

## 第四章：线性空间与线性变换

2.

解. 令  $a_1 f_1 + a_2 f_2 + a_3 f_3 = 0$ , 则  $a_1 + a_2(x-1) + a_3(x-1)^2 = 0$

$$a_1 - a_2 + a_3 + (a_2 - 2a_3)x + a_3x^2 = 0$$

因为  $1, x, x^2$  线性无关, 所以

$$\begin{cases} a_1 - a_2 + a_3 = 0 \\ a_2 - 2a_3 = 0 \\ a_3 = 0 \end{cases}$$

解得  $a_1 = a_2 = a_3 = 0$ , 所以  $f_1, f_2, f_3$  线性无关. 而  $P_2[x]$  的维数为 3, 所以  $f_1, f_2, f_3$  是一组基. 下面令

$$a_1 - a_2 + a_3 + (a_2 - 2a_3)x + a_3x^2 = 5x^2 + x + 3$$

得

$$\begin{cases} a_1 - a_2 + a_3 = 3 \\ a_2 - 2a_3 = 1 \\ a_3 = 5 \end{cases}$$

解得

$$\begin{cases} a_1 = 9 \\ a_2 = 11 \\ a_3 = 5 \end{cases}$$

所以  $g(x)$  在此基下的坐标:  $[9, 11, 5]^T$ .

□

3.

解. (1)

$$\begin{bmatrix} 1 & 1+x & (1+x)^2 & (1+x)^3 \end{bmatrix} = \begin{bmatrix} 1 & x & x^2 & x^3 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

所以过渡矩阵为  $M = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

(2)  $f(x)$  在基  $1, x, x^2, x^3$  下坐标为  $[a_0, a_1, a_2, a_3]^T$ , 在基  $1, 1+x, (1+x)^2, (1+x)^3$  下坐标为

$$M^{-1} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} a_0 - a_1 + a_2 - a_3 \\ a_1 - 2a_2 + 3a_3 \\ a_2 - 3a_3 \\ a_3 \end{bmatrix}$$

□

6.

解. (1)

$$[\xi_1 \quad \xi_2 \quad \xi_3] = [\epsilon_1 \quad \epsilon_2 \quad \epsilon_3] \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 1 & 0 & 2 \end{bmatrix} = [\epsilon_1 \quad \epsilon_2 \quad \epsilon_3] M_1$$

$$[\eta_1 \quad \eta_2 \quad \eta_3] = [\epsilon_1 \quad \epsilon_2 \quad \epsilon_3] \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = [\epsilon_1 \quad \epsilon_2 \quad \epsilon_3] M_2$$

$M_1, M_2$  都可逆, 且该线性空间维数为 3, 所以  $\xi_1, \xi_2, \xi_3$  以及  $\eta_1, \eta_2, \eta_3$  都是一组基.

(2) 令

$$\begin{aligned} [\eta_1 \quad \eta_2 \quad \eta_3] &= [\xi_1 \quad \xi_2 \quad \xi_3] A \\ [\epsilon_1 \quad \epsilon_2 \quad \epsilon_3] M_2 &= [\epsilon_1 \quad \epsilon_2 \quad \epsilon_3] M_1 A \\ A &= M_1^{-1} M_2 = \begin{bmatrix} 2 & 2 & 1 \\ 2 & 3 & 1 \\ -1 & -1 & 0 \end{bmatrix} \end{aligned}$$

□

7.

解. (1)

$$\begin{aligned} [\beta_1 \quad \beta_2 \quad \beta_3 \quad \beta_4] &= [\alpha_1 \quad \alpha_2 \quad \alpha_3 \quad \alpha_4] \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} \\ &= [\alpha_2 + \alpha_3 + \alpha_4 \quad \alpha_1 + \alpha_3 + \alpha_4 \quad \alpha_1 + \alpha_2 + \alpha_4 \quad \alpha_1 + \alpha_2 + \alpha_3] \end{aligned}$$

所以, 另一组基为:

$$\begin{aligned} \beta_1 &= \alpha_2 + \alpha_3 + \alpha_4 = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \\ \beta_2 &= \alpha_1 + \alpha_3 + \alpha_4 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \end{aligned}$$

$$\beta_3 = \alpha_1 + \alpha_2 + \alpha_4 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

$$\beta_4 = \alpha_1 + \alpha_2 + \alpha_3 = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

(2)

矩阵在  $\alpha_1, \alpha_2, \alpha_3, \alpha_4$  下的坐标为  $(0, 1, 2, -3)^T$ . 在  $\beta_1, \beta_2, \beta_3, \beta_4$  下坐标为

$$A^{-1}(0, 1, 2, -3)^T = (0, -1, -2, 3)^T$$

□

8.

解. 令

$$e_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, e_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, e_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, e_4 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

则,  $e_1, e_2, e_3, e_4$  为  $R^{2 \times 2}$  的一组基, 且

$$[\alpha_1 \quad \alpha_2 \quad \alpha_3 \quad \alpha_4] = [e_1 \quad e_2 \quad e_3 \quad e_4] \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} = [e_1 \quad e_2 \quad e_3 \quad e_4] M_1$$

$$[\beta_1 \quad \beta_2 \quad \beta_3 \quad \beta_4] = [e_1 \quad e_2 \quad e_3 \quad e_4] \begin{bmatrix} -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{bmatrix} = [e_1 \quad e_2 \quad e_3 \quad e_4] M_2$$

显然  $M_1, M_2$  都可逆, 且该线性空间维数为 4, 所以这两组向量都是基.

$$\text{令 } \alpha = \begin{bmatrix} a_0 & a_1 \\ a_2 & a_3 \end{bmatrix}$$

$$\alpha \text{ 在 } e_1, e_2, e_3, e_4 \text{ 下的坐标为 } \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix}$$

所以,  $\alpha$  在  $\alpha_1, \alpha_2, \alpha_3, \alpha_4$  下的坐标为

$$M_1^{-1} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} a_0 - a_1 \\ a_1 - a_2 \\ a_2 - a_3 \\ a_3 \end{bmatrix}$$

(2)

$$[\beta_1 \quad \beta_2 \quad \beta_3 \quad \beta_4] = [e_1 \quad e_2 \quad e_3 \quad e_4] M_2$$

$$= [\alpha_1 \quad \alpha_2 \quad \alpha_3 \quad \alpha_4] M_1^{-1} M_2 = [\alpha_1 \quad \alpha_2 \quad \alpha_3 \quad \alpha_4] \begin{bmatrix} -2 & 2 & 0 & 0 \\ 0 & -2 & 2 & 0 \\ 0 & 0 & -2 & 2 \\ 1 & 1 & 1 & -1 \end{bmatrix}$$

即, 过渡矩阵为 
$$\begin{bmatrix} -2 & 2 & 0 & 0 \\ 0 & -2 & 2 & 0 \\ 0 & 0 & -2 & 2 \\ 1 & 1 & 1 & -1 \end{bmatrix}$$
 □

9.

解. (1) 方程组通解为  $k_1(1, 0, \dots, -1)^T + k_2(0, 1, \dots, -1)^T + \dots + k_{n-1}(0, 0, \dots, 1, -1)^T$ , 所以解空间的维数为  $n-1$ , 基为  $(1, 0, \dots, -1)^T, (0, 1, \dots, -1)^T, \dots, (0, 0, \dots, 1, -1)^T$

(2)  $A = \begin{bmatrix} 2 & -4 & 5 & 3 \\ 3 & -6 & 4 & 2 \\ 4 & -8 & 17 & 11 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -2 & -1 & -1 \\ 2 & -4 & 5 & 3 \\ 4 & -8 & 17 & 11 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -2 & -1 & -1 \\ 0 & 0 & 7 & 5 \\ 0 & 0 & 21 & 15 \end{bmatrix} \rightarrow$   

$$\begin{bmatrix} 1 & -2 & -1 & -1 \\ 0 & 0 & 7 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

得到方程组通解为  $k_1(2, 1, 0, 0)^T + k_2(2, 0, -5, 7)^T$   
 所以解空间维数为 2, 基为  $(2, 1, 0, 0)^T, (2, 0, -5, 7)^T$  □

10.

解.

$$[\alpha_1 \quad \dots \quad \alpha_{i-1} \quad \alpha_{i+1} \quad \dots \quad \alpha_{n+1}] = [\alpha_1 \quad \dots \quad \alpha_n] \begin{bmatrix} 1 & \dots & 0 & 0 & \dots & 0 & x_1 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & 1 & 0 & \dots & 0 & x_{i-1} \\ 0 & \dots & 0 & 0 & \dots & 0 & x_i \\ 0 & \dots & 0 & 1 & \dots & 0 & x_{i+1} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & 0 & 0 & \dots & 1 & x_n \end{bmatrix}$$

该矩阵对应行列式值为  $(-1)^{n+i}x_i$ , 不为 0, 即矩阵可逆, 所以  $\alpha_1, \dots, \alpha_{i-1}, \alpha_{i+1}, \dots, \alpha_{n+1}$  线性无关, 又线性空间为  $n$  维, 所以, 任取  $n$  个向量都是一组基.

$$\begin{aligned} \alpha_{n+1} &= x_1\alpha_1 + x_2\alpha_2 + \dots + x_n\alpha_n \\ \frac{1}{x_1}\alpha_{n+1} &= \alpha_1 + \frac{x_2}{x_1}\alpha_2 + \dots + \frac{x_n}{x_1}\alpha_n \\ \alpha_1 &= -\frac{x_2}{x_1}\alpha_2 - \dots - \frac{x_n}{x_1}\alpha_n + \frac{1}{x_1}\alpha_{n+1} \end{aligned}$$

所以  $\alpha_1$  在这组基下坐标为  $[-\frac{x_2}{x_1}, -\frac{x_3}{x_1}, \dots, -\frac{x_n}{x_1}, \frac{1}{x_1}]^T$  □

13.

解.  $c_1c_3 \neq 0$ , 所以  $\alpha = -\frac{1}{c_1}(c_2\beta + c_3\gamma)$   
 $\forall v \in \text{span}(\alpha, \beta), \exists \lambda_1, \lambda_2 \in R, v = \lambda_1\alpha + \lambda_2\beta$   
 $v = (\lambda_2 - \frac{c_2\lambda_1}{c_1})\beta - \frac{c_3\lambda_1}{c_1}\gamma \in \text{span}(\beta, \gamma)$   
 $\text{span}(\alpha, \beta) \subseteq \text{span}(\beta, \gamma)$   
 同理可证  $\text{span}(\alpha, \beta) \supseteq \text{span}(\beta, \gamma)$ , 得到  $\text{span}(\alpha, \beta) = \text{span}(\beta, \gamma)$  □

15.

解. (1).

充分性:

令  $B = [\beta_1, \beta_2, \dots, \beta_n]$ , 则  $C(B) = \text{span}(\beta_1, \dots, \beta_n)$ .

而  $C(B) \subseteq N(A)$ .

所以  $\forall \beta_i \in C(B), \beta_i \in N(A)$

即  $A\beta_i = 0$ , 也就是说  $A\beta_1 = 0, \dots, A\beta_n = 0$ .

所以  $AB = A[\beta_1, \dots, \beta_n] = 0$ .

必要性:

$AB = 0$ , 则  $A[\beta_1, \dots, \beta_n] = 0$

即  $\forall \beta_i \in C(B), A\beta_i = 0, \beta_i \in N(A)$

所以  $C(B) \subseteq N(A)$ . 充要性得证.

(2).

由 (1) 得,  $AB = 0 \Rightarrow C(B) \subseteq N(A)$

所以  $\dim(C(B)) \leq \dim(N(A))$

而  $N(A)$  就是  $Ax = 0$  的解空间, 所以  $\dim(N(A)) = n - r(A)$

所以  $r(B) = \dim(C(B)) \leq \dim(N(A)) = n - r(A)$ , 即  $r(A) + r(B) \leq n$   $\square$

16.

解.  $W_1 + W_2 = \text{span}(\alpha_1, \alpha_2, \beta_1, \beta_2)$

$$[\alpha_1, \alpha_2, \beta_1, \beta_2] = \begin{bmatrix} 2 & 0 & 1 & 1 \\ 0 & -2 & 1 & -3 \\ 1 & 1 & 0 & 2 \\ 3 & 5 & -1 & 0 \\ -1 & -3 & 1 & 5 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & 0 & 1 & 0 \\ 0 & -2 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

所以  $W_1 + W_2$  的维数为 3, 一组基为  $\alpha_1, \alpha_2, \beta_2$ .

根据维数公式  $\dim(W_1 \cap W_2) = \dim W_1 + \dim W_2 - \dim(W_1 + W_2) = 1$

注意到  $\beta_1 \in W_2, \beta_1 = \frac{1}{2}\alpha_1 - \frac{1}{2}\alpha_2 \in W_1$ , 所以  $\beta_1 \in W_1 \cap W_2$ , 即  $\beta_1$  可以作为  $W_1 \cap W_2$  的一组基  $\square$

18.

解. (1) 当  $A = A_1$  时,

$$\langle \alpha, \beta \rangle = \alpha^T A_1 \beta = a_1 b_1 + 2a_2 b_2 + a_3 b_3 = \beta^T A_1 \alpha = \langle \beta, \alpha \rangle$$

$$\langle k\alpha, \beta \rangle = k\alpha^T A_1 \beta = k \langle \alpha, \beta \rangle$$

$$\langle \alpha_1 + \alpha_2, \beta \rangle = (\alpha_1 + \alpha_2)^T A_1 \beta = (\alpha_1^T + \alpha_2^T) A_1 \beta = \alpha_1^T A_1 \beta + \alpha_2^T A_1 \beta = \langle \alpha_1, \beta \rangle + \langle \alpha_2, \beta \rangle$$

$$\langle \alpha, \alpha \rangle = a_1^2 + 2a_2^2 + a_3^2 \geq 0 \text{ 恒成立, 且当且仅当 } \alpha = 0 \text{ 时, } \langle \alpha, \alpha \rangle = 0$$

所以, 当  $A = A_1$  时为内积.

(2) 当  $A = A_2$  时,

$$\langle \alpha, \alpha \rangle = a_1^2 + a_2^2 - 2a_1 a_2 + 2a_1 a_3 = (a_1 - a_2)^2 + 2a_1 a_3$$

取  $a_1 = a_2 > 0, a_3 < 0$ , 则  $\langle \alpha, \alpha \rangle < 0$ .

所以, 当  $A = A_2$  时不为内积.

(3) 当  $A = A_3$  时,

$$\langle \alpha, \beta \rangle = a_1 b_1 - a_2 b_1 + a_3 b_1 - a_1 b_2 + 2a_2 b_2 + 3a_1 b_3 + 3a_3 b_3$$

$$\langle \beta, \alpha \rangle = a_1 b_1 - a_2 b_1 + 3a_3 b_1 - a_1 b_2 + 2a_2 b_2 + a_1 b_3 + 3a_3 b_3$$

$$\langle \alpha, \beta \rangle \neq \langle \beta, \alpha \rangle$$

所以, 当  $A = A_3$  时不为内积.  $\square$

19.

证. (1)  $\langle \beta, \alpha \rangle = b_1 a_1 - b_2 a_1 - b_1 a_2 + 3b_2 a_2 = a_1 b_1 - a_2 b_1 - a_1 b_2 + 3a_2 b_2 = \langle \alpha, \beta \rangle$

(2)  $\langle k\alpha, \beta \rangle = k a_1 b_1 - k a_2 b_1 - k a_1 b_2 + 3k a_2 b_2 = k(a_1 b_1 - a_2 b_1 - a_1 b_2 + 3a_2 b_2) = k \langle \alpha, \beta \rangle$

(3) 取  $\gamma = [c_1, c_2]^T$ ,  $\langle \alpha + \beta, \gamma \rangle = (a_1 + b_1)c_1 - (a_2 + b_2)c_1 - (a_1 + b_1)c_2 + 3(a_2 + b_2)c_2 = a_1 c_1 - a_2 c_1 - a_1 c_2 + 3a_2 c_2 + b_1 c_1 - b_2 c_1 - b_1 c_2 + 3b_2 c_2 = \langle \alpha, \gamma \rangle + \langle \beta, \gamma \rangle$

(4)  $\langle \alpha, \alpha \rangle = a_1^2 - a_2 a_1 - a_1 a_2 + 3a_2^2 = a_1^2 - 2a_1 a_2 + 3a_2^2 = (a_1 - a_2)^2 + 2a_2^2 \geq 0$ , 当且仅当  $a_1 = a_2 = 0$  时  $\langle \alpha, \alpha \rangle = 0$ .

综上, 这是  $R^2$  的一个内积.  $\square$

21.

证. 假设  $\beta_1, \beta_2, \alpha_1, \alpha_2, \dots, \alpha_r$  线性相关, 则存在不全为 0 的数  $l_1, l_2, k_1, k_2, \dots, k_r$ , 使得

$$l_1 \beta_1 + l_2 \beta_2 + k_1 \alpha_1 + k_2 \alpha_2 + \dots + k_r \alpha_r = 0$$

首先  $l_1, l_2$  不全为 0, 否则  $\alpha_1, \dots, \alpha_r$  线性相关, 矛盾.

在等式两边与  $l_1 \beta_1 + l_2 \beta_2$  做内积. 得

$$\langle l_1 \beta_1 + l_2 \beta_2, l_1 \beta_1 + l_2 \beta_2 + k_1 \alpha_1 + k_2 \alpha_2 + \dots + k_r \alpha_r \rangle = 0$$

$$\langle l_1 \beta_1 + l_2 \beta_2, l_1 \beta_1 + l_2 \beta_2 \rangle = 0$$

$$l_1 \beta_1 + l_2 \beta_2 = 0$$

所以  $\beta_1, \beta_2$  线性相关, 矛盾.

所以,  $\beta_1, \beta_2, \alpha_1, \alpha_2, \dots, \alpha_r$  线性无关.  $\square$

22.

证.  $\alpha_1, \alpha_2$  已经正交, 只需标准化.

$$q_1 = \frac{\alpha_1}{\|\alpha_1\|} = \frac{1}{\sqrt{2}}[1, 0, 1, 0]^T$$

$$q_2 = \frac{\alpha_2}{\|\alpha_2\|} = \frac{1}{\sqrt{5}}[0, 1, 0, 2]^T$$

设  $\beta = [x_1, x_2, x_3, x_4]^T \in R^4$ , 且  $\beta \perp \alpha_1, \beta \perp \alpha_2$

得到

$$\begin{cases} x_1 + x_3 = 0 \\ x_2 + 2x_4 = 0 \end{cases}$$

得到基础解系为

$$\alpha_3 = [-1, 0, 1, 0]^T, \alpha_4 = [0, -2, 0, 1]^T$$

已正交, 只需标准化.

$$q_3 = \frac{\alpha_3}{\|\alpha_3\|} = \frac{1}{\sqrt{2}}[-1, 0, 1, 0]^T$$

$$q_4 = \frac{\alpha_4}{\|\alpha_4\|} = \frac{1}{\sqrt{5}}[0, -2, 0, 1]^T$$

所以  $q_1, q_2, q_3, q_4$  为  $R^4$  的一组标准正交基.  $\square$

24.

证. (1) 必要性:

令

$$A = [\alpha_1, \dots, \alpha_n], B = [\beta_1, \dots, \beta_n]$$

则  $A, B$  为正交矩阵,  $A^T A = I_n, B^T B = I_n$

令过渡矩阵为  $M$ , 即  $B = AM$ , 则  $B^T = M^T A^T$

相乘有  $I_n = B^T B = M^T A^T A M = M^T M$ , 所以  $M$  为正交矩阵.

(2) 充分性:

$$B^T B = M^T A^T A M = M^T M = I_n$$

所以  $\beta_1, \beta_2, \dots, \beta_n$  为标准正交基. □

25.

证. 记方程组为  $Ax = b, A = \begin{bmatrix} 1 & 1 \\ 2 & -1 \\ -2 & 4 \end{bmatrix}, b = \begin{bmatrix} 1 \\ 2 \\ 7 \end{bmatrix}$

$$A^T A = \begin{bmatrix} 1 & 2 & -2 \\ 1 & -1 & 4 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & -1 \\ -2 & 4 \end{bmatrix} = \begin{bmatrix} 9 & -9 \\ -9 & 18 \end{bmatrix}$$

$$(A^T A)^{-1} = \begin{bmatrix} 2/9 & 1/9 \\ 1/9 & 1/9 \end{bmatrix}$$

最小二乘解为

$$x = (A^T A)^{-1} A^T b = \begin{bmatrix} 2/9 & 1/9 \\ 1/9 & 1/9 \end{bmatrix} \begin{bmatrix} 1 & 2 & -2 \\ 1 & -1 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 7 \end{bmatrix} = \begin{bmatrix} 2/9 & 1/9 \\ 1/9 & 1/9 \end{bmatrix} \begin{bmatrix} -9 \\ 27 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

□

26.

解.  $[A|b] \rightarrow \begin{bmatrix} 1 & 2 & -1 & -1 \\ 2 & 0 & 1 & 1 \\ 2 & -4 & 2 & 1 \\ 4 & 0 & 0 & -2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & -1 & -1 \\ 0 & 4 & -3 & -3 \\ 0 & 8 & -4 & -3 \\ 0 & 8 & -4 & -2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & -1 & -1 \\ 0 & 4 & -3 & -3 \\ 0 & 0 & 2 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

$\text{rank}(A) < \text{rank}(A|b)$ , 所以  $Ax = b$  无解.

$$A^T A = \begin{bmatrix} 1 & 2 & 2 & 4 \\ 2 & 0 & -4 & 0 \\ -1 & 1 & 2 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & -1 \\ 2 & 0 & 1 \\ 2 & -4 & 2 \\ 4 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 25 & -6 & 5 \\ -6 & 20 & -10 \\ 5 & -10 & 6 \end{bmatrix}$$

$$(A^T A)^{-1} = \begin{bmatrix} 5/96 & -7/192 & -5/48 \\ -7/192 & 125/384 & 55/96 \\ -5/48 & 55/96 & 29/24 \end{bmatrix}$$

最小二乘解为

$$x = (A^T A)^{-1} A^T b = \begin{bmatrix} -11/24 \\ 25/48 \\ 23/12 \end{bmatrix}$$

□

28.

解.  $\begin{bmatrix} 2 & 1 & -1 & 1 & -3 \\ 1 & 1 & -1 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & 1 & -4 \\ 0 & 1 & -1 & -1 & 5 \end{bmatrix}$   
 得到解空间的一组基为  $(0, 1, 1, 0, 0)^T, (-1, 1, 0, 1, 0)^T, (4, -5, 0, 0, 1)^T$   
 将  $\alpha_1, \alpha_2, \alpha_3$  Schmidt 正交化, 即令

$$q_1 = \frac{\alpha_1}{\|\alpha_1\|} = \frac{1}{\sqrt{2}}(0, 1, 1, 0, 0)^T$$

$$q_2 = \frac{\alpha_2 - q_1 q_1^T \alpha_2}{\|\alpha_2 - q_1 q_1^T \alpha_2\|} = \frac{1}{\sqrt{10}}(-2, 1, -1, 2, 0)^T$$

$$q_3 = \frac{\alpha_3 - q_1 q_1^T \alpha_3 - q_2 q_2^T \alpha_3}{\|\alpha_3 - q_1 q_1^T \alpha_3 - q_2 q_2^T \alpha_3\|} = \frac{1}{3\sqrt{35}}(7, -6, 6, 13, 5)^T$$

□

29.

解. 令  $A = [\alpha_1, \alpha_2]$ , 则  $C(A)^\perp = N(A^T) = \{x | A^T x = 0\}$   
 得到齐次方程组

$$\begin{cases} x_1 + x_2 + 2x_3 = 0 \\ x_1 + x_3 = 0 \end{cases}$$

得到一个基础解系为  $\alpha_3 = [-1, -1, 1]^T$

所以  $C(A)^\perp = \text{span}(\alpha_3)$

$$[\alpha_1, \alpha_2, \alpha_3, \alpha] = \begin{bmatrix} 1 & 1 & -1 & 3 \\ 1 & 0 & -1 & 2 \\ 2 & 1 & 1 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & -1 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 3 & -4 \end{bmatrix}$$

解得唯一解  $(2/3, 1, -4/3)^T$ , 所以,  $\alpha = \frac{2}{3}\alpha_1 + \alpha_2 - \frac{4}{3}\alpha_3$

其中  $\frac{2}{3}\alpha_1 + \alpha_2 \in C(A)$ ,  $-\frac{4}{3}\alpha_3 \in C(A)^\perp$

所以,  $\alpha$  在  $C(A)$  中的正交投影为

$$\frac{2}{3}\alpha_1 + \alpha_2 = [5/3, 2/3, 7/3]^T$$

在  $C(A)^\perp$  中的投影为

$$-\frac{4}{3}\alpha_3 = [4/3, 4/3, -4/3]^T$$

□

31.



解. 令  $\epsilon_1, \epsilon_2, \epsilon_3$  到  $\eta_1, \eta_2, \eta_3$  的过渡矩阵为  $M$ .  
则

$$\begin{aligned} [\eta_1, \eta_2, \eta_3] &= [\epsilon_1, \epsilon_2, \epsilon_3]M \\ M &= [\epsilon_1, \epsilon_2, \epsilon_3]^{-1}[\eta_1, \eta_2, \eta_3] \\ &= \begin{bmatrix} 8 & -16 & 9 \\ -6 & 7 & -3 \\ 7 & -13 & 7 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 3 & 2 \\ -2 & -1 & 1 \\ 1 & 2 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & -3 \\ 1 & 2 & -5 \\ 1 & 3 & -6 \end{bmatrix} \end{aligned}$$

所以,  $T$  在  $\eta_1, \eta_2, \eta_3$  下的表示矩阵为

$$M^{-1}AM = \begin{bmatrix} 1 & 2 & 2 \\ 3 & -1 & -2 \\ 2 & -3 & 1 \end{bmatrix}$$

□

33.

解. 令  $\epsilon_1, \epsilon_2, \epsilon_3$  到  $\eta_1, \eta_2, \eta_3$  的过渡矩阵为  $M$ .  
则

$$\begin{aligned} [\eta_1, \eta_2, \eta_3] &= [\epsilon_1, \epsilon_2, \epsilon_3]M \\ T[\epsilon_1, \epsilon_2, \epsilon_3] &= [T\epsilon_1, T\epsilon_2, T\epsilon_3] = [\eta_1, \eta_2, \eta_3] = [\epsilon_1, \epsilon_2, \epsilon_3]M \end{aligned}$$

所以,  $T$  在  $\epsilon_1, \epsilon_2, \epsilon_3$  下的表示矩阵为  $M$ .

$T$  在  $\eta_1, \eta_2, \eta_3$  下的表示矩阵为  $M' = M^{-1}MM = M$

$$\begin{aligned} M &= [\epsilon_1, \epsilon_2, \epsilon_3]^{-1}[\eta_1, \eta_2, \eta_3] \\ &= \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 2 & 2 \\ 2 & 2 & -1 \\ -1 & -1 & -1 \end{bmatrix} = \begin{bmatrix} -2 & -3/2 & 3/2 \\ 1 & 3/2 & 3/2 \\ 1 & 1/2 & -5/2 \end{bmatrix} \end{aligned}$$

□