MA470 Assignment 2

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The final page of this submission consists of an appendix with derivations of general reusable results that are used throughout my solutions. (Note that $d^*_\pm$ takes q into account.)

Exercise 2.1 (d)

$$\widetilde{P}(S^\alpha(T_2) > S^\beta(T_1)) = \widetilde{P}(\frac{S^\alpha(T_2)}{S^\beta(T_1)} > 1)$$

Where

$$S^\alpha(T_2) = S^\alpha e^{\alpha(r-q-\sigma^2/2)T_2+\alpha\sigma\widetilde{W}(T_2)}$$

$$S^\beta(T_1) = S^\beta e^{\beta(r-q-\sigma^2/2)T_1+\beta\sigma\widetilde{W}(T_1)}$$

The probability is thus

$$= \widetilde{P}(S^{\alpha-\beta}e^{(r-q-\sigma^2/2)(\alpha T_2-\beta T_1)+\sigma(\alpha\widetilde{W}(T_2)-\beta\widetilde{W}(T_1))} > 1)$$

$$\text{(where } Var(\sigma(\alpha\widetilde{W}(T_2) - \beta\widetilde{W}(T_1))) = \sigma^2(\alpha^2T_2 + \beta^2T_1 - 2\alpha\beta T_1))$$

$$= \widetilde{P}(\widetilde{Z} > \frac{-(\alpha-\beta)ln(S) - (r-q-\sigma^2/2)(\alpha T_2 - \beta T_1)}{\sigma\sqrt{\alpha^2T_2 + \beta^2T_1 - 2\alpha\beta T_1}})$$

$$= N(\frac{(\alpha-\beta)ln(S) + (r-q-\sigma^2/2)(\alpha T_2 - \beta T_1)}{\sigma\sqrt{\alpha^2T_2 + \beta^2T_1 - 2\alpha\beta T_1}})$$

(e) I assume that $T_2 > T_1 > t$. The other cases can be obtained by relabelling the time values.

$$\widetilde{P}(S(T_2) > S(T_1) > S(t)) = \widetilde{P}(\frac{S(T_2)}{S(T_1)} > 1 \text{ and } \frac{S(T_1)}{S(t)} > 1) = \widetilde{P}(\frac{S(T_2)}{S(T_1)} > 1)\cdot\widetilde{P}(\frac{S(T_1)}{S(t)} > 1)$$

(because $\frac{S(T_2)}{S(T_1)}$ and $\frac{S(T_1)}{S(t)}$ are independent.)

By **Result 1**, the desired probability is thus

$$= N(d^*_-(1, T_2 - T_1)) \cdot N(d^*_-(1, T_1 - t))$$

Exercise 2.1 (f)

$$\widetilde{P}(S(T_1) < K_1, S(T_2) > K_2) = \widetilde{P}(S(T_1) < K_1 \text{ and } S(T_2) > K_2)$$

Case $T_1 < T_2$:

$$= \widetilde{P}(S(T_1) < K_1) \cdot \widetilde{P}(S(T_2) > K_2 | S(T_1) < K_1)$$

$$= \widetilde{P}(S(T_1) < K_1) \cdot \int_0^{K_1} \widetilde{P}(S(T_2) > K_2 | S(T_1) = s) ds$$

$$= \widetilde{P}(S(T_1) < K_1) \cdot \int_0^{K_1} \widetilde{P}(\frac{S(T_2)}{S(T_1)} > \frac{K_2}{s}) ds$$

By **Result 1** and **Result 2**

$$= (1 - N(d_-^*(S/K_1, T_1)) \cdot \int_0^{K_1} N(d_-^*(s/K_2, T_2 - T_1)) ds$$

Case $T_2 < T_1$:

$$= \widetilde{P}(S(T_2) > K_2) \cdot \widetilde{P}(S(T_1) < K_1 | S(T_2) > K_2)$$

$$= N(d_-^*(S/K_2, T_2)) \cdot \int_{K_2}^{\infty} 1 - N(d_-^*(s/K_1, T_1 - T_2)) ds$$

Exercise 2.2 (c)

$$V(t, S) = e^{-r\tau} \widetilde{E}_{t,S} \left[a | S^\alpha(T) - K | \right]$$

$$= a e^{-r\tau} (2 \widetilde{E}_{t,S} \left[(S^\alpha(T) - K) \mathbb{1}_{\{S^\alpha(T) > K\}} \right] + \widetilde{E}_{t,S} \left[K - S^\alpha(T) \right])$$

Where

$$\widetilde{E}_{t,S} [K] = K$$

$$\widetilde{E}_{t,S} [S^\alpha(T)] = S^\alpha e^{\alpha(r - q - \sigma^2(1-\alpha)/2)\tau}$$

(by **Result 3**)

Case $\alpha > 0$:

$$\widetilde{E}_{t,S} \left[K \cdot \mathbb{1}_{\{S^\alpha(T) > K\}} \right] = K \cdot \widetilde{P}_{t,S}(S(T) > K^{1/\alpha}) = K \cdot N(d_-^*(S/K^{1/\alpha}, \tau))$$

(by **Result 1**)

$$\widetilde{E}_{t,S} \left[S^\alpha(T) \cdot \mathbb{1}_{\{S^\alpha(T) > K\}} \right] = S^\alpha e^{\alpha(r - q - \sigma^2/2)\tau} \cdot \widetilde{E} \left[e^{\alpha\sigma\sqrt{\tau}\widetilde{Z}} \mathbb{1}_{\{\widetilde{Z} > -d_-^*(S/K^{1/\alpha}, \tau)\}} \right]$$

$$= S^\alpha e^{\alpha(r - q - \sigma^2/2)\tau} \cdot e^{(\alpha\sigma)^2\tau/2} (1 - N(-d_-^*(S/K^{1/\alpha}, \tau) - \alpha\sigma\sqrt{\tau}))$$

(by **Result 4**)

$$= S^\alpha e^{\alpha(r - q - \sigma^2(1-\alpha)/2)\tau} \cdot N(d_-^*(S/K^{1/\alpha}, \tau) + \alpha\sigma\sqrt{\tau})$$

Therefore:

$$V(\tau, S) = a e^{-r\tau} (S^\alpha e^{\alpha(r - q - (1-\alpha)\sigma^2/2)\tau} \cdot (2 \cdot N(d_-^*(S/K^{1/\alpha}, \tau) + \alpha\sigma\sqrt{\tau}) - 1) + K \cdot (2 \cdot N(d_-^*(S/K^{1/\alpha}, \tau)) - 1))$$

Case $\alpha < 0$:

$$\widetilde{E}_{t,S}\left[K\mathbb{1}_{\{S^\alpha(T)>K\}}\right] = K \cdot \widetilde{P}_{t,S}(S(T) < K^{1/\alpha}) = K \cdot (1 - N(d_-(S/K^{1/\alpha}, \tau)))$$

(by **Result 1**)

$$\widetilde{E}_{t,S}\left[S^\alpha(T) \cdot \mathbb{1}_{\{S^\alpha(T)>K\}}\right] = S^\alpha e^{\alpha(r-q-\sigma^2/2)\tau} \cdot \widetilde{E}\left[e^{\alpha\sigma\sqrt{\tau}\widetilde{Z}}(1 - \mathbb{1}_{\{\widetilde{Z}>-d_-^*(S/K^{1/\alpha},\tau)\}})\right]$$

$$= S^\alpha e^{\alpha(r-q-\sigma^2/2)\tau} \cdot (e^{(\alpha\sigma)^2\tau/2} - e^{(\alpha\sigma)^2\tau/2}(1 - N(-d_-^*(S/K^{1/\alpha}, \tau) - \alpha\sigma\sqrt{\tau})))$$

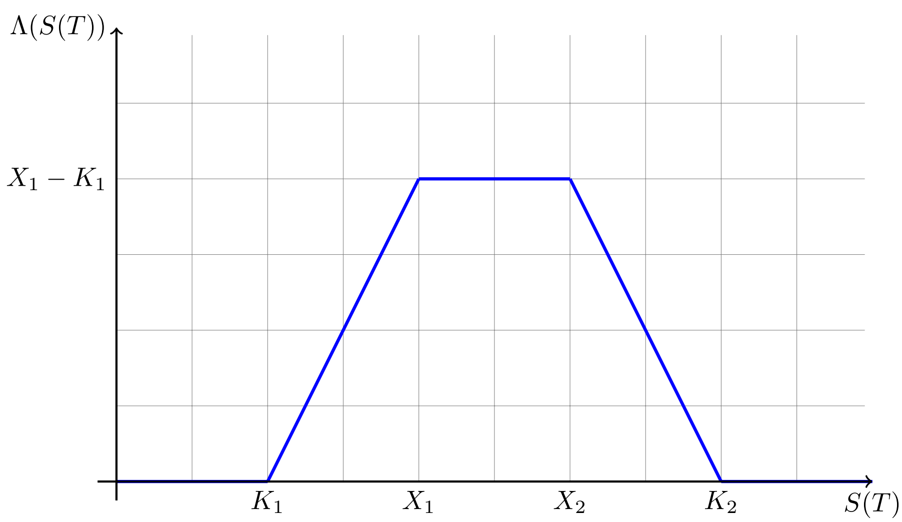
(by **Result 4**)

$$= S^\alpha e^{\alpha(r-q-\sigma^2(1-\alpha)/2)\tau} \cdot (1 - N(d_-^*(S/K^{1/\alpha}, \tau) + \alpha\sigma\sqrt{\tau}))$$

Therefore:

$$V(\tau, S) = ae^{-r\tau}(S^\alpha e^{\alpha(r-q-(1-\alpha)\sigma^2/2)\tau} \cdot (1 - 2 \cdot N(d_-^*(S/K^{1/\alpha}, \tau) + \alpha\sigma\sqrt{\tau})) + K \cdot (1 - 2 \cdot N(d_-^*(S/K^{1/\alpha}, \tau))))$$

Exercise 2.3 (a)



(strike values not necessarily evenly spaced out.)

This European option can be replicated by purchasing a European call with strike K_1, writing a European call with strike X_1, purchasing a European call with strike K_2, and writing a European call with strike X_2.

(b) Using the fact that

$$C(\tau, S, K, \sigma, r, q) = e^{-q\tau}C(\tau, S, K, \sigma, r - q, 0)$$

The value of the portfolio is

$$V(\tau, S, \sigma, r, q, K_1, X_1, K_2, X_2) = C(\tau, S, K_1, \sigma, r, q) - C(\tau, S, X_1, \sigma, r, q) + C(\tau, S, K_2, \sigma, r, q) - C(\tau, S, X_2, \sigma, r, q)$$

$$= e^{-q\tau}(C(\tau, S, K_1, \sigma, r-q, 0) - C(\tau, S, X_1, \sigma, r-q, 0) + C(\tau, S, K_2, \sigma, r-q, 0) - C(\tau, S, X_2, \sigma, r-q, 0))$$

$$= Se^{-q\tau}(N(d_+^*(S/K_1, \tau)) - N(d_+^*(S/X_1, \tau)) + N(d_+^*(S/K_2, \tau)) - N(d_+^*(S/X_2, \tau)))$$

$$-e^{-r\tau}(K_1 \cdot N(d_-^*(S/K_1, \tau)) - X_1 \cdot N(d_-^*(S/X_1, \tau)) + K_2 \cdot N(d_-^*(S/K_2, \tau)) - X_2 \cdot N(d_-^*(S/X_2, \tau)))$$

(c) Using the following fact about hedging European calls

$$\Delta_C(t, S; r, q) = e^{-q\tau}\Delta_C(t, S; r - q, 0)$$

And letting $\Delta_K(t, S; r, q)$ be the delta position for a European call with strike K, The desired delta position is

$$\Delta(t, S; r, q) = \Delta_{K_1}(t, S; r, q) - \Delta_{X_1}(t, S; r, q) + \Delta_{K_2}(t, S; r, q) - \Delta_{X_2}(t, S; r, q)$$

$$= e^{-q\tau}(N(d_+^*(S/K_1, \tau)) - N(d_+^*(S/X_1, \tau)) + N(d_+^*(S/K_2, \tau)) - N(d_+^*(S/X_2, \tau)))$$

Important: $d_\pm^*(m, \tau)$ is defined in the appendix and is slightly different from the function in the lecture notes in that it takes q into consideration.

Exercise 2.4 (a)

$$V(t, S) = e^{-r\tau}\widetilde{E}_{t,S}\left[\Sigma_{n=0}^N a_n S^n(T)\right] = e^{-r\tau}(a_0 + \Sigma_{n=1}^N a_n \widetilde{E}_{t,S}\left[S^n(T)\right])$$

By **Result 3** this is

$$= e^{-r\tau}(a_0 + \Sigma_{n=1}^N a_n S^n e^{n(r-q-\sigma^2(1-n)/2)\tau})$$

Or more compactly

$$= e^{-r\tau}(a_0 + \Sigma_{n=1}^N a_n \Theta^n e^{(n\sigma)^2\tau/2})$$

(where $\Theta := Se^{(r-q-\sigma^2/2)\tau}$)

Appendix

Throughout this assignment I make use of

$$d^*_{\pm}(m,\tau) := \frac{ln(m) + (r - q \pm \sigma^2/2)\tau}{\sigma\sqrt{\tau}}$$

which is different from the usual definition since it takes the continuous dividend yield q into account.

Result 1:

$$\widetilde{P}(\frac{S(T)}{S(t)} > \frac{K}{S}) = \widetilde{P}(\widetilde{Z} > \frac{-ln(S/K) - (r - q - \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}) = N(d^*_{-}(S/K, T-t))$$

Result 2:

$$\widetilde{P}(S(t) > K) = \widetilde{P}(\frac{S(t)}{S(0)} > \frac{K}{S(0)}) = N(d^*_{-}(S(0)/K, t))$$

Result 3:

$$\widetilde{E}_{t,S}\left[S^{\alpha}(T)\right] = S(t)^{\alpha} e^{\alpha(r-q-\sigma^2/2)\tau}\widetilde{E}\left[e^{\alpha\sigma\sqrt{\tau}\widetilde{Z}}\right] = S(t)^{\alpha} e^{\alpha(r-q-\sigma^2(1-\alpha)/2)\tau}$$

Result 4:

$$E\left[e^{aZ}\mathbb{1}_{\{Z>b\}}\right] = \int_b^{\infty} e^{az}\frac{1}{\sqrt{2\pi}}e^{-z^2/2}dz = e^{a^2/2}\int_b^{\infty}\frac{1}{\sqrt{2\pi}}e^{-(z-a)^2/2}dz = e^{a^2/2}\cdot(1 - N(b-a))$$