

# Nairian models and forcing axioms

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# The Proper Forcing Axiom

## Conjecture

PFA implies there is an inner model with a supercompact cardinal.

## Theorem (Todorćević)

*Assume PFA. Then  $\neg \square(\kappa)$  for all  $\kappa > \omega_1$ .*

- ▶ Lower bound computations for PFA go through failures of square
  - ▶ Schimmerling, Steel, Jensen-Schimmerling-Schindler-Steel
  - ▶ Sargsyan-Trang have obtained a model of LSA

## Conjecture

1. (Zeman)  $\neg \square_\kappa$  for  $\kappa$  singular is equiconsistent with a subcompact
2. (Steel)  $\neg \square_\kappa$  for  $\kappa$  singular strong limit requires a superstrong

# The Proper Forcing Axiom

## Theorem (Viale-Weiss)

*Suppose  $\kappa$  is an inaccessible cardinal and PFA is forced by an iteration  $\mathbb{P}$  collapsing  $\kappa$  to  $\omega_2$  such that*

- 1.  $\mathbb{P}$  is the direct limit of an iteration  $\langle \mathbb{P}_\alpha : \alpha < \kappa \rangle$  which takes direct limits stationarily often, and*
- 2.  $|\mathbb{P}_\alpha| < \kappa$  for all  $\alpha < \kappa$ .*

*Then  $\kappa$  is strongly compact. If  $\mathbb{P}$  is proper, then  $\kappa$  is supercompact.*

# The Proper Forcing Axiom

- ▶ (Woodin's reformulation) If PFA holds and  $N$  is an inner model with the  $\omega_2$ -cover and  $\omega_2$ -approximation properties in which  $\omega_2$  is inaccessible, then  $\omega_2^V$  is strongly compact.
- ▶ (Woodin) If  $\delta$  is weakly compact and  $V[G]$  is a  $\delta$ -cc forcing extension, then  $V$  has the  $\delta$ -cover and  $\delta$ -approximation properties in  $V[G]$ .

## Key question

Are there any other methods for building models of PFA/MM?

# Woodin's consistency proof for Martin's Maximum

## Theorem (Woodin)

*Assume there is a Vopěnka cardinal  $\delta$  and there is an elementary embedding  $j : V_\delta \rightarrow V_\delta$  with  $V_\kappa \prec V_\delta$ , where  $\kappa = \text{crit}(j)$ . Then there is a revised countable support iteration  $\mathbb{P}$  of semiproper forcings such that if  $g \subseteq \mathbb{P}$  is  $V$ -generic, then*

$$V[g]_\delta \models \text{ZFC} + \text{MM}^{++}.$$

*Moreover, in  $V[g]_\delta$  there is no proper inner model of ZFC with the  $\omega_2$ -cover and  $\omega_2$ -approximation properties in which  $\omega_2$  is inaccessible.*

- Vacuous if the HOD conjecture is true

# Axiom $(*)^{++}$

## Theorem (Aspero-Schindler)

Assume  $\text{MM}^{++}$ . Then axiom  $(*)$  holds.

## Definition (Axiom $(*)^{++}$ )

There is a pointclass  $\Gamma \subset \wp(\mathbb{R})$  and  $g \subseteq \mathbb{P}_{\max}$  such that

1.  $L(\Gamma, \mathbb{R}) \models \text{AD}^+$ ,
2.  $g$  is  $L(\Gamma, \mathbb{R})$ -generic, and
3.  $\wp(\mathbb{R}) \in L(\Gamma, \mathbb{R})[g]$ .

## Question (Woodin)

Is  $\text{MM}^{++}$  consistent with  $(*)^{++}$ ? Is SRP consistent with  $(*)^{++}$ ?

# The cofinality of $\Theta^{L(\Gamma^\infty, \mathbb{R})}$

- ▶  $\Theta$  is the least ordinal which is not the surjective image of  $\mathbb{R}$
- ▶  $\Gamma^\infty$  denotes the collection of universally Baire sets

## Theorem (Woodin)

1.  $(*)^{++}$  implies  $\Theta^{L(\Gamma^\infty, \mathbb{R})} = \omega_3$
2. Suppose  $\delta$  is a supercompact cardinal. If there are class many Woodin cardinals and  $V[g]$  is a  $\delta$ -cc forcing extension in which  $\delta = \omega_2$ , then  $V[g] \models \Theta^{L(\Gamma^\infty, \mathbb{R})} < \omega_3$ .

# Forcing over models of $\text{AD}_{\mathbb{R}} + “\Theta \text{ is regular.}”$

## Theorem (Woodin)

Assume  $\text{AD}_{\mathbb{R}} + “\Theta \text{ is regular.}”$  Then  $\mathbb{P}_{\max} * \text{Add}(\omega_3, 1) \Vdash \text{MM}^{++}(c)$ .

## Theorem (Caicedo-Larson-Sargsyan-Schindler-Steel-Zeman)

Assume  $\text{AD}_{\mathbb{R}} + “\Theta \text{ is regular}”$ . Suppose the set of  $\kappa$  which are regular in  $\text{HOD}$  and have cofinality  $\omega_1$  is stationary in  $\Theta$ . Then

$$\mathbb{P}_{\max} * \text{Add}(\omega_3, 1) \Vdash \text{MM}^{++}(c) + \neg \square(\omega_2) + \neg \square(\omega_3).$$

## Theorem (Larson-Sargsyan)

Assume  $\text{AD}_{\mathbb{R}} + \exists \lambda \bowtie_{\lambda}$ . Then

$$\mathbb{P}_{\max} * \text{Add}(\omega_3, 1) * \text{Add}(\omega_4, 1) \Vdash \neg \square(\omega_3) + \neg \square(\omega_4).$$

- (Woodin)  $\text{MM}^{++}$  cannot be forced over a determinacy model of the form  $L(S, \wp(\mathbb{R}))$  for  $S \subset \text{Ord}$ .



# Nairian models

## Definition

Assume LSA and let  $(\theta_\gamma : \gamma \leq \Omega)$  be the Solovay sequence. Suppose  $\alpha + 1 \leq \Omega$  is such that

$$\text{HOD} \models \text{“}\theta_{\alpha+1} \text{ is a limit of Woodin cardinals.”}$$

Let  $M = V_{\theta_{\alpha+1}}^{\text{HOD}}$  and  $N = L_{\theta_{\alpha+1}}(\bigcup_{\eta < \theta_{\alpha+1}} (M|_\eta)^\omega)$ .

- ▶ (Woodin, building on Steel)  $N \models \text{ZF}$ .
- ▶ For this talk, a *Nairian model* is an initial segment  $N_\gamma$  of  $N$  such that  $N_\gamma \models \text{ZF}$ .
- ▶ Nairian models exist assuming less than a Woodin limit of Woodin cardinals

$$\Theta^{L(\Gamma^\infty, \mathbb{R})}$$

## Theorem (B.-Sargsyan)

For  $i \in \{1, 2, 3\}$ , the theory

1. *there are class many Woodin cardinals,*
2.  $\Gamma^\infty$  *is sealed, and*
3.  $\Theta^{L(\Gamma^\infty, \mathbb{R})} = \omega_i$

*is consistent.*

- For  $\omega_2$  and  $\omega_3$ , uses forcing over Nairian models

# Failures of square

## Theorem (B.-Larson-Sargsyan)

Fix  $n < \omega$ . In a forcing extension of a Nairian model,  $\neg \square(\aleph_i)$  holds for all  $i \in [2, n]$ .

## Theorem (B.-Larson-Sargsyan)

Let  $N_\gamma$  be the least initial segment of  $N$  such that  $N_\gamma \models \text{ZF}$ . Then in a forcing extension  $N_\gamma[g]$  of  $N_\gamma$ ,  $\neg \square(\kappa)$  holds for all  $\kappa > \omega_1$ .

## Corollary

$\text{ZFC} + \forall \kappa > \omega_1 \neg \square(\kappa) <_{\text{Con}} \text{ZFC} + \text{WLW}$ .

- (Neeman-Steel) Assuming an iterability hypothesis, if  $\delta$  is a Woodin cardinal such that  $\neg \square_\delta + \neg \square(\delta)$ , then there is an inner model of  $\text{ZFC} +$  “there is a subcompact cardinal.”

# The HOD conjecture I

## Definition

A regular cardinal  $\kappa > \omega_1$  is  $\omega$ -strongly measurable in HOD if there is  $\gamma < \kappa$  such that

1.  $(2^\gamma)^{\text{HOD}} < \kappa$
2.  $\{\alpha < \kappa : \text{cof}(\alpha) = \omega\}$  cannot be definably partitioned into  $\gamma$  sets.

## Theorem (HOD Dichotomy theorem; Woodin, Goldberg)

If  $\delta$  is a supercompact cardinal, then exactly one of the following hold.

1. No regular cardinal  $\gamma \geq \delta$  is  $\omega$ -strongly measurable in HOD.
2. Class many regular cardinals are  $\omega$ -strongly measurable in HOD.

- **HOD Hypothesis:** There is a class of regular cardinals which are not  $\omega$ -strongly measurable in HOD.

# The HOD conjecture II

## Definition (HOD conjecture)

ZFC + “there is a supercompact cardinal”  $\vdash$  the HOD Hypothesis.

## Theorem (Ben Neria-Hayut)

*It is consistent relative to an inaccessible cardinal  $\kappa$  with  $\kappa = \sup_{\alpha < \kappa} o(\alpha)$  that every successor of a regular cardinal is  $\omega$ -strongly measurable in HOD.*

## Theorem (B.-Larson-Sargsyan)

*In  $N_\gamma[g]$ , successors of singular cardinals are  $\omega$ -strongly measurable in HOD.*

- The HOD Hypothesis is not provable in ZFC.

# Stationary Set Reflection I

## Definition (SRP; Todorćević)

Suppose  $\lambda > \omega_1$  and  $S \subseteq \wp_{\omega_1}(\lambda)$  is projective stationary. Then for every  $X \subseteq \lambda$  such that  $\omega_1 \subseteq X$  and  $|X| = \omega_1$ , there is  $X \subseteq Y \subseteq \lambda$  of size  $\omega_1$  such that  $S \cap \wp_{\omega_1}(Y)$  contains a club in  $\wp_{\omega_1}(Y)$ .

## Theorem (Woodin)

*Assume SRP. Then exactly one of the following hold.*

1.  $\text{Ord}^\omega \subset \text{HOD}$ .
2. *There is an ordinal  $\alpha$  such that every regular cardinal  $\kappa > \alpha$  is  $\omega$ -strongly measurable in HOD.*

## Definition (The (HOD+SRP) conjecture)

$\text{ZFC} + \text{SRP}$  proves that  $\text{Ord}^\omega \subset \text{HOD}$ .

# Stationary Set Reflection II

## Definition (SRP\*; Woodin)

Suppose  $\lambda > \omega_1$ . There is a normal fine ideal  $I \subset \wp(\wp_{\omega_1}(\lambda))$  such that

1. for every stationary  $T \subset \omega_1$ , the set  $\{\sigma \in \wp_{\omega_1}(\lambda) \mid \sigma \cap \omega_1 \in T\} \notin I$
2. if  $S \subset \wp_{\omega_1}(\lambda)$  is such that for each stationary  $T \subset \omega_1$ ,  $\{X \in S : X \cap \omega_1 \in T\} \notin I$ , there is  $\omega_1 < \gamma < \lambda$  such that  $S \cap \wp_{\omega_1}(\gamma)$  contains a club in  $\wp_{\omega_1}(\gamma)$ .

## Theorem (Steel-Zoble)

Suppose  $NS_{\omega_1}$  is saturated,  $2^\omega \leq \omega_2$ , and  $\text{SRP}^*(\omega_2)$ . Then  $L(\mathbb{R}) \models \text{AD}$ .

## Theorem (B.-Sargsyan)

$\text{SRP}^*$  holds in  $N_\gamma[g]$ .

# Stationary Set Reflection III

## Quasi-Club Conjecture

In Nairian models, the quasi-club filter on  $\wp_{\omega_1}(\lambda)$  is an ultrafilter for all  $\lambda \in \text{Ord}$ .

## Theorem (B. Sargsyan)

*Assuming the Quasi-Club conjecture,  $N_\gamma[g] \models \text{SRP}$ .*

## Corollary

*The  $(\text{HOD} + \text{SRP})$  conjecture is false, assuming the Quasi-Club conjecture.*

- ▶  $\omega_2$  is a supercompact cardinal in  $N_\gamma$

## Theorem (B.-Sargsyan)

$\text{SRP}^*$  is consistent with  $(*)^{++}$ . If the Quasi-Club conjecture is true, then  $\text{SRP}$  is consistent with  $(*)^{++}$ .



# Consequences of MM

## Theorem (B.-Sargsyan)

*The following hold in  $N_\gamma[g]$ .*

- ▶ *Moore's Open Mapping Reflection, and*
- ▶  $\text{MA}^+(\sigma\text{-closed})$ .

## Question

Does the P-Ideal Dichotomy hold in  $N_\gamma[g]$ ?

## Question

1. Is MM equivalent to some conjunction of these principles?
2. Could some fragment of  $\text{MM}^{++}$  suffice to produce a “minimal” model of  $\text{MM}^{++}$ ?

# Conclusion: Further Questions

## Question

Is  $\omega_1$  a  $\Theta^+$ -Berkeley cardinal in the full Nairian model?<sup>1</sup>

## Question

Can a Woodin cardinal exist in a forcing extension of a Nairian model?

## Question

Are the universally Baire sets sealed in full Nairian models? In their ZFC forcing extensions?

## Question

What is the axiomatic theory of  $N$ ?

Thank you

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<sup>1</sup>A cardinal  $\kappa$  is  $\eta$ -Berkeley if for every transitive  $M$  with  $|M| < \eta$  and every  $\alpha < \kappa$ , there is an elementary embedding  $j : M \rightarrow M$  with  $\text{crit}(j) \in (\alpha, \kappa)$ .