

MA372 Submission Problems

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1. Consider a linear program of the form

$$\text{Maximize } z = \mathbf{c} \cdot \mathbf{x}$$

subject to

$$A\mathbf{x} \leq \mathbf{b}$$

$$\mathbf{x} \geq 0,$$

where $\mathbf{b} \geq \mathbf{0}$. Prove that if $\mathbf{x}^{(1)}, \mathbf{x}^{(2)}, \dots, \mathbf{x}^{(k)}$ are optimum solutions to this linear program, then any convex combination of them is also an optimum solution.

2. Follow the directions of the January 17 discussion problem, but use the following tableaux instead: (tableaux included at start of each part of my solution for Question 2.)

3. Let (LP1) denote the following program:

$$\text{Maximize } z = \mathbf{c} \cdot \mathbf{x}$$

subject to

$$A\mathbf{x} \leq \mathbf{b}$$

$$\mathbf{x} \geq 0,$$

where $\mathbf{b} \geq \mathbf{0}$.

- (a) Let (LP2) denote the linear program obtained from (LP1) by replacing b_i with $b_i + \delta_i$, where $\delta_i \in \mathbb{R}$ for each $1 \leq i \leq m$. Determine conditions under which an optimum basis for (LP1) is also an optimum basis for (LP2), and determine the new optimum solution in this case.
- (b) Let (LP3) denote the linear program obtained from (LP1) by replacing c_j with $c_j + \Delta_j$, where $\Delta_j \in \mathbb{R}$ for each $1 \leq j \leq n$. Determine conditions under which an optimum basis for (LP1) is also an optimum basis for (LP3), and determine the value of the new objective function in this case.

1. Let $k \in \mathbb{N}$ be arbitrary and suppose that $\mathbf{x}^{(1)}, \mathbf{x}^{(2)}, \dots, \mathbf{x}^{(k)}$ are optimum solutions to the linear program. Define z^* to be the optimum objective value. That is to say,

$$z^* = z(\mathbf{x}^{(1)}) = z(\mathbf{x}^{(2)}) = \dots = z(\mathbf{x}^{(k)})$$

Let \mathbf{x}' be any convex combination of $\mathbf{x}^{(1)}, \mathbf{x}^{(2)}, \dots$, and $\mathbf{x}^{(k)}$. That is, \mathbf{x}' is of form

$$\mathbf{x}' = \sum_{i=1}^k t_i \mathbf{x}^{(i)} \text{ where } 0 \leq t_i \leq 1 \text{ and } \sum_{i=1}^k t_i = 1$$

\mathbf{x}' is feasible:

- Clearly $\mathbf{x}' \geq 0$ because $\mathbf{x}^{(i)} \geq 0$ and $t_i \geq 0$ for all i
- $A\mathbf{x}' = A(t_1\mathbf{x}^{(1)} + t_2\mathbf{x}^{(2)} + \dots + t_k\mathbf{x}^{(k)}) = t_1A\mathbf{x}^{(1)} + t_2A\mathbf{x}^{(2)} + \dots + t_kA\mathbf{x}^{(k)}$
- Since $\mathbf{x}^{(i)}$ is feasible for all i ($\Rightarrow A\mathbf{x}^{(i)} \leq b$),

$$A\mathbf{x}' \leq t_1b + t_2b + \dots + t_kb = (t_1 + t_2 + \dots + t_k)b = b$$

- Thus \mathbf{x}' is feasible.

\mathbf{x}' maximizes the objective function:

- Since the objective function is linear,

$$z(\mathbf{x}') = z(\sum_{i=1}^k t_i \mathbf{x}^{(i)}) = \sum_{i=1}^k t_i z(\mathbf{x}^{(i)}) = \sum_{i=1}^k t_i z^* = z^* \sum_{i=1}^k t_i = z^*$$

- z^* is the optimum objective value and thus \mathbf{x}' maximizes z .

\mathbf{x}' is feasible and maximizes z . Therefore, by definition, \mathbf{x}' is an optimum solution. Since k and \mathbf{x}' were arbitrary, we have that any convex combination of any finite number of optimum solutions is an optimum solution.

2. (a)

Basis	z	x_1	x_2	x_3	x_4	x_5	x_6	RHS
	1	0	0	-2	0	-3	5	15
x_4	0	0	0	1	1	0	-2	5
x_1	0	1	0	2	0	-1	3	2
x_2	0	0	1	-1	0	-5	1	4

This LP is unbounded because the objective row entry in column 5 is negative and all of the entries in the column are nonpositive. An infinite set of feasible solutions for which the objective function can become arbitrarily large is as follows:

$$\{(x_1 = 2 + t, x_2 = 4 + 5t, x_3 = 0) : t \geq 0\}$$

For a given choice of t , this causes the objective value to become $15 + 3t$. The objective value can thus be made as large as we like by choosing t sufficiently large.

(b)

Basis	z	x_1	x_2	x_3	x_4	x_5	x_6	RHS
	1	0	0	-2	0	-3	5	15
x_4	0	0	0	1	1	0	-2	1
x_1	0	1	0	2	0	1	3	2
x_2	0	0	1	-1	0	2	1	4

Pivoting at row 1, column 3 gives us the tableau

Basis	z	x_1	x_2	x_3	x_4	x_5	x_6	RHS
	1	0	0	0	2	-3	1	17
x_3	0	0	0	1	1	0	-2	1
x_1	0	1	0	0	-2	1	7	0
x_2	0	0	1	0	1	2	-1	5

If we now pivot at row 2, column 5

Basis	z	x_1	x_2	x_3	x_4	x_5	x_6	RHS
	1	3	0	0	-4	0	22	17
x_3	0	0	0	1	1	0	-2	1
x_5	0	1	0	0	-2	1	7	0
x_2	0	-2	1	0	5	0	-15	5

At this point it is clear that the LP is degenerate since the bases $\{x_1, x_2, x_3\}$ and $\{x_2, x_3, x_5\}$ both give us the same basic solution of $x_1 = 0, x_2 = 5, x_3 = 1, x_4 = 0, x_5 = 0, x_6 = 0$

(c)

Basis	z	x_1	x_2	x_3	x_4	x_5	x_6	RHS
	1	-4	0	1	0	0	2	7
x_4	0	0	0	1	1	0	-2	3
x_5	0	1	0	2	0	1	3	2
x_2	0	-1	1	-3	0	0	-2	2

Pivoting at row 2, column 1 gives us

Basis	z	x_1	x_2	x_3	x_4	x_5	x_6	RHS
	1	0	0	9	0	4	14	15
x_4	0	0	0	1	1	0	-2	3
x_1	0	1	0	2	0	1	3	2
x_2	0	0	1	-1	0	1	1	4

The entries of the objective row are now all nonnegative, and there is no way to create another optimum solution (as the objective entries in the nonbasic variable columns are positive.) The unique optimum solution is thus $x_1 = 2, x_2 = 4, x_3 = 0$ where the maximum objective value is 15.

(d)

Basis	z	x_1	x_2	x_3	x_4	x_5	x_6	RHS
	1	0	0	0	2	4	0	20
x_3	0	0	2	1	3	0	0	5
x_1	0	1	-3	0	0	-1	0	1
x_6	0	0	1	0	-1	2	1	2

From the tableau, we see that $x_1 = 1, x_2 = 0, x_3 = 5$ is an optimum solution. However if we pivot at row 3, column 2 we get

Basis	z	x_1	x_2	x_3	x_4	x_5	x_6	RHS
	1	0	0	0	2	4	0	20
x_3	0	0	0	1	5	-4	-2	1
x_1	0	1	0	0	-3	5	3	7
x_2	0	0	1	0	-1	2	1	2

And so $x_1 = 7, x_2 = 2, x_3 = 1$ is also an optimum solution. An infinite set of optimum solutions is thus given by the set of convex combinations

$$\begin{aligned} & \{(x_1 = t + 7(1 - t), x_2 = 0t + 2(1 - t), x_3 = 5t + 1(1 - t)) : 0 \leq t \leq 1\} \\ &= \{(x_1 = 7 - 6t, x_2 = 2(1 - t), x_3 = 4t + 1) : 0 \leq t \leq 1\} \end{aligned}$$

3. (a) Let $x_{i_1}, x_{i_2}, \dots, x_{i_n}$ be an ordered optimum basis for (LP1), and let B be the matrix formed by the columns of (LP1)'s tableau that correspond to $x_{i_1}, x_{i_2}, \dots, x_{i_n}$ with the columns of B arranged in that order.

Since only b is changing, the operations which need to be performed in order to arrive at that basis (B^{-1}) for (LP2) remain the same. By applying the Fundamental Insight, the objective row entries all remain the same (meaning they all remain nonnegative), so the only thing that needs to be checked is if the basic solution is still feasible.

This is true if and only if

$$B^{-1}(b + \delta) \geq 0$$

(Where δ is the column vector formed by δ_i for $i = 1, 2, \dots, n$)

If this condition holds then the optimum solution for (LP2) is

$$(x_{i_1}, x_{i_2}, \dots, x_{i_n}) = B^{-1}(b + \delta)$$

(b) When c is replaced by $c + \Delta$, the situation is similar to as in (a) except now feasibility is unaffected and what needs to be checked is that the basic solution is still optimal.

Letting $c_B^\Delta = [c_{i_1} + \Delta_{i_1}, c_{i_2} + \Delta_{i_2}, \dots, c_{i_n} + \Delta_{i_n}]$

Applying the Fundamental Insight gives us that the basic solution is still optimal if the objective row is still nonnegative which is true if and only if

$$c_B^\Delta B^{-1}A - (c + \Delta) \geq 0 \text{ and } c_B^\Delta B^{-1}b \geq 0$$

In which case the new objective value is $c_B^\Delta B^{-1}b$