MA372 Submission Problems

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1. (a) We can model this problem as a knapsack problem

   Let

   - x_1 be the number of frozen pizzas purchased
   - x_2 be the number of bags of popcorn purchased
   - x_3 be the number of bags of chips purchased

   Then $W = 15$ and

   - $w_1 = 6$
   - $w_2 = 4$
   - $w_3 = 3$

   - $c_1 = 12$
   - $c_2 = 7$
   - $c_3 = 5$

   Let stage i correspond to choosing the value for x_i.

   The following tables showcase the dynamic programming algorithm being carried out:

s_3	$f_3^*(s_3)$	x_3^*
0	0	0
1	0	0
2	0	0
3	5	1
4	5	1
5	5	1
6	10	2
7	10	2
8	10	2
9	15	3
10	15	3
11	15	3
12	20	4
13	20	4
14	20	4
15	25	5

s_2	$x_2 = 0$	$x_2 = 1$	$x_2 = 2$	$x_2 = 3$	$f_2^*(s_2)$	x_2^*
0	0	–	–	–	0	0
1	0	–	–	–	0	0
2	0	–	–	–	0	0
3	5	–	–	–	5	0
4	5	$7+0$	–	–	7	1
5	5	$7+0$	–	–	7	1
6	10	$7+0$	–	–	10	0
7	10	$7+5$	–	–	12	1
8	10	$7+5$	$14+0$	–	14	2
9	15	$7+5$	$14+0$	–	15	0
10	15	$7+10$	$14+0$	–	17	1
11	15	$7+10$	$14+5$	–	19	2
12	20	$7+10$	$14+5$	$21+0$	21	3
13	20	$7+15$	$14+5$	$21+0$	22	1
14	20	$7+15$	$14+10$	$21+0$	24	2
15	25	$7+15$	$14+10$	$21+5$	26	3

   At stage 1 the state is 15 and we have the row of values

s_1	$x_1 = 0$	$x_1 = 1$	$x_1 = 2$	$f_1^*(s_1)$	x_1^*
15	$0+26$	$12+15$	$24+5$	29	2

The optimum solution is $x_1 = 2, x_2 = 0, x_3 = 1$ with optimal objective value 29.

Or in terms of the problem, buy 2 frozen pizzas and 1 bag of chips to have 29 servings

(b) The purchasing of the (rather expensive) coffee changes W to 11. We need only look at the new row of values formed for stage 1.

s_1	$x_1 = 0$	$x_1 = 1$	$x_1 = 2$	$f_1^*(s_1)$	x_1^*
11	$0 + 19$	$12 + 7$	–	19	$0, 1$

The problem now has two optimum solutions $x_1 = 0, x_2 = 2, x_3 = 1$ and $x_1 = 1, x_2 = 1, x_3 = 0$ which give us optimal objective value 19. However, in the second case there will be $1 leftover. Since this is a real-world problem, you may prefer the option of buying 1 frozen pizza and 1 bag of popcorn so that you have 1 extra dollar that could go towards buying coffee.

2. Let x_i be the number of commercials run in area i ($i \in \{1, 2, 3, 4\}$) and let stage i correspond to choosing the value for x_i.The state corresponds to the number of commercials that are still able to be produced.

   The following tables showcase the dynamic programming algorithm being carried out:

   Stage 4:

s_4	$f_4^*(s_4)$	x_4^*
0	0	0
1	3	1
2	7	2
3	12	3
4	14	4
5	16	5

   Stage 3:

s_3	$x_3 = 0$	$x_3 = 1$	$x_3 = 2$	$x_3 = 3$	$x_3 = 4$	$x_3 = 5$	$f_3^*(s_3)$	x_3^*
0	0	–	–	–	–	–	0	0
1	3	$5+0$	–	–	–	–	5	1
2	7	$5+3$	$9+0$	–	–	–	9	2
3	12	$5+7$	$9+3$	$11+0$	–	–	12	$0, 1, 2$
4	14	$5+12$	$9+7$	$11+3$	$10+0$	–	17	1
5	16	$5+14$	$9+12$	$11+7$	$10+3$	$9+0$	21	2

   Stage 2:

s_2	$x_2 = 0$	$x_2 = 1$	$x_2 = 2$	$x_2 = 3$	$x_2 = 4$	$x_2 = 5$	$f_2^*(s_2)$	x_2^*
0	0	–	–	–	–	–	0	0
1	5	$6+0$	–	–	–	–	6	1
2	9	$6+5$	$8+0$	–	–	–	11	1
3	12	$6+9$	$8+5$	$10+0$	–	–	15	1
4	17	$6+12$	$8+9$	$10+5$	$11+0$	–	18	1
5	21	$6+17$	$8+12$	$10+9$	$11+5$	$12+0$	23	1

   Stage 1:

s_1	$x_1 = 0$	$x_1 = 1$	$x_1 = 2$	$x_1 = 3$	$x_1 = 4$	$x_1 = 5$	$f_1^*(s_1)$	x_1^*
5	$0+23$	$4+18$	$7+15$	$9+11$	$12+6$	$15+0$	23	0

   Conclusion: The optimum strategy that maximizes the number of votes is to run 0 commercials in Area 1, 1 commercial in Area 2, 1 commercial in Area 3, and 3 commercials in Area 4. This strategy is estimated to win 23 (thousand) additional votes.

3. (a) The objective function we wish to maximize with integer values is

$$z = 10x_1 - x_1^2 + 14x_2 + 5x_3$$

In modeling this problem using dynamic programming, let stage i correspond to choosing the value for x_i. From the constraint

$$4x_1 + 8x_2 + 3x_3 \leq 22$$

we see that the total allowed weight is 22 and x_1, x_2, x_3 have weights 4, 8, 3 respectively.

The following tables showcase the dynamic programming algorithm being carried out:

s_3	$f_3^*(s_3)$	x_3^*
0	0	0
1	0	0
2	0	0
3	5	1
4	5	1
5	5	1
6	10	2
7	10	2
8	10	2
9	15	3
10	15	3
11	15	3
12	20	4
13	20	4
14	20	4
15	25	5
16	25	5
17	25	5
18	30	6
19	30	6
20	30	6
21	35	7
22	35	7

s_2	$x_2 = 0$	$x_2 = 1$	$x_2 = 2$	$f_2^*(s_2)$	x_2^*
0	0	–	–	0	0
1	0	–	–	0	0
2	0	–	–	0	0
3	5	–	–	5	0
4	5	–	–	5	0
5	5	–	–	5	0
6	10	–	–	10	0
7	10	–	–	10	0
8	10	$14 + 0$	–	14	1
9	15	$14 + 0$	–	15	0
10	15	$14 + 0$	–	15	0
11	15	$14 + 5$	–	19	1
12	20	$14 + 5$	–	20	0
13	20	$14 + 5$	–	20	0
14	20	$14 + 10$	–	24	1
15	25	$14 + 10$	–	25	0
16	25	$14 + 10$	$28 + 0$	28	2
17	25	$14 + 15$	$28 + 0$	29	1
18	30	$14 + 15$	$28 + 0$	30	0
19	30	$14 + 15$	$28 + 5$	33	2
20	30	$14 + 20$	$28 + 5$	34	1
21	35	$14 + 20$	$28 + 5$	35	0
22	35	$14 + 20$	$28 + 10$	38	2

At stage 1 the state is 22 and we have the row of values

s_1	$x_1 = 0$	$x_1 = 1$	$x_1 = 2$	$x_1 = 3$	$x_1 = 4$	$x_1 = 5$	$f_1^*(s_1)$	x_1^*
22	$0 + 38$	$9 + 30$	$16 + 24$	$21 + 15$	$24 + 10$	$25 + 0$	40	2

The optimum solution is $x_1 = 2, x_2 = 1, x_3 = 2$ with optimal objective value 40.

(b) The change to the constraint merely changes the initial state to 21. The row for stage 1 becomes

s_1	$x_1 = 0$	$x_1 = 1$	$x_1 = 2$	$x_1 = 3$	$x_1 = 4$	$x_1 = 5$	$f_1^*(s_1)$	x_1^*
21	$0 + 35$	$9 + 29$	$16 + 20$	$21 + 15$	$24 + 5$	$25 + 0$	36	$2, 3$

The problem now has two optimum solutions $x_1 = 2, x_2 = 0, x_3 = 4$ and $x_1 = 3, x_2 = 0, x_3 = 3$ which give us optimal objective value 36.