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LATTICE SIMPLICES IN $[0,k]^d$

BY

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ABSTRACT

Consider a lattice simplex S contained in the cube $[0,k]^d$ where k is a positive integer and d is at least 2. It is shown in this note that there exists a lattice point $v \in [0,k]^d$ such that the convex hull of $S \cup \{v\}$ admits, as vertices, v and all the vertices of S.

In this note, a lattice (d,k)-polytope is a polytope $P \subset \mathbb{R}^d$ whose coordinates of vertices belong to $\{0,...,k\}$. Note that, by this definition, the dimension of a lattice (d,k)-polytope is possibly less than d.

Consider a d-dimensional lattice (d,k)-polytope S and assume that S is a simplex. Note that we restrict the claim in the abstract to S being d-dimensional, but this will be shown to be without loss of generality later (see Lemma 0.4). Let $\mathcal{V}$ be the set of vertices of S. For any $u \in \mathcal{V}$, denote by H_u the affine hull of the facet of S opposite u, and by H_u^- the closed half-space of $\mathbb{R}^d$ limited by H_u that does not contain u. For any $v \in \mathcal{V}$, the set

$$C_v = \bigcap_{u \in \mathcal{V} \setminus \{v\}} H_u^-$$

is a d-dimensional cone pointed at v. This cone is exactly the set of the points $x \in \mathbb{R}^d$ such that the convex hull of $S \cup \{x\}$ does not admit v as a vertex. By this remark, we have the following lemma.

LEMMA 0.1: *Let x be a lattice point in $[0,k]^d$. The convex hull of $S \cup \{x\}$ admits, as vertices, x and all the vertices of S if and only if x does not belong to S and, for all $v \in \mathcal{V}$, x does not belong to C_v.*

Call $\gamma = \min\{x_1 : x \in S\}$. Consider the following hyperplane of $\mathbb{R}^d$:

$$X = \{x \in \mathbb{R}^d : x_1 = \gamma\}$$

Denote by X^- the open half-space of $\mathbb{R}^d$ bounded by X and that does not contain S. By construction, the intersection $S \cap X$ is a non-empty, proper face of S. This face will be denoted by F in the following. Since S is a simplex, it admits another, non-empty face $F^\star$ whose vertices are exactly the vertices of S that do not belong to F.

By construction,

$$\dim(F) + \dim(F^\star) = d - 1.$$

In particular, there exists a vector c that is orthogonal to both F and $F^\star$. Consider the hyperplane Y of $\mathbb{R}^d$ that admits c as a normal vector and such that $F^\star \subset Y$. The intersection $S \cap Y$ is precisely $F^\star$. Denote by Y^- the closed half-space of $\mathbb{R}^d$ bounded by Y that does not contain F. It will be assumed that c has norm 1 and that it points towards Y^-. Let

$$\delta = \min\{c{\cdot}x : x \in X \cap [0, k]^d\}. \tag{1}$$

Further denote $G = \{x \in X \cap [0, k]^d : c{\cdot}x = \delta\}$. In the statement of the following lemma, $\mathrm{aff}(F)$ denotes the affine hull of F.

Lemma 0.2: *If v is a vertex of S, then*

$$C_v \subset \mathrm{aff}(F) \cup X^- \cup Y^-.$$

Proof. First observe that, if s is a face of S and H is a hyperplane of $\mathbb{R}^d$ that intersects S exactly along s, then

$$\bigcap_{u \in \mathcal{V} \setminus s} H_u^- \subset H^-,$$

where H^- is the closed half space of $\mathbb{R}^d$ bounded by H and disjoint from the interior of S. Further observe that the intersection of H with

$$\bigcap_{u \in \mathcal{V} \setminus s} H_u^-$$

is precisely the affine hull of s. As a direct consequence, taking in turn $s = F^\star$ and $s = F$, one obtains that, if v is a vertex of F, then $C_v \subset \mathrm{aff}(F) \cup X^-$ and if v is a vertex of $F^\star$, then $C_v \subset Y^-$. The result therefore holds because any vertex of S is either a vertex of F or a vertex of $F^\star$. ∎

As an immediate consequence of this and Lemma 0.1, for any lattice point $x \in [0,k]^d$ that does not belong to X^-, to Y^-, or to the affine hull of F, the convex hull of $S \cup \{x\}$ admits, as vertices, x and all the vertices of S.

Recall that c is orthogonal to F and $F^\star$. As a consequence, the map $x \mapsto c{\cdot}x$ is constant within F and within $F^\star$. Call ε the value of $c{\cdot}x$ when $x \in F$ and $\varepsilon^\star$ the value of $c{\cdot}x$ when $x \in F^\star$. Since F and Y^- are disjoint, $\varepsilon < \varepsilon^\star$. Moreover, by (1), $\delta \leq \varepsilon$. Observe that if the latter inequality is strict, then G is disjoint from both $\mathrm{aff}(F)$ and Y^-. By definition, it is also disjoint from X^- and the following lemma is then obtained as a consequence of Lemmas 0.1 and 0.2.

LEMMA 0.3: *If $\delta < \varepsilon$, then for any lattice point $x \in G$, the convex hull of $S \cup \{x\}$ admits, as vertices, x and all the vertices of S.*

If, on the contrary, δ and ε coincide, then $F \subset G$. This situation is familiar: we are looking at a lattice simplex F contained in a (possibly degenerate) lattice cube G. If the dimension of G is greater than the dimension of F, then the following lemma provides the desired result.

LEMMA 0.4: *If k and d are positive and if P is a lattice (d,k)-polytope of dimension less than d then there exists a lattice point x that belongs to $[0,k]^d$ but that does not belong to the affine hull of P.*

Proof. If P is a lattice (d,k)-polytope of dimension less than d, then the intersection I of its affine hull with $[0,k]^d$ cannot contain more than $(k+1)^{d-1}$ lattice points. Indeed, one can always project I orthogonally on a facet of $[0,k]^d$ in such a way that the dimension of the projection is not less than that of I. Such a projection induces an injection from the lattice points in I into the lattice points in the facet on which the projection is made.

Now observe that $[0,k]^d$ contains $(k+1)^d$ lattice points. Since k is positive, $(k+1)^{d-1} < (k+1)^d$ and the lemma is proven. ∎

Hence, it remains to solve the case when F is a subset of G and both have the same dimension. If this dimension is at least 2, then the strategy is to argue by induction on d. The base case of the induction is given by the following lemma.

LEMMA 0.5: *If $d = 2$ then there exists a lattice point $x \in [0,k]^2$ such that the convex hull of $S \cup \{x\}$ admits, as vertices, x and all the vertices of S.*

Proof. Probably just a careful, hopefully short disjunction. Rado, Julien? ∎

The case when F is a subset of G and their common dimension is either 0 or 1 has to be treated separately.

LEMMA 0.6: *Assume that d is greater than 2. If G has dimension at most 1 and admits F as a subset, then there exists at least one lattice point x in $[0,k]^d$ that does not belong to $\mathrm{aff}(F) \cup X^- \cup Y^-$.*

Proof. Assume that the dimension of G is 0 or 1 and that F is a subset of G. Because of the latter assumption, $\delta = \varepsilon$. Further assume that every lattice point in $X \cap [0,k]^d$ belongs to either $\mathrm{aff}(F)$ or Y^-. Since Y^- is the set of points $x \in \mathbb{R}^d$ such that $c{\cdot}x \geq \varepsilon^\star$, this assumption yields that any lattice point x in $X \cap [0,k]^d$ such that $c{\cdot}x < \varepsilon^\star$ satisfies $c{\cdot}x = \delta$.

In particular, the only lattice points in $X \cap [0,k]^d$ that may belong to Y have distance exactly 1 to some lattice point in G.

Now consider the set N of the points x in $[0,k]^d$ whose orthogonal projection on X belongs to G and such that $x_i = \gamma + 1$. By construction, $\gamma < k$ and therefore, N is non-empty. More precisely, N is made up of a single point if G has dimension 0, and N is a line segment parallel to G if G has dimension 1. In particular, the map $x \mapsto c{\cdot}x$ is constant within N. Call this constant η. If $\eta \geq \varepsilon^\star$, then the only lattice points x in $[0,k]^d$ such that $x_i \geq \gamma$ that may belong to Y are the ones whose distance to some lattice point in G is exactly 1. Among these candidates, the only ones that can also be vertices of S are the lattice point in N. Indeed, all the other candidates belong to X. This is impossible because S would then be contained in the convex hull of $G \cup N$ whose dimension is either 1 or 2 depending on the dimension of G, whereas the dimension of S is at least 3.

It follows from this contradiction that $\eta < \varepsilon^\star$. In other words, N is disjoint from Y^-. By construction, N is also disjoint from X^- and from the affine hull of F. As N contains lattice points, this proves the lemma. ∎

We are now ready to prove the desired result.

THEOREM 0.7: *There exists a lattice point x in the cube $[0,k]^d$ such that the convex hull of $S \cup \{x\}$ admits, as vertices, x and all the vertices of S.*

Proof. We need to write the induction carefully. Rado, Julien? ∎