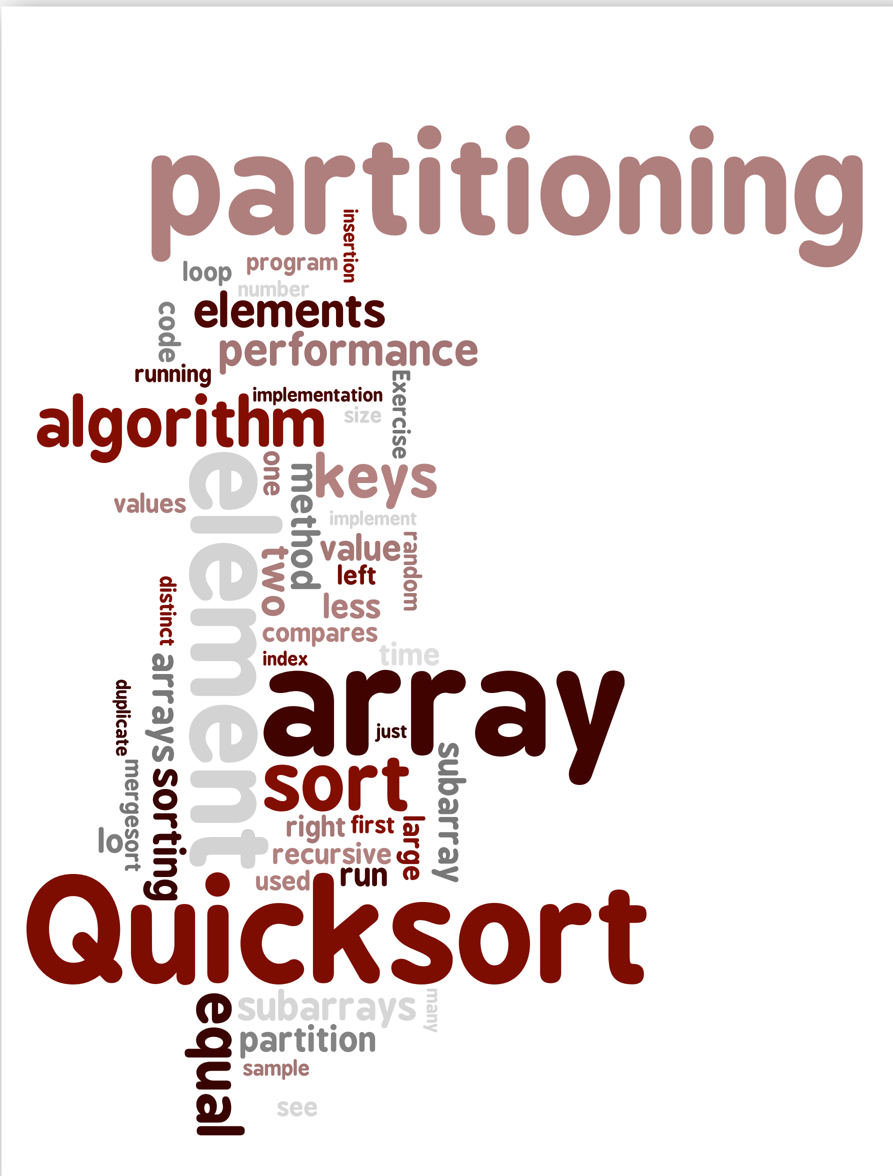
2.3 Quicksort



- quicksort
- selection
- duplicate keys

Two classic sorting algorithms

Critical components in the world's computational infrastructure.

- Full scientific understanding of their properties has enabled us to develop them into practical system sorts.
- Quicksort honored as one of top 10 algorithms of 20^{th} century in science and engineering.

Mergesort.

← last lecture

- Java sort for objects.
- Perl, C++ stable sort, Python stable sort, Firefox JavaScript, ...

Quicksort.

← this lecture

- Java sort for primitive types.
- C qsort, Unix, Visual C++, Python, Matlab, Chrome JavaScript, ...

- quicksort
- selection
- duplicate keys

Quicksort

Basic plan.

- Shuffle the array.
- Partition so that, for some `j`
  - element `a[j]` is in place
  - no larger element to the left of `j`
  - no smaller element to the right of `j`
- Sort each piece recursively.

Sir Charles Antony Richard Hoare
1980 Turing Award

input	Q	U	I	C	K	S	O	R	T	E	X	A	M	P	L	E
shuffle	K	R	A	T	E	L	E	P	U	I	M	Q	C	X	O	S
partition	E	C	A	I	E	K	L	P	U	T	M	Q	R	X	O	S
sort left	A	C	E	E	I	K	L	P	U	T	M	Q	R	X	O	S
sort right	A	C	E	E	I	K	L	M	O	P	Q	R	S	T	U	X
result	A	C	E	E	I	K	L	M	O	P	Q	R	S	T	U	X

partitioning element

not greater

not less

Basic plan.

- Scan `i` from left for an item that belongs on the right.
- Scan `j` from right for item that belongs on the left.
- Exchange `a[i]` and `a[j]`.
- Repeat until pointers cross.

			a[i]															
	i	j	0 (v)	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
initial values	0	16	K	R	A	T	E	L	E	P	U	I	M	Q	C	X	O	S
scan left, scan right	1	12	K	R	A	T	E	L	E	P	U	I	M	Q	C	X	O	S
exchange	1	12	K	C	A	T	E	L	E	P	U	I	M	Q	R	X	O	S
scan left, scan right	3	9	K	C	A	T	E	L	E	P	U	I	M	Q	R	X	O	S
exchange	3	9	K	C	A	I	E	L	E	P	U	T	M	Q	R	X	O	S
scan left, scan right	5	6	K	C	A	I	E	L	E	P	U	T	M	Q	R	X	O	S
exchange	5	6	K	C	A	I	E	E	L	P	U	T	M	Q	R	X	O	S
scan left, scan right	6	5	K	C	A	I	E	E	L	P	U	T	M	Q	R	X	O	S
final exchange	6	5	E	C	A	I	E	K	L	P	U	T	M	Q	R	X	O	S
result	6	5	E	C	A	I	E	K	L	P	U	T	M	Q	R	X	O	S

Partitioning trace (array contents before and after each exchange)

Quicksort: Java code for partitioning

```
private static int partition(Comparable[] a, int lo, int hi)
{
   int i = lo, j = hi+1;
   while (true)
   {
      while (less(a[++i], a[lo]))
         if (i == hi) break;

      while (less(a[lo], a[--j]))
         if (j == lo) break;

      if (i >= j) break;
      exch(a, i, j);
   }

   exch(a, lo, j);
   return j;
}
```

find item on left to swap

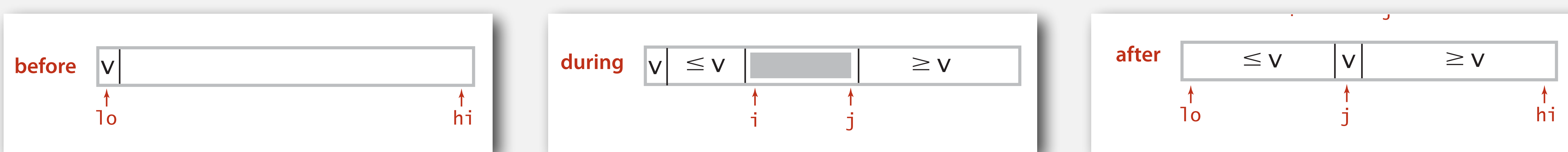
find item on right to swap

check if pointers cross

swap

swap with partitioning item

return index of item now known to be in place



Quicksort: Java implementation

```
public class Quick
{
   private static int partition(Comparable[] a, int lo, int hi)
   {  /* see previous slide */  }

   public static void sort(Comparable[] a)
   {
      StdRandom.shuffle(a);
      sort(a, 0, a.length - 1);
   }

   private static void sort(Comparable[] a, int lo, int hi)
   {
      if (hi <= lo) return;
      int j = partition(a, lo, hi);
      sort(a, lo, j-1);
      sort(a, j+1, hi);
  }
}
```

shuffle needed for performance guarantee (stay tuned)

Quicksort trace

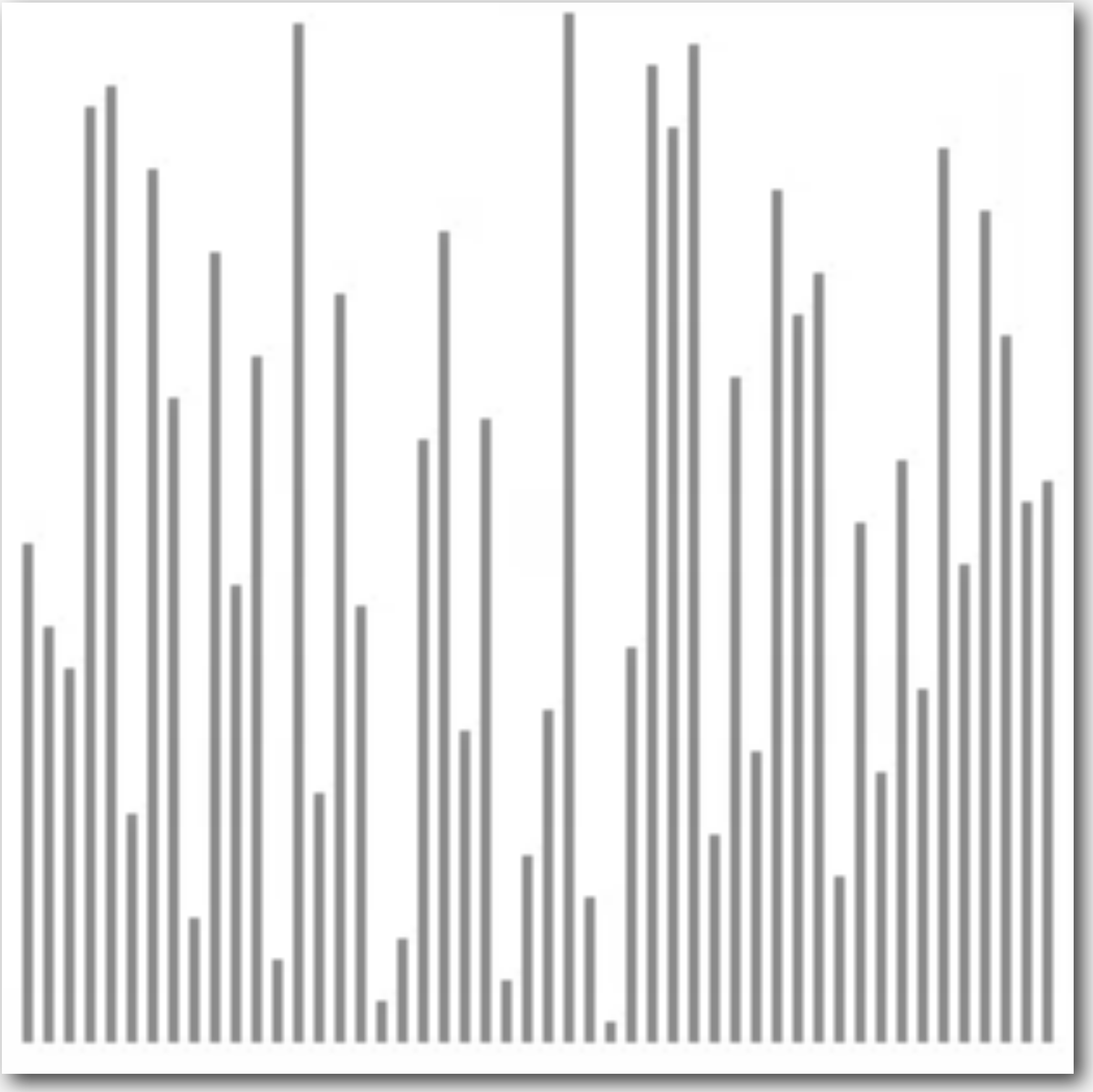
	lo	j	hi	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
initial values				Q	U	I	C	K	S	O	R	T	E	X	A	M	P	L	E
random shuffle				K	R	A	T	E	L	E	P	U	I	M	Q	C	X	O	S
	0	5	15	E	C	A	I	E	K	L	P	U	T	M	Q	R	X	O	S
	0	3	4	E	C	A	E	I	K	L	P	U	T	M	Q	R	X	O	S
	0	2	2	A	C	E	E	I	K	L	P	U	T	M	Q	R	X	O	S
	0	0	1	A	C	E	E	I	K	L	P	U	T	M	Q	R	X	O	S
	1		1	A	C	E	E	I	K	L	P	U	T	M	Q	R	X	O	S
	4		4	A	C	E	E	I	K	L	P	U	T	M	Q	R	X	O	S
	6	6	15	A	C	E	E	I	K	L	P	U	T	M	Q	R	X	O	S
	7	9	15	A	C	E	E	I	K	L	M	O	P	T	Q	R	X	U	S
	7	7	8	A	C	E	E	I	K	L	M	O	P	T	Q	R	X	U	S
	8		8	A	C	E	E	I	K	L	M	O	P	T	Q	R	X	U	S
	10	13	15	A	C	E	E	I	K	L	M	O	P	S	Q	R	T	U	X
	10	12	12	A	C	E	E	I	K	L	M	O	P	R	Q	S	T	U	X
	10	11	11	A	C	E	E	I	K	L	M	O	P	Q	R	S	T	U	X
	10		10	A	C	E	E	I	K	L	M	O	P	Q	R	S	T	U	X
	14	14	15	A	C	E	E	I	K	L	M	O	P	Q	R	S	T	U	X
	15		15	A	C	E	E	I	K	L	M	O	P	Q	R	S	T	U	X
result				A	C	E	E	I	K	L	M	O	P	Q	R	S	T	U	X

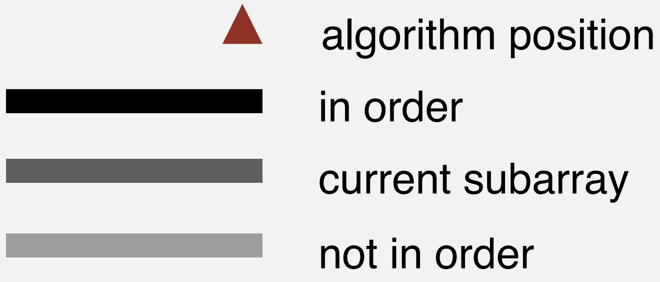
no partition for subarrays of size 1

Quicksort trace (array contents after each partition)

Quicksort animation

50 random elements

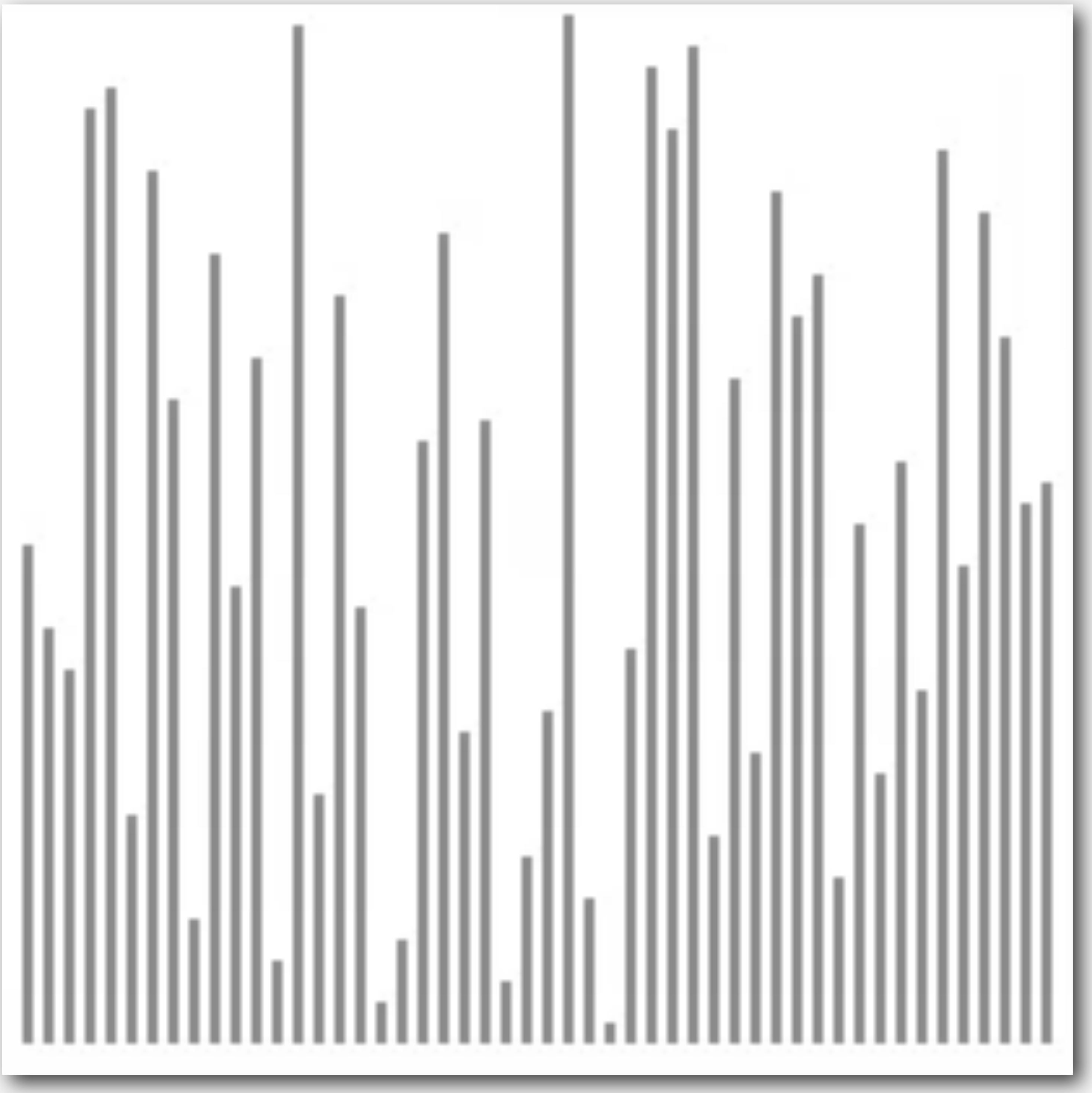




https://www.cs.purdue.edu/homes/cs251/slides/media/quick-sort.mov

Quicksort animation

50 random elements



algorithm position

in order

current subarray

not in order

https://www.cs.purdue.edu/homes/cs251/slides/media/quick-sort.mov

Quicksort: implementation details

Partitioning in-place. Using an extra array makes partitioning easier (and stable), but is not worth the cost.

Terminating the loop. Testing whether the pointers cross is a bit trickier than it might seem.

Staying in bounds. The `(j == lo)` test is redundant (why?), but the `(i == hi)` test is not.

Preserving randomness. Shuffling is needed for performance guarantee.

Equal keys. When duplicates are present, it is (counter-intuitively) better to stop on elements equal to the partitioning element.

Quicksort: empirical analysis

Running time estimates:

- Home PC executes 10^8 compares/second.
- Supercomputer executes 10^{12} compares/second.

	insertion sort (N^2)			mergesort (N log N)			quicksort (N log N)		
computer	thousand	million	billion	thousand	million	billion	thousand	million	billion
home	instant	2.8 hours	317 years	instant	1 second	18 min	instant	0.6 sec	12 min
super	instant	1 second	1 week	instant	instant	instant	instant	instant	instant

Lesson 1. Good algorithms are better than supercomputers.

Lesson 2. Great algorithms are better than good ones.

Quicksort: best-case analysis

Best case. Number of compares is $\sim N \lg N$.

			a[ ]														
lo	j	hi	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
initial values			H	A	C	B	F	E	G	D	L	I	K	J	N	M	O
random shuffle			H	A	C	B	F	E	G	D	L	I	K	J	N	M	O
0	7	14	D	A	C	B	F	E	G	H	L	I	K	J	N	M	O
0	3	6	B	A	C	D	F	E	G	H	L	I	K	J	N	M	O
0	1	2	A	B	C	D	F	E	G	H	L	I	K	J	N	M	O
0		0	A	B	C	D	F	E	G	H	L	I	K	J	N	M	O
2		2	A	B	C	D	F	E	G	H	L	I	K	J	N	M	O
4	5	6	A	B	C	D	E	F	G	H	L	I	K	J	N	M	O
4		4	A	B	C	D	E	F	G	H	L	I	K	J	N	M	O
6		6	A	B	C	D	E	F	G	H	L	I	K	J	N	M	O
8	11	14	A	B	C	D	E	F	G	H	J	I	K	L	N	M	O
8	9	10	A	B	C	D	E	F	G	H	I	J	K	L	N	M	O
8		8	A	B	C	D	E	F	G	H	I	J	K	L	N	M	O
10		10	A	B	C	D	E	F	G	H	I	J	K	L	N	M	O
12	13	14	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
12		12	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
14		14	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
			A	B	C	D	E	F	G	H	I	J	K	L	M	N	O

Quicksort: worst-case analysis

Worst case. Number of compares is $\sim \frac{1}{2} N^2$.

			a[ ]														
lo	j	hi	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
initial values			A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
random shuffle			A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
0	0	14	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
1	1	14	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
2	2	14	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
3	3	14	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
4	4	14	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
5	5	14	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
6	6	14	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
7	7	14	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
8	8	14	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
9	9	14	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
10	10	14	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
11	11	14	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
12	12	14	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
13	13	14	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
14		14	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
			A	B	C	D	E	F	G	H	I	J	K	L	M	N	O

Quicksort: average-case analysis

Proposition. The average number of compares C_N to quicksort an array of N distinct keys is $\sim 2N \ln N$ (and the number of exchanges is $\sim \frac{1}{3} N \ln N$).

Pf. C_N satisfies the recurrence $C_0 = C_1 = 0$ and for $N \geq 2$:

$$C_N = \underbrace{(N+1)}_{\text{partitioning}} + \underbrace{\frac{C_0 + C_1 + \ldots + C_{N-1}}{N}}_{\text{left}} + \underbrace{\frac{C_{N-1} + C_{N-2} + \ldots + C_0}{N}}_{\text{right}}$$

partitioning probability

- Multiply both sides by N and collect terms:

$$NC_N = N(N+1) + 2(C_0 + C_1 + \ldots + C_{N-1})$$

- Subtract this from the same equation for $N - 1$:

$$NC_N - (N-1)C_{N-1} = 2N + 2C_{N-1}$$

- Rearrange terms and divide by $N(N+1)$:

$$\frac{C_N}{N+1} = \frac{C_{N-1}}{N} + \frac{2}{N+1}$$

Quicksort: average-case analysis

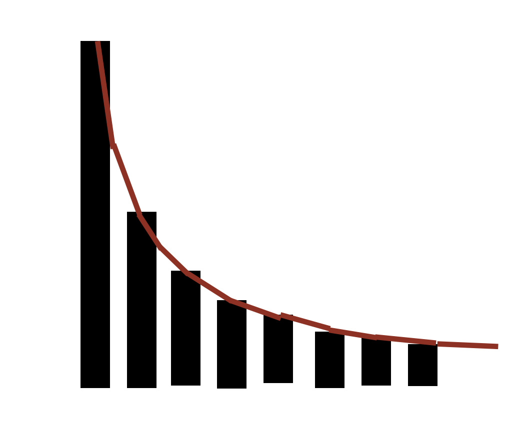
- Repeatedly apply above equation:

$$\begin{aligned}\frac{C_N}{N+1} &= \frac{C_{N-1}}{N} + \frac{2}{N+1}\\ &= \frac{C_{N-2}}{N-1} + \frac{2}{N} + \frac{2}{N+1}\\ &= \frac{C_{N-3}}{N-2} + \frac{2}{N-1} + \frac{2}{N} + \frac{2}{N+1}\\ &= \frac{2}{3} + \frac{2}{4} + \frac{2}{5} + \ldots + \frac{2}{N+1}\end{aligned}$$

previous equation

- Approximate sum by an integral:

$$\begin{aligned}C_N &= 2(N+1)\left(\frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \ldots \frac{1}{N+1}\right)\\ &\sim 2(N+1)\int_3^{N+1} \frac{1}{x}\,dx\end{aligned}$$



- Finally, the desired result:

$$C_N \sim 2(N+1)\ln N \approx 1.39 N \lg N$$

Quicksort: summary of performance characteristics

Worst case. Number of compares is quadratic.

- $N + (N - 1) + (N - 2) + \ldots + 1 \sim \frac{1}{2} N^2$.
- More likely that your computer is struck by lightning bolt.

Average case. Number of compares is $\sim 1.39\, N \lg N$.

- 39% more compares than mergesort.
- But faster than mergesort in practice because of less data movement.

Random shuffle.

- Probabilistic guarantee against worst case.
- Basis for math model that can be validated with experiments.

Caveat emptor. Many textbook implementations go quadratic if array

- Is sorted or reverse sorted.
- Has many duplicates (even if randomized!)

Quicksort: practical improvements

Insertion sort small subarrays.

- Even quicksort has too much overhead for tiny subarrays.
- Can delay insertion sort until end.

```
private static void sort(Comparable[] a, int lo, int hi)
{
   if (hi <= lo + CUTOFF - 1)
   {
      Insertion.sort(a, lo, hi);
      return;
   }
   int j = partition(a, lo, hi);
   sort(a, lo, j-1);
   sort(a, j+1, hi);
}
```

Quicksort: practical improvements

Insertion sort small subarrays.

- Even quicksort has too much overhead for tiny subarrays.
- Can delay insertion sort until end.

Median of sample.

- Best choice of pivot element = median.
- Estimate true median by taking median of sample.

```
private static void sort(Comparable[] a, int lo, int hi)
{
   if (hi <= lo) return;

   int m = medianOf3(a, lo, lo + (hi - lo)/2, hi);
   swap(a, lo, m);

   int j = partition(a, lo, hi);
   sort(a, lo, j-1);
   sort(a, j+1, hi);
}
```

Quicksort: practical improvements

Insertion sort small subarrays.

- Even quicksort has too much overhead for tiny subarrays.
- Can delay insertion sort until end.

Median of sample.

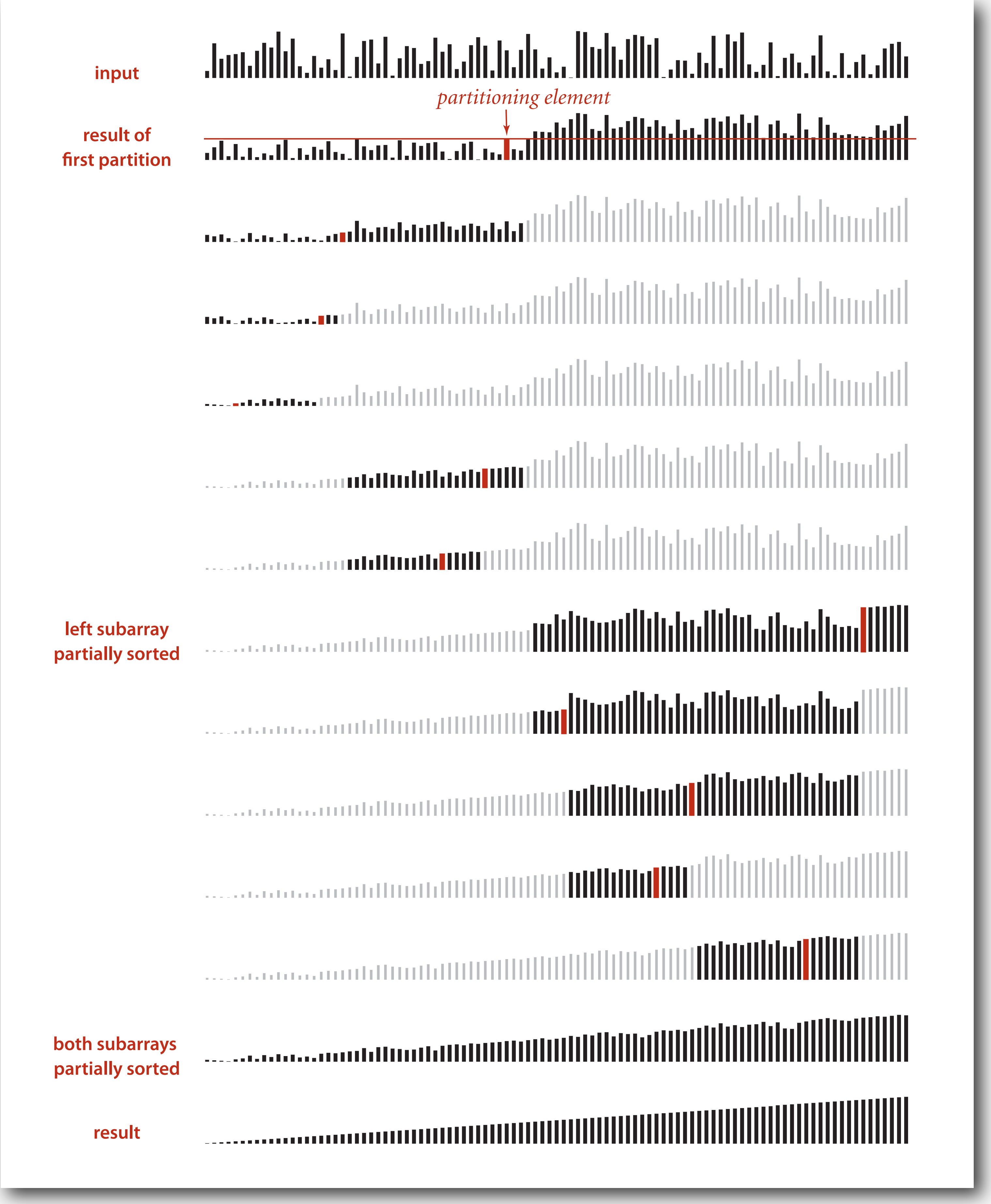
- Best choice of pivot element = median.
- Estimate true median by taking median of sample.

~ 12/7 N ln N compares (slightly fewer)
~ 12/35 N ln N exchanges (slightly more)

Optimize parameters.

- Median-of-3 (random) elements.
- Cutoff to insertion sort for ≈ 10 elements.

Quicksort with median-of-3 and cutoff to insertion sort: visualization



- quicksort
- **selection**
- duplicate keys

Selection

Goal. Find the k^{th} largest element.

Ex. Min $(k = 0)$, max $(k = N - 1)$, median $(k = N/2)$.

Applications.

- Order statistics.
- Find the "top k."

Use theory as a guide.

- Easy $O(N \log N)$ upper bound. How?
- Easy $O(N)$ upper bound for $k = 1, 2, 3$. How?
- Easy $\Omega(N)$ lower bound. Why?

Which is true?

- $\Omega(N \log N)$ lower bound? ← is selection as hard as sorting?
- $O(N)$ upper bound? ← is there a linear-time algorithm for all k?

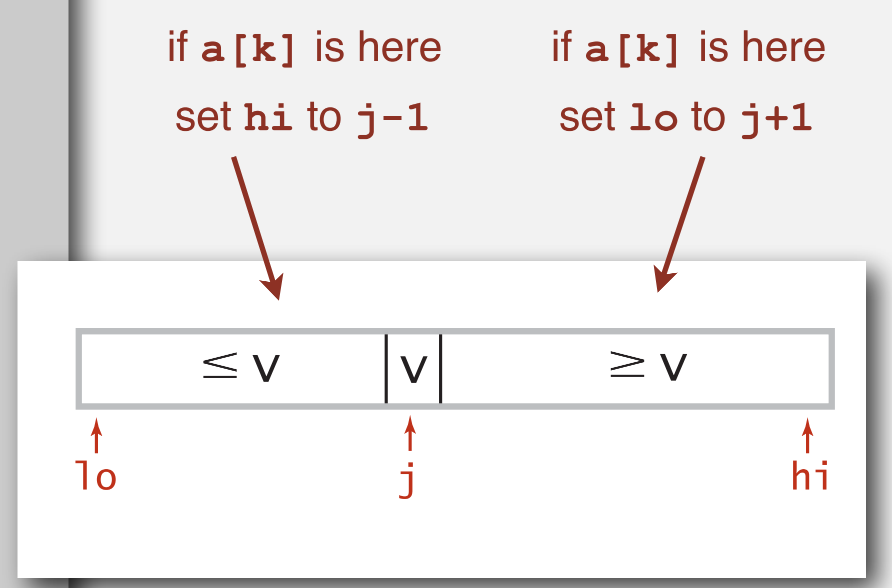
Quick-select

Partition array so that:

- Element `a[j]` is in place.
- No larger element to the left of `j`.
- No smaller element to the right of `j`.

Repeat in one subarray, depending on `j`; finished when `j` equals `k`.

```
public static Comparable select(Comparable[] a, int k)
{
    StdRandom.shuffle(a);
    int lo = 0, hi = a.length - 1;
    while (hi > lo)
    {
       int j = partition(a, lo, hi);
       if       (j < k) lo = j + 1;
       else if (j > k) hi = j - 1;
       else              return a[k];
    }
    return a[k];
}
```



Quick-select: mathematical analysis

Proposition. Quick-select takes linear time on average.

Pf sketch.

- Intuitively, each partitioning step splits array approximately in half: $N + N/2 + N/4 + \ldots + 1 \sim 2N$ compares.
- Formal analysis similar to quicksort analysis yields:

$$C_N = 2\,N + k \ln (N/k) + (N-k) \ln (N/(N-k))$$

Ex. $(2 + 2 \ln 2)\,N$ compares to find the median.

Remark. Quick-select uses $\sim \frac{1}{2} N^2$ compares in the worst case, but (as with quicksort) the random shuffle provides a probabilistic guarantee.

Theoretical context for selection

Proposition. [Blum, Floyd, Pratt, Rivest, Tarjan, 1973] There exists a compare-based selection algorithm whose worst-case running time is linear.

> Time Bounds for Selection
>
> by .
>
> Manuel Blum, Robert W. Floyd, Vaughan Pratt,
> Ronald L. Rivest, and Robert E. Tarjan
>
> <u>Abstract</u>
>
> The number of comparisons required to select the i-th smallest of n numbers is shown to be at most a linear function of n by analysis of a new selection algorithm -- PICK. Specifically, no more than 5.4305 n comparisons are ever required. This bound is improved for

Remark. But, constants are too high $\Rightarrow$ not used in practice.

Use theory as a guide.

- Still worthwhile to seek practical linear-time (worst-case) algorithm.
- Until one is discovered, use quick-select if you don't need a full sort.

In our `select()` implementation, client needs a cast.

```
Double[] a = new Double[N];
for (int i = 0; i < N; i++)
   a[i] = StdRandom.uniform();
Double median = (Double) Quick.select(a, N/2);
```

unsafe cast required in client

The compiler complains.

```
% javac Quick.java
Note: Quick.java uses unchecked or unsafe operations.
Note: Recompile with -Xlint:unchecked for details.
```

Q. How to fix?

Generic methods

Pedantic (safe) version. Compiles cleanly, no cast needed in client.

```
public class QuickPedantic
{
    public  static <Key extends Comparable<Key>> Key select(Key[] a, int k)
    {  /* as before */  }

    public  static <Key extends Comparable<Key>> void sort(Key[] a)
    {  /* as before */  }

    private static <Key extends Comparable<Key>> int partition(Key[] a, int lo, int hi)
    {  /* as before */  }

    private static <Key extends Comparable<Key>> boolean less(Key v, Key w)
    {  /* as before */  }

    private static <Key extends Comparable<Key>> void exch(Key[] a, int i, int j)
    {  Key swap = a[i]; a[i] = a[j]; a[j] = swap;  }

}
```

generic type variable
(value inferred from argument `a[]`)

return type matches array type

can declare variables of generic type

http://www.cs.princeton.edu/algs4/23quicksort/QuickPedantic.java.html

Remark. Obnoxious code needed in system sort; not in this course (for brevity).

- quicksort
- selection
- **duplicate keys**

Duplicate keys

Often, purpose of sort is to bring records with duplicate keys together.

- Sort population by age.
- Find collinear points. ← see Assignment 2
- Remove duplicates from mailing list.
- Sort job applicants by college attended.

Typical characteristics of such applications.

- Huge array.
- Small number of key values.

```
Chicago 09:25:52
Chicago 09:03:13
Chicago 09:21:05
Chicago 09:19:46
Chicago 09:19:32
Chicago 09:00:00
Chicago 09:35:21
Chicago 09:00:59
Houston 09:01:10
Houston 09:00:13
Phoenix 09:37:44
Phoenix 09:00:03
Phoenix 09:14:25
Seattle 09:10:25
Seattle 09:36:14
Seattle 09:22:43
Seattle 09:10:11
Seattle 09:22:54
```

key

Duplicate keys

Mergesort with duplicate keys. Always between $\frac{1}{2} N \lg N$ and $N \lg N$ compares.

Quicksort with duplicate keys.

- Algorithm goes quadratic unless partitioning stops on equal keys!
- 1990s C user found this defect in `qsort()`.

several textbook and system implementation also have this defect

S T O P O N E Q U A L K E Y S

swap

if we don't stop on equal keys

if we stop on equal keys

Duplicate keys: the problem

Mistake. Put all keys equal to the partitioning element on one side.

Consequence. $\sim \frac{1}{2} N^2$ compares when all keys equal.

`B A A B A B B B C C C`   `A A A A A A A A A A A`

Recommended. Stop scans on keys equal to the partitioning element.

Consequence. $\sim N \lg N$ compares when all keys equal.

`B A A B A B C C B C B`   `A A A A A A A A A A A`

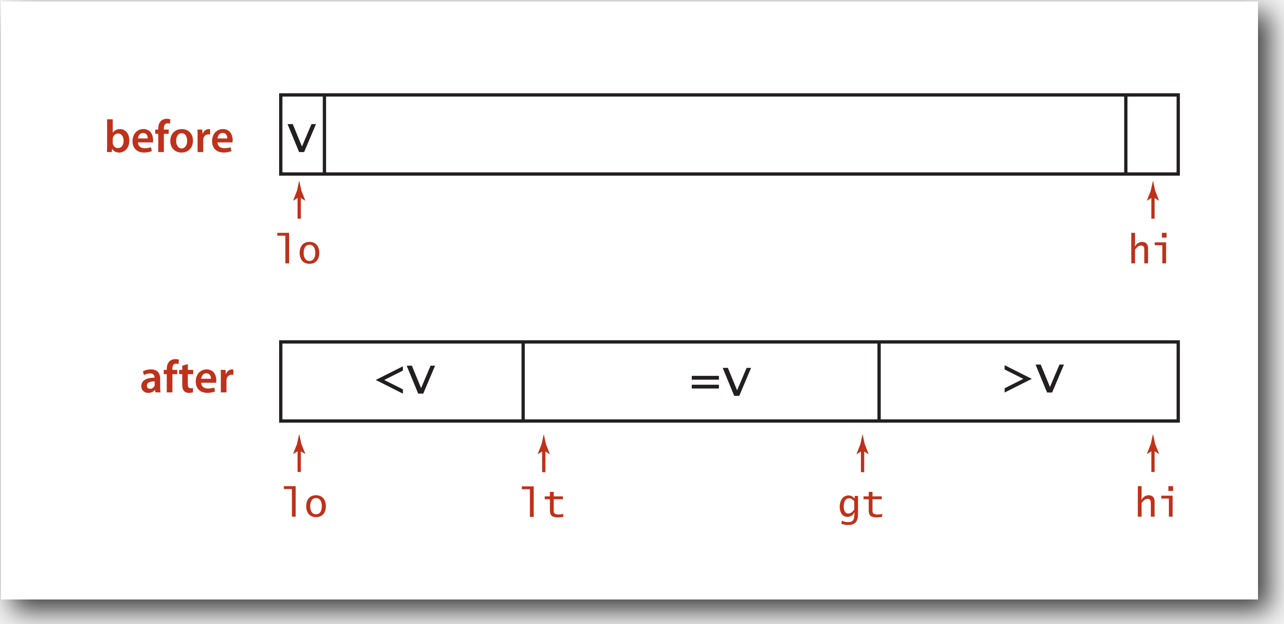
Desirable. Put all keys equal to the partitioning element in place.

`A A A B B B B B C C C`   `A A A A A A A A A A A`

3-way partitioning

Goal. Partition array into 3 parts so that:

- Elements between `lt` and `gt` equal to partition element `v`.
- No larger elements to left of `lt`.
- No smaller elements to right of `gt`.



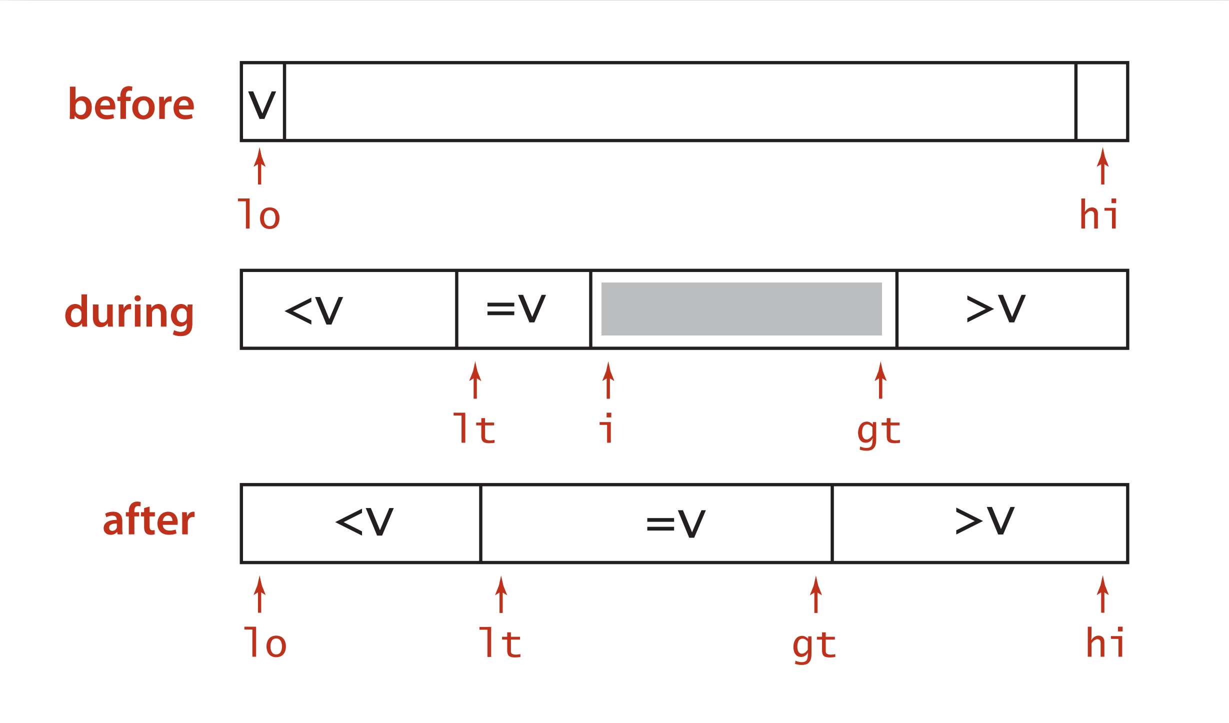
Dutch national flag problem. [Edsger Dijkstra]

- Conventional wisdom until mid 1990s: not worth doing.
- New approach discovered when fixing mistake in C library `qsort()`.
- Now incorporated into `qsort()` and Java system sort.

Dijkstra 3-way partitioning algorithm

3-way partitioning.

- Let `v` be partitioning element `a[lo]`.
- Scan `i` from left to right.
  - `a[i]` less than `v`: exchange `a[lt]` with `a[i]` and increment both `lt` and `i`
  - `a[i]` greater than `v`: exchange `a[gt]` with `a[i]` and decrement `gt`
  - `a[i]` equal to `v`: increment `i`



All the right properties.

- In-place.
- Not much code.
- Small overhead if no equal keys.

3-way partitioning: trace

v

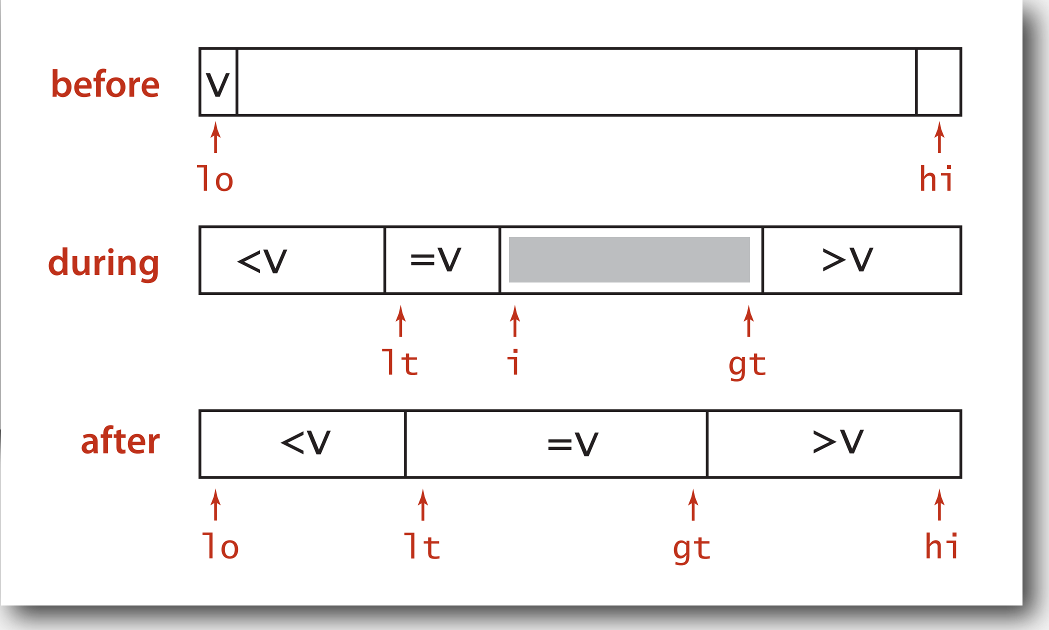
lt	i	gt	a[] 0	1	2	3	4	5	6	7	8	9	10	11
0	0	11	R	B	W	W	R	W	B	R	R	W	B	R
0	1	11	R	B	W	W	R	W	B	R	R	W	B	R
1	2	11	B	R	W	W	R	W	B	R	R	W	B	R
1	2	10	B	R	R	W	R	W	B	R	R	W	B	W
1	3	10	B	R	R	W	R	W	B	R	R	W	B	W
1	3	9	B	R	R	B	R	W	B	R	R	W	W	W
2	4	9	B	B	R	R	R	W	B	R	R	W	W	W
2	5	9	B	B	R	R	R	W	B	R	R	W	W	W
2	5	8	B	B	R	R	R	W	B	R	R	W	W	W
2	5	7	B	B	R	R	R	R	B	R	W	W	W	W
2	6	7	B	B	R	R	R	R	B	R	W	W	W	W
3	7	7	B	B	B	R	R	R	R	R	W	W	W	W
3	8	7	B	B	B	R	R	R	R	R	W	W	W	W
3	8	7	B	B	B	R	R	R	R	R	W	W	W	W

3-way partitioning trace (array contents after each loop iteration)

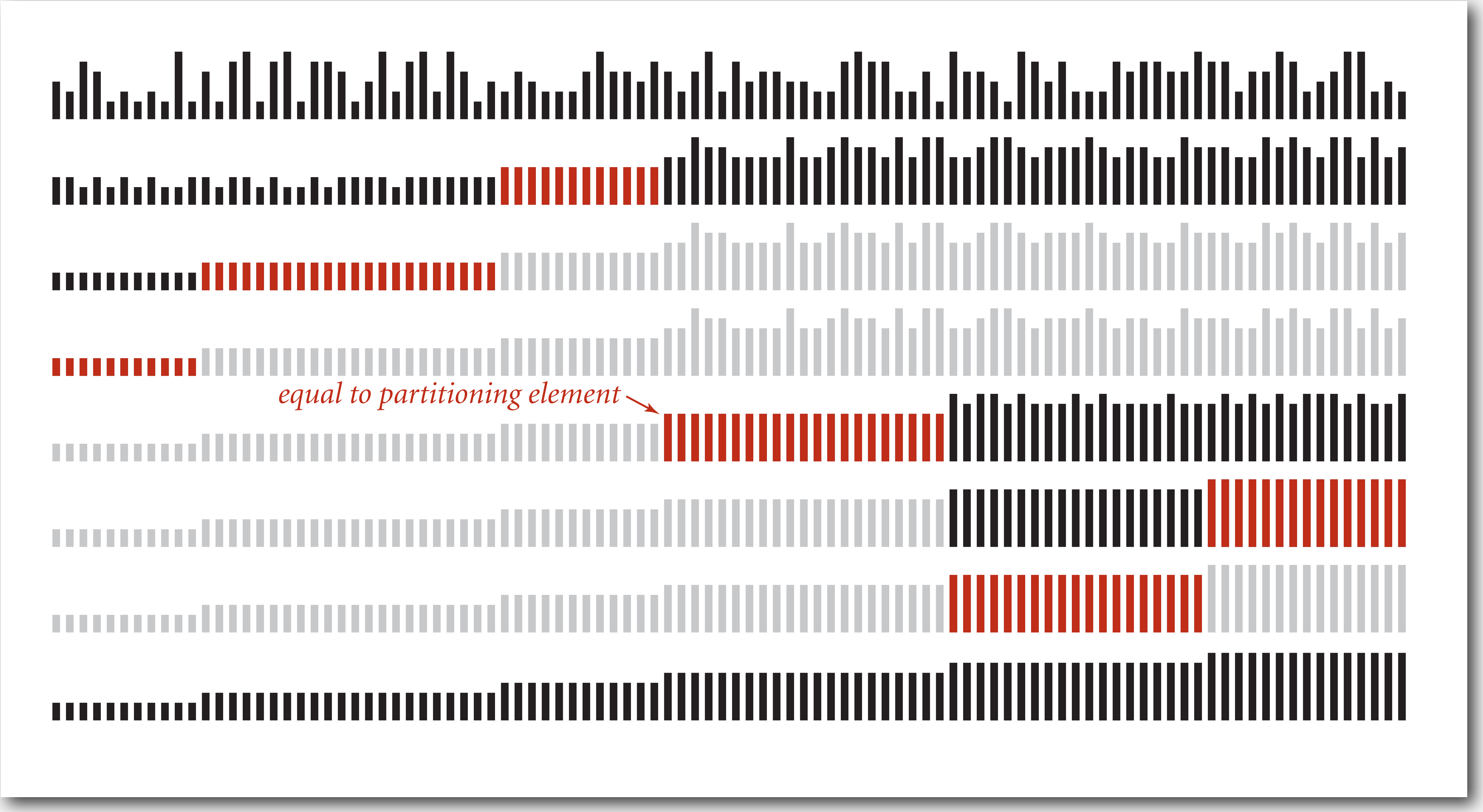
3-way quicksort: Java implementation

```
private static void sort(Comparable[] a, int lo, int hi)
{
   if (hi <= lo) return;
   int lt = lo, gt = hi;
   Comparable v = a[lo];
   int i = lo;
   while (i <= gt)
   {
      int cmp = a[i].compareTo(v);
      if      (cmp < 0) exch(a, lt++, i++);
      else if (cmp > 0) exch(a, i, gt--);
      else              i++;
   }

   sort(a, lo, lt - 1);
   sort(a, gt + 1, hi);
}
```



3-way quicksort: visual trace



Duplicate keys: lower bound

Sorting lower bound. If there are n distinct keys and the i^{th} one occurs x_i times, any compare-based sorting algorithm must use at least

$$\lg\left(\frac{N!}{x_1!\; x_2! \cdots x_n!}\right) \sim -\sum_{i=1}^{n} x_i \lg \frac{x_i}{N}$$

$N \lg N$ when all distinct;
linear when only a constant number of distinct keys

compares in the worst case.

proportional to lower bound

Proposition. [Sedgewick-Bentley, 1997]
Quicksort with 3-way partitioning is entropy-optimal.
Pf. [beyond scope of course]

Bottom line. Randomized quicksort with 3-way partitioning reduces running time from linearithmic to linear in broad class of applications.

Sorting summary

	inplace?	stable?	worst	average	best	remarks
selection	x		$N^2 / 2$	$N^2 / 2$	$N^2 / 2$	*N* exchanges
insertion	x	x	$N^2 / 2$	$N^2 / 4$	N	use for small *N* or partially ordered
shell	x		?	?	N	tight code, subquadratic
merge		x	N lg N	N lg N	N lg N	*N* log *N* guarantee, stable
quick	x		$N^2 / 2$	2 N ln N	N lg N	*N* log *N* probabilistic guarantee fastest in practice
3-way quick	x		$N^2 / 2$	2 N ln N	N	improves quicksort in presence of duplicate keys
???	x	x	N lg N	N lg N	N lg N	holy sorting grail