Vector Integral Calculus

Line Integrals

The concept of a line integral is a simple and natural generalization of a definite integral $\int_a^b f(x)\,dx$. Recall that, we integrate the function $f(x)$, also known as the integrand, from $x = a$ along the x-axis to $x = b$.

Now, in a line integral, we shall integrate a given function, also called the **integrand**, along a curve C in space or in the plane.

This requires that we represent the curve C by a parametric representation

$$\mathbf{r}(t) = [x(t), y(t), z(t)] = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k} \qquad (a \leqq t \leqq b).$$

The curve C is called the **path of integration**.

Vector Integral Calculus

Line Integrals

A **line integral** of a vector function $\mathbf{F}(\mathbf{r})$ over a curve C: $\mathbf{r}(t)$ is defined by

$$\int_C \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt$$

where $\mathbf{r}(t)$ is the parametric representation of C.

We see that the integral on the right is a definite integral of a function of t taken over the interval $a \leqq t \leqq b$ on the t-axis in the ***positive*** direction. This definite integral exists for continuous $\mathbf{F}$ and piecewise smooth C, because this makes $\mathbf{F} \cdot \mathbf{r}'$ piecewise continuous.

If the path of integration C is a *closed* curve, then instead of $\int_C$ we write $\oint_C$.

Oriented curve

In Fig. 219a the path of integration goes from A to B.

Thus A: $\mathbf{r}(a)$ is its initial point and B: $\mathbf{r}(b)$ is its terminal point.

C is now *oriented*.

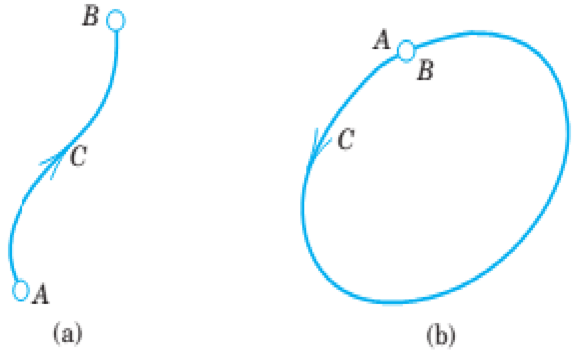


Fig. 219. Oriented curve

The direction from A to B, in which t increases is called the positive direction on C. We mark it by an arrow.

closed path.

The points A and B may coincide, then C is called a **closed path**.

smooth curve

C is called a **smooth curve** if it has at each point a unique tangent whose direction varies continuously as we move along C.

EXAMPLE Evaluation of a Line Integral in the Plane

Find the value of the line integral when $\mathbf{F}(\mathbf{r}) = [-y, -xy] = -y\mathbf{i} - xy\mathbf{j}$ and C is the circular arc as in Fig. 220 from A to B.

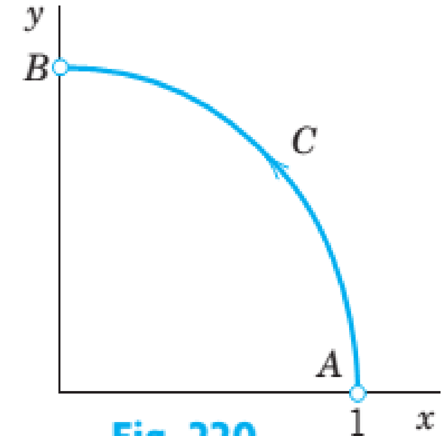


Fig. 220.

Solution. We may represent C by

$$\mathbf{r}(t) = [\cos t, \sin t] = \cos t\,\mathbf{i} + \sin t\,\mathbf{j},$$

where $0 \leqq t \leqq \pi/2$.

By differentiation, $\mathbf{r}'(t) = [-\sin t, \quad \cos t]$

$$\boxed{\mathbf{r}'(t) = -\sin t\,\mathbf{i} + \cos t\,\mathbf{j}}$$

Then $x(t) = \cos t$, $y(t) = \sin t$, and

$$\mathbf{F}(\mathbf{r}(t)) = -y(t)\mathbf{i} - x(t)y(t)\mathbf{j}$$

$$= [-\sin t, \quad -\cos t \sin t]$$

$$\boxed{\mathbf{F}(\mathbf{r}(t)) = -\sin t\,\mathbf{i} - \cos t \sin t\,\mathbf{j}.}$$

so that $\mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) = [-\sin t, -\cos t \sin t] \cdot [-\sin t, \cos t]$

$$\mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) = (\sin^2 t - \cos^2 t \sin t)$$

Thus

$$\int_C \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r} = \int_0^{\pi/2} \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt$$

$$= \int_0^{\pi/2} (\sin^2 t - \cos^2 t \sin t)\, dt \quad [\text{set } \cos t = u \text{ in the second term}]$$

$$= \int_0^{\pi/2} \frac{1}{2}(1 - \cos 2t)\, dt - \int_1^0 u^2(-du)$$

$$= \frac{\pi}{4} - 0 - \frac{1}{3}$$

$$\boxed{\int_C \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r} \approx 0.4521.}$$

EXAMPLE Line Integral in Space

Find the value of the line integral when $\mathbf{F}(\mathbf{r}) = [z, x, y] = z\mathbf{i} + x\mathbf{j} + y\mathbf{k}$ and C is the helix

$$\mathbf{r}(t) = [\cos t, \sin t, 3t] = \cos t\,\mathbf{i} + \sin t\,\mathbf{j} + 3t\,\mathbf{k} \quad (0 \leqq t \leqq 2\pi).$$

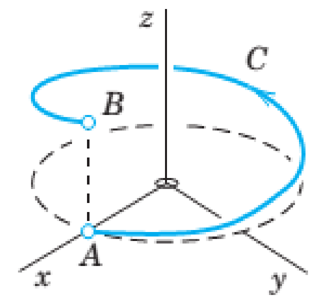


Fig.

Solution. We have

$$x(t) = \cos t, \quad y(t) = \sin t, \quad z(t) = 3t.$$

Thus

$$\mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) = (3t\,\mathbf{i} + \cos t\,\mathbf{j} + \sin t\,\mathbf{k}) \cdot (-\sin t\,\mathbf{i} + \cos t\,\mathbf{j} + 3\mathbf{k}).$$

$$\boxed{\mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) = 3t(-\sin t) + \cos^2 t + 3 \sin t.}$$

Hence

$$\int_C \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r} = \int_0^{2\pi} (-3t \sin t + \cos^2 t + 3 \sin t)\, dt$$

$$= 6\pi + \pi + 0$$

$$\boxed{\int_C \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r} = 7\pi \approx 21.99.}$$

Calculate $\int_C \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r}$ for the given data. If $\mathbf{F}$ is a force, this gives the work done by the force in the displacement along C. Show the details.

2. $\mathbf{F} = [y^2, -x^2]$, C: $y = 4x^2$ from (0, 0) to (1, 4)
3. $\mathbf{F}$ as in Prob. 2, C from (0, 0) straight to (1, 4). Compare.
4. $\mathbf{F} = [xy, x^2y^2]$, C from (2, 0) straight to (0, 2)
5. $\mathbf{F}$ as in Prob. 4, C the quarter-circle from (2, 0) to (0, 2) with center (0, 0)
6. $\mathbf{F} = [x - y, y - z, z - x]$, C: $\mathbf{r} = [2 \cos t, t, 2 \sin t]$ from (2, 0, 0) to $(2, 2\pi, 0)$
7. $\mathbf{F} = [x^2, y^2, z^2]$, C: $\mathbf{r} = [\cos t, \sin t, e^t]$ from (1, 0, 1) to $(1, 0, e^{2\pi})$. Sketch C.

Simple general properties of the line integral

Simple general properties of the line integral follow directly from corresponding properties of the definite integral in calculus, namely,

(a) $\displaystyle\int_C k\mathbf{F} \cdot d\mathbf{r} = k\int_C \mathbf{F} \cdot d\mathbf{r}$ (k constant)

(b) $\displaystyle\int_C (\mathbf{F} + \mathbf{G}) \cdot d\mathbf{r} = \int_C \mathbf{F} \cdot d\mathbf{r} + \int_C \mathbf{G} \cdot d\mathbf{r}$ (orientation of C is same in all three integrals)

(c) $\displaystyle\int_C \mathbf{F} \cdot d\mathbf{r} = \int_{C_1} \mathbf{F} \cdot d\mathbf{r} + \int_{C_2} \mathbf{F} \cdot d\mathbf{r}$ (the path C is subdivided into two arcs C_1 and C_2 that have the same orientation as C)

If the sense of integration along C is reversed, the value of the integral is multiplied by -1.

THEOREM **Direction-Preserving Parametric Transformations**

Any representations of C that give the same positive direction on C also yield the same value of the line integral.

Motivation of the Line Integral: Work Done by a Variable Force

The work W done by a *constant* force $\mathbf{F}$ in the displacement along a *straight* segment $\mathbf{d}$ is $W = \mathbf{F} \cdot \mathbf{d}$

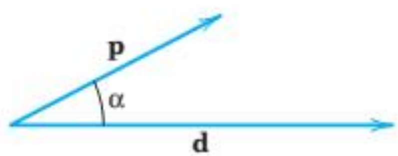
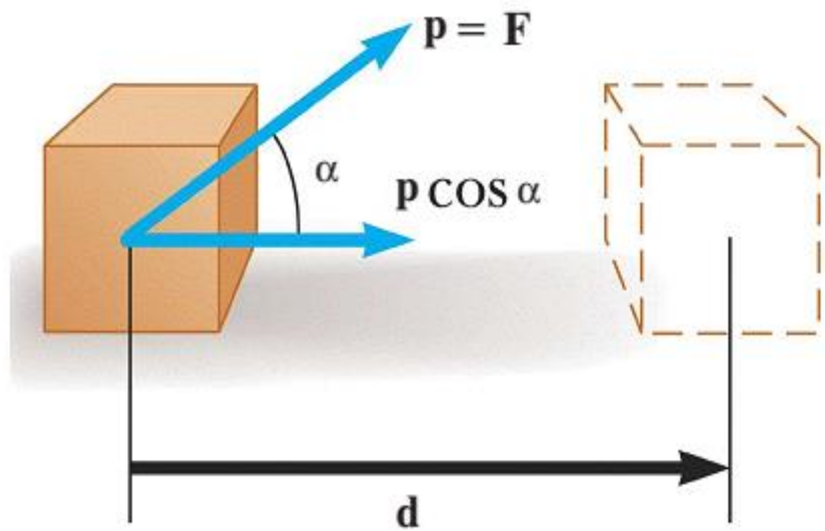


Fig. Work done by a force

$W = |\mathbf{p}||\mathbf{d}| \cos \alpha = \mathbf{p} \cdot \mathbf{d},$

$$\boxed{W = \mathbf{F} \cdot \mathbf{d}}$$



This suggests that we define the work W done by a *variable* force $\mathbf{F}$ in the displacement along a curve C: $\mathbf{r}(t)$ as the limit of sums of works done in displacements along small chords of C.

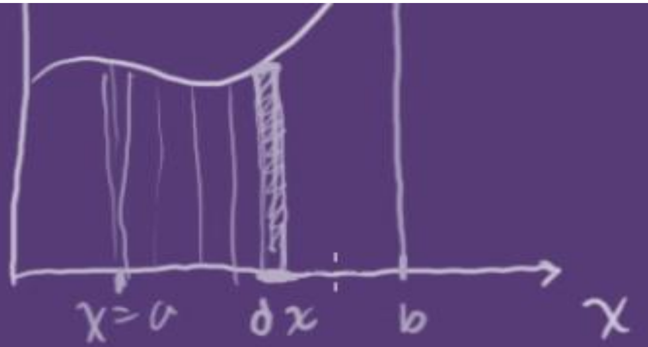
For this we choose points $t_0\,(=a) < t_1 < \cdots < t_n\,(=b)$. Then the work ΔW_m done by $\mathbf{F}(\mathbf{r}(t_m))$ in straight displacement from $\mathbf{r}(t_m)$ to $\mathbf{r}(t_{m+1})$ is

$$\boxed{\Delta W_m = \mathbf{F}(\mathbf{r}(t_m)) \bullet [\mathbf{r}(t_{m+1}) - \mathbf{r}(t_m)]}$$

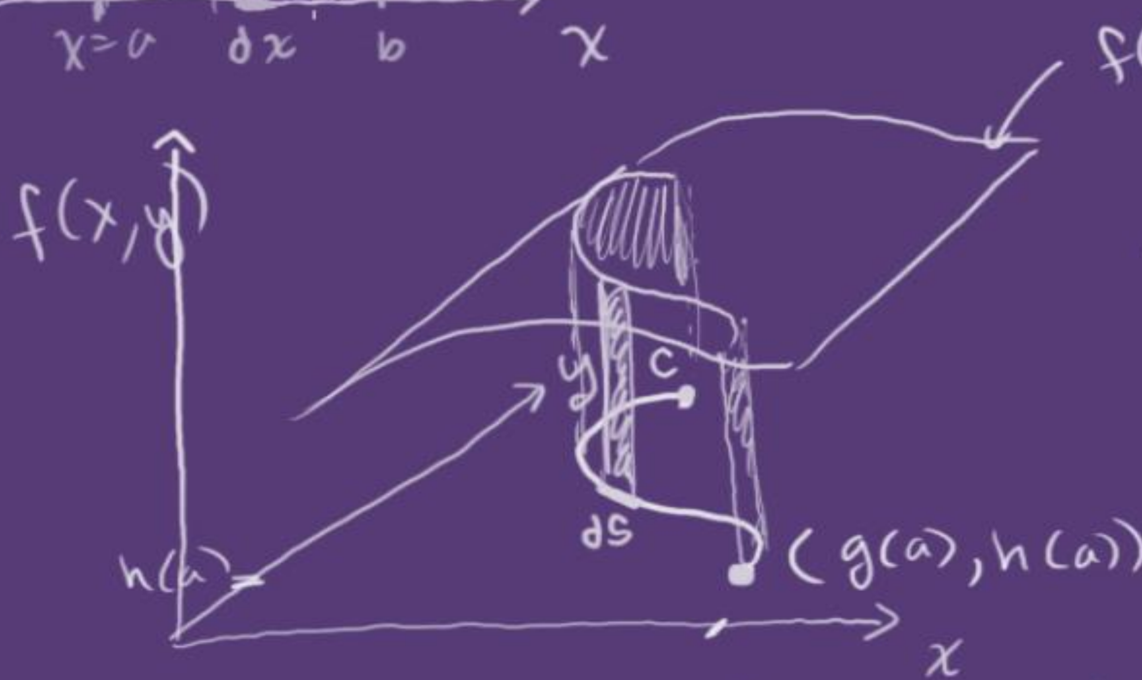
$$\approx \mathbf{F}(\mathbf{r}(t_m)) \bullet \mathbf{r}'(t_m)\Delta t_m \quad (\Delta t_m = \Delta t_{m+1} - t_m).$$

The sum of these n works is $W_n = \Delta W_0 + \cdots + \Delta W_{n-1}$.
If we choose points and consider W_n for every n arbitrarily but so that the greatest Δt_m approaches zero as $n \to \infty$, then the limit of W_n as $n \to \infty$ is the line integral

$$\int_C \mathbf{F}(\mathbf{r}) \bullet d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \bullet \mathbf{r}'(t)\,dt$$



$$\int_a^b f(x)\,dx$$

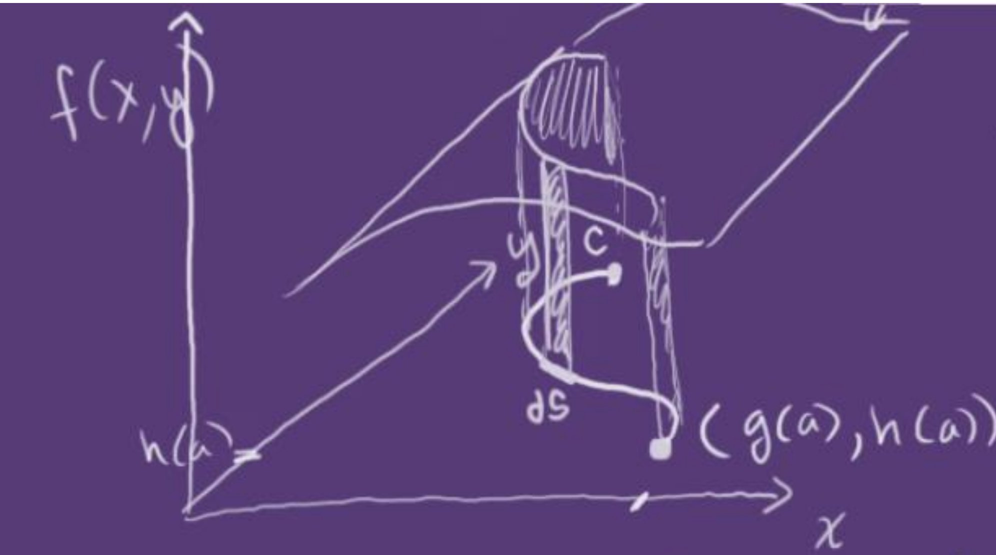


$x = g(t)$

$y = h(t)$

ds= super small change in arc length

$$\int_{t=a}^{t=b} f(x,y)\,dS$$



$$x = g(t) \qquad a \le t \le b$$

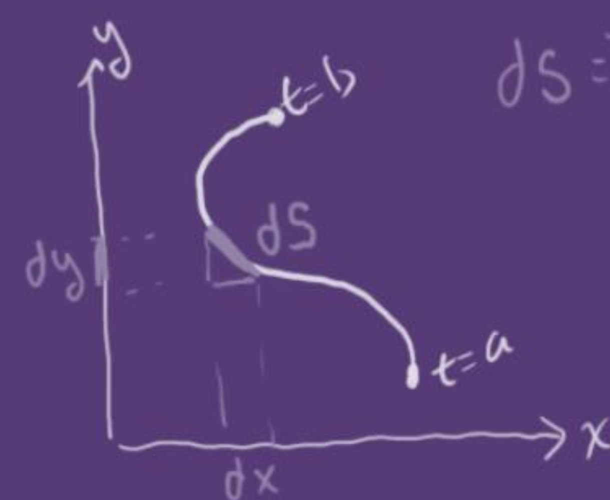
$$y = h(t)$$

$$x = g(a)$$

$$y = h(a)$$

ds= super small change in arc length

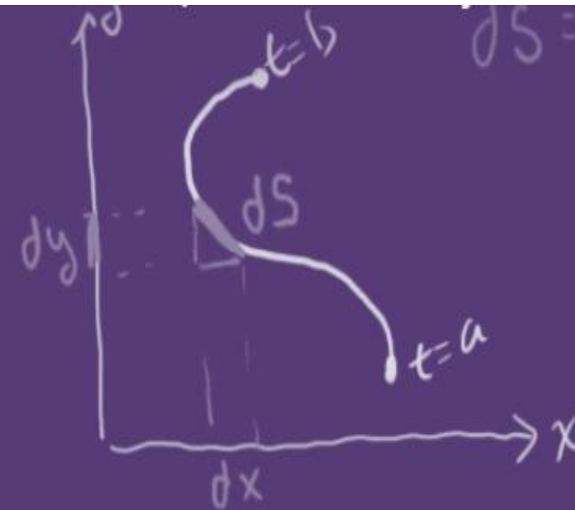
$$\int_{t=a}^{t=b} f(x,y)\, dS$$



$$ds = \sqrt{dx^2 + dy^2}$$

$$\int_{t=a}^{t=b} f(x,y) \sqrt{dx^2 + dy^2}$$

$$ds = \sqrt{dx^2 + dy^2}$$



$$\int_{t=a}^{t=b} f(x,y)\sqrt{dx^2 + dy^2}$$

$$\frac{dt}{dt}\sqrt{dx^2 + dy^2}$$

$$\int_{t=a}^{t=b} f(x(t),y(t))\sqrt{dx^2 + dy^2}$$

$$= \frac{1}{dt}\sqrt{dx^2 + dy^2}\;dt$$

$$\int^{t=b} f(x(t),y(t))\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\;dt$$

$$= \sqrt{\frac{1}{dt^2}\left(dx^2 + dy^2\right)}\;dt$$

Calculate $\int_C F(\bar{r}) \cdot d\bar{r}$ $= \int_C \bar{F} \cdot d\bar{r}$ for the following data. If F is a force, this gives the work done in the displacement along C.

① $F = [y^3, x^3]$, C the parabola $y = 5x^2$ from A : (0, 0) to B : (2, 20)

Sol

$\bar{F} = y^3\hat{i} + x^3\hat{j}$

$\bar{r}(t) : [t, 5t^2], 0 \le t \le 2$

$\bar{r}(t) = t\hat{i} + 5t^2\hat{j}$

$\bar{r}'(t) = \hat{i} + 10t\hat{j}$

$x = t$; $y = 5t^2$

$F(\bar{r}(t)) = 125t^6\hat{i} + t^3\hat{j}$

$\int_C \bar{F} \cdot d\bar{r} = \int_0^2 (125t^6\hat{i} + t^3\hat{j}) \cdot (\hat{i} + 10t\hat{j})\, dt$

$x = t$

$\boxed{x = 0 \Rightarrow t = 0 \quad x = 2 \Rightarrow t = 2}$

$= \int_0^2 (125t^6 + 10t^4)\, dt$

$y = 5t^2$

$y = 0 \Rightarrow \boxed{t = 0}$

$y = 20 \Rightarrow 20 = 5t^2$, $4 = t^2$, $\boxed{t = 2}$

$= \left[\frac{125t^7}{7} + \frac{10t^5}{5}\right]_0^2$

$= \frac{125(2)^7}{7} + 2(2)^5$

$= 2^5\left[\frac{125(2)^2}{7} + 2\right]$

$= 2^5\left[\frac{508}{7} ...\right]$

$= 2^6\left[\frac{254}{7} - 1\right]$

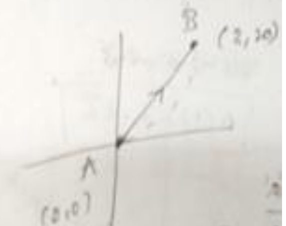
$= 2^6\left[\frac{254 - 7}{7}\right]$

$= 2^6\left(\frac{147}{7}\right)$

$= 2^6(21)$

$= 64(21)$

$\int_C \bar{F} \cdot d\bar{r} = 1344.$



$\bar{r} = (0,0) + t[2, 20]$

$\bar{r} = [2t, 20t]$

$0 \le t \le 1$

$\boxed{2349.7}$

$\boxed{\int_C \bar{F} \cdot d\bar{r} = \int_C \bar{F}(\bar{r}(t)) \cdot \bar{r}'(t)\, dt}$

(3) $F = [z\ x\ y]$, $C: \bar r = [\cos t, \sin t, t]$ from $(1,0,0)$ to $(1,0,4\pi)$

$\bar r(t) = \cos t\,\hat i + \sin t\,\hat j + t\hat k \quad 0 \le t \le 4\pi$

$\bar r'(t) = -\sin t\,\hat i + \cos t\,\hat j + \hat k$

$\bar F(\bar r(t)) = t\hat i + \cos t\,\hat j + \sin t\,\hat k$

$\bar F(\bar r)\cdot \bar r' = -t\sin t + \cos^2 t + \sin t$

$\int_C \bar F \cdot d\bar r = \int_0^{4\pi} (-t\sin t + \cos^2 t + \sin t)\,dt$

$= -\int_0^{4\pi} t\sin t\,dt + \int_0^{4\pi} \cos^2 t\,dt + \int_0^{4\pi} \sin t\,dt$

$= -\left[\left[t\cos t\right]_0^{4\pi} + \int_0^{4\pi} \cos t\,dt\right]$

$+ \int_0^{4\pi} \frac{1+\cos 2t}{2}\,dt + \left[\cos t\right]_0^{4\pi}$

$= -\left[(4\pi) + (\sin t)_0^{4\pi}\right] + \frac{1}{2}4\pi + \left[\frac{\sin 2t}{2}\right]_0^{4\pi}$

$- \cos 4\pi + 1$

$= +4\pi - 0 + 2\pi$

6π

check

$x = \cos t$

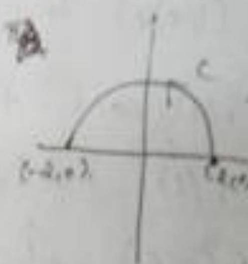
$x = 1 \Rightarrow 1 = \cos t$

$t = 0$

$z = t$

(8) $F = [x^2, y^2, 0]$, C the semicircle from $(2,0)$ to $(-2,0)$, $y \ge 0$.

$\bar r(t) = [2\cos t, 2\sin t, 0]$; $0 \le t \le \pi$



$\bar r(t) = [2\cos t, 2\sin t]$

$\bar r'(t) = [-2\sin t, 2\cos t]$

$\bar F(\bar r(t)) = [4\cos^2 t, 4\sin^2 t, 0]$

$\bar F(\bar r(t))\cdot \bar r' = -8\sin t\cos^2 t + 8\cos t\sin^2 t$

$\int_C \bar F\, d\bar r = \int_0^{2\pi\ \pi} (-8\sin t\cos^2 t + 8\cos t \sin^2 t)\,dt$

$x = 2\cos t$

$2 = 2\cos t$

$1 = \cos t$

$\boxed{t = 1}$

$= \int_0^{2\pi} -8\sin t(1-\sin^2 t)\,dt + 8\int_0^{2\pi} \cos t(1-\cos^2 t)\,dt$

$\sin 3\theta = 3\sin\theta - 4\sin^3\theta$

$\cos 3\theta = 4\cos^3\theta - 3\cos\theta$

$= -8\int_0^{2\pi} \sin t\,dt + 8\int_0^{2\pi} \sin^3 t\,dt + 8\int_0^{2\pi} \cos t\,dt - 8\int_0^{2\pi} \cos^3 t\,dt$

(8) Evaluate the line integral with

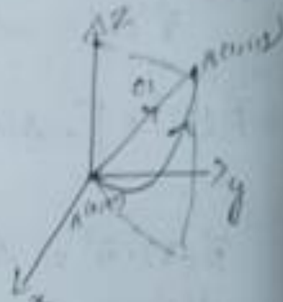
$\vec{F}(\vec{r}) = [5z,\ xy,\ x^2 z]$ along two different paths with the same initial pt. $A(0,0,0)$ and the same terminal point $B(1,1,1)$ namely

C_1: St line segment

$\vec{r}_1(t) = [t, t, t]$, $0 \le t \le 1$

and C_2: the parabolic arc

$\vec{r}_2(t) = [t, t, t^2]$, $0 \le t \le 1$

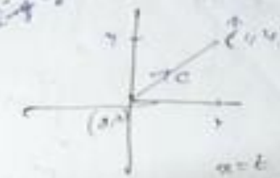


Ans: (i) $37/12$ (ii) $7/3$.

Line integral depends on the path of integration eventhough the endpts are same.

(1) $\vec{F} = [y^2, -x^2]$ $C \to$ St line segment from $(0,0)$ to $(1,4)$

Ans: 4.

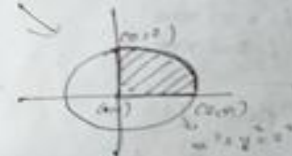


$x = t$, $y = 4t$

$\boxed{(3), (4), (10)}$

(3) $\vec{F} = [xy,\ x^2 y^2]$, C is the quarter-circle from $(2,0)$ to $(0,2)$ with centre $(0,0)$

Ans: $= +8/5$



(4) $\vec{F} = [xy,\ x^2y^2]$; C, the st line seg from $(2,0)$ to $(0,2)$.

$\vec{r}(t) = 2\hat{i} + t(-2\hat{i} + 2\hat{j})$
$= (2-2t)\hat{i} + 2t\hat{j}$.

$\vec{r} = \vec{r}_1 + t(\vec{r}_2 - \vec{r}_1)$

Ans: $= -8/30$.

(10) $\vec{F} = [\cosh x,\ \sinh y,\ e^z]$,
$C: \vec{r} = [t, t^2, t^3]$ from $(0,0,0)$ to $(2,4,8)$

$\vec{r}'(t) = 20\vec{j}$

$\vec{F}(\vec{r}(t)) = 8000t^3\,\vec{i} + 8\vec{j}$

$\vec{F}(\vec{r}(t)) \cdot \vec{r}'(t) = (8000t^3\,\vec{i} + 8\vec{j}) \cdot (20\vec{j})$

$\vec{F}(\vec{r}(t)) \cdot \vec{r}'(t) = 160$

$\int_C \vec{F} \cdot d\vec{r} = \int_0^1 160\,dt = 160[t]_0^1$
$= 160$

$$\boxed{\int_C \vec{F} \cdot d\vec{r} = 0 + 160 = 160}$$