第三章：行列式

1.

解. 根据行列式的定义当 $(i_1i_2i_3i_4)=(1234)$ 时可以构成 x^4

$$(-1)^{\tau(1234)}a_{11}a_{22}a_{33}a_{44}=2x^4$$

所以，x^4 的系数为 2.
同理，当 $(i_1i_2i_3i_4)=(2134)$ 时可以构成 x^3

$$(-1)^{\tau(2134)}a_{21}a_{12}a_{33}a_{44}=-x^3$$

所以，x^4 的系数为-1.

□

2.

解. 将行列式按定义展开每一项是

$$(-1)^{\tau(i_1i_2i_3i_4i_5)}(a_{i_11}a_{i_22}a_{i_33}a_{i_44}a_{i_55})$$

而且 $a_{i_33},a_{i_44},a_{i_55}$ 至少有一个为 0.
所以行列式展开每一项都是 0. 行列式值为 0. □

3.

解. 如果 n 阶行列式中的 0 的个数比 n^2-n 多，那么至少存在一行（列）全为 0. 所以行列式必等于 0. □

4.

解.

$$D=\begin{vmatrix}1&1&\cdots&1&1\\1&1&\cdots&1&1\\\vdots&\vdots&\ddots&\vdots&\vdots\\1&1&\cdots&1&1\\1&1&\cdots&1&1\end{vmatrix}=\sum_{i_1i_2\cdots i_n}(-1)^{\tau(i_1i_2\cdots i_n)}=0$$

奇排列下每一项值为 -1，偶排列下为 1，所以令奇排列有 n 个，偶排列 m 个，则 $m-n=0,m=n$
所以奇偶排列各半. □

5.

解. (1)$D = 0$, 因为第二列为 0.

(2)$D = \begin{vmatrix} 1 & 1 & 1 & 3 \\ 0 & 3 & 1 & 1 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 5 \end{vmatrix} = 30$ □

6.

解. (1) 当 $x \neq 0$ 时

$$D_{2n} = \begin{vmatrix} x & 0 & \cdots & 0 & y \\ 0 & x & \cdots & y & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & y & \cdots & x & 0 \\ y & 0 & \cdots & 0 & x \end{vmatrix}$$

$$= \begin{vmatrix} x & 0 & \cdots & 0 & y \\ 0 & x & \cdots & y & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & x - y^2/x & 0 \\ 0 & 0 & \cdots & 0 & x - y^2/x \end{vmatrix}$$

$$= x^n (x - y^2/x)^n = (x^2 - y^2)^n$$

(2) 当 $x = 0$ 时

$$D_{2n} = \begin{vmatrix} 0 & 0 & \cdots & 0 & y \\ 0 & 0 & \cdots & y & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & y & \cdots & 0 & 0 \\ y & 0 & \cdots & 0 & 0 \end{vmatrix}$$

$$= (-1)^{\tau(2n,2n-1,\ldots,2,1)} y^{2n} = (-1)^{2n(2n-1)/2} y^{2n} = (-y^2)^n$$

□

7.

解.

$$D_n = \begin{vmatrix} x & y & 0 & \cdots & 0 & 0 \\ 0 & x & y & \cdots & 0 & 0 \\ 0 & 0 & x & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & x & y \\ y & 0 & 0 & \cdots & 0 & x \end{vmatrix}$$

$$= x \begin{vmatrix} x & y & \cdots & 0 & 0 \\ 0 & x & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & x & y \\ 0 & 0 & \cdots & 0 & x \end{vmatrix} + (-1)^{1+n} y \begin{vmatrix} y & 0 & \cdots & 0 & 0 \\ x & y & \cdots & 0 & 0 \\ 0 & x & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & x & y \end{vmatrix}$$

$$= x^n + (-1)^{1+n} y^n$$

$\square$

8.

解.

$$D_n = \begin{vmatrix} 2 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 1 & 2 & 1 & \cdots & 0 & 0 & 0 \\ 0 & 1 & 2 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 2 & 1 & 0 \\ 0 & 0 & 0 & \cdots & 1 & 2 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 1 & 2 \end{vmatrix}$$

$$= 2\begin{vmatrix} 2 & 1 & \cdots & 0 & 0 & 0 \\ 1 & 2 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 2 & 1 & 0 \\ 0 & 0 & \cdots & 1 & 2 & 1 \\ 0 & 0 & \cdots & 0 & 1 & 2 \end{vmatrix} - \begin{vmatrix} 1 & 0 & \cdots & 0 & 0 & 0 \\ 1 & 2 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 2 & 1 & 0 \\ 0 & 0 & \cdots & 1 & 2 & 1 \\ 0 & 0 & \cdots & 0 & 1 & 2 \end{vmatrix}$$

$$= 2D_{n-1} - D_{n-2}$$

$$D_n - D_{n-1} = D_{n-1} - D_{n-2} = ... = D_2 - D_1 = 3 - 2 = 1$$

$$D_n - D_{n-1} + D_{n-1} - D_{n-2} + ... + D_2 - D_1 = n - 1$$

$$D_n = n - 1 + D_1 = n - 1 + 2 = n + 1$$

$\square$

9.

解. 将行列式按第一列展开:

$$B_n = \begin{vmatrix} a+b & ab & 0 & \cdots & 0 & 0 & 0 & 0 \\ 1 & a+b & ab & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & a+b & ab & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 & a+b & ab \\ 0 & 0 & 0 & \cdots & 0 & 0 & 1 & a+b \end{vmatrix}$$

$$(a+b)\begin{vmatrix} a+b & ab & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & a+b & ab & 0 \\ 0 & 0 & \cdots & 0 & 1 & a+b & ab \\ 0 & 0 & \cdots & 0 & 0 & 1 & a+b \end{vmatrix}$$

$$+ (-1)^{1+2}\begin{vmatrix} ab & 0 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & a+b & ab & 0 \\ 0 & 0 & \cdots & 0 & 1 & a+b & ab \\ 0 & 0 & \cdots & 0 & 0 & 1 & a+b \end{vmatrix}$$

$$= (a+b)B_{n-1} - abB_{n-2}$$

我们可以得到:
$B_n - bB_{n-1} = aB_{n-1} - abB_{n-2} = a(B_{n-1} - bB_{n-2})$
$B_1 = a+b$, $B_2 = \begin{vmatrix} a+b & ab \\ 1 & a+b \end{vmatrix} = a^2 + ab + b^2$
$B_2 - bB_1 = a^2 + ab + b^2 - ab - b^2 = a^2$
所以 $B_n - bB_{n-1} = a^{n-2}a^2 = a^n$
同理可得 $B_n - aB_{n-1} = b^{n-2}b^2 = b^n$
当 $a \neq b$ 时:
得到 $aB_n - bB_n = a^{n+1} - b^{n+1}$
即 $B_n = \frac{a^{n+1}-b^{n+1}}{a-b}$
当 $a = b$ 时:
得到 $B_n - aB_{n-1} = a^n$ 即 $\frac{B_n}{a^n} - \frac{B_{n-1}}{a^{n-1}} = 1$
又 $\frac{B_1}{a} = 2$, 所以 $\frac{B_n}{a^n} = n+1$, 即 $B_n = (n+1)a^n$. □

10.

解.

$$D_n = \begin{vmatrix} 2cos\theta & 1 & 0 & \cdots & 0 & 0 & 0 \\ 1 & 2cos\theta & 1 & \cdots & 0 & 0 & 0 \\ 0 & 1 & 2cos\theta & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 2cos\theta & 1 & 0 \\ 0 & 0 & 0 & \cdots & 1 & 2cos\theta & 1 \\ 0 & 0 & 0 & \cdots & 0 & 1 & 2cos\theta \end{vmatrix}$$

$$= 2cos\theta D_{n-1} - D_{n-2}$$

$$D_1 = 2cos\theta = \frac{sin2\theta}{sin\theta}$$

$$D_2 = 4cos^2\theta - 1 = \frac{sin3\theta}{sin\theta}$$

$$D_3 = 8cos^3\theta - 4cos\theta = \frac{sin4\theta}{sin\theta}$$

我们大胆猜想

$$D_n = \frac{sin(n+1)\theta}{sin\theta}$$

用数学归纳法证明
当 $n = 1, 2, 3$ 时显然成立.
假设 $D_{n-2} = \frac{sin(n-1)\theta}{sin\theta}, D_{n-1} = \frac{sinn\theta}{sin\theta}$.
则

$$D_n = 2cos\theta D_{n-1} - D_{n-2} = 2cos\theta\frac{sinn\theta}{sin\theta} - \frac{sin(n-1)\theta}{sin\theta}$$

$$= \frac{cos\theta sinn\theta + cosn\theta sin\theta + cos\theta sinn\theta - cosn\theta sin\theta - sin(n-1)\theta}{sin\theta}$$

$$= \frac{sin(n+1)\theta + sin(n-1)\theta - sin(n-1)\theta}{sin\theta} = \frac{sin(n+1)\theta}{sin\theta}$$

所以，假设成立. □

17.

解. 当 $n=1$ 时

$$D_1 = a_1 + b_1$$

当 $n=2$ 时

$$D_2 = a_1b_2 + a_2b_1 - a_1b_1 - a_2b_2$$

当 $n \geq 3$ 时

$$D_n = \begin{vmatrix} a_1+b_1 & a_1+b_2 & \cdots & a_1+b_n \\ a_2+b_1 & a_2+b_2 & \cdots & a_2+b_n \\ \vdots & \vdots & \ddots & \vdots \\ a_n+b_1 & a_n+b_2 & \cdots & a_n+b_n \end{vmatrix}$$

$$= \begin{vmatrix} a_1 & a_1 & \cdots & a_1 \\ a_2+b_1 & a_2+b_2 & \cdots & a_2+b_n \\ \vdots & \vdots & \ddots & \vdots \\ a_n+b_1 & a_n+b_2 & \cdots & a_n+b_n \end{vmatrix} + \begin{vmatrix} b_1 & b_2 & \cdots & b_n \\ a_2+b_1 & a_2+b_2 & \cdots & a_2+b_n \\ \vdots & \vdots & \ddots & \vdots \\ a_n+b_1 & a_n+b_2 & \cdots & a_n+b_n \end{vmatrix}$$

$$= \begin{vmatrix} a_1 & 0 & \cdots & 0 \\ a_2+b_1 & b_2-b_1 & \cdots & b_n-b_1 \\ \vdots & \vdots & \ddots & \vdots \\ a_n+b_1 & b_2-b_1 & \cdots & b_n-b_1 \end{vmatrix} + \begin{vmatrix} b_1 & b_2 & \cdots & b_n \\ a_2 & a_2 & \cdots & a_2 \\ \vdots & \vdots & \ddots & \vdots \\ a_n & a_n & \cdots & a_n \end{vmatrix}$$

$$a_1 \begin{vmatrix} b_2-b_1 & \cdots & b_n-b_1 \\ \vdots & \ddots & \vdots \\ b_2-b_1 & \cdots & b_n-b_1 \end{vmatrix} + 0 = 0$$

□

27.

解. 当 $n=1$ 时

$$D_1 = 1 + x_1y_1$$

当 $n=2$ 时

$$D_2 = x_1y_1 + x_2y_2 - x_1y_2 - x_2y_1$$

当 $n \geq 3$ 时

$$D_n = \begin{vmatrix} 1+x_1y_1 & 1+x_1y_2 & \cdots & 1+x_1y_n \\ 1+x_2y_1 & 1+x_2y_2 & \cdots & 1+x_2y_n \\ \vdots & \vdots & \ddots & \vdots \\ 1+x_ny_1 & 1+x_ny_2 & \cdots & 1+x_ny_n \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 1+x_1y_2 & \cdots & 1+x_1y_n \\ 1 & 1+x_2y_2 & \cdots & 1+x_2y_n \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1+x_ny_2 & \cdots & 1+x_ny_n \end{vmatrix} + \begin{vmatrix} x_1y_1 & 1+x_1y_2 & \cdots & 1+x_1y_n \\ x_2y_1 & 1+x_2y_2 & \cdots & 1+x_2y_n \\ \vdots & \vdots & \ddots & \vdots \\ x_ny_1 & 1+x_ny_2 & \cdots & 1+x_ny_n \end{vmatrix}$$

$$\begin{vmatrix}1 & x_1y_2 & \cdots & x_1y_n \\ 1 & x_2y_2 & \cdots & x_2y_n \\ \vdots & \vdots & \ddots & \vdots \\ 1 & x_ny_2 & \cdots & x_ny_n\end{vmatrix} + \begin{vmatrix}x_1y_1 & 1 & \cdots & 1 \\ x_2y_1 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ x_ny_1 & 1 & \cdots & 1\end{vmatrix}$$
$$= 0 + 0 = 0$$

□

28(3).

解.

$$A^{-1} = A^*/|A| = A^* = \begin{bmatrix}1 & 0 & -1 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & -1\end{bmatrix}$$

□

29.

解. $AA^* = |A|E$
A 可逆，则 $A^* = |A|A^{-1}$
所以，$|(\frac{1}{4}A)^{-1} - 2A^*|$
$= |4A^{-1} - 2A^*|$
$= |4A^{-1} - 2|A|A^{-1}|$
$= |(4 - 2|A|)A^{-1}|$
$= |(-2)A^{-1}|$
$= (-2)^n|A|^{-1}$
$= \frac{(-2)^n}{3}$ □

30.

解. $ABA^{-1} = BA^{-1} + 3I$
$\Rightarrow AB = B + 3A$
$\Rightarrow (A - I)B = 3A$
$\Rightarrow (I - A^{-1})B = 3I$
$|A^*| = ||A|A^{-1}| = |A|^{n-1}$
$|A| = |A^*|^{\frac{1}{n-1}} = |A^*|^{\frac{1}{3}} = 2$

$I - A^{-1} = I - \frac{A^*}{|A|} = \begin{bmatrix}1/2 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 0 \\ -1/2 & 0 & 1/2 & 0 \\ 0 & 3/2 & 0 & -3\end{bmatrix}$ 可逆

所以,$B = 3(I - A^{-1})^{-1} = \begin{bmatrix}6 & 0 & 0 & 0 \\ 0 & 6 & 0 & 0 \\ 6 & 0 & 6 & 0 \\ 0 & 3 & 0 & -1\end{bmatrix}$ □

34.

解. (1)(2) 的秩都是 2. □