Queen's University
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Nonlinear System Identification

Implementation of Fast Orthogonal Search

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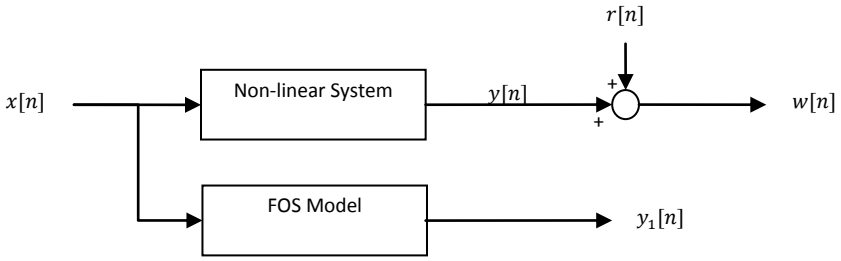
Contents

Introduction

This is a report showing my implementation of Fast Orthogonal Search (FOS) and discusses its results when applied to various systems:

- a model represented by a simple linear difference equation,
- a model represented by a more complex non-linear difference equation,
- a Linear/Non-linear/Linear (LNL) cascade model,
- a model represented by a non-polynomial equation,
- bias drift of a stationary gyroscope, and
- bias drift of s stationary accelerometer.

Methodology



Finding the Best Model for Noise-Free Output

1. Generate $x[n]$, for $n = 0, \cdots, 2999$ where $x[n]$ is sufficiently variable (we shall use MATLAB's function `rand` for uniform random distribution).
2. Use the assumed non-linear model to generate resulting output, $y[n]$, for $n = 0, \cdots, 2999$.
3. **Training Phase**: Use FOS to identify several possible non-linear difference equations over the record from $n = N_0, \cdots, 999$, where $N_0 = \max(K, L)$:

   $$y_1[n] = F(y[n-1], \cdots y[n-K], x[n], \cdots x[n-L]) + e[n]$$

   Assume various values for K and L.

4. **Selection Phase**: Compare the various models' ability to predict the true output, $y[n]$, over $n = 1000 + N_0, \cdots, 1999$ and choose the best, by obtaining %MSE=$\frac{\sum_{n=N_O}^{N} \overline{(y[n]-y_1[n])^2}}{\text{var}(y[n])}$.
5. **Evaluation Phase**: Compute the %MSE of the chosen model over $n = 2000 + N_0, \cdots, 2999$.

Finding the Best Model for Noisy Output

1. Generate zero-mean white noise, $r[n]$, independent of $x[n]$ with variance equivalent to: $\text{var}(r[n]) = \frac{P}{100}\text{var}(y[n])$. Try different values of $P = 150, 100, 50, 25$.
2. **Training Phase:** Use $x[n]$ and $w[n] = y[n] + r[n]$ to find various models over $n = N_0, \cdots, 999$.
3. **Selection Phase**: Compare the various models over $n = 1000 + N_0, \cdots, 1999$ by ability to predict $w[n]$ and compare %MSE with respect to noisy output=$\frac{\sum_{n=N_O}^{N} \overline{(w[n]-y_1[n])^2}}{\text{var}(w[n])}$.
4. **Evaluation Phase**: Compute the noisy-free %MSE=$\frac{\sum_{n=N_O}^{N} \overline{(y[n]-y_1[n])^2}}{\text{var}(y[n])}$ of the chosen model over $n = 2000 + N_0, \cdots, 2999$.

Various Input Generations

Try different methods to generate input $x[n]$ and re-compute the corresponding $y[n]$. Using the chosen values of K and L in the Selection Phase, apply the FOS algorithm to the 1st 1000 samples of data. Analyze the effect of type of input generation on %MSE.

Analyze Various Orders

Using the chosen values of K and L in the Selection Phase, apply the FOS algorithm to the 1st 1000 samples of data but with maximum order of cross-products ranging from 2 to 5. Analyze the effect of maximum cross-products on %MSE.

Results

This section explains the methodology of analyzing each model and a description of its results.

Model Name

Description of the model in mathematical or physical terms.

```
Code which generates the model output.
```

Total Elapsed Time = Total computation time for all 3 phases (Training Phase + Selection Phase + Evaluation Phase)

Chosen values of K & L: The values of K & L for the selected model in the Selection Phase.

Figure showing: $y[n]$: the true output of the model $y1[n]$: the output of the FOS model	Figure showing $Q[m]$ vs. m

Equation representing the selected FOS model in the form: $\sum_{m=0}^{M} a_m p_m[n]$

%MSE = Percentage Mean Square Error of the chosen model, which is equivalent to: $\frac{\sum_{n=N_O}^{N} \overline{(y[n]-y_1[n])^2}}{\mathrm{var}(y[n])}$

Table to show the computation time needed to implement the FOS algorithm for various values of K & L.

K	L	Computation Time

Noise Inserted

This table analyzes the effect of adding noise to the output and re-runs the whole algorithm over the 3 phases (Training Phase + Selection Phase + Evaluation Phase).
We shall try different values of P: 150, 100, 50, 25.

P = $\frac{\mathrm{var}(r[n])}{\mathrm{var}(y[n])} \times 100\%$ %MSE = $\frac{\sum_{n=N_O}^{N} \overline{(y[n]-y_1[n])^2}}{\mathrm{var}(y[n])}$	Total Elapsed Time = Total computation time for all 3 phases %MSE w.r.t. noisy output = $\frac{\sum_{n=N_O}^{N} \overline{(w[n]-y_1[n])^2}}{\mathrm{var}(w[n])}$ Ideal %MSE = $\frac{\mathrm{var}(r[n])}{\mathrm{var}(w[n])}$
Figure showing: $y[n]$: the true output of the model $y1[n]$: the output of the FOS model	Figure showing $Q[m]$ vs. m

Different Input Generations

Here we set K & L to the chosen values of the Selection Phase and apply FOS to the 1st 1000 samples only and evaluate the computation time and %MSE for various generation types for input, $x[n]$.
The different generations we use:

- Uniform Distribution: Using MATLAB's `rand` function and trying different standard deviations.
- Normal Distribution: Using MATLAB's `randn` function and trying different standard deviations.
- Sinusoidal: Using MATLAB's `sin` function and trying different frequencies and amplitudes.
- Triangular: Using MATLAB's `sawtooth` function and trying different widths.

Varying Orders

Here we set K & L to the chosen values of the Selection Phase and apply FOS to the 1st 1000 samples only but using various orders of cross products.

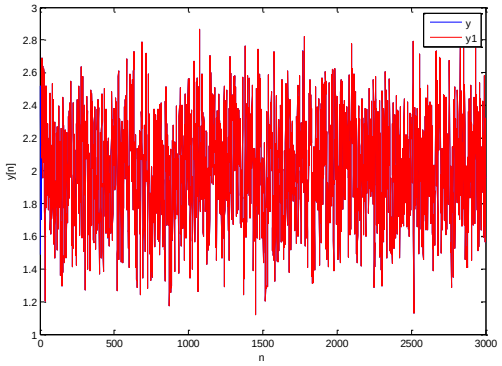
Linear Difference Equation

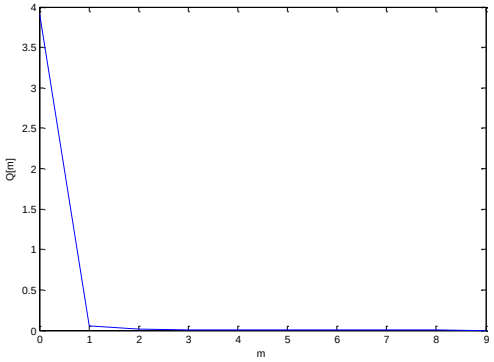
$$y[n] = 1 + 0.6x[n] + 0.3x[n-1] + 0.4x[n-2] + 0.7x[n-3]$$

```
% Simple Difference Equation 1
y = 1 +  0.6*x  + 0.3*delay(x,1) + 0.4*delay(x,2) + 0.7*delay(x,3);
```

Total Elapsed Time = 58.313930 seconds

Chosen values of K & L: $K = 6, \qquad L = 5$





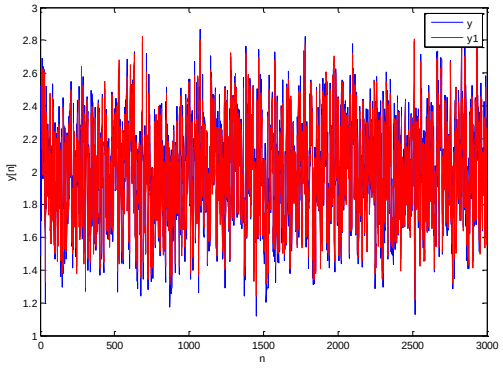
$$y_1[n] = 0.999999999999951 + 4.546054288220858 \times 10^{-14}x[n]x[n-3] - 1.382782777170633 \times 10^{-13}x[n-1]x[n-2] + 1.276756478318930 \times 10^{-14}x[n-2]x[n-3] - 5.928590951498336 \times 10^{-14}x[n]x[n-1] + 0.699999999999959x[n-3] + 0.600000000000023x[n] + 0.400000000000077x[n-2] + 0.300000000000107x[n-1]$$

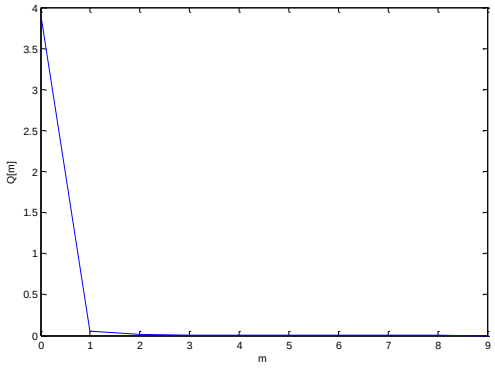
%MSE = 2.5117×10^{-25} %

K	L	Computation Time
10	12	4.344746 seconds
3	15	4.629001 seconds
12	8	5.378242 seconds
10	12	6.895127 seconds
10	19	10.559044 seconds
8	19	8.900025 seconds
6	5	1.871609 seconds
14	6	5.230363 seconds
8	3	1.889446 seconds
7	10	4.031555 seconds

Noise Inserted

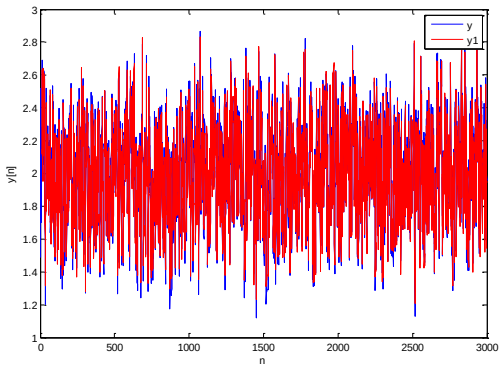
P = 150	Total Elapsed Time = 22.151039 seconds
%MSE = 3.078%	%MSE w.r.t. noisy output = 62.19%
	Ideal %MSE = 61.52%

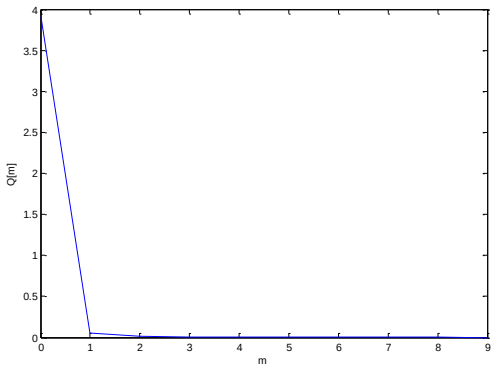




P = 100
%MSE = 2.0079%

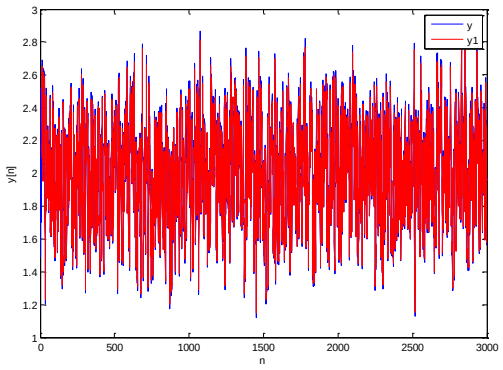
Total Elapsed Time = 23.151971 seconds
%MSE w.r.t. noisy output = 51.69%
Ideal %MSE = 51.30%

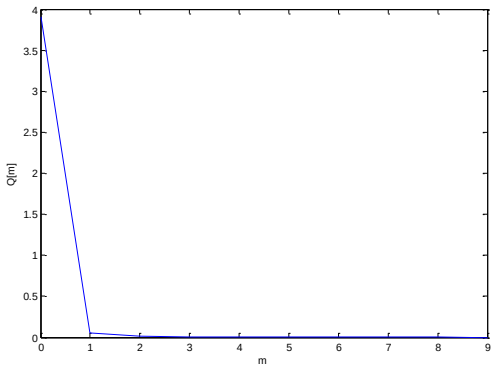




P = 50
%MSE = 0.45%

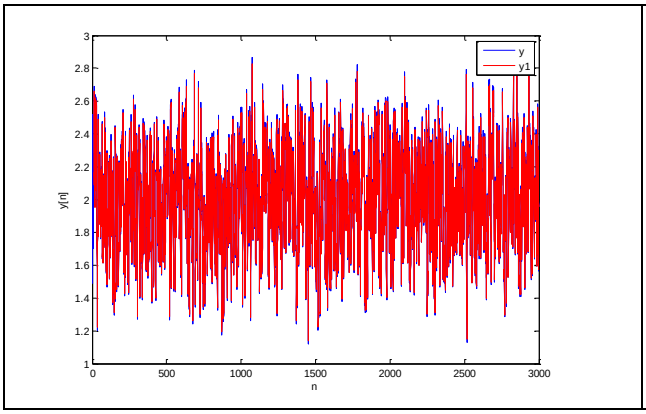
Total Elapsed Time = 26.159095 seconds
%MSE w.r.t. noisy output = 34.41%
Ideal %MSE = 34.15%

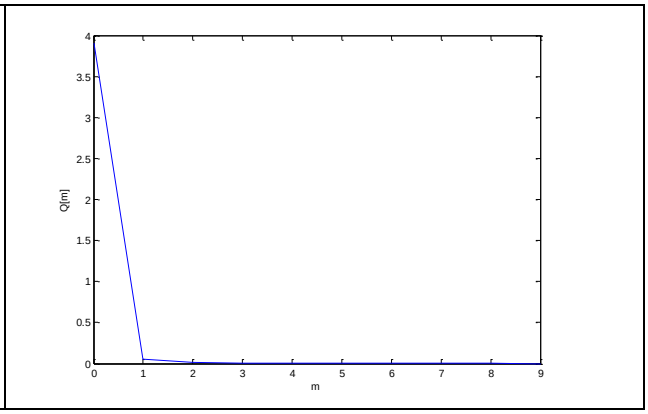




P = 25
%MSE = 0.2255%

Total Elapsed Time = 24.730101 seconds
%MSE w.r.t. noisy output = 20.77%
Ideal %MSE = 20.41%





Different Input Generations

Setting $K = 6, L = 5$

Input Type	Parameter	Computation Time	%MSE
Uniform Distribution	Standard Deviation = 1	1.531016 seconds	2.67×10^{-25}%
Uniform Distribution	Standard Deviation = 5	16.078507 seconds	6.43×10^{-21}%
Uniform Distribution	Standard Deviation = 10	1.569202 seconds	5.01×10^{-25}%
Normal Distribution	Standard Deviation = 1	0.941653 seconds	6.11×10^{-28}%
Normal Distribution	Standard Deviation = 5	0.891351 seconds	1.28×10^{-28}%
Normal Distribution	Standard Deviation = 10	1.464059 seconds	1.54×10^{-28}%
Sinusoidal	Frequency = 1 Hz, Amplitude = 1	10.395199 seconds	39814%
Sinusoidal	Frequency = 10 Hz, Amplitude = 1	12.403733 seconds	1993%
Sinusoidal	Frequency = 100 Hz, Amplitude = 1	5.686532 seconds	46.47%
Sinusoidal	Frequency = 1 kHz, Amplitude = 1	11.499738 seconds	2582%
Sinusoidal	Frequency = 10 kHz, Amplitude = 1	11.721615 seconds	41948%
Sinusoidal	Frequency = 1 Hz, Amplitude = 10	11.594711 seconds	2272%
Sinusoidal	Frequency = 10 Hz, Amplitude = 10	11.043747 seconds	398%
Sinusoidal	Frequency = 100 Hz, Amplitude = 10	8.798550 seconds	960150%
Sinusoidal	Frequency = 1 kHz, Amplitude = 10	10.548092 seconds	5965%
Sinusoidal	Frequency = 10 kHz, Amplitude = 10	2.166551 seconds	2.65×10^{8}%
Triangular	Width = 2π, Amplitude = 1	4.902682 seconds	1.77×10^{-19}%
Triangular	Width = 0.1×2π, Amplitude = 1	2.264347 seconds	1.81×10^{-24}%
Triangular	Width = 0.01×2π, Amplitude = 1	14.424037 seconds	6.05×10^{-18}%
Triangular	Width = 2π, Amplitude = 10	5.365355 seconds	1.84×10^{-18}%
Triangular	Width = 0.1×2π, Amplitude = 10	9.810197 seconds	2.37×10^{-18}%
Triangular	Width = 0.01×2π, Amplitude = 10	1.229538 seconds	1.07×10^{-27}%

Varying Orders

Setting $K = 6, L = 5$

Order	Computation Time	%MSE
2	1.907617 seconds	2.64×10^{-25}%
3	5.331890 seconds	2.68×10^{-25}%
4	26.402480 seconds	2.68×10^{-25}%
5	67.471670 seconds	2.68×10^{-25}%

Analysis

For a simple linear difference equation we may conclude that:

- %MSE was almost zero because the system model consists of polynomial terms which are candidates of the FOS algorithm.
- %MSE w.r.t to original input improves with the decrease of noise at the output.
- %MSE w.r.t to noisy output is very close to ideal %MSE, therefore the FOS algorithm removes out the effect of noise due to the averaging which appears in the algorithm.
- Generating input randomly using normal distribution seems to have better results than using uniform distribution.
- Sinusoidal input generated very high error rates probably because limited number of values of $x[n]$ and $y[n]$ are generated. Increasing the frequency of sinusoidal input reduced %MSE up to a certain limit and then increased again. This is probably due to the effect of aliasing which means that the same sequence of inputs are generated when frequency increases.
- Triangular input surprisingly showed low %MSE although they have the same repetitive nature of sinusoidal signals. Further investigation is needed to evaluate whether the FOS model concluded using a triangular input shall give good results for different types of input. The effect of aliasing also appears for triangular input.
- Increasing the maximum-cross order of candidates shall not improve the %MSE but rather only increase computation time because the original model consist only of 1^{st} order terms.

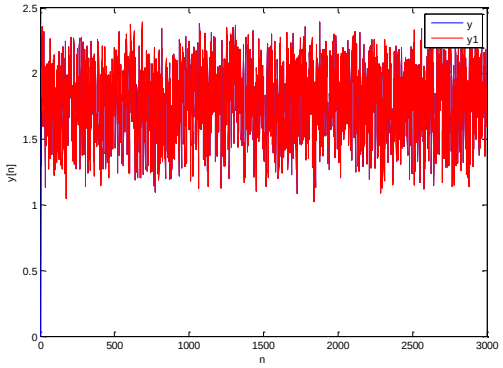
Non-Linear Difference Equation

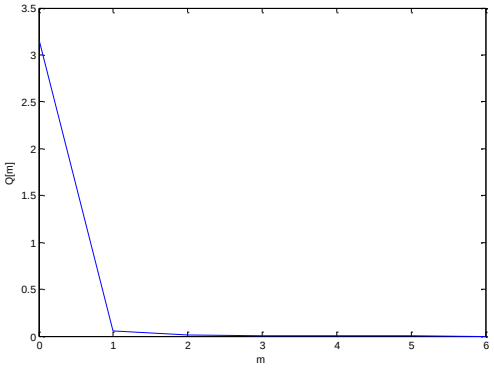
$$y[n] = 1 + 0.7x[n] + 0.8x[n-1] + 0.1x[n-1]y[n-1] - 0.4x[n]x[n-1] + 0.2x[n-4]x[n-5]$$

```
% Complex Difference Equation
y = zeros(3000, 1);

a0 = 1;
b0 = 0.7;
b1 = 0.8;
c1 = 0.1;
c2 = 0.4;
c4 = 0.2;

for n = 6:3000
    y(n) = a0 + b0*x(n) + b1*x(n-1) + c1*x(n-1)*y(n-1) - c2*x(n)*x(n-1)+ c4*x(n-4)*x(n-
5);
end
```

Total Elapsed Time = 32.708566 seconds

Chosen values of K & L: $K = 8, \qquad L = 3$



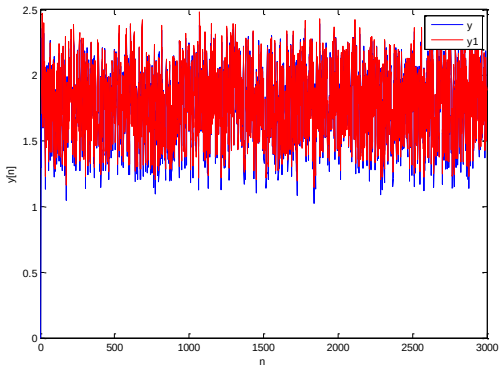
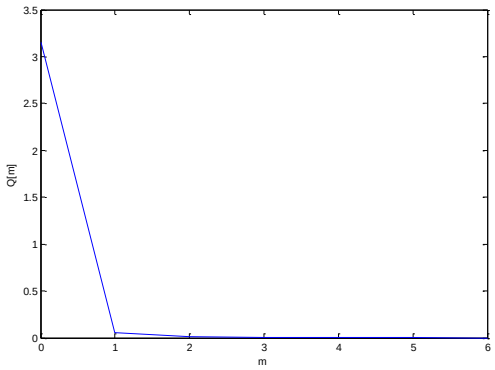
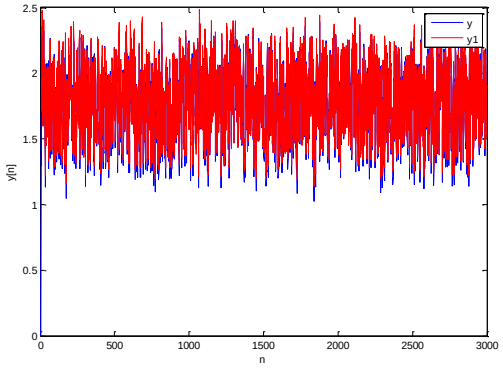
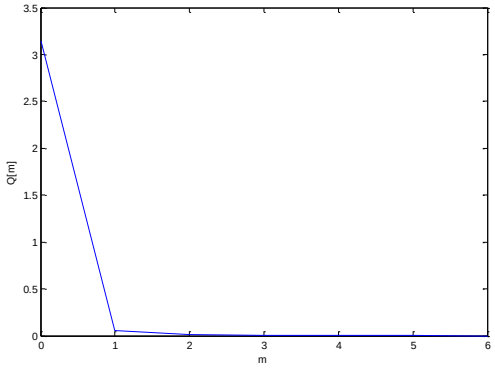
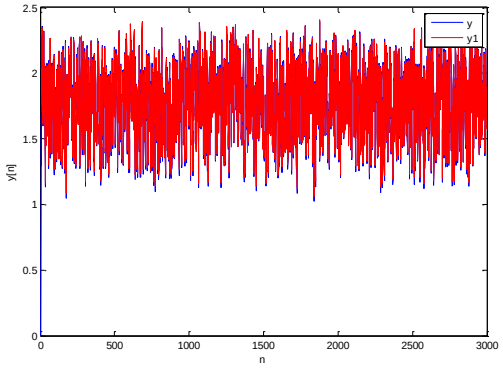
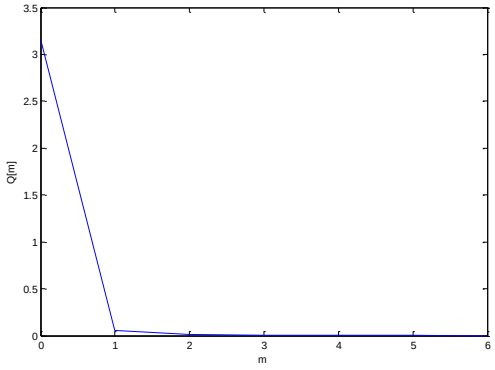


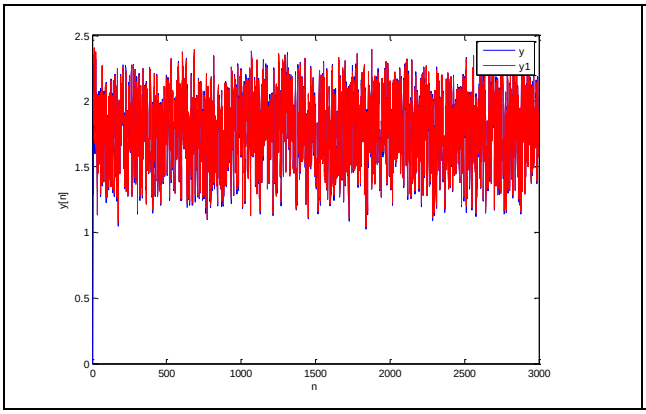
$$y_1[n] = 1.00000000000028 - 0.399999999999855x[n]x[n-1] + 0.100000000000215x[n-1]y[n-1] + 0.699999999999899x[n] + 0.799999999999558x[n-1] + 0.200000000000109x[n-4]x[n-5] - 8.39989949488176 \times 10^{-14}y[n-1]y[n-3]$$

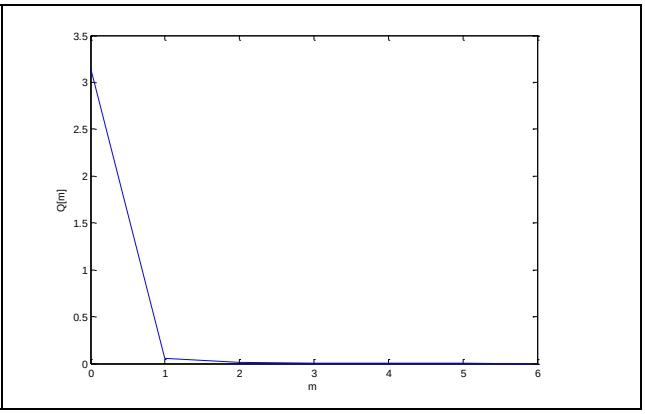
%MSE = 2.397×10^{-24} %

K	L	Computation Time
10	12	3.087012 seconds
3	15	34.769338 seconds
12	8	4.524928 seconds
10	12	4.338774 seconds
10	19	6.453342 seconds
8	19	5.733319 seconds
6	5	1.272192 seconds
14	6	3.317925 seconds
8	3	1.227670 seconds
7	10	2.901604 seconds

Noise Inserted

P = 150 %MSE = 3.048%	Total Elapsed Time = 14.707611 seconds %MSE w.r.t. noisy output = 65.51% Ideal %MSE = 61.36%
	
P = 100 %MSE = 2.70%	Total Elapsed Time = 14.885791 seconds %MSE w.r.t. noisy output = 55.39% Ideal %MSE = 51.16%
	
P = 50 %MSE = 0.48%	Total Elapsed Time = 20.216341 seconds %MSE w.r.t. noisy output = 36.60% Ideal %MSE = 34.06%
	
P = 25 %MSE = 0.45%	Total Elapsed Time = 22.122402 seconds %MSE w.r.t. noisy output = 22.38% Ideal %MSE = 20.37%





Different Input Generations

Setting $K = 8, L = 3$

Input Type	Parameter	Computation Time	%MSE
Uniform Distribution	Standard Deviation = 1	1.404847 seconds	6.56×10^{-25}%
Uniform Distribution	Standard Deviation = 5	1.064718 seconds	2.01×10^{-27}%
Uniform Distribution	Standard Deviation = 10	N/A[1]	
Normal Distribution	Standard Deviation = 1	1.292744 seconds	2.28×10^{-28}%
Normal Distribution	Standard Deviation = 5	1.135890 seconds	4.70×10^{-28}%
Normal Distribution	Standard Deviation = 10	1.126074 seconds	2.28×10^{-29}%
Sinusoidal	Frequency = 1 Hz, Amplitude = 1	2.128849 seconds	0.239%
Sinusoidal	Frequency = 10 Hz, Amplitude = 1	7.927907 seconds	95.67%
Sinusoidal	Frequency = 100 Hz, Amplitude = 1	2.761670 seconds	0.0046%
Sinusoidal	Frequency = 1 kHz, Amplitude = 1	23.944826 seconds	1.72×10^{5}%
Sinusoidal	Frequency = 10 kHz, Amplitude = 1	2.219971 seconds	0.92%
Sinusoidal	Frequency = 1 Hz, Amplitude = 10	1.313247 seconds	0.49%
Sinusoidal	Frequency = 10 Hz, Amplitude = 10	1.833226 seconds	0.81%
Sinusoidal	Frequency = 100 Hz, Amplitude = 10	1.631035 seconds	0.01%
Sinusoidal	Frequency = 1 kHz, Amplitude = 10	1.319631 seconds	2.76×10^{-7}%
Sinusoidal	Frequency = 10 kHz, Amplitude = 10	0.955011 seconds	2.04×10^{-8}%
Triangular	Width = 2π, Amplitude = 1	1.598169 seconds	3.98×10^{-26}%
Triangular	Width = 0.1×2π, Amplitude = 1	19.331429 seconds	1.06×10^{-12}%
Triangular	Width = 0.01×2π, Amplitude = 1	1.560098 seconds	6.19×10^{-26}%
Triangular	Width = 2π, Amplitude = 10	11.058648 seconds	2.85×10^{-15}%
Triangular	Width = 0.1×2π, Amplitude = 10	1.312038 seconds	7.62×10^{-28}%
Triangular	Width = 0.01×2π, Amplitude = 10	19.017779 seconds	1.55×10^{-17}%

Varying Orders

Setting $K = 8, L = 3$

Order	Computation Time	%MSE
2	1.346171 seconds	2.50×10^{-24}%
3	5.354506 seconds	2.50×10^{-24}%
4	17.021489 seconds	2.50×10^{-24}%
5	58.637289 seconds	2.50×10^{-24}%

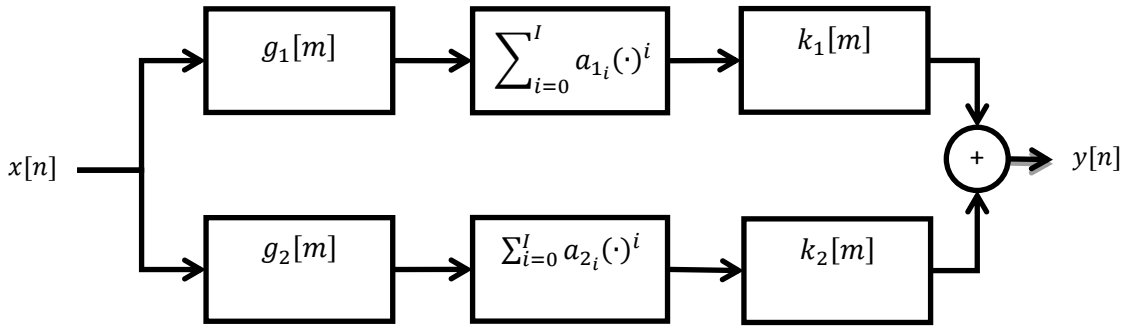
[1] $y[n]$ diverged to $-\infty$

Analysis

For a non-linear difference equation we may conclude that:

- %MSE was almost zero because the system model consists of polynomial terms which are candidates of the FOS algorithm.
- %MSE w.r.t to original input improves with the decrease of noise at the output.
- %MSE w.r.t to noisy output is very close to ideal %MSE, therefore the FOS algorithm removes out the effect of noise due to the averaging which appears in the algorithm.
- Generating input randomly using normal distribution seems to have better results than using uniform distribution.
- Generating input of uniform distribution with high standard deviation may result in high divergence due to the feedback effect (i.e. current output depends on previous outputs) of the model.
- Sinusoidal and triangular inputs resulted in better results than for linear difference equations. Further investigation is needed to evaluate this.
- Increasing the maximum-cross order of candidates shall not improve the %MSE but rather only increase computation time because the original model consist only of 2^{nd} order terms.

LNL Cascade Model



$$g_1[m] = \sum_{i=0}^{4} \left(e^{-i} + e^{-2i}\right)\delta[m-i]$$

$$k_1[m] = \sum_{i=0}^{4} 3e^{-i}\delta[m-i]$$

$$a_{1_i} = 0.5 + 2e^{-i}; \quad i = 0, \dots, 4$$

$$g_2[m] = \sum_{i=0}^{4} \left(e^{-i} + 3e^{-2i}\right)\delta[m-i]$$

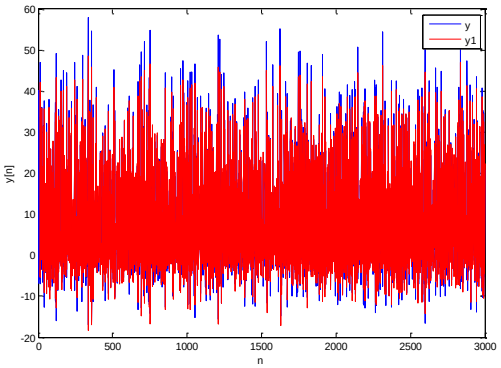
$$k_2[m] = \sum_{i=0}^{4} \left(0.1e^{-i} + 0.9e^{-2i}\right)\delta[m-i]$$

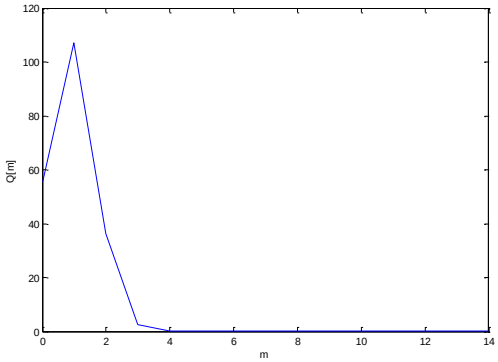
$$a_{2_i} = 0.2 + 3e^{-i}; \quad i = 0, \dots, 4$$

```
% LNL Cascade
t = 1:5;
g1 = exp(-t) + exp(-2*t);
a1 = 0.5 + 2*exp(-t);
k1 = 3*exp(-t);

g2 = exp(-t) + 3*exp(-2*t);
a2 = 0.2 + 3*exp(-t);
k2 = 0.1*exp(-t) + 0.9*exp(-2*t);

y = lnl(g1, a1, k1, x) + lnl(g2, a2, k2, x);
```

Total Elapsed Time = 54.206952 seconds

Chosen values of K & L: $K = 6, \qquad L = 5$



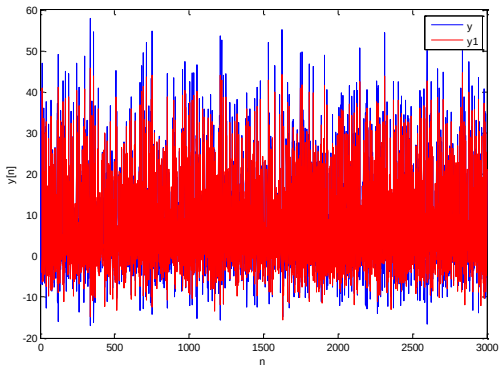
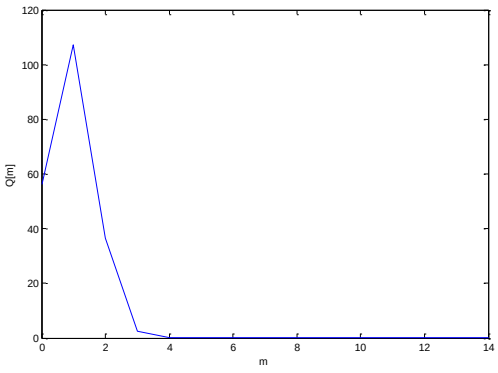
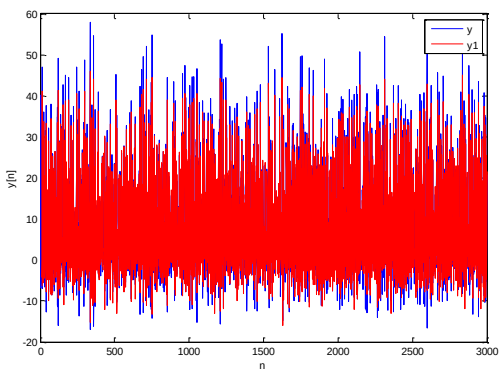
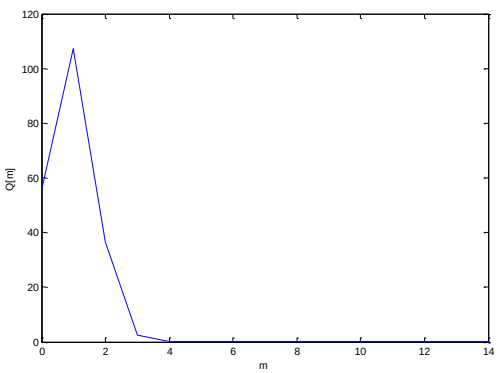


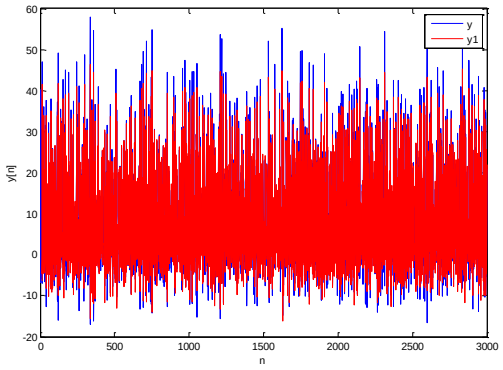
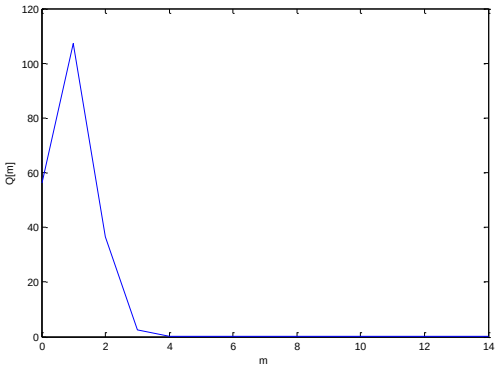
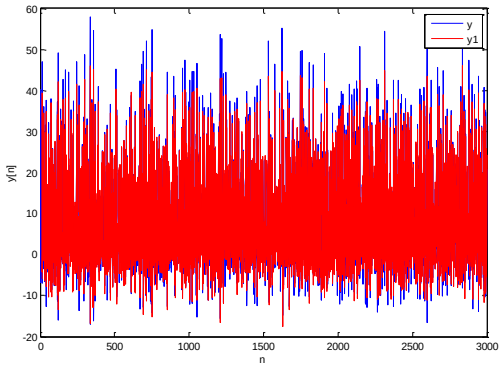
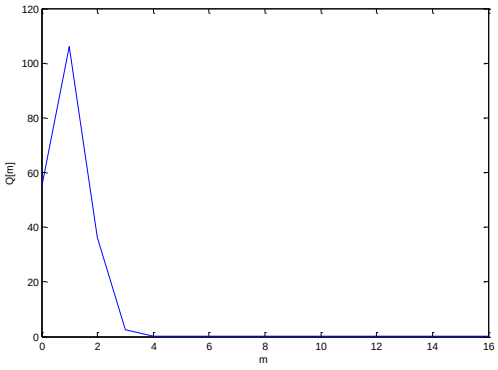
$$y_1[n] = 0.282599899504132 + 47.7354618228480x2[n] - 34.9751007376709x[n]x[n-1] - 0.133415188730252x[n-1]y[n-1] - 0.00112090835658793y2[n-1] - 1.63541987950650x[n-1] + 5.90529341665448x2[n-1] + 0.195475096560701x[n]y[n-1] + 1.90162632049297x[n] + 3.10448279699245x[n]y[n-2] - 2.55594690102701x2[n-2] - 0.156345383248782y[n-1] + 0.000741968612804866y[n-4]y[n-5]$$

%MSE = 1.3856 %

K	L	Computation Time
10	12	11.016234 seconds
3	15	10.190549 seconds
12	8	11.830497 seconds
10	12	15.097112 seconds
10	19	25.064469 seconds
8	19	24.624407 seconds
6	5	3.742241 seconds
14	6	14.313596 seconds
8	3	4.309165 seconds
7	10	7.572814 seconds

Noise Inserted

P = 150 %MSE = 3.46%	Total Elapsed Time = 28.024301 seconds %MSE w.r.t. noisy output = 60.99% Ideal %MSE = 62.91%
	
P = 100 %MSE = 2.95%	Total Elapsed Time = 89.228064 seconds %MSE w.r.t. noisy output = 50.50% Ideal %MSE = 52.48%
	
P = 50 %MSE = 2.33%	Total Elapsed Time = 33.752851 seconds %MSE w.r.t. noisy output = 33.45%

Ideal %MSE = 34.88%	
	
P = 25 %MSE = 2.22%	Total Elapsed Time = 29.121019 seconds %MSE w.r.t. noisy output = 20.88% Ideal %MSE = 20.78%
	

Different Input Generations

Setting $K = 6, L = 5$

Input Type	Parameter	Computation Time	%MSE
Uniform Distribution	Standard Deviation = 1	2.097548 seconds	7.57%
Uniform Distribution	Standard Deviation = 5	2.768679 seconds	8.91%
Uniform Distribution	Standard Deviation = 10	2.759290 seconds	9.03%
Normal Distribution	Standard Deviation = 1	3.719084 seconds	43.83%
Normal Distribution	Standard Deviation = 5	5.288471 seconds	46.77%
Normal Distribution	Standard Deviation = 10	4.395752 seconds	47.48%
Sinusoidal	Frequency = 1 Hz, Amplitude = 1	0.819467 seconds	0.0025%
Sinusoidal	Frequency = 10 Hz, Amplitude = 1	1.084350 seconds	1.52×10^{-5}%
Sinusoidal	Frequency = 100 Hz, Amplitude = 1	1.339380 seconds	1.69×10^{-6}%
Sinusoidal	Frequency = 1 kHz, Amplitude = 1	1.762254 seconds	1.439×10^{-10}%
Sinusoidal	Frequency = 10 kHz, Amplitude = 1	2.536762 seconds	7.21×10^{-11}%
Sinusoidal	Frequency = 1 Hz, Amplitude = 10	1.005178 seconds	1.30×10^{-5}%
Sinusoidal	Frequency = 10 Hz, Amplitude = 10	1.099383 seconds	2.76×10^{-6}%
Sinusoidal	Frequency = 100 Hz, Amplitude = 10	2.392238 seconds	7.11×10^{-11}%
Sinusoidal	Frequency = 1 kHz, Amplitude = 10	2.336597 seconds	7.00×10^{-11}%
Sinusoidal	Frequency = 10 kHz, Amplitude = 10	2.097544 seconds	2.84×10^{-11}%
Triangular	Width = 2π, Amplitude = 1	12.876216 seconds	5.77×10^{-4}%
Triangular	Width = 0.1×2π, Amplitude = 1	15.964115 seconds	0.05%
Triangular	Width = 0.01×2π, Amplitude = 1	4.545124 seconds	0.19%

Triangular	Width = 2π, Amplitude = 10	14.295613 seconds	3.64×10^{-5}%
Triangular	Width = 0.1×2π, Amplitude = 10	9.972019 seconds	0.093%
Triangular	Width = 0.01×2π, Amplitude = 10	11.586914 seconds	0.50%

Varying Orders

Setting $K = 8, L = 1$ (close to the values which we know previously).

Order	Computation Time	%MSE
2	2.338557 seconds	1.44%
3	35.752206 seconds	0.082%
4	1096.8528 seconds	9.56×10^{-4}%
5	23159.321 seconds	1.46×10^{-7}%

Analysis

For a LNL cascade model we may conclude that:

- %MSE was relatively high (more than 1%) because the system model consists of polynomial terms of higher order terms which were not candidates of the FOS algorithm when using maximum order of 2 only.
- %MSE w.r.t to original input improves with the decrease of noise at the output.
- %MSE w.r.t to noisy output is very close to ideal %MSE, therefore the FOS algorithm removes out the effect of noise due to the averaging which appears in the algorithm.
- Generating input randomly using normal distribution and uniform distributions with higher standard deviations increases the %MSE. Therefore, increasing the variance of the input makes the FOS algorithm more vulnerable to errors if it does not contain the required candidates.
- Sinusoidal and triangular inputs resulted in better results than random data. But this is probably due to the repetitive nature of the inputs and outputs and the resulting FOS models are likely to result in erroneous results for non-repetitive inputs.
- Increasing the maximum-cross order of candidates improves the %MSE to levels of almost zero since the LNL model contains terms of higher order than 2.

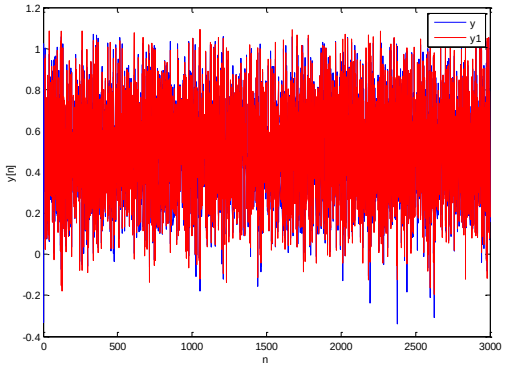
Non-Polynomial Equation

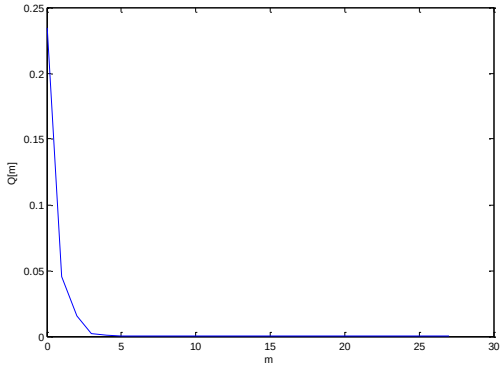
$$y[n] = \sin x[n-1]\cos x[n] + e^{-3x[n]}\sqrt{|x[n]|} + 0.1y[n-1]\ln|y[n-2] + 0.01|$$

```
% Non-Polynomial Equation
y = zeros(3*N, 1);
for n = 3:3*N
    y(n) = sin(x(n-1))*cos(x(n)) + exp(-3*x(n))*sqrt(abs(x(n))) + 0.1*log(abs(y(n-
2)+0.01))*y(n-1);
end
```

Total Elapsed Time = 91.730930 seconds

Chosen values of K & L: $K = 6, \qquad L = 5$





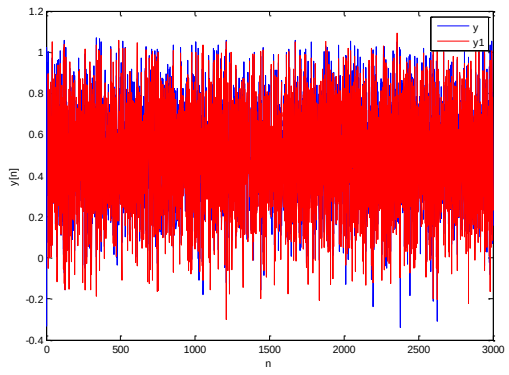
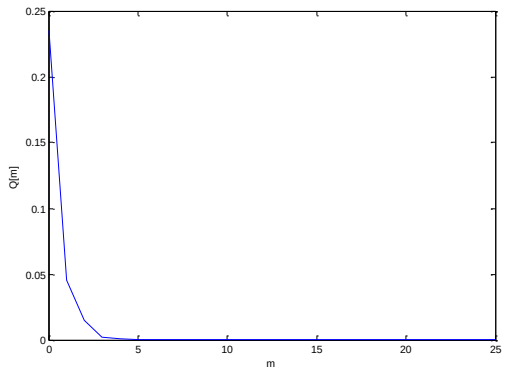
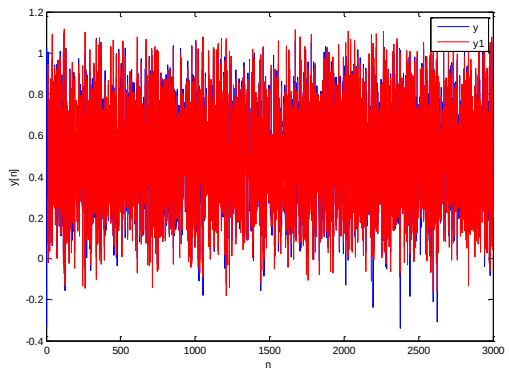
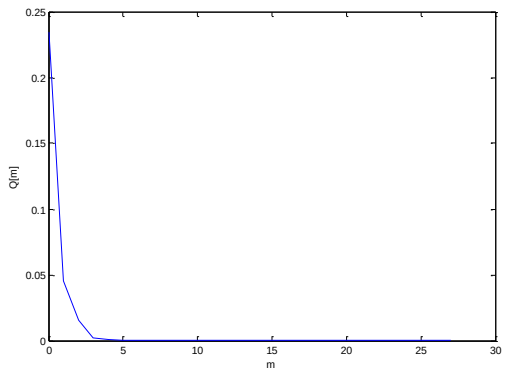
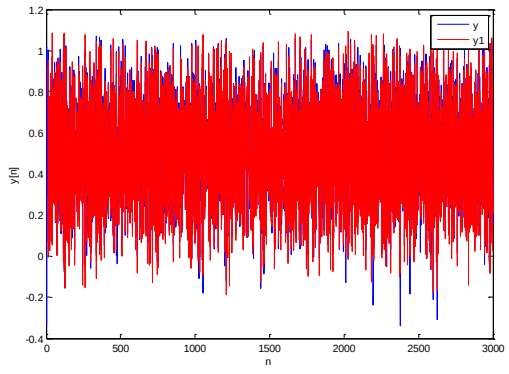
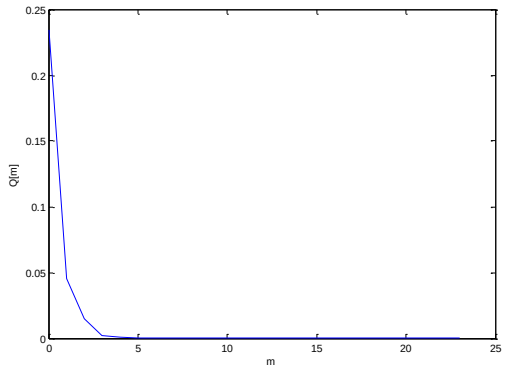
$$\begin{aligned} y_1[n] = {} & 0.200968068292994 + 1.09173048872365x[n-1] - 0.378350758079972x^2[n] \\ & + 0.151182847461841y[n-2] - 0.374021088225236x[n]x[n-1] - 0.163602046405864y^2[n \\ & - 2] + 0.0600825128503481y^2[n-1] - 0.223658319998891x^2[n-1] \\ & + 0.149992839811412x[n] + 0.125966428136127x[n-3]y[n-1] \\ & + 0.00728621143271623x[n-1]x[n-2] - 0.152073114982888x[n-4]y[n-1] \\ & + 0.0879734059763351y[n-3]y[n-4] - 0.0205838654356548x[n-5]x[n-6] \\ & - 0.0422496867726727x[n]x[n-3] - 0.0219634539132460x[n-1]x[n-5] \\ & + 0.0172663756710060x[n-3]x[n-3] - 0.121843283748814x[n-2]x[n-2] \\ & + 0.0431824254047976x[n-2]x[n-5] - 0.000105273324198171y[n-1]y[n-4] \\ & + 0.0563395591834832x[n-3]x[n-4] - 0.352383067594532y[n-1] \\ & + 0.134893881955249x[n-2] - 0.0347448220268434x[n-5] + 0.154716906997142y[n \\ & - 1]y[n-2] + 0.0859304798905644y[n-1]y[n-3] + 0.0259980056671551x[n]x[n-5] \end{aligned}$$

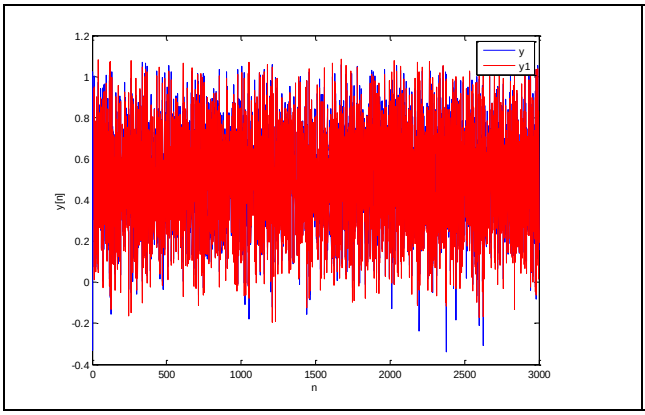
%MSE = 2.0276 %

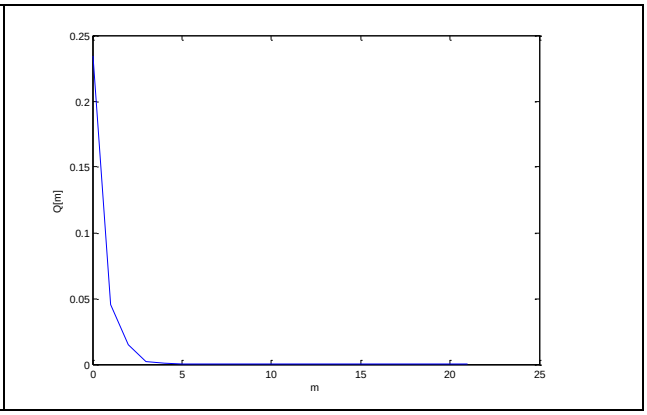
K	L	Computation Time
10	12	22.192264 seconds
3	15	9.406721 seconds
12	8	19.798458 seconds
10	12	23.021391 seconds
10	19	38.117116 seconds
8	19	27.037631 seconds
6	5	8.068722 seconds
14	6	13.414465 seconds
8	3	6.237757 seconds

7	10	9.920265 seconds

Noise Inserted

P = 150 %MSE = 6.81%	Total Elapsed Time = 17.353620 seconds %MSE w.r.t. noisy output = 59.92% Ideal %MSE = 50.81%
	
P = 100 %MSE = 3.48%	Total Elapsed Time = 19.811687 seconds %MSE w.r.t. noisy output = 48.13% Ideal %MSE = 48.99%
	
P = 50 %MSE = 2.85%	Total Elapsed Time = 26.531140 seconds %MSE w.r.t. noisy output = 32.33% Ideal %MSE = 32.70%
	
P = 25 %MSE = 2.63%	Total Elapsed Time = 25.260854 seconds %MSE w.r.t. noisy output = 20.29% Ideal %MSE = 19.67%





Different Input Generations

Setting $K = 6, L = 5$

Input Type	Parameter	Computation Time	%MSE
Uniform Distribution	Standard Deviation = 1	2.942042 seconds	20.56%
Uniform Distribution	Standard Deviation = 5	0.563647 seconds	96.62%
Uniform Distribution	Standard Deviation = 10	0.704365 seconds	94.37%
Normal Distribution	Standard Deviation = 1	4.095902 seconds	58.75%
Normal Distribution	Standard Deviation = 5	N/A[2]	
Normal Distribution	Standard Deviation = 10		
Sinusoidal	Frequency = 1 Hz, Amplitude = 1	19.752543 seconds	1.29%
Sinusoidal	Frequency = 10 Hz, Amplitude = 1	15.155718 seconds	2.13%
Sinusoidal	Frequency = 100 Hz, Amplitude = 1	24.461126 seconds	2.23%
Sinusoidal	Frequency = 1 kHz, Amplitude = 1	5.441387 seconds	0.07%
Sinusoidal	Frequency = 10 kHz, Amplitude = 1	14.903009 seconds	2.26%
Sinusoidal	Frequency = 1 Hz, Amplitude = 10	18.959170 seconds	1.29%
Sinusoidal	Frequency = 10 Hz, Amplitude = 10	14.603828 seconds	2.13%
Sinusoidal	Frequency = 100 Hz, Amplitude = 10	24.214600 seconds	2.23%
Sinusoidal	Frequency = 1 kHz, Amplitude = 10	5.369926 seconds	0.068%
Sinusoidal	Frequency = 10 kHz, Amplitude = 10	18.129356 seconds	2.21%
Triangular	Width = 2π, Amplitude = 1	12.826004 seconds	0.0032%
Triangular	Width = 0.1×2π, Amplitude = 1	5.200054 seconds	0.187%
Triangular	Width = 0.01×2π, Amplitude = 1	4.808909 seconds	0.385%
Triangular	Width = 2π, Amplitude = 10	N/A[3]	
Triangular	Width = 0.1×2π, Amplitude = 10		
Triangular	Width = 0.01×2π, Amplitude = 10		

Varying Orders

Setting $K = 6, L = 5$

Order	Computation Time	%MSE
2	6.928569 seconds	1.41%
3	33.842255 seconds	0.56%
4	447.53317 seconds	0.22%
5	4475.6902 seconds	0.106%

[2] $y[n]$ diverged to $-\infty$
[3] $y[n]$ diverged to $-\infty$

Analysis

For a non-polynomial model we may conclude that:

- %MSE was relatively high (more than 2%) because the system model may be expanded using Taylor series to higher order terms which were not candidates of the FOS algorithm when using maximum order of 2 only.
- %MSE w.r.t to original input improves with the decrease of noise at the output.
- %MSE w.r.t to noisy output is very close to ideal %MSE, therefore the FOS algorithm removes out the effect of noise due to the averaging which appears in the algorithm.
- Generating input randomly using normal distribution and uniform distributions with higher standard deviations increases the %MSE. Therefore, increasing the variance of the input makes the FOS algorithm more vulnerable to errors if it does not contain the required candidates.
- Sinusoidal and triangular inputs resulted in better results than random data. But this is probably due to the repetitive nature of the inputs and outputs and the resulting FOS models are likely to result in erroneous results for non-repetitive inputs.
- The higher the maximum order of the cross-product terms of the FOS algorithm, the lower %MSE we get. However, we don't reach the almost-zero %MSE we got in previous models, because the non-polynomial model is equivalent to a model of infinite order.

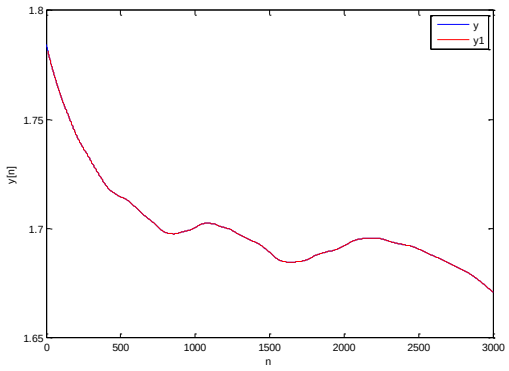
Static Gyroscope Data

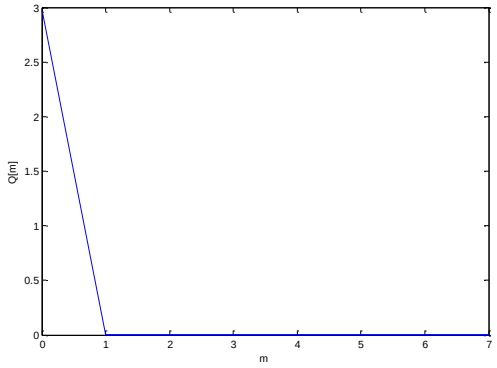
A gyroscope is a device rate of angular rotation. I used readings from a stationary MEMS grade gyroscope. When stationary, its reading should be zero. However, there is a usually white noise as well as bias error which drifts with time. I here try to model this bias drift, which we labeled as y, with time, which we labeled as x. I start with removing noise using wavelet denoising.

```
% Gyroscope Data
load wz.mat;
wzd = wden(wz, 'heursure', 's', 'one', 15, 'db4');
x = (1:3*N)';
y = wzd(1:3*N);
```

Total Elapsed Time = 35.985119 seconds

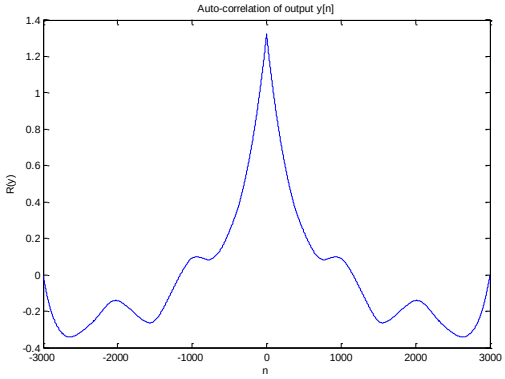
Chosen values of K & L: $K = 6$, $L = 5$





We also plot the auto-correlation of the output $y[n]$ using the following code:

```
figure(3);
Rw = xcorr(w - mean(w));
plot(-length(Rw)/2:length(Rw)/2-1 , Rw);
```



$$y[n] = 8.388889477426577 \times 10^{-6} + 2.752693099157840y[n-1] - 2.945459931486186y[n-2] + 2.010277952619016y[n-3] - 1.199348512911429y[n-4] + 0.381832274453565y[n-5] + 4.921580147312959 \times 10^{-13}x^2[n-6]$$

%MSE = 1.897×10^{-5}%

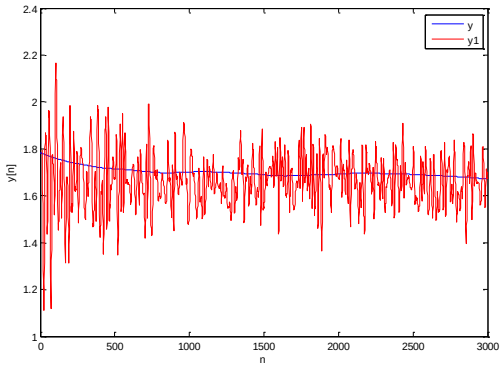
K	L	Computation Time
10	12	3.913472 seconds

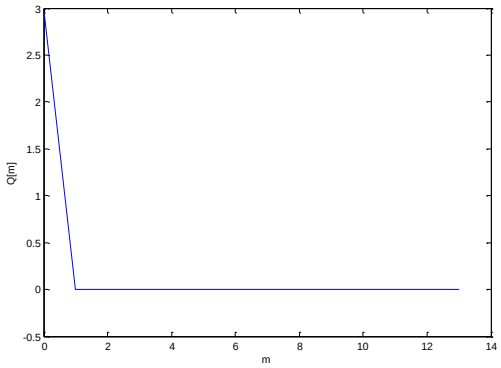
3	15	4.546578 seconds
12	8	3.950725 seconds
10	12	5.179663 seconds
10	19	5.546014 seconds
8	19	4.829279 seconds
6	5	0.691572 seconds
14	6	2.847456 seconds
8	3	0.748498 seconds
7	10	2.982398 seconds

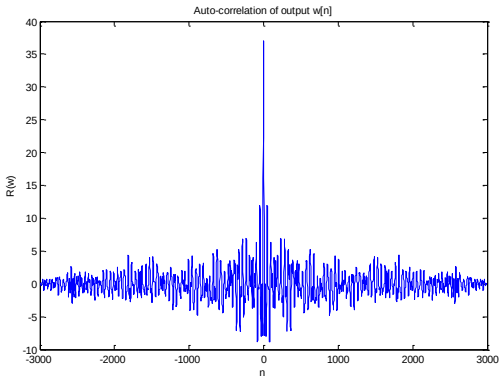
Reduce De-Noising

Wavelet Scale = 3, P = 2872
%MSE = 15846%

Total Elapsed Time = 123.977470 seconds
%MSE w.r.t. noisy output = 0.39%
Ideal %MSE = 102.5%

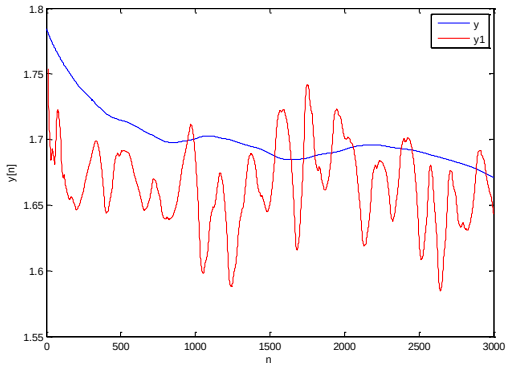


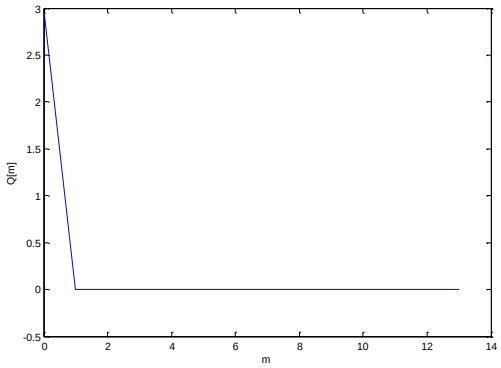


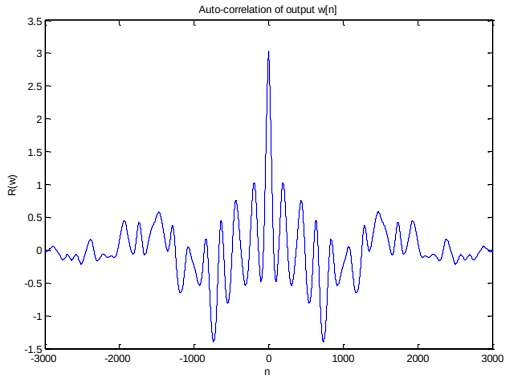


Wavelet Scale = 6, P = 278.1
%MSE = 3288%

Total Elapsed Time = 72.600251 seconds
%MSE w.r.t. noisy output = 8.86×10^{-4}%
Ideal %MSE = 121.5%

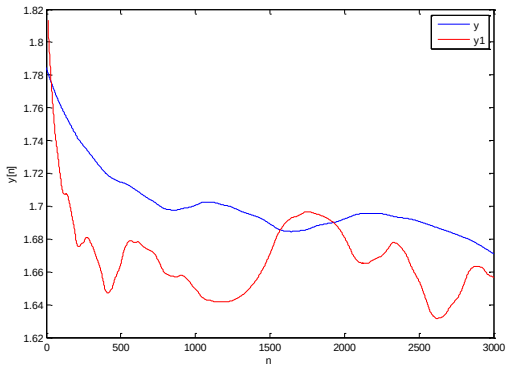


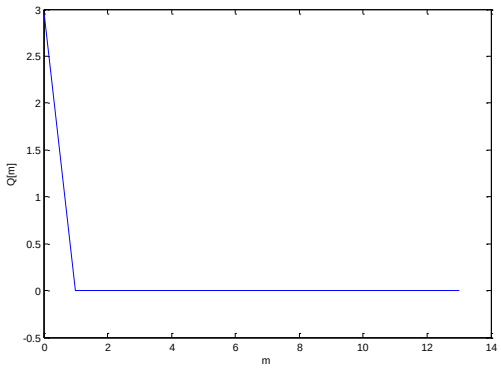


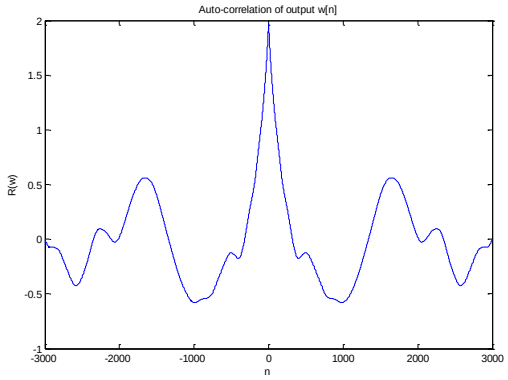


Wavelet Scale = 8, P = 113.08
%MSE = 2032%

Total Elapsed Time = 42.758778 seconds
%MSE w.r.t. noisy output = 6.98×10^{-6}%
Ideal %MSE = 76.1%

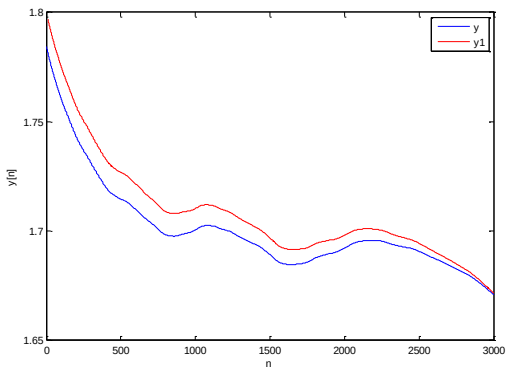


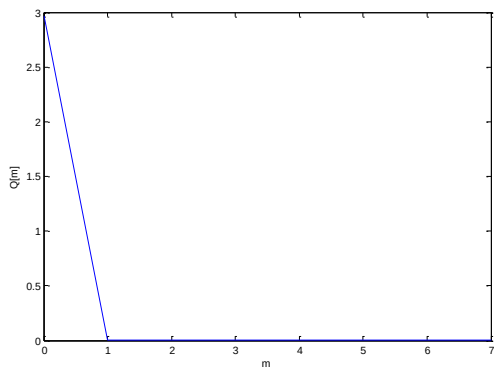


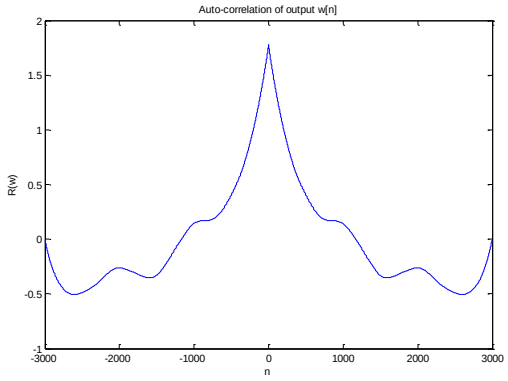


Wavelet Scale = 12, P = 3.38
%MSE = 29.03%

Total Elapsed Time = 50.976234 seconds
%MSE w.r.t. noisy output = 2.68×10^{-4}%
Ideal %MSE = 2.52%

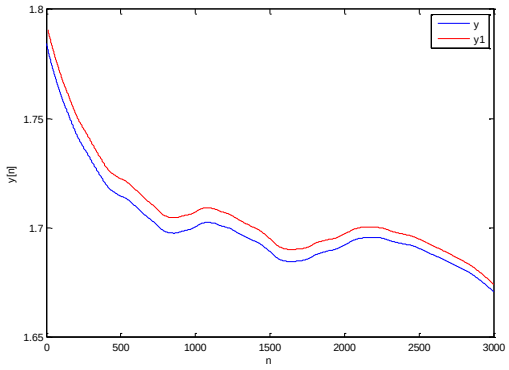


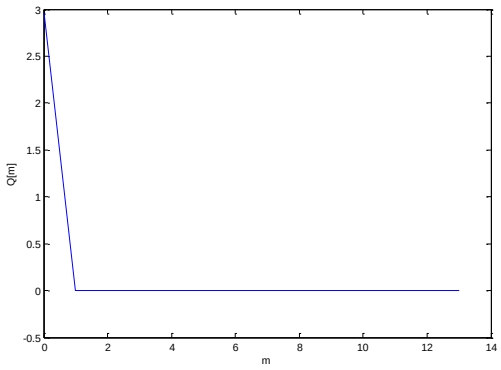


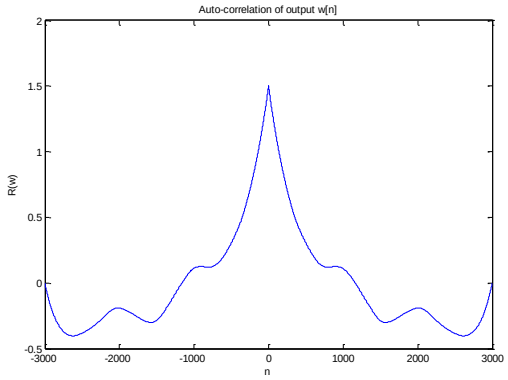


Wavelet Scale = 13, P = 0.59
%MSE = 35.48%

Total Elapsed Time = 57.308801 seconds
%MSE w.r.t. noisy output = 2.21×10^{-7}%
Ideal %MSE = 0.517%

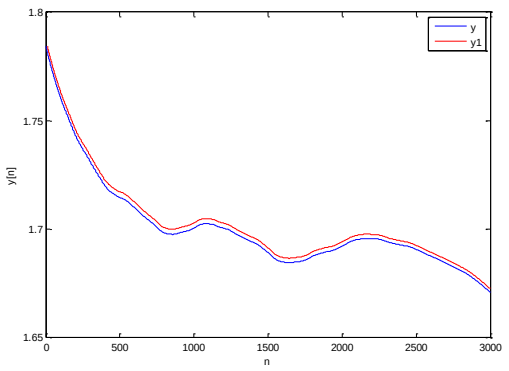


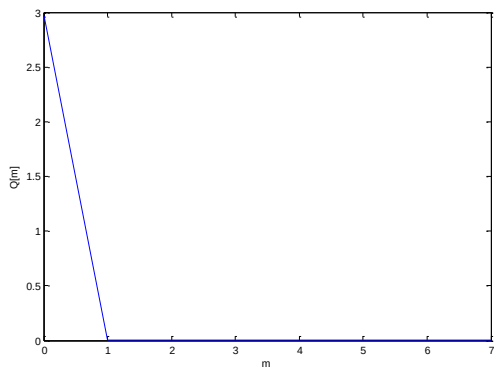


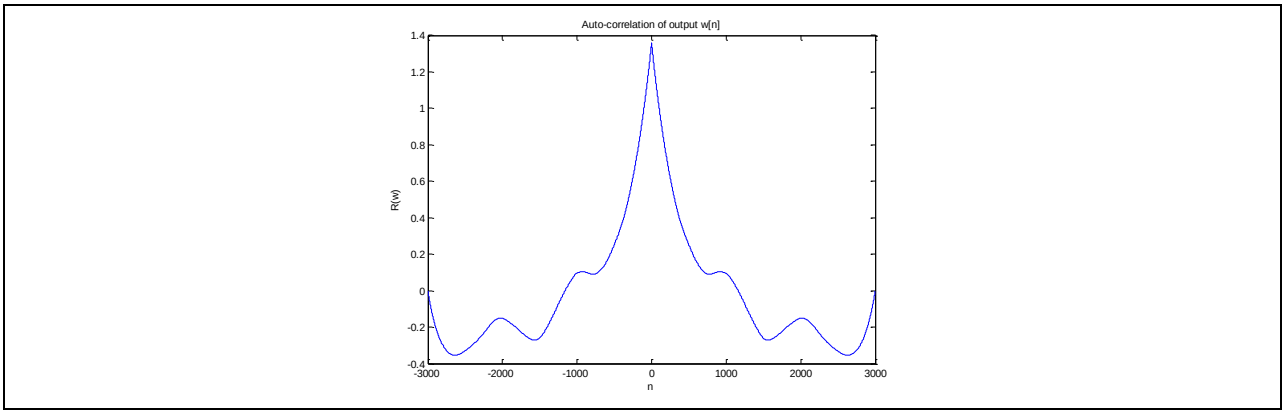


Wavelet Scale = 14, P = 0.0256
%MSE = 6.6%

Total Elapsed Time = 47.901612 seconds
%MSE w.r.t. noisy output = 1.58×10^{-4}%
Ideal %MSE = 0.0249%







Different Input Generations

This test is not applicable because the input here is time which has a constant representation and cannot be generated differently.

Varying Orders

Setting $K = 6, L = 5$

Order	Computation Time	%MSE
2	0.617821 seconds	1.92×10^{-7}%
3	3.627746 seconds	1.92×10^{-7}%
4	13.204463 seconds	1.92×10^{-7}%
5	44.273138 seconds	1.92×10^{-7}%

Analysis

For modeling a gyroscope we may conclude that:

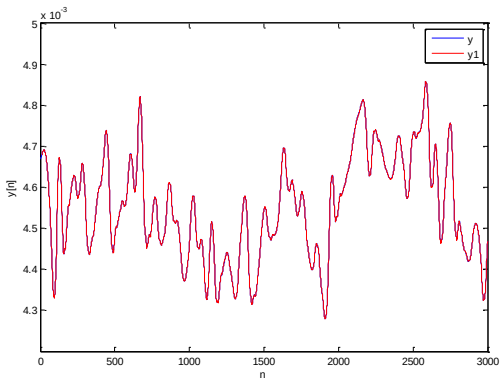
- %MSE was relatively low and therefore it seems that the cross-product terms of order 2 containing $y[n-l]$ and $x[n-k]$ are enough.
- %MSE w.r.t to original input improves with the increase of denoising. However, the results are not similar as to noise additions shown in previous examples because the noise which is being removed by wavelet denoising are probably not zero-mean white noise.
- Increasing the maximum order of cross-product terms did not improve %MSE and therefore it seems that the cross-product terms of order 2 containing $y[n-l]$ and $x[n-k]$ are enough.

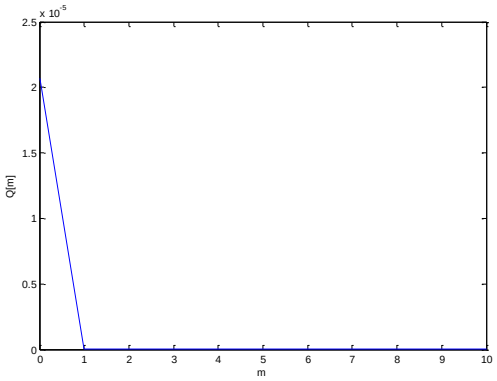
Static Accelerometer Data

```
% Gyroscope Data
load 'F:\EE517 - Project 3 Data\Xbow_stat_data_other.mat';
x = (1:3*N)';
y = f.x(1:3*N);
clear f w interp_info denoising_info denoising_info_ orig_data_info;
```

Total Elapsed Time = 31.910843 seconds

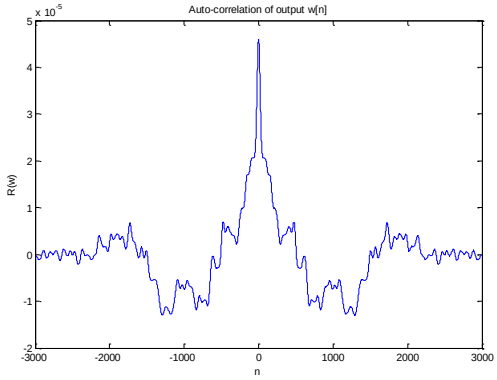
Chosen values of K & L: $K = 7$, $L = 10$





We also plot the auto-correlation of the output $y[n]$ using the following code:

```
figure(3);
Rw = xcorr(w - mean(w));
plot(-length(Rw)/2:length(Rw)/2-1 , Rw);
```



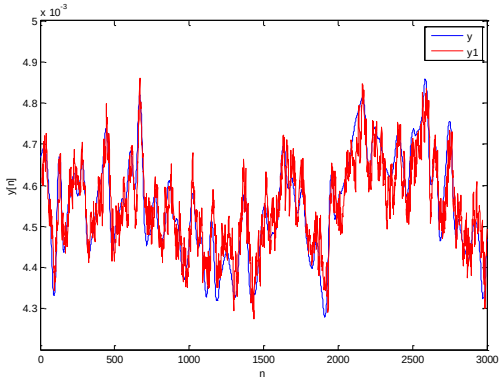
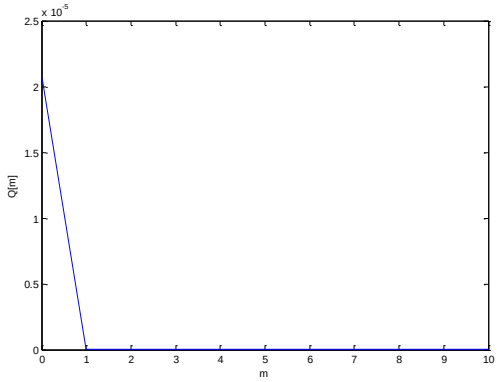
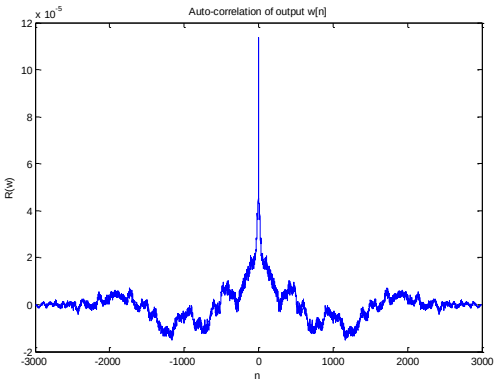
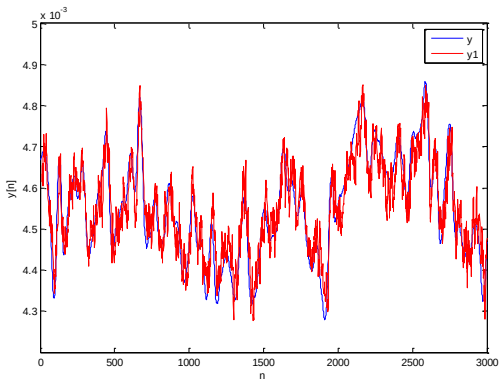
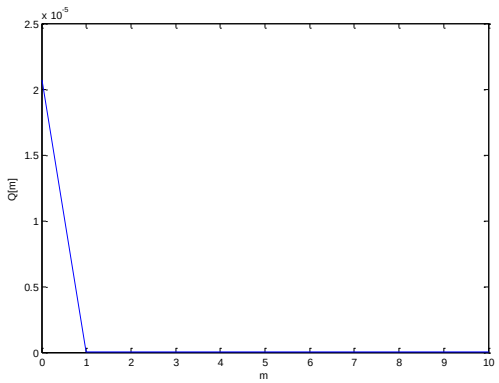
$$y_1[n] = 6.036871067146074 \times 10^{-6} + 2.256999195289575y[n-1] - 1.208845728945833y[n-2] + 1.021246182729992y[n-4] - 0.674366590246582y[n-3] - 0.265408926025274y[n-5] + 0.628453091656779y[n-7] - 0.518938020038324y[n-6] - 0.323561112222304y[n-8] + 0.083095305802417y[n-9]$$

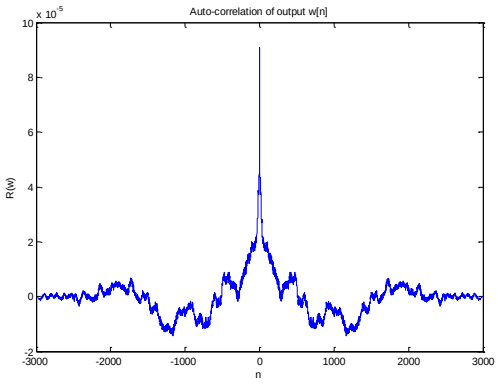
%MSE = 0.0017%

K	L	Computation Time
10	12	2.519522 seconds
3	15	3.236106 seconds
12	8	2.509121 seconds
10	12	3.290113 seconds

10	19	7.709763 seconds
8	19	6.706057 seconds
6	5	0.610797 seconds
14	6	1.822470 seconds
8	3	0.613365 seconds
7	10	2.087462 seconds

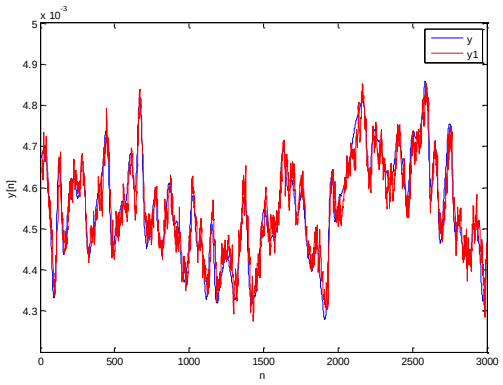
Noise Inserted

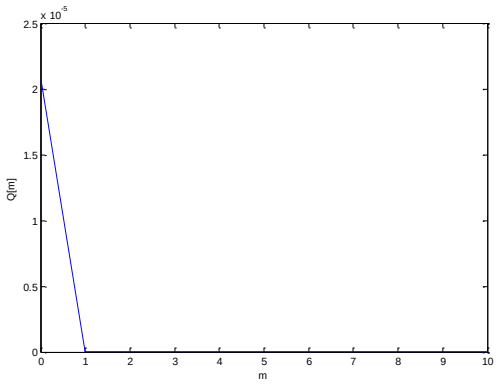
P = 150 %MSE = 22.31%	Total Elapsed Time = 15.706235 seconds %MSE w.r.t. noisy output = 69.26% Ideal %MSE = 60.77%
	
	
P = 100 %MSE = 16.72%	Total Elapsed Time = 16.757993 seconds %MSE w.r.t. noisy output = 58.86% Ideal %MSE = 50.65%
	

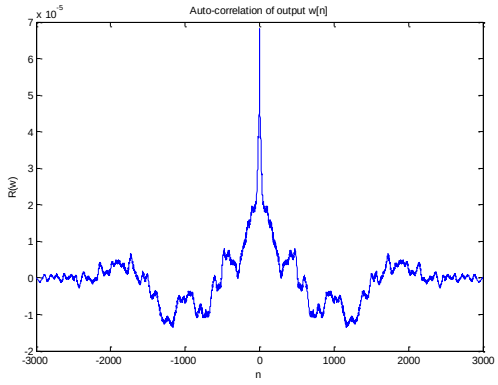


P = 50
%MSE = 9.66%

Total Elapsed Time = 16.855352 seconds
%MSE w.r.t. noisy output = 40.25%
Ideal %MSE = 33.74%

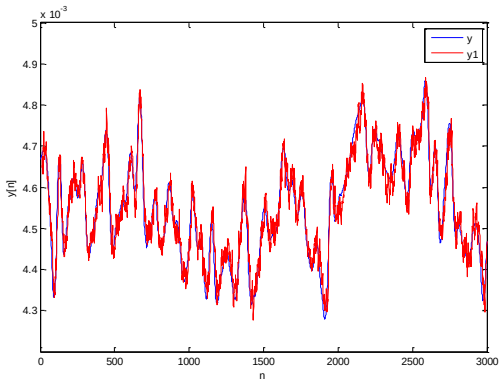


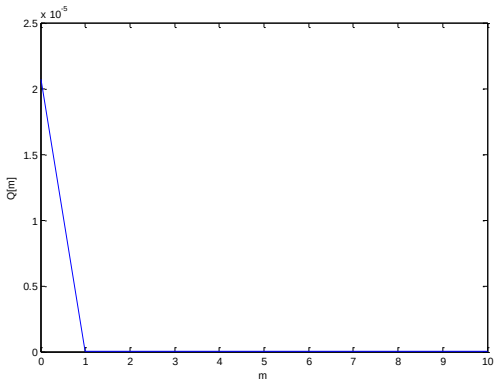


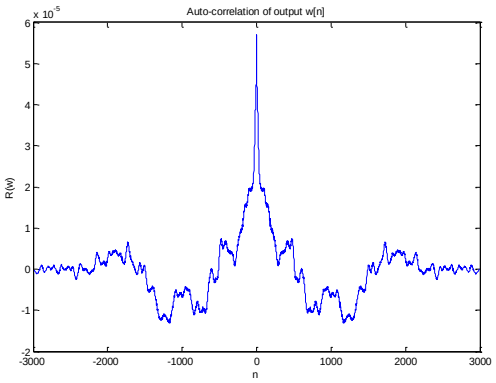


P = 25
%MSE = 5.88%

Total Elapsed Time = 16.007723 seconds
%MSE w.r.t. noisy output = 25.03%
Ideal %MSE = 20.21%







Different Input Generations

This test is not applicable because the input here is time which has a constant representation and cannot be generated differently.

Varying Orders

Setting $K = 7, L = 10$

Order	Computation Time	%MSE
2	2.757120 seconds	0.0043%
3	18.079261 seconds	0.0043%
4	110.92006 seconds	0.0043%
5	508.46559 seconds	0.0043%

Analysis

For modeling an accelerometer we may conclude that:

- The higher peak of the auto-correlation function around zero value, means that there is correlation between the outputs of the accelerometer and therefore there is more randomness in the output which cannot be modeled by FOS.
- %MSE was relatively low and therefore it seems that the cross-product terms of order 2 containing $y[n-l]$ and $x[n-k]$ are enough.
- %MSE is much higher when noise is inserted when compared with previous models. This is probably due to the fact that the model of the accelerometer is partially stochastic and cannot be totally modeled deterministically.
- Increasing the maximum order of cross-product terms did not improve %MSE and therefore it seems that the cross-product terms of order 2 containing $y[n-l]$ and $x[n-k]$ are enough.

Conclusion

The FOS algorithm proved to be excellent in modeling deterministic models resulting in %MSE which are almost zeros. However, some real-life models seem to have some stochastic nature and therefore the accuracy of FOS reduces to 1-3%. For complex deterministic models, FOS results in better results the higher the maximum cross-order and lags of its candidates. However, a balance has to be made between seeking higher accuracy and seeking less computation time.

Normal distribution followed by uniform distribution of random inputs seems to result in highest accuracies. They are better than periodic inputs since they cover more combination of values and more combination of orders of values due to their random nature.

Code

Main Code

Training Phase

```
%% Training Phase
tic
% Apply FOS for 1st 1000 samples of data
rng(2);
K = randi([3,20], 10, 1);
rng(3);
L = randi([3,20], 10, 1);
for i = 1:length(K)
    order = 2;
    N0 = max(K(i),L(i));
    %tic
    [at{i}, pt{i}] = fos( x(1:N), w(1:N), K(i), L(i), order );
    %toc

    y1 = evalfunct( x(1:N), w(1:N), pt{i}, at{i} );
    e = y1 - w(1:N);
    MSEpercent(i) = mean(e(N0+1:N).^2)/var(w(N0+1:N)) *100;
end
```

Selection Phase

```
%% Selection Phase
clear MSEpercent;
% Apply FOS for 2nd 1000 samples of data
for i = 1:1:length(K)
    y1 = evalfunct( x(N+1:2*N), w(N+1:2*N), pt{i}, at{i} );
    e = y1 - w(N+1:2*N);
    MSEpercent(i) = mean(e(N0+1:N).^2)/var(w(N+N0+1:2*N)) *100;
end

% Choose the best model over the 2nd 1000 samples
index = find(MSEpercent == min(MSEpercent));
as = at{index(1)};
ps = pt{index(1)};
```

Evaluation Phase

```
%% Evaluation Phase
clear MSEpercent;
% Apply FOS for 3nd 1000 samples of data
y1 = evalfunct( x(2*N+1:3*N), w(2*N+1:3*N), ps, as);
e = y1 - y(2*N+1:3*N);
MSEpercent = mean(e(N0+1:N).^2)/var(y(2*N+N0+1:3*N)) *100;
IdealMSEPercent = var(r) / var(w) * 100;

e = y1 - w(2*N+1:3*N);
MSEpercent1 = mean(e(N0+1:N).^2)/var(w(2*N+N0+1:3*N)) *100;
```

Plot Results

```
%% Plot
y1 = evalfunct( x, w, ps, as);
```

```
figure(1);
plot(y, 'b'); hold on;
plot(N0+1:3*N, y1(N0+1:3*N),'r');
xlabel('n');
ylabel('y[n]');
legend('y', 'y1');

figure(3);
Rw = xcorr(w - mean(w));
plot(-length(Rw)/2:length(Rw)/2-1 , Rw);
title('Auto-correlation of output w[n]');
xlabel('n');
ylabel('R(w)');

toc

%% Plot Q[m] for best K & L
order = 2;
N0 = max( K(index(1)) , L(index(1)) );
[a, p] = fos( x(1:N), y(1:N), K(index(1)), L(index(1)), order );
```

Input Generation

Default generation

```
% Uniformly distributed pseudorandom numbers [0,1]
rng(0);
x = rand(3*N, 1);
```

Uniform Distribution

```
% Uniform distribution
sigmax = 1;
rng(0);
x1 = rand(3*N, 1);
x = x1 / std(x1) * sigmax;
```

Normal Distribution

```
% Normal distribution
sigmax = 1;
rng(0);
x1 = randn(3*N, 1);
x = x1 / std(x1) * sigmax;
```

Sinusoidal Input

```
% Sinusoidal
f = 1;
A = 1;
t = (1:3*N)';
x = A*sin(2*pi*f*t);
```

Triangular Input

```
% Triangular
width = 1;
A = 1;
t = (1:3*N)';
x = A*sawtooth(t,width);
```

System Models

Linear Difference Equation

```
% Simple Difference Equation 1
y = 1 +  0.6*x  + 0.3*delay(x,1) + 0.4*delay(x,2) + 0.7*delay(x,3);
```

Complex Difference Equation

```
% Complex Difference Equation
y = zeros(3*N, 1);

a0 = 1;
b0 = 0.7;
b1 = 0.8;
c1 = 0.1;
c2 = 0.4;
c4 = 0.2;

for n = 6:3*N
    y(n) = a0 + b0*x(n) + b1*x(n-1) + c1*x(n-1)*y(n-1) - c2*x(n)*x(n-1)+ c4*x(n-4)*x(n-
5);
end
```

LNL Cascade

```
% LNL Cascade
i = 1:5;
g1 = exp(-i) + exp(-2*i);
a1 = 0.5 + 2*exp(-i);
k1 = 3*exp(-i);

g2 = exp(-i) + 3*exp(-2*i);
a2 = 0.2 + 3*exp(-i);
k2 = 0.1*exp(-i) + 0.9*exp(-2*i);

y = lnl(g1, a1, k1, x) + lnl(g2, a2, k2, x);
```

Non-Polynomial Equation

```
% Non-Polynomial Equation
y = zeros(3*N, 1);
for n = 3:3*N
    y(n) = sin(x(n-1))*cos(x(n)) + exp(-3*x(n))*sqrt(abs(x(n))) + 0.1*log(abs(y(n-
2)+0.01))*y(n-1);
end
```

Gyroscope Data

```
% Real Data 1 - Project 2
load wz.mat;
wzd = wden(wz, 'heursure', 's', 'one', 15, 'db4');
x = (1:3*N)';
y = wzd(1:3*N);
w = wden(wz, 'heursure', 's', 'one', 14, 'db4'); w=w(1:length(x));
r = w - y;
```

```
P = 100 * var(r)/var(y)
```

Accelerometer Data

```
% Real Data 2 - Project 3
load 'C:\Data Logging\EE 517 Winter 2012\Project 3\Xbow_stat_data_other.mat';
x = (1:3*N)';
y = f.x(1:3*N);
clear f w interp_info denoising_info denoising_info_ orig_data_info;
```

Noise Insertion

```
% Noise Insertion
rng(1);
r1 = wgn(3*N, 1, 0);
r1 = r1 - mean(r1);
r = r1 / std(r1) * sqrt(P/100) * std(y);
w = r + y;
```

FOS Algorithm

```
function [ a, p ] = fos( x, y, K, L, order )
%FOS Summary of this function goes here
%   Detailed explanation goes here
N = length(x);
N0 = max(K,L);

h = waitbar(0,'1','Name','FOS Calculation...',...
    'CreateCancelBtn',...
    'setappdata(gcbf,''canceling'',1)');
setappdata(h,'canceling',0)
% Structure of p:
% p.x = delays of different x terms.
% p.y = delays of different y terms.
% p.x + p.y <= order
p = struct('const', 1, 'x', [], 'y', []);
P = struct(p);

g(0 +1) = mean(y(N0+1:N));
D(0 +1,0 +1) = 1;
C(0 +1) = mean(y(N0+1:N));
P(0 +1) = [];
Q(0 +1) = g(0 +1)^2 * D(0 +1,0 +1);

waitbar(0, h,'Generating Candidates...');

% generate all candidates
i = 1;
for torder = 1 : order
    waitbar(torder / order, h);
    for xorder = 0:torder
        if getappdata(h,'canceling')
            delete(h);
            return;
        end

        yorder = torder - xorder;

        xdelays = combsrep(0:K, xorder);
        ydelays = combsrep(1:L, yorder);

        if (size(xdelays,1) >= 1)
            for j = 1:size(xdelays,1)
                P(i).x = xdelays(j, :);
                if (size(ydelays,1) >= 1)
                    for k = 1:size(ydelays,1)
                        P(i).y = ydelays(k, :);

                        i = i + 1;
                    end
                else
                    i = i+1;
                end
            end
        else
            for k = 1:size(ydelays,1)
                P(i).y = ydelays(k, :);

                i = i + 1;
            end
        end
```

```
    end
end

waitbar(0, h, 'Evaluating Candidates...');

M = 1;
while (true)
    if getappdata(h,'canceling')
        delete(h);
        return;
    end

    waitbar(0, h, sprintf('Evaluating Candidate %d...', M));
    m = M;

    % Evaluate Q for each candidate
    clear Qc;
    if (isempty(P))
        break;
    end
    for i=1:length(P)
        if getappdata(h,'canceling')
            delete(h);
            return;
        end

        waitbar(i / length(P), h);
        Pval = evalterm(x, y, P(i));

        D(m+1,1) = mean(Pval(N0+1:N));
        for j=1:m
            if getappdata(h,'canceling')
                delete(h);
                return;
            end

            alpha(m+1, j) = D(m+1, j) ./ D(j, j);
            if (j < M)
                pval = evalterm(x, y, p(j+1));
            else
                pval = Pval;
            end
            D(m+1, j+1) = mean(Pval(N0+1:N) .* pval(N0+1:N)) - sum(alpha(j+1,
1:j) .* D(m+1, 1:j));
        end
        C(m+1) = mean(y(N0+1:N) .* Pval(N0+1:N)) - sum(alpha(m+1, 1:m) .*
C(1:m));

        g(m+1) = C(m+1)/D(m+1, m+1);
        Qc(i) = g(m+1)^2 * D(m+1, m+1);
    end

    % Find index of maximum Q
    index = find(Qc == max(Qc));
    Pval = evalterm(x, y, P(index(1)));

    for j=1:m
        if getappdata(h,'canceling')
            delete(h);
            return;
        end
```

```
        D(m+1,1) = mean(Pval(N0+1:N));
        for j=1:m
            if getappdata(h,'canceling')
                delete(h);
                return;
            end

            alpha(m+1, j) = D(m+1, j) ./ D(j, j);
            if (j < M)
                pval = evalterm(x, y, p(j+1));
            else
                pval = Pval;
            end
            D(m+1, j+1) = mean(Pval(N0+1:N) .* pval(N0+1:N)) - sum(alpha(j+1,
1:j) .* D(m+1, 1:j));
        end

    end
    C(m+1) = mean(y(N0+1:N) .* Pval(N0+1:N)) - sum(alpha(m+1, 1:m) .*
C(1:m));
    Q(m+1) = max(Qc);

    diagD = diag(D)';
    if ( Q(M+1) < 4/(N - N0 + 1) *(mean(y(N0+1:N).^2) - sum(Q(1:M))))
        M = M - 1;
        break;
    end

    p(m+1) = P(index(1));
    P(index(1)) = []; % remove it from the P

    % find the coefficient of the chosen candidate
    g(m+1) = C(m+1) / D(m+1,m+1);

    if (isempty(P) || Q(m+1)<1e-26 )
        break;
    end

    M = M + 1;
end

figure(2);
plot(0:length(Q)-1, Q(1:end));
ylabel('Q[m]');
xlabel('m');

% Obtain a
for i=0:M
    v(i +1)=1;
    for j = i+1:M
        v(j +1) = -sum(alpha(j +1,i +1 : j-1 +1) .* v(i +1 : j-1 +1));
    end
    a(i +1) = sum(g(i +1:M +1).*v(i +1:M +1));
end

delete(h);

end
```

Helper Functions

Function to evaluate term.

```
function [ p ] = evalterm( x, y, lags)
%GETFUNCTION Summary of this function goes here
%   Detailed explanation goes here

p = ones(length(x),1);

for i=1:length(lags.x)
    k = lags.x(i);
    p = p.*delay(x,k);
end

for i=1:length(lags.y)
    l = lags.y(i);
    p = p.*delay(y,l);
end

end
```

Function to evaluate function.

```
function [ y1 ] = evalfunct( x, y, p, a )
%EVALFUNCT Summary of this function goes here
%   Detailed explanation goes here

y1 = ones(length(x),1);
y1 = y1 * a(1);
for j=2:length(p)
    y1 = y1 + a(j) * evalterm(x, y, p(j));
end

end
```

Function to delay a signal.

```
function [ x1 ] = delay( x, k )
%DELAY Summary of this function goes here
%   Detailed explanation goes here

x1 = [zeros(k,1); x(1:end-k)];

end
```

Function to implement LNL system.

```
function [y, u] = lnl( g, a, k, x )
%LNL Summary of this function goes here
%   Detailed explanation goes here
u = filter(1, g, x);

v = zeros(size(u));
for i=1:length(a)
    v = v + a(i)*u.^i;
end
```

```
y = filter(1, k, v);

end
```