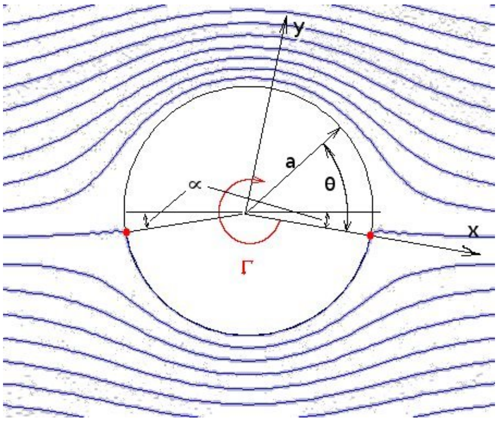
Lift and moment coefficients of a flat plate aerofoil.

A Joukowski mapping can be used of known flow around a circular cylinder to predict flow over a flat plate aerofoil at angle of attack.

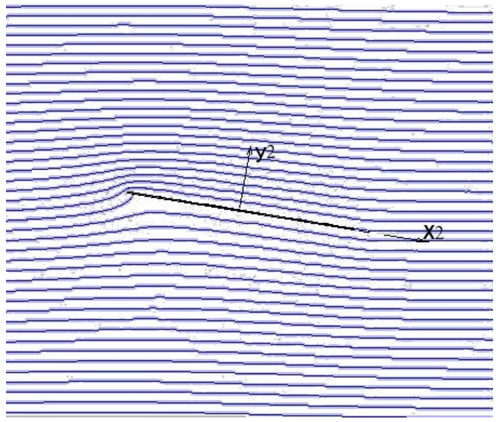


$$z_2 = z_1 + \frac{k^2}{z_1}$$

k = a
a – radius of circle

$$z_1 = x_1 + i.\, y_1$$

$$z_2 = x_2 + i.\, y_2$$



The equation for the surface of the circle can be set in polar coordinates as, $z_1 = a\,e^{i\theta}$

hence the equation for the surface of the plate will be found from the transformation,

$$z_2 = a\,e^{i\theta} + \frac{a^2}{a\,e^{i\theta}} \quad , \quad z_2 = a\,e^{i\theta} + a\,e^{-i\theta} \quad , \quad z_2 = a\left(\cos(\theta) + i.\sin(\theta) + \cos(\theta) - i\sin(\theta)\right)$$

$$z_2 = 2\mathrm{a}\cos(\theta) \quad \text{hence} \quad x_2 = 2\mathrm{a}\cos(\theta) \quad \text{and} \quad y_2 = 0 \quad .$$

To obtain a correct Kutta condition on the plate the circulation required will be $\Gamma = 4\pi a V_\infty \sin(\alpha)$

Velocities on the surface of the circle in z1 will be the sum of the standard circle flow, rotated by an angle α , and the contribution from the vortex on the origin.

$$V_1 = 2V_\infty \sin(\theta - \alpha) + \frac{\Gamma}{2\pi a} \quad . \quad V_1 = 2V_\infty \sin(\theta - \alpha) + \frac{4\pi a V_\infty \sin(\alpha)}{2\pi a}$$

$$V_1 = 2V_\infty \sin(\theta - \alpha) + 2V_\infty \sin(\alpha) \quad , \quad V_1 = 2V_\infty\left(\sin(\theta)\cos(\alpha) - \sin(\alpha)\cos(\theta) + \sin(\alpha)\right)$$

The velocity on the surface of the plate will be based on circle flow velocity and the absolute value of the derivative of the transformation function.

$$V_2 = \frac{V_1}{\left|\frac{dz_2}{dz_1}\right|} \quad .$$

The derivative can be evaluated as $\frac{dz_2}{dz_1} = 1 - \frac{k^2}{z_1^2}$ and on the surface or the circle this will be

$$\frac{dz_2}{dz_1} = 1 - \frac{a^2}{a^2 e^{2\mathrm{i}\theta}} \quad , \quad \frac{dz_2}{dz_1} = 1 - e^{-2\mathrm{i}\theta} \quad , \quad \frac{dz_2}{dz_1} = 1 - \left(\cos(2\theta) - i.\sin(2\theta)\right)$$

$$\frac{dz_2}{dz_1} = \cos^2(\theta) + \sin^2(\theta) - \cos^2(\theta) + \sin^2(\theta) + i.\,2\cos(\theta).\sin(\theta)$$

$$\frac{dz_2}{dz_1} = 2\sin(\theta).\left(\sin(\theta) + i.cos(\theta)\right) \quad \text{the magnitude is then} \quad \left|\frac{dz_2}{dz_1}\right| = 2\sin(\theta) \quad .$$

Velocity on the plate can then be calculated as,

$$V_2=\frac{2V_\infty\left(\sin(\theta)\cos(\alpha)-\sin(\alpha)\cos(\theta)+\sin(\alpha)\right)}{\left(2\sin(\theta)\right)}$$

$$V_2=V_\infty.\left(\cos(\alpha)+\sin(\alpha)\left(1-\cot(\theta)\right)\right)\quad,\quad V_2=V_\infty.\left(\cos(\alpha)+\sin(\alpha)\tan(\frac{\theta}{2})\right)$$

If only small angles of attack are considered so that no flow separation occurs,

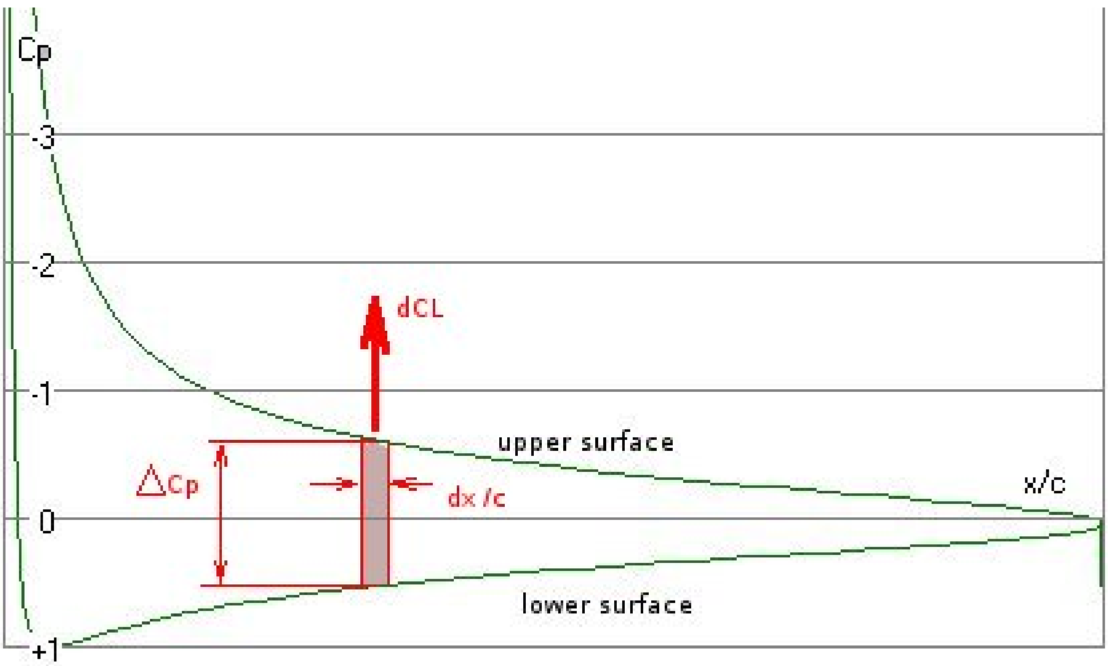
$$\alpha\rightarrow small\quad,\quad\cos(\alpha)\simeq 1\qquad\sin(\alpha)\simeq\alpha\quad\text{and}\quad V_2=V_\infty.\left(1+\alpha\tan(\frac{\theta}{2})\right)$$

Surface pressure coefficient will be found by applying the incompressible version of Bernouli equation,

$$C_P=1-\frac{V_2^2}{V_\infty^2}\quad,\quad C_p=1-\left(1+\alpha\tan(\frac{\theta}{2})\right)^2\quad,\quad C_p=1-1-2\alpha\tan(\frac{\theta}{2})-\alpha^2\tan(\frac{\theta}{2})^2$$

$$C_p=-2\alpha\tan(\frac{\theta}{2})-\alpha^2\tan(\frac{\theta}{2})^2$$

The pressure distribution over the aerofoil can be observed by looking at upper surface ($+\theta$) and the lower surface ($-\theta$) pressure coefficients. C_p At the trailing edge ($\theta=0$) is zero. On the upper surface the coefficient becomes more and more negative until the leading edge ($\theta=\pi$) where it produces infinite suction. On the lower surface going forward $\tan(-\frac{\theta}{2})$ is positive and $\tan(\frac{\theta}{2})>\tan(\frac{\theta}{2})^2$ so initially the coefficient becomes more positive. This continues up to the point when $V_2=0$, where $\alpha\tan(-\frac{\theta}{2})=-1$ and $C_p=1$. In front of this $\alpha\tan(\frac{\theta}{2})>1$ and $\alpha\tan(\frac{\theta}{2})<\alpha^2\tan(\frac{\theta}{2})^2$ so the coefficient reduces to zero and then becomes more and more negative, eventually equaling the infinite suction presssure at the leading edge.



The resultant of upper and lower surface pressures pushing on the surface of the plate will result in a normal force. The component of this normal force perpendicular to the stream will be the lift. . For small angles the normal force is the same as the lift. Lift can then be approximated as the sum of pressure forces acting normal to the surface.

$L=\int(P_{lower}-P_{upper}).dx.1$ Where for 2D sections the force acts on a 1 unit depth of wing.

The lift coefficient will then be,

$$C_L=\frac{L}{\frac{1}{2}\rho V_\infty^2 S}=\int\left(\frac{P_{lower}}{\frac{1}{2}\rho V_\infty^2}-\frac{P_{upper}}{\frac{1}{2}\rho V_\infty^2}\right).\frac{dx}{c.1} \quad , \quad C_L=\int_0^1 (C_P(lower)-C_P(upper)).\frac{dx}{c}$$

Substituting for the previous expression for pressure coefficient,

$$C_L=\int_0^1 \left(-\alpha\tan\left(\frac{-\theta}{2}\right)-\alpha^2\tan^2\left(\frac{-\theta}{2}\right)+\alpha\tan\left(\frac{\theta}{2}\right)+\alpha^2\tan^2\left(\frac{\theta}{2}\right)\right).\frac{dx}{c}$$

$$C_L=\int_0^1 \left(\alpha\tan\left(\frac{\theta}{2}\right)-\alpha^2\tan^2\left(\frac{\theta}{2}\right)+\alpha\tan\left(\frac{\theta}{2}\right)+\alpha^2\tan^2\left(\frac{\theta}{2}\right)\right).\frac{dx}{c}$$

$$C_L=\int_0^1 \left(2\alpha\tan\left(\frac{\theta}{2}\right)\right).\frac{dx}{c}$$

From the mapping geometry, $dx=dx_2$ and $dx_2=\frac{dx_2}{d\theta}.d\theta=-2\mathrm{a}\sin(\theta).d\theta$.

Also $c=4\mathrm{a}$ and when $\frac{x}{c}=0,\theta=\pi$ and $\frac{x}{c}=1,\theta=0$ so

$$C_L=\int_\pi^0 \left(2\alpha\tan\left(\frac{\theta}{2}\right)\frac{(-2\mathrm{a}\sin\theta)}{4\mathrm{a}}\right).d\theta \quad , \quad C_L=\int_0^\pi \left(\alpha\tan\left(\frac{\theta}{2}\right)(\sin\theta)\right).d\theta$$

$$C_L=2\pi\alpha \quad .$$

Pitching moment will be caused by the distribution of lift over the chord of the plate. By summing the moments caused by individual lift elements along the chord the moment at the leading edge will be found.

$$Mo=-\int (P_{lower}-P_{upper}).x.dx \quad \text{thus} \quad C_{Mo}=-\int_0^1 (C_p(lower)-C_P(upper)).\frac{x}{c}.\frac{dx}{c}$$

By definition, moment is position nose up, hence the negative sign for leading edge moment coefficient. In this case the moment arm can also be found from the geometry of the transformation,

$$x=x_2+2\mathrm{a} \quad , \quad x=2\mathrm{a}\cos(\theta)+2\mathrm{a}$$

so the substitution can be made for pressure coefficient, moment arm and integration variable.

$$C_{Mo}=-\int_0^1 \left(-\alpha\tan\left(\frac{-\theta}{2}\right)-\alpha^2\tan^2\left(\frac{-\theta}{2}\right)+\alpha\tan\left(\frac{\theta}{2}\right)+\alpha^2\tan^2\left(\frac{\theta}{2}\right)\right).\left(\frac{2\mathrm{a}+2\mathrm{a}\cos(\theta)}{c}\right).\frac{dx}{c}$$

$$C_{Mo}=-\int_\pi^0 \left(2\alpha\tan\left(\frac{\theta}{2}\right)\right).(1+\cos(\theta))\frac{2\mathrm{a}}{4\mathrm{a}}.\frac{(-2\mathrm{a}\sin(\theta))}{4\mathrm{a}}.d\theta$$

$$C_{Mo}=-\int_0^\pi \left(\frac{1}{2}\alpha\tan\left(\frac{\theta}{2}\right)\right).(1+\cos(\theta)).(\sin(\theta)).d\theta$$

$$C_{Mo}=\frac{-\pi}{2}\alpha$$

This can be used to predict pitching moment about the standard reference point at ¼ chord location.

$$C_{M\frac{1}{4}c}=C_{Mo}+C_L.\frac{0.25c}{c} \quad , \quad C_{M\frac{1}{4}c}=C_{Mo}+\frac{C_L}{4} \quad , \quad C_{M\frac{1}{4}c}=\frac{-\pi}{2}\alpha+\frac{2\pi\alpha}{4}$$

$$C_{M\frac{1}{4}c}=0$$