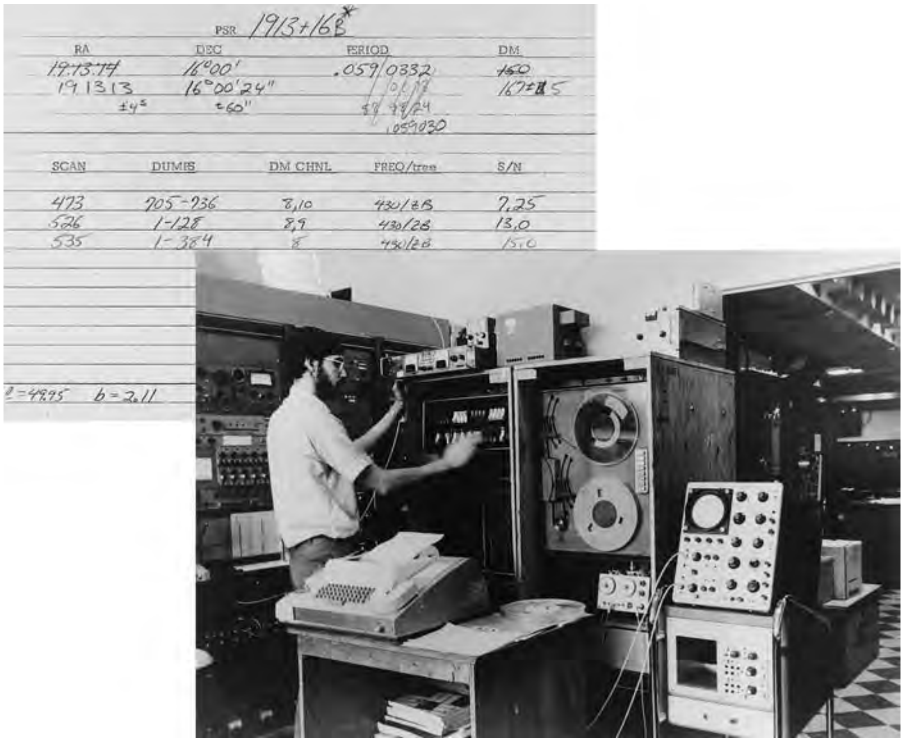


Figure 2.11 400 consecutive radio pulses from pulsar PSR 0950+08. Each line of the 400 represents a consecutive time interval of 0.253 s.

At this time, several "**l**aser-**i**nterferometric" **g**ravitational-wave **o**bservatories (LIGO) are in operation or under construction in the United States and Europe. These reflect laser beams along perpendicular arms to monitor tiny variations in length between mirrors spaced several kilometers apart in a giant Michelson–Morley apparatus. (See Figures 2.10a and b.) The variations in arm length should occur when a gravitational wave passes through the apparatus. Two LIGO sites with 4-km arms are currently in operation in the United States in Livingston, Louisiana and Hanford, Washington. The two sites, separated by about 2000 miles, search for signals that appear simultaneously at both sites. Such coincidences are more likely to be gravity waves from a distant star rather than local noise signals.

Although gravitational radiation has not been detected directly, we know that it exists through the observations of a remarkable system known as the binary pulsar. Discovered in 1974 by radio astronomers Russell Hulse and Joseph Taylor, it consists of a pulsar (which is a rapidly spinning neutron star) and a companion star in orbit around each other. Although the companion has not been seen directly, it is also believed to be a neutron star. The pulsar acts as an extremely stable clock, its pulse period of approximately 59 milliseconds drifting by only 0.25 ns/year. Figure 2.11 shows the remarkable regularity of 400 consecutive radio pulses from a pulsar. By measuring the arrival times of radio pulses at Earth, observers were able to determine the motion of the pulsar about its companion with amazing accuracy. For example, the accurate value for the orbital period is 27906.980 895 s, and the orbital eccentricity is 0.617|131. Like a rotating dumbbell, an orbiting binary system should emit gravitational radiation and, in the process, lose some of its orbital energy. This energy loss will cause the pulsar and its companion to spiral in toward each other and the orbital period to shorten. According to general relativity, the predicted decrease in the orbital period is 75.8 μs/year. The observed decrease in orbital period is in agreement with the prediction to better than 0.5%. This confirms the existence of gravitational radiation and the general relativistic equations that describe it.



(a)

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Figure 2.12 (a) Russell Hulse shown in 1974 operating his computer and teletype at Arecibo observatory in Puerto Rico. The form records the “fantastic” detection of PSR 1913+16, with its ever-changing periods scratched out by Hulse in frustration.

(*a: Photo Courtesy of Russell Hulse. © The Nobel Foundation, 1993;*

Hulse and Taylor (Figure 2.12) received the Nobel prize in 1993 for this discovery.

SUMMARY

The relativistic expression for the **linear momentum** of a particle moving with a velocity $\mathbf{u}$ is

$$\mathbf{p} \equiv \frac{m\mathbf{u}}{\sqrt{1 - (u^2/c^2)}} = \gamma m\mathbf{u} \tag{2.1}$$

where γ is given by

$$\gamma = \frac{1}{\sqrt{1 - (u^2/c^2)}}$$

The relativistic expression for the **kinetic energy** of a particle is

$$K = \gamma mc^2 - mc^2 \tag{2.9}$$

where mc^2 is called the **rest energy** of the particle.

The total energy E of a particle is related to the mass through the expression

$$E = \gamma mc^2 = \frac{mc^2}{\sqrt{1 - (u^2/c^2)}} \tag{2.10}$$

The total energy of a particle of mass m is related to the momentum through the equation

$$E^2 = p^2c^2 + (mc^2)^2 \tag{2.11}$$

Finally, the law of the conservation of mass–energy states that *the sum of the mass–energy of a system of particles before interaction must equal the sum of the mass–energy of the system after interaction where the mass–energy of the ith particle is defined as*

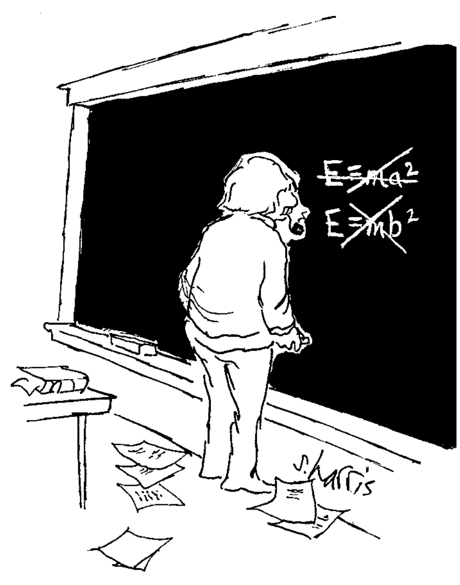
$$E_i = \frac{m_ic^2}{\sqrt{1 - (u_i^2/c^2)}}$$

Application of the principle of conservation of mass–energy to the specific cases of (1) the fission of a heavy nucleus at rest and (2) the fusion of several particles into a composite nucleus with less total mass allows us to define (1) the energy released per fission, Q, and (2) the binding energy of a composite system, BE.

The two postulates of Einstein's **general theory of relativity** are

- The laws of nature have the same form for observers in any frame of reference, whether accelerated or not.
- In the vicinity of any point, a gravitational field is equivalent to an accelerated frame of reference in the absence of gravitational effects. (This is the **principle of equivalence**.)

The field equations of general relativity predict gravitational waves, and a worldwide search is currently in progress to detect these elusive waves.



(© *S. Harris*)

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QUESTIONS

1. A particle is moving at a speed of less than $c/2$. If the speed of the particle is doubled, what happens to its momentum?
2. Give a physical argument showing that it is impossible to accelerate an object of mass m to the speed of light, even with a continuous force acting on it.
3. The upper limit of the speed of an electron is the speed of light, c. Does that mean that the momentum of the electron has an upper limit?
4. Because mass is a measure of energy, can we conclude that the mass of a compressed spring is greater than the mass of the same spring when it is not compressed?
5. Photons of light have zero mass. How is it possible that they have momentum?
6. "Newtonian mechanics correctly describes objects moving at ordinary speeds, and relativistic mechanics correctly describes objects moving very fast." "Relativistic mechanics must make a smooth transition as it reduces to Newtonian mechanics in a case where the speed of an object becomes small compared to the speed of light." Argue for or against each of these two statements.
7. Two objects are identical except that one is hotter than the other. Compare how they respond to identical forces.
8. With regard to reference frames, how does general relativity differ from special relativity?
9. Two identical clocks are in the same house, one upstairs in a bedroom, and the other downstairs in the kitchen. Which clock runs more slowly? Explain.
10. *A thought experiment.* Imagine ants living on a merry-go-round, which is their two-dimensional world. From measurements on small circles they are thoroughly familiar with the number π. When they measure the circumference of their world, and divide it by the diameter, they expect to calculate the number $\pi = 3.14159. . . .$ We see the merry-go-round turning at relativistic speed. From our point of view, the ants' measuring rods on the circumference are experiencing Lorentz contraction in the tangential direction; hence the ants will need some extra rods to fill that entire distance. The rods measuring the diameter, however, do not contract, because their motion is perpendicular to their lengths. As a result, the computed ratio does not agree with the number π. If you were an ant, you would say that the rest of the universe is spinning in circles, and your disk is stationary. What possible explanation can you then give for the discrepancy, in view of the general theory of relativity?

PROBLEMS

2.1 Relativistic Momentum and the Relativistic Form of Newton's Laws

1. Calculate the momentum of a proton moving with a speed of (a) $0.010c$, (b) $0.50c$, (c) $0.90c$. (d) Convert the answers of (a)–(c) to MeV/c.
2. An electron has a momentum that is 90% larger than its classical momentum. (a) Find the speed of the electron. (b) How would your result change if the particle were a proton?
3. Consider the relativistic form of Newton's second law. Show that when **F** is parallel to **v**,

$$F = m\left(1 - \frac{v^2}{c^2}\right)^{-3/2} \frac{dv}{dt}$$

where m is the mass of an object and v is its speed.

4. A charged particle moves along a straight line in a uniform electric field E with a speed v. If the motion and the electric field are both in the x direction, (a) show

that the magnitude of the acceleration of the charge q is given by

$$a = \frac{dv}{dt} = \frac{qE}{m}\left(1 - \frac{v^2}{c^2}\right)^{3/2}$$

(b) Discuss the significance of the dependence of the acceleration on the speed. (c) If the particle starts from rest at $x = 0$ at $t = 0$, find the speed of the particle and its position after a time t has elapsed. Comment on the limiting values of v and x as $t \to \infty$.

5. Recall that the magnetic force on a charge q moving with velocity $\mathbf{v}$ in a magnetic field $\mathbf{B}$ is equal to $q\mathbf{v} \times \mathbf{B}$. If a charged particle moves in a circular orbit with a fixed speed v in the presence of a constant magnetic field, use the relativistic form of Newton's second law to show that the frequency of its orbital motion is

$$f = \frac{qB}{2\pi m}\left(1 - \frac{v^2}{c^2}\right)^{1/2}$$

6. Show that the momentum of a particle having charge e moving in a circle of radius R in a magnetic field B is given by $p = 300BR$, where p is in MeV/c, B is in teslas, and R is in meters.

2.2 Relativistic Energy

7. Show that the energy–momentum relationship given by $E^2 = p^2c^2 + (mc^2)^2$ follows from the expressions $E = \gamma mc^2$ and $p = \gamma mu$.
8. A proton moves at a speed of $0.95c$. Calculate its (a) rest energy, (b) total energy, and (c) kinetic energy.
9. An electron has a kinetic energy 5 times greater than its rest energy. Find (a) its total energy and (b) its speed.
10. Find the speed of a particle whose total energy is 50% greater than its rest energy.
11. A proton in a high-energy accelerator is given a kinetic energy of 50 GeV. Determine the (a) momentum and (b) speed of the proton.
12. An electron has a speed of $0.75c$. Find the speed of a proton that has (a) the same kinetic energy as the electron and (b) the same momentum as the electron.
13. Protons in an accelerator at the Fermi National Laboratory near Chicago are accelerated to an energy of 400 times their rest energy. (a) What is the speed of these protons? (b) What is their kinetic energy in MeV?
14. *How long will the Sun shine, Nellie*? The Sun radiates about 4.0×10^{26} J of energy into space each second. (a) How much mass is released as radiation each second? (b) If the mass of the Sun is 2.0×10^{30} kg, how long can the Sun survive if the energy release continues at the present rate?
15. Electrons in projection television sets are accelerated through a total potential difference of 50,000 V. (a) Calculate the speed of the electrons using the relativistic form of kinetic energy assuming the electrons start from rest. (b) Calculate the speed of the electrons using the classical form of kinetic energy. (c) Is the difference in speed significant in the design of this set in your opinion?
16. As noted in Section 2.2, the quantity $E - p^2c^2$ is an *invariant* in relativity theory. This means that the quantity $E^2 - p^2c^2$ has the same value in all inertial frames even though E and p have different values in different frames. Show this explicitly by considering the following case. A particle of mass m is moving in the $+x$ direction with speed u and has momentum p and energy E in the frame S. (a) If S′ is moving at speed v in the standard way, find the momentum p' and energy E' observed in S′. (*Hint:* Use the Lorentz velocity transformation to find p' and E'. Does $E = E'$ and $p = p'$? (b) Show that $E^2 - p^2c^2$ is equal to $E'^2 - p'^2c^2$.

2.3 Mass as a Measure of Energy

17. A radium isotope decays to a radon isotope, ^{222}Rn, by emitting an α particle (a helium nucleus) according to the decay scheme $^{226}\text{Ra} \to {}^{222}\text{Rn} + {}^{4}\text{He}$. The masses of the atoms are 226.0254 (Ra), 222.0175 (Rn), and 4.0026 (He). How much energy is released as the result of this decay?
18. Consider the decay ${}^{55}_{24}\text{Cr} \to {}^{55}_{25}\text{Mn} + e^-$, where e^- is an electron. The ^{55}Cr nucleus has a mass of 54.9279 u, and the ^{55}Mn nucleus has a mass of 54.9244 u. (a) Calculate the mass difference in MeV. (b) What is the maximum kinetic energy of the emitted electron?
19. Calculate the binding energy in MeV per nucleon in the isotope ${}^{12}_{6}$C. Note that the mass of this isotope is exactly 12 u, and the masses of the proton and neutron are 1.007276 u and 1.008665 u, respectively.
20. The free neutron is known to decay into a proton, an electron, and an antineutrino $\bar{v}$ (of negligible rest mass) according to

$$\text{n} \longrightarrow \text{p} + \text{e}^- + \bar{v}$$

This is called *beta decay* and will be discussed further in Chapter 13. The decay products are measured to have a total kinetic energy of 0.781 MeV ± 0.005 MeV. Show that this observation is consistent with the excess energy predicted by the Einstein mass–energy relationship.

2.4 Conservation of Relativistic Momentum and Energy

21. An electron having kinetic energy $K = 1.000$ MeV makes a head-on collision with a positron at rest. (A positron is an antimatter particle that has the same mass as the electron but opposite charge.) In the collision the two particles annihilate each other and are replaced by two γ rays of equal energy, each traveling at equal angles θ with the electron's direction of motion. (Gamma rays are massless particles of elec-

tromagnetic radiation having energy $E = pc$.) Find the energy E, momentum p, and angle of emission θ of the γ rays.

22. The K^0 meson is an uncharged member of the particle "zoo" that decays into two charged pions according to $K^0 \rightarrow \pi^+ + \pi^-$. The pions have opposite charges, as indicated, and the same mass, $m_\pi = 140\ \text{MeV}/c^2$. Suppose that a K^0 at rest decays into two pions in a bubble chamber in which a magnetic field of 2.0 T is present (see Fig. P2.22). If the radius of curvature of the pions is 34.4 cm, find (a) the momenta and speeds of the pions and (b) the mass of the K^0 meson.
23. An unstable particle having a mass of 3.34×10^{-27} kg is initially at rest. The particle decays into two fragments that fly off with velocities of $0.987c$ and $-0.868c$. Find the rest masses of the fragments.

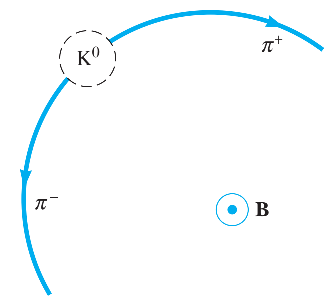

Figure P2.22 A sketch of the tracks made by the π^+ and π^- in the decay of the K^0 meson at rest. The pion motion is perpendicular to **B**. (**B** is directed out of the page.)

ADDITIONAL PROBLEMS

24. As measured by observers in a reference frame S, a particle having charge q moves with velocity $\mathbf{v}$ in a magnetic field $\mathbf{B}$ and an electric field $\mathbf{E}$. The resulting force on the particle is then measured to be $\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$. Another observer moves along with the charged particle and measures its charge to be q also but measures the electric field to be $\mathbf{E}'$. If both observers are to measure the same force, $\mathbf{F}$, show that $\mathbf{E}' = \mathbf{E} + \mathbf{v} \times \mathbf{B}$.
25. *Classical deflection of light by the Sun* Estimate the deflection of starlight grazing the surface of the Sun. Assume that light consists of particles of mass m traveling with velocity c and that the deflection is small. (a) Use $\Delta p_x = \int_{-\infty}^{+\infty} F_x\,dt$ to show that the angle of deflection θ is given by $\theta \cong \dfrac{2GM_s}{bc^2}$ where Δp_x is the total change in momentum of a light particle grazing the Sun. See Figures P2.25a and b. (b) For $b = R_s$, show that $\theta = 4.2 \times 10^{-6}$ rad.
26. An object having mass of 900 kg and traveling at a speed of $0.850c$ collides with a stationary object having mass 1400 kg. The two objects stick together. Find (a) the speed and (b) the mass of the composite object.
27. Imagine that the entire Sun collapses to a sphere of radius R_g such that the work required to remove a small mass m from the surface would be equal to its rest energy mc^2. This radius is called the *gravitational radius* for the Sun. Find R_g. (It is believed that the ultimate fate of very massive stars is to collapse beyond their gravitational radii into black holes.)
28. A rechargeable AA battery with a mass of 25.0 g can supply a power of 1.20 W for 50.0 min. (a) What is the difference in mass between a charged and an uncharged battery? (b) What fraction of the total mass is this mass difference?

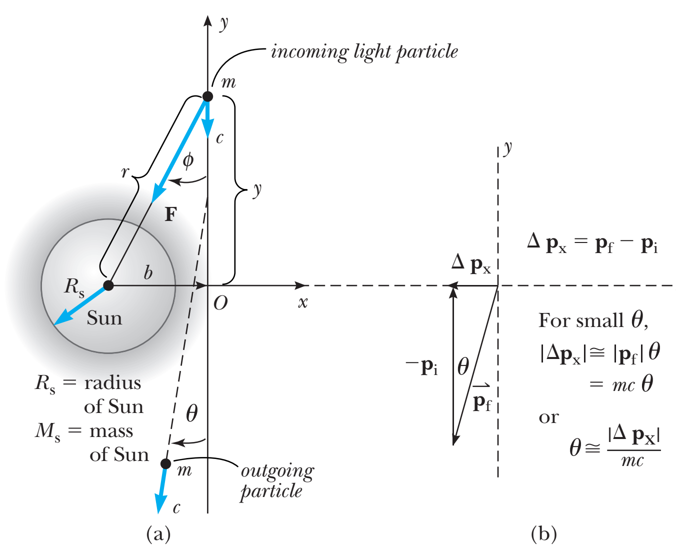

Figure P2.25 The classical deflection of starlight grazing the sun.

29. An object disintegrates into two fragments. One of the fragments has mass $1.00\ \text{MeV}/c^2$ and momentum $1.75\ \text{MeV}/c$ in the positive x direction. The other fragment has mass $1.50\ \text{MeV}/c^2$ and momentum $2.005\ \text{MeV}/c$ in the positive y direction. Find (a) the mass and (b) the speed of the original object.
30. The creation and study of new elementary particles is an important part of contemporary physics. Especially

interesting is the discovery of a very massive particle. To create a particle of mass M requires an energy Mc^2. With enough energy, an exotic particle can be created by allowing a fast-moving particle of ordinary matter, such as a proton, to collide with a similar target particle. Let us consider a perfectly inelastic collision between two protons: An incident proton with mass m, kinetic energy K, and momentum magnitude p joins with an originally stationary target proton to form a single product particle of mass M. You might think that the creation of a new product particle, 9 times more massive than in a previous experiment, would require just 9 times more energy for the incident proton. Unfortunately, not all of the kinetic energy of the incoming proton is available to create the product particle, since conservation of momentum requires that after the collision the system as a whole still must have some kinetic energy. Only a fraction of the energy of the incident particle is thus available to create a new particle. You will determine how the energy available for particle creation depends on the energy of the moving proton. Show that the energy available to create a product particle is given by

$$Mc^2 = 2mc^2\sqrt{1 + \frac{K}{2mc^2}}$$

From this result, when the kinetic energy K of the incident proton is large compared to its rest energy mc^2, we see that M approaches $(2mK)^{1/2}/c$. Thus if the energy of the incoming proton is increased by a factor of 9, the mass you can create increases only by a factor of 3. This disappointing result is the main reason that most modern accelerators, such as those at CERN (in Europe), at Fermilab (near Chicago), at SLAC (at Stanford), and at DESY (in Germany), use *colliding beams*. Here the total momentum of a pair of interacting particles can be zero. The center of mass can be at rest after the collision, so in principle all of the initial kinetic energy can be used for particle creation, according to

$$Mc^2 = 2mc^2 + K = 2mc^2\left(1 + \frac{K}{2mc^2}\right)$$

where K is the total kinetic energy of two identical colliding particles. Here, if $K >> mc^2$, we have M directly proportional to K, as we would desire. These machines are difficult to build and to operate, but they open new vistas in physics.

31. A particle of mass m moving along the x-axis with a velocity component $+u$ collides head-on and sticks to a particle of mass $m/3$ moving along the x-axis with the velocity component $-u$. What is the mass M of the resulting particle?
32. Compact high-power lasers can produce a 2.00-J light pulse of duration 100 fs focused to a spot 1 μm in diameter. (See Mourou and Umstader, "Extreme Light," *Scientific American*, May 2002, p. 81.) The electric field in the light accelerates electrons in the target material to near the speed of light. (a) What is the average power of the laser during the pulse? (b) How many electrons can be accelerated to $0.9999c$ if 0.0100% of the pulse energy is converted into energy of electron motion?
33. Energy reaches the upper atmosphere of the Earth from the Sun at the rate of 1.79×10^{17} W. If all of this energy were absorbed by the Earth and not re-emitted, how much would the mass of the Earth increase in 1.00 yr?