3.4 Hash Tables

- hash functions
- separate chaining
- linear probing
- applications

Optimize judiciously

" More computing sins are committed in the name of efficiency (without necessarily achieving it) than for any other single reason—including blind stupidity. " — *William A. Wulf*

" We should forget about small efficiencies, say about 97% of the time: premature optimization is the root of all evil. " — *Donald E. Knuth*

" We follow two rules in the matter of optimization:
Rule 1: Don't do it.
Rule 2 (for experts only). Don't do it yet - that is, not until you have a perfectly clear and unoptimized solution. " — *M. A. Jackson*

Reference: Effective Java by Joshua Bloch

ST implementations: summary

implementation	guarantee			average case			ordered iteration?	operations on keys
	search	insert	delete	search hit	insert	delete		
sequential search (linked list)	N	N	N	N/2	N	N/2	no	`equals()`
binary search (ordered array)	lg N	N	N	lg N	N/2	N/2	yes	`compareTo()`
BST	N	N	N	1.38 lg N	1.38 lg N	?	yes	`compareTo()`
red-black tree	2 lg N	2 lg N	2 lg N	1.00 lg N	1.00 lg N	1.00 lg N	yes	`compareTo()`

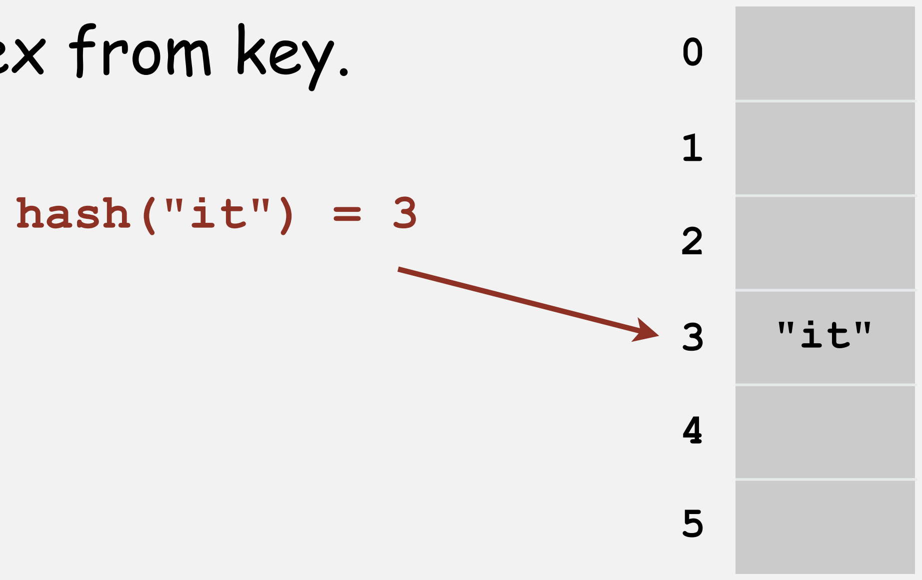
Q. Can we do better?

A. Yes, but with different access to the data.

Hashing: basic plan

Save items in a key-indexed table (index is a function of the key).

Hash function. Method for computing array index from key.



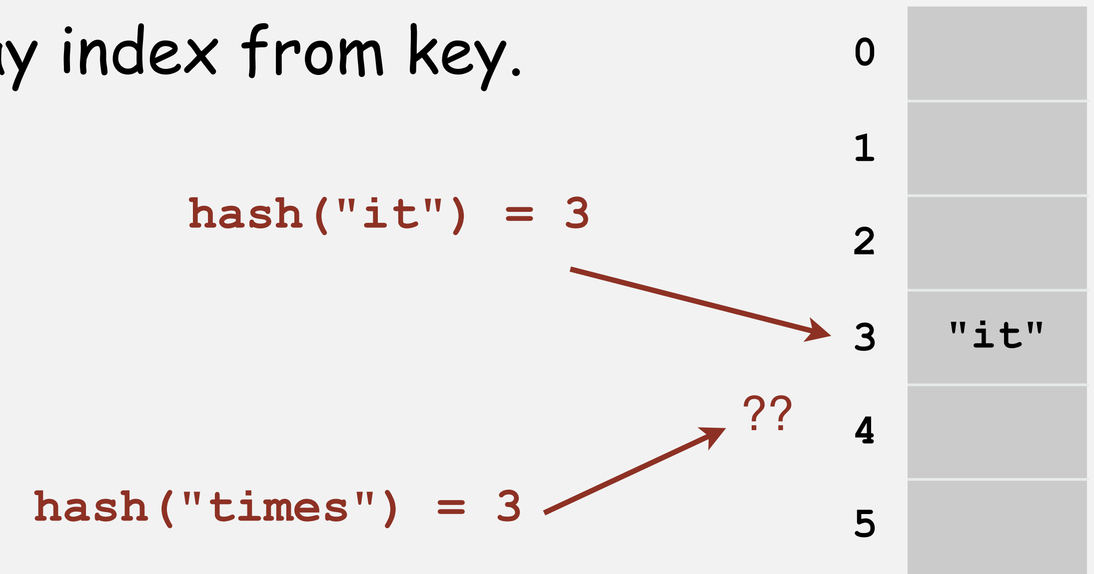
Issues.

- Computing the hash function.
- Equality test: Method for checking whether two keys are equal.

Hashing: basic plan

Save items in a key-indexed table (index is a function of the key).

Hash function. Method for computing array index from key.



Issues.

- Computing the hash function.
- Equality test: Method for checking whether two keys are equal.
- Collision resolution: Algorithm and data structure to handle two keys that hash to the same array index.

Classic space-time tradeoff.

- No space limitation: trivial hash function with key as index.
- No time limitation: trivial collision resolution with sequential search.
- Space and time limitations: hashing (the real world).

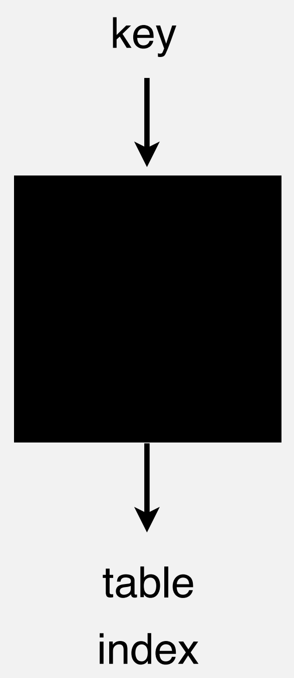
- **hash functions**
- separate chaining
- linear probing
- applications

Computing the hash function

Idealistic goal. Scramble the keys uniformly to produce a table index.

- Efficiently computable.
- Each table index equally likely for each key.

thoroughly researched problem,
still problematic in practical applications



Ex 1. Phone numbers.

- Bad: first three digits.
- Better: last three digits.

Ex 2. Social Security numbers.

573 = California, 574 = Alaska
(assigned in chronological order within geographic region)

- Bad: first three digits.
- Better: last three digits.

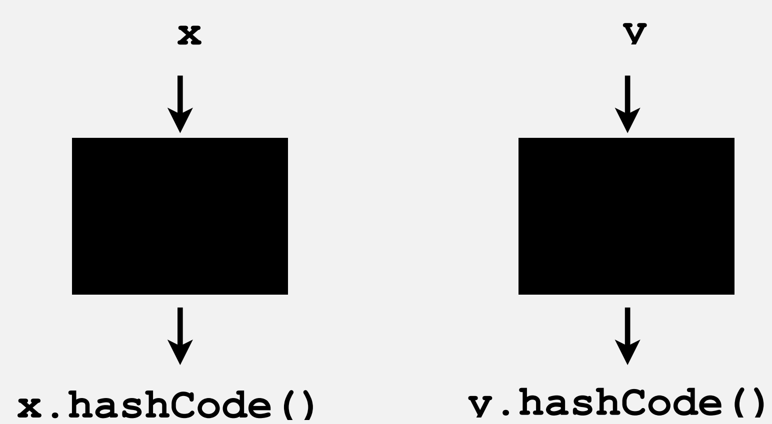
Practical challenge. Need different approach for each key type.

Java's hash code conventions

All Java classes inherit a method `hashCode()`, which returns a 32-bit `int`.

Requirement. If `x.equals(y)`, then `(x.hashCode() == y.hashCode())`.

Highly desirable. If `!x.equals(y)`, then `(x.hashCode() != y.hashCode())`.



Default implementation. Memory address of `x`.

Trivial (but poor) implementation. Always return `17`.

Customized implementations. `Integer`, `Double`, `String`, `File`, `URL`, `Date`, …

User-defined types. Users are on their own.

Implementing hash code: integers, booleans, and doubles

```
public final class Integer
{
   private final int value;
   ...

   public int hashCode()
   {  return value;  }
}
```

```
public final class Boolean
{
   private final boolean value;
   ...

   public int hashCode()
   {
      if (value) return 1231;
      else       return 1237;
   }
}
```

```
public final class Double
{
   private final double value;
   ...

   public int hashCode()
   {
      long bits = doubleToLongBits(value);
      return (int) (bits ^ (bits >>> 32));
   }
}
```

convert to IEEE 64-bit representation;
xor most significant 32-bits
with least significant 32-bits

Implementing hash code: strings

```
public final class String
{
   private final char[] s;
   ...

   public int hashCode()
   {
      int hash = 0;
      for (int i = 0; i < length(); i++)
         hash = s[i] + (31 * hash);
      return hash;
   }
}
```

i^{th} character of s

char	Unicode
…	…
'a'	97
'b'	98
'c'	99
…	...

- Horner's method to hash string of length L: L multiplies/adds.
- Equivalent to $h = 31^{L-1} \cdot s^0 + \ldots + 31^2 \cdot s^{L-3} + 31^1 \cdot s^{L-2} + 31^0 \cdot s^{L-1}$.

Ex.

```
String s = "call";
int code = s.hashCode();
```

$3045982 = 99 \cdot 31^3 + 97 \cdot 31^2 + 108 \cdot 31^1 + 108 \cdot 31^0$

$= 108 + 31 \cdot (108 + 31 \cdot (97 + 31 \cdot (99)))$

War story: String hashing in Java

String `hashCode()` in Java 1.1.

- For long strings: only examine 8-9 evenly spaced characters.
- Benefit: saves time in performing arithmetic.

```
public int hashCode()
{
   int hash = 0;
   int skip = Math.max(1, length() / 8);
   for (int i = 0; i < length(); i += skip)
      hash = s[i] + (37 * hash);
   return hash;
}
```

- Downside: great potential for bad collision patterns.

War story: String hashing in Java

String `hashCode()` in Java 1.1.

- For long strings: only examine 8-9 evenly spaced characters.
- Benefit: saves time in performing arithmetic.

```
public int hashCode()
{
   int hash = 0;
   int skip = Math.max(1, length() / 8);
   for (int i = 0; i < length(); i += skip)
      hash = s[i] + (37 * hash);
   return hash;
}
```

- Downside: great potential for bad collision patterns.

```
http://www.cs.princeton.edu/introcs/13loop/Hello.java
http://www.cs.princeton.edu/introcs/13loop/Hello.class
http://www.cs.princeton.edu/introcs/13loop/Hello.html
http://www.cs.princeton.edu/introcs/12type/index.html
↑     ↑     ↑     ↑     ↑     ↑     ↑     ↑
```

Implementing hash code: user-defined types

```
public final class Transaction
{
   private final long who;
   private final Date when;
   private final String where;

   public Transaction(long who, Date when, String where)
   {  /* as before */  }

   ...

   public boolean equals(Object y)
   {  /* as before */  }

   public int hashCode()
   {
      int hash = 17;
      hash = 31*hash + ((Long) who).hashCode();
      hash = 31*hash + when.hashCode();
      hash = 31*hash + where.hashCode();
      return hash;
   }
}
```

nonzero constant

for primitive types, use **hashCode()** of wrapper type

for reference types, use **hashCode()**

typically a small prime

Hash code design

"Standard" recipe for user-defined types.

- Combine each significant field using the $31x + y$ rule.
- If field is a primitive type, use wrapper type `hashCode()`.
- If field is an array, apply to each element. ← or use `Arrays.deepHashCode()`
- If field is a reference type, use `hashCode()`. ← applies rule recursively

In practice. Recipe works reasonably well; used in Java libraries.

In theory. Need a theorem for each type to ensure reliability.

Basic rule. Need to use the whole key to compute hash code; consult an expert for state-of-the-art hash codes.

Modular hashing

Hash code. An `int` between -2^{31} and $2^{31}-1$.

Hash function. An `int` between `0` and `M-1` (for use as array index).

typically a prime or power of 2

```
private int hash(Key key)
{  return key.hashCode() % M;  }
```

bug

Modular hashing

Hash code. An `int` between -2^{31} and $2^{31}-1$.

Hash function. An `int` between `0` and `M-1` (for use as array index).

typically a prime or power of 2

```
private int hash(Key key)
{  return key.hashCode() % M;   }
```

bug

```
private int hash(Key key)
{  return Math.abs(key.hashCode()) % M;   }
```

1-in-a-billion bug

Modular hashing

Hash code. An `int` between -2^{31} and $2^{31}-1$.

Hash function. An `int` between `0` and `M-1` (for use as array index).

typically a prime or power of 2

```
private int hash(Key key)
{  return key.hashCode() % M;  }
```

bug

```
private int hash(Key key)
{  return Math.abs(key.hashCode()) % M;  }
```

1-in-a-billion bug

hashCode() of "polygenelubricants" is -2^{31}

Modular hashing

Hash code. An `int` between -2^{31} and $2^{31}-1$.

Hash function. An `int` between `0` and `M-1` (for use as array index).

typically a prime or power of 2

```
private int hash(Key key)
{  return key.hashCode() % M;  }
```

bug

```
private int hash(Key key)
{  return Math.abs(key.hashCode()) % M;  }
```

1-in-a-billion bug

hashCode() of "polygenelubricants" is -2^{31}

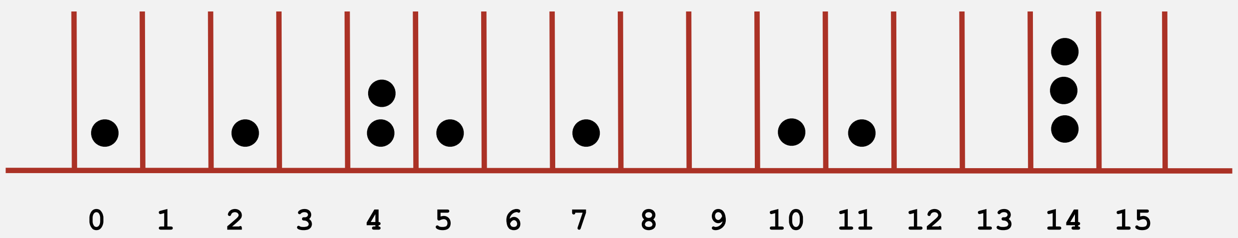
```
private int hash(Key key)
{  return (key.hashCode() & 0x7fffffff) % M;  }
```

correct

Uniform hashing assumption

Uniform hashing assumption. Each key is equally likely to hash to an integer between 0 and $M - 1$.

Bins and balls. Throw balls uniformly at random into M bins.



Birthday problem. Expect two balls in the same bin after $\sim \sqrt{\pi M / 2}$ tosses.

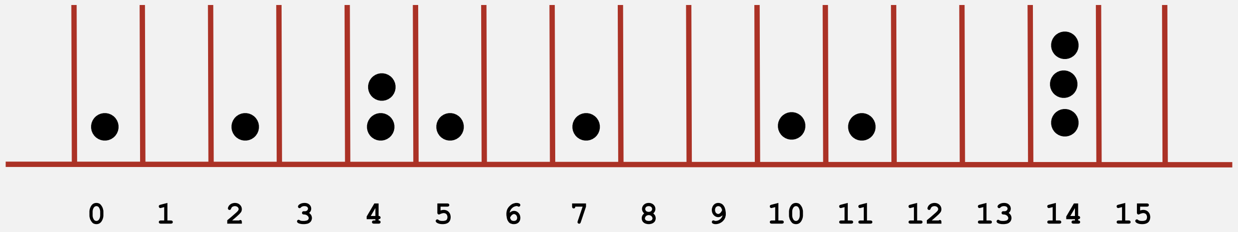
Coupon collector. Expect every bin has ≥ 1 ball after $\sim M \ln M$ tosses.

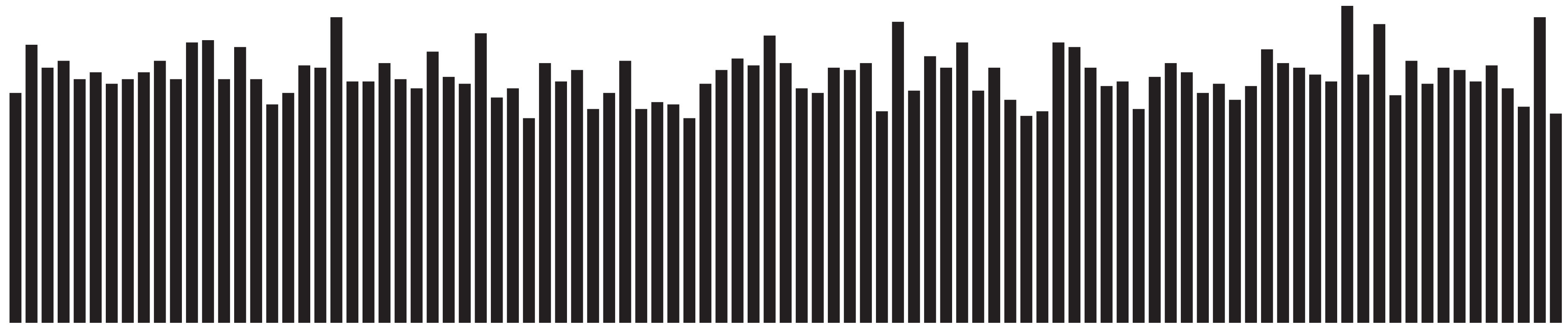
Load balancing. After M tosses, expect most loaded bin has $\Theta(\log M / \log \log M)$ balls.

Uniform hashing assumption

Uniform hashing assumption. Each key is equally likely to hash to an integer between 0 and $M - 1$.

Bins and balls. Throw balls uniformly at random into M bins.





Hash value frequencies for words in Tale of Two Cities (M = 97)

Java's `String` data uniformly distribute the keys of Tale of Two Cities

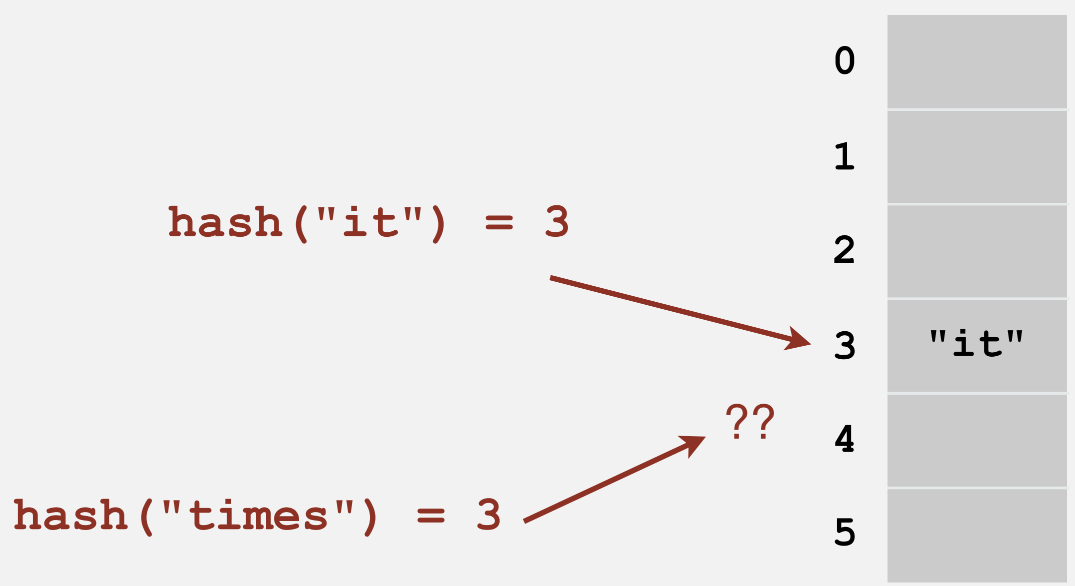
- hash functions
- **separate chaining**
- linear probing
- applications

Collisions

Collision. Two distinct keys hashing to same index.

- Birthday problem $\Rightarrow$ can't avoid collisions unless you have a ridiculous (quadratic) amount of memory.
- Coupon collector + load balancing $\Rightarrow$ collisions will be evenly distributed.

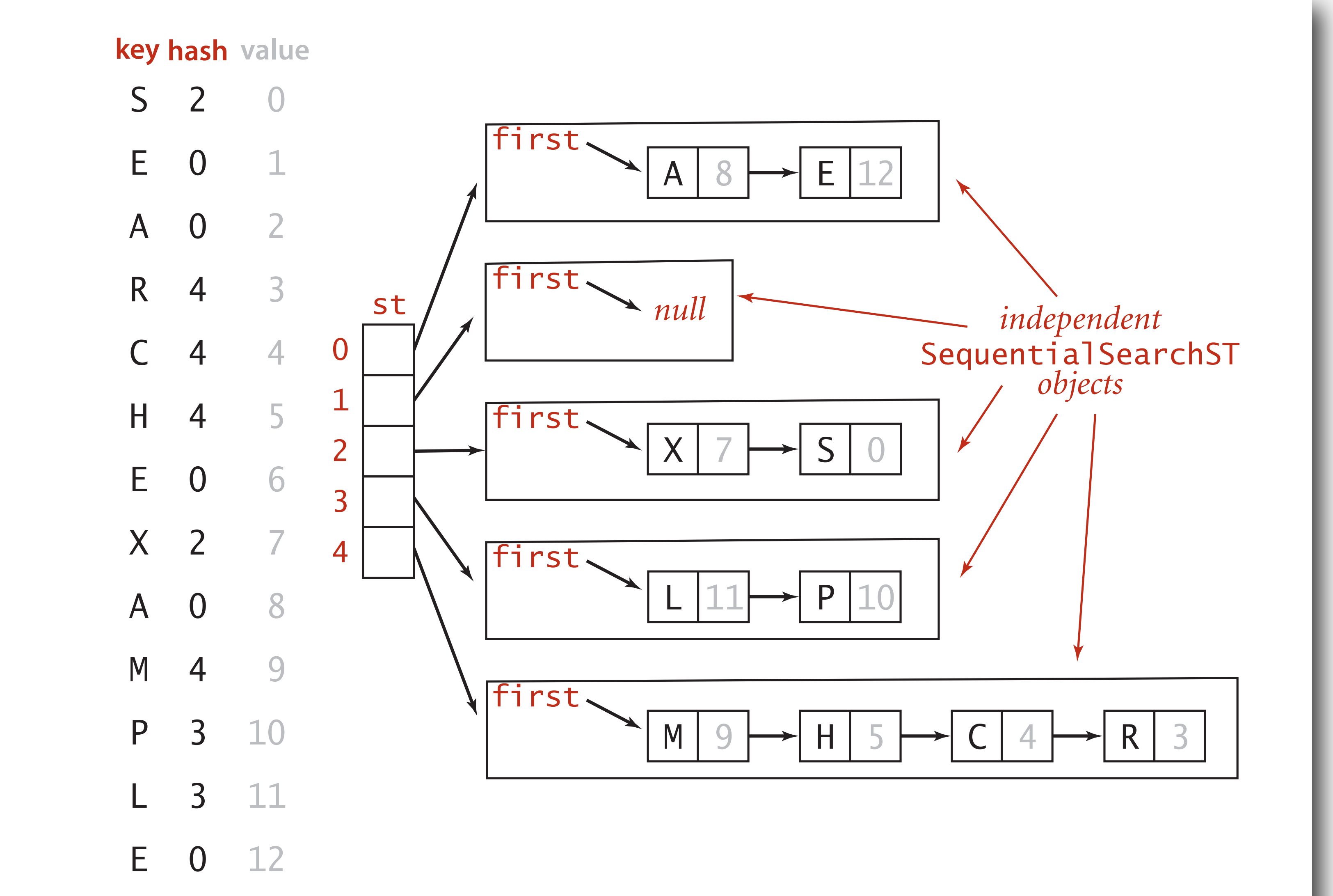
Challenge. Deal with collisions efficiently.



Separate chaining ST

Use an array of $M < N$ linked lists. [H. P. Luhn, IBM 1953]

- Hash: map key to integer i between 0 and $M - 1$.
- Insert: put at front of i^{th} chain (if not already there).
- Search: only need to search i^{th} chain.



Hashing with separate chaining for standard indexing client

Separate chaining ST: Java implementation

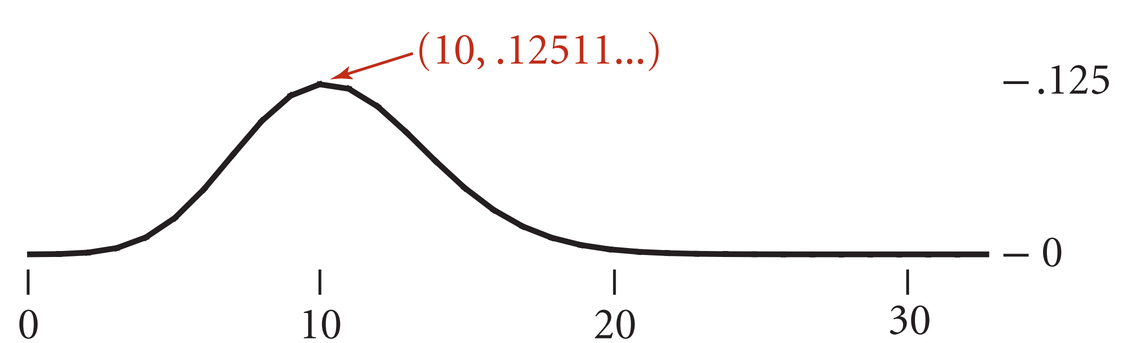
```
public class SeparateChainingHashST<Key, Value>
{
   private int N;      // number of key-value pairs
   private int M;      // hash table size
   private SequentialSearchST<Key, Value> [] st;   // array of STs

   public SeparateChainingHashST()
   {  this(997);  }

   public SeparateChainingHashST(int M)
   {
      this.M = M;
      st = (SequentialSearchST<Key, Value>[]) new SequentialSearchST[M];
      for (int i = 0; i < M; i++)
         st[i] = new SequentialSearchST<Key, Value>();
   }
   private int hash(Key key)
   {  return (key.hashCode() & 0x7fffffff) % M;  }

   public Value get(Key key)
   {  return st[hash(key)].get(key);  }

   public void put(Key key, Value val)
   {  st[hash(key)].put(key, val);  }
}
```

array doubling and halving code omitted

Analysis of separate chaining

Proposition. Under uniform hashing assumption, probability that the number of keys in a list is within a constant factor of N / M is extremely close to 1.

Pf sketch. Distribution of list size obeys a binomial distribution.



Binomial distribution $(N = 10^4, M = 10^3, \alpha = 10)$

`equals()` and `hashCode()`

Consequence. Number of probes for search/insert is proportional to N / M.

- M too large $\Rightarrow$ too many empty chains.
- M too small $\Rightarrow$ chains too long.
- Typical choice: $M \sim N / 5$ $\Rightarrow$ constant-time ops.

M times faster than sequential search

ST implementations: summary

implementation	guarantee			average case			ordered iteration?	operations on keys
	search	insert	delete	search hit	insert	delete		
sequential search (linked list)	N	N	N	N/2	N	N/2	no	`equals()`
binary search (ordered array)	lg N	N	N	lg N	N/2	N/2	yes	`compareTo()`
BST	N	N	N	1.38 lg N	1.38 lg N	?	yes	`compareTo()`
red-black tree	2 lg N	2 lg N	2 lg N	1.00 lg N	1.00 lg N	1.00 lg N	yes	`compareTo()`
separate chaining	lg N *	lg N *	lg N *	3-5 *	3-5 *	3-5 *	no	`equals()`

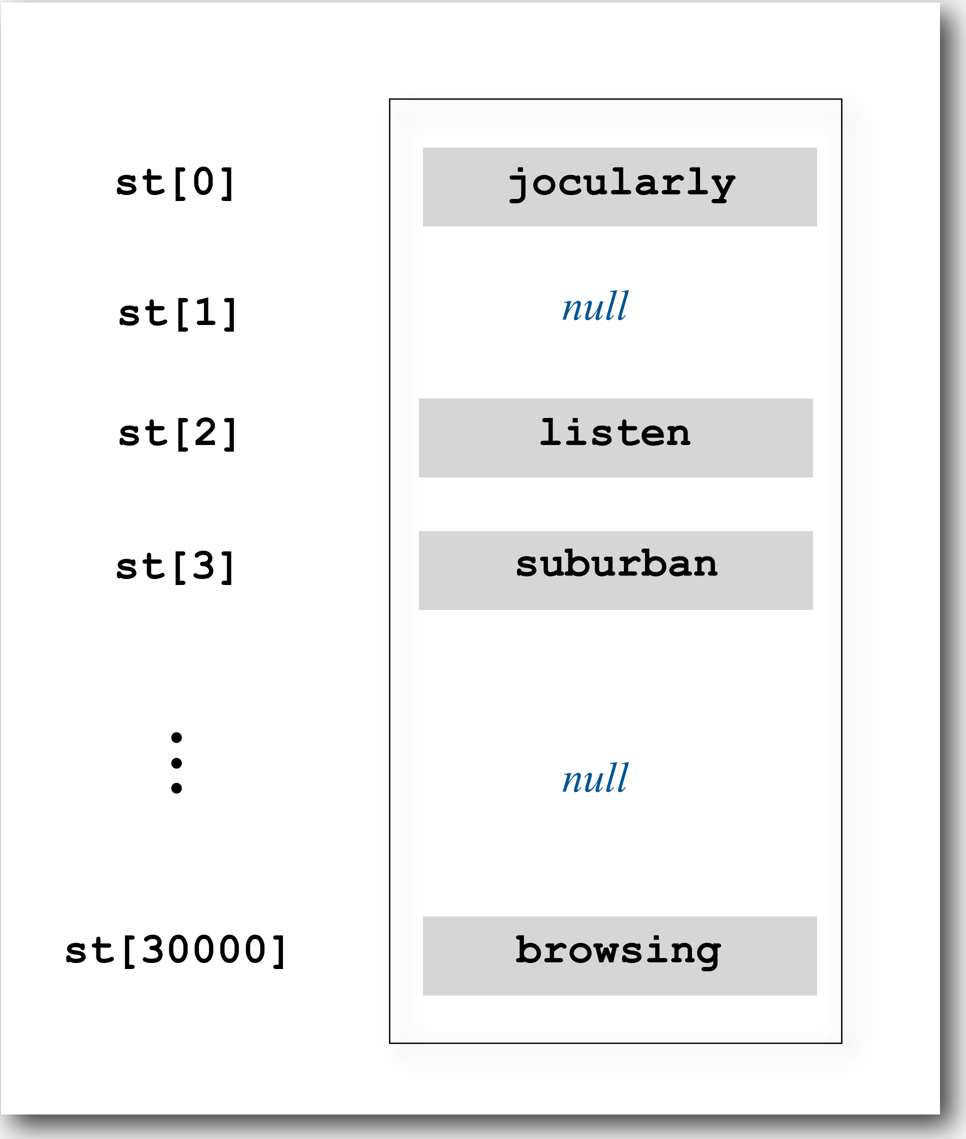
* under uniform hashing assumption

- hash functions
- separate chaining
- **linear probing**
- applications

Collision resolution: open addressing

Open addressing. [Amdahl-Boehme-Rocherster-Samuel, IBM 1953]

When a new key collides, find next empty slot, and put it there.



linear probing (M = 30001, N = 15000)

Linear probing

Use an array of size $M > N$.

- Hash: map key to integer i between 0 and $M - 1$.
- Insert: put at table index i if free; if not try $i + 1$, $i + 2$, etc.
- Search: search table index i; if occupied but no match, try $i + 1$, $i + 2$, etc.

-	-	-	S	H	-	-	A	C	E	R	-	-
0	1	2	3	4	5	6	7	8	9	10	11	12

Linear probing

Use an array of size $M > N$.

- Hash: map key to integer i between 0 and $M - 1$.
- Insert: put at table index i if free; if not try $i + 1$, $i + 2$, etc.
- Search: search table index i; if occupied but no match, try $i + 1$, $i + 2$, etc.

-	-	-	S	H	-	-	A	C	E	R	-	-
0	1	2	3	4	5	6	7	8	9	10	11	12

-	-	-	S	H	-	-	A	C	E	R	I	-
0	1	2	3	4	5	6	7	8	9	10	11	12

insert I
hash(I) = 11

Linear probing

Use an array of size $M > N$.

- Hash: map key to integer i between 0 and $M - 1$.
- Insert: put at table index i if free; if not try $i + 1$, $i + 2$, etc.
- Search: search table index i; if occupied but no match, try $i + 1$, $i + 2$, etc.

-	-	-	S	H	-	-	A	C	E	R	-	-
0	1	2	3	4	5	6	7	8	9	10	11	12

-	-	-	S	H	-	-	A	C	E	R	I	-
0	1	2	3	4	5	6	7	8	9	10	11	12

insert I
hash(I) = 11

-	-	-	S	H	-	-	A	C	E	R	I	N
0	1	2	3	4	5	6	7	8	9	10	11	12

insert N
hash(N) = 8

Linear probing: trace of standard indexing client

key	hash	value	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
S	6	0							S 0									
E	10	1							S 0				E 1					
A	4	2					A 2		S 0				E 1					
R	14	3					A 2		S 0				E 1				R 3	
C	5	4					A 2	C 5	S 0				E 1				R 3	
H	4	5					A 2	C 5	S 0	H 5			E 1				R 3	
E	10	6					A 2	C 5	S 0	H 5			E 6				R 3	
X	15	7					A 2	C 5	S 0	H 5			E 6				R 3	X 7
A	4	8					A 8	C 5	S 0	H 5			E 6				R 3	X 7
M	1	9		M 9			A 8	C 5	S 0	H 5			E 6				R 3	X 7
P	14	10	P 10	M 9			A 8	C 5	S 0	H 5			E 6				R 3	X 7
L	6	11	P 10	M 9			A 8	C 5	S 0	H 5	L 11		E 6				R 3	X 7
E	10	12	P 10	M 9			A 8	C 5	S 0	H 5	L 11		E 12				R 3	X 7

entries in red are new

entries in gray are untouched

keys in black are probes

probe sequence wraps to 0

keys[]

vals[]

Linear probing ST implementation

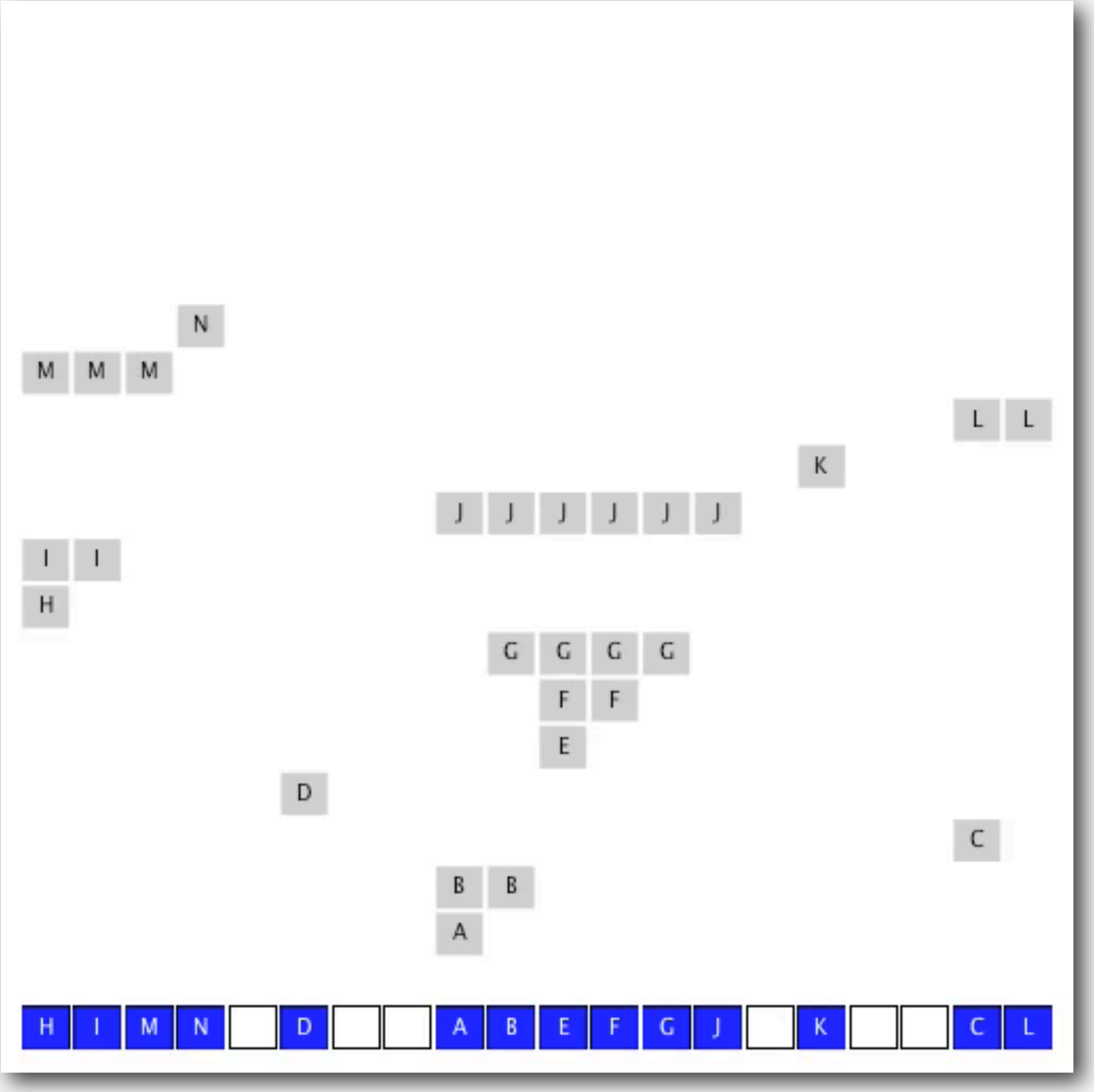
```
public class LinearProbingHashST<Key, Value>
{
   private int M = 30001;
   private Value[] vals = (Value[]) new Object[M];
   private Key[]   keys = (Key[])   new Object[M];

   private int hash(Key key) {  /* as before */  }

   public void put(Key key, Value val)
   {
      int i;
      for (i = hash(key); keys[i] != null; i = (i+1) % M)
         if (keys[i].equals(key))
             break;
      keys[i] = key;
      vals[i] = val;
   }

   public Value get(Key key)
   {
      for (int i = hash(key); keys[i] != null; i = (i+1) % M)
         if (key.equals(keys[i]))
             return vals[i];
      return null;
   }
}
```

array doubling and halving code omitted

Clustering

Cluster. A contiguous block of items.

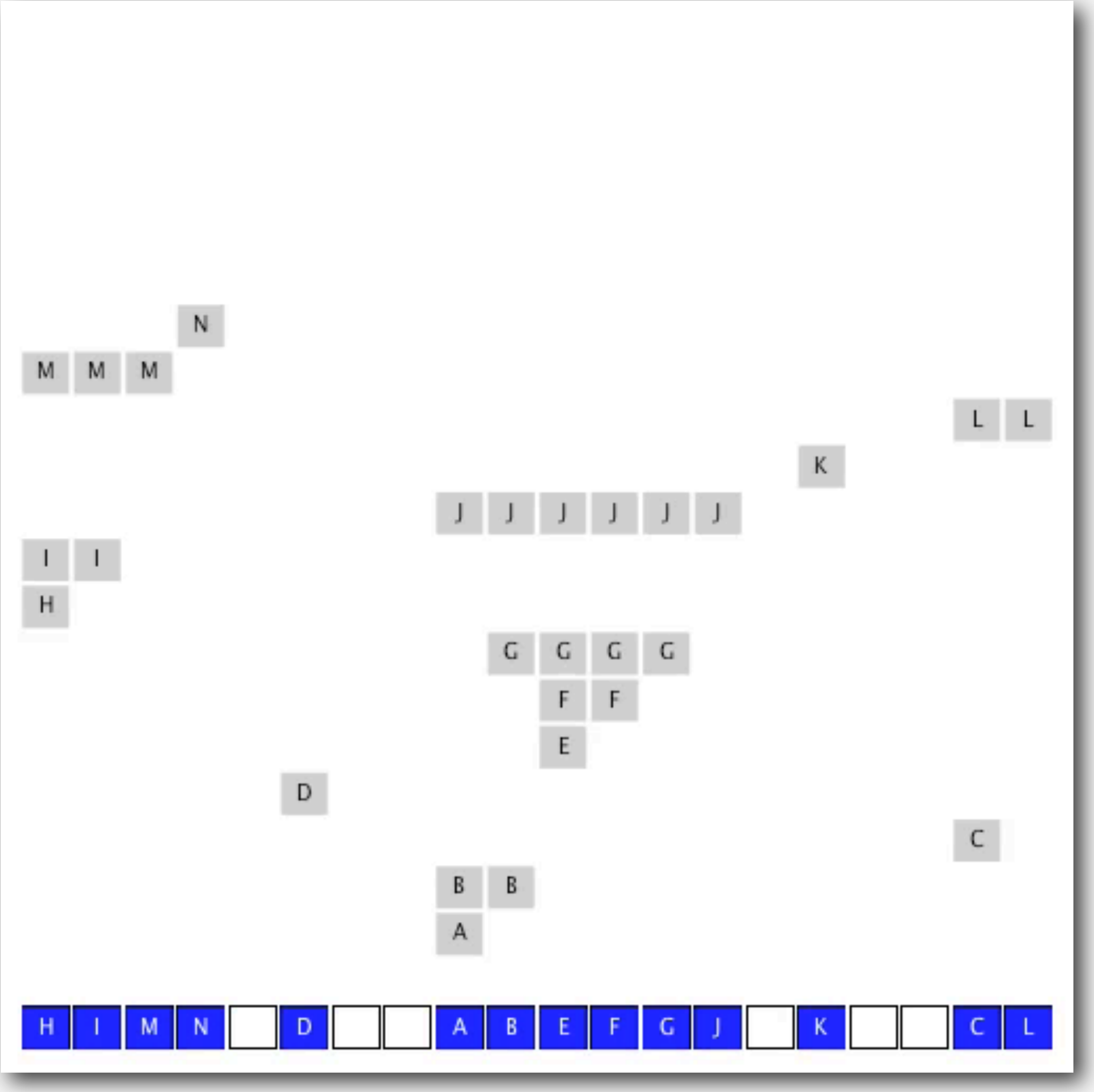
Observation. New keys likely to hash into middle of big clusters.



Clustering

Cluster. A contiguous block of items.

Observation. New keys likely to hash into middle of big clusters.

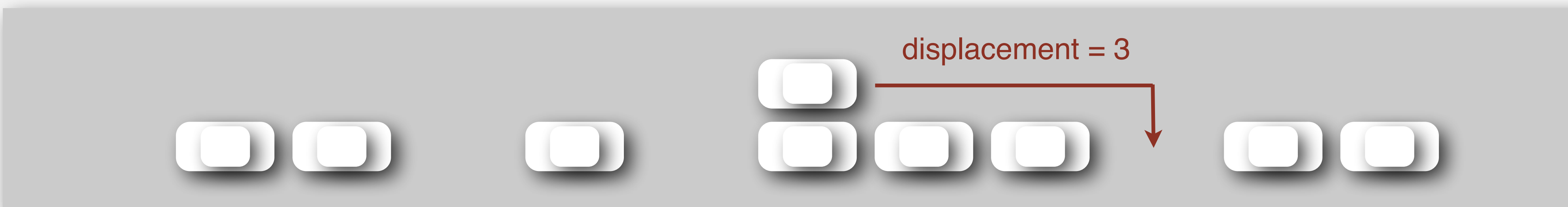


Knuth's parking problem

Model. Cars arrive at one-way street with M parking spaces.

Each desires a random space i : if space i is taken, try $i + 1$, $i + 2$, etc.

Q. What is mean displacement of a car?



Half-full. With $M / 2$ cars, mean displacement is $\sim 3 / 2$.

Full. With M cars, mean displacement is $\sim \sqrt{\pi M / 8}$

Analysis of linear probing

Proposition. Under uniform hashing assumption, the average number of probes in a hash table of size M that contains $N = \alpha M$ keys is:

$$\sim \frac{1}{2}\left(1 + \frac{1}{1-\alpha}\right) \qquad \sim \frac{1}{2}\left(1 + \frac{1}{(1-\alpha)^2}\right)$$

search hit   search miss / insert

Pf. [Knuth 1962] A landmark in analysis of algorithms.

Parameters.

- M too large $\Rightarrow$ too many empty array entries.
- M too small $\Rightarrow$ search time blows up.
- Typical choice: $\alpha = N / M \sim ½$.

probes for search hit is about 3/2

probes for search miss is about 5/2

ST implementations: summary

implementation	guarantee			average case			ordered iteration?	operations on keys
	search	insert	delete	search hit	insert	delete		
sequential search (linked list)	N	N	N	N/2	N	N/2	no	`equals()`
binary search (ordered array)	lg N	N	N	lg N	N/2	N/2	yes	`compareTo()`
BST	N	N	N	1.38 lg N	1.38 lg N	?	yes	`compareTo()`
red-black tree	2 lg N	2 lg N	2 lg N	1.00 lg N	1.00 lg N	1.00 lg N	yes	`compareTo()`
separate chaining	lg N *	lg N *	lg N *	3-5 *	3-5 *	3-5 *	no	`equals()`
linear probing	lg N *	lg N *	lg N *	3-5 *	3-5 *	3-5 *	no	`equals()`

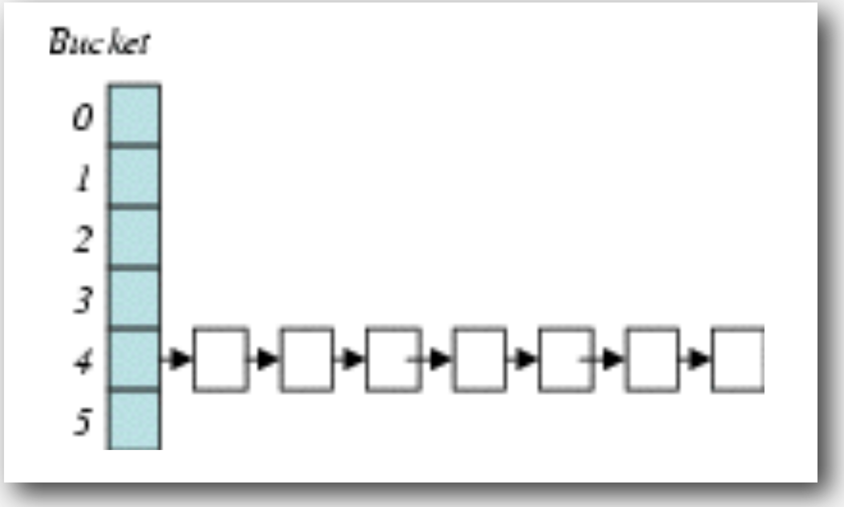
* under uniform hashing assumption

War story: algorithmic complexity attacks

Q. Is the uniform hashing assumption important in practice?

A. Obvious situations: aircraft control, nuclear reactor, pacemaker.

A. Surprising situations: denial-of-service attacks.



malicious adversary learns your hash function (e.g., by reading Java API) and causes a big pile-up in single slot that grinds performance to a halt

Real-world exploits. [Crosby-Wallach 2003]

- Bro server: send carefully chosen packets to DOS the server, using less bandwidth than a dial-up modem.
- Perl 5.8.0: insert carefully chosen strings into associative array.
- Linux 2.4.20 kernel: save files with carefully chosen names.

Algorithmic complexity attack on Java

Goal. Find family of strings with the same hash code.

Solution. The base-31 hash code is part of Java's string API.

key	hashCode()
"Aa"	2112
"BB"	2112

Algorithmic complexity attack on Java

Goal. Find family of strings with the same hash code.

Solution. The base-31 hash code is part of Java's string API.

key	hashCode()
"Aa"	2112
"BB"	2112

key	hashCode()
"AaAaAaAa"	-540425984
"AaAaAaBB"	-540425984
"AaAaBBAa"	-540425984
"AaAaBBBB"	-540425984
"AaBBAaAa"	-540425984
"AaBBAaBB"	-540425984
"AaBBBBAa"	-540425984
"AaBBBBBB"	-540425984

key	hashCode()
"BBAaAaAa"	-540425984
"BBAaAaBB"	-540425984
"BBAaBBAa"	-540425984
"BBAaBBBB"	-540425984
"BBBBAaAa"	-540425984
"BBBBAaBB"	-540425984
"BBBBBBAa"	-540425984
"BBBBBBBB"	-540425984

2^N strings of length $2N$ that hash to same value!

Diversion: one-way hash functions

One-way hash function. "Hard" to find a key that will hash to a desired value (or two keys that hash to same value).

Ex. $\underbrace{\text{MD4, MD5, SHA-0, SHA-1,}}_{\text{known to be insecure}}$ SHA-2, WHIRLPOOL, RIPEMD-160, ...

```
String password = args[0];
MessageDigest sha1 = MessageDigest.getInstance("SHA1");
byte[] bytes = sha1.digest(password);

/* prints bytes as hex string */
```

Applications. Digital fingerprint, message digest, storing passwords.

Caveat. Too expensive for use in ST implementations.

Separate chaining vs. linear probing

Separate chaining.

- Easier to implement delete.
- Performance degrades gracefully.
- Clustering less sensitive to poorly-designed hash function.

Linear probing.

- Less wasted space.
- Better cache performance.

Hashing: variations on the theme

Many improved versions have been studied.

Two-probe hashing. (separate-chaining variant)

- Hash to two positions, put key in shorter of the two chains.
- Reduces expected length of the longest chain to $\log \log N$.

Double hashing. (linear-probing variant)

- Use linear probing, but skip a variable amount, not just 1 each time.
- Effectively eliminates clustering.
- Can allow table to become nearly full.
- Difficult to implement delete.

Hashing vs. balanced search trees

Hashing.

- Simpler to code.
- No effective alternative for unordered keys.
- Faster for simple keys (a few arithmetic ops versus $\log N$ compares).
- Better system support in Java for strings (e.g., cached hash code).

Balanced search trees.

- Stronger performance guarantee.
- Support for ordered ST operations.
- Easier to implement `compareTo()` correctly than `equals()` and `hashCode()`.

Java system includes both.

- Red-black trees: `java.util.TreeMap`, `java.util.TreeSet`.
- Hashing: `java.util.HashMap`, `java.util.IdentityHashMap`.

- sets
- dictionary clients
- indexing clients
- sparse vectors
- **challenges**

ST implementations: summary

implementation	guarantee			average case			ordered iteration?	operations on keys
	search	insert	delete	search hit	insert	delete		
sequential search (linked list)	N	N	N	N/2	N	N/2	no	`equals()`
binary search (ordered array)	lg N	N	N	lg N	N/2	N/2	yes	`compareTo()`
BST	N	N	N	1.38 lg N	1.38 lg N	?	yes	`compareTo()`
red-black tree	2 lg N	2 lg N	2 lg N	1.00 lg N	1.00 lg N	1.00 lg N	yes	`compareTo()`
separate chaining	lg N *	lg N *	lg N *	3-5 *	3-5 *	3-5 *	no	`equals()`
linear probing	lg N *	lg N *	lg N *	3-5 *	3-5 *	3-5 *	no	`equals()`

* under uniform hashing assumption

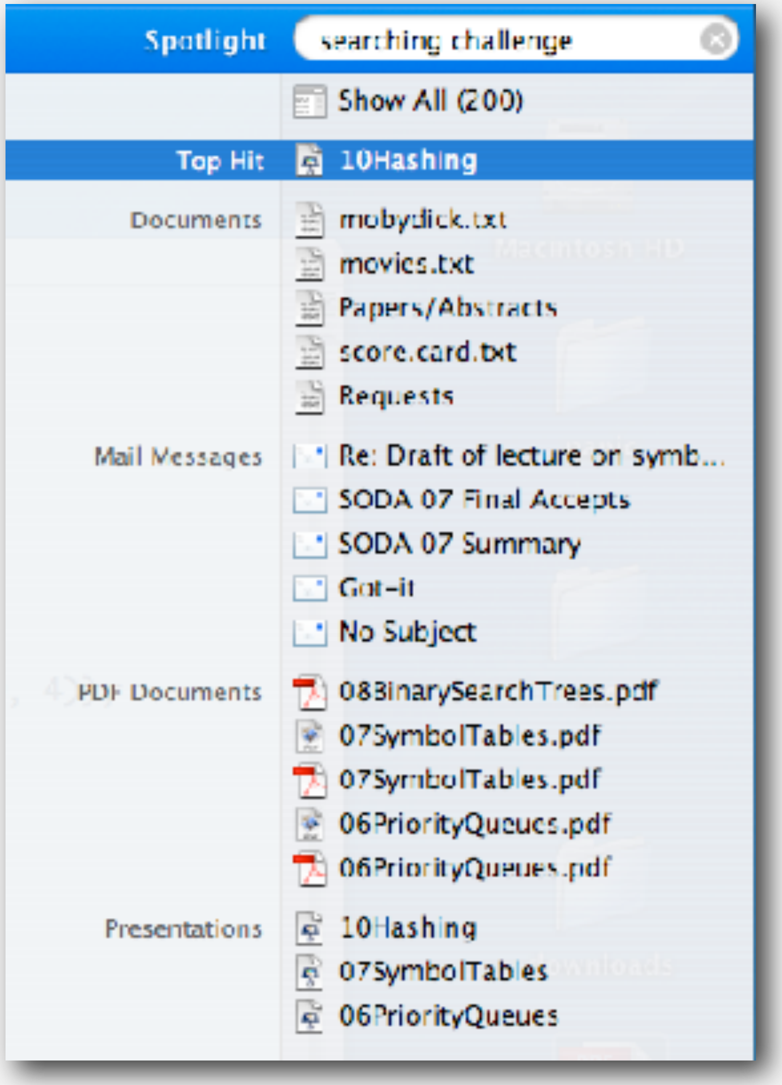
Searching challenge 1

Problem. Index for a PC or the web.

Assumptions. 1 billion++ words to index.

Which searching method to use?

- Hashing
- Red-black-trees
- Doesn't matter much.



Searching challenge 1

Problem. Index for a PC or the web.

Assumptions. 1 billion++ words to index.

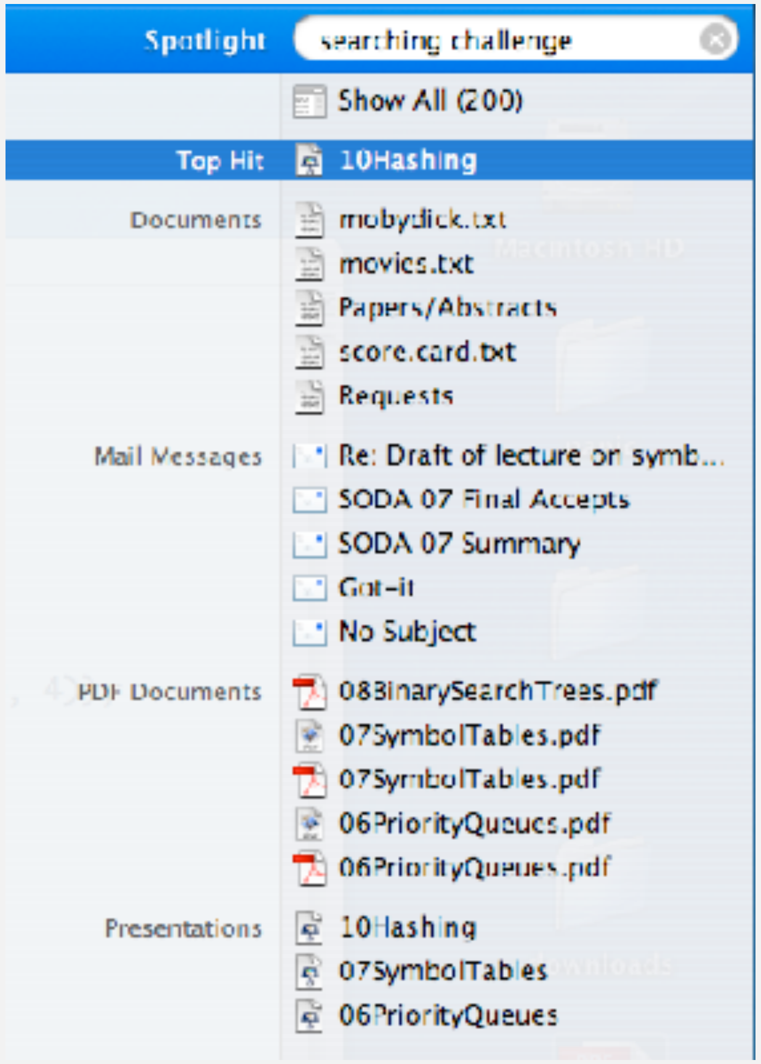
Which searching method to use?

- ✓ Hashing
- Red-black-trees ← too much space
- Doesn't matter much.

Solution. Symbol table with:

- Key = query string.
- Value = set of pointers to files.

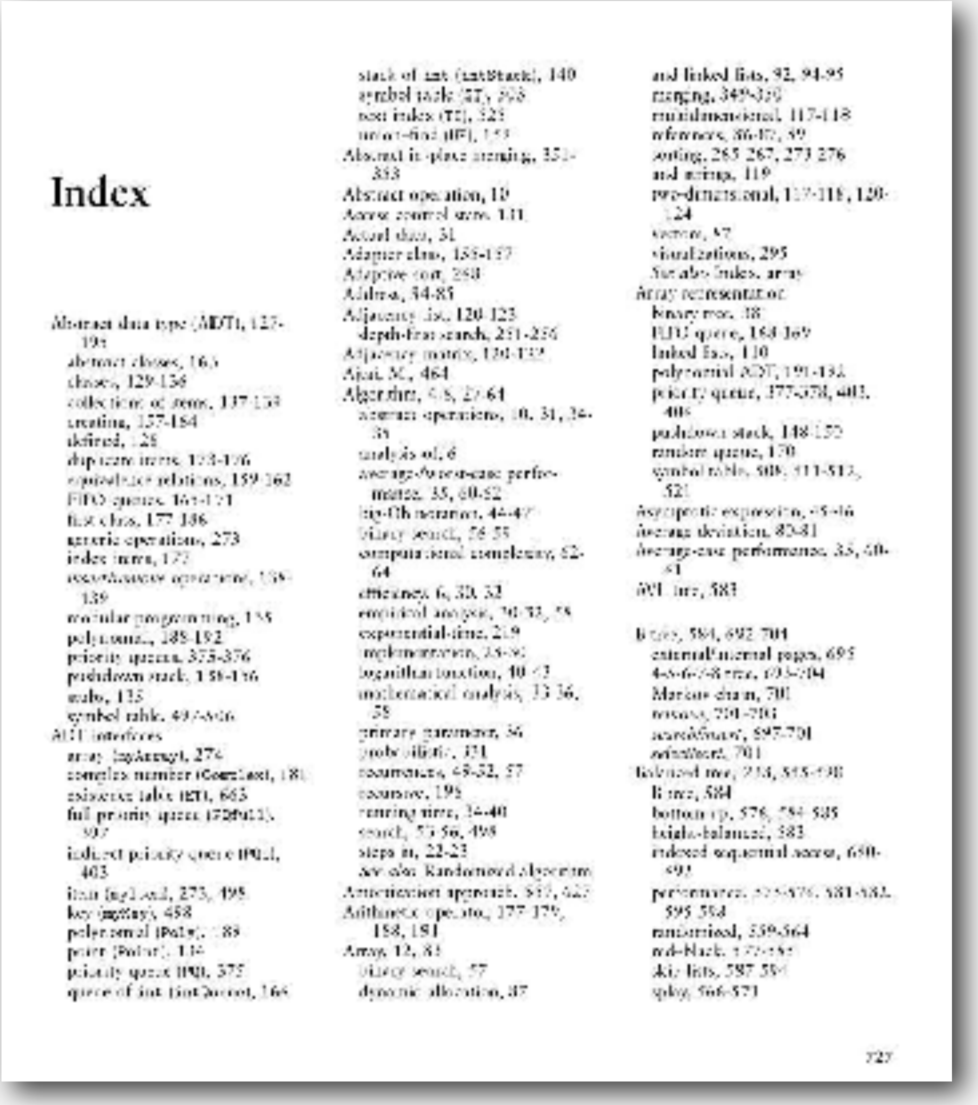
sort the (relatively few) search hits



Searching challenge 2

Problem. Index for an e-book.

Assumptions. Book has 100,000+ words.


Index

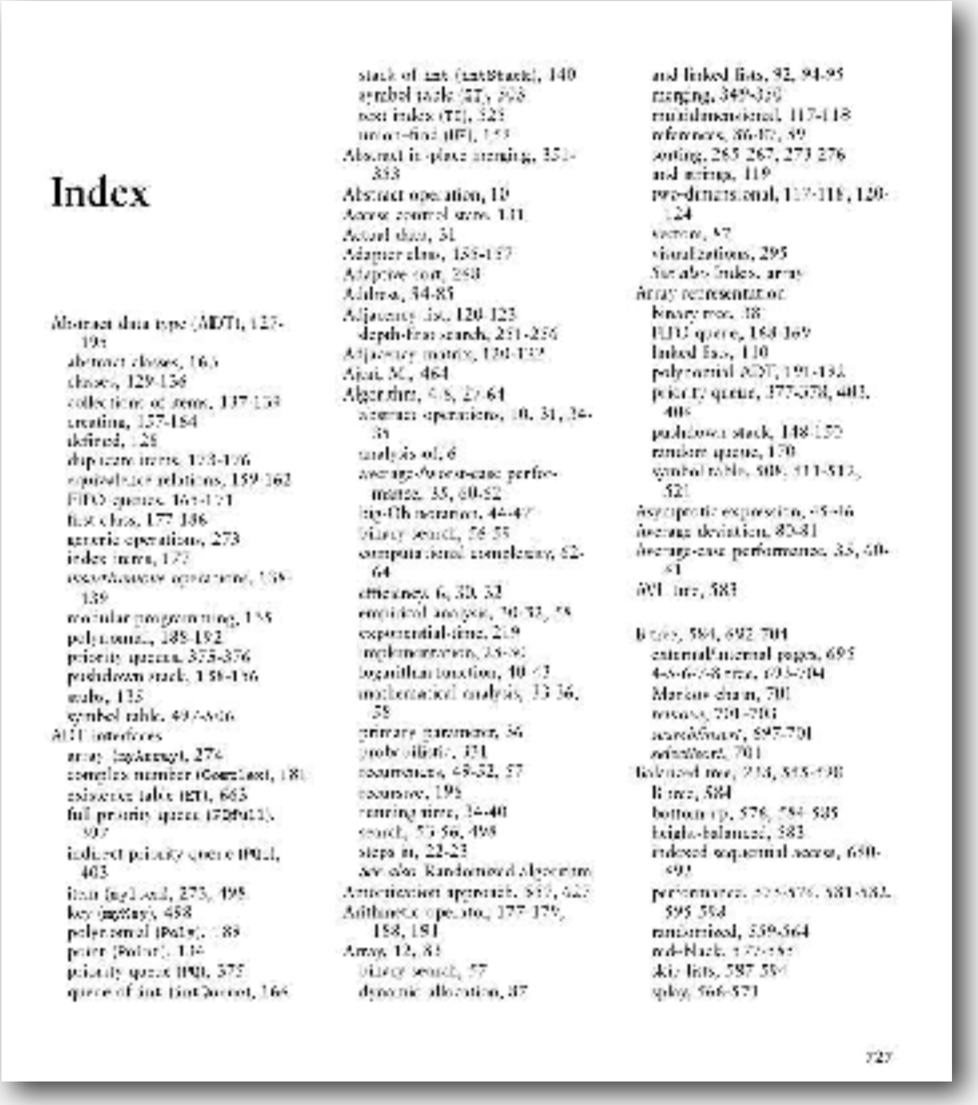
Which searching method to use?

1. Hashing
2. Red-black-tree
3. Doesn't matter much.

Searching challenge 2

Problem. Index for an e-book.

Assumptions. Book has 100,000+ words.



Which searching method to use?

1. Hashing
2. ✓ Red-black-tree ← need ordered iteration
3. Doesn't matter much.

Solution. Symbol table with:

- Key = index term.
- Value = ordered set of pages on which term appears.