3.2 Binary Search Trees

- Trees
- BSTs
- ordered operations
- deletion

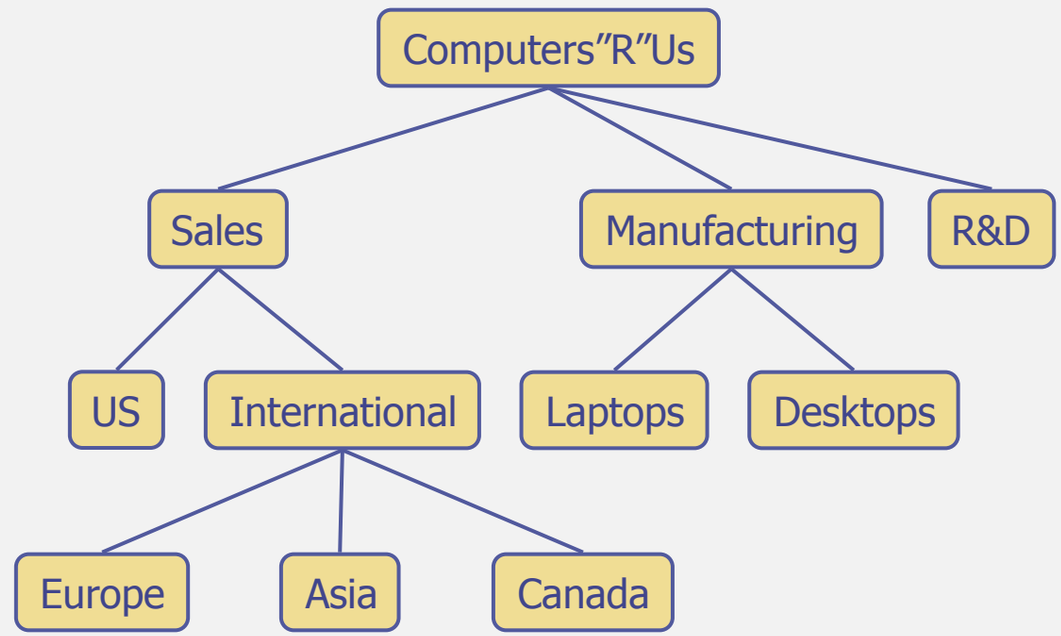
What is a tree?

In computer science, a tree is an
abstract model of a hierarchical structure

A tree consists of nodes with a parent-child relation

Applications:

- Organization charts
- File systems
- Programming environments



Tree terminology

Root:

- Node without parent (e.g., A)

Internal node:

- Node with at least one child (e.g., A, B, C, F)

External node:

- Node without children (e.g., E, I, J, K, G, H, D)
- Also known as **leaf node**

Ancestors of a node:

- Parent, grandparent, grand-grandparent, etc.

Descendant of a node:

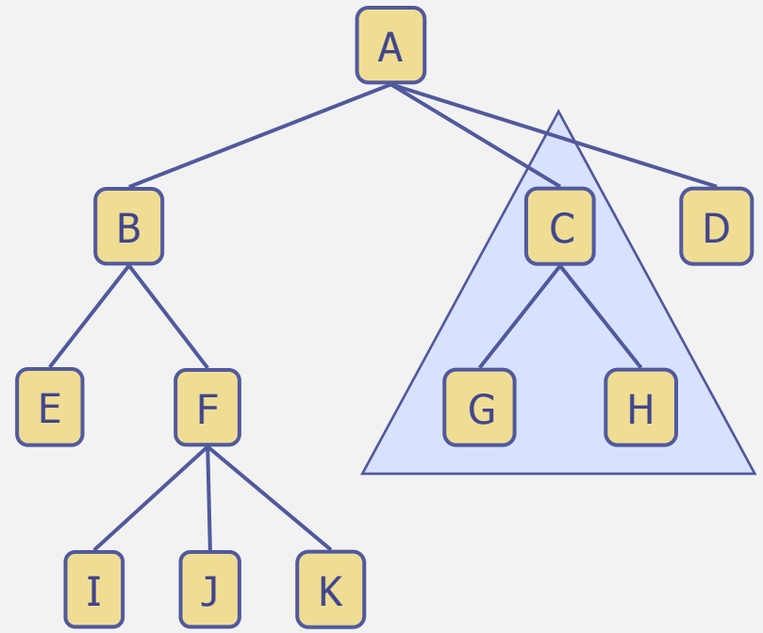
- Child, grandchild, grand-grandchild, etc.

Depth of a node:

- Number of ancestors

Height of a tree:

- Maximum depth of any node (e.g., 3)



Subtree:

- Tree consisting of a node and its descendants

Tree: abstract data type

We use positions to abstract nodes

Generic methods:

- integer size()
- boolean isEmpty()
- Iterator iterator()
- Iterable positions()

Accessor methods:

- position root()
- position parent(p)
- Iterable children(p)

Query methods:

- boolean isInternal(p)
- boolean isExternal(p)
- boolean isRoot(p)

Update method:

- element replace (p, o)

Additional update methods may be defined by data structures implementing the Tree ADT

Depth and height

Define depth recursively

- If v is the root, then the depth of v is 0
- Otherwise, the depth of v is one plus the depth of the parent of v.

Define height recursively

- If v is an external node, then the height of v is 0
- Otherwise, the height of v is one plus the maximum height of a child of v.

Preorder traversal

A traversal visits the nodes of a tree in a systematic manner

In a **preorder** traversal, a node is visited **before** its descendants

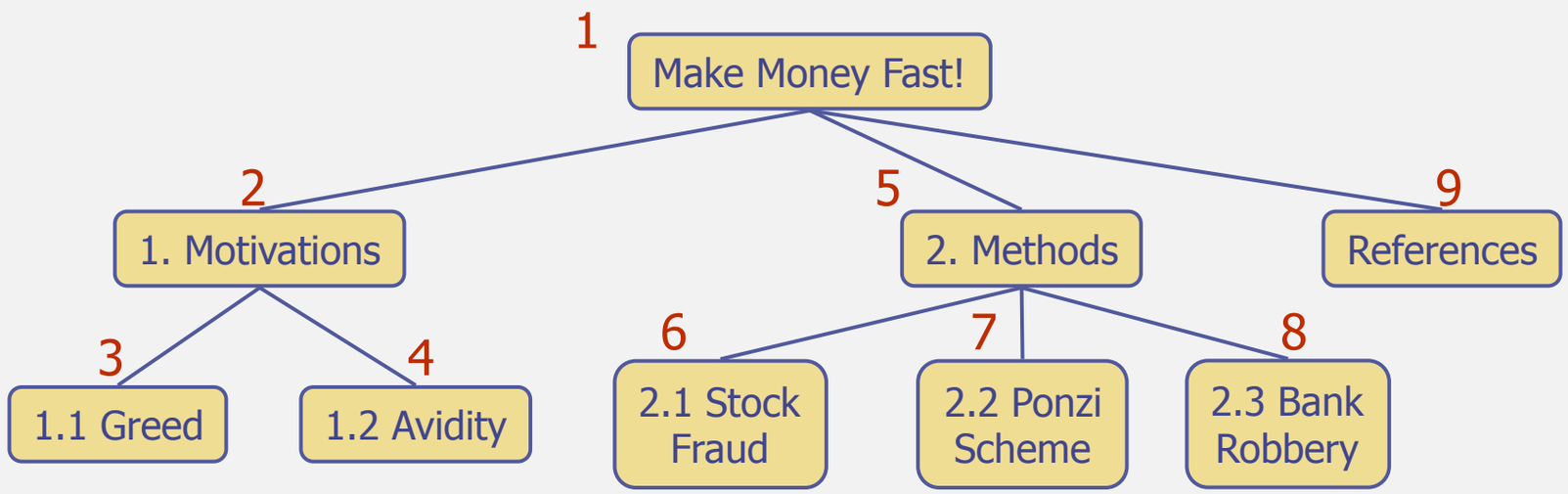
- Application: print a structured document

Algorithm preOrder(v)

visit(v)

for each child w of v

preorder (w)



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Algorithm preOrder(v)
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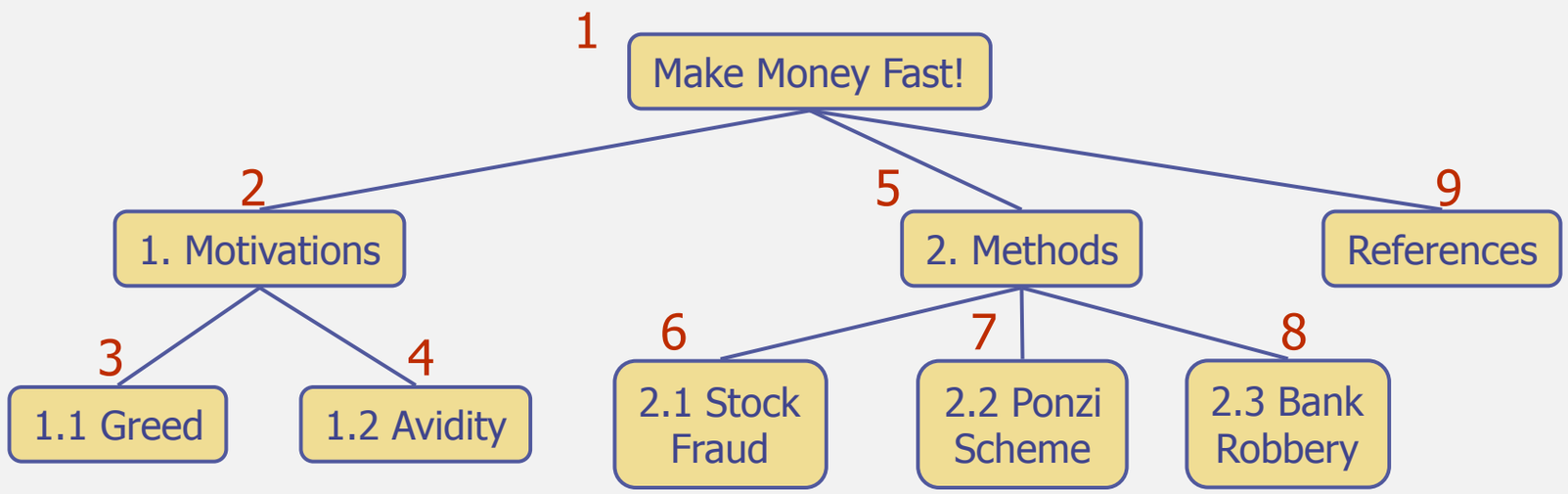
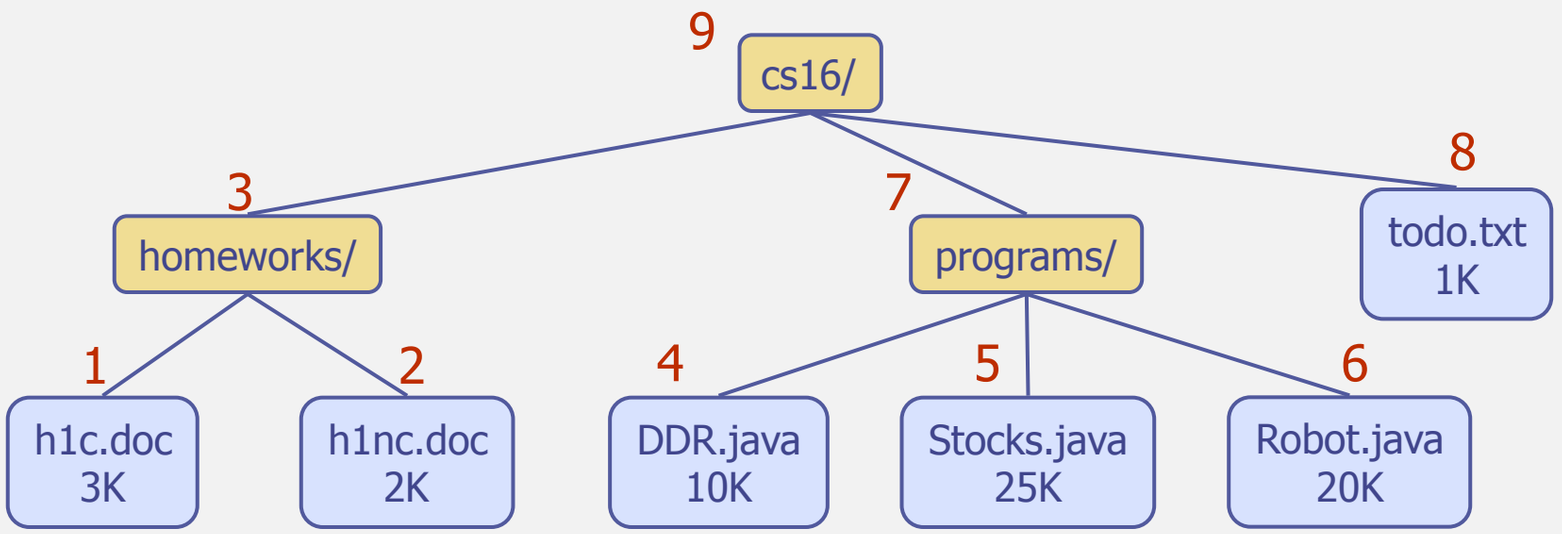


Postorder traversal

In a **postorder** traversal, a node is visited **after** its descendants

- Application: compute space used by files in a directory and its subdirectories

Algorithm postOrder(v)
    for each child w of v
        postOrder (w)
    visit(v)



Postorder traversal

In a **postorder** traversal, a node is visited **after** its descendants

- Application: compute space used by files in a directory and its subdirectories

```
Algorithm postOrder(v)
    for each child w of v
            postOrder (w)
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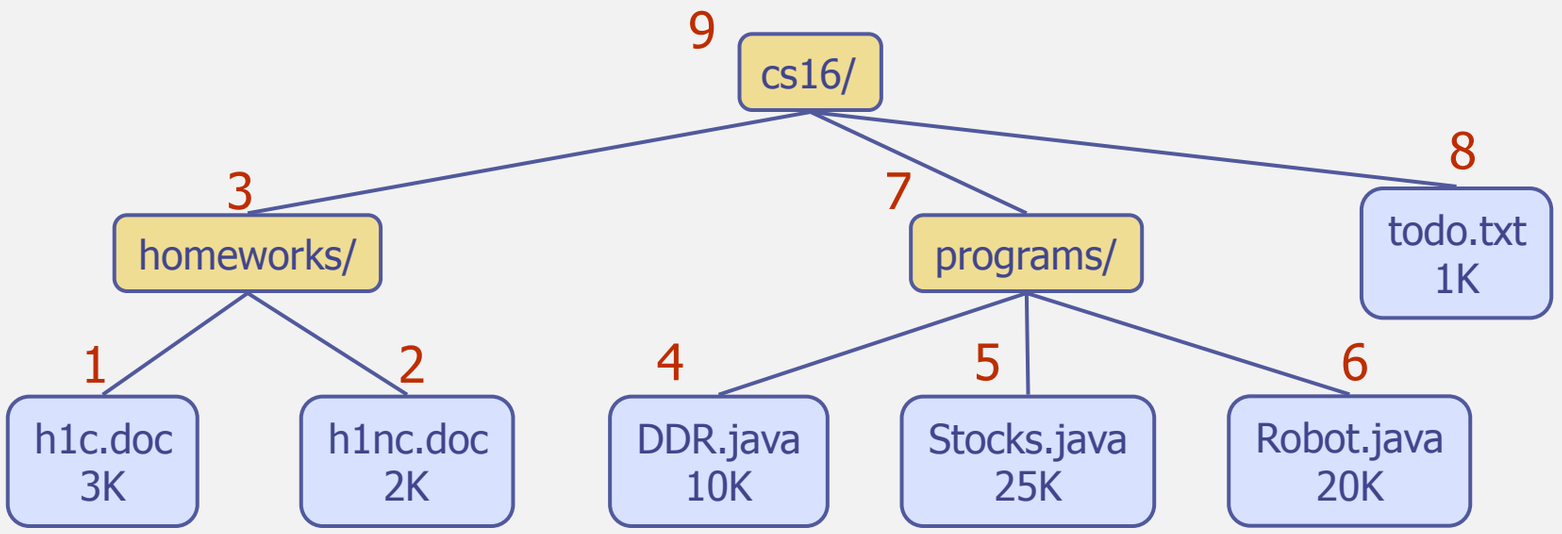


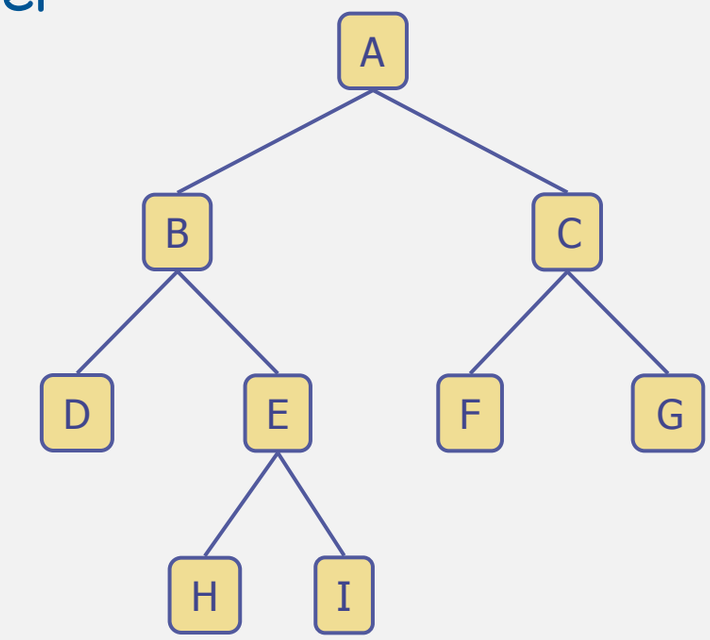
Binary trees

A binary tree is a tree with the following properties:

- Each internal node has at most two children (exactly two for proper binary trees)
- The children of a node are an ordered pair
- We call the children of an internal node left child and right child

Alternative recursive definition: a binary tree is either

- Tree consisting of a single node, or
- Tree whose root has an ordered pair of children, each of which is a binary tree



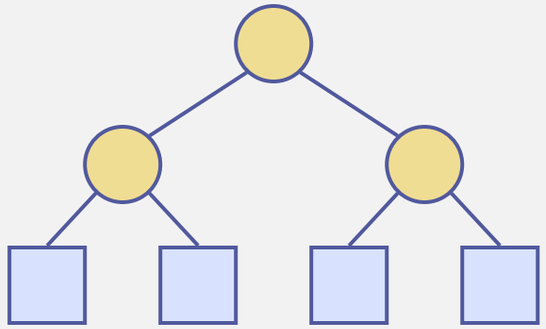
Applications:

- Arithmetic expressions
- Decision processes
- Searching

Properties of binary trees

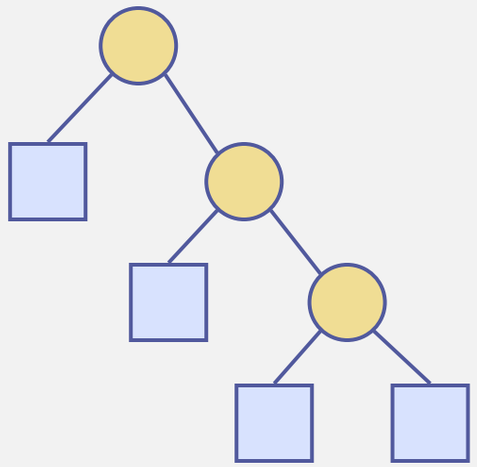
Notation

- n: number of nodes
- e: number of external nodes
- i: number of internal nodes
- h: height



Properties:

- e = i + 1 ← proof by induction
- n = 2e - 1
- h ≤ i
- h ≤ (n - 1)/2
- $e \leq 2^h$
- $h \geq \log_2 e$
- $h \geq \log_2(n + 1) - 1$



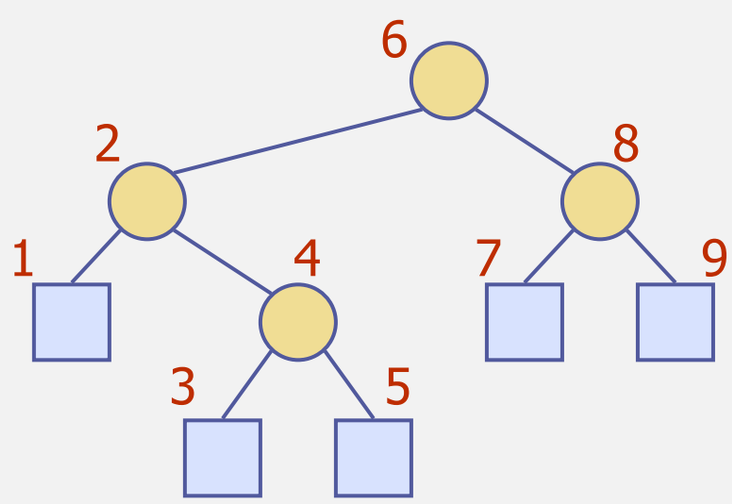
Inorder traversal

In an **inorder** traversal a node is visited **after** its left subtree and **before** its right subtree

- Application: draw a binary tree
  - x(v) = inorder rank of v
  - y(v) = depth of v

Algorithm inOrder(v)

```
    if hasLeft (v)
        inOrder (left (v))
    visit(v)
    if hasRight (v)
        inOrder (right (v))
```



Inorder traversal

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- Application: draw a binary tree
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```
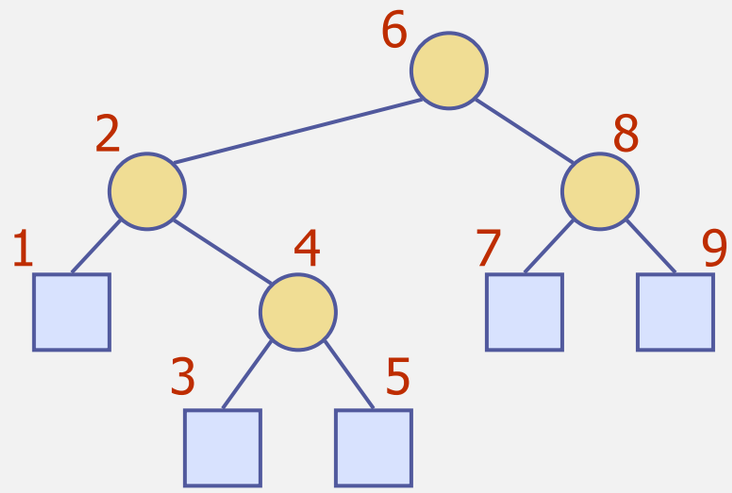
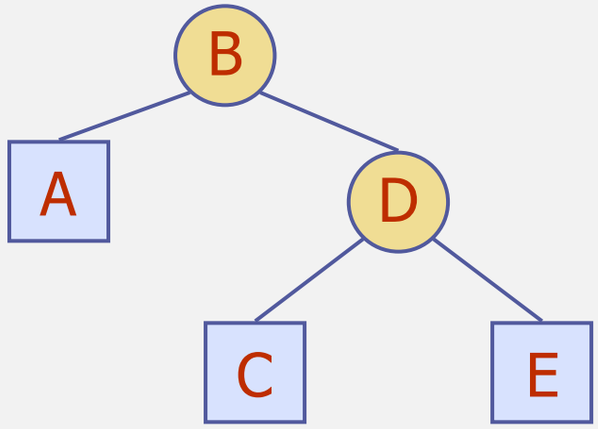



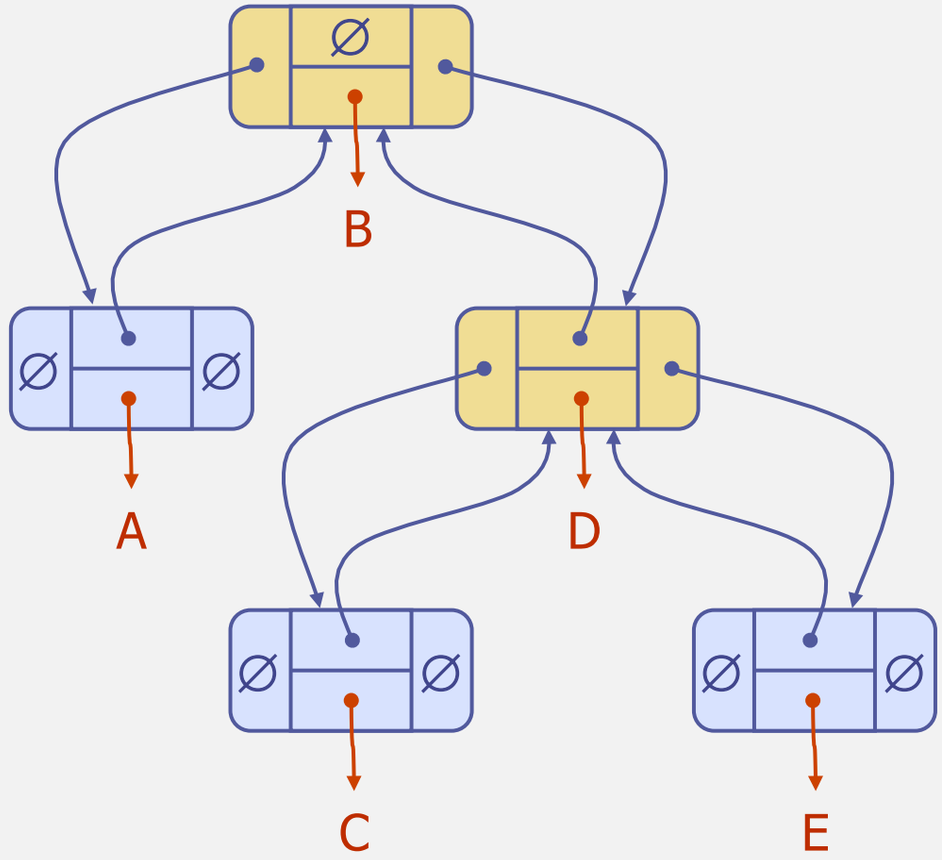
Linked structure for binary trees

A node is represented by an object storing

- Element
- Parent node
- Left child node
- Right child node

Node objects implement the Position ADT



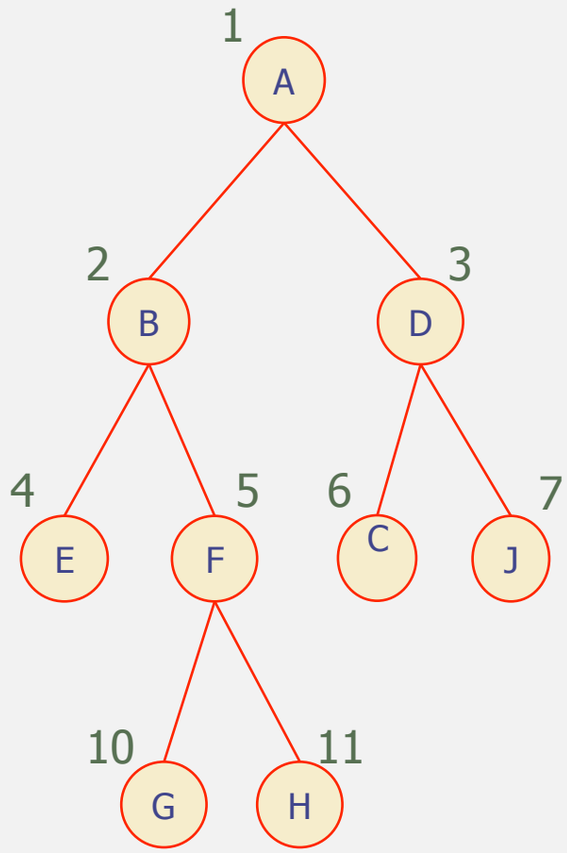


Array-based representation of binary trees

Nodes are stored in an array *A*

Node v is stored at *A*[rank(v)]

- rank(root) = 1
- if node is the left child of parent(node)
  rank(node) = 2 · rank(parent(node))
- if node is the right child of parent(node)
  rank(node) = 2 · rank(parent(node)) + 1



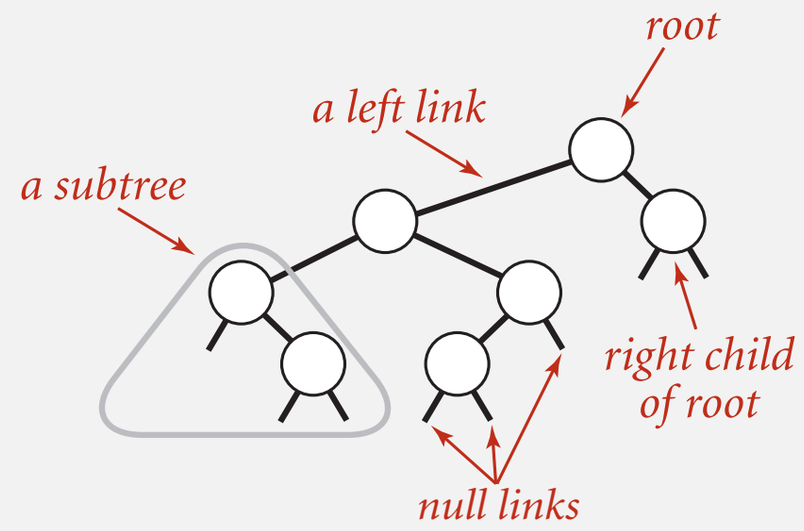
- Trees
- **BSTs**
- ordered operations
- deletion

Binary search trees

Definition. A BST is a binary tree in symmetric order.

A binary tree is either:

- Empty.
- Two disjoint binary trees (left and right).

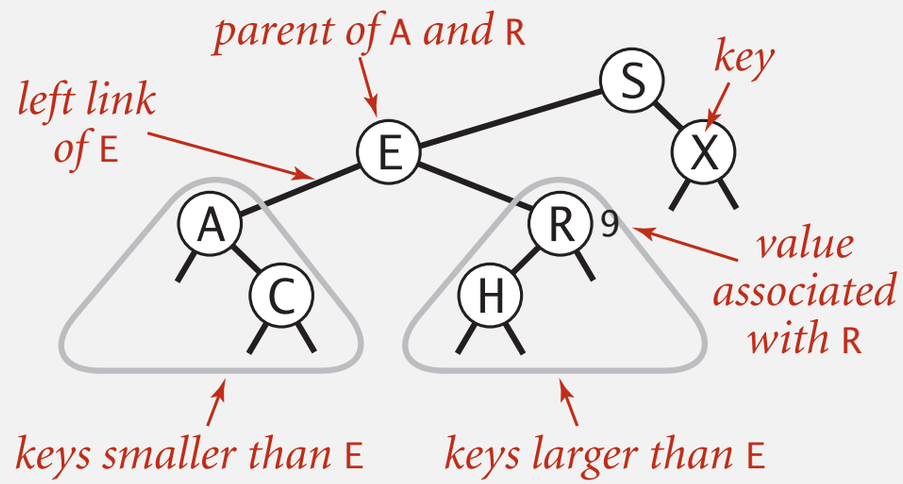


Anatomy of a binary tree

Symmetric order.

Each node has a key, and every node's key is:

- Larger than all keys in its left subtree.
- Smaller than all keys in its right subtree.



Anatomy of a binary search tree

BST representation in Java

Java definition. A BST is a reference to a root `Node`.

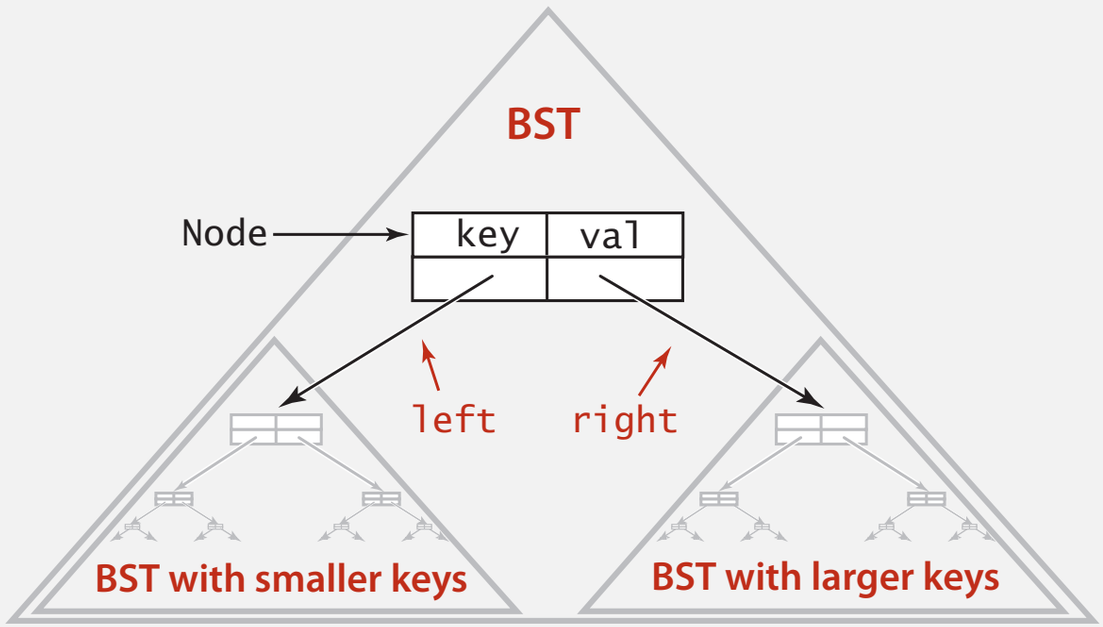
A `Node` is comprised of four fields:

- A `Key` and a `Value`.
- A reference to the left and right subtree.

smaller keys larger keys

```
private class Node
{
   private Key key;
   private Value val;
   private Node left, right;
   public Node(Key key, Value val)
   {
      this.key = key;
      this.val = val;
   }
}
```

`Key` and `Value` are generic types; `Key` is `Comparable`



Binary search tree

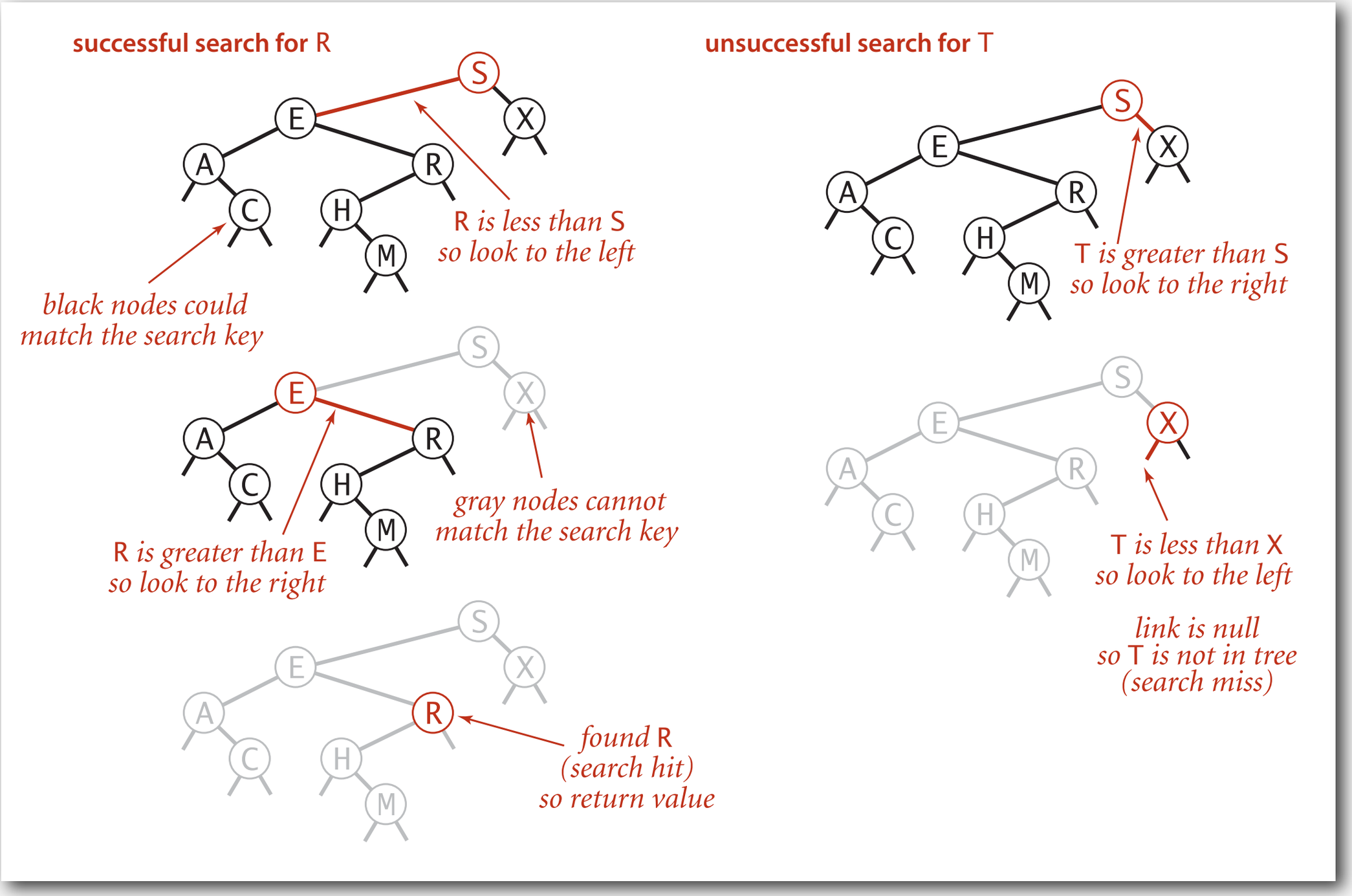
BST implementation (skeleton)

```
public class BST<Key extends Comparable<Key>, Value>
{
    private Node root;

   private class Node
   {  /* see previous slide */  }

   public void put(Key key, Value val)
   {  /* see next slides */  }

   public Value get(Key key)
   {  /* see next slides */  }

   public void delete(Key key)
   {  /* see next slides */  }

   public Iterable<Key> iterator()
   {  /* see next slides */  }

}
```

root of BST

BST search

Get. Return value corresponding to given key, or `null` if no such key.



BST search: Java implementation

Get. Return value corresponding to given key, or `null` if no such key.

```
public Value get(Key key)
{
   Node x = root;
   while (x != null)
   {
      int cmp = key.compareTo(x.key);
      if      (cmp  < 0) x = x.left;
      else if (cmp  > 0) x = x.right;
      else if (cmp == 0) return x.val;
   }
   return null;
}
```

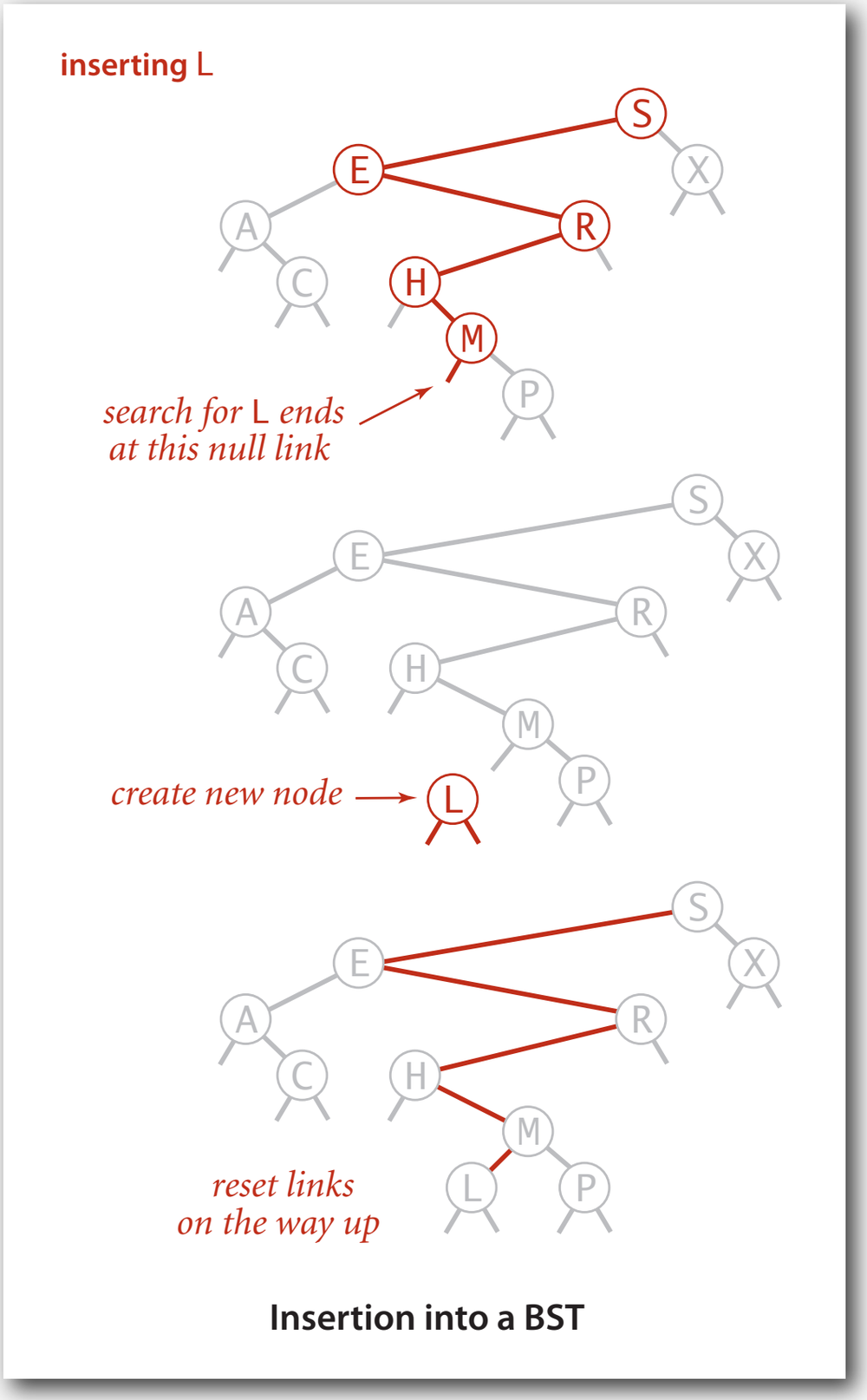
Cost. Number of compares is equal to depth of node.

BST insert

Put. Associate value with key.

Search for key, then two cases:

- Key in tree ⇒ reset value.
- Key not in tree ⇒ add new node.



Insertion into a BST

BST insert: Java implementation

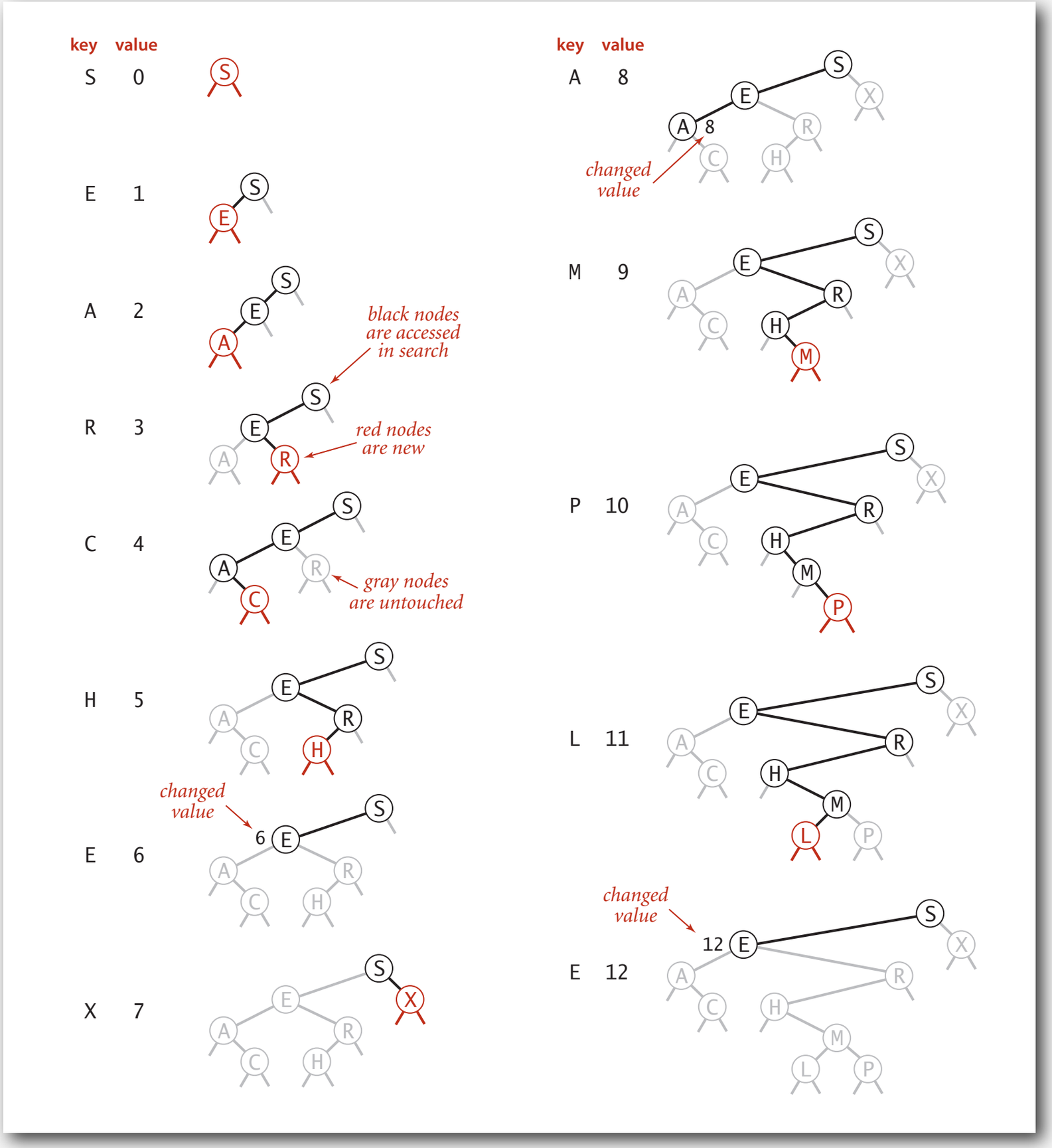
Put. Associate value with key.

```
public void put(Key key, Value val)
{  root = put(root, key, val);  }

private Node put(Node x, Key key, Value val)
{
   if (x == null) return new Node(key, val);
   int cmp = key.compareTo(x.key);
   if      (cmp  < 0)
      x.left  = put(x.left,  key, val);
   else if (cmp  > 0)
      x.right = put(x.right, key, val);
   else if (cmp == 0)
      x.val = val;
   return x;
}
```

concise, but tricky, recursive code; read carefully!

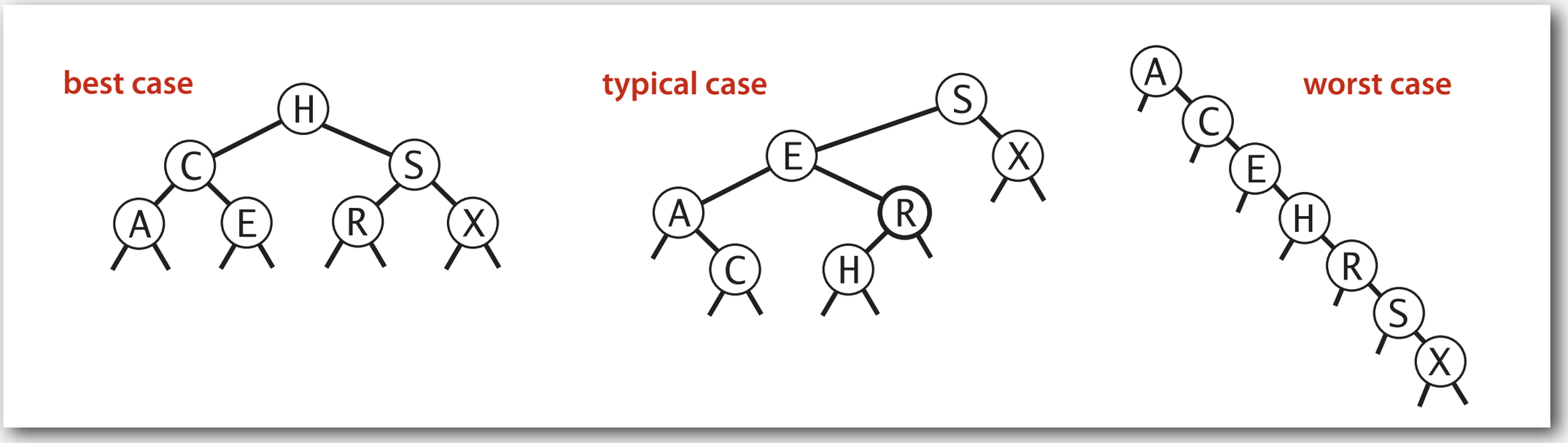
Cost. Number of compares is equal to depth of node.

BST trace: standard indexing client



Tree shape

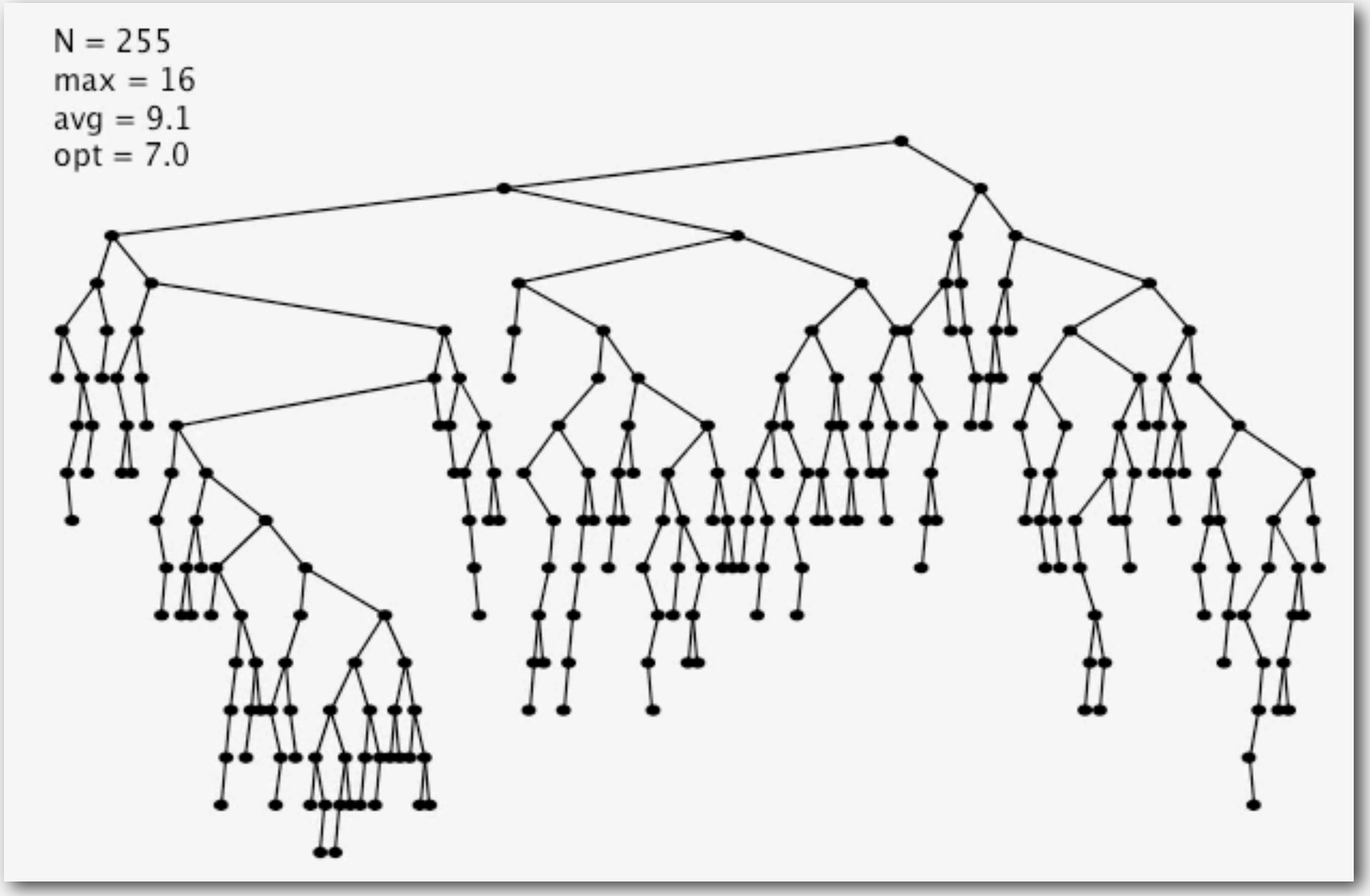
- Many BSTs correspond to same set of keys.
- Number of compares for search/insert is equal to depth of node.



Remark. Tree shape depends on order of insertion.

BST insertion: random order

Observation. If keys inserted in random order, tree stays relatively flat.



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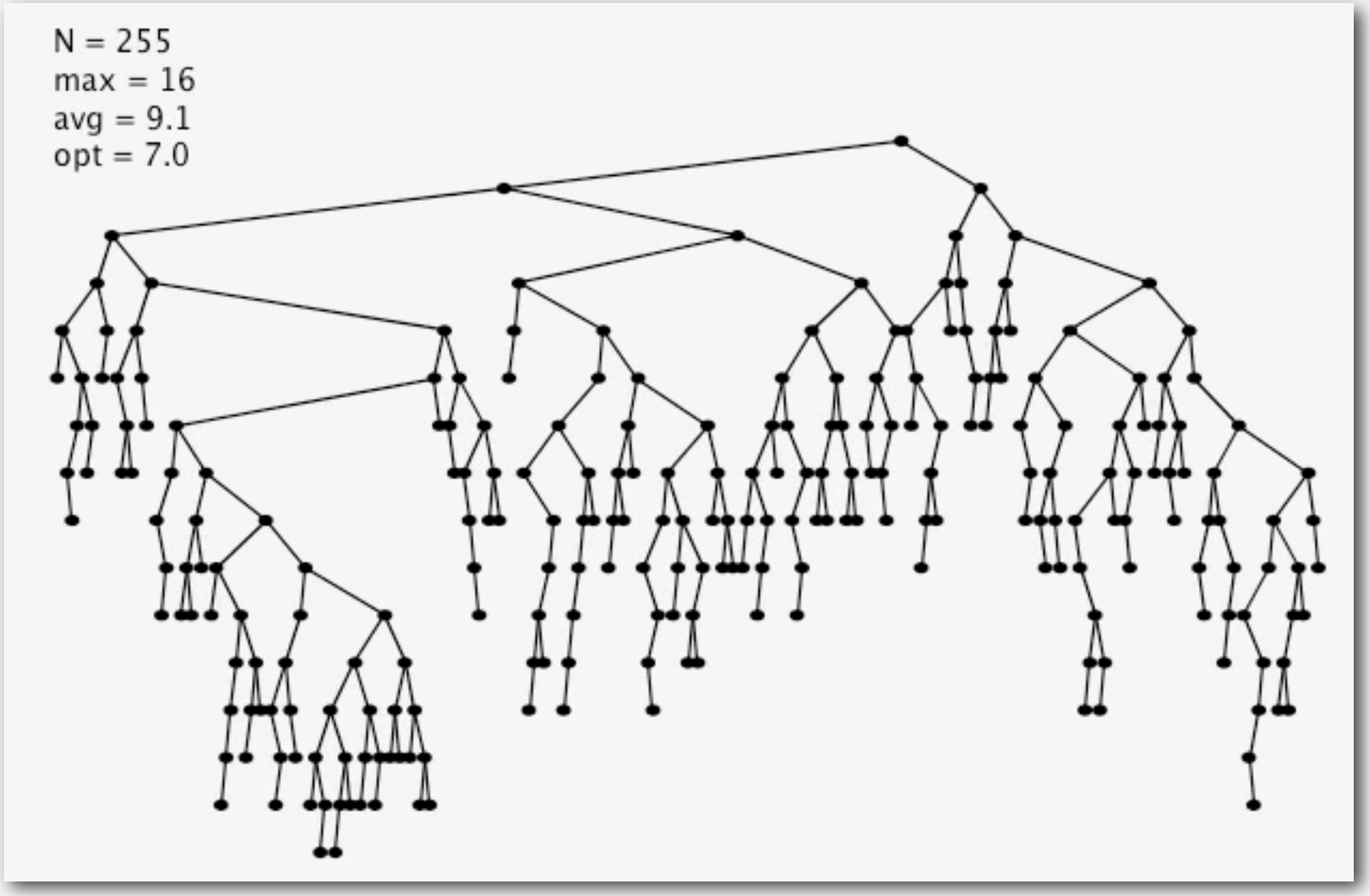

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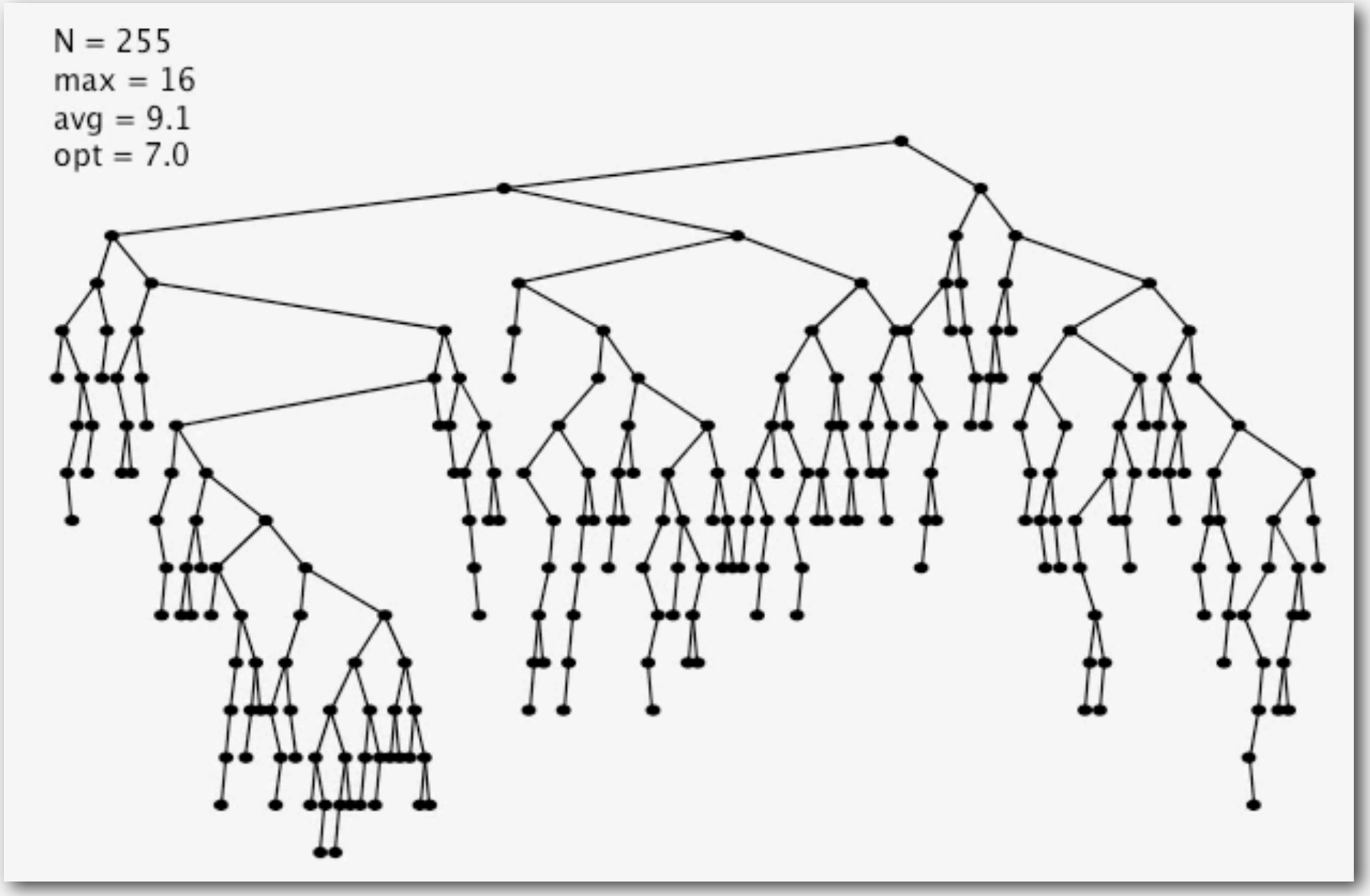
BST insertion: random order visualization

Ex. Insert keys in random order.



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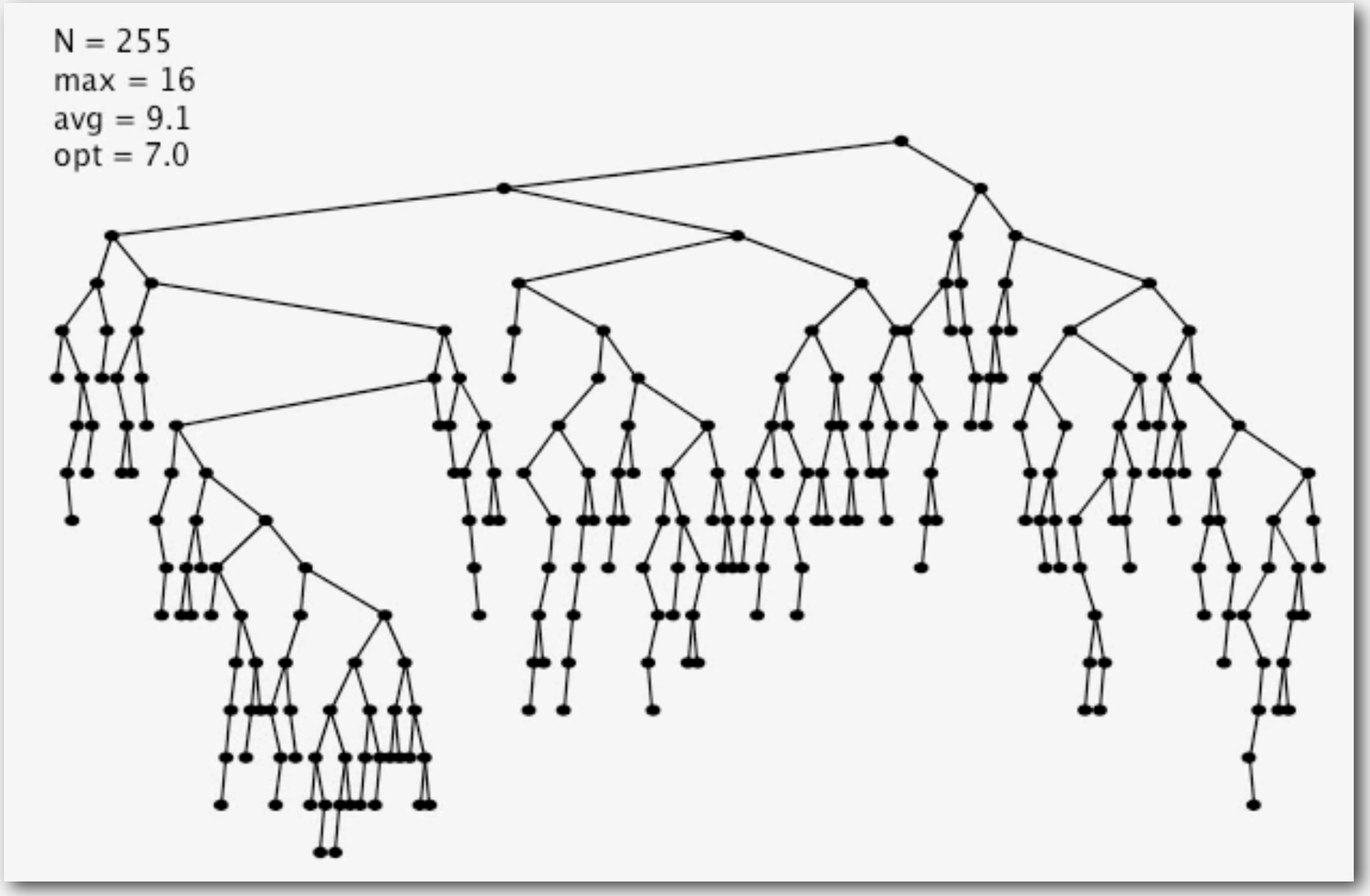

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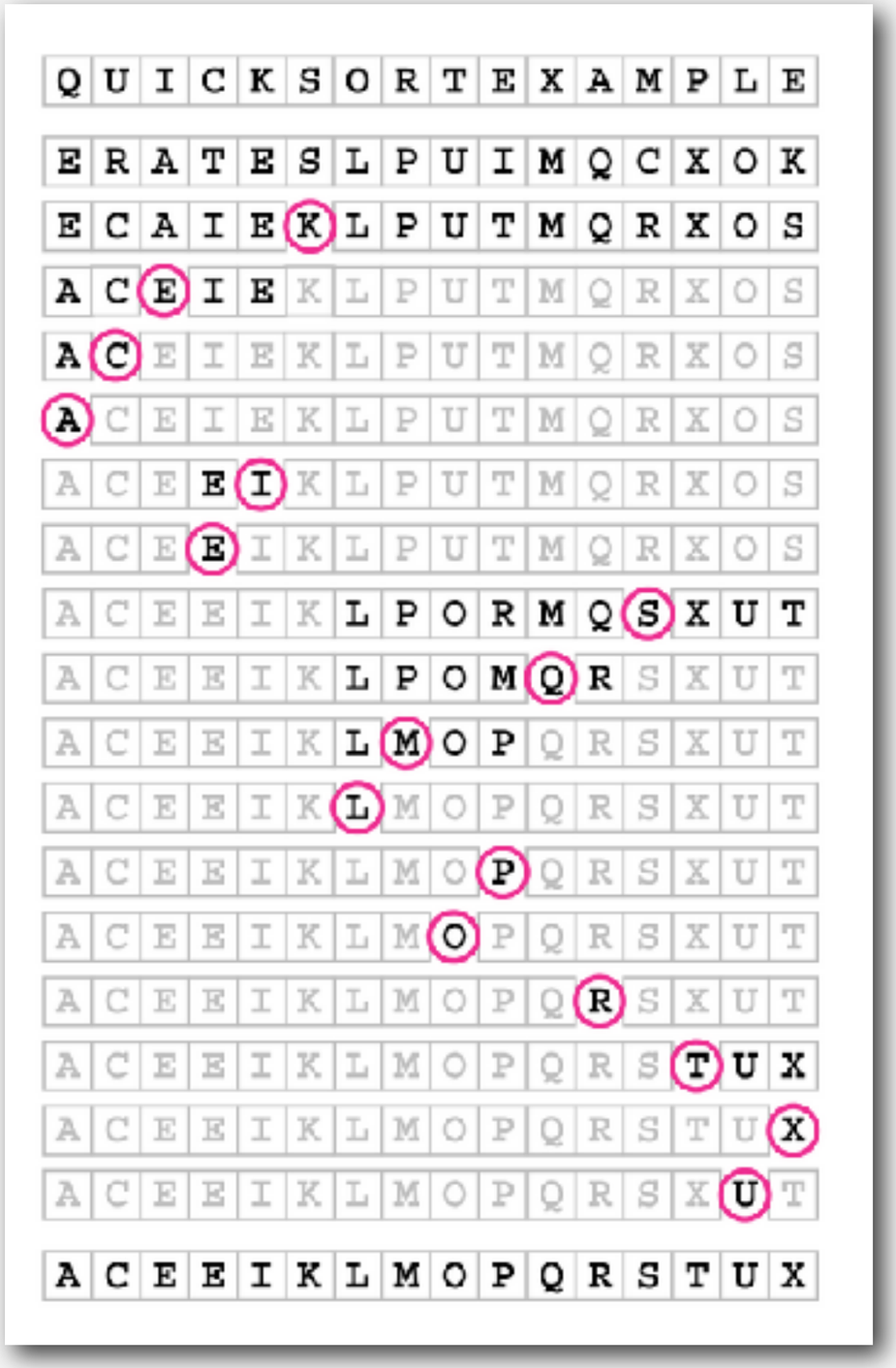
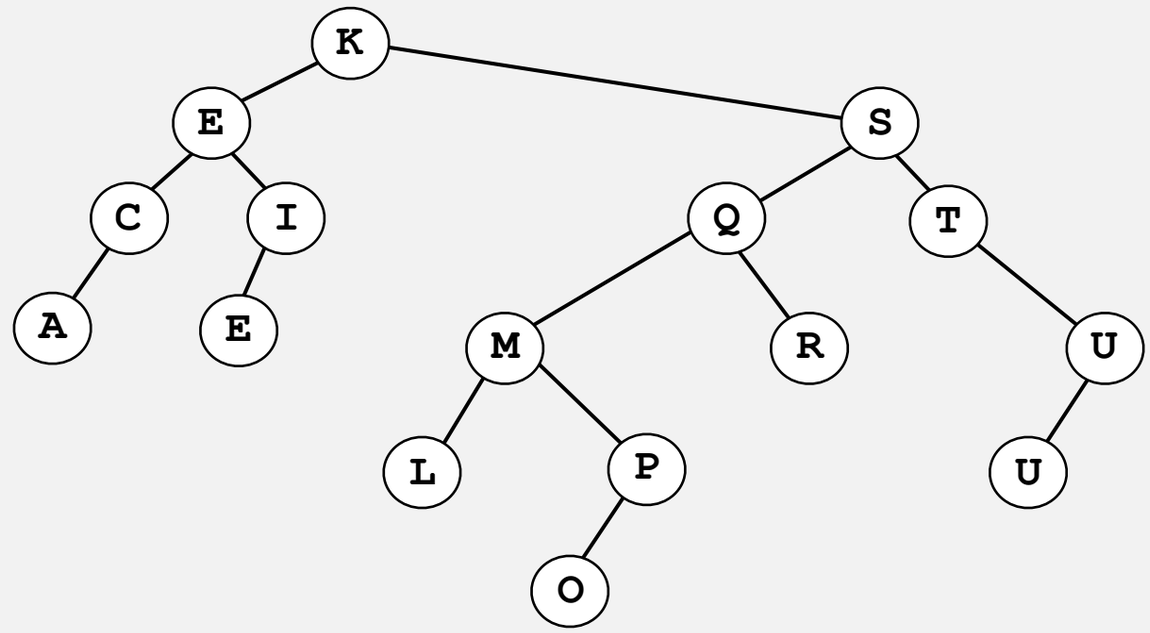
Correspondence between BSTs and quicksort partitioning





Remark. Correspondence is 1-1 if array has no duplicate keys.

BSTs: mathematical analysis

Proposition. If keys are inserted in random order, the expected number of compares for a search/insert is $\sim 2 \ln N$.

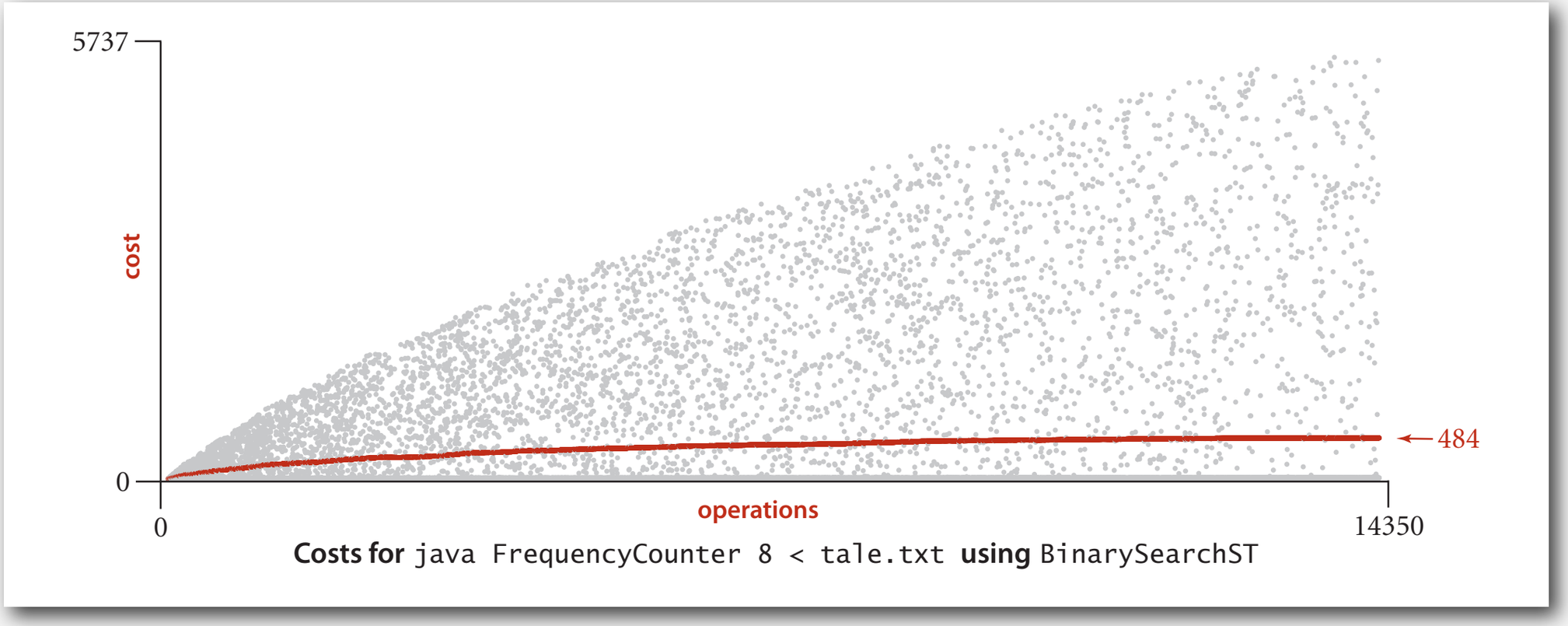
Pf. 1-1 correspondence with quicksort partitioning.

Proposition. [Reed, 2003] If keys are inserted in random order, expected height of tree is $\sim 4.311 \ln N$.

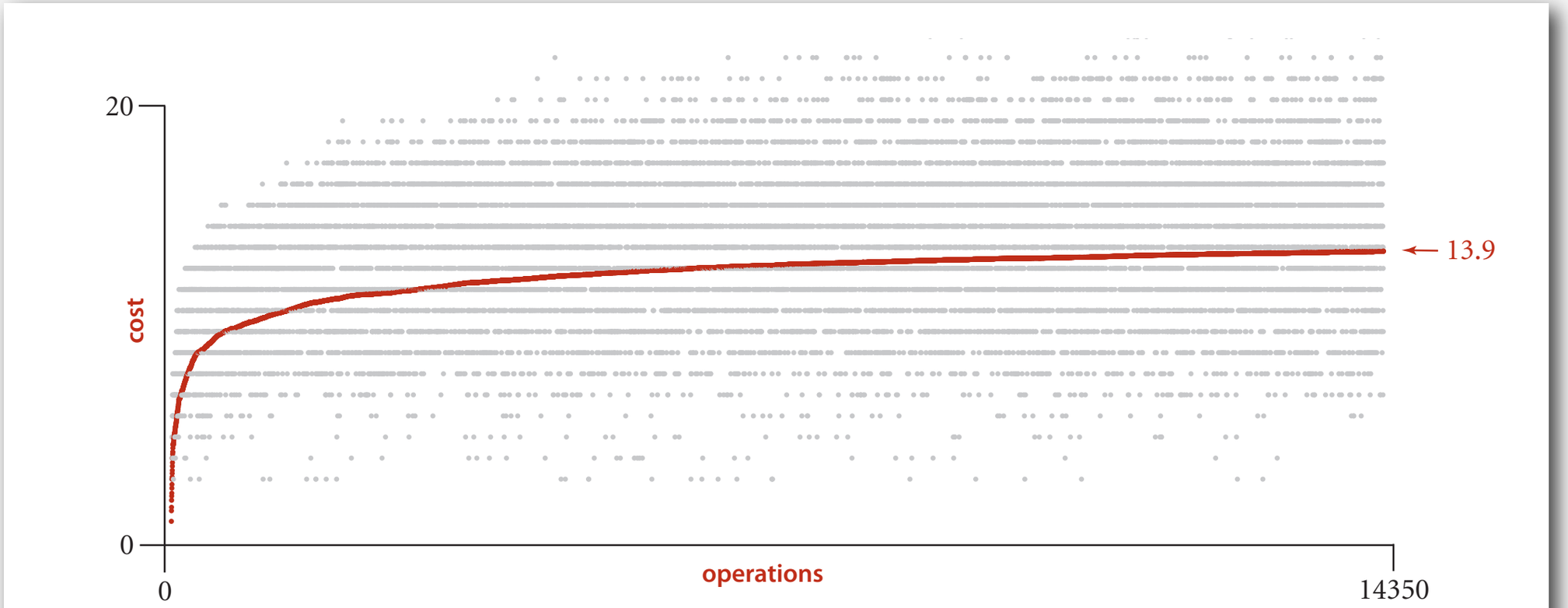
But... Worst-case height is N.

(exponentially small chance when keys are inserted in random order)

ST implementations: frequency counter



Costs for `java FrequencyCounter 8 < tale.txt` **using** `BinarySearchST`



Costs for `java FrequencyCounter 8 < tale.txt` **using** `BST`

ST implementations: summary

implementation	guarantee		average case		ordered ops?	operations on keys
	search	insert	search hit	insert		
sequential search (unordered list)	N	N	N/2	N	no	`equals()`
binary search (ordered array)	lg N	N	lg N	N/2	yes	`compareTo()`
BST	N	N	1.39 lg N	1.39 lg N	?	`compareTo()`

- Trees
- BSTs
- **ordered operations**
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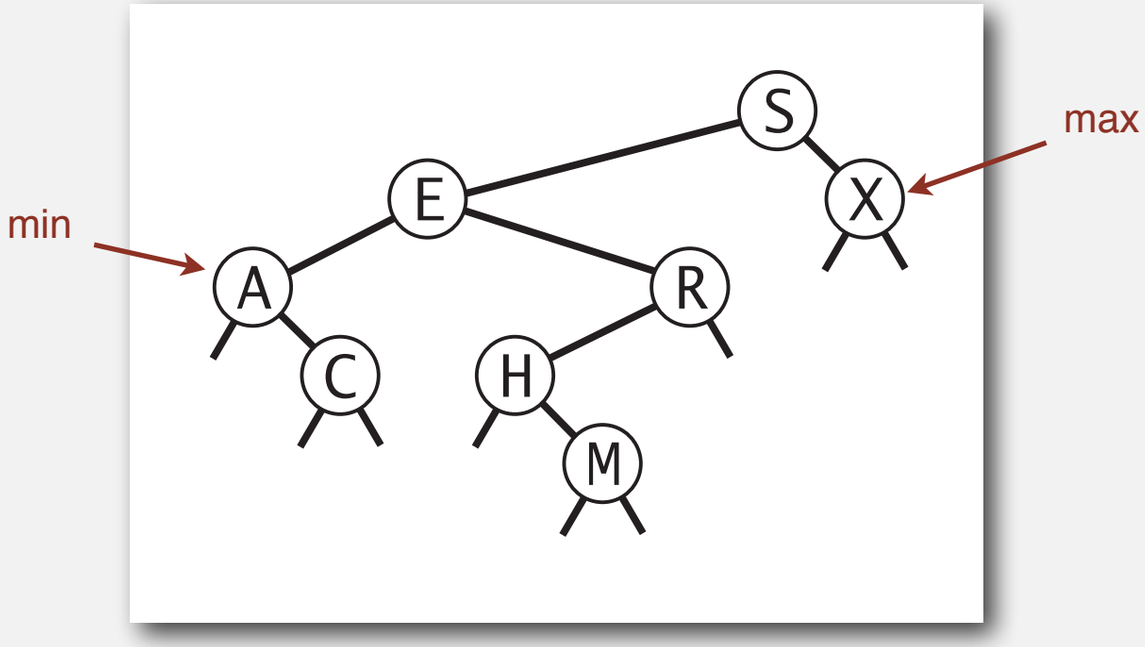
Minimum and maximum

Minimum. Smallest key in table.

Maximum. Largest key in table.

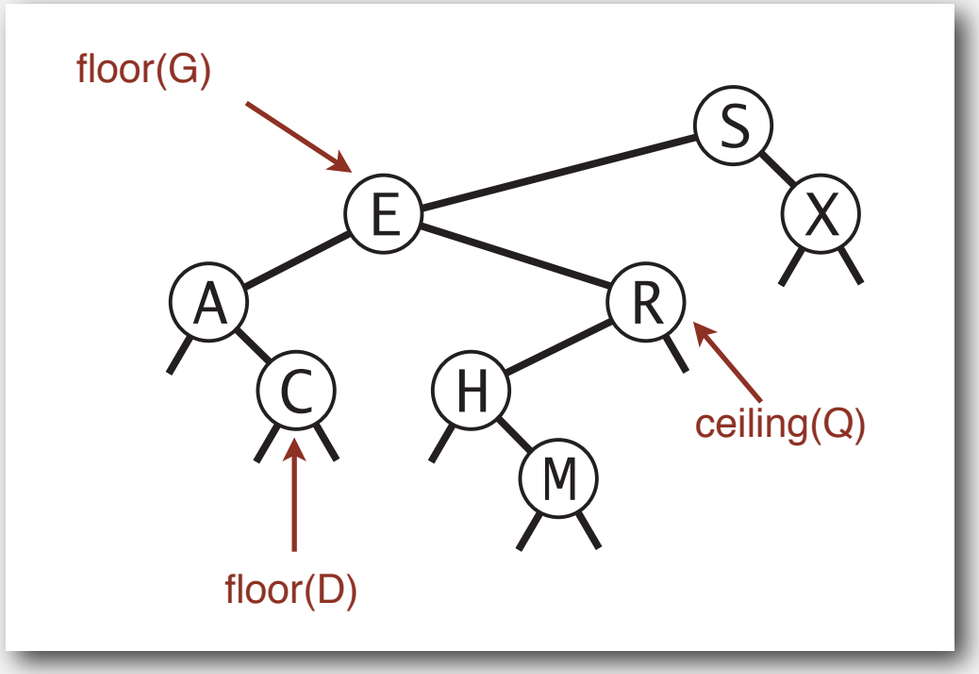


Q. How to find the min / max?

Floor and ceiling

Floor. Largest key ≤ to a given key.

Ceiling. Smallest key ≥ to a given key.



Q. How to find the floor /ceiling?

Computing the floor

Case 1. [k equals the key at root]

The floor of k is k.

Case 2. [k is less than the key at root]

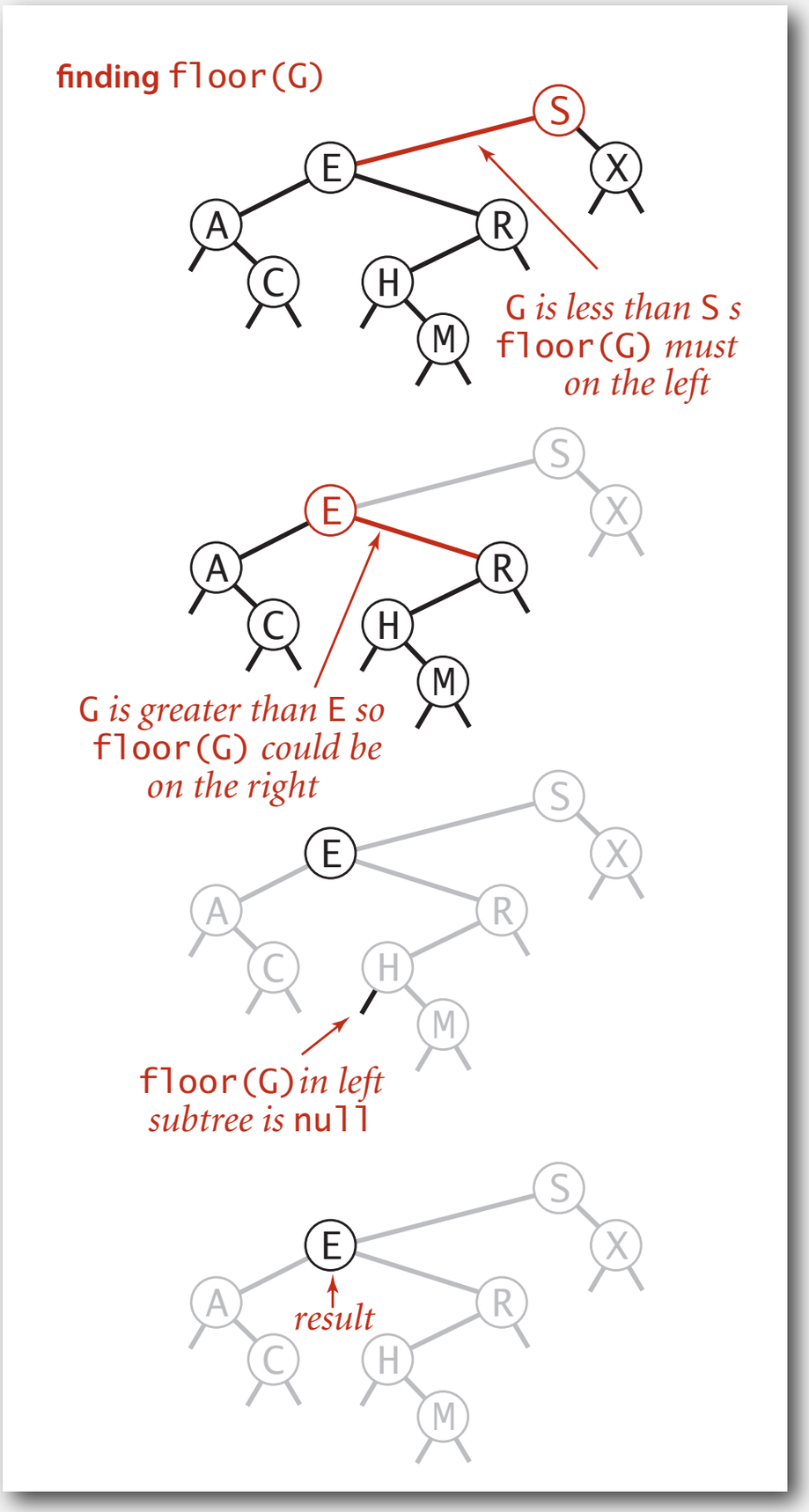
The floor of k is in the left subtree.

Case 3. [k is greater than the key at root]

The floor of k is in the right subtree

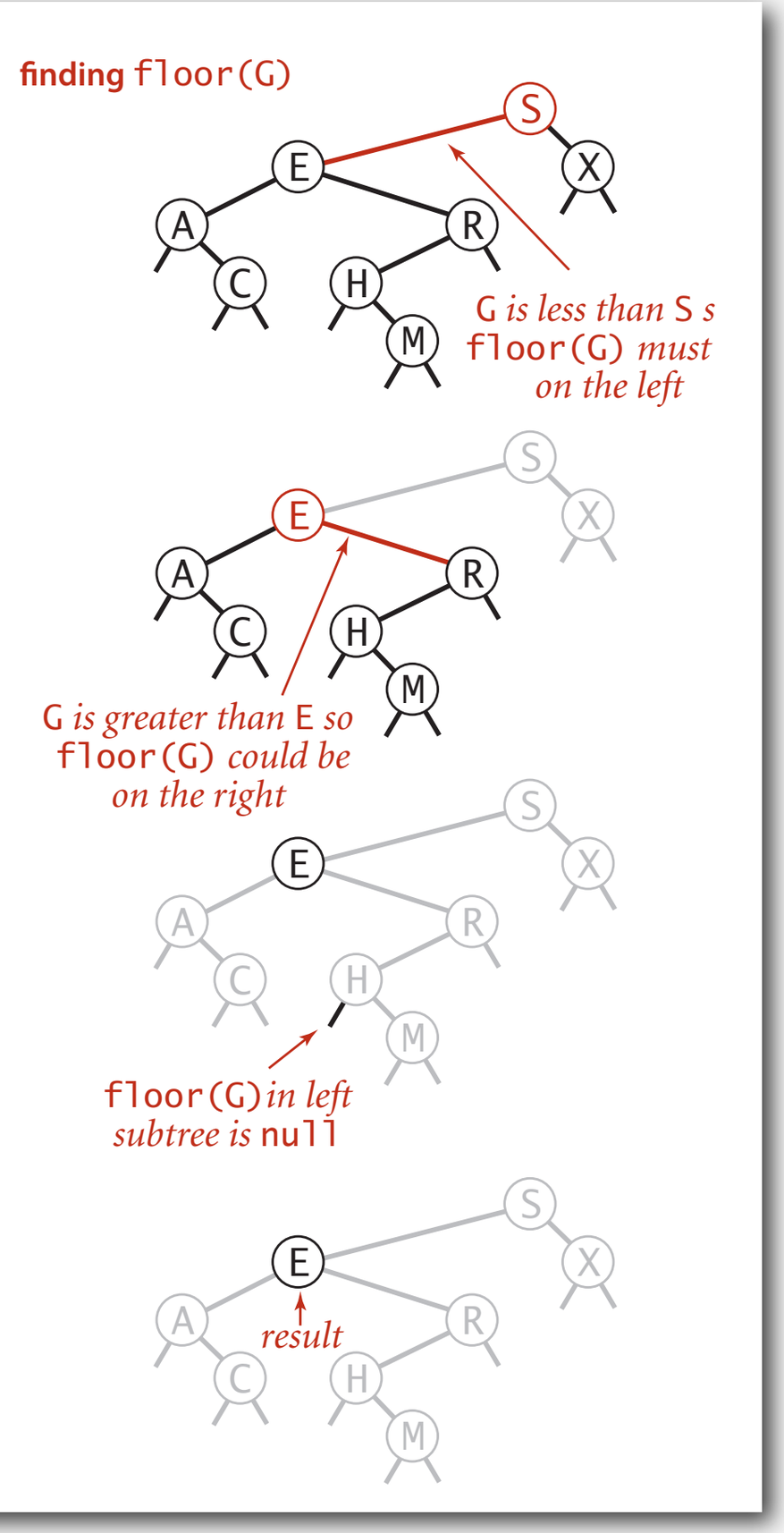
(if there is any key $\leq k$ in right subtree);

otherwise it is the key in the root.



Computing the floor

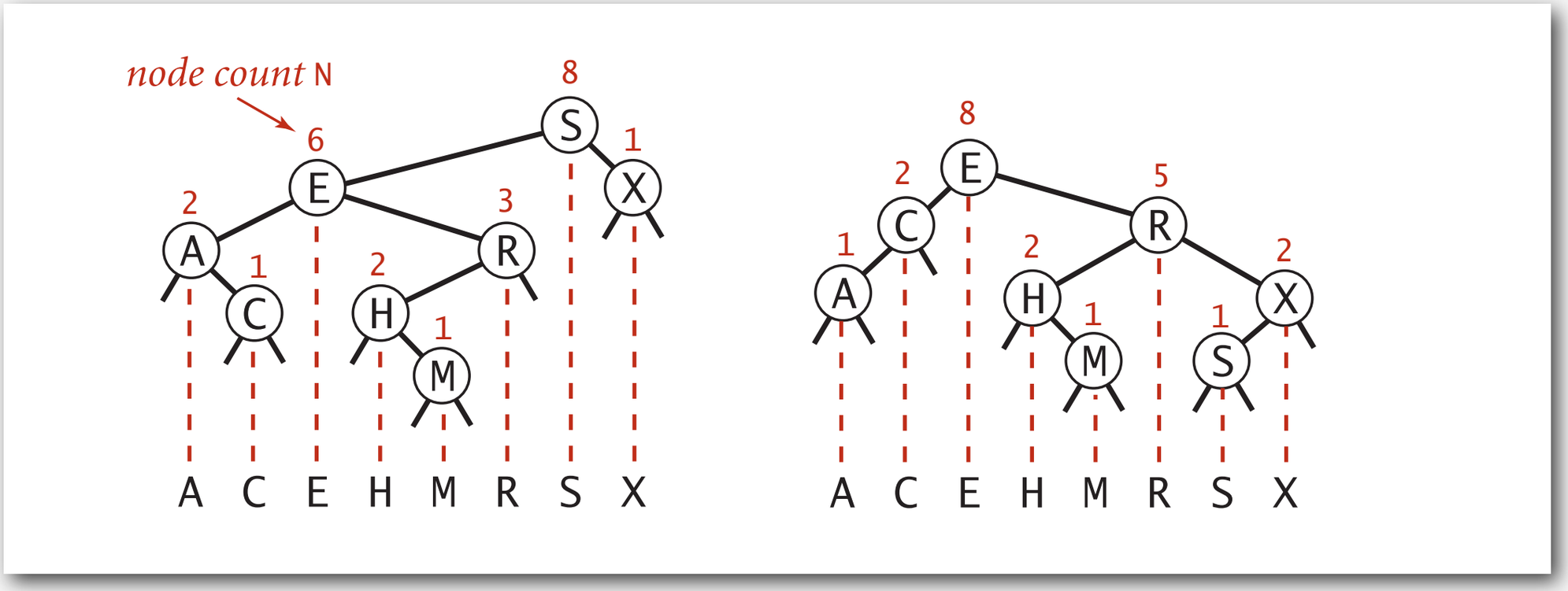
```
public Key floor(Key key)
{
   Node x = floor(root, key);
   if (x == null) return null;
   return x.key;
}
private Node floor(Node x, Key key)
{
   if (x == null) return null;
   int cmp = key.compareTo(x.key);

   if (cmp == 0) return x;

   if (cmp < 0)  return floor(x.left, key);

   Node t = floor(x.right, key);
   if (t != null) return t;
   else           return x;

}
```



Subtree counts

In each node, we store the number of nodes in the subtree rooted at that node.

To implement `size()`, return the count at the root.



Remark. This facilitates efficient implementation of `rank()` and `select()`.

BST implementation: subtree counts

```
private class Node
{
   private Key key;
   private Value val;
   private Node left;
   private Node right;
   private int N;
}
```

nodes in subtree

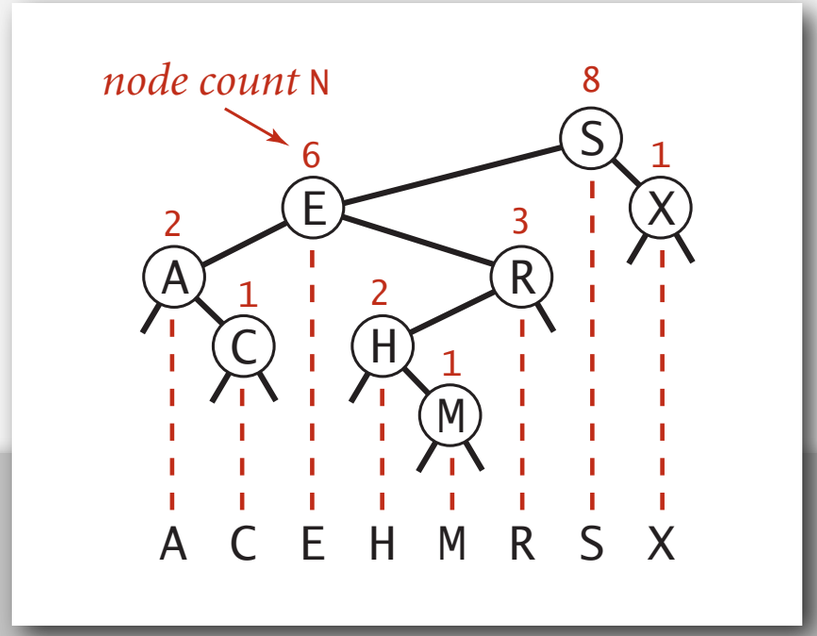
```
public int size()
{  return size(root);  }

private int size(Node x)
{
   if (x == null) return 0;
   return x.N;
}
```

```
private Node put(Node x, Key key, Value val)
{
   if (x == null) return new Node(key, val);
   int cmp = key.compareTo(x.key);
   if      (cmp  < 0) x.left  = put(x.left,  key, val);
   else if (cmp  > 0) x.right = put(x.right, key, val);
   else if (cmp == 0) x.val = val;
   x.N = 1 + size(x.left) + size(x.right);
   return x;
}
```

Rank

Rank. How many keys < k ?

Easy recursive algorithm (4 cases!)

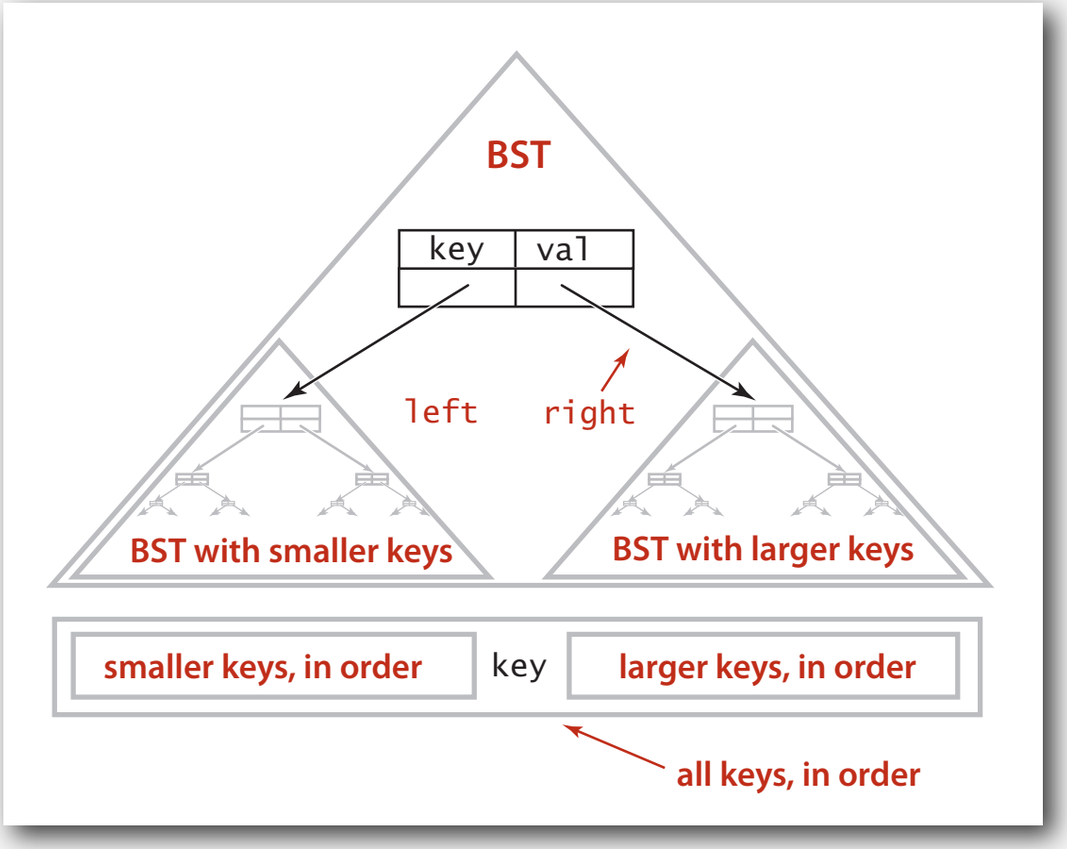



```
public int rank(Key key)
{  return rank(key, root);  }

private int rank(Key key, Node x)
{
   if (x == null) return 0;
   int cmp = key.compareTo(x.key);
   if      (cmp  < 0) return rank(key, x.left);
   else if (cmp  > 0) return 1 + size(x.left) + rank(key, x.right);
   else if (cmp == 0) return size(x.left);
}
```

Inorder traversal

- Traverse left subtree.
- Enqueue key.
- Traverse right subtree.

```
public Iterable<Key> keys()
{
    Queue<Key> q = new Queue<Key>();
    inorder(root, q);
    return q;
}

private void inorder(Node x, Queue<Key> q)
{
   if (x == null) return;
   inorder(x.left, q);
   q.enqueue(x.key);
   inorder(x.right, q);
}
```



Property. Inorder traversal of a BST yields keys in ascending order.

Inorder traversal

- Traverse left subtree.
- Enqueue key.
- Traverse right subtree.

```
inorder(S)
  inorder(E)
    inorder(A)
      enqueue A
      inorder(C)
        enqueue C
    enqueue E
    inorder(R)
      inorder(H)
        enqueue H
        inorder(M)
          enqueue M
      print R
enqueue S
inorder(X)
  enqueue X
```

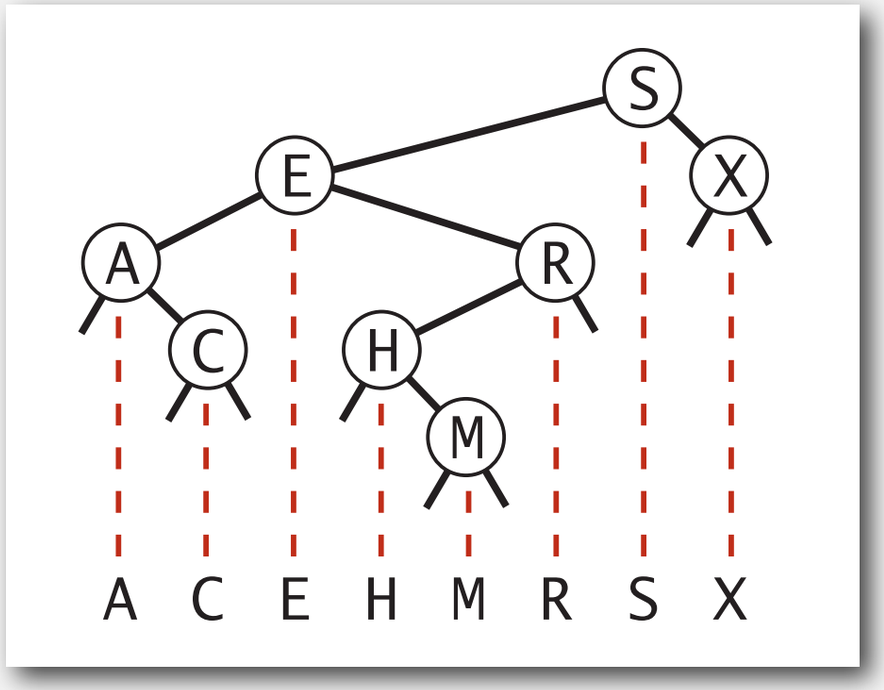
recursive calls

```
A
C
E
H
M
R
S
X
```

queue

```
S
S E
S E A

S E A C

S E R
S E R H

S E R H M

S X
```

function call stack



BST: ordered symbol table operations summary

	sequential search	binary search	BST
search	N	lg N	h
insert	1	N	h
min / max	N	1	h
floor / ceiling	N	lg N	h
rank	N	lg N	h
select	N	1	h
ordered iteration	N log N	N	N

h = height of BST
(proportional to log N
if keys inserted in random order)

worst-case running time of ordered symbol table operations

- Trees
  - BSTs
- ordered operations
- **deletion**

ST implementations: summary

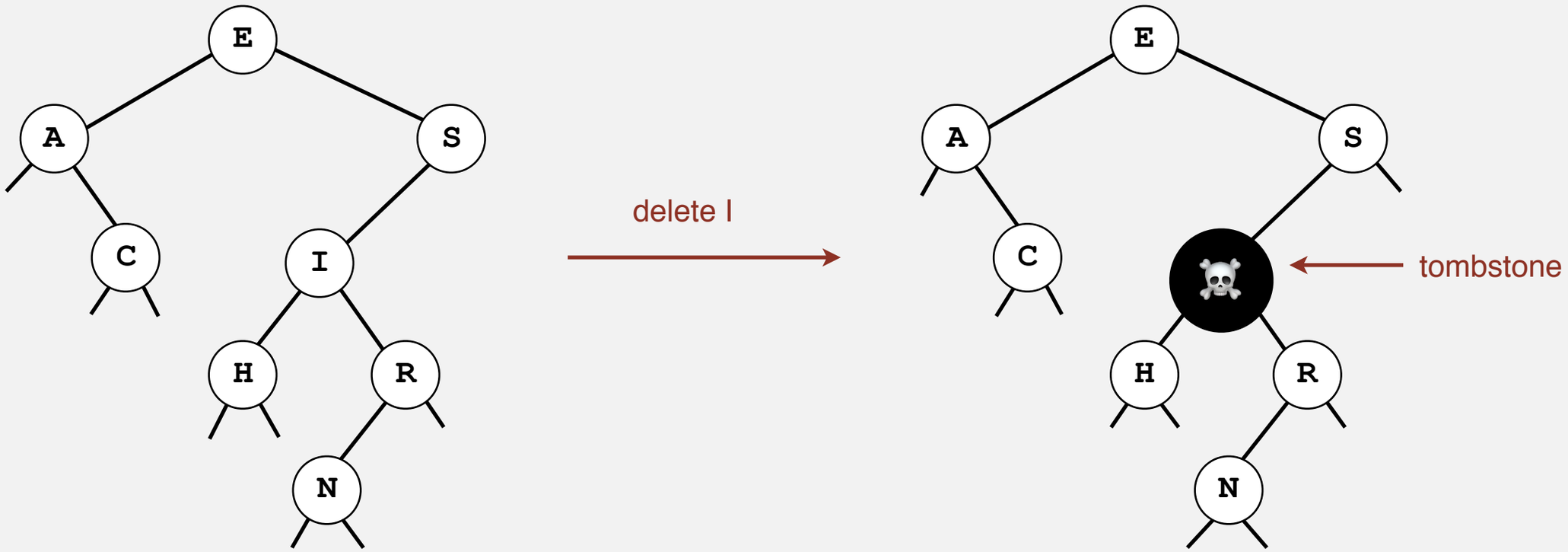
implementation	guarantee			average case			ordered iteration?	operations on keys
	search	insert	delete	search hit	insert	delete		
sequential search (linked list)	N	N	N	N/2	N	N/2	no	`equals()`
binary search (ordered array)	lg N	N	N	lg N	N/2	N/2	yes	`compareTo()`
BST	N	N	N	1.39 lg N	1.39 lg N	? ? ?	yes	`compareTo()`

Next. Deletion in BSTs.

BST deletion: lazy approach

To remove a node with a given key:

- Set its value to `null`.
- Leave key in tree to guide searches (but don't consider it equal to search key).



Cost. $2 \ln N'$ per insert, search, and delete (if keys in random order), where N' is the number of key-value pairs ever inserted in the BST.

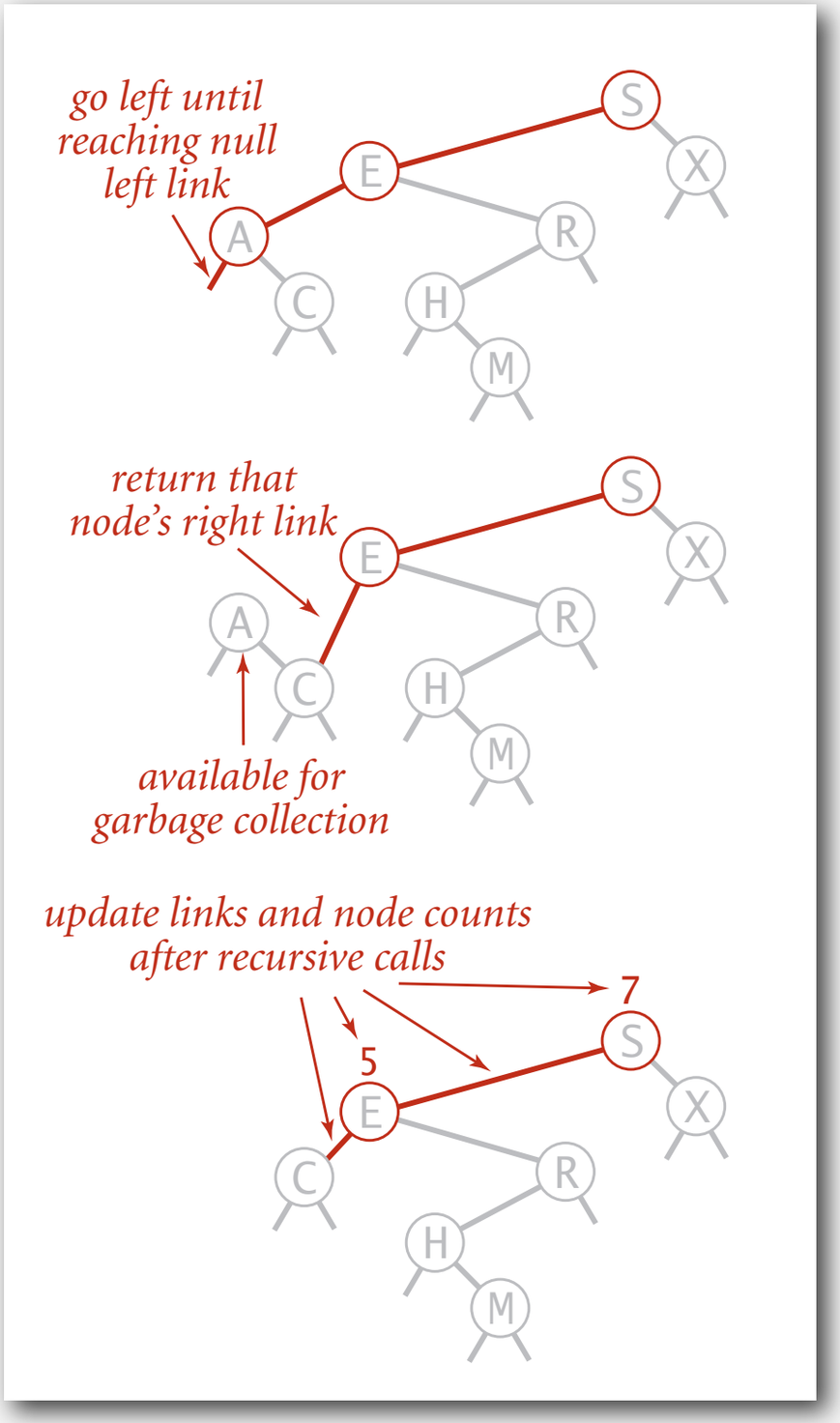
Unsatisfactory solution. Tombstone overload.

Deleting the minimum

To delete the minimum key:

- Go left until finding a node with a null left link.
- Replace that node by its right link.
- Update subtree counts.

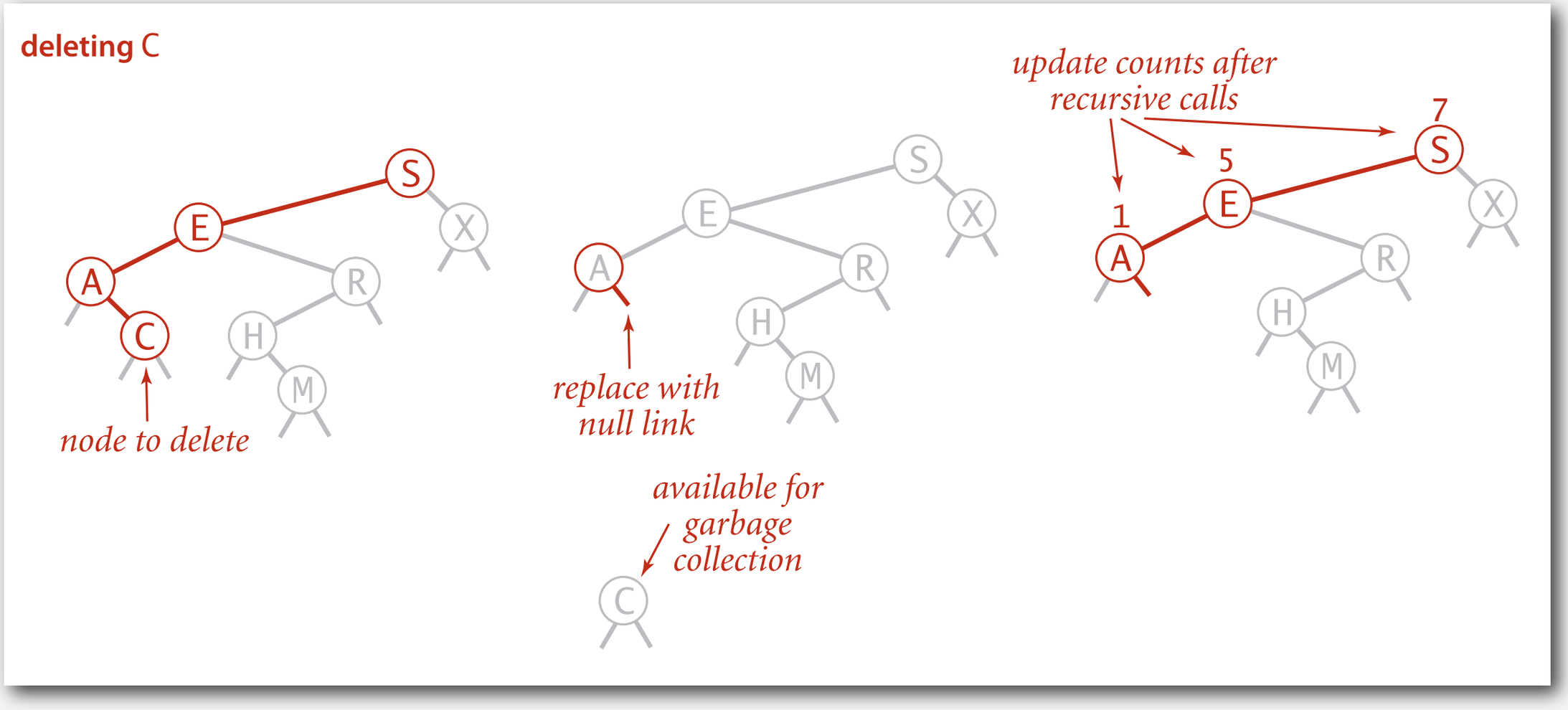
```
public void deleteMin()
{  root = deleteMin(root);  }

private Node deleteMin(Node x)
{
   if (x.left == null) return x.right;
   x.left = deleteMin(x.left);
   x.N = 1 + size(x.left) + size(x.right);
   return x;
}
```



Hibbard deletion

To delete a node with key k: search for node t containing key k.

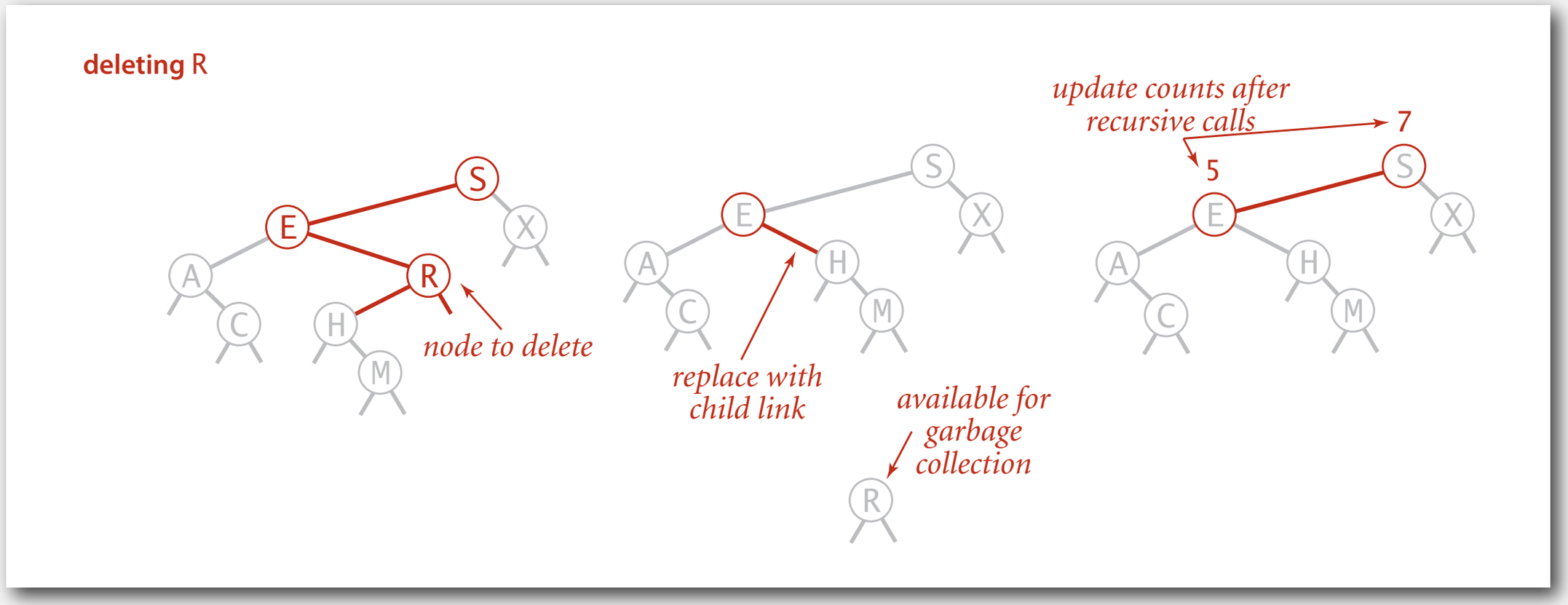
Case 0. [0 children] Delete t by setting parent link to null.



Hibbard deletion

To delete a node with key k: search for node t containing key k.

Case 1. [1 child] Delete t by replacing parent link.

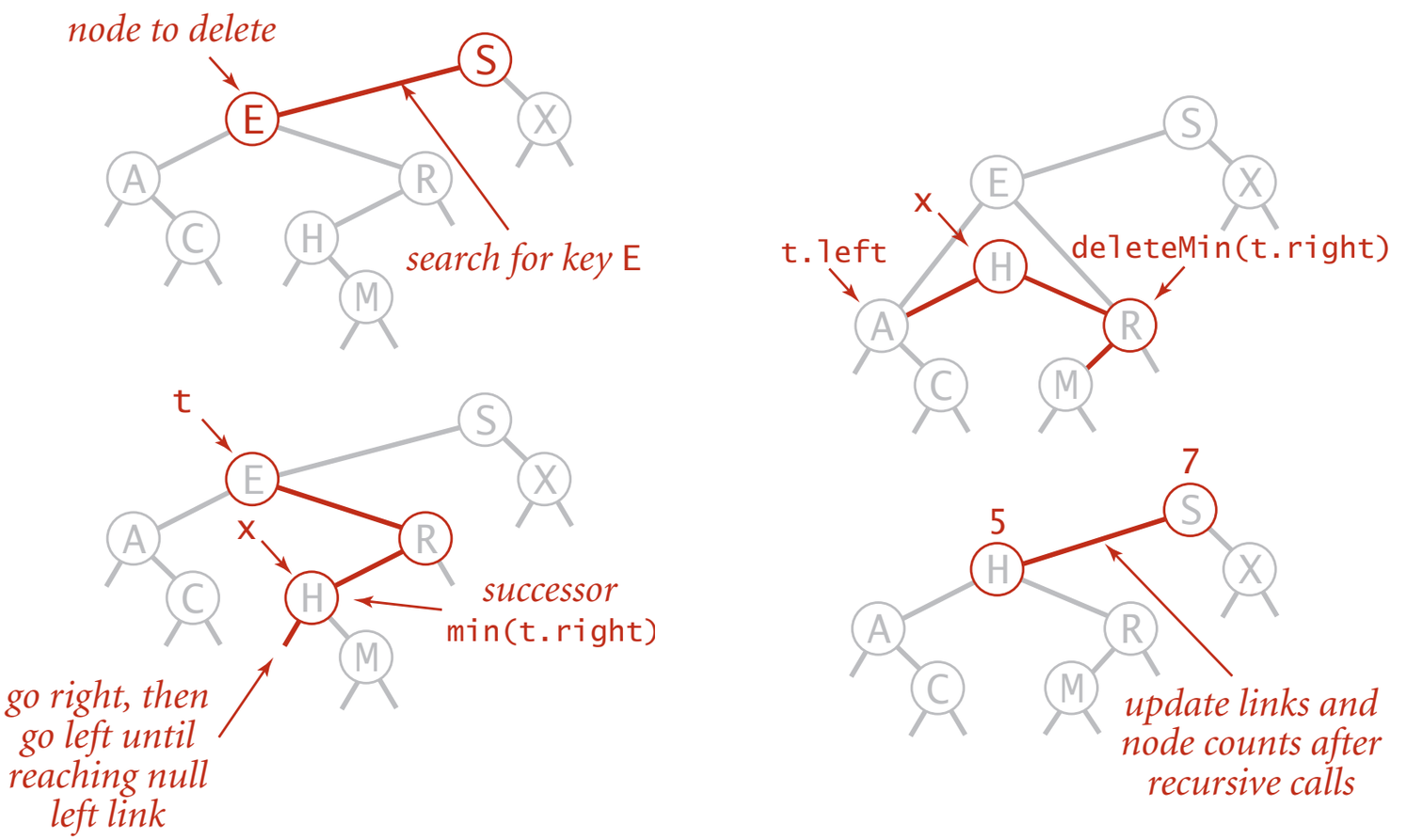


Hibbard deletion

To delete a node with key k: search for node t containing key k.

Case 2. [2 children]

- Find successor x of t. ← x has no left child
- Delete the minimum in t's right subtree. ← but don't garbage collect x
- Put x in t's spot. ← still a BST



Hibbard deletion: Java implementation

```
public void delete(Key key)
{  root = delete(root, key);  }

private Node delete(Node x, Key key) {
   if (x == null) return null;
   int cmp = key.compareTo(x.key);
   if      (cmp < 0) x.left  = delete(x.left,  key);
   else if (cmp > 0) x.right = delete(x.right, key);
   else {
      if (x.right == null) return x.left;

      Node t = x;
      x = min(t.right);
      x.right = deleteMin(t.right);
      x.left = t.left;
   }
   x.N = size(x.left) + size(x.right) + 1;
   return x;
}
```

search for key

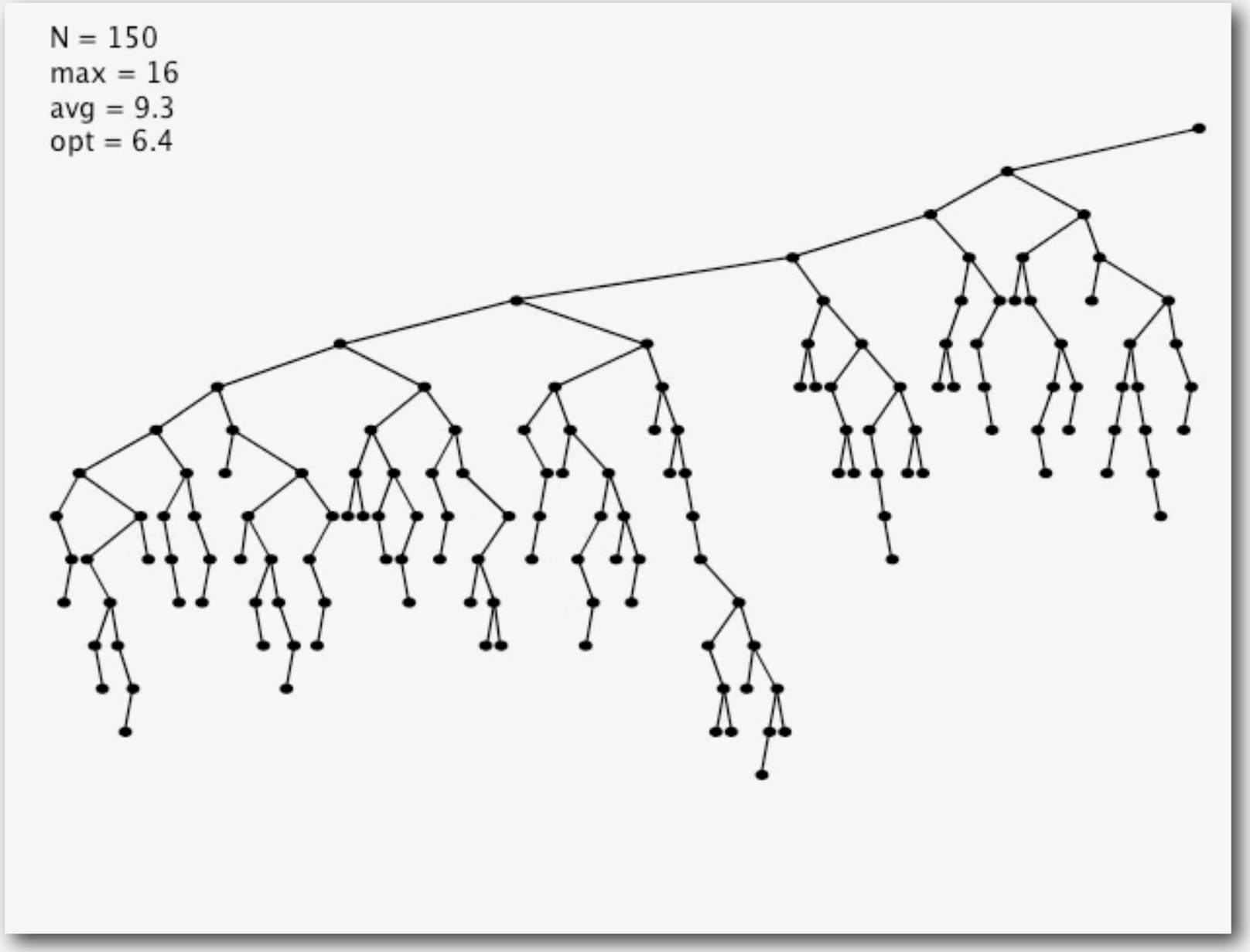
no right child

replace with successor

update subtree counts

Hibbard deletion: analysis

Unsatisfactory solution. Not symmetric.



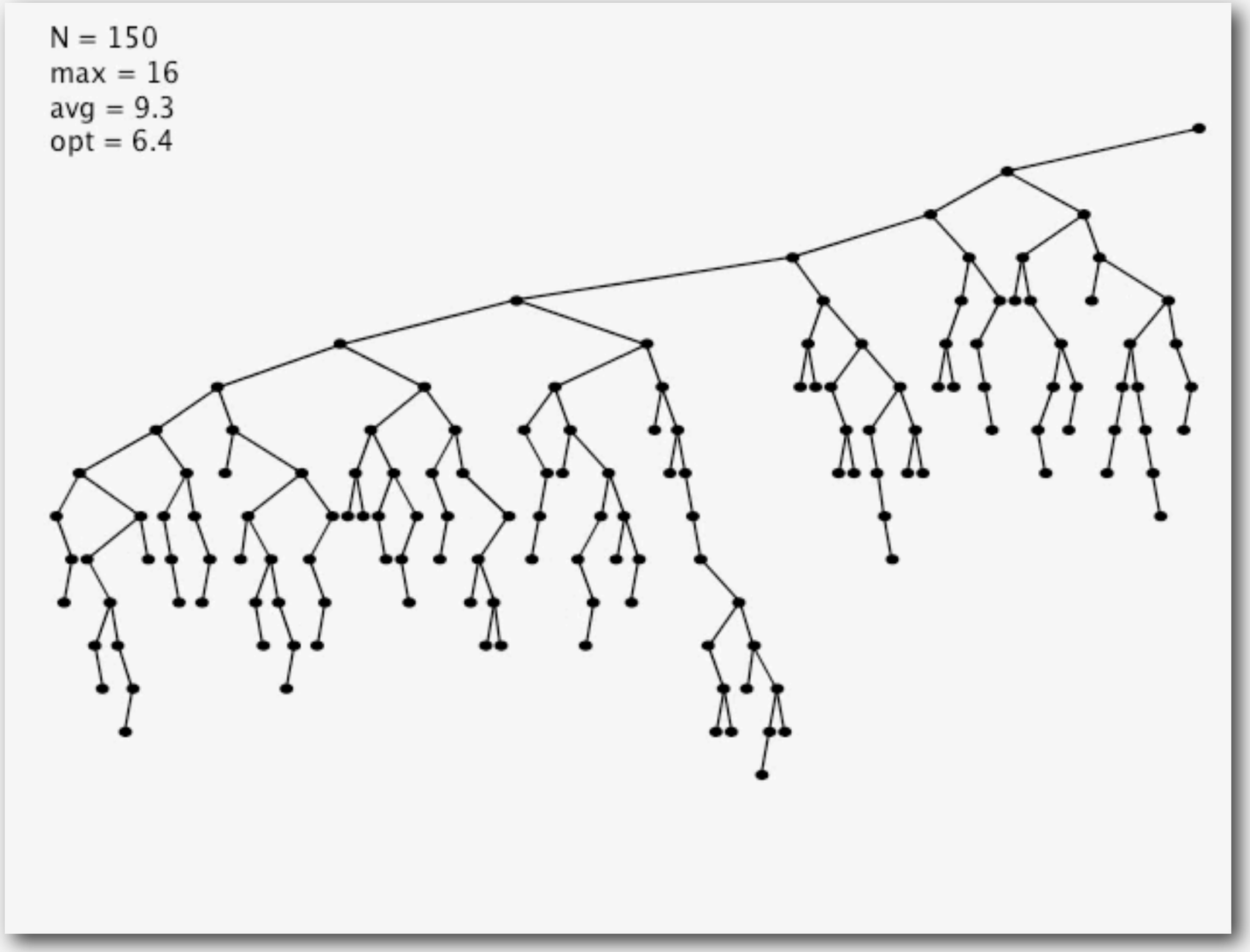
https://www.cs.purdue.edu/homes/cs251/slides/media/hibbard-random.mov

Surprising consequence. Trees not random (!) $\Rightarrow$ $\mathrm{sqrt}\,(N)$ per op.

Longstanding open problem. Simple and efficient delete for BSTs.

Hibbard deletion: analysis

Unsatisfactory solution. Not symmetric.



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BST	N	N	N	1.39 lg N	1.39 lg N	√N	yes	`compareTo()`

other operations also become √N if deletions allowed

Next lecture. Guarantee logarithmic performance for all operations.