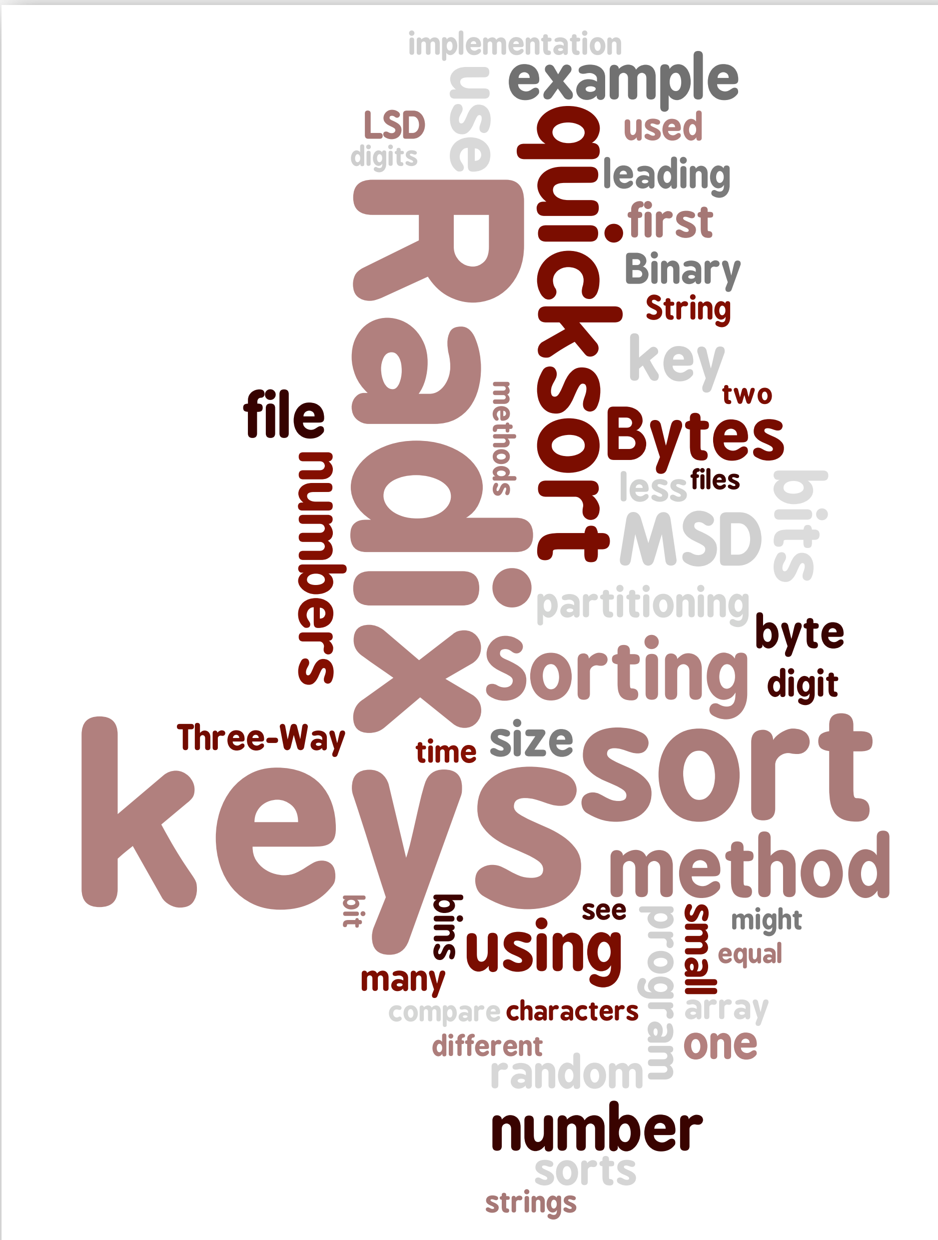
5.1 String Sorts



- key-indexed counting
- LSD string sort
- MSD string sort
- 3-way string quicksort
- suffix arrays

Review: summary of the performance of sorting algorithms

Frequency of operations = key compares.

algorithm	guarantee	random	extra space	stable?	operations on keys
insertion sort	$N^2/2$	$N^2/4$	no	yes	`compareTo()`
mergesort	N lg N	N lg N	N	yes	`compareTo()`
quicksort	1.39 N lg N *	1.39 N lg N	c lg N	no	`compareTo()`
heapsort	2 N lg N	2 N lg N	no	no	`compareTo()`

* probabilistic

Lower bound. $\sim N \lg N$ compares are required by any compare-based algorithm.

Q. Can we do better (despite the lower bound)?

A. Yes, if we don't depend on compares.

- key-indexed counting
- LSD string sort
- MSD string sort
- 3-way string quicksort
- longest repeated substring

Key-indexed counting: assumptions about keys

Assumption. Keys are integers between 0 and $R - 1$.

Implication. Can use key as an array index.

Applications.

- Sort string by first letter.
- Sort class roster by section.
- Sort phone numbers by area code.
- Subroutine in a sorting algorithm.

Remark. Keys may have associated data $\Rightarrow$ can't just count up number of keys of each value.

input *name*	*section*	sorted result (*by section*)	
Anderson	2	Harris	1
Brown	3	Martin	1
Davis	3	Moore	1
Garcia	4	Anderson	2
Harris	1	Martinez	2
Jackson	3	Miller	2
Johnson	4	Robinson	2
Jones	3	White	2
Martin	1	Brown	3
Martinez	2	Davis	3
Miller	2	Jackson	3
Moore	1	Jones	3
Robinson	2	Taylor	3
Smith	4	Williams	3
Taylor	3	Garcia	4
Thomas	4	Johnson	4
Thompson	4	Smith	4
White	2	Thomas	4
Williams	3	Thompson	4
Wilson	4	Wilson	4

keys are small integers

Key-indexed counting

Goal. Sort an array `a[]` of N integers between 0 and $R - 1$.

- Count frequencies of each letter using key as index.

count frequencies

```
int N = a.length;
int[] count = new int[R+1];

for (int i = 0; i < N; i++)
   count[a[i]+1]++;

for (int r = 0; r < R; r++)
   count[r+1] += count[r];

for (int i = 0; i < N; i++)
   aux[count[a[i]]++] = a[i];

for (int i = 0; i < N; i++)
   a[i] = aux[i];
```

i	a[i]
0	d
1	a
2	c
3	f
4	f
5	b
6	d
7	b
8	f
9	b
10	e
11	a

offset by 1
[stay tuned]

r	count[r]
a	0
b	2
c	3
d	1
e	2
f	1
-	3

Key-indexed counting

Goal. Sort an array `a[]` of N integers between 0 and $R - 1$.

- Count frequencies of each letter using key as index.
- Compute frequency cumulates which specify destinations.
-

```
int N = a.length;
int[] count = new int[R+1];

for (int i = 0; i < N; i++)
   count[a[i]+1]++;

for (int r = 0; r < R; r++)
   count[r+1] += count[r];

for (int i = 0; i < N; i++)
   aux[count[a[i]]++] = a[i];

for (int i = 0; i < N; i++)
   a[i] = aux[i];
```

compute cumulates

i	a[i]
0	d
1	a
2	c
3	f
4	f
5	b
6	d
7	b
8	f
9	b
10	e
11	a

r	count[r]
a	0
b	2
c	5
d	6
e	8
f	9
-	12

6 keys < **d**, 8 keys < **e**

so **d**'s go in **a[6]** and **a[7]**

Key-indexed counting

Goal. Sort an array `a[]` of N integers between 0 and $R - 1$.

- Count frequencies of each letter using key as index.
- Compute frequency cumulates which specify destinations.
- Access cumulates using key as index to move records.

```
int N = a.length;
int[] count = new int[R+1];

for (int i = 0; i < N; i++)
   count[a[i]+1]++;

for (int r = 0; r < R; r++)
   count[r+1] += count[r];

for (int i = 0; i < N; i++)
   aux[count[a[i]]++] = a[i];

for (int i = 0; i < N; i++)
   a[i] = aux[i];
```

move records

i	a[i]
0	d
1	a
2	c
3	f
4	f
5	b
6	d
7	b
8	f
9	b
10	e
11	a

r	count[r]
a	0
b	2
c	5
d	6
e	8
f	9
-	12

i	aux[i]
0	
1	
2	
3	
4	
5	
6	
7	
8	
9	
10	
11	

Key-indexed counting

Goal. Sort an array `a[]` of N integers between 0 and $R - 1$.

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int[] count = new int[R+1];

for (int i = 0; i < N; i++)
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   count[r+1] += count[r];

for (int i = 0; i < N; i++)
   aux[count[a[i]]++] = a[i];

for (int i = 0; i < N; i++)
   a[i] = aux[i];
```

move records

i	a[i]
0	d
1	a
2	c
3	f
4	f
5	b
6	d
7	b
8	f
9	b
10	e
11	a

r	count[r]
a	0
b	2
c	5
d	7
e	8
f	9
-	12

i	aux[i]
0	
1	
2	
3	
4	
5	
6	d
7	
8	
9	
10	
11	

Key-indexed counting

Goal. Sort an array `a[]` of N integers between 0 and $R-1$.

- Count frequencies of each letter using key as index.
- Compute frequency cumulates which specify destinations.
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for (int i = 0; i < N; i++)
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```

move records

i	a[i]
0	d
1	a
2	c
3	f
4	f
5	b
6	d
7	b
8	f
9	b
10	e
11	a

r	count[r]
a	1
b	2
c	5
d	7
e	8
f	9
-	12

i	aux[i]
0	a
1	
2	
3	
4	
5	
6	d
7	
8	
9	
10	
11	

Key-indexed counting

Goal. Sort an array `a[]` of N integers between 0 and $R - 1$.

- Count frequencies of each letter using key as index.
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```

move records

i	a[i]
0	d
1	a
2	c
3	f
4	f
5	b
6	d
7	b
8	f
9	b
10	e
11	a

r	count[r]
a	1
b	2
c	6
d	7
e	8
f	9
-	12

i	aux[i]
0	a
1	
2	
3	
4	
5	c
6	d
7	
8	
9	
10	
11	

Key-indexed counting

Goal. Sort an array `a[]` of N integers between 0 and $R - 1$.

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move records

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0	d
1	a
2	c
3	f
4	f
5	b
6	d
7	b
8	f
9	b
10	e
11	a

r	count[r]
a	1
b	2
c	6
d	7
e	8
f	10
-	12

i	aux[i]
0	a
1	
2	
3	
4	
5	c
6	d
7	
8	
9	f
10	
11	

Key-indexed counting

Goal. Sort an array `a[]` of N integers between 0 and $R - 1$.

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1	a
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6	d
7	b
8	f
9	b
10	e
11	a

r	count[r]
a	1
b	2
c	6
d	7
e	8
f	11
-	12

i	aux[i]
0	a
1	
2	
3	
4	
5	c
6	d
7	
8	
9	f
10	f
11	

Key-indexed counting

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move records

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0	d
1	a
2	c
3	f
4	f
5	b
6	d
7	b
8	f
9	b
10	e
11	a

r	count[r]
a	1
b	3
c	6
d	7
e	8
f	11
-	12

i	aux[i]
0	a
1	
2	b
3	
4	
5	c
6	d
7	
8	
9	f
10	f
11	

Key-indexed counting

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move records

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0	d
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6	d
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8	f
9	b
10	e
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r	count[r]
a	1
b	3
c	6
d	8
e	8
f	11
-	12

i	aux[i]
0	a
1	
2	b
3	
4	
5	c
6	d
7	d
8	
9	f
10	f
11	

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move records

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0	d
1	a
2	c
3	f
4	f
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6	d
7	b
8	f
9	b
10	e
11	a

r	count[r]
a	1
b	4
c	6
d	8
e	8
f	11
-	12

i	aux[i]
0	a
1	
2	b
3	b
4	
5	c
6	d
7	d
8	
9	f
10	f
11	

Key-indexed counting

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move records

i	a[i]
0	d
1	a
2	c
3	f
4	f
5	b
6	d
7	b
8	f
9	b
10	e
11	a

r	count[r]
a	1
b	4
c	6
d	8
e	8
f	12
-	12

i	aux[i]
0	a
1	
2	b
3	b
4	
5	c
6	d
7	d
8	
9	f
10	f
11	f

Key-indexed counting

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move records

i	a[i]
0	d
1	a
2	c
3	f
4	f
5	b
6	d
7	b
8	f
9	b
10	e
11	a

r	count[r]
a	1
b	5
c	6
d	8
e	8
f	12
-	12

i	aux[i]
0	a
1	
2	b
3	b
4	b
5	c
6	d
7	d
8	
9	f
10	f
11	f

Key-indexed counting

Goal. Sort an array `a[]` of N integers between 0 and $R - 1$.

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move records →

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2	c
3	f
4	f
5	b
6	d
7	b
8	f
9	b
10	e
11	a

r	count[r]
a	1
b	5
c	6
d	8
e	9
f	12
-	12

i	aux[i]
0	a
1	
2	b
3	b
4	b
5	c
6	d
7	d
8	e
9	f
10	f
11	f

Key-indexed counting

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for (int i = 0; i < N; i++)
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move records

i	a[i]
0	d
1	a
2	c
3	f
4	f
5	b
6	d
7	b
8	f
9	b
10	e
11	a

r	count[r]
a	2
b	5
c	6
d	8
e	9
f	12
-	12

i	aux[i]
0	a
1	a
2	b
3	b
4	b
5	c
6	d
7	d
8	e
9	f
10	f
11	f

Key-indexed counting

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for (int i = 0; i < N; i++)
   a[i] = aux[i];
```

move records

i	a[i]
0	d
1	a
2	c
3	f
4	f
5	b
6	d
7	b
8	f
9	b
10	e
11	a

r	count[r]
a	2
b	5
c	6
d	8
e	9
f	12
-	12

i	aux[i]
0	a
1	a
2	b
3	b
4	b
5	c
6	d
7	d
8	e
9	f
10	f
11	f

Key-indexed counting

Goal. Sort an array `a[]` of N integers between 0 and $R - 1$.

- Count frequencies of each letter using key as index.
- Compute frequency cumulates which specify destinations.
- Access cumulates using key as index to move records.
- Copy back into original array.

```
int N = a.length;
int[] count = new int[R+1];

for (int i = 0; i < N; i++)
   count[a[i]+1]++;

for (int r = 0; r < R; r++)
   count[r+1] += count[r];

for (int i = 0; i < N; i++)
   aux[count[a[i]]++] = a[i];

for (int i = 0; i < N; i++)
   a[i] = aux[i];
```

copy back

i	a[i]
0	a
1	a
2	b
3	b
4	b
5	c
6	d
7	d
8	e
9	f
10	f
11	f

r	count[r]
a	2
b	5
c	6
d	8
e	9
f	12
-	12

i	aux[i]
0	a
1	a
2	b
3	b
4	b
5	c
6	d
7	d
8	e
9	f
10	f
11	f

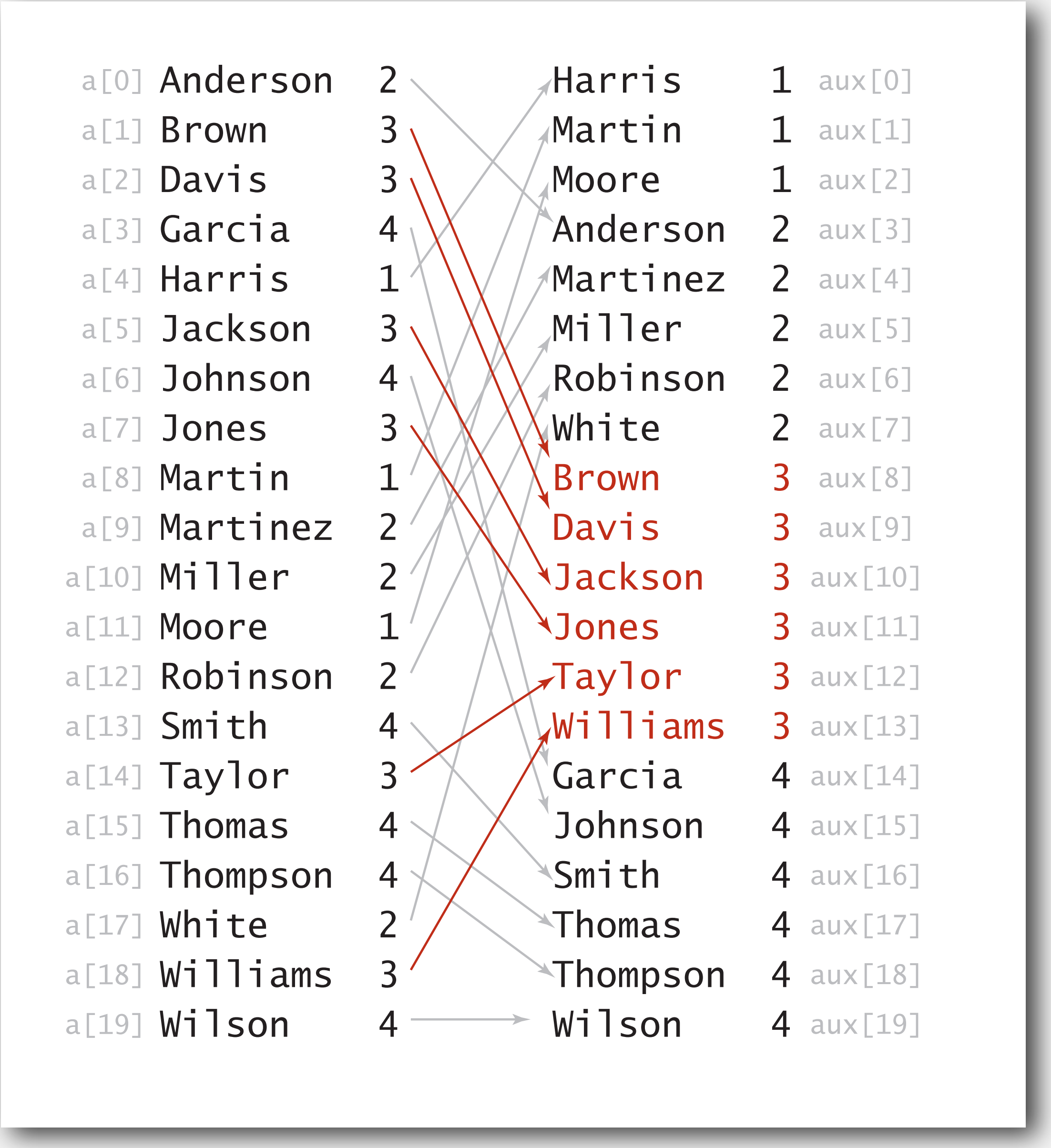
Key-indexed counting: analysis

Proposition. Key-indexed counting uses $8N + 3R$ array accesses to sort N records whose keys are integers between 0 and $R - 1$.

Proposition. Key-indexed counting uses extra space proportional to $N + R$.

Stable? Yes!

In-place? No.



- key-indexed counting
- **LSD string sort**
- MSD string sort
- 3-way string quicksort
- suffix arrays

LSD string sort.

- Consider characters from right to left.
- Stably sort using d^{th} character as the key (using key-indexed counting).

0	d	a	b
1	a	d	d
2	c	a	b
3	f	a	d
4	f	e	e
5	b	a	d
6	d	a	d
7	b	e	e
8	f	e	d
9	b	e	d
10	e	b	b
11	a	c	e

Least-significant-digit-first string sort

LSD string sort.

- Consider characters from right to left.
- Stably sort using d^{th} character as the key (using key-indexed counting).

0	d	a	b
1	a	d	d
2	c	a	b
3	f	a	d
4	f	e	e
5	b	a	d
6	d	a	d
7	b	e	e
8	f	e	d
9	b	e	d
10	e	b	b
11	a	c	e

sort key

0	d	a	b
1	c	a	b
2	e	b	b
3	a	d	d
4	f	a	d
5	b	a	d
6	d	a	d
7	f	e	d
8	b	e	d
9	f	e	e
10	b	e	e
11	a	c	e

sort must be stable
(arrows do not cross)

Least-significant-digit-first string sort

LSD string sort.

- Consider characters from right to left.
- Stably sort using d^{th} character as the key (using key-indexed counting).

sort key

sort key

0	d	a	b	0	d	a	b	0	d	a	b
1	a	d	d	1	c	a	b	1	c	a	b
2	c	a	b	2	e	b	b	2	f	a	d
3	f	a	d	3	a	d	d	3	b	a	d
4	f	e	e	4	f	a	d	4	d	a	d
5	b	a	d	5	b	a	d	5	e	b	b
6	d	a	d	6	d	a	d	6	a	c	e
7	b	e	e	7	f	e	d	7	a	d	d
8	f	e	d	8	b	e	d	8	f	e	d
9	b	e	d	9	f	e	e	9	b	e	d
10	e	b	b	10	b	e	e	10	f	e	e
11	a	c	e	11	a	c	e	11	b	e	e

sort must be stable
(arrows do not cross)

Least-significant-digit-first string sort

LSD string sort.

- Consider characters from right to left.
- Stably sort using d^{th} character as the key (using key-indexed counting).

sort key | sort key | sort key

0	d	a	b	0	d	a	b	0	d	a	b	0	a	c	e
1	a	d	d	1	c	a	b	1	c	a	b	1	a	d	d
2	c	a	b	2	e	b	b	2	f	a	d	2	b	a	d
3	f	a	d	3	a	d	d	3	b	a	d	3	b	e	d
4	f	e	e	4	f	a	d	4	d	a	d	4	b	e	e
5	b	a	d	5	b	a	d	5	e	b	b	5	c	a	b
6	d	a	d	6	d	a	d	6	a	c	e	6	d	a	b
7	b	e	e	7	f	e	d	7	a	d	d	7	d	a	d
8	f	e	d	8	b	e	d	8	f	e	d	8	e	b	b
9	b	e	d	9	f	e	e	9	b	e	d	9	f	a	d
10	e	b	b	10	b	e	e	10	f	e	e	10	f	e	d
11	a	c	e	11	a	c	e	11	b	e	e	11	f	e	e

sort must be stable
(arrows do not cross)

LSD string sort: correctness proof

Proposition. LSD sorts fixed-length strings in ascending order.

Pf. [thinking about the future]

- If the characters not yet examined differ, it doesn't matter what we do now.
- If the characters not yet examined agree, stability ensures later pass won't affect order.

0	d	a	b
1	c	a	b
2	f	a	d
3	b	a	d
4	d	a	d
5	e	b	b
6	a	c	e
7	a	d	d
8	f	e	d
9	b	e	d
10	f	e	e
11	b	e	e

in order by previous passes

sort key

0	a	c	e
1	a	d	d
2	b	a	d
3	b	e	d
4	b	e	e
5	c	a	b
6	d	a	b
7	d	a	d
8	e	b	b
9	f	a	d
10	f	e	d
11	f	e	e

LSD string sort: Java implementation

```
public class LSD
{
   public static void sort(String[] a, int W)
   {
      int R = 256
      int N = a.length;
      String[] aux = new String[N];

      for (int d = W-1; d >= 0; d--)
      {
         int[] count = new int[R+1];
         for (int i = 0; i < N; i++)
            count[a[i].charAt(d) + 1]++;
         for (int r = 0; r < R; r++)
            count[r+1] += count[r];
         for (int i = 0; i < N; i++)
            aux[count[a[i].charAt(d)]++] = a[i];
         for (int i = 0; i < N; i++)
            a[i] = aux[i];
      }
   }
}
```

fixed-length W strings

radix R

do key-indexed counting
for each digit from right to left

key-indexed counting

LSD string sort: example

Input	d = 6	d = 5	d = 4	d = 3	d= 2	d= 1	d = 0	Output
4PGC938	2IYE230	3CIO720	2IYE230	2RLA629	1ICK750	3ATW723	1ICK750	1ICK750
2IYE230	3CIO720	3CIO720	4JZY524	2RLA629	1ICK750	3CIO720	1ICK750	1ICK750
3CIO720	1ICK750	3ATW723	2RLA629	4PGC938	4PGC938	3CIO720	1OHV845	1OHV845
1ICK750	1ICK750	4JZY524	2RLA629	2IYE230	1OHV845	1ICK750	1OHV845	1OHV845
1OHV845	3CIO720	2RLA629	3CIO720	1ICK750	1OHV845	1ICK750	1OHV845	1OHV845
4JZY524	3ATW723	2RLA629	3CIO720	1ICK750	1OHV845	2IYE230	2IYE230	2IYE230
1ICK750	4JZY524	2IYE230	3ATW723	3CIO720	3CIO720	4JZY524	2RLA629	2RLA629
3CIO720	1OHV845	4PGC938	1ICK750	3CIO720	3CIO720	1OHV845	2RLA629	2RLA629
1OHV845	1OHV845	1OHV845	1ICK750	1OHV845	2RLA629	1OHV845	3ATW723	3ATW723
1OHV845	1OHV845	1OHV845	1OHV845	1OHV845	2RLA629	1OHV845	3CIO720	3CIO720
2RLA629	4PGC938	1OHV845	1OHV845	1OHV845	3ATW723	4PGC938	3CIO720	3CIO720
2RLA629	2RLA629	1ICK750	1OHV845	3ATW723	2IYE230	2RLA629	4JZY524	4JZY524
3ATW723	2RLA629	1ICK750	4PGC938	4JZY524	4JZY524	2RLA629	4PGC938	4PGC938

Frequency of operations.

algorithm	guarantee	random	extra space	stable?	operations on keys
insertion sort	$N^2/2$	$N^2/4$	1	yes	`compareTo()`
mergesort	N lg N	N lg N	N	yes	`compareTo()`
quicksort	1.39 N lg N *	1.39 N lg N	c lg N	no	`compareTo()`
heapsort	2 N lg N	2 N lg N	1	no	`compareTo()`
LSD †	2 W N	2 W N	N + R	yes	`charAt()`

* probabilistic

† fixed-length W keys

Q. What if strings do not have same length?

String sorting challenge 1

Problem. Sort a huge commercial database on a fixed-length key field.

Ex. Account number, date, SS number, ...

Which sorting method to use?

- Insertion sort.
- Mergesort.
- Quicksort.
- Heapsort.
- LSD string sort.

B14-99-8765
756-12-AD46
CX6-92-0112
332-WX-9877
375-99-QWAX
CV2-59-0221
[illegible]87-SS-0321
KJ-0[illegible]12388
715-YT-013C
MJ0-PP-983F
908-KK-33TY
BBN-63-23RE
48G-BM-912D
982-ER-9P1B
WBL-37-PB81
810-F4-J87Q
LE9-N8-XX76
908-KK-33TY
B14-99-8765
CX6-92-0112
CV2-59-0221
332-WX-23SQ
332-6A-9877

String sorting challenge 1

Problem. Sort a huge commercial database on a fixed-length key field.

Ex. Account number, date, SS number, ...

Which sorting method to use?

- Insertion sort.
- Mergesort.
- Quicksort.
- Heapsort.
- ✓ LSD string sort.

256 (or 65,536) counters;

Fixed-length strings sort in W passes.

B14-99-8765
756-12-AD46
CX6-92-0112
332-WX-9877
375-99-QWAX
CV2-59-0221
[illegible]87-SS-0321
KJ-0[illegible]12388
715-YT-013C
MJ0-PP-983F
908-KK-33TY
BBN-63-23RE
48G-BM-912D
982-ER-9P1B
WBL-37-PB81
810-F4-J87Q
LE9-N8-XX76
908-KK-33TY
B14-99-8765
CX6-92-0112
CV2-59-0221
332-WX-23SQ
332-6A-9877

String sorting challenge 2

Problem. Sort 1 million 32-bit integers.

Ex. Google interview (or presidential interview).

Which sorting method to use?

- Insertion sort.
- Mergesort.
- Quicksort.
- Heapsort.
- LSD string sort.

Google CEO Eric Schmidt interviews Barack Obama

https://www.cs.purdue.edu/homes/cs251/slides/media/barack-bubblesort.mov

Problem. Sort 1 million 32-bit integers.

Ex. Google interview (or presidential interview).

Which sorting method to use?

- Insertion sort.
- Mergesort.
- Quicksort.
- Heapsort.
- LSD string sort.

Google CEO Eric Schmidt interviews Barack Obama

https://www.cs.purdue.edu/homes/cs251/slides/media/barack-bubblesort.mov

- key-indexed counting
- LSD string sort
- **MSD string sort**
- 3-way string quicksort
- suffix arrays

Most-significant-digit-first string sort

MSD string sort.

- Partition file into R pieces according to first character (use key-indexed counting).
- Recursively sort all strings that start with each character (key-indexed counts delineate subarrays to sort).

0	d	a	b
1	a	d	d
2	c	a	b
3	f	a	d
4	f	e	e
5	b	a	d
6	d	a	d
7	b	e	e
8	f	e	d
9	b	e	d
10	e	b	b
11	a	c	e

MSD string sort.

- Partition file into R pieces according to first character (use key-indexed counting).
- Recursively sort all strings that start with each character (key-indexed counts delineate subarrays to sort).

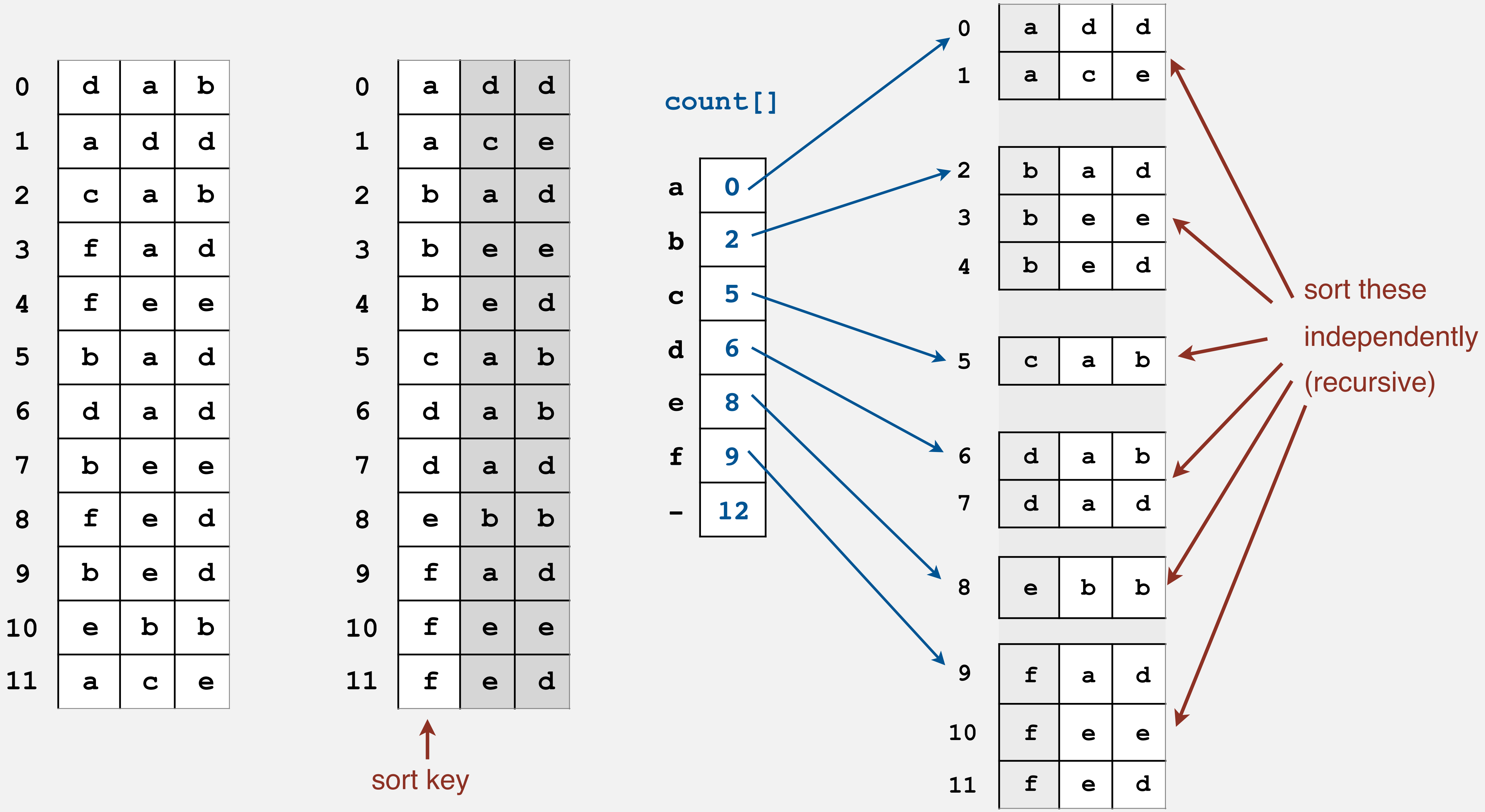
0	d	a	b
1	a	d	d
2	c	a	b
3	f	a	d
4	f	e	e
5	b	a	d
6	d	a	d
7	b	e	e
8	f	e	d
9	b	e	d
10	e	b	b
11	a	c	e

0	a	d	d
1	a	c	e
2	b	a	d
3	b	e	e
4	b	e	d
5	c	a	b
6	d	a	b
7	d	a	d
8	e	b	b
9	f	a	d
10	f	e	e
11	f	e	d

sort key

MSD string sort.

- Partition file into R pieces according to first character (use key-indexed counting).
- Recursively sort all strings that start with each character (key-indexed counts delineate subarrays to sort).



Variable-length strings

Treat strings as if they had an extra char at end (smaller than any char).

why smaller?

0	s	e	a	-1						
1	s	e	a	s	h	e	l	l	s	-1
2	s	e	l	l	s	-1				
3	s	h	e	-1						
4	s	h	e	-1						
5	s	h	e	l	l	s	-1			
6	s	h	o	r	e	-1				
7	s	u	r	e	l	y	-1			

`she` before **`shells`**

```
private static int charAt(String s, int d)
{
   if (d < s.length()) return s.charAt(d);
   else return -1;
}
```

C strings. Have extra char `'\0'` at end $\Rightarrow$ no extra work needed.

MSD string sort: top-level trace

use key-indexed counting on first character | **recursively sort subarrays**

r	char	*count frequencies*	*transform counts to indices*	*indices at completion of distribute phase*	
0		0	0	0	sort(a, 0, 0);
1	a	0	0	1	sort(a, 1, 1);
2	b	1	1	2	sort(a, 2, 1);
3	c	1	2	2	sort(a, 2, 1);
4	d	0	2	2	sort(a, 2, 1);
5	e	0	2	2	sort(a, 2, 1);
6	f	0	2	2	sort(a, 2, 1);
7	g	0	2	2	sort(a, 2, 1);
8	h	0	2	2	sort(a, 2, 1);
9	i	0	2	2	sort(a, 2, 1);
10	j	0	2	2	sort(a, 2, 1);
11	k	0	2	2	sort(a, 2, 1);
12	l	0	2	2	sort(a, 2, 1);
13	m	0	2	2	sort(a, 2, 1);
14	n	0	2	2	sort(a, 2, 1);
15	o	0	2	2	sort(a, 2, 1);
16	p	0	2	2	sort(a, 2, 1);
17	q	0	2	2	sort(a, 2, 1);
18	r	0	2	2	sort(a, 2, 11);
19	s	0	2	12	sort(a, 12, 13);
20	t	10	12	14	sort(a, 14, 13);
21	u	2	14	14	sort(a, 14, 13);
22	v	0	14	14	sort(a, 14, 13);
23	w	0	14	14	sort(a, 14, 13);
24	x	0	14	14	sort(a, 14, 13);
25	y	0	14	14	sort(a, 14, 13);
26	z	0	14	14	sort(a, 14, 13);
27		0	14	14	sort(a, 14, 13);

start of s subarray

1 + end of s subarray

i	input	*distribute and copy back*	sorted
0	she	are	are
1	sells	by	by
2	seashells	she	sea
3	by	sells	seashells
4	the	seashells	seashells
5	sea	sea	sells
6	shore	shore	sells
7	the	shells	she
8	shells	she	she
9	she	sells	shells
10	sells	surely	shore
11	are	seashells	surely
12	surely	the	the
13	seashells	the	the

MSD string sort: example

input		d						
she	are	are	are	are	are	are	are	are
sells	by	by	by	by	by	by	by	by
seashells	she	sells	seashells	sea	sea	sea	seas	sea
by	sells	seashells	sea	seashells	seashells	seashells	seashells	seashells
the	seashells	sea	seashells	seashells	seashells	seashells	seashells	seashells
sea	sea	sells	sells	sells	sells	sells	sells	sells
shore	shore	seashells	sells	sells	sells	sells	sells	sells
the	shells	she	she	she	she	she	she	she
shells	she	shore	shore	shore	shore	shore	shells	shells
she	sells	shells	shells	shells	shells	shells	shore	shore
sells	surely	she	she	she	she	she	she	she
are	seashells	surely	surely	surely	surely	surely	surely	surely
surely	the	the	the	the	the	the	the	the
seashells	the	the	the	the	the	the	the	the

lo hi

need to examine every character in equal keys

end-of-string goes before any char value

							output
are	are	are	are	are	are	are	are
by	by	by	by	by	by	by	by
sea	sea	sea	sea	sea	sea	sea	sea
seashells	seashells	seashells	seashells	seashells	seashells	seashells	seashells
seashells	seashells	seashells	seashells	seashells	seashells	seashells	seashells
sells	sells	sells	sells	sells	sells	sells	sells
sells	sells	sells	sells	sells	sells	sells	sells
she	she	she	she	she	she	she	she
shells	shells	shells	shells	she	she	she	she
she	she	she	she	shells	shells	shells	shells
shore	shore	shore	shore	shore	shore	shore	shore
surely	surely	surely	surely	surely	surely	surely	surely
the	the	the	the	the	the	the	the
the	the	the	the	the	the	the	the

Trace of recursive calls for MSD string sort (no cutoff for small subarrays, subarrays of size 0 and 1 omitted)

MSD string sort: Java implementation

```
public static void sort(String[] a)
{
   aux = new String[a.length];
   sort(a, aux, 0, a.length, 0);
}

private static void sort(String[] a, String[] aux, int lo, int hi, int d)
{
   if (hi <= lo) return;
   int[] count = new int[R+2];
   for (int i = lo; i <= hi; i++)
      count[charAt(a[i], d) + 2]++;
   for (int r = 0; r < R+1; r++)
      count[r+1] += count[r];
   for (int i = lo; i <= hi; i++)
      aux[count[charAt(a[i], d) + 1]++] = a[i];
   for (int i = lo; i <= hi; i++)
      a[i] = aux[i - lo];

   for (int r = 0; r < R; r++)
      sort(a, aux, lo + count[r], lo + count[r+1] - 1, d+1);
}
```

can recycle `aux[]`

but not `count[]`

key-indexed counting

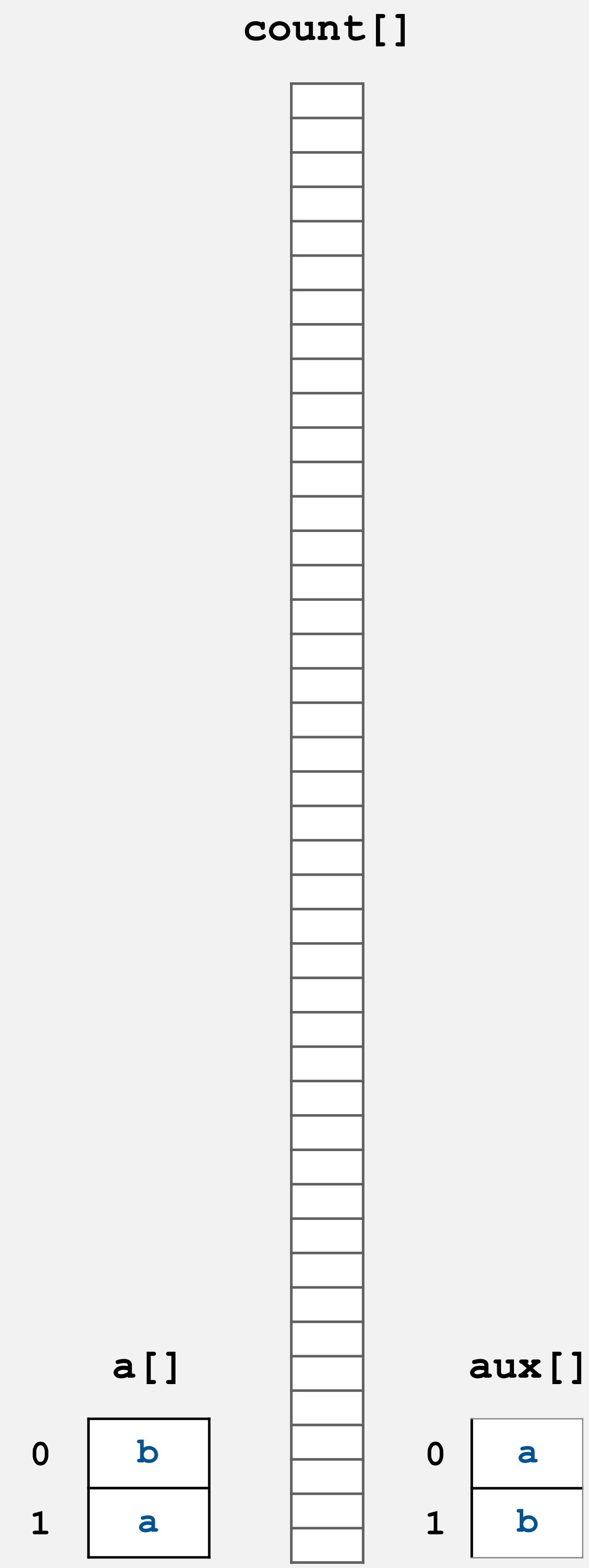
recursively sort subarrays

MSD string sort: potential for disastrous performance

Observation 1. Much too slow for small subarrays.

- Each function call needs its own `count[]` array.
- ASCII (256 counts): 100x slower than copy pass for $N = 2$.
- Unicode (65,536 counts): 32,000x slower for $N = 2$.

Observation 2. Huge number of small subarrays because of recursion.



Cutoff to insertion sort

Solution. Cutoff to insertion sort for small N.

- Insertion sort, but start at d^{th} character.
- Implement `less()` so that it compares starting at d^{th} character.

```
public static void sort(String[] a, int lo, int hi, int d)
{
   for (int i = lo; i <= hi; i++)
      for (int j = i; j > lo && less(a[j], a[j-1], d); j--)
         exch(a, j, j-1);
}
```

```
private static boolean less(String v, String w, int d)
{  return v.substring(d).compareTo(w.substring(d)) < 0;  }
```

in Java, forming and comparing substrings is faster than directly comparing chars with charAt()

MSD string sort: performance

Number of characters examined.

- MSD examines just enough characters to sort the keys.
- Number of characters examined depends on keys.
- Can be sublinear!

Random (sublinear)	Non-random with duplicates (nearly linear)	Worst case (linear)
1EIO402	are	1DNB377
1HYL490	by	1DNB377
1ROZ572	sea	1DNB377
2HXE734	seashells	1DNB377
2IYE230	seashells	1DNB377
2XOR846	sells	1DNB377
3CDB573	sells	1DNB377
3CVP720	she	1DNB377
3IGJ319	she	1DNB377
3KNA382	shells	1DNB377
3TAV879	shore	1DNB377
4CQP781	surely	1DNB377
4QGI284	the	1DNB377
4YHV229	the	1DNB377

Characters examined by MSD string sort

Summary of the performance of sorting algorithms

Frequency of operations.

algorithm	guarantee	random	extra space	stable?	operations on keys
insertion sort	$N^2/2$	$N^2/4$	1	yes	`compareTo()`
mergesort	N lg N	N lg N	N	yes	`compareTo()`
quicksort	1.39 N lg N *	1.39 N lg N	c lg N	no	`compareTo()`
heapsort	2 N lg N	2 N lg N	1	no	`compareTo()`
LSD †	2 N W	2 N W	N + R	yes	`charAt()`
MSD ‡	2 N W	$N \log_R N$	N + D R	yes	`charAt()`

stack depth D = length of longest prefix match

* probabilistic

† fixed-length W keys

‡ average-length W keys

MSD string sort vs. quicksort for strings

Disadvantages of MSD string sort.

- Accesses memory "randomly" (cache inefficient).
- Inner loop has a lot of instructions.
- Extra space for `count[]`.
- Extra space for `aux[]`.

Disadvantage of quicksort.

- Linearithmic number of string compares (not linear).
- Has to rescan long keys for compares.

Goal. Combine advantages of MSD and quicksort.

3-way string quicksort (Bentley and Sedgewick, 1997)

Overview. Do 3-way partitioning on the d^{th} character.

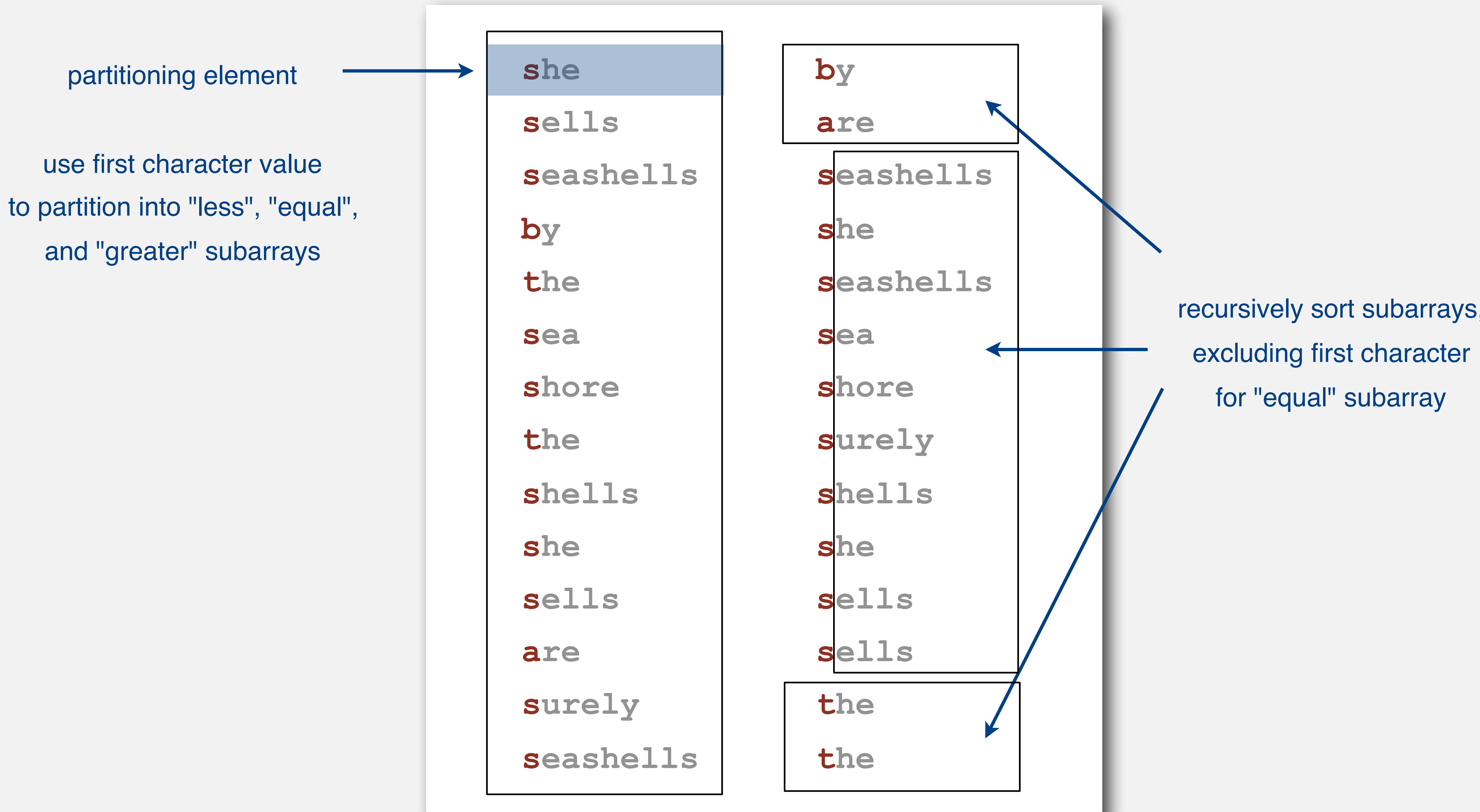
- Cheaper than R-way partitioning of MSD string sort.
- Need not examine again characters equal to the partitioning char.

Overview of 3-way string Quicksort

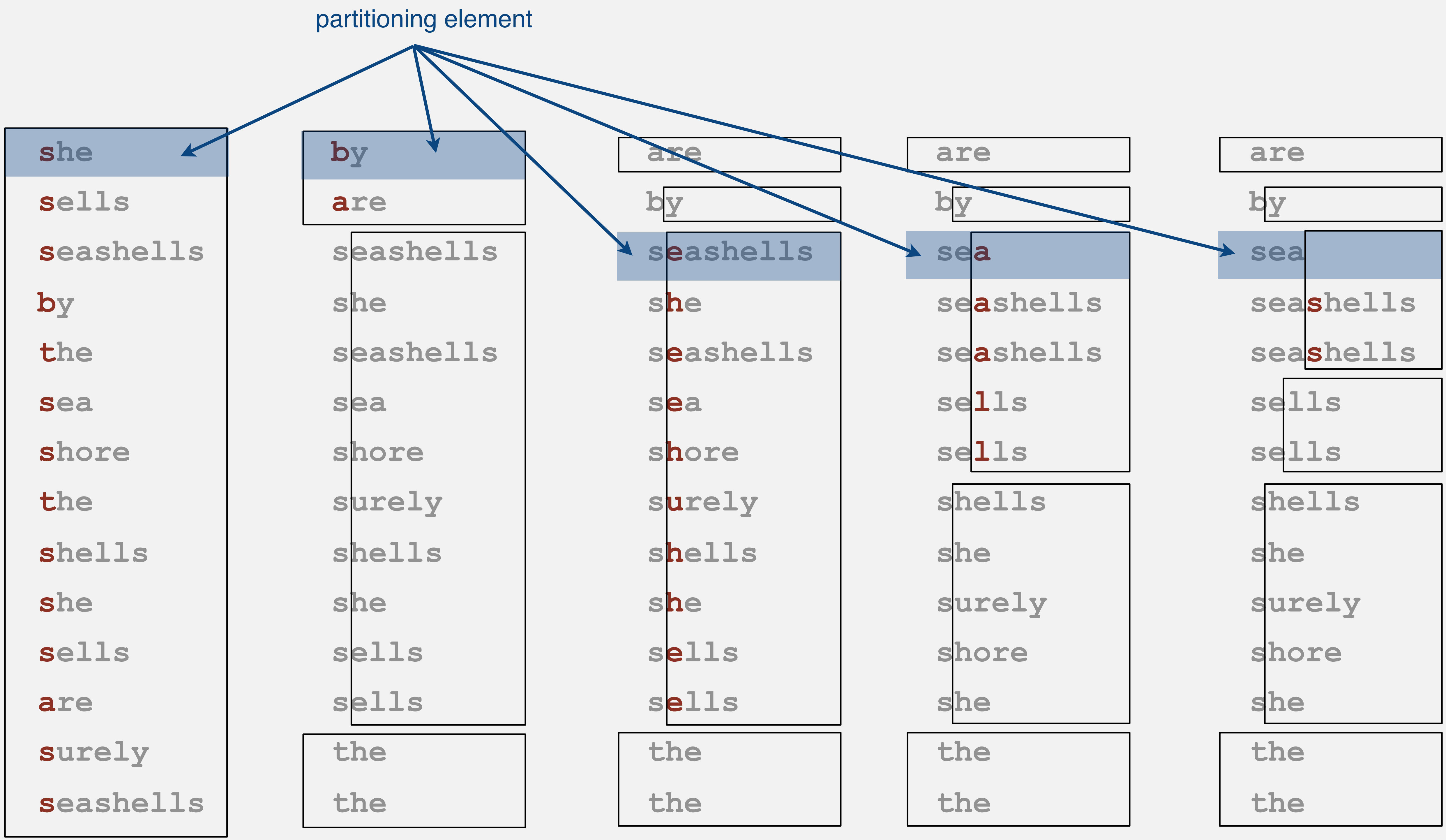
3-way string quicksort (Bentley and Sedgewick, 1997)

Overview. Do 3-way partitioning on the d^{th} character.

- Cheaper than R-way partitioning of MSD string sort.
- Need not examine again characters equal to the partitioning char.



3-way string quicksort: trace of recursive calls



Trace of first few recursive calls for 3-way string quicksort (subarrays of size 1 not shown)

3-way string quicksort: Java implementation

```
private static void sort(String[] a)
{  sort(a, 0, a.length - 1, 0);  }

private static void sort(String[] a, int lo, int hi, int d)
{
   if (hi <= lo) return;
   int lt = lo, gt = hi;
   int v = charAt(a[lo], d);
   int i = lo + 1;
   while (i <= gt)
   {
      int t = charAt(a[i], d);
      if      (t < v) exch(a, lt++, i++);
      else if (t > v) exch(a, i, gt--);
      else            i++;
   }

   sort(a, lo, lt-1, d);
   if (v >= 0) sort(a, lt, gt, d+1);
   sort(a, gt+1, hi, d);
}
```

3-way partitioning (using d^{th} character)

to handle variable-length strings

sort 3 pieces recursively

3-way string quicksort vs. standard quicksort

Standard quicksort.

- Uses $2N \ln N$ string compares on average.
- Costly for long keys that differ only at the end (and this is a common case!)

3-way string quicksort.

- Uses $2N \ln N$ character compares on average for random strings.
- Avoids recomparing initial parts of the string.
- Adapts to data: uses just "enough" characters to resolve order.
- Sublinear when strings are long.

Proposition. 3-way string quicksort is optimal (to within a constant factor); no sorting algorithm can (asymptotically) examine fewer chars.

Pf. Ties cost to entropy. Beyond scope of CS251.

3-way string quicksort vs. MSD string sort

MSD string sort.

- Has a long inner loop.
- Is cache-inefficient.
- Too much overhead reinitializing `count[]` and `aux[]`.

3-way string quicksort.

- Has a short inner loop.
- Is cache-friendly.
- Is in-place.

library call numbers

```
WUS-------10706-----7---10
WUS-------12692-----4---27
WLSOC------2542----30
LTK--6015-P-63-1988
LDS---361-H-4
...
```

Bottom line. 3-way string quicksort is the method of choice for sorting strings.

Summary of the performance of sorting algorithms

Frequency of operations.

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insertion sort	$N^2 /2$	$N^2 /4$	1	yes	`compareTo()`
mergesort	N lg N	N lg N	N	yes	`compareTo()`
quicksort	1.39 N lg N *	1.39 N lg N	c lg N	no	`compareTo()`
heapsort	2 N lg N	2 N lg N	1	no	`compareTo()`
LSD †	2 N W	2 N W	N + R	yes	`charAt()`
MSD ‡	2 N W	$N \log_R N$	N + D R	yes	`charAt()`
3-way string quicksort	1.39 W N lg N *	1.39 N lg N	log N + W	no	`charAt()`

* probabilistic

† fixed-length W keys

‡ average-length W keys

- key-indexed counting
- LSD string sort
- MSD string sort
- 3-way string quicksort
- **suffix arrays**

Warmup: longest common prefix

LCP. Given two strings, find the longest substring that is a prefix of both.

p	r	e	f	e	t	c	h
0	1	2	3	4	5	6	7
p	r	e	f	i	x		

```
public static String lcp(String s, String t)
{
   int n = Math.min(s.length(), t.length());
   for (int i = 0; i < n; i++)
   {
      if (s.charAt(i) != t.charAt(i))
         return s.substring(0, i);
   }
   return s.substring(0, n);
}
```

Running time. Linear-time in length of prefix match.

Space. Constant extra space.

Longest repeated substring

Given a string of N characters, find the longest repeated substring.

Ex.

```
a a c a a g t t t a c a a g c a t g a t g c t g t a c t a
g g a g a g t t a t a c t g g t c g t c a a a c c t g a a
c c t a a t c c t t g t g t g t a c a c a c a c t a c t a
c t g t c g t c g t c a t a t a t c g a g a t c a t c g a
a c c g g a a g g c c g g a c a a g g c g g g g g g t a t
a g a t a g a t a g a c c c c t a g a t a c a c a t a c a
t a g a t c t a g c t a g c t a g c t c a t c g a t a c a
c a c t c t c a c a c t c a a g a g t t a t a c t g g t c
a a c a c a c t a c t a c g a c a g a c g a c c a a c c a
g a c a g a a a a a a a a c t c t a t a t c t a t a a a a
```

Applications. Bioinformatics, cryptanalysis, data compression, ...

Given a string of N characters, find the longest repeated substring.

Ex.

```
a a c a a g t t t a c a a g c a t g a t g c t g t a c t a
g g a g a g t t a t a c t g g t c g t c a a a c c t g a a
c c t a a t c c t t g t g t g t a c a c a c a c t a c t a
c t g t c g t c g t c a t a t a t c g a g a t c a t c g a
a c c g g a a g g c c g g a c a a g g c g g g g g g t a t
a g a t a g a t a g a c c c c t a g a t a c a c a t a c a
t a g a t c t a g c t a g c t a g c t c a t c g a t a c a
c a c t c t c a c a c t c a a g a g t t a t a c t g g t c
a a c a c a c t a c t a c g a c a g a c g a c c a a c c a
g a c a g a a a a a a a a c t c t a t a t c t a t a a a a
```

Applications. Bioinformatics, cryptanalysis, data compression, ...

Longest repeated substring: a musical application

Visualize repetitions in music. `http://www.bewitched.com`

Mary Had a Little Lamb

Bach's Goldberg Variations

Longest repeated substring

Given a string of N characters, find the longest repeated substring.

Brute-force algorithm.

- Try all indices i and j for start of possible match.
- Compute longest common prefix (LCP) for each pair.

Analysis. Running time $\leq M N^2$, where M is length of longest match.

Longest repeated substring: a sorting solution

input string

a	a	c	a	a	g	t	t	t	a	c	a	a	g	c
0	1	2	3	4	5	6	7	8	9	10	11	12	13	14

Longest repeated substring: a sorting solution

input string

a	a	c	a	a	g	t	t	t	a	c	a	a	g	c
0	1	2	3	4	5	6	7	8	9	10	11	12	13	14

form suffixes

0	a a c a a g t t t a c a a g c
1	a c a a g t t t a c a a g c
2	c a a g t t t a c a a g c
3	a a g t t t a c a a g c
4	a g t t t a c a a g c
5	g t t t a c a a g c
6	t t t a c a a g c
7	t t a c a a g c
8	t a c a a g c
9	a c a a g c
10	c a a g c
11	a a g c
12	a g c
13	g c
14	c

Longest repeated substring: a sorting solution

input string

a	a	c	a	a	g	t	t	t	a	c	a	a	g	c
0	1	2	3	4	5	6	7	8	9	10	11	12	13	14

form suffixes

0	a a c a a g t t t a c a a g c
1	a c a a g t t t a c a a g c
2	c a a g t t t a c a a g c
3	a a g t t t a c a a g c
4	a g t t t a c a a g c
5	g t t t a c a a g c
6	t t t a c a a g c
7	t t a c a a g c
8	t a c a a g c
9	a c a a g c
10	c a a g c
11	a a g c
12	a g c
13	g c
14	c

sort suffixes to bring repeated substrings together

0	a a c a a g t t t a c a a g c
11	a a g c
3	a a g t t t a c a a g c
9	a c a a g c
1	a c a a g t t t a c a a g c
12	a g c
4	a g t t t a c a a g c
14	c
10	c a a g c
2	c a a g t t t a c a a g c
13	g c
5	g t t t a c a a g c
8	t a c a a g c
7	t t a c a a g c
6	t t t a c a a g c

Longest repeated substring: a sorting solution

input string

a	a	c	a	a	g	t	t	t	a	c	a	a	g	c
0	1	2	3	4	5	6	7	8	9	10	11	12	13	14

form suffixes

0	a a c a a g t t t a c a a g c
1	a c a a g t t t a c a a g c
2	c a a g t t t a c a a g c
3	a a g t t t a c a a g c
4	a g t t t a c a a g c
5	g t t t a c a a g c
6	t t t a c a a g c
7	t t a c a a g c
8	t a c a a g c
9	a c a a g c
10	c a a g c
11	a a g c
12	a g c
13	g c
14	c

sort suffixes to bring repeated substrings together

0	a a c a a g t t t a c a a g c
11	a a g c
3	a a g t t t a c a a g c
9	a c a a g c
1	a c a a g t t t a c a a g c
12	a g c
4	a g t t t a c a a g c
14	c
10	c a a g c
2	c a a g t t t a c a a g c
13	g c
5	g t t t a c a a g c
8	t a c a a g c
7	t t a c a a g c
6	t t t a c a a g c

compute longest prefix between adjacent suffixes

a	a	c	a	a	g	t	t	t	a	c	a	a	g	c
0	1	2	3	4	5	6	7	8	9	10	11	12	13	14

Longest repeated substring: Java implementation

```
public String lrs(String s)
{
   int N = s.length();

   String[] suffixes = new String[N];
   for (int i = 0; i < N; i++)
      suffixes[i] = s.substring(i, N);

   Arrays.sort(suffixes);

   String lrs = "";
   for (int i = 0; i < N-1; i++)
   {
      String x = lcp(suffixes[i], suffixes[i+1]);
      if (x.length() > lrs.length()) lrs = x;
   }
   return lrs;
}
```

create suffixes (linear time and space)

sort suffixes

find LCP between suffixes that are adjacent after sorting

```
% java LRS < mobydick.txt
,- Such a funny, sporty, gamy, jesty, joky, hoky-poky lad, is the Ocean, oh! Th
```

Sorting challenge

Problem. Five scientists A, B, C, D, and E are looking for long repeated substring in a genome with over 1 billion nucleotides.

- A has a grad student do it by hand.
- B uses brute force (check all pairs).
- C uses suffix sorting solution with insertion sort.
- D uses suffix sorting solution with LSD string sort.
- E uses suffix sorting solution with 3-way string quicksort.

Q. Which one is more likely to lead to a cure cancer?

Sorting challenge

Problem. Five scientists A, B, C, D, and E are looking for long repeated substring in a genome with over 1 billion nucleotides.

- A has a grad student do it by hand.
- B uses brute force (check all pairs).
- C uses suffix sorting solution with insertion sort.
- D uses suffix sorting solution with LSD string sort.
- ✓ E uses suffix sorting solution with 3-way string quicksort.

only if LRS is not long (!)

Q. Which one is more likely to lead to a cure cancer?

Longest repeated substring: empirical analysis

input file	characters	brute	suffix sort	length of LRS
`LRS.java`	2,162	0.6 sec	0.14 sec	73
`amendments.txt`	18,369	37 sec	0.25 sec	216
`aesop.txt`	191,945	1.2 hours	1.0 sec	58
`mobydick.txt`	1.2 million	43 hours †	7.6 sec	79
`chromosome11.txt`	7.1 million	2 months †	61 sec	12,567
`pi.txt`	10 million	4 months †	84 sec	14

† estimated