MA470 Assignment 4

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The final page of this submission consists of an appendix with derivations of general reusable results that are used throughout my solutions.

Exercise 2.7 (a) At time T, the payoff of the European contract is

$$\Lambda(S(T)) = \begin{cases} S(T) - K_1 & : S(T) \leq K_1, \\ 0 & : K_1 < S(T) < K_2, \\ S(T) - K_2 & : S(T) \geq K_2. \end{cases}$$

In the first and third scenarios, the holder receives a stock worth $S(T)$ but must give up the respective fixed strike amount. In the second scenario, the holder is forced to buy a stock for $S(T)$ but can immediately sell it back to the market for that amount since the model assumes that a seller can always find a buyer.

Clearly this contract can be replicated by a portfolio consisting of a short European put with strike K_1 and a long European call with strike K_2.

The value of the contract at time t with spot S is thus

$$V(t,S) = C(t,S,K_2;r,q) - P(t,S,K_1;r,q)$$

$$= e^{-q\tau}S \cdot (N(d_+^*(S/K_2,\tau)) + N(-d_+^*(S/K_1,\tau)))$$
$$-e^{-r\tau} \cdot (K_2 \cdot N(d_-^*(S/K_2,\tau) + K_1 \cdot N(-d_-^*(S/K_1,\tau)))$$

(b) In general, to have a present value of 0, K_1 and K_2 must satisfy the relation

$$Se^{(r-q)\tau} = \frac{K_2 \cdot N(d_-^*(S/K_2,\tau) + K_1 \cdot N(-d_-^*(S/K_1,\tau))}{N(d_+^*(S/K_2,\tau)) + N(-d_+^*(S/K_1,\tau))}$$

In particular, if $K_1 = K_2$, then we have an ordinary forward contract with fair strike $Se^{(r-q)\tau}$.

Exercise 2.21 (a)

$$V^{cp}(t,S) = e^{-r(T_1-t)}\widetilde{E}_{t,S}\left[(P_{T_1}-K_1)^+\right]$$

where P_{T_1} is a strictly decreasing continuous function of random variable $S(T_1)$.

$$\lim_{S(T_1)\to 0} S(T_2) = \lim_{S(T_1)\to 0} S(T_1)e^{\mu(T_2-T_1)+\sigma(W(T_2)-W(T_1))} = 0$$

$$\therefore \lim_{S(T_1)\to 0} P_{T_1}(S(T_1)) = K_2 e^{-r(T_2-T_1)}$$

Similarly,

$$\lim_{S(T_1)\to\infty} S(T_2) = \infty$$

$$\therefore \lim_{S(T_1)\to\infty} P_{T_1}(S(T_1)) = 0$$

Thus $P_{T_1}(S(T_1)) \in (0, K_2e^{-r(T_2-T_1)})$ so the call on a put is worthless if $K_1 \geq K_2 e^{-r(T_2-T_1)}$.

Assuming that $K_1 \in (0, K_2e^{-r(T_2-T_1)})$, there exists a unique value S_1^* such that

$$P_{T_1}(S_1^*) = K_1$$

Thus,

$$V^{cp}(t,S) = e^{-r(T_1-t)}\widetilde{E}_{t,S}\left[(P_{T_1}-K_1)\mathbb{1}_{\{S(T_1)<S_1^*\}}\right]$$

$$= e^{-r(T_1-t)}(\widetilde{E}_{t,S}\left[P_{T_1}\mathbb{1}_{\{S(T_1)<S_1^*\}}\right] - K_1\cdot\widetilde{P}_{t,S}(S(T_1)<S_1^*))$$

Applying the Tower Property,

$$= e^{-r(T_1-t)}(\widetilde{E}_{t,S}\left[e^{-r(T_2-T_1)}\widetilde{E}_{T_1,S(T_1)}\left[(K_2-S(T_2))\mathbb{1}_{\{S(T_2)<K_2\}}\mathbb{1}_{\{S(T_1)<S_1^*\}}\right]\right] - K_1\cdot\widetilde{P}_{t,S}(S(T_1)<S_1^*))$$

$$= \mathbf{e^{-r(T_2-t)}\widetilde{E}_{t,S}\left[(K_2-S(T_2))\mathbb{1}_{\{S(T_1)<S_1^*,S(T_2)<K_2\}}\right] - e^{-r(T_1-t)}K_1\cdot\widetilde{P}_{t,S}(S(T_1)<S_1^*)}$$

By **Result 6**,

$$e^{-r(T_2-t)}K_2\widetilde{E}_{t,S}\left[\mathbb{1}_{\{S(T_1)<S_1^*,S(T_2)<K_2\}}\right] = e^{-r(T_2-t)}K_2\widetilde{P}_{t,S}(S(T_1)<S_1^*, S(T_2)<K_2)$$

$$= e^{-r(T_2-t)}K_2\cdot N_2(-d_-^*(S/S_1^*, T_1-t), -d_-^*(S/K_2, T_2-t); \sqrt{\frac{T_1-t}{T_2-t}})$$

By **Result 8**,

$$e^{-r(T_2-t)}\widetilde{E}_{t,S}\left[S(T_2)\mathbb{1}_{\{S(T_1)<S_1^*,S(T_2)<K_2\}}\right]$$

$$= e^{-r(T_2-t)}Se^{(r-q)(T_2-t)}N_2(-d_+^*(S/S_1^*, T_1-t), -d_+^*(S/K_2, T_2-t), \sqrt{\frac{T_1-t}{T_2-t}})$$

$$= Se^{-q(T_2-t)}N_2(-d_+^*(S/S_1^*, T_1-t), -d_+^*(S/K_2, T_2-t), \sqrt{\frac{T_1-t}{T_2-t}})$$

By **Result 1**,

$$e^{-r(T_1-t)}K_1\cdot\widetilde{P}_{t,S}(S(T_1)<S_1^*) = e^{-r(T_1-t)}K_1\cdot(1-N(d_-(S/S_1^*, T_1-t)))$$

$$= e^{-r(T_1-t)}K_1\cdot N(-d_-(S/S_1^*, T_1-t))$$

Define

$$a_{\pm} := d^*_{\pm}(S/S^*_1, T_1 - t)$$
$$b_{\pm} := d^*_{\pm}(S/K_2, T_2 - t)$$

Then

$$V^{cp}(t,S) = e^{-r(T_2-t)}K_2 \cdot N_2(-a_-, -b_-; \sqrt{\frac{T_1-t}{T_2-t}})$$
$$-Se^{-q(T_2-t)}N_2(-a_+, -b_+, \sqrt{\frac{T_1-t}{T_2-t}})$$
$$-e^{-r(T_1-t)}K_1 \cdot N(-a_-)$$

(assuming that $K_1 \in (0, K_2 e^{-r(T_2-T_1)})$ and is 0 otherwise.)

(b) For $t < T_1$,

$$\Delta(t,S) = \frac{\partial V^{cp}}{\partial S}$$
$$= -N_2(-a_+, -b_+, \sqrt{\frac{T_1-t}{T_2-t}})$$
$$+(e^{-r(T_2-t)}K_2 \frac{\partial}{\partial S} N_2(-a_-, -b_-; \sqrt{\frac{T_1-t}{T_2-t}}) - Se^{-q(T_2-t)}\frac{\partial}{\partial S} N_2(-a_+, -b_+, \sqrt{\frac{T_1-t}{T_2-t}}))$$
$$-\frac{\partial}{\partial S} e^{-r(T_1-t)}K_1 \cdot N(-a_-)$$

Applying **Result 10**, we have

$$\Delta(t,S) = e^{-r(T_1-t)}K_1 \cdot \frac{n(-a_-)}{S\sigma\sqrt{T_1-t}} - N_2(-a_+, -b_+, \sqrt{\frac{T_1-t}{T_2-t}})$$
$$+(e^{-r(T_2-t)}K_2 \frac{\partial}{\partial S} N_2(-a_-, -b_-; \sqrt{\frac{T_1-t}{T_2-t}}) - Se^{-q(T_2-t)}\frac{\partial}{\partial S} N_2(-a_+, -b_+, \sqrt{\frac{T_1-t}{T_2-t}}))$$

(assuming that $K_1 \in (0, K_2 e^{-r(T_2-T_1)})$ and is 0 otherwise.)

Exercise 4.2 The righthandside is equal to

$$C(S;K,r,q) = \begin{cases} \frac{K}{\lambda_+-1}(\frac{\lambda_+-1}{\lambda_+})^{\lambda_+}(\frac{S}{K})^{\lambda_+} = \frac{S^*}{\lambda_+}(\frac{S}{S^*})^{\lambda_+} & : 0 < S \leq S^*, \\ S - K & : S^* < S. \end{cases}$$

(The location of equality can be changed because the price function is continuous at the point $S = S^*$.)

where

$$S^* = \frac{K\lambda_+}{\lambda_+ - 1}$$

and

$$\lambda_+ = \frac{-(r-q-\sigma^2/2) + \sqrt{(r-q-\sigma^2/2)^2 + 2\sigma^2 r}}{\sigma^2}$$

whereas the lefthandside is equal to

$$P(K;S,q,r) = \begin{cases} \frac{S}{1-\lambda'_-}(\frac{\lambda'_- -1}{\lambda'_-})^{\lambda'_-}(\frac{K}{S})^{\lambda'_-} = -\frac{K^*}{\lambda'_-}(\frac{K}{K^*})^{\lambda'_-} & : K^* \leq K, \\ S - K & : 0 < K < K^* \end{cases}$$

where

$$K^* = \frac{S\lambda'_-}{\lambda'_- - 1}$$

and

$$\lambda'_- = \frac{-(q-r-\sigma^2/2) - \sqrt{(q-r-\sigma^2/2)^2 + 2\sigma^2 q}}{\sigma^2}$$

Lemma 1: $\lambda_+ + \lambda'_- = 1$

Proof:

$$\lambda_+ + \lambda'_- = 1 + \frac{\sqrt{(r-q-\sigma^2/2)^2 + 2\sigma^2 r} - \sqrt{(q-r-\sigma^2/2)^2 + 2\sigma^2 q}}{\sigma^2}$$

Note that

$$\begin{aligned} &((r-q-\sigma^2/2)^2 + 2\sigma^2 r) - ((q-r-\sigma^2/2)^2 + 2\sigma^2 q) \\ &= (r-q)^2 - (r-q)\sigma^2 + \sigma^4/4 + 2\sigma^2 r - (q-r)^2 + (q-r)\sigma^2 - \sigma^4/4 - 2\sigma^2 q \\ &= -(r-q)\sigma^2 + 2\sigma^2 r + (q-r)\sigma^2 - 2\sigma^2 q = 0 \end{aligned}$$

Thus

$$\lambda_+ + \lambda'_- = 1 + 0 = 1$$

Lemma 2: $K < K^* \iff S > S^*$

Proof:

$$K < K^* \iff K < \frac{S\lambda'_-}{\lambda'_- - 1} \iff \frac{S}{K} > \frac{\lambda'_- - 1}{\lambda'_-}$$

(Applying **Lemma 1**.)

$$\iff \frac{S}{K} > \frac{-\lambda_+}{1-\lambda_+} \iff S > \frac{K\lambda_+}{\lambda_+ - 1} \iff S > S^*$$

(Note that it is assumed that $S > 0$ and $K > 0$.)

(Lemma 2 also implies that $K \geq K^* \iff S \leq S^*$)

After applying **Lemma 2**, the only thing left to show is that

$$\frac{S^*}{\lambda_+}(\frac{S}{S^*})^{\lambda_+} = -\frac{K^*}{\lambda'_-}(\frac{K}{K^*})^{\lambda'_-}$$

This follows since

$$\frac{S^*}{\lambda_+}(\frac{S}{S^*})^{\lambda_+} = \frac{K}{\lambda_+ - 1}(\frac{S(\lambda_+ - 1)}{K\lambda_+})^{\lambda_+} = \frac{K}{-\lambda'_-}(\frac{S(-\lambda'_-)}{K(1-\lambda'_-)})^{1-\lambda'_-}$$

(by **Lemma 1**.)

$$= -\frac{K}{\lambda'_-}(\frac{S(\lambda'_-)}{K(\lambda'_- - 1)})^{1-\lambda'_-} = -\frac{K}{\lambda'_-}(\frac{K^*}{K})^{1-\lambda'_-}$$

$$= -\frac{K}{\lambda'_-}(\frac{K^*}{K})(\frac{K}{K^*})^{\lambda'_-} = -\frac{K^*}{\lambda'_-}(\frac{K}{K^*})^{\lambda'_-}$$

Thus we can conclude that

$$P(K; S, q, r) = C(S; K, r, q)$$

Exercise 4.6 We have the payoff function

$$\Lambda(S) = (K_1 - S)^+ + (S - K_2)^+$$

Moreover we know that the value of the option satisfies the Black-Scholes PDE

$$\mathcal{L}V + \frac{\partial V}{\partial t} = 0 \text{ (assuming that (t, S) is in the continuation domain,)}$$

but $V = V(S)$ does not depend on t so $\frac{\partial V}{\partial t} = 0$ and we have the ODE

$$\frac{1}{2}\sigma^2 S^2 \frac{d^2V}{dS^2} + (r-q)S\frac{dV}{dS} - rV = 0$$

There are two scenarios in which the option should be exercised right away:

i) S is such that $K_1 - S$ is sufficiently large, or

ii) S is such that $S - K_2$ is sufficiently large.

Thus there exist S_1^* and S_2^* such that the stopping domain is

$$\mathcal{D} = \{S \in (0,\infty) : S \leq S_1^* \text{ or } S \geq S_2^*\}$$

We thus have the ODE

$$\frac{1}{2}\sigma^2 S^2 \frac{d^2V}{dS^2} + (r-q)S\frac{dV}{dS} - rV = 0 \text{ for all } S \in \mathcal{D}^c$$

with boundary conditions

$$V(S_1^*) = \Lambda(S_1^*) = K_1 - S_1^*, \text{ and}$$

$$V(S_2^*) = \Lambda(S_2^*) = S_2^* - K_2.$$

The general solution to this ODE is

$$V(S) = a_+ S^{\lambda_+} + a_- S^{\lambda_-}$$

where

$$\lambda_\pm = \frac{-(r-q-\sigma^2/2) + \sqrt{(r-q-\sigma^2/2) + 2\sigma^2 r}}{\sigma^2}.$$

We can find a_+ and a_- by solving the linear system:

$$V(S_1^*) = a_+ S_1^{*\lambda_+} + a_- S_1^{*\lambda_-} = K_1 - S_1^*$$

$$V(S_2^*) = a_+ S_2^{*\lambda_+} + a_- S_2^{*\lambda_-} = S_2^* - K_2$$

The above linear system has solution

$$a_+ = \frac{S_2^{*\lambda_-}(K_1 - S_1^*) - S_1^{*\lambda_-}(S_2^* - K_2)}{S_1^{*\lambda_+} S_2^{*\lambda_-} - S_1^{*\lambda_-} S_2^{*\lambda_+}}$$

$$a_- = \frac{S_1^{*\lambda_+}(S_2^* - K_2) - S_2^{*\lambda_+}(K_1 - S_1^*)}{S_1^{*\lambda_+} S_2^{*\lambda_-} - S_1^{*\lambda_-} S_2^{*\lambda_+}}$$

To find S_1^* and S_2^* we can apply the smooth pasting condition.

First we need to find

$$V'(S) = a_+\lambda_+ S^{(\lambda_+ - 1)} + a_-\lambda_- S^{(\lambda_- - 1)}$$

Near $S = S_1^*$, $\Lambda(S) = K_1 - S$ and so

$$\Lambda'(S_1^*) = -1$$

Similarly,

$$\Lambda'(S_2^*) = 1.$$

By the smooth pasting condition, S_1^* and S_2^* satisfy the nonlinear system:

$$V'(S_1^*) = a_+\lambda_+ S_1^{*(\lambda_+ - 1)} + a_-\lambda_- S_1^{*(\lambda_- - 1)} = -1$$

$$V'(S_2^*) = a_+\lambda_+ S_2^{*(\lambda_+ - 1)} + a_-\lambda_- S_2^{*(\lambda_- - 1)} = 1$$

It is unclear if we can proceed analytically at this point.

Appendix

Generalized d function:

$$d_{\pm}^{*}(m,\tau) := \frac{ln(m) + (r-q \pm \sigma^2/2)\tau}{\sigma\sqrt{\tau}}$$

European Call Without Dividend:

$$C(t,S) = S \cdot N(d_{+}(S/K,\tau)) - e^{-r\tau}K \cdot N(d_{-}(S/K,\tau))$$

European Call With Continuous Dividend:

$$C(t,S;r,q) = e^{-q\tau}S \cdot N(d_{+}^{*}(S/K,\tau)) - e^{-r\tau}K \cdot N(d_{-}^{*}(S/K,\tau))$$

European Put-Call Parity

$$C(t,S;r,q) - P(t,S;r,q) = e^{-q\tau}S - e^{-r\tau}K$$

European Put Without Dividend:

$$P(t,S) = C(t,S) + e^{-r\tau}K - S = e^{-r\tau}K \cdot N(-d_{-}(S/K,\tau)) - S \cdot N(-d_{+}(S/K,\tau))$$

European Put With Dividend:

$$P(t,S;r,q) = e^{-r\tau}K \cdot N(-d_{-}^{*}(S/K,\tau)) - e^{-q\tau}S \cdot N(-d_{+}^{*}(S/K,\tau))$$

Result 1:

$$\widetilde{P}_{t,S}(S(t) > K) = \widetilde{P}(\frac{S(T)}{S(t)} > \frac{K}{S}) = \widetilde{P}(\widetilde{Z} > \frac{-ln(S/K) - (r-q-\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}) = N(d_{-}^{*}(S/K,T-t))$$

Result 2:

$$\widetilde{P}(S(t) > K) = \widetilde{P}(\frac{S(t)}{S(0)} > \frac{K}{S(0)}) = N(d_{-}^{*}(S(0)/K,t))$$

Result 3:

$$\widetilde{E}_{t,S}\left[S^{\alpha}(T)\right] = S(t)^{\alpha}e^{\alpha(r-q-\sigma^2/2)\tau}\widetilde{E}\left[e^{\alpha\sigma\sqrt{\tau}\widetilde{Z}}\right] = S(t)^{\alpha}e^{\alpha(r-q-\sigma^2(1-\alpha)/2)\tau}$$

Result 4:

$$E\left[e^{aZ}\mathbb{1}_{\{Z>b\}}\right] = \int_b^{\infty} e^{az}\frac{1}{\sqrt{2\pi}}e^{-z^2/2}dz = e^{a^2/2}\int_b^{\infty}\frac{1}{\sqrt{2\pi}}e^{-(z-a)^2/2}dz = e^{a^2/2}\cdot(1-N(b-a)) = e^{a^2/2}\cdot N(a-b)$$

Result 5:

$$\widetilde{P}_{t,S}(S(T_1) > K_1, S(T_2) > K_2) = N_2(d_{-}^{*}(S/K_1,T_1-t), d_{-}^{*}(S/K_2,T_2-t); \sqrt{\frac{T_1-t}{T_2-t}})$$

(derived in MA470 notes, equation 2.88)

Result 6:

$$\widetilde{P}_{t,S}(S(T_1) < K_1, S(T_2) < K_2) = \widetilde{P}_{t,S}(\widetilde{Z}_1 < -d_{-}(S/K_1,T_1-t), \widetilde{Z}_2 < -d_{-}(S/K_2,T_2-t))$$

$$= N_2(-d_{-}^{*}(S/K_1,T_1-t), -d_{-}^{*}(S/K_2,T_2-t); \sqrt{\frac{T_1-t}{T_2-t}})$$

Result 7:

$$\widetilde{E}_{t,S}\left[S(T_2)\mathbb{1}_{\{S(T_1)>K_1,S(T_2)>K_2\}}\right] = Se^{(r-q)(T_2-t)}N_2(d_+^*(S/K_1,T_1-t),d_+^*(S/K_2,T_2-t),\sqrt{\frac{T_1-t}{T_2-t}})$$

(derived in MA470 notes, equation 2.89)

Result 8:

$$\widetilde{E}_{t,S}\left[S(T_2)\mathbb{1}_{\{S(T_1)<K_1,S(T_2)<K_2\}}\right] = \widetilde{E}_{t,S}\left[S(T_2)(1-\mathbb{1}_{\{S(T_1)>K_1\}})(1-\mathbb{1}_{\{S(T_2)>K_2\}})\right]$$

$$= \widetilde{E}_{t,S}\left[S(T_2)\right] - \widetilde{E}_{t,S}\left[S(T_2)\mathbb{1}_{\{S(T_1)>K_1\}}\right] - \widetilde{E}_{t,S}\left[S(T_2)\mathbb{1}_{\{S(T_2)>K_2\}}\right] + \widetilde{E}_{t,S}\left[S(T_2)\mathbb{1}_{\{S(T_1)>K_1,S(T_2)>K_2\}}\right]$$

$$= Se^{(r-q)(T_2-t)} - \widetilde{E}_{t,S}\left[S(T_1)\mathbb{1}_{\{S(T_1)>K_1\}}\right]\widetilde{E}\left[\frac{S(T_2)}{S(T_1)}\right] - Se^{(r-q)(T_2-t)}N(d_+^*(S/K_2,T_2-t))$$

$$+Se^{(r-q)(T_2-t)}N_2(d_+^*(S/K_1,T_1-t),d_+^*(S/K_2,T_2-t),\sqrt{\frac{T_1-t}{T_2-t}})$$

$$(\widetilde{E}_{t,S}\left[S(T_1)\mathbb{1}_{\{S(T_1)>K_1\}}\right] = Se^{(r-q)(T_1-t)}N(d_+^*(S/K_1,T_1-t)), \widetilde{E}\left[\tfrac{S(T_2)}{S(T_1)}\right] = e^{(r-q)(T_2-T_1)})$$

$$= \mathbf{Se^{(r-q)(T_2-t)}(1-N(d_+^*(S/K_1,T_1-t))-N(d_+^*(S/K_2,T_2-t))}$$

$$\mathbf{+N_2(d_+^*(S/K_1,T_1-t),d_+^*(S/K_2,T_2-t),\sqrt{\frac{T_1-t}{T_2-t}}))}$$

(The idea is to now use

$$P(X\le x,Y\le y) = 1-P(X>x)-P(Y>y)+P(X>x,Y>y)$$

with $-\widetilde{Z}_1$ and $-\widetilde{Z}_2$.)

In short, Result 8 states that

$$\widetilde{E}_{t,S}\left[S(T_2)\mathbb{1}_{\{S(T_1)<K_1,S(T_2)<K_2\}}\right] = Se^{(r-q)(T_2-t)}N_2(-d_+^*(S/K_1,T_1-t),-d_+^*(S/K_2,T_2-t),\sqrt{\frac{T_1-t}{T_2-t}})$$

Result 9:

$$e^{-r\tau}K\frac{\partial}{\partial S}N(-d_-^*(S/K,\tau)) - Se^{-q\tau}\frac{\partial}{\partial S}N(-d_+^*(S/K,\tau)) = 0$$

(this follows from the Delta of a European Put option.)

Result 10:

$$\frac{\partial}{\partial S}N(-d_\pm^*(S/K,\tau)) = \frac{\partial}{\partial S}N(-(\frac{ln(S/K)+(r-q\pm\sigma^2/2)\tau}{\sigma\sqrt{\tau}}))$$

$$= \frac{-n(d_\pm^*(S/K,\tau))}{S\sigma\sqrt{\tau}}$$

(where n is the pdf of a standard Normal distribution.)