# MA470 Assignment 3

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The final page of this submission consists of an appendix with derivations of general reusable results that are used throughout my solutions.

**Exercise 2.9**

$$V(t,S) = e^{-r\tau}\widetilde{E}_{t,S}\left[(S(T)-K-X)\mathbb{1}_{\{S(T)\geq K\}}\right] = e^{-r\tau}(\widetilde{E}_{t,S}\left[(S(T)-K)^+\right] - \widetilde{E}_{t,S}\left[X\cdot\mathbb{1}_{\{S(T)\geq K\}}\right])$$

$$= C(t,S;r,q) - e^{-r\tau}X\cdot\widetilde{P}_{t,S}(S(T)\geq K)$$

$$= e^{-q\tau}S\cdot N(d_+^*(S/K,\tau)) - e^{-r\tau}K\cdot N(d_-^*(S/K,\tau)) - e^{-r\tau}X\cdot N(d_-^*(S/K,\tau))$$

$$= e^{-q\tau}S\cdot N(d_+^*(S/K,\tau)) - e^{-r\tau}(K+X)\cdot N(d_-^*(S/K,\tau))$$

Moreover, X is such that

$$V(t,S) = 0$$

Therefore

$$X = \frac{e^{(r-q)\tau}S\cdot N(d_+^*(S/K,\tau))}{N(d_-^*(S/K,\tau))} - K$$

More explicitly

$$X = \frac{e^{(r-q)\tau}S\cdot N(\frac{ln(S/K)+(r-q+\sigma^2/2)\tau}{\sigma\sqrt{\tau}})}{N(\frac{ln(S/K)+(r-q-\sigma^2/2)\tau}{\sigma\sqrt{\tau}})} - K$$

The fair value for X as the stock's volatility parameter becomes arbitrarily large is

$$\lim_{\sigma\to\infty} X = \lim_{\sigma\to\infty}\frac{e^{(r-q)\tau}S\cdot N(\frac{ln(S/K)+(r-q+\sigma^2/2)\tau}{\sigma\sqrt{\tau}})}{N(\frac{ln(S/K)+(r-q-\sigma^2/2)\tau}{\sigma\sqrt{\tau}})} - K$$

$$= \lim_{\sigma\to\infty} e^{(r-q)\tau}S\cdot\frac{N(\sigma\sqrt{\tau}/2)}{N(-\sigma\sqrt{\tau}/2)} - K = \infty$$

Thus we see that the fair value for X also becomes arbitrarily large.

**Exercise 2.17** (a)

$$dC_t = \mu_c C_t dt + \sigma_c C_t d\widetilde{W}(t)$$

$$d\frac{1}{B(t)} = -re^{-rt}dt$$

$$d\bar{C}_t = d\frac{C_t}{B(t)} = C_t d\frac{1}{B(t)} + \frac{1}{B(t)}dC_t + dC_t d\frac{1}{B(t)}$$
$$= (\mu_c - r)e^{-rt}C_t dt + \sigma_c C_t e^{-rt} d\widetilde{W}(t)$$

but in general for any portfolio in the (B, S) economy with nondividend paying stock,

$$d\bar{\Pi}_t = \Delta_t \sigma \bar{S}(t) d\widetilde{W}(t)$$

It follows that $\mu_c = r$ and $\sigma_c = \frac{\Delta_t \sigma S(t)}{C_t}$

More explicitly,

$$\sigma_c = \frac{N(d_+(S(t)/K,\tau))\sigma S(t)}{S(t)N(d_+(S(t)/K,\tau)) - Ke^{-r\tau}N(d_-(S(t)/K,\tau))}$$
$$= \frac{\sigma}{1 - \frac{Ke^{-r\tau}N(d_-(S(t)/K,\tau))}{S(t)N(d_+(S(t)/K,\tau))}}$$

(b)

$$\lim_{K\searrow 0} \sigma_c = \frac{\sigma}{1-0} = \sigma$$

The reason why this is so is because a European call option where you pay \$0 for a stock at time T can be replicated by a portfolio consisting of a share of stock bought at an earlier date (e.g. the present date.) Since a share of stock has volatility parameter $\sigma$ and the same payoff as the option in all scenarios, the European call option with strike $K = 0$ must also have $\sigma$ as its volatility parameter.

**Exercise 2.19** (a) For all $t \leq T_1$,

$$C(t,S;T_1,T) = e^{-r\tau}\widetilde{E}_{t,S}\left[(S(T)-\alpha S(T_1))^+\right] = e^{-r\tau}\widetilde{E}_{t,S}\left[\widetilde{E}_{T_1,S(T_1)}\left[(S(T)-\alpha S(T_1))^+\right]\right]$$

$$= e^{-r\tau}\widetilde{E}_{t,S}\left[\widetilde{E}_{T_1,S(T_1)}\left[S(T_1)\cdot(\frac{S(T)}{S(T_1)}-\alpha)^+\right]\right] = e^{-r\tau}\widetilde{E}_{t,S}\left[S(T_1)\cdot\widetilde{E}_{T_1,S(T_1)}\left[(\frac{S(T)}{S(T_1)}-\alpha)^+\right]\right]$$

(where $S(T_1)$ and $\frac{S(T)}{S(T_1)}$ are independent.)

Thus we have

$$C(t,S;T_1,T) = e^{-r\tau}\widetilde{E}_{t,S}\left[S(T_1)\right]\cdot\widetilde{E}\left[(\frac{S(T)}{S(T_1)}-\alpha)^+\right]$$

$$= S\cdot e^{-r(T-T_1)}\widetilde{E}\left[(\frac{S(T)}{S(T_1)}-\alpha)^+\right]$$

Note that

$$\frac{S(T)}{S(T_1)} \stackrel{d}{=} e^{(r-q-\sigma^2/2)(T-T_1)+\sigma(\widetilde{W}(T)-\widetilde{W}(T_1))}$$

Thus the term in the above expectation can be thought of as the payoff for a European call option with strike $K=\alpha$ purchased at time 0 with maturity date $\tau_1 := T-T_1$ and initial stock price $S(0)=1$.

Thus

$$C(t,S;T_1,T) = S\cdot(e^{-q\tau_1}N(d_+^*(1/\alpha,\tau_1)) - \alpha e^{-r\tau_1}N(d_-^*(1/\alpha,\tau_1)))$$

Note that for all $t \leq T_1$,

$$\Delta(t,S) = \frac{\partial C}{\partial S} = e^{-q\tau_1}N(d_+^*(1/\alpha,\tau_1)) - \alpha e^{-r\tau_1}N(d_-^*(1/\alpha,\tau_1))$$

Thus up until time $T_1$, the forward starting call can be hedged with a portfolio of

$$e^{-q\tau_1}N(d_+^*(1/\alpha,\tau_1)) - \alpha e^{-r\tau_1}N(d_-^*(1/\alpha,\tau_1))$$

units of stock and no investment in the risk-free asset. This strategy is static.

(b)

$$\lim_{T_1\to t} C(t,S;T_1,T) = \lim_{T_1\to t} S\cdot(e^{-q\tau_1}N(d_+^*(1/\alpha,\tau_1)) - \alpha e^{-r\tau_1}N(d_-^*(1/\alpha,\tau_1)))$$

$$= \lim_{\tau_1\to\tau} S\cdot(e^{-q\tau_1}N(d_+^*(1/\alpha,\tau_1)) - \alpha e^{-r\tau_1}N(d_-^*(1/\alpha,\tau_1)))$$

$$= Se^{-q\tau}N(d_+^*(1/\alpha,\tau)) - \alpha Se^{-r\tau}N(d_-^*(1/\alpha,\tau))$$

This is in fact the price of a European call option with strike $K=\alpha S$.

$$\lim_{T_1\to T} C(t,S;T_1,T) = \lim_{T_1\to T} S\cdot(e^{-q\tau_1}N(d_+^*(1/\alpha,\tau_1)) - \alpha e^{-r\tau_1}N(d_-^*(1/\alpha,\tau_1)))$$

$$= \lim_{\tau_1\to 0} S\cdot(e^{-q\tau_1}N(d_+^*(1/\alpha,\tau_1)) - \alpha e^{-r\tau_1}N(d_-^*(1/\alpha,\tau_1)))$$

$$= \lim_{\tau_1\to 0} S\cdot(N(d_+^*(1/\alpha,\tau_1)) - \alpha N(d_-^*(1/\alpha,\tau_1)))$$

$$= \lim_{\tau_1\to 0} S\cdot(N(-ln(\alpha)/\tau_1) - \alpha N(-ln(\alpha)/\tau_1)) = S\cdot(1-\alpha)^+$$

(Some unnecessary terms ommitted on the last line. The key point is that the Normal CDF values go to 0 if $\alpha >= 1$ ($\Rightarrow -ln(\alpha) < 0$) and go to 1 if $\alpha < 1$ ($\Rightarrow -ln(\alpha) > 0$).)

**Exercise 2.20** (a) For all $t \le T_0$

$$V = V(t,S) = e^{-r\tau}\widetilde{E}_{t,S}\left[\min\{S(T_0), S(T)\}\right]$$
$$= e^{-r\tau}\widetilde{E}_{t,S}\left[S(T) - (S(T) - S(T_0))\mathbb{1}_{\{S(T)\ge S(T_0)\}}\right]$$
$$= e^{-r\tau}\widetilde{E}_{t,S}\left[S(T)\right] - e^{-r(T_0-t)}\widetilde{E}_{t,S}\left[e^{-r(T-T_0)}\widetilde{E}_{T_0,S(T_0)}\left[(S(T)-S(T_0))\mathbb{1}_{\{S(T)\ge S(T_0)\}}\right]\right]$$
$$= S - e^{-r(T_0-t)}\widetilde{E}_{t,S}\left[C(T_0, S(T_0))\right]$$

(where the European call option has strike price $K = S(T_0)$.)

The above can be simplified

$$V = S - e^{-r(T_0-t)}\widetilde{E}_{t,S}\left[S(T_0)\cdot N(d_+(S(T_0)/S(T_0), T-T_0)) - e^{-r(T-T_0)}S(T_0)\cdot N(d_-(S(T_0)/S(T_0), T-T_0))\right]$$
$$= S - e^{-r(T_0-t)}\widetilde{E}_{t,S}\left[S(T_0)(N(d_+) - e^{-r(T-T_0)}N(d_-))\right]$$

(note that $d_+$ and $d_-$ are deterministic.)

$$= S - S\cdot(N(d_+) - e^{-r(T-T_0)}N(d_-))$$
$$= S\cdot(1 - N(d_+) + e^{-r(T-T_0)}N(d_-))$$
$$= S\cdot(N(-d_+) + e^{-r(T-T_0)}N(d_-))$$

(b) For all $t \le T_0$

$$\Delta(t,S) = \frac{\partial V}{\partial S} = N(-d_+) + e^{-r(T-T_0)}N(d_-)$$

That is, the position in the stock is constant. Moreover the value of the portfolio corresponding to the stock is

$$S(t)\cdot\Delta(t, S(t)) = V(t, S(t))$$

so we see that investing in the risk-free asset is not necessary. Thus until $T_0$ the payoff can be replicated using a static portfolio with a position of

$$N(-d_+) + e^{-r(T-T_0)}N(d_-)$$

units of stock and 0 units of the risk-free asset.

**Aside:** At time $T_0$ however, changes to the portfolio will occur. For example, a European call option with strike $S(T_0)$ might be written to another investor. This guarantees that at time $T$ the payoff will be the minimum of $S(T_0)$ and $S(T)$.

## Appendix

**Generalized $d$ function:**

$$d^*_{\pm}(m,\tau) := \frac{ln(m) + (r - q \pm \sigma^2/2)\tau}{\sigma\sqrt{\tau}}$$

**Call Without Dividend:**

$$C(t,S) = S \cdot N(d_+(S/K,\tau)) - e^{-r\tau}K \cdot N(d_-(S/K,\tau))$$

**Call With Dividend:**

$$C(t,S;r,q) = e^{-q\tau}S \cdot N(d^*_+(S/K,\tau)) - e^{-r\tau}K \cdot N(d^*_-(S/K,\tau))$$

**Put-Call Parity**

$$C(t,S;r,q) - P(t,S;r,q) = e^{-q\tau}S - e^{-r\tau}K$$

**Result 1:**

$$\widetilde{P}(\frac{S(T)}{S(t)} > \frac{K}{S}) = \widetilde{P}(\widetilde{Z} > \frac{-ln(S/K) - (r - q - \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}) = N(d^*_-(S/K, T-t))$$

**Result 2:**

$$\widetilde{P}(S(t) > K) = \widetilde{P}(\frac{S(t)}{S(0)} > \frac{K}{S(0)}) = N(d^*_-(S(0)/K, t))$$

**Result 3:**

$$\widetilde{E}_{t,S}\left[S^\alpha(T)\right] = S(t)^\alpha e^{\alpha(r-q-\sigma^2/2)\tau}\widetilde{E}\left[e^{\alpha\sigma\sqrt{\tau}\widetilde{Z}}\right] = S(t)^\alpha e^{\alpha(r-q-\sigma^2(1-\alpha)/2)\tau}$$

**Result 4:**

$$E\left[e^{aZ}\mathbb{1}_{\{Z>b\}}\right] = \int_b^\infty e^{az}\frac{1}{\sqrt{2\pi}}e^{-z^2/2}dz = e^{a^2/2}\int_b^\infty \frac{1}{\sqrt{2\pi}}e^{-(z-a)^2/2}dz = e^{a^2/2}\cdot(1-N(b-a)) = e^{a^2/2}\cdot N(a-b)$$