MA372 Submission Problems

Richard Douglas

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1. For an LP of the form maximize $z = cx$ subject to $Ax \leq b$, $x \geq 0$, with n decision variables and m constraints, prove the *Complementary Slackness Theorem*: let x be a feasible solution to the LP, and let y be a feasible solution to its dual. Then x and y are optimum solutions to their respective problems if and only if

$$x_j(\sum_{1 \leq i \leq m} a_{ij}y_i - c_j) = 0$$

for all $1 \leq j \leq m$, and

$$y_i(\sum_{1 \leq j \leq n} a_{ij}x_j - b_i) = 0$$

for all $1 \leq i \leq n$.

2. (a) In part (a) of the Submission Problem in the January 22 Study Guide, you saw that the change in resources resulted in a tableau with an infeasible basic solution:

Basis	z	x_1	x_2	x_3	x_4	x_5	x_6	RHS
	1	0	4	0	0	2	2	72
x_4	0	0	2	0	1	$-3/2$	1	4
x_1	0	1	$-1/2$	0	0	$7/4$	$-5/2$	-5
x_3	0	0	$1/2$	1	0	$-1/4$	$1/2$	3

Use the Dual Simplex Algorithm to return to a tableau with a feasible solution, and find the new optimum solution.

(b) Apply the Dual Simplex Algorithm to solve the following Linear Program:

$$\text{Minimize } 6x_1 + x_2 + 2x_3$$

subject to

$$3x_1 + x_2 + x_3 \geq 2$$

$$x_1 - x_2 + x_3 \geq -1$$

$$x_1 + 2x_2 - x_3 \geq 1$$

$$x_1, x_2, x_3 \geq 0$$

3. (a) Solve the following LP using the "Big M" method.

$$\text{Minimize } 40x_1 + 30x_2 + 10x_3$$

subject to

$$x_1 + x_2 + x_3 = 6$$
$$x_1 + 2x_2 + 4x_3 \leq 12$$
$$x_1, x_2, x_3 \geq 0$$

(b) Use the "Big M" method to demonstrate that the following LP has no feasible solutions:

$$\text{Maximize } z = 5x_1 + 4x_2$$

subject to

$$3x_1 + 2x_2 \leq 6$$
$$2x_1 - x_2 \geq 6$$
$$x_1, x_2 \geq 0$$

Note: In my tableaux I put a -1 in the z entry of the objective row whenever I am maximizing $-z$. I do this so that I remember to multiply the objective value for $-z$ by -1 at the end of the algorithm. I also denote artificial variables as a_1, a_2, a_3, etc.... This is to remind me that we don't want such variables to be in the optimum basis. I will change notation if it would be preferred.

1. **Proof:**
"$\Leftarrow$"
The sums can be rewritten as
$$\sum_{1\le i\le m} (x_j a_{ij} y_i - c_j x_j) = 0$$
for all $1 \le j \le m$, and
$$\sum_{1\le j\le n} (x_j a_{ij} y_i - b_i y_i) = 0$$
for all $1 \le i \le n$.

It follows then that if we sum over all j for the first sum equations and all i for the second sum equations then the righthandside will still be 0.
$$\sum_{1\le j\le n}\sum_{1\le i\le m} (x_j a_{ij} y_i - c_j x_j) = 0$$
$$\sum_{1\le i\le m}\sum_{1\le j\le n} (x_j a_{ij} y_i - b_i y_i) = 0$$
Simplifying gives us
$$\sum_{1\le j\le n}\sum_{1\le i\le m} x_j a_{ij} y_i - \sum_{1\le j\le n} c_j x_j = 0$$
$$\sum_{1\le i\le m}\sum_{1\le j\le n} x_j a_{ij} y_i - \sum_{1\le i\le m} b_i y_i = 0$$
The order of summation in the double sum does not change its value. Thus
$$\sum_{1\le j\le n} c_j x_j = \sum_{1\le i\le m} b_i y_i$$
By the Fundamental Theorem of Optimization, x and y are solutions to their respective problems.

"$\Rightarrow$"
Since x and y are optimal, we can use the fact that
$$\sum_{1\le j\le n} c_j x_j = \sum_{1\le i\le m} b_i y_i$$
It follows that
$$-\sum_{1\le j\le n}\sum_{1\le i\le m} x_j a_{ij} y_i + \sum_{1\le j\le n} c_j x_j + \sum_{1\le i\le m}\sum_{1\le j\le n} x_j a_{ij} y_i - \sum_{1\le i\le m} b_i y_i = 0$$
Factoring gives us
$$\sum_{1\le j\le n} x_j (\sum_{1\le i\le m} c_j - y_i a_{ij}) + \sum_{1\le i\le m} y_i (\sum_{1\le j\le n} a_{ij} x_j - b_i) = 0$$

x and y are feasible in their respective problems. Therefore

$$Ax \leq b \Rightarrow \sum_{1 \leq j \leq n} a_{ij}x_j \leq b_j \text{ for all } j \Rightarrow b_j - \sum_{1 \leq j \leq n} a_{ij}x_j \geq 0 \text{ for all } j \leq 0$$

$$yA \geq c \Rightarrow \sum_{1 \leq i \leq m} y_i a_{ij} \geq c_i \text{ for all } i \Rightarrow \sum_{1 \leq i \leq m} y_i a_{ij} - c_i \geq 0 \text{ for all } i$$

Now in the previous equation

$$\sum_{1 \leq j \leq n} x_j (\sum_{1 \leq i \leq m} c_j - y_i a_{ij}) + \sum_{1 \leq i \leq m} y_i (\sum_{1 \leq j \leq n} a_{ij}x_j - b_i) = 0$$

x_j, y_i, $\sum_{1 \leq i \leq m}(y_i a_{ij} - c_j)$, and $\sum_{1 \leq j \leq n}(a_{ij}x_j - b_i)$ are all nonnegative. Therefore it must be the case that each of the individual summations holds.

That is we must have

$$x_j (\sum_{1 \leq i \leq m} c_j - a_{ij}y_i) = x_j (\sum_{1 \leq i \leq m} a_{ij}y_i - c_j) = 0$$

for all $1 \leq j \leq m$, and

$$y_i (\sum_{1 \leq j \leq n} a_{ij}x_j - b_i) = 0$$

for all $1 \leq i \leq n$.

With "⇐' and "⇒" proven, the result follows.

2. (a) Starting with

Basis	z	x_1	x_2	x_3	x_4	x_5	x_6	RHS
	1	0	4	0	0	2	2	72
x_4	0	0	2	0	1	$-3/2$	1	4
x_1	0	1	$-1/2$	0	0	$7/4$	$-5/2$	-5
x_3	0	0	$1/2$	1	0	$-1/4$	$1/2$	3

For the Dual Simplex Algorithm, we wish to pivot on row 2 due to the negative entry in the b column. The Dual Minimum Ratio Test says to pivot at column 6 since $|\frac{2}{-5/2}| < |\frac{4}{-1/2}|$.
The resulting tableau is

Basis	z	x_1	x_2	x_3	x_4	x_5	x_6	RHS
	1	$4/5$	$18/5$	0	0	$17/5$	0	68
x_4	0	$2/5$	$9/5$	0	1	$-4/5$	0	2
x_6	0	$-2/5$	$1/5$	0	0	$-7/10$	1	2
x_3	0	$1/5$	$2/5$	1	0	$1/10$	0	2

Thus the new optimum solution is $\{x_1 = 0,\ x_2 = 0,\ x_3 = 2\}$ with objective value 68.

(b) The linear program is first converted to the maximization problem

$$\text{Maximize } -6x_1 - x_2 - 2x_3$$

subject to

$$-3x_1 - x_2 - x_3 + x_4 = -2$$
$$-x_1 + x_2 - x_3 + x_5 = 1$$
$$-x_1 - 2x_2 + x_3 + x_6 = -1$$
$$x_1, x_2, x_3, x_4, x_5, x_6 \geq 0$$

where x_4, x_5, x_6 are slack variables.

The starting tableau is

Basis	z	x_1	x_2	x_3	x_4	x_5	x_6	RHS
	-1	6	1	2	0	0	0	0
x_4	0	-3	-1	-1	1	0	0	-2
x_5	0	-1	1	-1	0	1	0	1
x_6	0	-1	-2	1	0	0	1	-1

The basic solution is not feasible because of the negative entry in the final column of row 3. If we pivot on row 3, the Dual Minimum Ratio Test says to only consider columns with a negative entry in that row. Thus column 3 is not considered. $|\frac{1}{-2}| < |\frac{6}{-1}|$ therefore column 2 is selected.
Pivot at row 3, column 2

Basis	z	x_1	x_2	x_3	x_4	x_5	x_6	RHS
	-1	$11/2$	0	$5/2$	0	0	$1/2$	$-1/2$
x_4	0	$-5/2$	0	$-3/2$	1	0	$-1/2$	$-3/2$
x_5	0	$-3/2$	0	$-1/2$	0	1	$1/2$	$1/2$
x_2	0	$1/2$	1	$-1/2$	0	0	$-1/2$	$1/2$

The basic solution still is not feasible because of the negative entry in the final column of row 1. For pivoting on row 1, the Dual Minimum Ratio Test says to choose column 6 since $|\frac{1/2}{-1/2}| < |\frac{5/2}{-3/2}|$ and $|\frac{1/2}{-1/2}| < |\frac{11/2}{-5/2}|$.

Pivot at row 1, column 6

Basis	z	x_1	x_2	x_3	x_4	x_5	x_6	RHS
	-1	3	0	1	1	0	0	-2
x_6	0	5	0	3	-2	0	1	3
x_5	0	-4	0	-2	1	1	0	-1
x_2	0	3	1	1	-1	0	0	2

Applying similar reasoning,

Pivot at row 2, column 3

Basis	z	x_1	x_2	x_3	x_4	x_5	x_6	RHS
	-1	1	0	0	$3/2$	$1/2$	0	$-5/2$
x_6	0	-1	0	0	$-1/2$	$3/2$	1	$3/2$
x_3	0	2	0	1	$-1/2$	$-1/2$	0	$1/2$
x_2	0	1	1	0	$-1/2$	$1/2$	0	$3/2$

All entries in the final column are nonnegative. This tells us that the basic solution is now feasible and thus also optimum (as the objective row entries are also nonnegative.) The Dual Simplex Algorithm terminates.

The optimum solution is $\{x_1 = 0,\ x_2 = 3/2,\ x_3 = 1/2\}$ with objective value $5/2$.

3. (a) The LP can be viewed as the maximization problem:

$$\text{Maximize } -z = -40x_1 - 30x_2 - 10x_3 - 1000a_1$$

subject to

$$x_1 + 2x_2 + 4x_3 + x_4 = 12$$

$$x_1 + x_2 + x_3 + a_1 = 6$$

$$x_1, x_2, x_3, x_4, a_1 \geq 0$$

where x_4 is a slack variable and a_1 is an artificial variable introduced so that we may have a valid starting basis for the simplex algorithm. The current tableau is thus

Basis	z	x_1	x_2	x_3	x_4	a_1	RHS
	-1	40	30	10	0	1000	0
x_4	0	1	2	4	1	0	12
a_1	0	1	1	1	0	1	6

The basis $\{x_4, a_1\}$ is not valid until after we perform Gaussian elimination to remove the 1000 from the objective row. The starting tableau is thus

Basis	z	x_1	x_2	x_3	x_4	a_1	RHS
	-1	-960	-970	-990	0	0	-6000
x_4	0	1	2	4	1	0	12
a_1	0	1	1	1	0	1	6

Pivot at row 2, column 1

Basis	z	x_1	x_2	x_3	x_4	a_1	RHS
	-1	0	-10	-30	0	960	-240
x_4	0	0	1	3	1	-1	6
x_1	0	1	1	1	0	1	6

Pivot at row 1, column 3

Basis	z	x_1	x_2	x_3	x_4	a_1	RHS
	-1	0	0	0	10	950	-180
x_3	0	0	$1/3$	1	$1/3$	$-1/3$	2
x_1	0	1	$2/3$	0	$-1/3$	$4/3$	4

This solution is optimal. However if we pivot at row 1, column 2, we get

Basis	z	x_1	x_2	x_3	x_4	a_1	RHS
	-1	0	0	0	10	950	-180
x_2	0	0	1	3	1	-1	6
x_1	0	1	0	-2	-1	2	0

The LP is degenerate with two possible solutions being $\{x_1 = 4,\ x_2 = 0,\ x_3 = 2\}$ and $\{x_1 = 0,\ x_2 = 6,\ x_3 = 0\}$. In any case, the objective value is 180.

3. (b) The problem can be viewed as

$$\text{Maximize } z = 5x_1 + 4x_2 - Ma_1$$

subject to

$$3x_1 + 2x_2 + x_3 = 6$$

$$2x_1 - x_2 - x_4 + a_1 = 6$$

$$x_1, x_2, x_3, x_4, a_1 \geq 0$$

where x_3 is a slack variable, x_4 is a surplus variable, a_1 is an artificial variable, and M is a large positive constant. This gives us the tableau

Basis	z	x_1	x_2	x_3	x_4	a_1	RHS
	1	-5	-4	0	0	M	0
x_3	0	3	2	1	0	0	6
a_1	0	2	-1	0	-1	1	6

Performing Gaussian elimination to get a valid basis gives us the starting tableau

Basis	z	x_1	x_2	x_3	x_4	a_1	RHS
	1	$-5-2M$	$M-4$	0	M	0	$-6M$
x_3	0	3	2	1	0	0	6
a_1	0	2	-1	0	-1	1	6

Since M is a large positive constant, only the objective entry in column 1 is negative. Thus the next step is to pivot at row 1, column 1 which gives us

Basis	z	x_1	x_2	x_3	x_4	a_1	RHS
	1	0	$(7M-2)/3$	$(5+2M)/3$	M	0	$10-2M$
x_1	0	1	$2/3$	$1/3$	0	0	2
a_1	0	0	$-7/3$	$-2/3$	-1	1	2

At this point the simplex algorithm would terminate since a large value for M would cause all entries in the objective row (with the exception of the objective value) to be nonnegative. However the artificial variable a_1 is part of the basis if M is sufficiently large. This implies that there are no feasible solutions to the LP.