Steel's program and Nairian models

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Plan

New axioms

Steel's program

Nairian models

New axioms

Gödel's Program

Continuum Hypothesis

There are exactly ω_1-many real numbers.

Theorem

If ZFC *is consistent, then*

1. *(Gödel)* $\mathrm{ZFC} + CH$ *is consistent, and*
2. *(Cohen)* $\mathrm{ZFC} + \neg CH$ *is consistent.*

Gödel's Program

Decide ZFC-undecidable problems in well-justified extensions of ZFC.

Large cardinals

- No precise definition; strong versions of the axiom of infinity which imply the consistency of ZFC and more
- Widely accepted
- Decide problems about definable sets of reals, e.g. Projective Determinacy (Martin-Steel)
- (Levy-Solovay) Do not decide CH
- Used primarily to show that various states of affairs are consistent

The HOD Dichotomy

Theorem (HOD Dichotomy; Woodin, Goldberg)

If δ is a supercompact cardinal, then exactly one of the following hold.

1. *Every singular cardinal $\lambda > \delta$ is singular in HOD and $(\lambda^+)^{\mathrm{HOD}} = \lambda^+$.*
2. *Every regular cardinal greater than δ is measurable in HOD.*

HOD Hypothesis

There are class many regular cardinals which are not measurable in HOD.

HOD Conjecture

ZFC + "there is a supercompact cardinal" $\vdash$ the HOD Hypothesis.

Suslin's Hypothesis

Definition
A tree is a *Suslin tree* if it has height ω_1 but no uncountable branches or antichains.

Suslin's hypothesis
There are no Suslin trees.

Theorem (Jech, Tennenbaum, Solovay-Tennenbaum)
ZFC does not decide Suslin's hypothesis.

Forcing axioms

- Martin and Solovay isolated **Martin's Axiom** from the Solovay-Tennenbaum proof
- Baumgartner defined the **Proper Forcing Axiom**
- Foreman, Magidor, and Shelah defined **Martin's Maximum**

Theorem
If there is a supercompact cardinal, then

1. *(Baumgartner) PFA holds in a forcing extension.*
2. *(Foreman-Magidor-Shelah) MM holds in a forcing extension.*

Theorem (Todorcevic, Velickovic)
PFA implies $\mathfrak{c} = \omega_2$.

New axioms derived from MM

- Weak Reflection Principle
- Strong Reflection Principle
- Open Graph Axiom
- P-Ideal Dichotomy
- MA^{+}(σ-closed)
- Mapping Reflection Principle

SRP

SRP (Todorcevic)

Suppose $\lambda > \omega_1$ and $S \subset \wp_{\omega_1}(\lambda)$ is projective stationary. Then for every $X \subseteq \lambda$ such that $\omega_1 \subseteq X$ and $|X| = \omega_1$, there is $X \subseteq Y \subseteq \lambda$ of size ω_1 such that $S \cap \wp_{\omega_1}(Y)$ contains a club in $\wp_{\omega_1}(Y)$.

Theorem (Woodin)

Assume SRP. Then exactly one of the following hold.

1. $\mathrm{Ord}^\omega \subset \mathrm{HOD}$.
2. *There is an ordinal α such that every regular cardinal $\kappa > \alpha$ is measurable in* HOD.

The (HOD+SRP) conjecture

$\mathrm{ZFC} + \mathrm{SRP} \vdash \mathrm{Ord}^\omega \subset \mathrm{HOD}.$

Determinacy and forcing axioms

Definition

For $A \subseteq \mathbb{R}$, G_A is the game in which players I and II alternately play natural numbers, producing $x \in \mathbb{R}$. Player I wins if and only if $x \in A$.

Axiom of Determinacy

For all $A \subseteq \mathbb{R}$, either player I or player II has a winning strategy in G_A.

Woodin's Axiom $(*)$

1. $L(\mathbb{R}) \models \mathrm{AD}$
2. $\wp(\omega_1) \subset L(\mathbb{R})^{\mathbb{P}_{\max}}$

- Implies there is a definable counterexample to CH
- Effectively settles the theory of $H(\omega_2)$

Determinacy and forcing axioms

Theorem (Aspero-Schindler)

Assume MM^{++}. Then axiom $(*)$ holds.

- What about $H(\omega_3)$?

Axiom $(*)^{++}$

There is a pointclass $\Gamma \subset \wp(\mathbb{R})$ and $g \subseteq \mathbb{P}_{\max}$ such that

1. $L(\Gamma, \mathbb{R}) \models \mathrm{AD}^+$,
2. g is $L(\Gamma, \mathbb{R})$-generic, and
3. $\wp(\mathbb{R}) \in L(\Gamma, \mathbb{R})[g]$.

Question (Woodin)

Is MM^{++} consistent with $(*)^{++}$? Is SRP consistent with $(*)^{++}$?

Steel's program

Relative consistencies

The zoo of new axioms is more organized than it appears a priori

Definition

Let $T, U \supseteq \mathrm{ZFC}$. Then

1. $T \leq_{\mathrm{Con}} U$ if $\mathrm{ZFC} \vdash \mathrm{Con}(U) \Rightarrow \mathrm{Con}(T)$.
2. T, U are equiconsistent if $T \leq_{\mathrm{Con}} U$ and $U \leq_{\mathrm{Con}} T$.

In practice, given $T \supset \mathrm{ZFC}$:

- $T \leq_{\mathrm{Con}} H$, for some large cardinal hypothesis H, via forcing
- Show H is optimal: $H \leq_{\mathrm{Con}} T$, via inner model theory

The concrete

Suppose T, U are natural extensions of ZFC.

- If $T \leq_{\mathrm{Con}} U$, then the arithmetical consequences of T are consequences of U
- If $\mathrm{ZFC} +$ "there are infinitely many Woodin cardinals" $\leq_{\mathrm{Con}} T \leq_{\mathrm{Con}} U$, then the second order number theoretic consequences of T are consequences of U

Our current understanding of the possibilities for maximizing interpretative power has led us to **one theory of the concrete***, and a family of theoretical superstructures for it, each containing all the large cardinal hypotheses.* **These different theories are logically related in a way that enables us to use them all together.**

–Steel

Logical relationships

> *If truly distinct frameworks emerged, the first order of business would be to unify them.*
>
> *In fact, the different natural theories we have found in our hierarchy are not independent of one another. Their common theory of the concrete stems from logical relationships that go deeper, and are brought out in our relative consistency proofs. These logical relationships may suggest a unifying framework.*
>
> *–Steel*

- Never mind "first order" problems like CH or SH for now
- Develop methods for understanding how natural extensions of ZFC relate to one another

Translations

The goal of our framework theory is to maximize interpretative power, to provide a language and theory in which all mathematics, of today, and of the future so far as we can anticipate it today, can be developed. **Maximizing interpretative power entails maximizing consistency strength, but it requires more, in that we want to be able to translate other theories/languages into our framework theory/language in such a way that we preserve their meaning**. *The way we interpret set theories today is to think of them as theories of inner models of generic extensions of models satisfying some large cardinal hypothesis, and this method has had amazing success. We don't seem to lose any meaning this way.*

–Steel

Steel's program & PFA

Natural consistency strengths wellordered: *If T is a natural extension of ZFC, then there is an extension H axiomatized by large cardinal hypotheses such that $T \equiv_{\mathrm{Con}} H$. Moreover, $\leq_{\mathrm{Con}}$ is a prewellorder of the natural extensions of ZFC. In particular, if T and U are natural extensions of ZFC, then either $T \leq_{\mathrm{Con}} U$ or $U \leq_{\mathrm{Con}} T$.*

Perhaps the main thrust of this vague conjecture at the moment is programmatic: **understand better, and develop further, our methods for comparing consistency strengths**. *At present, this devolves at once into: understand better, and develop further, the theory of canonical inner models satisfying large cardinal hypotheses. One very ambitious conjecture here is that* **Con(PFA)** $\Rightarrow$ **Con(there is a supercompact cardinal)**. *This has been a target of inner model theory for about 30 years.*

–Steel

Lower bounds of PFA

Theorem (Todorčević)

Assume PFA. Then $\neg\Box_\kappa$ *for all* $\kappa > \omega$.

Lower bound computations for PFA go through failures of square

- Schimmerling, Steel, Jensen-Schimmerling-Schindler-Steel
- Sargsyan-Trang have obtained a model of the Largest Suslin Axiom

Conjecture

1. (Zeman) $\neg\Box_\kappa$ for κ singular is equiconsistent with a subcompact
2. (Steel) $\neg\Box_\kappa$ for κ singular strong limit requires a superstrong

Models of PFA

Definition

Suppose δ is a regular uncountable cardinal and N is a transitive inner model of ZFC containing the ordinals. Then N has the

1. δ-*cover property* if for all $\sigma \subset N$ with $|\sigma| < \delta$, there is $\tau \in N$ such that $\sigma \subset \tau$ and $|\tau| < \delta$
2. δ-*approximation property* if for all $X \subset N$, the following are equivalent:
   2.1 $X \in N$
   2.2 $X \cap \tau \in N$ for all $\tau \in N$ with $|\tau| < \delta$.

Models of PFA

Theorem (Viale-Weiss)

If PFA holds and N is an inner model with the ω_2-cover and ω_2-approximation properties in which ω_2 is inaccessible, then ω_2^V is strongly compact in N.

Theorem (Usuba)

If δ is weakly compact and $V[G]$ is a δ-cc forcing extension, then V has the δ-cover and δ-approximation properties in $V[G]$.

Woodin's exotic model of MM

Theorem (Woodin)

Assume there is a Vopěnka cardinal δ and there is an elementary embedding $j : V_\delta \to V_\delta$ with $V_\kappa \prec V_\delta$, where $\kappa = \mathrm{crit}(j)$. Then there is a revised countable support iteration $\mathbb{P}$ of semiproper forcings such that if $g \subseteq \mathbb{P}$ is V-generic, then

$$V[g]_\delta \models \mathrm{ZFC} + \mathrm{MM}^{++}.$$

Moreover, in $V[g]_\delta$ there is no proper inner model of ZFC with the $\omega_2^{V[g]}$-cover and $\omega_2^{V[g]}$-approximation properties in $V[g]_\delta$.

- Vacuous if the HOD conjecture is true
- Interpret forcing axioms in a forcing extension of a model of AD instead

Nairian models

An obstacle/opportunity

Θ

$\sup\{\alpha :$ there is a surjection $f : \mathbb{R} \to \alpha\}$

Γ^∞

The collection of universally Baire sets of reals

Theorem (Woodin)

1. $(*)^{++}$ *implies* $\Theta^{L(\Gamma^\infty,\mathbb{R})} = \omega_3$
2. *Suppose* δ *is a supercompact cardinal. If there are class many Woodin cardinals and* $V[g]$ *is a* δ*-cc forcing extension in which* $\delta = \omega_2$*, then* $V[g] \models \Theta^{L(\Gamma^\infty,\mathbb{R})} < \omega_3$.

Forcing over models of determinacy

Theorem (Woodin)

Assume $\mathrm{AD}_{\mathbb{R}}$ + *"Θ is regular." Then* $\mathbb{P}_{\max} * Add(\omega_3, 1) \Vdash \mathrm{MM}^{++}(c)$.

Theorem (Caicedo-Larson-Sargsyan-Schindler-Steel-Zeman)

Assume $\mathrm{AD}_{\mathbb{R}}$ + *"Θ is regular". Suppose the set of* κ *which are regular in HOD and have cofinality* ω_1 *is stationary in Θ. Then*

$$\mathbb{P}_{\max} * Add(\omega_3, 1) \Vdash \mathrm{MM}^{++}(c) + \neg\square_{\omega_1} + \neg\square_{\omega_2}.$$

- The ground models of these theorems satisfy $V = L(\wp(\mathbb{R}))$
- They satisfy $\square_\Theta$, but $\mathbb{P}_{\max}$ collapses Θ to ω_3

Forcing MM over a model of determinacy

Theorem (Larson-Sargsyan)

Assume $\mathrm{AD}_{\mathbb{R}} + \exists\lambda \bowtie_\lambda$. *Then*

$$\mathbb{P}_{\max} * Add(\omega_3, 1) * Add(\omega_4, 1) \Vdash \neg\Box_{\omega_2} + \neg\Box_{\omega_3}.$$

- Ground model satisfies $V = L(\wp(\wp(\mathbb{R})))$

> *. . . note that important parts of the theory [of forcing axioms] are not really about just structures from $L(\wp(\omega_1))$ but about structures that could not be captured by inner models of the form $L(X)$ for X a set. So it is quite unclear that the theory could be captured by a homogeneous forcing extension of an inner model of determinacy.*
>
> *–Todorčević*

Nairian models

Definition

Assume LSA and suppose

$$\mathrm{HOD} \models \text{“}\Theta \text{ is a limit of Woodin cardinals.”}$$

Let $M = V_\Theta^{\mathrm{HOD}}$ and $N = L_\Theta(\bigcup_{\eta<\Theta}(M|\eta)^\omega)$.

- (Woodin, building on work by Steel) $N \models \mathrm{ZF}$.
- Think of a *Nairian model* is a proper initial segment N_γ of N such that $N_\gamma \models \mathrm{ZF}$.
- Not of the form $L(X)$ for any set X
- Exist assuming less than a Woodin limit of Woodin cardinals

Failure of square everywhere

Theorem (B.-Larson-Sargsyan)

Let N_γ be the least initial segment of N such that $N_\gamma \models \mathrm{ZF}$. Then in a forcing extension $N_\gamma[g]$ of N_γ, $\neg\Box_\kappa$ holds for all $\kappa > \omega$.

Corollary

$\mathrm{ZFC} + \forall\kappa > \omega\ \neg\Box_\kappa <_{\mathrm{Con}}$
$\mathrm{ZFC} +$ *"there is a Woodin limit of Woodin cardinals."*

Failure of square everywhere

Theorem (Neeman-Steel)

If δ is a Woodin cardinal such that $\neg\Box_\delta + \neg\Box(\delta)$ and the Strategic Branch Hypothesis holds at δ, then there is an inner model with a subcompact cardinal.

- No Woodin cardinals in $N_\gamma[g]$

The HOD Hypothesis

Theorem (Ben Neria-Hayut)

It is consistent relative to mild large cardinals that every successor of a regular cardinal is measurable in HOD.

Theorem (B.-Larson-Sargsyan)

In $N_\gamma[g]$, successors of singular cardinals are measurable in HOD.

- The HOD Hypothesis is not provable in ZFC.

Stationary Set Reflection

Definition (SRP^*; Woodin)

Suppose $\lambda > \omega_1$. There is a normal fine ideal $I \subset \wp(\wp_{\omega_1}(\lambda))$ such that

1. for every stationary $T \subset \omega_1$, the set $\{\sigma \in \wp_{\omega_1}(\lambda) | \sigma \cap \omega_1 \in T\} \notin I$
2. if $S \subset \wp_{\omega_1}(\lambda)$ is such that for each stationary $T \subset \omega_1$, $\{X \in S : X \cap \omega_1 \in T\} \notin I$, there is $\omega_1 < \gamma < \lambda$ such that $S \cap \wp_{\omega_1}(\gamma)$ contains a club in $\wp_{\omega_1}(\gamma)$.

Theorem (Steel-Zoble)

Suppose NS_{ω_1} is saturated, $2^\omega \leq \omega_2$, and $\mathrm{SRP}^*(\omega_2)$. Then $L(\mathbb{R}) \models \mathrm{AD}$.

Theorem (B.-Sargsyan)

SRP^* holds in $N_\gamma[g]$.

Stationary Set Reflection

Theorem (Woodin)

Assume $\mathrm{AD}_{\mathbb{R}}$. Then the quasi-club filter on $\wp_{\omega_1}(\lambda)$ is an ultrafilter for $\lambda < \Theta$.

Quasi-Club Conjecture

In Nairian models, the quasi-club filter on $\wp_{\omega_1}(\lambda)$ is an ultrafilter for all $\lambda \in \mathrm{Ord}$.

Theorem (B.-Sargsyan)

Assuming the Quasi-Club conjecture, $N_\gamma[g] \models \mathrm{SRP}$.

Corollary

The (HOD+SRP) conjecture is false, assuming the Quasi-Club conjecture.

Stationary Set Reflection

Theorem (B.-Sargsyan)

1. SRP^* *is consistent with* $(*)^{++}$.
2. *If the Quasi-Club conjecture is true, then* SRP *is consistent with* $(*)^{++}$.

Further consequences of MM

Theorem (B.-Sargsyan)

The following hold in $N_\gamma[g]$.

- PFA *for (ω, ∞)-distributive forcings.*
- *P-Ideal Dichotomy.*

Theorem (B.-Sargsyan)

Assume the Quasi-Club conjecture. The following hold in $N_\gamma[g]$.

- *The Mapping Reflection Principle,*
- MA^+*(σ-closed), and*
- MM *for (ω, ∞)-distributive forcings.*

Steel's program

- Close to achieving one of the program's test questions (or showing that MM is incompatible with $(*)^{++}$)
  - Already have the inner model theory
- Nairian models are important pieces of our framework theory for providing the intertranslations Steel's program calls for
  - Understanding the logical relationships between determinacy and forcing axioms
- Ideally work on Nairian models will connect with the forcing axioms, tree property, etc. research programs in ways Steel envisages

Questions

Question

Can a Woodin cardinal exist in a forcing extension of a Nairian model? A supercompact cardinal?

Revised HOD+SRP Conjecture (Woodin)

Assume SRP. Then provably either

1. $\mathrm{Ord}^\omega \subset \mathrm{HOD}$, or
2. There is no supercompact cardinal in HOD.

Question

Why are there "consistency strength gaps" when forcing over models of AD versus over ZFC models with large cardinals?

Revised PFA conjectures

Conjecture

Assume $\mathrm{ZFC} + \forall\kappa > \omega\ \neg\Box_\kappa$. Then there is an inner model of $\mathrm{AD}_\mathbb{R}$ + "ω_1 is supercompact."

Theorem (Neeman, Trang)

PFA + "there is a Woodin cardinal" implies there is an inner model with a Woodin limit of Woodin cardinals.

Conjecture

PFA + "there is a Woodin cardinal" implies there is an inner model of ZFC with a supercompact cardinal.

Thank you

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Failure of square everywhere is weaker than a Woodin limit of Woodin cardinals.
Coming Christmas 2025.

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