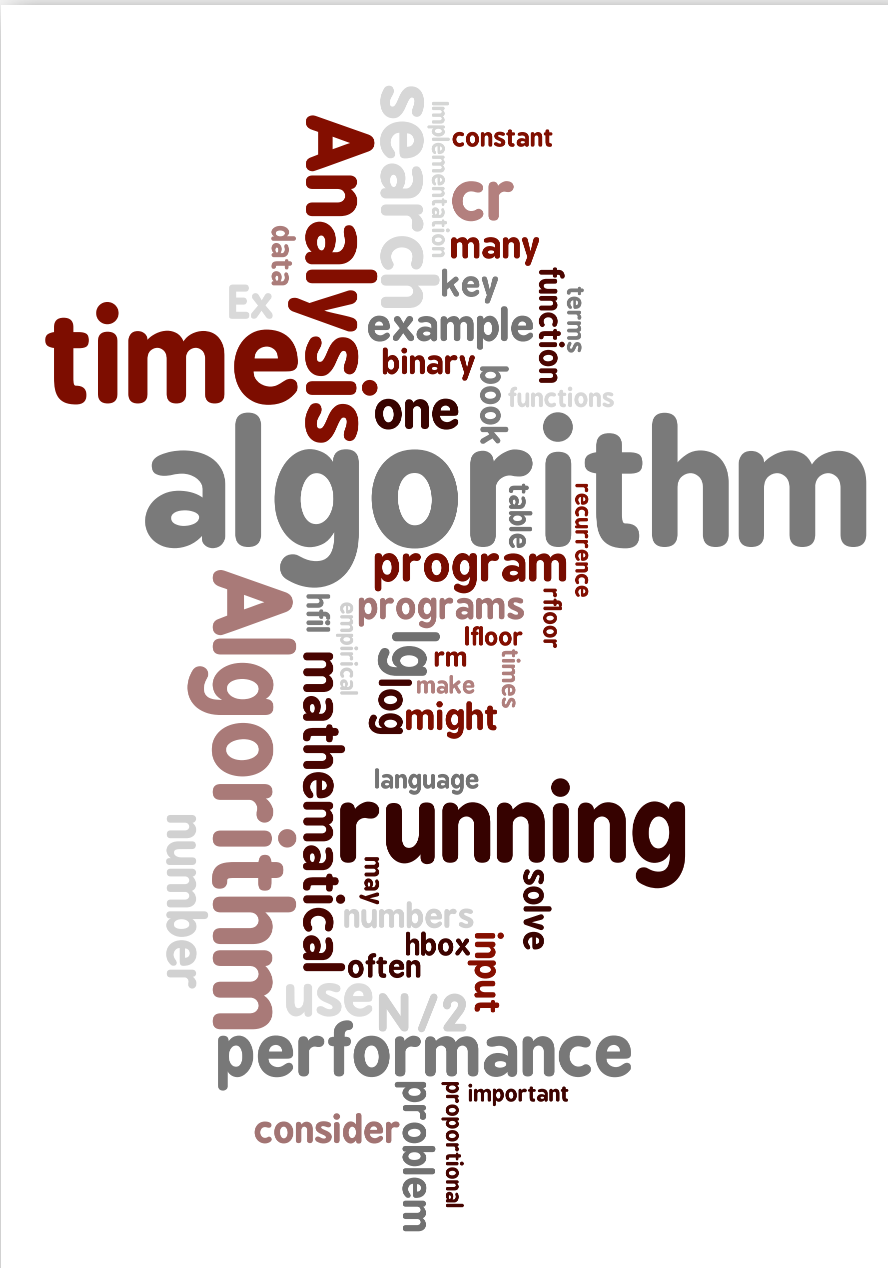
# 1.4 Analysis of Algorithms



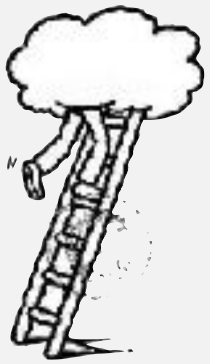

- observations
- mathematical models
- amortized analysis
- order-of-growth classifications
- dependencies on inputs
- memory

# Cast of characters

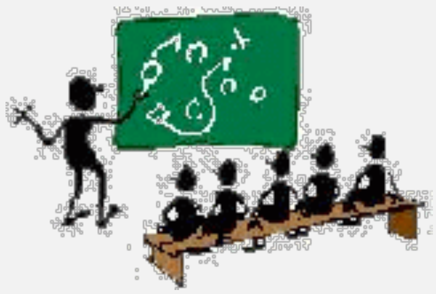
Programmer needs to develop a working solution.

Student might play any or all of these roles someday.

Client wants to solve problem efficiently.



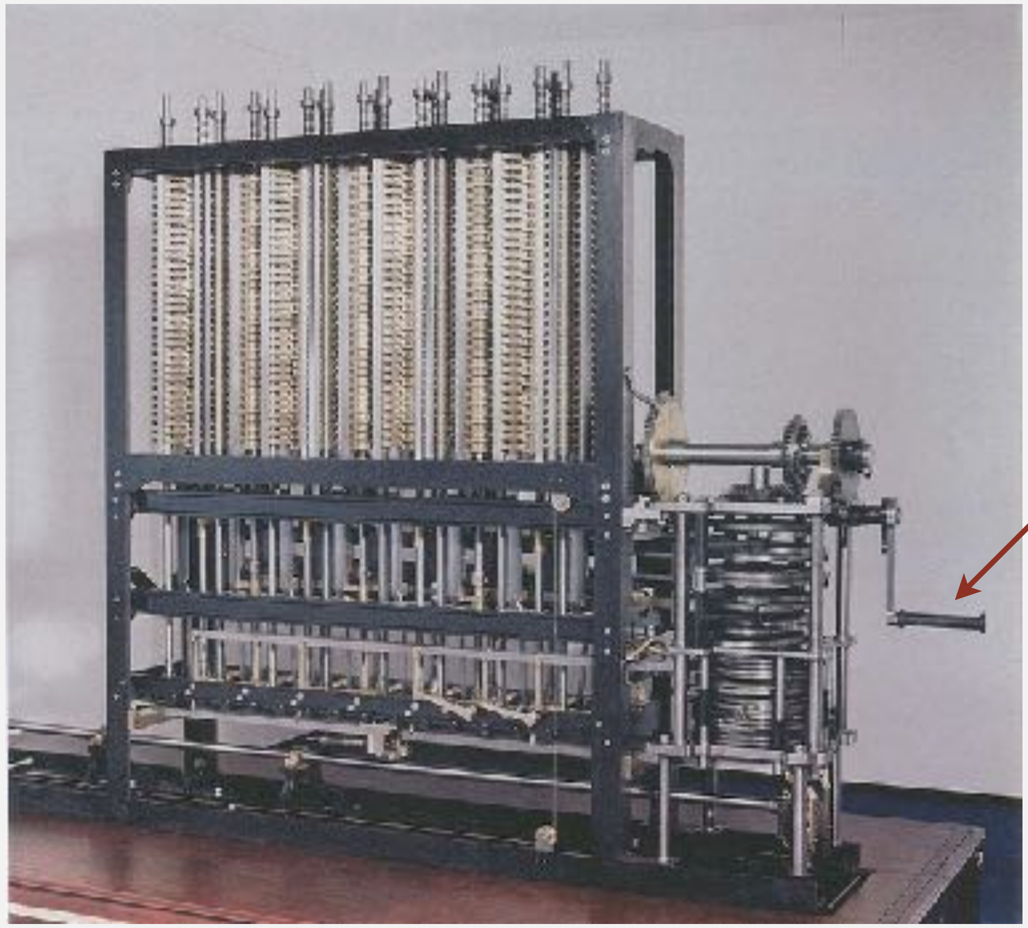
Theoretician wants to understand.



Basic blocking and tackling is sometimes necessary. [this lecture]

# Running time

*" As soon as an Analytic Engine exists, it will necessarily guide the future course of the science. Whenever any result is sought by its aid, the question will arise—By what course of calculation can these results be arrived at by the machine in the shortest time? "* — Charles Babbage (1864)



how many times do you have to turn the crank?

Analytic Engine

# Reasons to analyze algorithms

Predict performance.

Compare algorithms.

Provide guarantees.

Understand theoretical basis.

this course (CS 251)

theory of algorithms (CS 381)

Primary practical reason: avoid performance bugs.

client gets poor performance because programmer did not understand performance characteristics

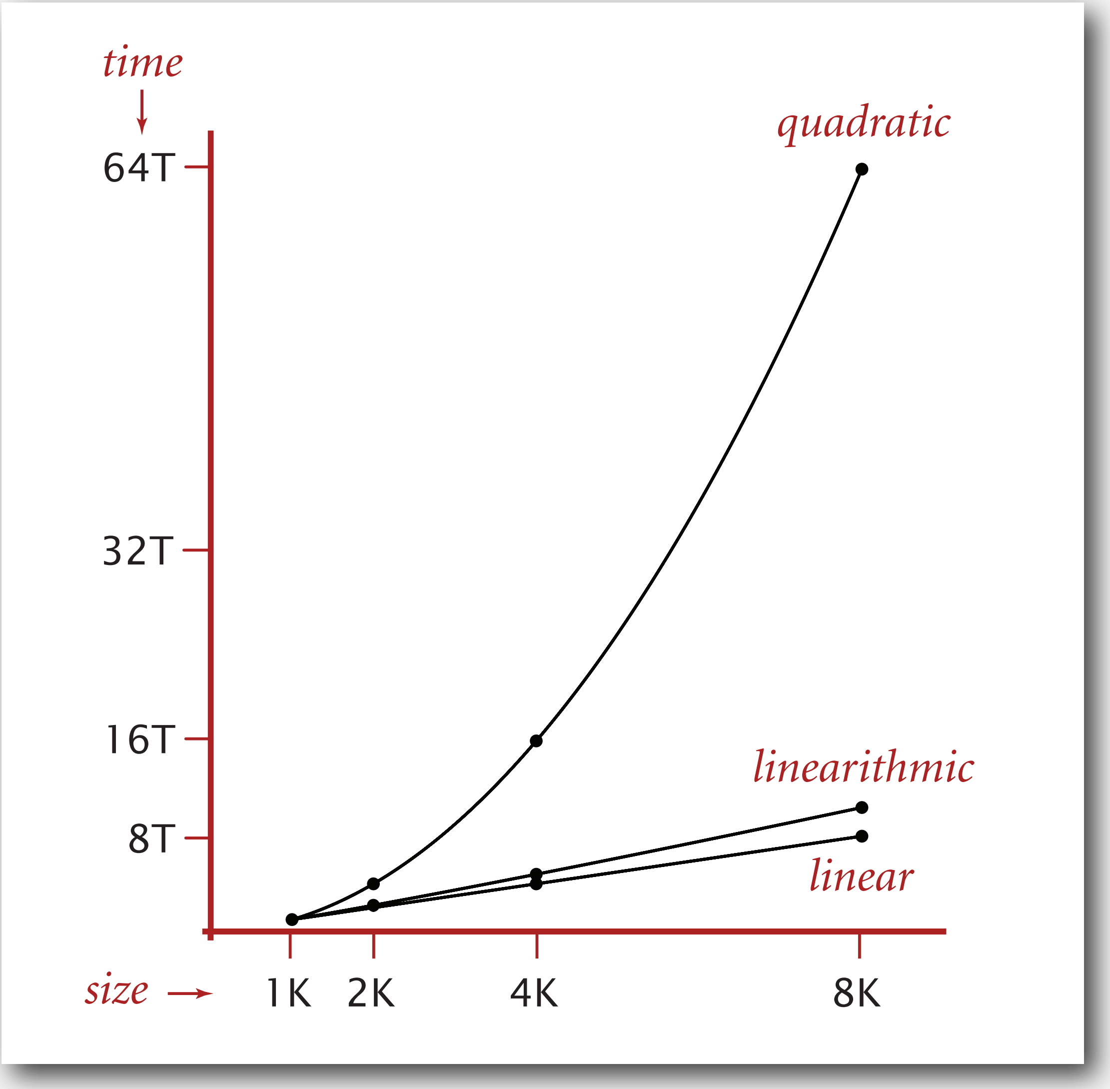
# Some algorithmic successes

Discrete Fourier transform.

- Break down waveform of $N$ samples into periodic components.
- Applications: DVD, JPEG, MRI, astrophysics, ….
- Brute force: $N^2$ steps.
- FFT algorithm: $N \log N$ steps, enables new technology.

Friedrich Gauss

1805

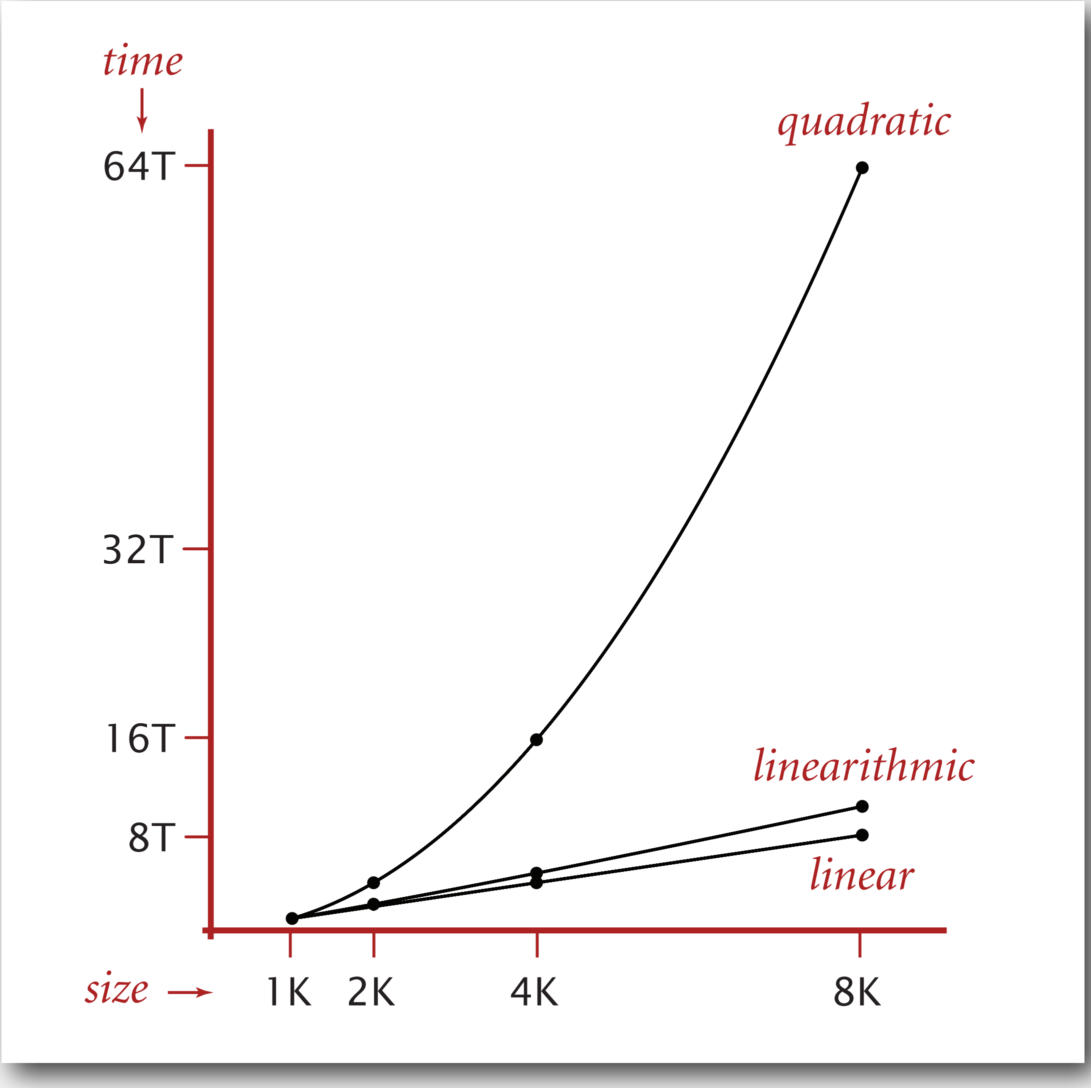


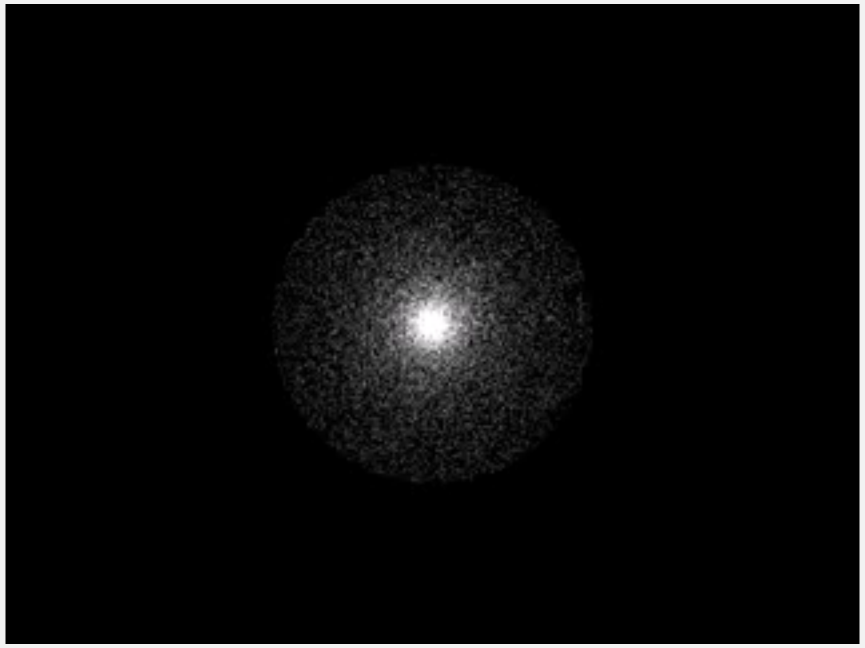

# Some algorithmic successes

N-body simulation.

- Simulate gravitational interactions among $N$ bodies.
- Brute force: $N^2$ steps.
- Barnes-Hut algorithm: $N \log N$ steps, enables new research.

Andrew Appel
PU '81



https://www.cs.purdue.edu/homes/cs251/slides/media/nbody.mov

# Some algorithmic successes

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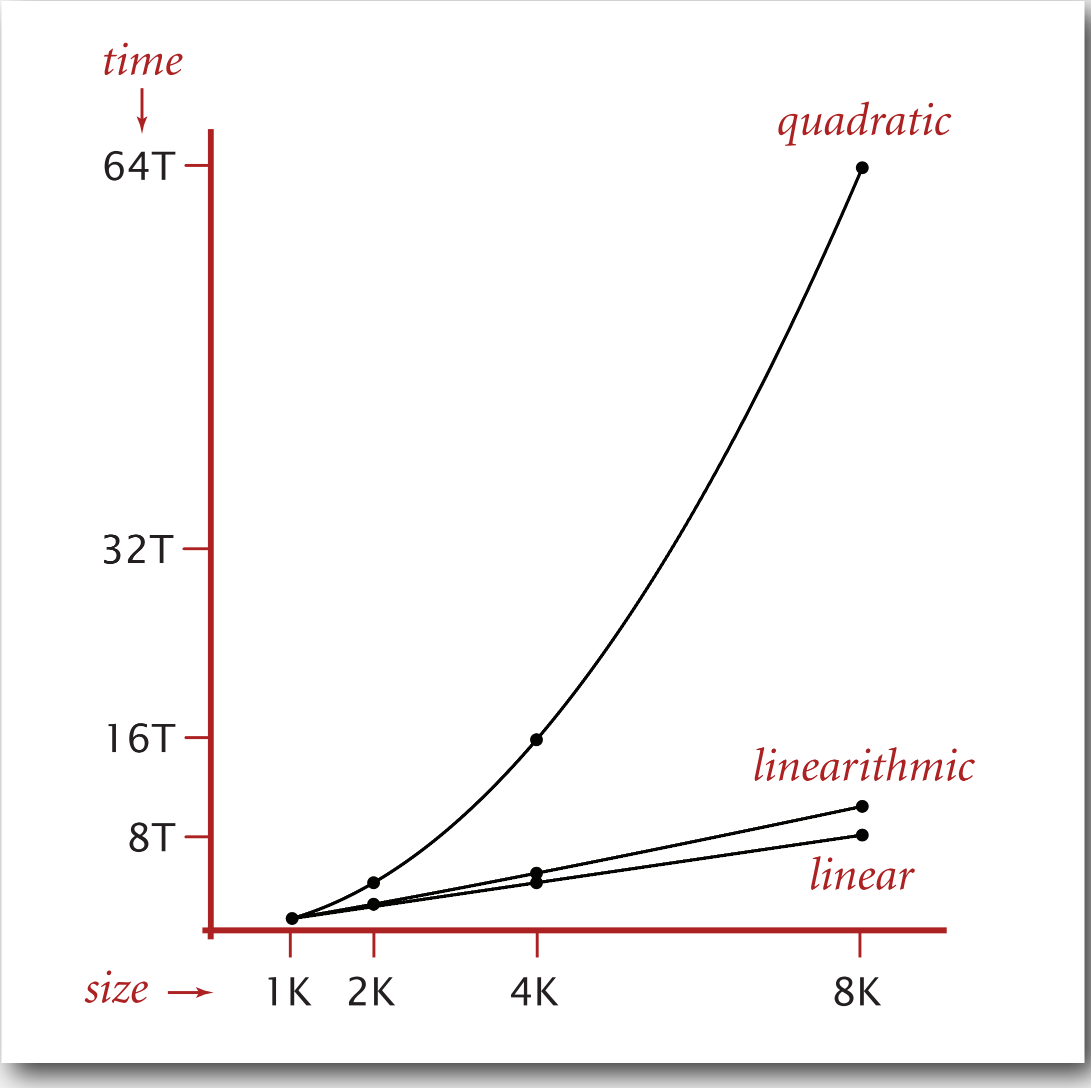
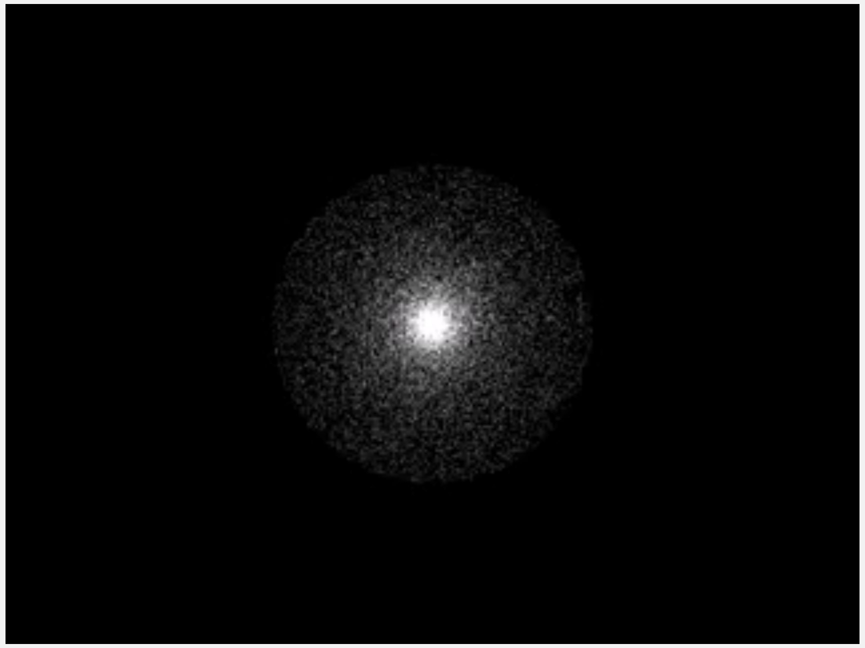
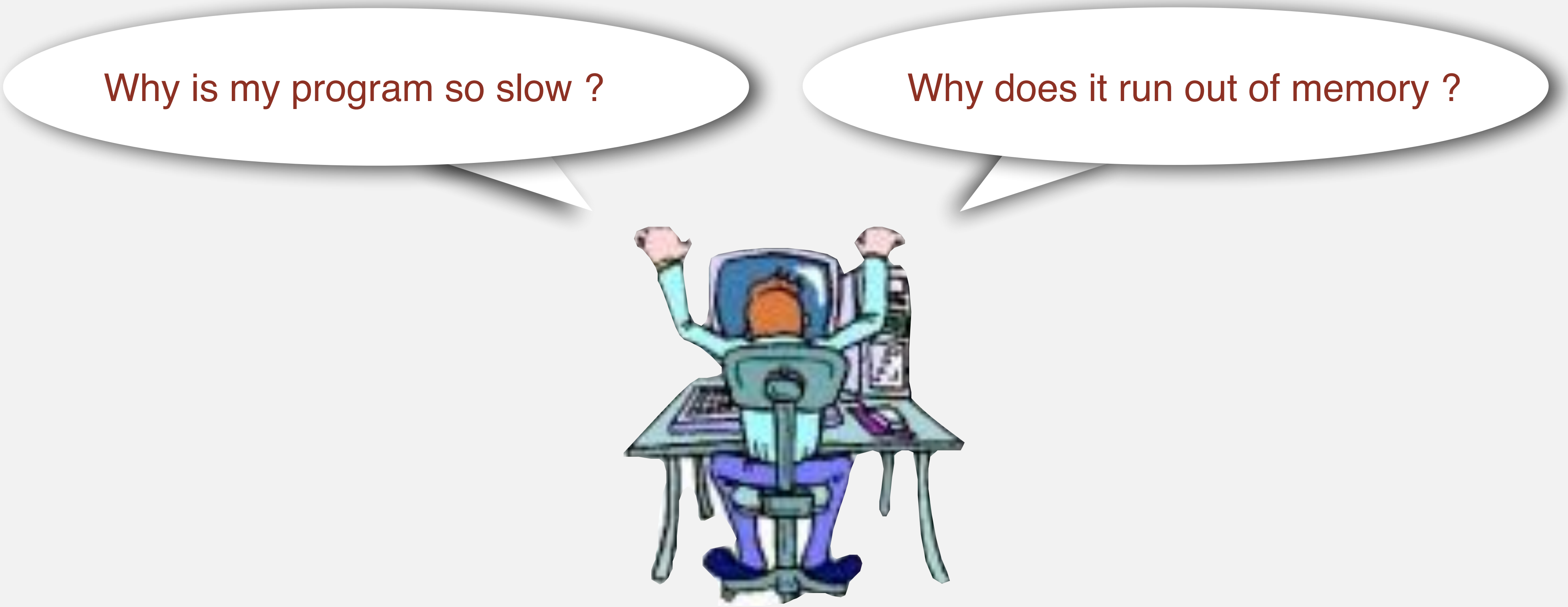


https://www.cs.purdue.edu/homes/cs251/slides/media/nbody.mov

# The challenge

Q. Will my program be able to solve a large practical input?




Key insight. [Knuth 1970s] Use scientific method to understand performance.

# Scientific method applied to analysis of algorithms

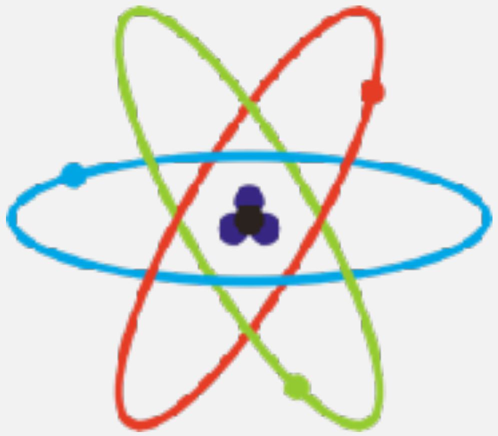
A framework for predicting performance and comparing algorithms.

Scientific method.

- Observe some feature of the natural world.
- Hypothesize a model that is consistent with the observations.
- Predict events using the hypothesis.
- Verify the predictions by making further observations.
- Validate by repeating until the hypothesis and observations agree.

Principles.

- Experiments must be reproducible.
- Hypotheses must be falsifiable.



Feature of the natural world = computer itself.

- **observations**
- mathematical models
- amortized analysis
- order-of-growth classifications
- dependencies on inputs
- memory

# Example: 3-sum

3-sum. Given $N$ distinct integers, how many triples sum to exactly zero?

```
% more 8ints.txt
8
30 -40 -20 -10 40 0 10 5

% java ThreeSum < 8ints.txt
4
```

| | a[i] | a[j] | a[k] | sum |
|---|---|---|---|---|
| 1 | 30 | -40 | 10 | 0 |
| 2 | 30 | -20 | -10 | 0 |
| 3 | -40 | 40 | 0 | 0 |
| 4 | -10 | 0 | 10 | 0 |

Context. Deeply related to problems in computational geometry.

# 3-sum: brute-force algorithm

```
public class ThreeSum
{
   public static int count(int[] a)
   {
      int N = a.length;
      int count = 0;
      for (int i = 0; i < N; i++)
         for (int j = i+1; j < N; j++)
            for (int k = j+1; k < N; k++)
               if (a[i] + a[j] + a[k] == 0)
                  count++;
      return count;
   }

   public static void main(String[] args)
   {
      int[] a = StdArrayIO.readInt1D();
      StdOut.println(count(a));
   }
}
```

check each triple

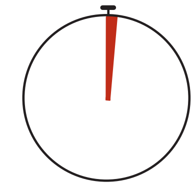
we ignore any integer overflow

# Measuring the running time

Q. How to time a program?

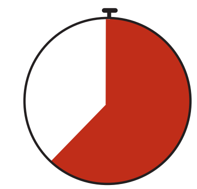
A. Manual.


```
% java ThreeSum < 1Kints.txt
```



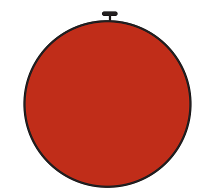
*tick tick tick*

```
70
% java ThreeSum < 2Kints.txt
```



*tick tick tick tick tick tick tick tick*
*tick tick tick tick tick tick tick tick*
*tick tick tick tick tick tick tick tick*

```
528
% java ThreeSum < 4Kints.txt
```



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```
4039
```

# Measuring the running time

Q. How to time a program?

A. Automatic.

| public class Stopwatch | | |
|---|---|---|
| | `Stopwatch()` | *create a new stopwatch* |
| `double` | `elapsedTime()` | *time since creation (in seconds)* |

```
public static void main(String[] args)
{
   int[] a = StdArrayIO.readInt1D();
   Stopwatch stopwatch = new Stopwatch();
   StdOut.println(ThreeSum.count(a));
   double time = stopwatch.elapsedTime();
}
```

# Measuring the running time

Q. How to time a program?

A. Automatic.

| public class Stopwatch | | |
|---|---|---|
| | `Stopwatch()` | *create a new stopwatch* |
| `double` | `elapsedTime()` | *time since creation (in seconds)* |

```
public class Stopwatch
{
   private final long start = System.currentTimeMillis();

   public double elapsedTime()
   {
      long now = System.currentTimeMillis();
      return (now - start) / 1000.0;
   }
}
```

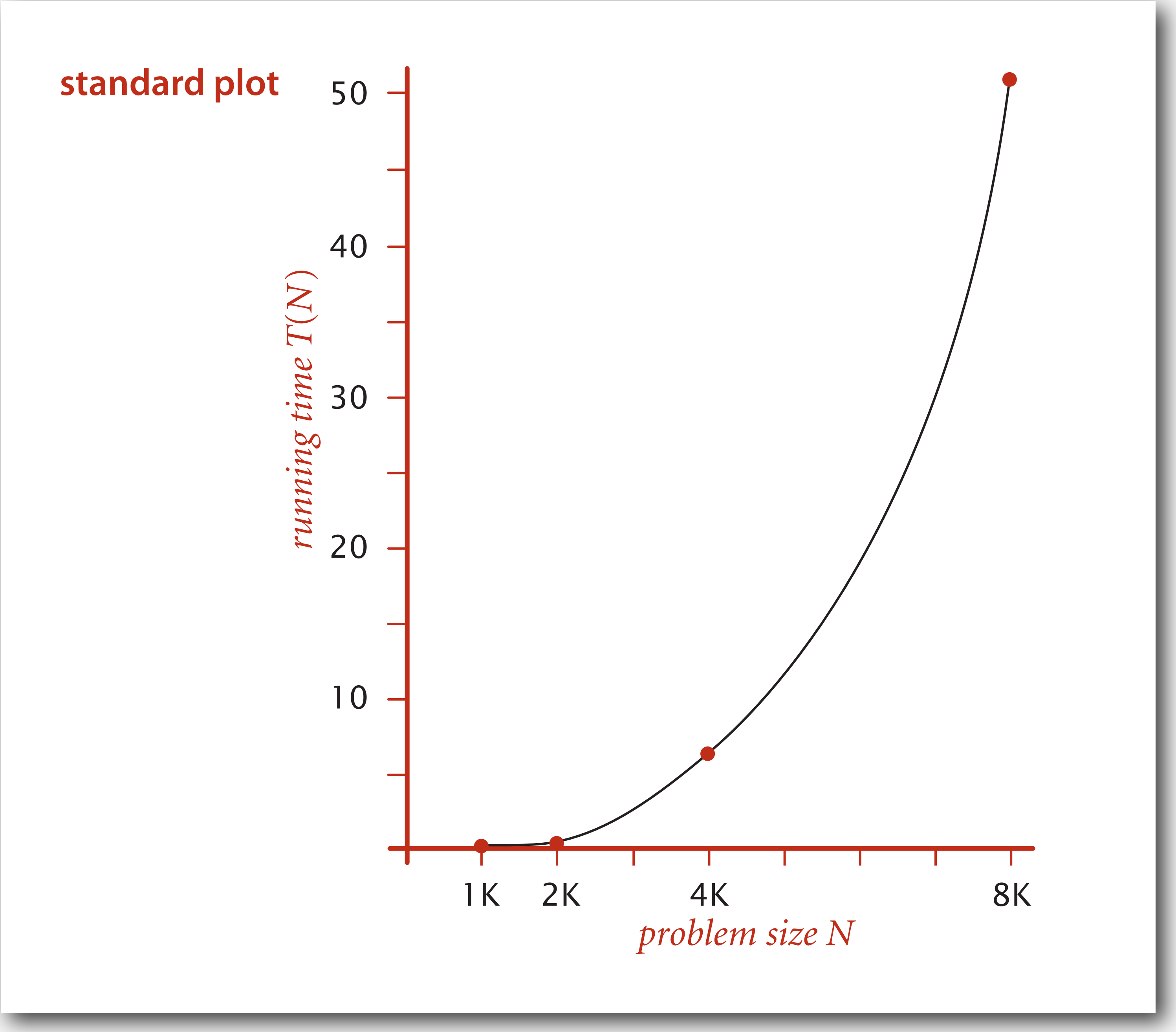
# Empirical analysis

Run the program for various input sizes and measure running time.

| N | time (seconds) † |
|---|---|
| 250 | 0 |
| 500 | 0 |
| 1,000 | 0.1 |
| 2,000 | 0.8 |
| 4,000 | 6.4 |
| 8,000 | 51.1 |
| 16,000 | ? |

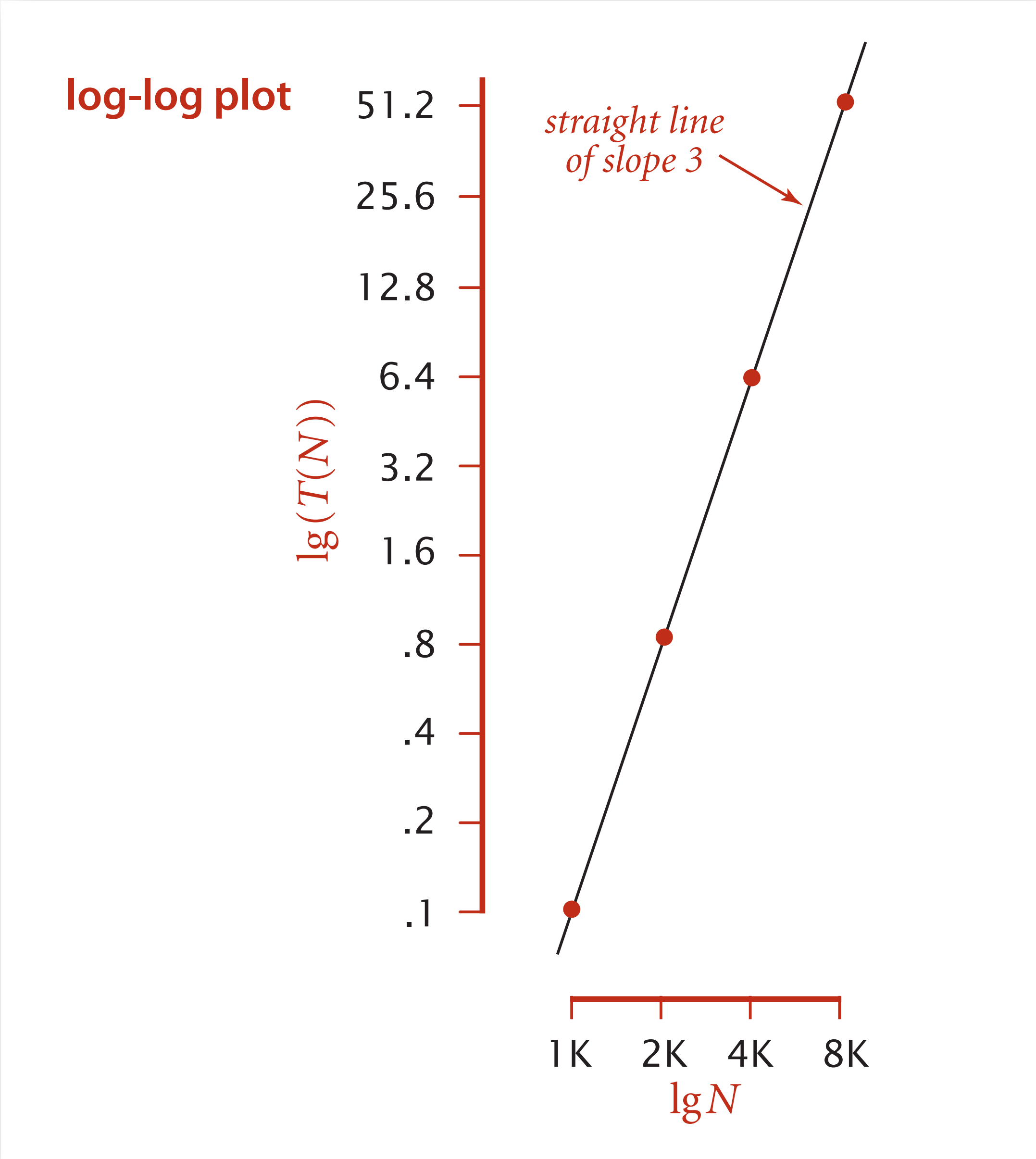
# Data analysis

**Standard plot.** Plot running time $T(N)$ vs. input size $N$.



# Data analysis

Log-log plot. Plot running time vs. input size $N$ using log-log scale.




$$\lg(T(N)) = b \lg N + c$$
$$b = 2.999$$
$$c = -33.2103$$

$$T(N) = a N^b, \text{ where } a = 2^c$$

power law

Regression. Fit straight line through data points: $a N^b$.

slope

Hypothesis. The running time is about $1.006 \times 10^{-10} \times N^{2.999}$ seconds.

# Prediction and validation

**Hypothesis.** The running time is about $1.006 \times 10^{-10} \times N^{2.999}$ seconds.

**Predictions.**

- $51.0$ seconds for $N = 8{,}000$.
- $408.1$ seconds for $N = 16{,}000$.

**Observations.**

| N | time (seconds) † |
|---|---|
| 8,000 | 51.1 |
| 8,000 | 51 |
| 8,000 | 51.1 |
| 16,000 | 410.8 |

validates hypothesis!

# Doubling hypothesis

Doubling hypothesis. Quick way to estimate $b$ in a power-law relationship.

Run program, doubling the size of the input.

| N | time (seconds) † | ratio | lg ratio |
|---|---|---|---|
| 250 | 0 | | – |
| 500 | 0 | 4.8 | 2.3 |
| 1,000 | 0.1 | 6.9 | 2.8 |
| 2,000 | 0.8 | 7.7 | 2.9 |
| 4,000 | 6.4 | 8 | 3 |
| 8,000 | 51.1 | 8 | 3 |

seems to converge to a constant $b \approx 3$

Hypothesis. Running time is about $a N^b$ with $b = \lg$ ratio.

Caveat. Cannot identify logarithmic factors with doubling hypothesis.

# Doubling hypothesis

Doubling hypothesis. Quick way to estimate $b$ in a power-law hypothesis.

Q. How to estimate $a$ ?

A. Run the program!

| N | time (seconds) † |
|---|---|
| 8,000 | 51.1 |
| 8,000 | 51 |
| 8,000 | 51.1 |

$$51.1 = a \times 8000^3$$
$$\Rightarrow \quad a = 9.98 \times 10^{-11}$$

Hypothesis. Running time is about $9.98 \times 10^{-11} \times N^3$ seconds.

almost identical hypothesis
to one obtained via linear regression

# Experimental algorithmics

System independent effects.

- Algorithm.
- Input data.

determines exponent b in power law

System dependent effects.

- Hardware: CPU, memory, cache, …
- Software: compiler, interpreter, garbage collector, …
- System: operating system, network, other applications, …

helps determines constant a in power law

Bad news. Difficult to get precise measurements.

Good news. Much easier and cheaper than other sciences.

e.g., can run huge number of experiments

# Example

Q. How long does this program take as a function of $N$ ?

```
String s = StdIn.readString();
int N = s.length();
...
for (int i = 0; i < N; i++)
   for (int j = 0; j < N; j++)
      distance[i][j] = ...
...
```

| N | time |
|---|---|
| 1,000 | 0.11 |
| 2,000 | 0.35 |
| 4,000 | 1.6 |
| 8,000 | 6.5 |

Jenny $\sim c_1 N^2$ seconds

| N | time |
|---|---|
| 250 | 0.5 |
| 500 | 1.1 |
| 1,000 | 1.9 |
| 2,000 | 3.9 |

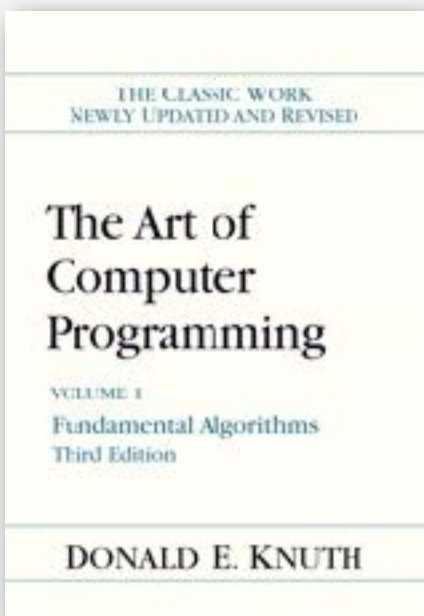
Kenny $\sim c_2 N$ seconds

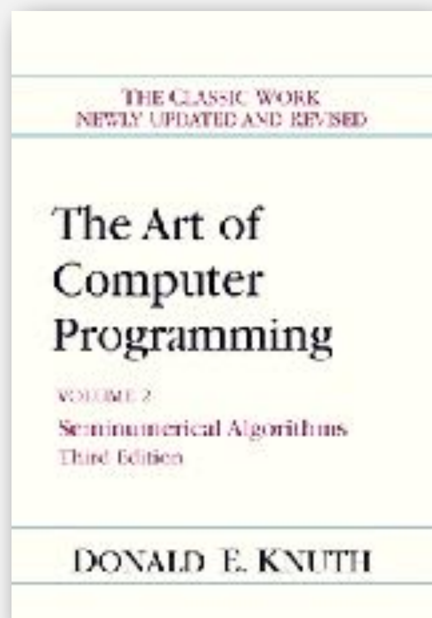
- observations
- **mathematical models**
- amortized analysis
- order-of-growth classifications
- dependencies on inputs
- memory

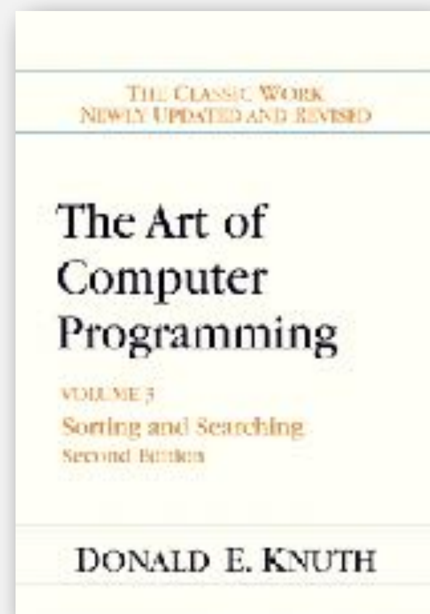
# Mathematical models for running time

Total running time: sum of cost × frequency for all operations.

- Need to analyze program to determine set of operations.
- Cost depends on machine, compiler.
- Frequency depends on algorithm, input data.




Donald Knuth
1974 Turing Award

In principle, accurate mathematical models are available.

# Cost of basic operations

| operation | example | nanoseconds † |
|---|---|---|
| integer add | `a + b` | 2.1 |
| integer multiply | `a * b` | 2.4 |
| integer divide | `a / b` | 5.4 |
| floating-point add | `a + b` | 4.6 |
| floating-point multiply | `a * b` | 4.2 |
| floating-point divide | `a / b` | 13.5 |
| sine | `Math.sin(theta)` | 91.3 |
| arctangent | `Math.atan2(y, x)` | 129 |
| ... | ... | ... |

† Running OS X on Macbook Pro 2.2GHz with 2GB RAM

# Cost of basic operations

| operation | example | nanoseconds † |
|---|---|---|
| variable declaration | `int a` | $c_1$ |
| assignment statement | `a = b` | $c_2$ |
| integer compare | `a < b` | $c_3$ |
| array element access | `a[i]` | $c_4$ |
| array length | `a.length` | $c_5$ |
| 1D array allocation | `new int[N]` | $c_6 N$ |
| 2D array allocation | `new int[N][N]` | $c_7 N^2$ |
| string length | `s.length()` | $c_8$ |
| substring extraction | `s.substring(N/2, N)` | $c_9$ |
| string concatenation | `s + t` | $c_{10} N$ |

Novice mistake. Abusive string concatenation.

# Example: 1-sum

Q. How many instructions as a function of input size $N$ ?

```
int count = 0;
for (int i = 0; i < N; i++)
   if (a[i] == 0)
      count++;
```

| operation | frequency |
|---|---|
| variable declaration | 2 |
| assignment statement | 2 |
| less than compare | N + 1 |
| equal to compare | N |
| array access | N |
| increment | N to 2 N |

# Example: 2-sum

Q. How many instructions as a function of input size $N$?

```
int count = 0;
for (int i = 0; i < N; i++)
   for (int j = i+1; j < N; j++)
      if (a[i] + a[j] == 0)
         count++;
```

$$0 + 1 + 2 + \ldots + (N-1) = \frac{1}{2} N(N-1) = \binom{N}{2}$$

| operation | frequency |
|---|---|
| variable declaration | N + 2 |
| assignment statement | N + 2 |
| less than compare | ½ (N + 1) (N + 2) |
| equal to compare | ½ N (N – 1) |
| array access | N (N – 1) |
| increment | N to 2 N |

tedious to count exactly

# Simplification 1: cost model

Cost model. Use some basic operation as a proxy for running time.

```
int count = 0;
for (int i = 0; i < N; i++)
   for (int j = i+1; j < N; j++)
      if (a[i] + a[j] == 0)
         count++;
```

$$0 + 1 + 2 + \ldots + (N-1) = \frac{1}{2} N (N-1) = \binom{N}{2}$$

| operation | frequency |
|---|---|
| variable declaration | N + 2 |
| assignment statement | N + 2 |
| less than compare | ½ (N + 1) (N + 2) |
| equal to compare | ½ N (N − 1) |
| array access | N (N − 1) |
| increment | N to 2 N |

cost model = array accesses

# Simplification 2: tilde notation

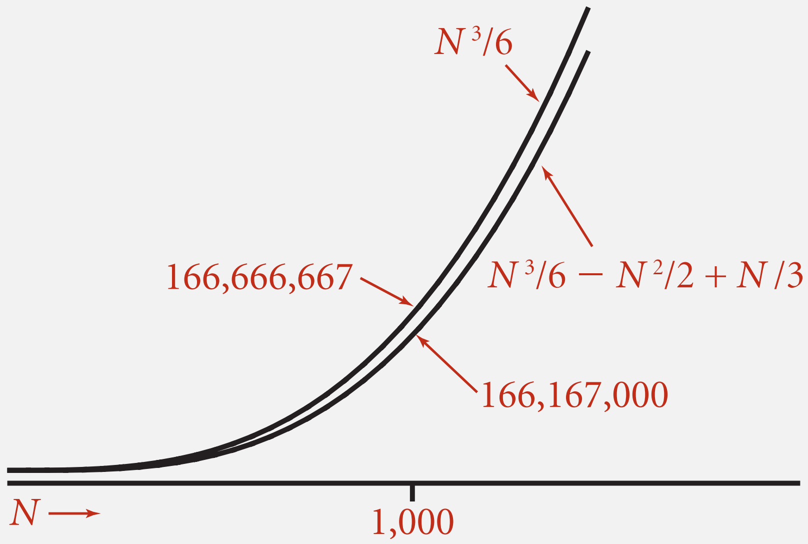
- Estimate running time (or memory) as a function of input size $N$.
- Ignore lower order terms.
  - when $N$ is large, terms are negligible
  - when $N$ is small, we don't care

Ex 1. $\frac{1}{6} N^3 + 20 N + 16 \quad \sim \frac{1}{6} N^3$

Ex 2. $\frac{1}{6} N^3 + 100 N^{4/3} + 56 \quad \sim \frac{1}{6} N^3$

Ex 3. $\frac{1}{6} N^3 - \underbrace{\frac{1}{2} N^2 + \frac{1}{3} N}_{\text{discard lower-order terms}} \quad \sim \frac{1}{6} N^3$

(e.g., N = 1000: 500 thousand vs. 166 million)



Leading-term approximation

**Technical definition.** $f(N) \sim g(N)$ means $\displaystyle \lim_{N \to \infty} \frac{f(N)}{g(N)} = 1$

# Simplification 2: tilde notation

- Estimate running time (or memory) as a function of input size $N$.
- Ignore lower order terms.
  - when $N$ is large, terms are negligible
  - when $N$ is small, we don't care

| operation | frequency | tilde notation |
|---|---|---|
| variable declaration | $N + 2$ | $\sim N$ |
| assignment statement | $N + 2$ | $\sim N$ |
| less than compare | $\frac{1}{2} (N + 1) (N + 2)$ | $\sim \frac{1}{2} N^2$ |
| equal to compare | $\frac{1}{2} N (N - 1)$ | $\sim \frac{1}{2} N^2$ |
| array access | $N (N - 1)$ | $\sim N^2$ |
| increment | $N$ to $2 N$ | $\sim N$ to $\sim 2 N$ |

# Example: 2-sum

Q. Approximately how many array accesses as a function of input size $N$ ?

```
int count = 0;
for (int i = 0; i < N; i++)
   for (int j = i+1; j < N; j++)
      if (a[i] + a[j] == 0)
         count++;
```

"inner loop"

$$0 + 1 + 2 + \ldots + (N-1) = \frac{1}{2} N (N-1) = \binom{N}{2}$$

A. $\sim N^2$ array accesses.

Bottom line. Use cost model and tilde notation to simplify frequency counts.

# Example: 3-sum

Q. Approximately how many array accesses as a function of input size $N$?

```
int count = 0;
for (int i = 0; i < N; i++)
   for (int j = i+1; j < N; j++)
      for (int k = j+1; k < N; k++)
         if (a[i] + a[j] + a[k] == 0)
            count++;
```

"inner loop"

$$\binom{N}{3} = \frac{N(N-1)(N-2)}{3!} \sim \frac{1}{6}N^3$$

A. $\sim \frac{1}{2} N^3$ array accesses.

Bottom line. Use cost model and tilde notation to simplify frequency counts.

# Estimating a discrete sum

Q. How to estimate a discrete sum?

A1. Take COS 340.

A2. Replace the sum with an integral, and use calculus!

Ex 1. $1 + 2 + \ldots + N.$

$$\sum_{i=1}^{N} i \ \sim \ \int_{x=1}^{N} x \, dx \ \sim \ \frac{1}{2} N^2$$

Ex 2. $1 + 1/2 + 1/3 + \ldots + 1/N.$

$$\sum_{i=1}^{N} \frac{1}{i} \ \sim \ \int_{x=1}^{N} \frac{1}{x} dx \ = \ \ln N$$

Ex 3. 3-sum triple loop.

$$\sum_{i=1}^{N} \sum_{j=i}^{N} \sum_{k=j}^{N} 1 \ \sim \ \int_{x=1}^{N} \int_{y=x}^{N} \int_{z=y}^{N} dz \, dy \, dx \ \sim \ \frac{1}{6} N^3$$

# Mathematical models for running time

In principle, accurate mathematical models are available.

In practice,

- Formulas can be complicated.
- Advanced mathematics might be required.
- Exact models best left for experts.

costs (depend on machine, compiler)

$$T_N = c_1 A + c_2 B + c_3 C + c_4 D + c_5 E$$

$A$ = array access

$B$ = integer add

$C$ = integer compare

$D$ = increment

$E$ = variable assignment

frequencies (depend on algorithm, input)

Bottom line. We use approximate models in this course: $T(N) \sim c N^3$.

- observations
- mathematical models
- **amortized analysis**
- order-of-growth classifications
- dependencies on inputs
- memory

# Recall: Stack dynamic-array implementation

Amortized analysis. Average running time per operation over a worst-case sequence of operations. [stay tuned]

Proposition. Starting from empty stack (with dynamic resizing), any sequence of $M$ push and pop operations takes time proportional to $M$.

| | best | worst | amortized |
|---|---|---|---|
| construct | 1 | 1 | 1 |
| push | 1 | N | 1 |
| pop | 1 | N | 1 |
| size | 1 | 1 | 1 |

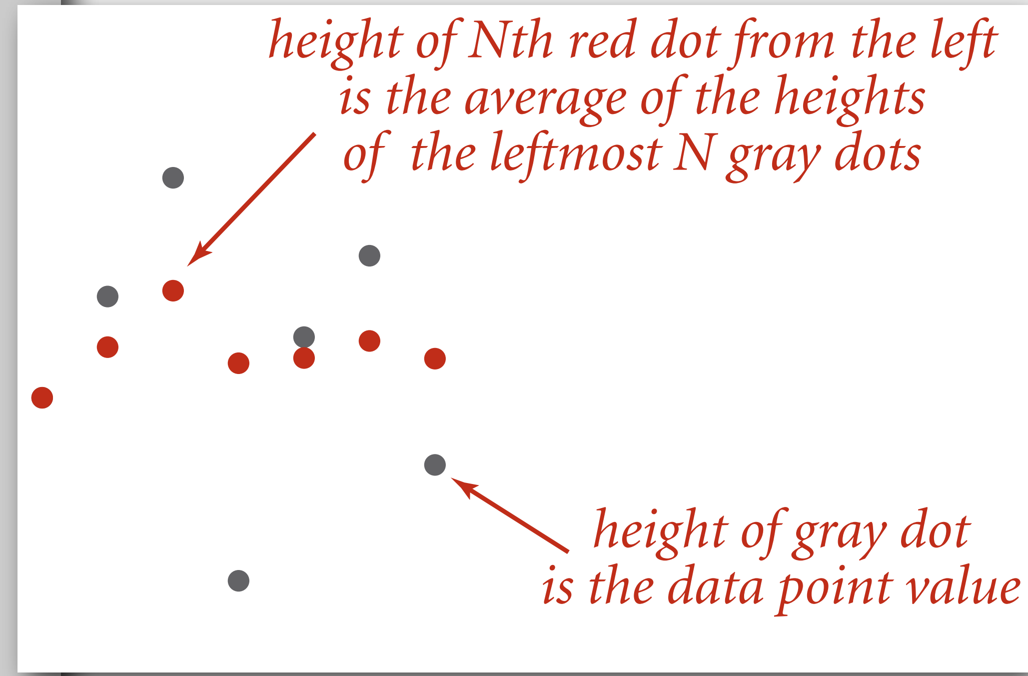
doubling and shrinking

**running time for doubling stack with N items**

# Amortized analysis

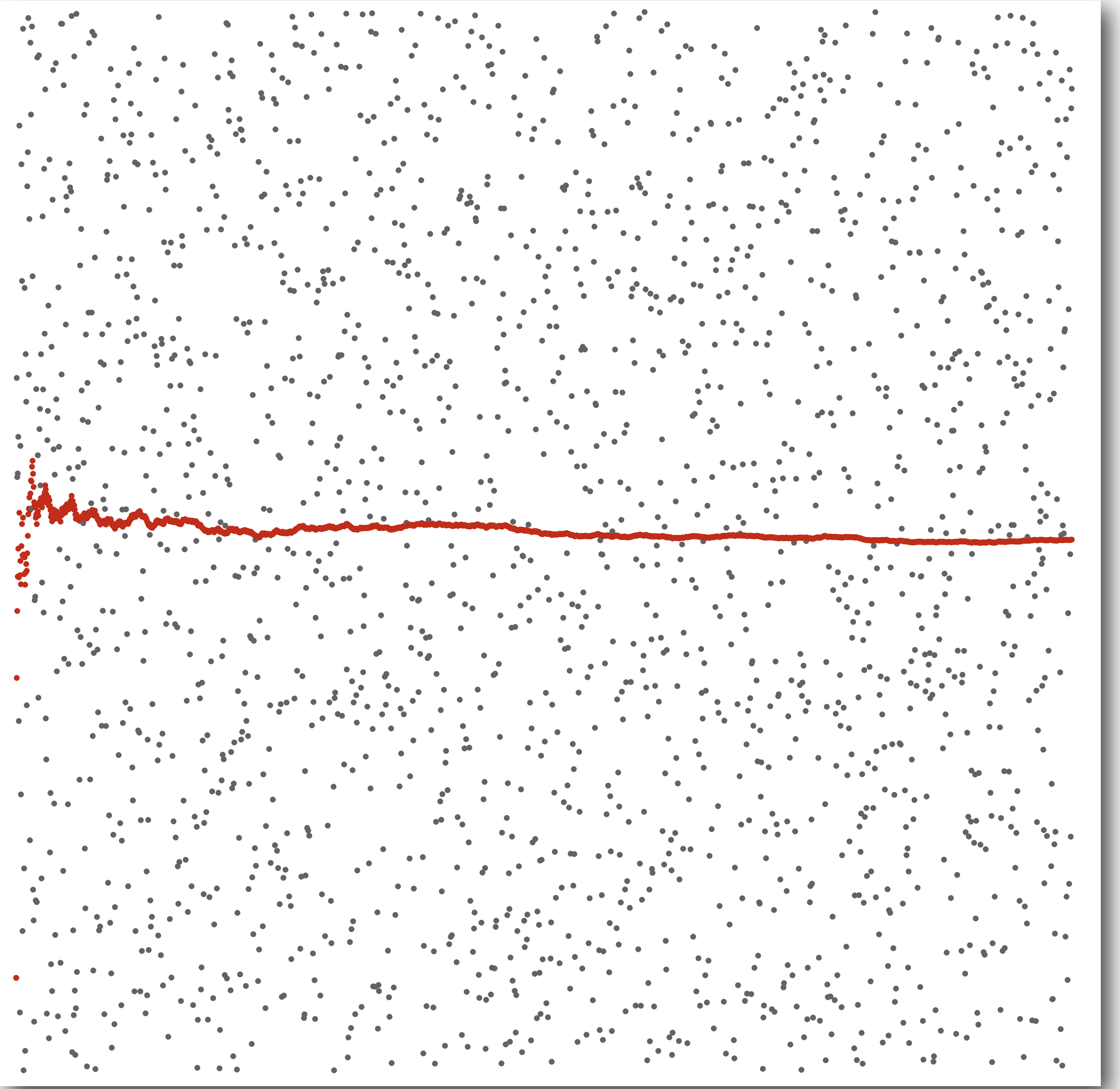
Often useful to compute average cost per operation over a sequence of ops.

```
public class VisualAccumulator
{
    private double total;
    private int N;

    public VisualAccumulator(int maxN, double max)
    {
        StdDraw.setXscale(0, maxN);
        StdDraw.setYscale(0, max);
        StdDraw.setPenRadius(.005);
    }

    public void addDataValue(double val)
    {
        N++;
        total += val;
        StdDraw.setPenColor(StdDraw.DARK_GRAY);
        StdDraw.point(N, val);
        StdDraw.setPenColor(StdDraw.RED);
        StdDraw.point(N, total/N);
    }
}
```



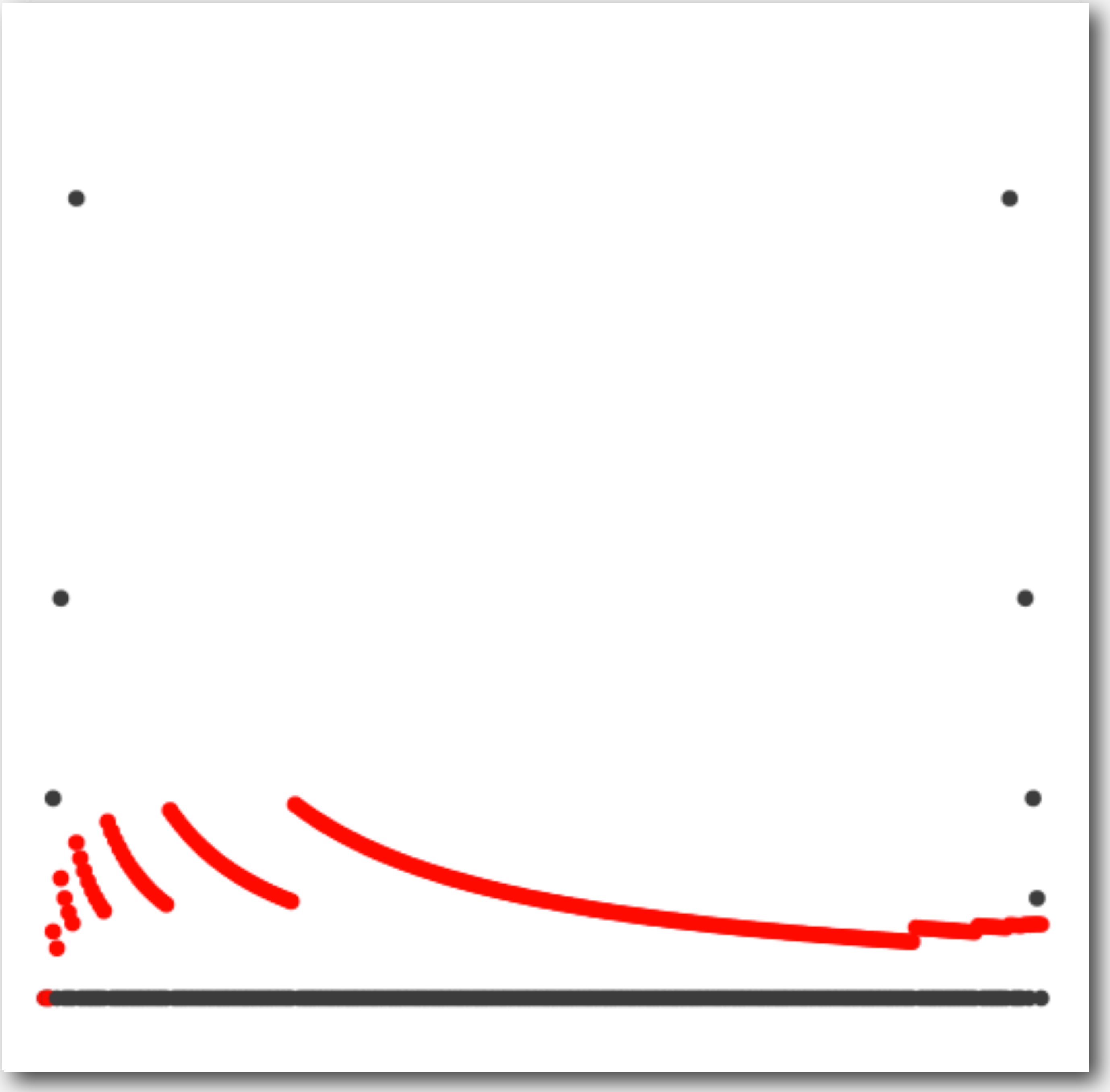

Visual accumulator plot

# Random data values

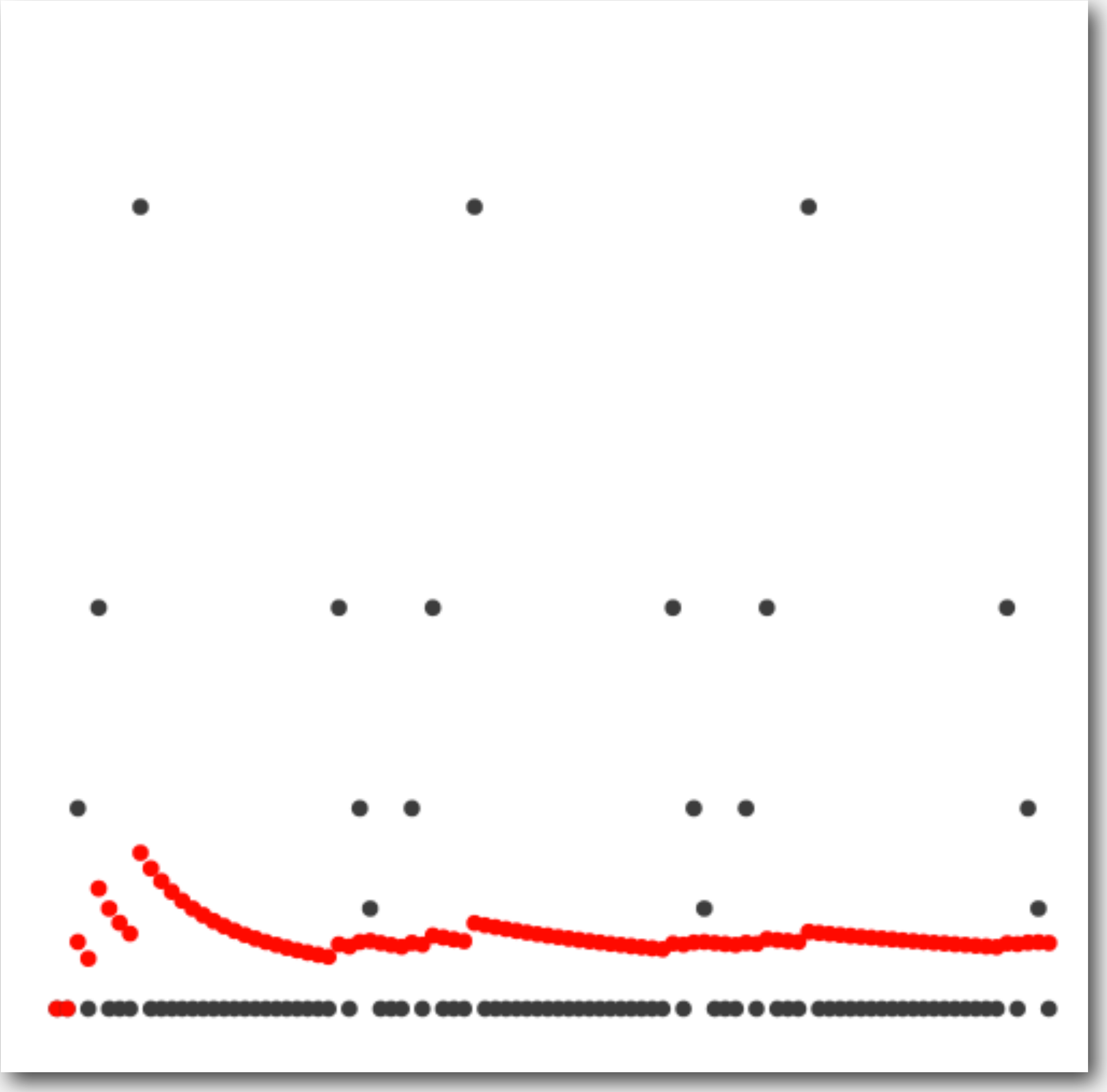


```
VisualAccumulator a;
a = new VisualAccumulator(2000, 1.0);
for (int i = 0; i < 2000; i++)
   a.addDataValue(Math.random());
```

# Doubling stack (N pushes followed by N pops)

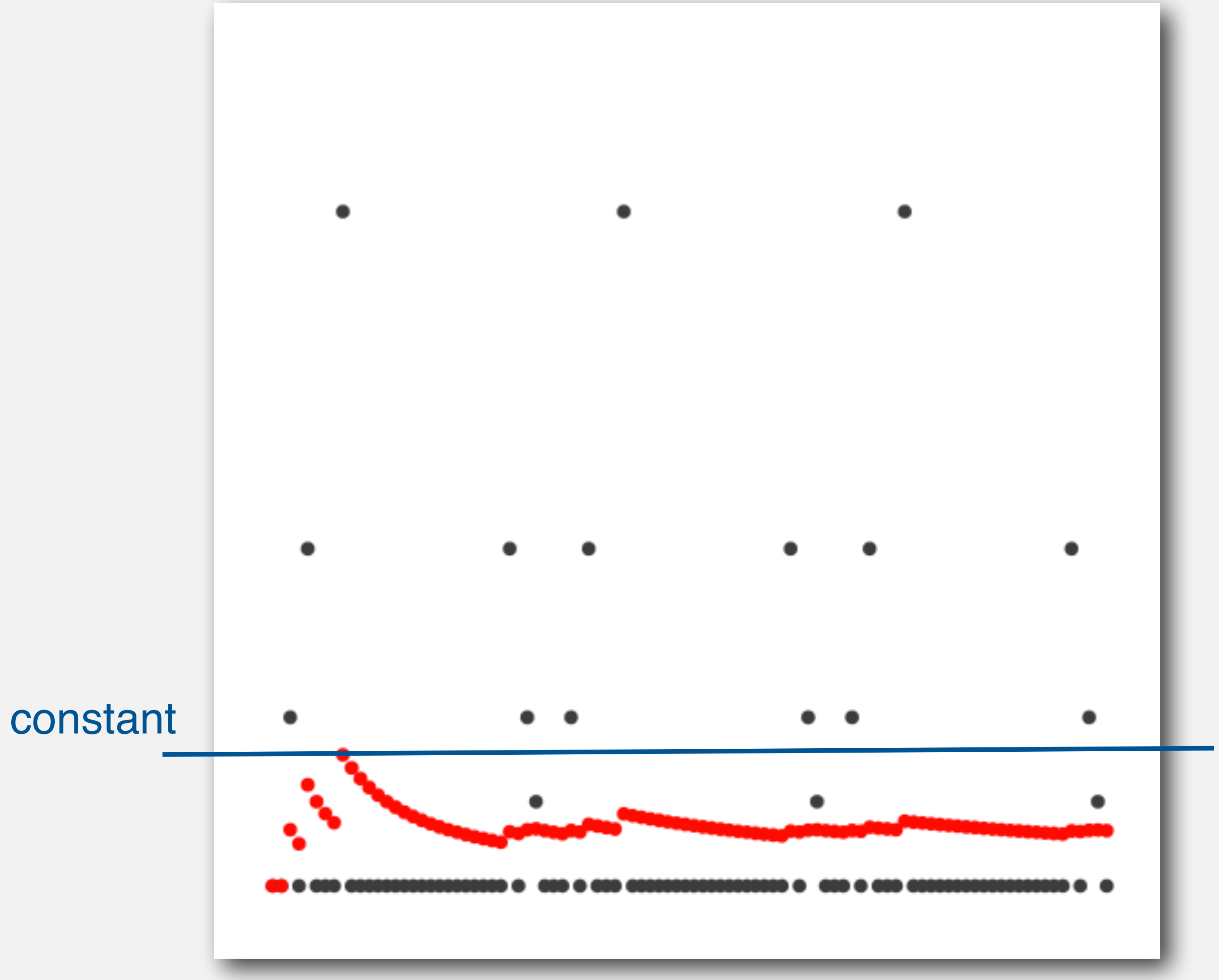


# Doubling stack (N pushes followed by N pops, three times)

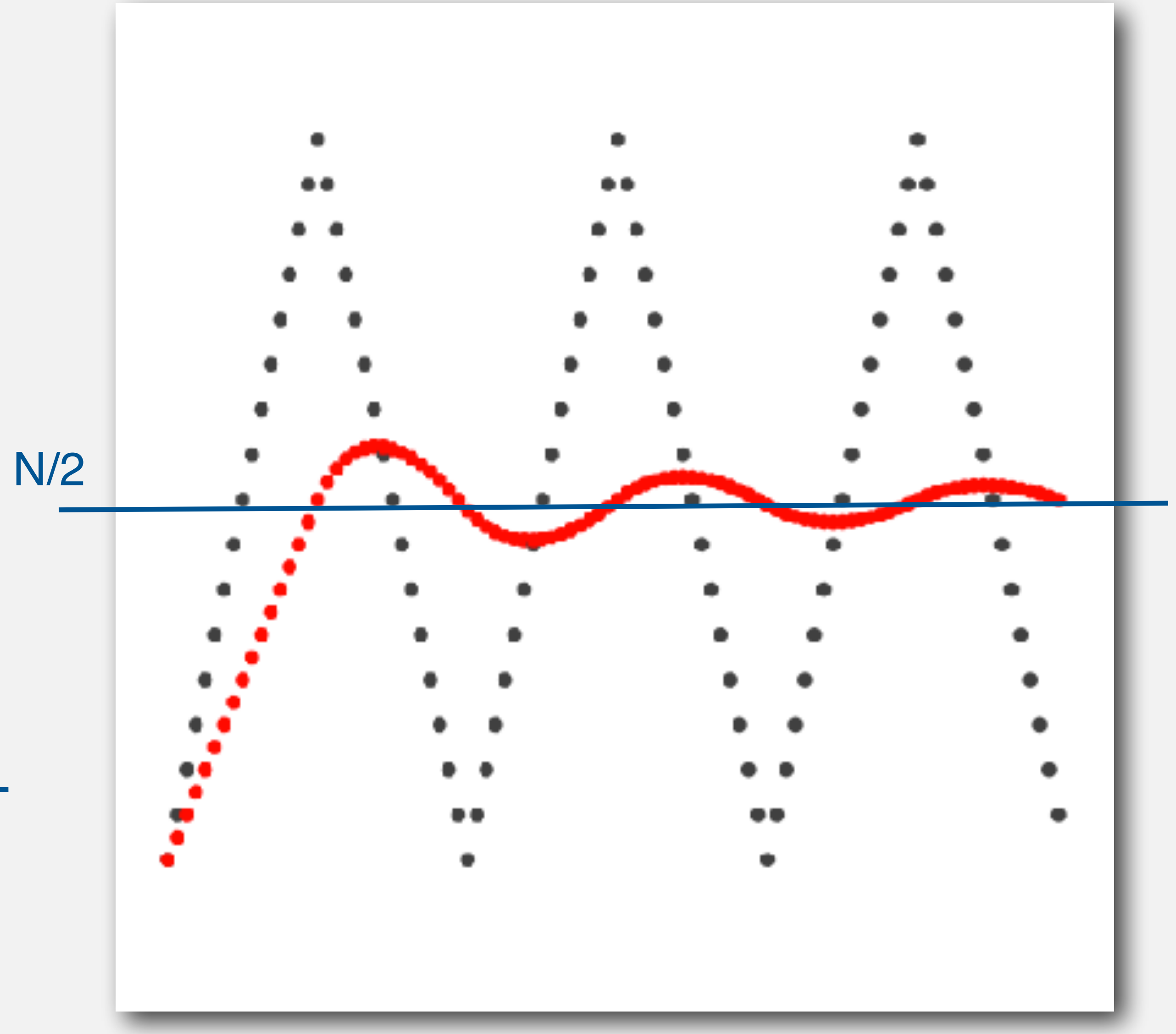


# Stack array implementation alternatives

Doubling




Resize after every op



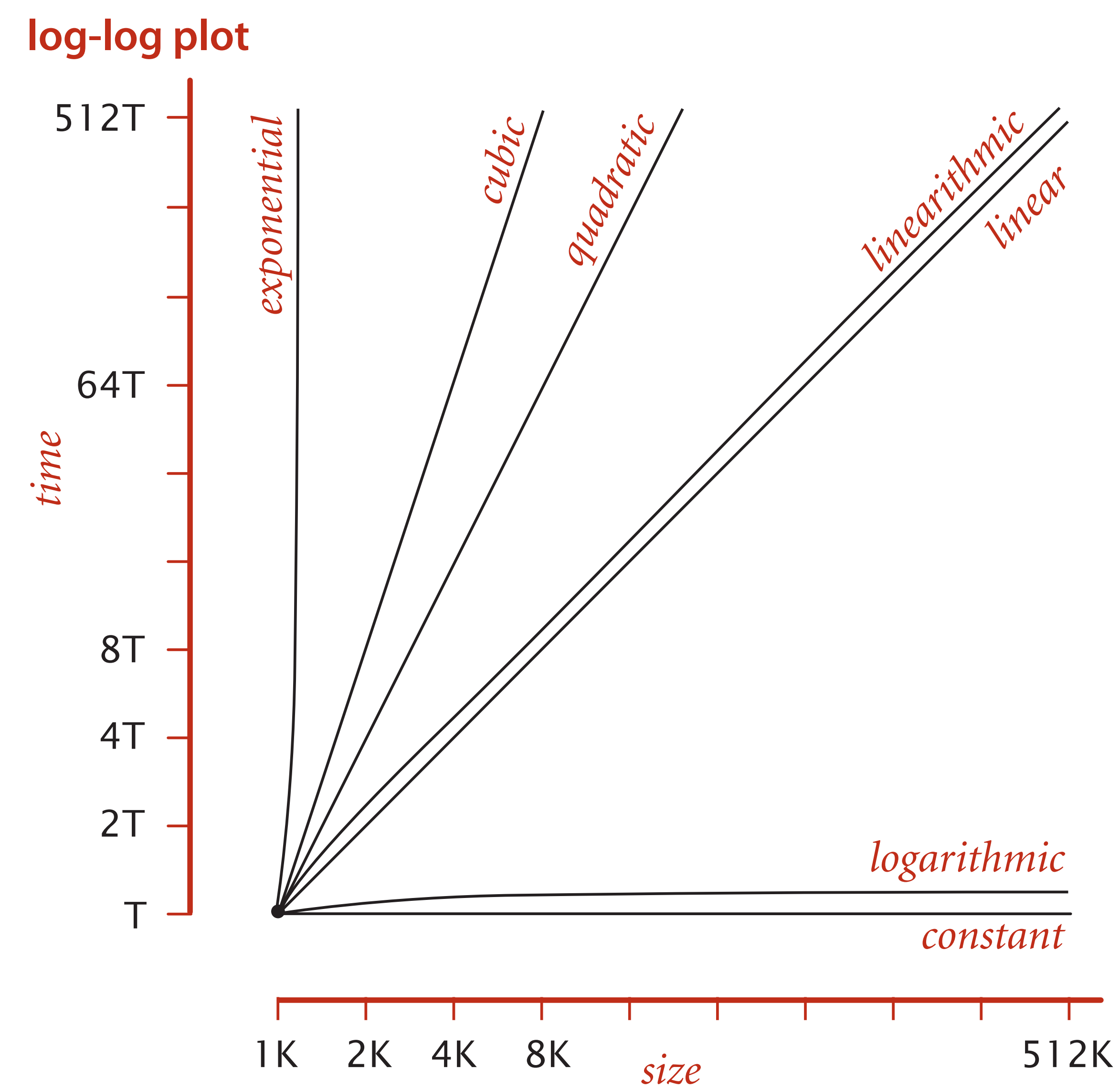
- observations
- mathematical models
- amortized analysis
- **order-of-growth classifications**
- dependencies on inputs
- memory

# Common order-of-growth classifications

Good news. the small set of functions

$$1, \ \log N, \ N, \ N \log N, \ N^2, \ N^3, \text{ and } 2^N$$

suffices to describe order-of-growth of typical algorithms.




Typical orders of growth

# Common order-of-growth classifications

| growth rate | name | typical code framework | description | example | T(2N) / T(N) |
|---|---|---|---|---|---|
| 1 | constant | `a = b + c;` | statement | add two numbers | 1 |
| log N | logarithmic | `while (N > 1)`<br>`{   N = N / 2;   ...   }` | divide in half | binary search | ~ 1 |
| N | linear | `for (int i = 0; i < N; i++)`<br>`{   ...           }` | loop | find the maximum | 2 |
| N log N | linearithmic | [see mergesort lecture] | divide and conquer | mergesort | ~ 2 |
| $N^2$ | quadratic | `for (int i = 0; i < N; i++)`<br>`  for (int j = 0; j < N; j++)`<br>`  {   ...           }` | double loop | check all pairs | 4 |
| $N^3$ | cubic | `for (int i = 0; i < N; i++)`<br>`  for (int j = 0; j < N; j++)`<br>`    for (int k = 0; k < N; k+`<br>`+)`<br>`{                }` | triple loop | check all triples | 8 |
| $2^N$ | exponential | [see combinatorial search lecture] | exhaustive search | check all subsets | T(N) |

# Practical implications of order-of-growth

| growth rate | problem size solvable in minutes | | | |
|---|---|---|---|---|
| | 1970s | 1980s | 1990s | 2000s |
| 1 | any | any | any | any |
| log N | any | any | any | any |
| N | millions | tens of millions | hundreds of millions | billions |
| N log N | hundreds of thousands | millions | millions | hundreds of millions |
| $N^2$ | hundreds | thousand | thousands | tens of thousands |
| $N^3$ | hundred | hundreds | thousand | thousands |
| $2^N$ | 20 | 20s | 20s | 30 |

Bottom line. Need linear or linearithmic alg to keep pace with Moore's law.

# Binary search

Goal. Given a sorted array and a key, find index of the key in the array?

Successful search. Binary search for 33.

| 6 | 13 | 14 | 25 | 33 | 43 | 51 | 53 | 64 | 72 | 84 | 93 | 95 | 96 | 97 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 |
| lo | | | | | | | | | | | | | | hi |

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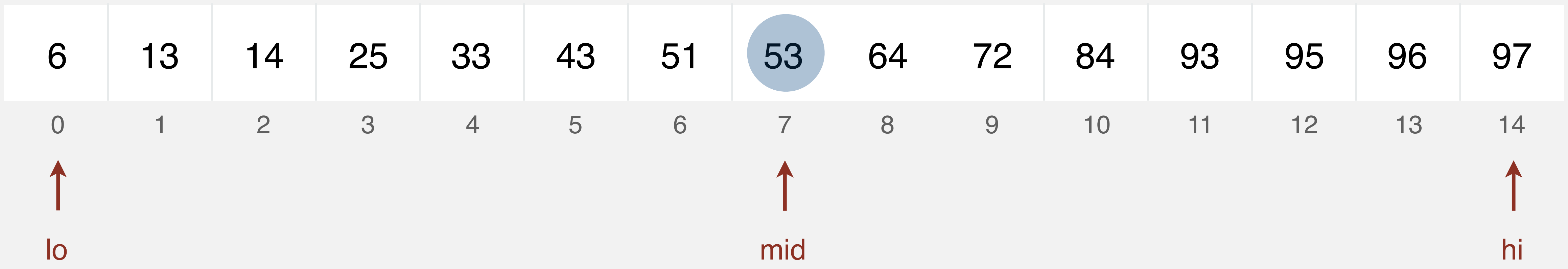

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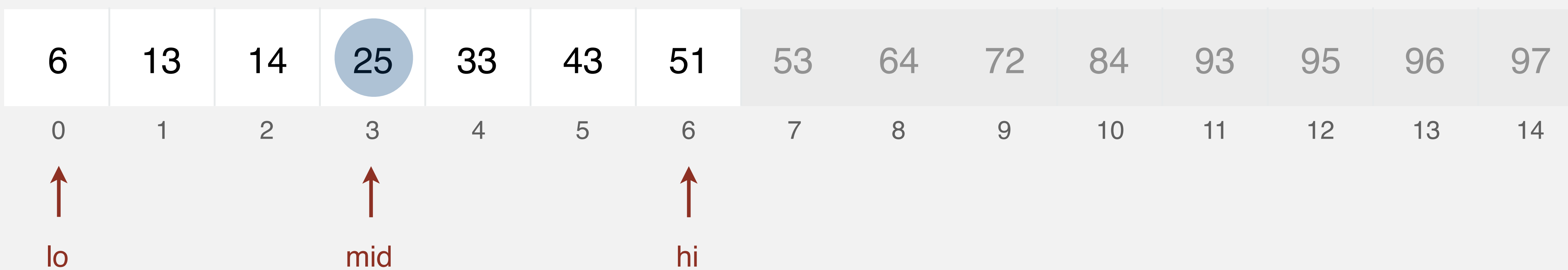

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| | | | | lo | | hi | | | | | | | | |

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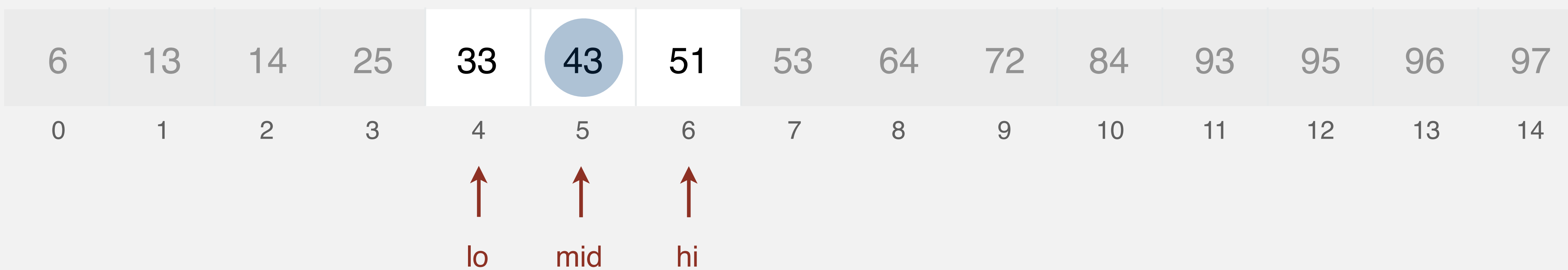

# Binary search

Goal. Given a sorted array and a key, find index of the key in the array?

Successful search. Binary search for 33.

lo = hi

| | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 6 | 13 | 14 | 25 | 33 | 43 | 51 | 53 | 64 | 72 | 84 | 93 | 95 | 96 | 97 |
| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 |

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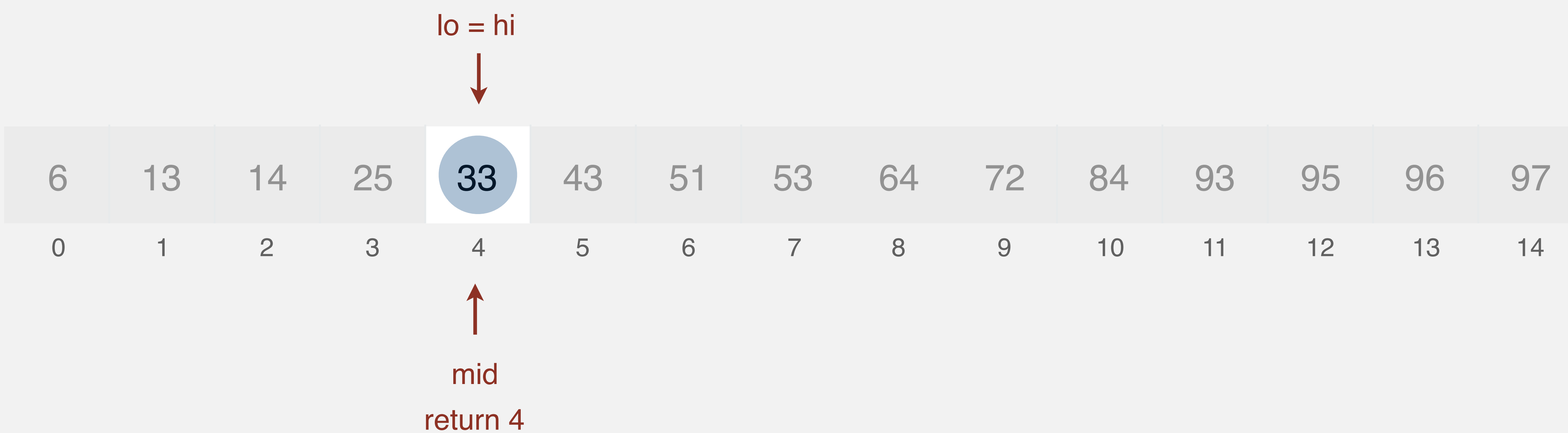

# Binary search: Java implementation

Trivial to implement?

- First binary search published in 1946; first bug-free one published in 1962.
- Java bug in `Arrays.binarySearch()` not fixed until 2006.

```
public static int binarySearch(int[] a, int key)
{
    int lo = 0, hi = a.length-1;
    while (lo <= hi)
    {
        int mid = lo + (hi - lo) / 2;
        if      (key < a[mid]) hi = mid - 1;
        else if (key > a[mid]) lo = mid + 1;
        else return mid;
    }
    return -1;
}
```

one 3-way compare

Invariant. If `key` appears in the array `a[]`, then `a[lo]` ≤ `key` ≤ `a[hi]`.

# Binary search: mathematical analysis

Proposition. Binary search uses at most $1 + \lg N$ compares to search in a sorted array of size $N$.

Def. $T(N) \equiv$ # compares to binary search in a sorted subarray of size $N$.

Binary search recurrence. $T(N) \le \underbrace{T(N/2)}_{\text{left or right half}} + 1$ for $N > 1$, with $T(1) = 1$.

Pf sketch.

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Pf sketch.

$$\begin{aligned}
T(N) &\le T(N/2) + 1 && \text{given}\\
&\le T(N/4) + 1 + 1 && \text{apply recurrence to first term}\\
&\le T(N/8) + 1 + 1 + 1 && \text{apply recurrence to first term}\\
&\dots\\
&\le T(N/N) + 1 + 1 + \dots + 1 && \text{stop applying, T(1) = 1}\\
&= 1 + \lg N
\end{aligned}$$

# An $N^2 \log N$ algorithm for 3-sum

**Step 1.** Sort the $N$ numbers.

**Step 2.** For each pair of numbers `a[i]` and `a[j]`, binary search for `-(a[i] + a[j])`.

**Analysis.** Order of growth is $N^2 \log N$.

- Step 1: $N^2$ with insertion sort.
- Step 2: $N^2 \log N$ with binary search.

input

```
 30 -40 -20 -10 40   0 10   5
```

sort

```
-40 -20 -10   0   5 10 30 40
```

binary search

```
(-40, -20)    60
(-40, -10)    30
(-40,   0)    40
(-40,   5)    35
(-40,  10)    30
...
(-40,  40)     0
...
(-10,   0)    10
...
(-20,  10)    10
...
( 10,  30)   -40
( 10,  40)   -50
( 30,  40)   -70
```

only count if a[i] < a[j] < a[k] to avoid double counting

# Comparing programs

Hypothesis. The $N^2 \log N$ three-sum algorithm is significantly faster in practice than the brute-force $N^3$ one.

| N | time (seconds) |
|---|---|
| 1,000 | 0.1 |
| 2,000 | 0.8 |
| 4,000 | 6.4 |
| 8,000 | 51.1 |

ThreeSum.java

| N | time (seconds) |
|---|---|
| 1,000 | 0.14 |
| 2,000 | 0.18 |
| 4,000 | 0.34 |
| 8,000 | 0.96 |
| 16,000 | 3.67 |
| 32,000 | 14.88 |
| 64,000 | 59.16 |

ThreeSumDeluxe.java

Bottom line. Typically, better order of growth $\Rightarrow$ faster in practice.

# Types of analyses

Best case. Lower bound on cost.

- Determined by "easiest" input.
- Provides a goal for all inputs.

Worst case. Upper bound on cost.

- Determined by "most difficult" input.
- Provides a guarantee for all inputs.

Average case. Expected cost for random input.

- Need a model for "random" input.
- Provides a way to predict performance.

Ex 1. Array accesses for brute-force 3 sum.

Best: $\sim \frac{1}{2} N^3$

Average: $\sim \frac{1}{2} N^3$

Worst: $\sim \frac{1}{2} N^3$

Ex 2. Compares for binary search.

Best: $\sim 1$

Average: $\sim \lg N$

Worst: $\sim \lg N$

# Types of analyses

Best case. Lower bound on cost.

Worst case. Upper bound on cost.

Average case. "Expected" cost.

Actual data might not match input model?

- Need to understand input to effectively process it.
- Approach 1: design for the worst case.
- Approach 2: randomize, depend on probabilistic guarantee.

# Commonly-used notations

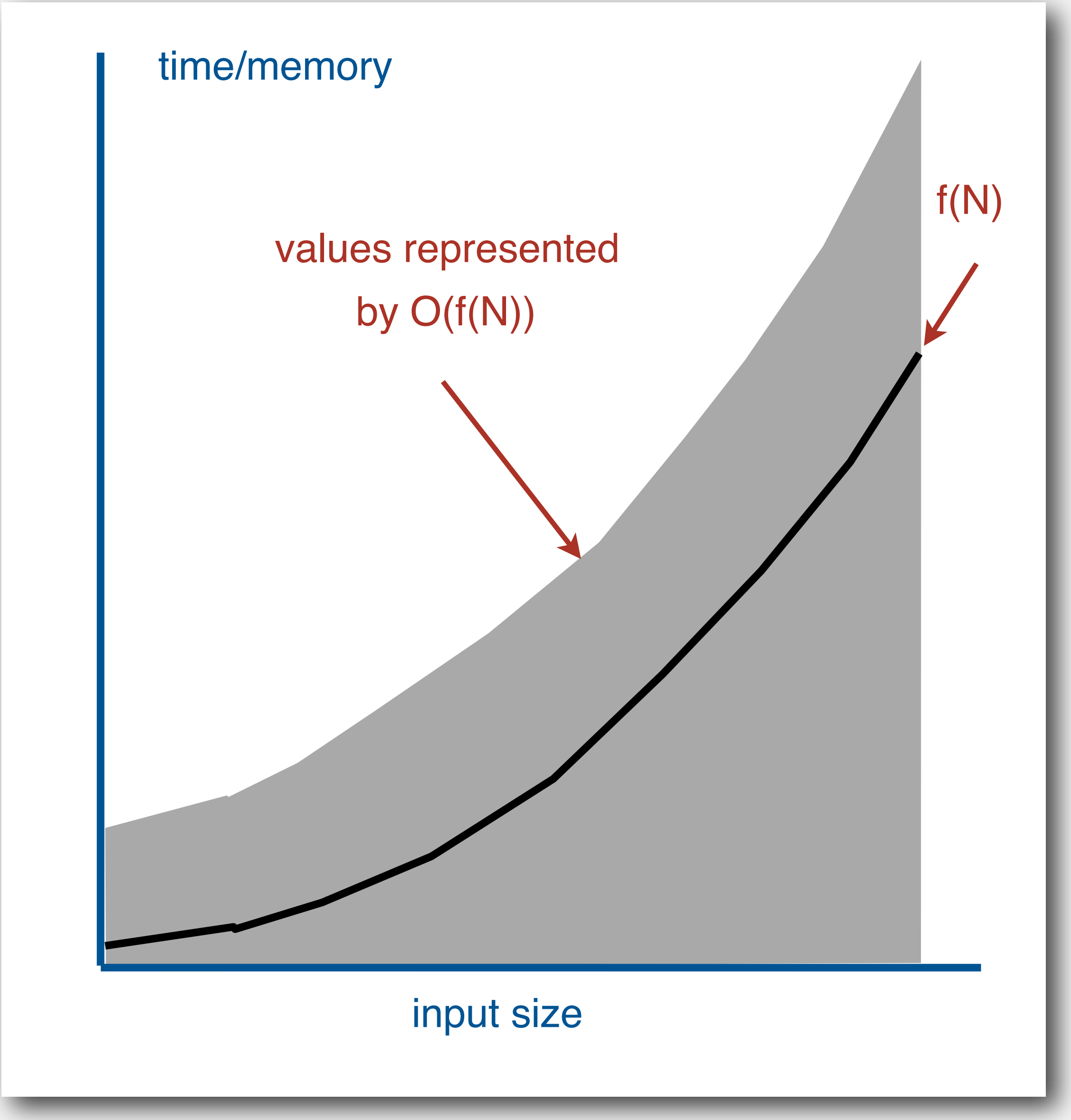
| notation | provides | example | shorthand for | used to |
| --- | --- | --- | --- | --- |
| Tilde | leading term | $\sim 10\ N^2$ | $10\ N^2$<br>$10\ N^2 + 22\ N \log N$<br>$10\ N^2 + 2\ N + 37$ | provide approximate model |
| Big Theta | asymptotic growth rate | $\Theta(N^2)$ | $\frac{1}{2}\ N^2$<br>$10\ N^2$<br>$5\ N^2 + 22\ N \log N + 3N$ | classify algorithms |
| Big Oh | $\Theta(N^2)$ and smaller | $O(N^2)$ | $10\ N^2$<br>$100\ N$<br>$22\ N \log N + 3\ N$ | develop upper bounds |
| Big Omega | $\Theta(N^2)$ and larger | $\Omega(N^2)$ | $\frac{1}{2}\ N^2$<br>$N^5$<br>$N^3 + 22\ N \log N + 3\ N$ | develop lower bounds |

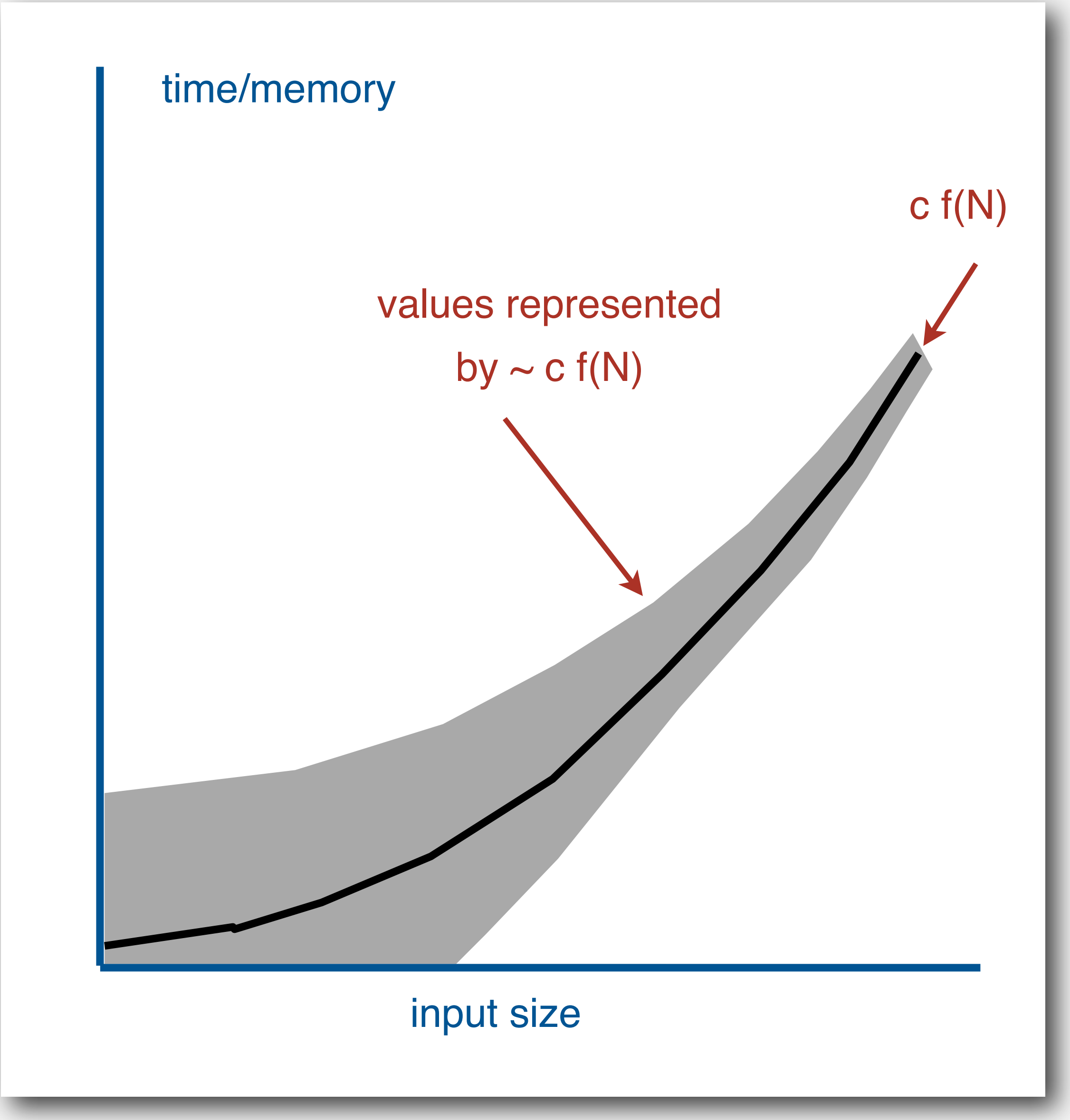
**Common mistake.** Interpreting big-Oh as an approximate model.

# Tilde notation vs. big-Oh notation

We use tilde notation whenever possible.

- Big-Oh notation suppresses leading constant.
- Big-Oh notation only provides upper bound (not lower bound).



- observations
- mathematical models
- amortized analysis
- order-of-growth classifications
- dependencies on inputs
- **memory**

# Typical memory requirements for primitive types in Java

Bit. 0 or 1.

Byte. 8 bits.

Megabyte (MB). 1 million bytes.

Gigabyte (GB). 1 billion bytes.

| type | bytes |
|---|---|
| `boolean` | 1 |
| `byte` | 1 |
| `char` | 2 |
| `int` | 4 |
| `float` | 4 |
| `long` | 8 |
| `double` | 8 |

**for primitive types**

# Typical memory requirements for arrays in Java

**Array overhead.** 16 bytes.

| type | bytes |
|---|---|
| `char[]` | 2N + 16 |
| `int[]` | 4N + 16 |
| `double[]` | 8N + 16 |

for one-dimensional arrays

| type | bytes |
|---|---|
| `char[][]` | ~ 2 M N |
| `int[][]` | ~ 4 M N |
| `double[][]` | ~ 8 M N |

for two-dimensional arrays

**Ex.** An $N$-by-$N$ array of doubles consumes $\sim 8N^2$ bytes of memory.

# Typical memory requirements for objects in Java

Object overhead. 8 bytes.

Reference. 4 bytes.

Ex 1. A `Complex` object consumes 24 bytes of memory.

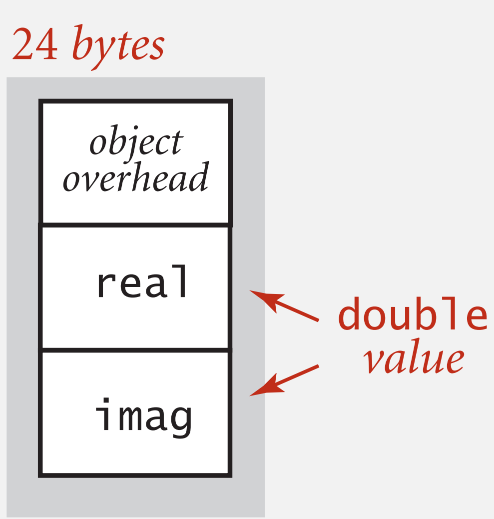
```
public class Complex
{
   private double re;
   private double im;
   ...
}
```

8 bytes (object overhead)

8 bytes (double)

8 bytes (double)

24 bytes



# Typical memory requirements for objects in Java

Object overhead. 8 bytes.

Reference. 4 bytes.

Ex 2. A virgin `String` of length $N$ consumes $\sim 2N$ bytes of memory.

```
public class String
{
   private int offset;
   private int count;
   private int hash;
   private char[] value;
   ...
}
```

8 bytes (object overhead)

4 bytes (int)

4 bytes (int)

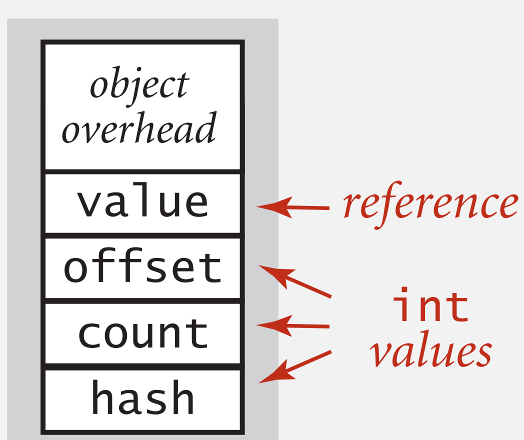
4 bytes (int)

4 bytes (reference to array)

2N + 16 bytes (char[] array)

---

2N + 40 bytes



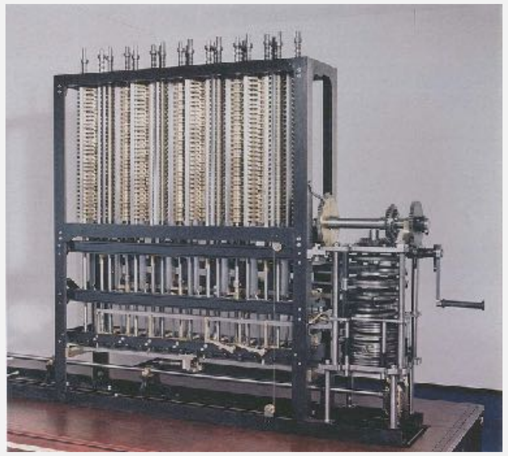
# Turning the crank: summary

Empirical analysis.

- Execute program to perform experiments.
- Assume power law and formulate a hypothesis for running time.
- Model enables us to make predictions.

Mathematical analysis.

- Analyze algorithm to count frequency of operations.
- Use tilde notation to simplify analysis.
- Model enables us to explain behavior.



Scientific method.

- Mathematical model is independent of a particular system; applies to machines not yet built.
- Empirical analysis is necessary to validate mathematical models and to make predictions.