

Nairian models and forcing axioms

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The Proper Forcing Axiom

Conjecture

PFA implies there is an inner model with a supercompact cardinal.

Theorem (Todorćević)

Assume PFA. Then $\neg \square(\kappa)$ for all $\kappa > \omega_1$.

- ▶ Lower bound computations for PFA go through failures of square
 - ▶ Schimmerling, Steel, Jensen-Schimmerling-Schindler-Steel
 - ▶ Sargsyan-Trang have obtained a model of LSA

Conjecture

1. (Zeman) $\neg \square_\kappa$ for κ singular is equiconsistent with a subcompact
2. (Steel) $\neg \square_\kappa$ for κ singular strong limit requires a superstrong

The Proper Forcing Axiom

Theorem (Viale-Weiss)

Suppose κ is an inaccessible cardinal and PFA is forced by an iteration \mathbb{P} collapsing κ to ω_2 such that

- 1. \mathbb{P} is the direct limit of an iteration $\langle \mathbb{P}_\alpha : \alpha < \kappa \rangle$ which takes direct limits stationarily often, and*
- 2. $|\mathbb{P}_\alpha| < \kappa$ for all $\alpha < \kappa$.*

Then κ is strongly compact. If \mathbb{P} is proper, then κ is supercompact.

The Proper Forcing Axiom

- ▶ (Woodin's reformulation) If PFA holds and N is an inner model with the ω_2 -cover and ω_2 -approximation properties in which ω_2 is inaccessible, then ω_2^V is strongly compact.
- ▶ (Woodin) If δ is weakly compact and $V[G]$ is a δ -cc forcing extension, then V has the δ -cover and δ -approximation properties in $V[G]$.

Key question

Are there any other methods for building models of PFA/MM?

Woodin's consistency proof for Martin's Maximum

Theorem (Woodin)

Assume there is a Vopěnka cardinal δ and there is an elementary embedding $j : V_\delta \rightarrow V_\delta$ with $V_\kappa \prec V_\delta$, where $\kappa = \text{crit}(j)$. Then there is a revised countable support iteration \mathbb{P} of semiproper forcings such that if $g \subseteq \mathbb{P}$ is V -generic, then

$$V[g]_\delta \models \text{ZFC} + \text{MM}^{++}.$$

Moreover, in $V[g]_\delta$ there is no proper inner model of ZFC with the ω_2 -cover and ω_2 -approximation properties in which ω_2 is inaccessible.

- Vacuous if the HOD conjecture is true

Axiom $(*)^{++}$

Theorem (Aspero-Schindler)

Assume MM^{++} . Then axiom $(*)$ holds.

Definition (Axiom $(*)^{++}$)

There is a pointclass $\Gamma \subset \wp(\mathbb{R})$ and $g \subseteq \mathbb{P}_{\max}$ such that

1. $L(\Gamma, \mathbb{R}) \models \text{AD}^+$,
2. g is $L(\Gamma, \mathbb{R})$ -generic, and
3. $\wp(\mathbb{R}) \in L(\Gamma, \mathbb{R})[g]$.

Question (Woodin)

Is MM^{++} consistent with $(*)^{++}$? Is SRP consistent with $(*)^{++}$?

The cofinality of $\Theta^{L(\Gamma^\infty, \mathbb{R})}$

- ▶ Θ is the least ordinal which is not the surjective image of \mathbb{R}
- ▶ Γ^∞ denotes the collection of universally Baire sets

Theorem (Woodin)

1. $(*)^{++}$ implies $\Theta^{L(\Gamma^\infty, \mathbb{R})} = \omega_3$
2. Suppose δ is a supercompact cardinal. If there are class many Woodin cardinals and $V[g]$ is a δ -cc forcing extension in which $\delta = \omega_2$, then $V[g] \models \Theta^{L(\Gamma^\infty, \mathbb{R})} < \omega_3$.

Forcing over models of $AD_{\mathbb{R}} + “\Theta \text{ is regular.}”$

Theorem (Woodin)

Assume $AD_{\mathbb{R}} + “\Theta \text{ is regular.}”$ Then $\mathbb{P}_{\max} * Add(\omega_3, 1) \Vdash MM^{++}(c)$.

Theorem (Caicedo-Larson-Sargsyan-Schindler-Steel-Zeman)

Assume $AD_{\mathbb{R}} + “\Theta \text{ is regular}”$. Suppose the set of κ which are regular in HOD and have cofinality ω_1 is stationary in Θ . Then

$$\mathbb{P}_{\max} * Add(\omega_3, 1) \Vdash MM^{++}(c) + \neg \square(\omega_2) + \neg \square(\omega_3).$$

Theorem (Larson-Sargsyan)

Assume $AD_{\mathbb{R}} + \exists \lambda \bowtie_{\lambda}$. Then

$$\mathbb{P}_{\max} * Add(\omega_3, 1) + Add(\omega_4, 1) \Vdash \neg \square(\omega_3) + \neg \square(\omega_4).$$

- (Woodin) MM^{++} cannot be forced over a determinacy model of the form $L(S, \wp(\mathbb{R}))$ for $S \subset \text{Ord}$.

Nairian models

Definition

Assume AD^+ and let $(\theta_\alpha : \alpha \leq \Omega)$ be the Solovay sequence. Suppose $\alpha + 1 \leq \Omega$ is such that

$$\text{HOD} \models \text{“}\theta_{\alpha+1} \text{ is a limit of Woodin cardinals.”}$$

Let $M = V_{\theta_{\alpha+1}}^{\text{HOD}}$ and $N = L_{\theta_{\alpha+1}}(\bigcup_{\eta < \theta_{\alpha+1}} (M|_\eta)^\omega)$.

- ▶ (Woodin-Steel) $N \models \text{ZF}$.
- ▶ For this talk, a *Nairian model* is an initial segment N_γ of N such that $N_\gamma \models \text{ZF}$.
- ▶ Nairian models exist assuming less than a Woodin limit of Woodin cardinals

$$\Theta^{L(\Gamma^\infty, \mathbb{R})}$$

Theorem (B.-Sargsyan)

For $i \in \{1, 2, 3\}$, the theory

1. *there are class many Woodin cardinals,*
2. Γ^∞ *is sealed, and*
3. $\Theta^{L(\Gamma^\infty, \mathbb{R})} = \omega_i$

is consistent.

- For ω_2 and ω_3 , uses forcing over Nairian models

Failures of square

Theorem (B.-Larson-Sargsyan)

Fix $n < \omega$. In a forcing extension of a Nairian model, $\neg \square(\aleph_i)$ holds for all $i \in [2, n]$.

Theorem (B.-Larson-Sargsyan)

Let N_γ be the least initial segment of N such that $N_\gamma \models \text{ZF}$. Then in a forcing extension $N_\gamma[g]$ of N_γ , $\neg \square(\kappa)$ holds for all $\kappa > \omega_1$.

Corollary

$\text{ZFC} + \forall \kappa > \omega_1 \neg \square(\kappa) <_{\text{Con}} \text{ZFC} + \text{WLW}$.

- (Neeman-Steel) Assuming an iterability hypothesis, if δ is a Woodin cardinal such that $\neg \square_\delta + \neg \square(\delta)$, then there is an inner model of $\text{ZFC} +$ “there is a subcompact cardinal.”

The HOD conjecture I

Definition

A regular cardinal $\kappa > \omega_1$ is ω -strongly measurable in HOD if there is $\gamma < \kappa$ such that

1. $(2^\gamma)^{\text{HOD}} < \kappa$
2. $\{\alpha < \kappa : \text{cof}(\alpha) = \omega\}$ cannot be definably partitioned into γ sets.

Theorem (HOD Dichotomy theorem; Woodin, Goldberg)

If δ is a supercompact cardinal, then exactly one of the following hold.

1. No regular cardinal $\gamma \geq \delta$ is ω -strongly measurable in HOD.
2. Class many regular cardinals are ω -strongly measurable in HOD.

- **HOD Hypothesis:** There is a class of regular cardinals which are not ω -strongly measurable in HOD.

The HOD conjecture II

Definition (HOD conjecture)

ZFC + “there is an extendible cardinal” \vdash the HOD Hypothesis.

Theorem (Ben Neria-Hayut)

It is consistent relative to an inaccessible cardinal κ with $\kappa = \sup_{\alpha < \kappa} o(\alpha)$ that every successor of a regular cardinal is ω -strongly measurable in HOD.

Theorem (B.-Larson-Sargsyan)

In $N_\gamma[g]$, successors of singular cardinals are ω -strongly measurable in HOD.

- The HOD Hypothesis is not provable in ZFC.

Stationary Set Reflection I

Definition (SRP; Todorćević)

Suppose $\lambda > \omega_1$ and $S \subseteq \wp_{\omega_1}(\lambda)$ is projective stationary. Then for every $X \subseteq \lambda$ such that $\omega_1 \subseteq X$ and $|X| = \omega_1$, there is $X \subseteq Y \subseteq \lambda$ of size ω_1 such that $S \cap \wp_{\omega_1}(Y)$ contains a club in $\wp_{\omega_1}(Y)$.

Theorem (Woodin)

Assume SRP. Then exactly one of the following hold.

1. $\text{Ord}^\omega \subset \text{HOD}$.
2. *There is an ordinal α such that every regular cardinal $\kappa > \alpha$ is ω -strongly measurable in HOD.*

Definition (The (HOD+SRP) conjecture)

$\text{ZFC} + \text{SRP}$ proves that $\text{Ord}^\omega \subset \text{HOD}$.

Stationary Set Reflection II

Definition (SRP*; Woodin)

Suppose $\lambda > \omega_1$. There is a normal fine ideal $I \subset \wp(\wp_{\omega_1}(\lambda))$ such that

1. for every stationary $T \subset \omega_1$, the set $\{\sigma \in \wp_{\omega_1}(\lambda) \mid \sigma \cap \omega_1 \in T\} \notin I$
2. if $S \subset \wp_{\omega_1}(\lambda)$ is such that for each stationary $T \subset \omega_1$, $\{X \in S : X \cap \omega_1 \in T\} \notin I$, there is $\omega_1 < \gamma < \lambda$ such that $S \cap \wp_{\omega_1}(\gamma)$ contains a club in $\wp_{\omega_1}(\gamma)$.

Theorem (Steel-Zoble)

Suppose NS_{ω_1} is saturated, $2^\omega \leq \omega_2$, and $\text{SRP}^*(\omega_2)$. Then $L(\mathbb{R}) \models \text{AD}$.

Theorem (B.-Sargsyan)

SRP^* holds in $N_\gamma[g]$.

Stationary Set Reflection III

Quasi-Club Conjecture

In Nairian models, the quasi-club filter on $\wp_{\omega_1}(\lambda)$ is an ultrafilter for all $\lambda \in \text{Ord}$.

Theorem (B. Sargsyan)

Assuming the Quasi-Club conjecture, $N_\gamma[g] \models \text{SRP}$.

Corollary

The $(\text{HOD} + \text{SRP})$ conjecture is false, assuming the Quasi-Club conjecture.

- ▶ ω_2 is a supercompact cardinal in N_γ

Theorem (B.-Sargsyan)

SRP^* is consistent with $(*)^{++}$. If the Quasi-Club conjecture is true, then SRP is consistent with $(*)^{++}$.

Consequences of MM

Theorem (B.-Sargsyan)

The following hold in $N_\gamma[g]$.

- ▶ *Moore's Open Mapping Reflection, and*
- ▶ $\text{MA}^+(\sigma\text{-closed})$.

Question

Does the P-Ideal Dichotomy hold in $N_\gamma[g]$?

Question

1. Is MM equivalent to some conjunction of these principles?
2. Could some fragment of MM^{++} suffice to produce a “minimal” model of MM^{++} ?

Conclusion: Further Questions

Question

Is ω_1 a Θ^+ -Berkeley cardinal in the full Nairian model?¹

Question

Can an extendible cardinal exist in a Nairian model?

Question

Are the universally Baire sets sealed in full Nairian models? In their ZFC forcing extensions?

Question

What is the axiomatic theory of N ?

Thank you

¹A cardinal κ is η -Berkeley if for every transitive M with $|M| < \eta$ and every $\alpha < \kappa$, there is an elementary embedding $j : M \rightarrow M$ with $\text{crit}(j) \in (\alpha, \kappa)$.