# MA372 Submission Problems

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February 7, 2014

1. This problem is a continuation of the Submission Problem from the January 10 Study Guide.

   (a) Several customers have complained that the ultra-condensed broth is too sweet, so the company has altered the recipe to use fewer carrots, and more onions. Specifically, the recipe for ultra-condensed broth now calls for 800 grams of each of the three vegetables. With this new recipe, is the optimum basis for the problem still optimum? If so, determine the new optimum solution and objective value.

   (b) The company has neglected to account for the need to package the broth in its original description of the problem. An automated canning machine is used to package both the three types of vegetable broth, as well as other products produced by the company. The company has allocated enough time on this machine to package 3 litres of broth per hour. Let $x_7$ denote the slack variable for this new constraint. Determine whether adding $x_7$ to the optimum basis for the original problem results in an optimum basis, and if so, determine the new optimum solution and objective value.

2. Solve the following linear program using the Revised Simplex Algorithm, showing all steps:

$$\text{Maximize } z = 6x_1 + 10x_2 + 34x_3$$

   subject to

$$2x_1 + 5x_2 + 8x_3 \leq 18$$
$$2x_1 + 4x_2 + 10x_3 \leq 16$$
$$x_1 + 3x_2 + 7x_3 \leq 10$$
$$x_i \geq 0 \text{ for } 1 \leq i \leq 3$$

3. (a) Let (P) be a linear program in standard form, with objective function $z$, and let (D) be its dual, with objective function $w$. Prove that if $\mathbf{x}$ is any feasible solution to (P), and $\mathbf{y}$ is any feasible solution to (D), then $z(\mathbf{x}) \leq w(\mathbf{y})$. (This is called the Weak LP Duality Property.)

   (b) Consider the following linear program:

$$\text{Maximize } z = 2x_1 + x_2$$

   subject to

$$-x_1 - 2x_2 \leq 2$$
$$-x_1 + 3x_2 \leq 6$$
$$x_1 - 4x_2 \leq 4$$
$$x_1, x_2 \geq 0$$

   Prove that there are no feasible solutions to the dual of this problem.

1. The starting tableau for the original problem is

| Basis | $z$ | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $x_6$ | RHS |
|---|---|---|---|---|---|---|---|---|
| | 1 | $-6$ | $-10$ | $-34$ | 0 | 0 | 0 | 0 |
| $x_4$ | 0 | 2 | 5 | 8 | 1 | 0 | 0 | 18 |
| $x_5$ | 0 | 2 | 4 | 10 | 0 | 1 | 0 | 16 |
| $x_6$ | 0 | 1 | 3 | 7 | 0 | 0 | 1 | 10 |

After the Simplex algorithm is performed, we reach the final tableau

| Basis | $z$ | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $x_6$ | RHS |
|---|---|---|---|---|---|---|---|---|
| | 1 | 0 | 4 | 0 | 0 | 2 | 2 | 52 |
| $x_4$ | 0 | 0 | 2 | 0 | 1 | $-3/2$ | 1 | 4 |
| $x_1$ | 0 | 1 | $-1/2$ | 0 | 0 | $7/4$ | $-5/2$ | 3 |
| $x_3$ | 0 | 0 | $1/2$ | 1 | 0 | $-1/4$ | $1/2$ | 1 |

With fundamental insight matrices

$$A = \begin{bmatrix} 2 & 5 & 8 \\ 2 & 4 & 10 \\ 1 & 3 & 7 \end{bmatrix}, M = \begin{bmatrix} 1 & -3/2 & 1 \\ 0 & 7/4 & -5/2 \\ 0 & -1/4 & 1/2 \end{bmatrix}$$

$$c = \begin{bmatrix} 6 & 10 & 34 \end{bmatrix}, c_B = \begin{bmatrix} 0 & 6 & 34 \end{bmatrix}, y = \begin{bmatrix} 0 & 2 & 2 \end{bmatrix}$$

(a) The change in the recipe for ultra-condensed broth changes the constraint matrix to

$$A' = \begin{bmatrix} 2 & 5 & 8 \\ 2 & 4 & 8 \\ 1 & 3 & 8 \end{bmatrix}$$

Still using the original $M$ matrix,

$$MA' = \begin{bmatrix} 0 & 2 & 4 \\ 1 & -1/2 & -6 \\ 0 & 1/2 & 2 \end{bmatrix}, c_B MA' - c = \begin{bmatrix} 0 & 4 & -2 \end{bmatrix}$$

Giving us the tableau

| Basis | $z$ | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $x_6$ | RHS |
|---|---|---|---|---|---|---|---|---|
| | 1 | 0 | 4 | $-2$ | 0 | 2 | 2 | 52 |
| $x_4$ | 0 | 0 | 2 | 4 | 1 | $-3/2$ | 1 | 4 |
| $x_1$ | 0 | 1 | $-1/2$ | $-6$ | 0 | $7/4$ | $-5/2$ | 3 |
| $x_3$ | 0 | 0 | $1/2$ | 2 | 0 | $-1/4$ | $1/2$ | 1 |

Pivoting at row 3, column 3 gives us

| Basis | $z$ | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $x_6$ | RHS |
|---|---|---|---|---|---|---|---|---|
| | 1 | 0 | $9/2$ | 0 | 0 | $7/4$ | $5/2$ | 53 |
| $x_4$ | 0 | 0 | 1 | 0 | 1 | $-1$ | 0 | 2 |
| $x_1$ | 0 | 1 | 1 | 0 | 0 | 1 | $-1$ | 6 |
| $x_3$ | 0 | 0 | $1/4$ | 1 | 0 | $-1/8$ | $1/4$ | $1/2$ |

The RHS column and objective coefficients are still nonnegative. Thus the optimum basis is still optimum. The new optimum solution is $x_1 = 6$, $x_2 = 0$, $x_3 = 1/2$ with objective value 53.

(b) The new constraint for canning is

$$x_1 + x_2 + x_3 \leq 3$$

Converted to equality with slack variable $x_7$, this constraint becomes

$$x_1 + x_2 + x_3 + x_7 = 3$$

The tableau becomes

| Basis | $z$ | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $x_6$ | $x_7$ | RHS |
|---|---|---|---|---|---|---|---|---|---|
| | 1 | 0 | 4 | 0 | 0 | 2 | 2 | 0 | 52 |
| $x_4$ | 0 | 0 | 2 | 0 | 1 | $-3/2$ | 1 | 0 | 4 |
| $x_1$ | 0 | 1 | $-1/2$ | 0 | 0 | $7/4$ | $-5/2$ | 0 | 3 |
| $x_3$ | 0 | 0 | $1/2$ | 1 | 0 | $-1/4$ | $1/2$ | 0 | 1 |
| $x_7$ | 0 | 1 | 1 | 1 | 0 | 0 | 0 | 1 | 3 |

Pivoting on columns 1 and 3 yields

| Basis | $z$ | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $x_6$ | $x_7$ | RHS |
|---|---|---|---|---|---|---|---|---|---|
| | 1 | 0 | 4 | 0 | 0 | 2 | 2 | 0 | 52 |
| $x_4$ | 0 | 0 | 2 | 0 | 1 | $-3/2$ | 1 | 0 | 4 |
| $x_1$ | 0 | 1 | $-1/2$ | 0 | 0 | $7/4$ | $-5/2$ | 0 | 3 |
| $x_3$ | 0 | 0 | $1/2$ | 1 | 0 | $-1/4$ | $1/2$ | 0 | 1 |
| $x_7$ | 0 | 0 | 1 | 0 | 0 | $-3/2$ | 2 | 1 | $-1$ |

The basic solution is no longer feasible as $x_7$ is negative. This means that this basis is not optimum.

2. **LP Matrices:**

$$A = \begin{bmatrix} 2 & 5 & 8 \\ 2 & 4 & 10 \\ 1 & 3 & 7 \end{bmatrix},\ b = \begin{bmatrix} 18 \\ 16 \\ 10 \end{bmatrix},\ c = \begin{bmatrix} 6 & 10 & 34 \end{bmatrix}$$

**Initially:**

$$M = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},\ c_B = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix},\ \text{Basis: } \{x_4, x_5, x_6\}$$

**Iteration 1:**

$$c_B MA - c = \begin{bmatrix} -6 & -10 & -34 \end{bmatrix},\ c_B M = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}$$

Entering variable: $x_1$

$$MA_1 = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix},\ Mb = \begin{bmatrix} 18 \\ 16 \\ 10 \end{bmatrix},$$

Leaving variable: $x_5$

$$\left[\begin{array}{c|ccc} 2 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{c|ccc} 0 & 1 & -1 & 0 \\ 1 & 0 & 1/2 & 0 \\ 0 & 0 & -1/2 & 1 \end{array}\right]$$

Result:

$$M = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1/2 & 0 \\ 0 & -1/2 & 1 \end{bmatrix},\ c_B = \begin{bmatrix} 0 & 6 & 0 \end{bmatrix},\ \text{Basis: } \{x_4, x_1, x_6\}$$

**Iteration 2:**

$$c_B MA - c = \begin{bmatrix} 0 & 2 & -4 \end{bmatrix},\ c_B M = \begin{bmatrix} 0 & 3 & 0 \end{bmatrix}$$

Entering variable: $x_3$

$$MA_3 = \begin{bmatrix} -2 \\ 5 \\ 2 \end{bmatrix},\ Mb = \begin{bmatrix} 2 \\ 8 \\ 2 \end{bmatrix},$$

Leaving variable: $x_6$

$$\left[\begin{array}{c|ccc} -2 & 1 & -1 & 0 \\ 5 & 0 & 1/2 & 0 \\ 2 & 0 & -1/2 & 1 \end{array}\right] \to \left[\begin{array}{c|ccc} 0 & 1 & -3/2 & 1 \\ 0 & 0 & 7/4 & -5/2 \\ 1 & 0 & -1/4 & 1/2 \end{array}\right]$$

Result:

$$M = \begin{bmatrix} 1 & -3/2 & 1 \\ 0 & 7/4 & -5/2 \\ 0 & -1/4 & 1/2 \end{bmatrix},\ c_B = \begin{bmatrix} 0 & 6 & 34 \end{bmatrix},\ \text{Basis: } \{x_4, x_1, x_3\}$$

**Iteration 3:**

$$c_B MA - c = \begin{bmatrix} 0 & 4 & 0 \end{bmatrix},\ c_B M = \begin{bmatrix} 0 & 2 & 2 \end{bmatrix}$$

Algorithm terminated.

**Optimum solution:**

$$Mb = \begin{bmatrix} 4 \\ 3 \\ 1 \end{bmatrix},\ x_1 = 3, x_2 = 0, x_3 = 1$$

**Objective value:**

$$c_B Mb = 52$$

3. (a) Suppose that $\mathbf{x}$ is a feasible solution for (P). This implies $A\mathbf{x} \leq b$ and $\mathbf{x} \geq 0$.

Suppose also that $\mathbf{y}$ is a feasible solution for (D). This implies $\mathbf{y}A \geq c$ and $\mathbf{y} \geq 0$.

Therefore

$$w(\mathbf{y}) = \mathbf{y}b \geq \mathbf{y}(A\mathbf{x}) = (\mathbf{y}A)\mathbf{x} \geq c\mathbf{x} = z(\mathbf{x})$$

Thus a feasible solution for (D) can be used to obtain an upper bound on the objective value for (P).

(b) The dual problem to this LP is

$$\text{Minimize } w = -2y_1 + 6y_2 + 4y_3$$

subject to

$$-y_1 - y_2 + y_3 \geq 2$$
$$-2y_1 + 3y_2 - 4y_3 \geq 1$$
$$y_1, y_2, y_3 \geq 0$$

If the first two inequalities hold, then we can add 3 times the first to the second to get

$$-5y_1 - y_3 \geq 7$$

but this is impossible since $y_1$ and $y_3$ are both nonnegative.

Thus the structural constraints imply that the nonnegativity constraints do not hold.

Thus there are no feasible solutions to the dual problem.