Analyzing Repeated Measures Data
Module 6: The Random Slope Model

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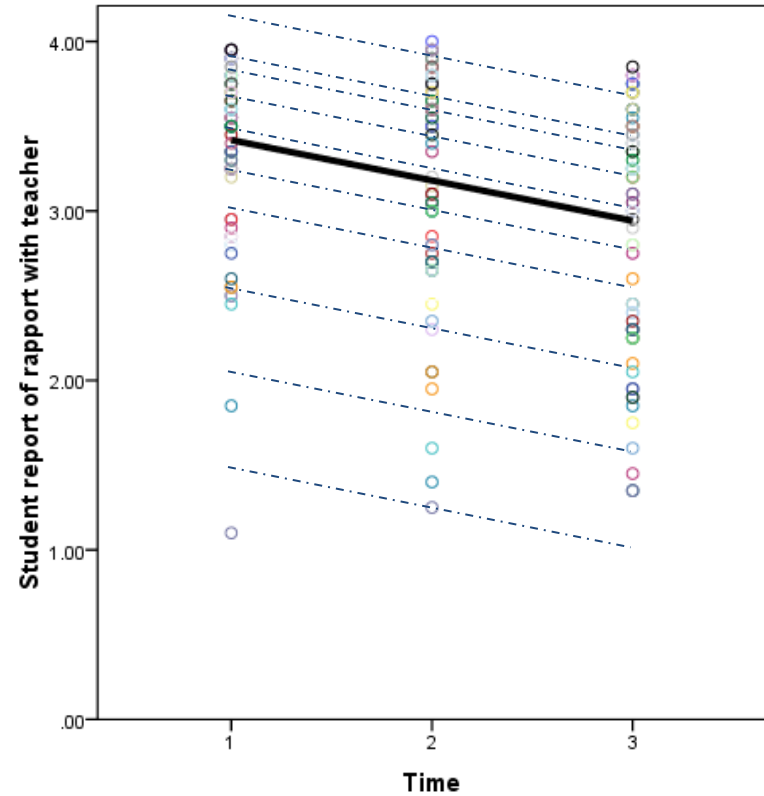
Workshop Outline: The Random Slope Model

1. The Model and Concepts
    a) Random Slopes
    b) Covariance Structures for Random Effects
    c) The Empty Random Slope model
    d) The Random Slope Model with Predictors
    e) Centering & Missing Data

2. Examples
    a) Physical Training
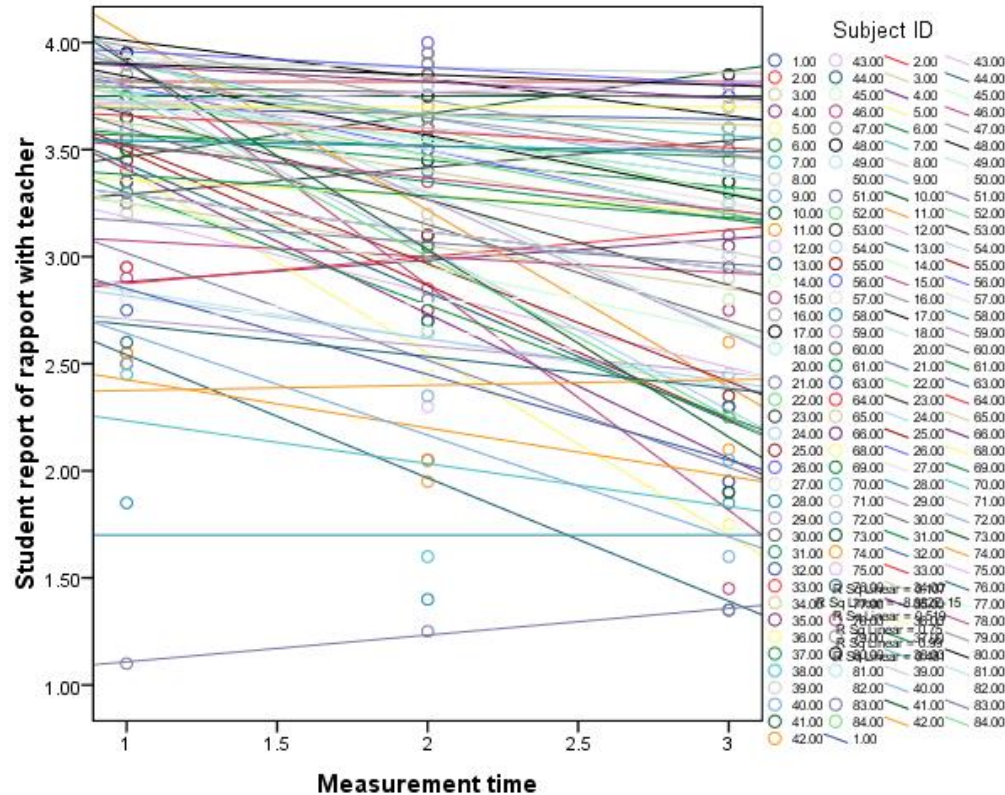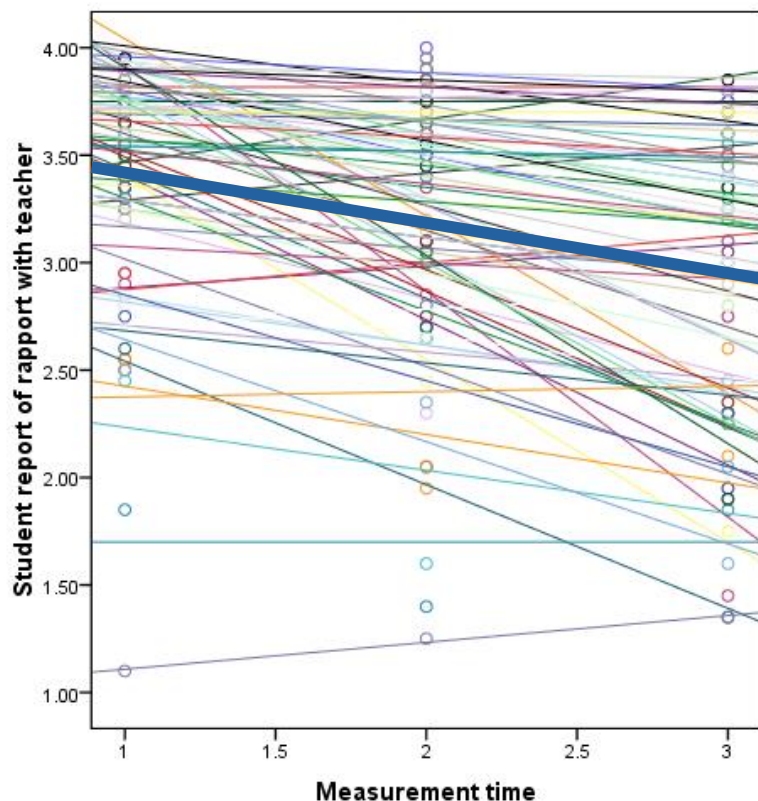   b) Swallowing

Random Slopes

The Random Intercept Model

The Random Slope for Time Model

The Random Slope for Time Model

Solution for Fixed Effects

Effect	Estimate	Standard Error	DF	t Value	Pr > |t|
Intercept	3.4144	0.06090	80	56.06	<.0001
time0	-0.2345	0.02891	80	-8.11	<.0001

Solution for Random Effects

Effect	Subject	Estimate	Std Err Pred	DF	t Value	Pr > |t|
Intercept	1	0.3470	0.1905	77	1.82	0.0724
time0	1	0.1011	0.1121	77	0.90	0.3698
Intercept	2	0.4626	0.1905	77	2.43	0.0175
time0	2	0.1213	0.1121	77	1.08	0.2825
Intercept	3	0.1755	0.1905	77	0.92	0.3597
time0	3	0.02207	0.1121	77	0.20	0.8444
Intercept	4	-0.2318	0.1905	77	-1.22	0.2275
time0	4	0.07544	0.1121	77	0.67	0.5030

The Empty Random Slope Model

Random Effect Covariance Structure (G)[a]

	Intercept | SubID	Time0 | SubID
Intercept | SubID	.224016	.015753
Time0 | SubID	.015753	.021831

$$G = Var(u_i) = \begin{bmatrix} \sigma^2_{uo} & \sigma_{uo,u1} \\ \sigma_{uo,u1} & \sigma^2_{u1} \end{bmatrix}$$

Residual Covariance (R) Matrix[a]

	Residual
Residual	.089727

$$\Sigma = Var(\varepsilon_i) = \begin{bmatrix} \sigma^2 & 0 & 0 \\ 0 & \sigma^2 & 0 \\ 0 & 0 & \sigma^2 \end{bmatrix}$$

General Specification of a Linear Mixed Model

$Rapport_{ij} = \beta_0 + \beta_1 Time + \beta_2 t0TchExp + \beta_3 Time*t0TchExp + u_{0i} + u_{1i}Time + \varepsilon_{ij}$

$$\mathbf{Y} = \mathbf{X\beta} + \mathbf{Zu} + \varepsilon$$

$$E\begin{bmatrix} u \\ \varepsilon \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$Var\begin{bmatrix} u \\ \varepsilon \end{bmatrix} = \begin{bmatrix} G & 0 \\ 0 & R \end{bmatrix}$$

$$u_i = \begin{bmatrix} u_{0i} \\ u_{1i} \end{bmatrix} \sim N(0, G)$$

$$G = Var(u_i) = \begin{bmatrix} \sigma^2_{uo} & \sigma_{uo,u1} \\ \sigma_{uo,u1} & \sigma^2_{u1} \end{bmatrix}$$
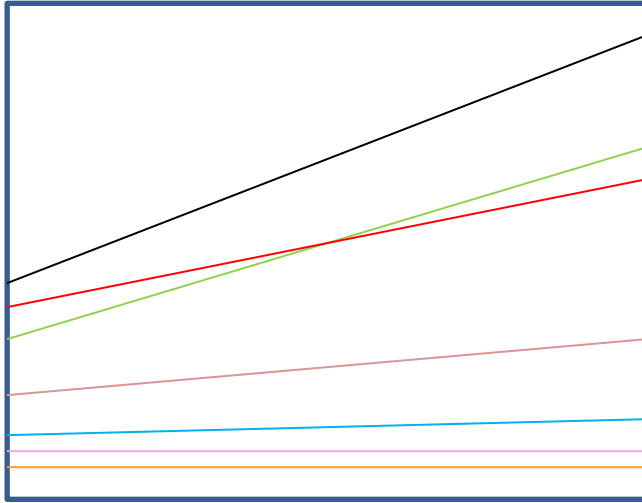
$$\varepsilon_{ij} \sim N(0, \Sigma)$$

$$\Sigma = Var(\varepsilon_i) = \begin{bmatrix} \sigma^2 & 0 & 0 \\ 0 & \sigma^2 & 0 \\ 0 & 0 & \sigma^2 \end{bmatrix}$$

Covariance Structures for G

Covariance Structures for the G Matrix

$$G = Var(u_i) = \begin{bmatrix} \sigma_{uo}^2 & \sigma_{uo,u1} \\ \sigma_{uo,u1} & \sigma_{u1}^2 \end{bmatrix}$$

Covariance Structures for the G Matrix

$$G = Var(u_i) = \begin{bmatrix} \sigma_{uo}^2 & \sigma_{uo,u1} \\ \sigma_{uo,u1} & \sigma_{u1}^2 \end{bmatrix}$$

$$G = Var(u_i) = \begin{bmatrix} \sigma_{uo}^2 & \sigma_{uo,u1} \\ \sigma_{uo,u1} & \sigma_{u1}^2 \end{bmatrix}$$

Covariance Structures for the G Matrix
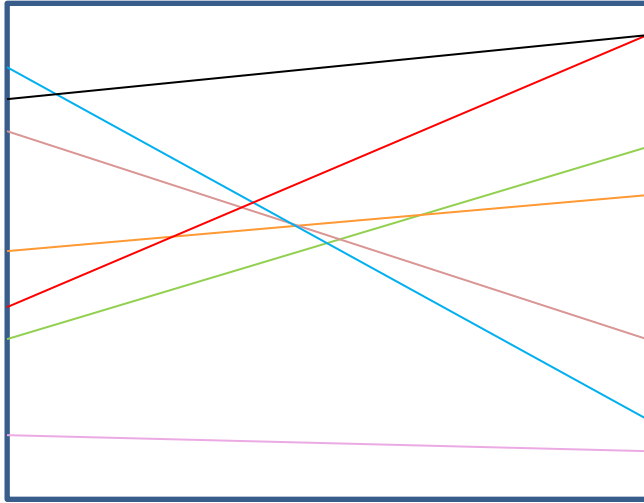
$$G = Var(u_i) = \begin{bmatrix} \sigma^2_{uo} & \sigma_{uo,u1} \\ \sigma_{uo,u1} & \sigma^2_{u1} \end{bmatrix}$$

The Empty Random Slope Model

The Empty Random Slope Model

Rapport$_{ij}$ = β_0 + β_1Time + u_{0i} + u_{1i}Time + ε_{ij}

$$u_i = \begin{bmatrix} u_{0i} \\ u_{1i} \end{bmatrix} \sim \text{N}(0, \text{G}) \qquad G = Var(u_i) = \begin{bmatrix} \sigma^2_{uo} & \sigma_{uo,u1} \\ \sigma_{uo,u1} & \sigma^2_{u1} \end{bmatrix}$$

$$\varepsilon_{ij} \sim \text{N}(0, \Sigma) \qquad \Sigma = Var(\varepsilon_i) = \begin{bmatrix} \sigma^2 & 0 & 0 \\ 0 & \sigma^2 & 0 \\ 0 & 0 & \sigma^2 \end{bmatrix}$$
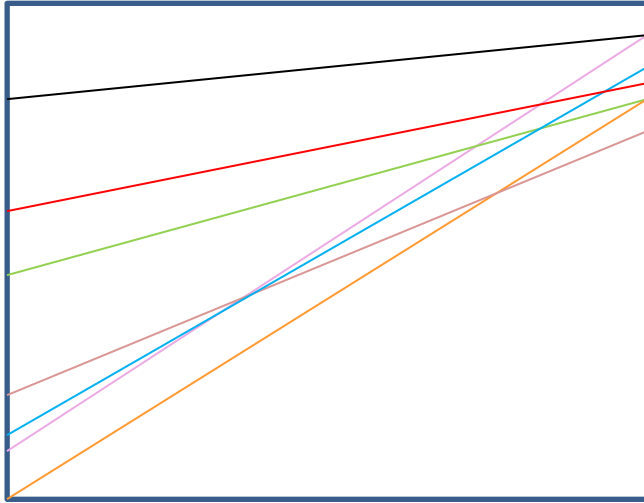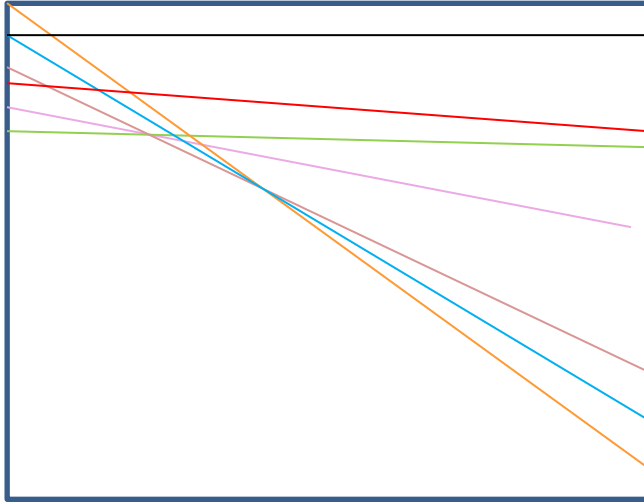
Specify the Empty Random Slope Model

$$\text{Rapport}_{ij} = \beta_0 + \beta_1 \text{Time} + u_{0i} + u_{1i}\text{Time} + \varepsilon_{ij}$$

$$u_i = \begin{bmatrix} u_{0i} \\ u_{1i} \end{bmatrix} \sim N(0, G) \qquad \varepsilon_{ij} \sim N(0, \Sigma)$$

$$G = Var(u_i) = \begin{bmatrix} \sigma^2_{uo} & \sigma_{uo,u1} \\ \sigma_{uo,u1} & \sigma^2_{u1} \end{bmatrix}$$

$$\Sigma = Var(\varepsilon_i) = \begin{bmatrix} \sigma^2 & 0 & 0 \\ 0 & \sigma^2 & 0 \\ 0 & 0 & \sigma^2 \end{bmatrix}$$
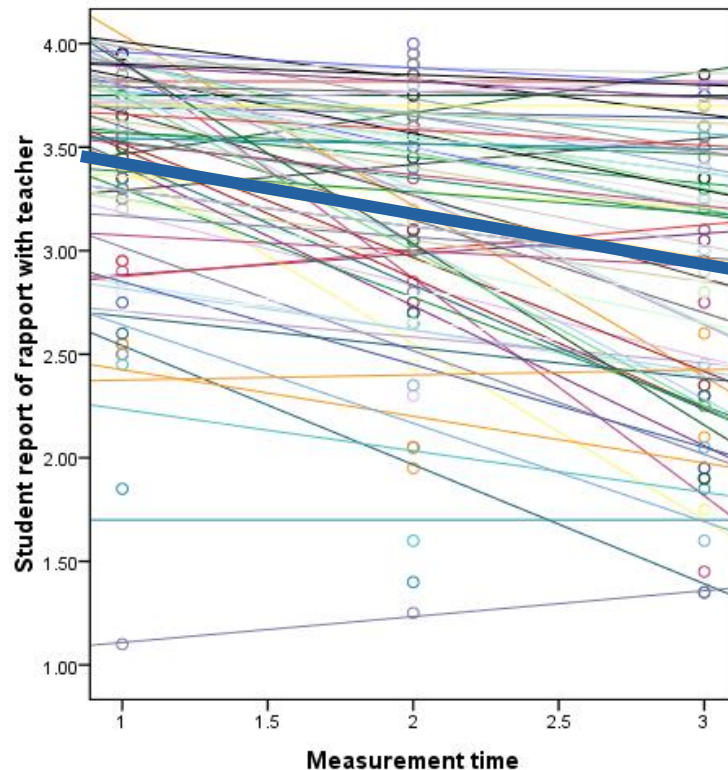
Specify the Empty Random Slope Model

Define Single Outcome Variable

Define Model Fixed Variables and whether each is categorical or continuous

$$\text{Rapport}_{ij} = \beta_0 + \beta_1 \text{Time} + u_{0i} + u_{1i}\text{Time} + \varepsilon_{ij}$$

Define the Subject i: who gets a unique trajectory

Define the random effects: which aspects of a subjects' trajectory vary—intercept and slope

$$\mathbf{u}_i = \begin{bmatrix} u_{0i} \\ u_{1i} \end{bmatrix} \sim N(0,\ G) \qquad \varepsilon_{ij} \sim N(0,\ \Sigma)$$

$$\mathbf{G} = Var(u_i) = \begin{bmatrix} \sigma_{uo}^2 & \sigma_{uo,u1} \\ \sigma_{uo,u1} & \sigma_{u1}^2 \end{bmatrix}$$

Define a covariance structure for G

$$\mathbf{\Sigma} = Var(\varepsilon_i) = \begin{bmatrix} \sigma^2 & 0 & 0 \\ 0 & \sigma^2 & 0 \\ 0 & 0 & \sigma^2 \end{bmatrix}$$

Define a covariance structure for Σ

The Empty Random Slope Model

Information Criteria[a]

-2 Restricted Log Likelihood	322.235
Akaike's Information Criterion (AIC)	330.235
Hurvich and Tsai's Criterion (AICC)	330.408
Bozdogan's Criterion (CAIC)	348.108
Schwarz's Bayesian Criterion (BIC)	344.108

The information criteria are displayed in smaller-is-better forms.

a. Dependent Variable: Rapport Student report of rapport with teacher.

The Empty Random Slope Model

Type III Tests of Fixed Effects[a]

Source	Numerator df	Denominator df	F	Sig.
Intercept	1	80.432	3142.882	.000
Time0	1	79.391	65.763	.000

a. Dependent Variable: Rapport Student report of rapport with teacher.

Estimates of Fixed Effects[a]

Parameter	Estimate	Std. Error	df	t	Sig.
Intercept	3.414383	.060904	80.432	56.061	.000
Time0	-.234455	.028911	79.391	-8.109	.000

a. Dependent Variable: Rapport Student report of rapport with teacher.

The Empty Random Slope Model

Estimates of Covariance Parameters[a]

Parameter		Estimate	Std. Error
Residual		.089727	.014313
Intercept + Time0 [subject = SubID]	UN (1,1)	.224016	.048749
	UN (2,1)	.015753	.017707
	UN (2,2)	.021831	.012843

a. Dependent Variable: Rapport.

Random Effect Covariance Structure (G)[a]

	Intercept | SubID	Time0 | SubID
Intercept | SubID	.224016	.015753
Time0 | SubID	.015753	.021831

Unstructured

a. Dependent Variable: Rapport.

Residual Covariance (R) Matrix[a]

	Residual
Residual	.089727

a. Dependent Variable: Rapport.

The Random Slope Model with Predictors

The Random Slope Model with Predictors

$$\text{Rapport}_{ij} = \beta_0 + \beta_1 \text{Time} + \beta_2 \text{t0TchExp} + \beta_3 \text{Time*t0TchExp}$$
$$+ u_{0i} + u_{1i}\text{Time} + \varepsilon_{ij}$$

The Random Slope Model with Predictors

Empty Random Slope Model

Information Criteria[a]

-2 Restricted Log Likelihood	322.235
Akaike's Information Criterion (AIC)	330.235
Hurvich and Tsai's Criterion (AICC)	330.408
Bozdogan's Criterion (CAIC)	348.108
Schwarz's Bayesian Criterion (BIC)	344.108

Random Slope Model with Predictors

Information Criteria[a]

-2 Restricted Log Likelihood	314.464
Akaike's Information Criterion (AIC)	322.464
Hurvich and Tsai's Criterion (AICC)	322.640
Bozdogan's Criterion (CAIC)	340.268
Schwarz's Bayesian Criterion (BIC)	336.268

The Random Slope Model with Predictors

Empty Random Slope Model

Type III Tests of Fixed Effects[a]

Source	Numerator df	Denominator df	F	Sig.
Intercept	1	80.432	3142.882	.000
Time0	1	79.391	65.763	.000

a. Dependent Variable: Rapport Student report of rapport with teacher.

Estimates of Fixed Effects[a]

Parameter	Estimate	Std. Error	df	t	Sig.
Intercept	3.414383	.060904	80.432	56.061	.000
Time0	-.234455	.028911	79.391	-8.109	.000

a. Dependent Variable: Rapport Student report of rapport with teacher.

Random Slope Model with Predictors

Type III Tests of Fixed Effects[a]

Source	Numerator df	Denominator df	F	Sig.
Intercept	1	78.601	20.619	.000
Time0	1	77.605	.875	.353
t0TchExp	1	78.510	11.262	.001
Time0 * t0TchExp	1	77.569	.017	.898

a. Dependent Variable: Rapport Student report of rapport with teacher.

Estimates of Fixed Effects[a]

Parameter	Estimate	Std. Error	df	t	Sig.
Intercept	1.971903	.434267	78.601	4.541	.000
Time0	-.203817	.217925	77.605	-.935	.353
t0TchExp	.444045	.132317	78.510	3.356	.001
Time0 * t0TchExp	-.008573	.066397	77.569	-.129	.898

a. Dependent Variable: Rapport Student report of rapport with teacher.

The Random Slope Model with Predictors

Empty Random Slope Model

Estimates of Covariance Parameters[a]

Parameter		Estimate	Std. Error
Residual		.089727	.014313
Intercept + Time0 [subject = SubID]	UN (1,1)	.224016	.048749
	UN (2,1)	.015753	.017707
	UN (2,2)	.021831	.012843

a. Dependent Variable: Rapport.

Random Effect Covariance Structure (G)[a]

	Intercept | SubID	Time0 | SubID
Intercept | SubID	.224016	.015753
Time0 | SubID	.015753	.021831

Unstructured

a. Dependent Variable: Rapport.

Residual Covariance (R) Matrix[a]

	Residual
Residual	.089727

a. Dependent Variable: Rapport.

Random Slope Model with Predictors

Estimates of Covariance Parameters[a]

Parameter		Estimate	Std. Error
Residual		.089937	.014417
Intercept + Time0 [subject = SubID]	UN (1,1)	.191068	.044357
	UN (2,1)	.016242	.017142
	UN (2,2)	.022339	.012976

a. Dependent Variable: Rapport.

Random Effect Covariance Structure (G)[a]

	Intercept | SubID	Time0 | SubID
Intercept | SubID	.191068	.016242
Time0 | SubID	.016242	.022339

Unstructured

a. Dependent Variable: Rapport.

Residual Covariance (R) Matrix[a]

	Residual
Residual	.089937

a. Dependent Variable: Rapport.

Model Comparison

Full Random Intercept Model

Information Criteria[a]

-2 Restricted Log Likelihood	324.102
Akaike's Information Criterion (AIC)	328.102
Hurvich and Tsai's Criterion (AICC)	328.154
Bozdogan's Criterion (CAIC)	337.004
Schwarz's Bayesian Criterion (BIC)	335.004

The information criteria are displayed in smaller-is-better forms.

a. Dependent Variable: Rapport Student report of rapport with teacher.

Covariance Parameters

Estimates of Covariance Parameters[a]

Parameter		Estimate	Std. Error
Residual		.112200	.012744
Intercept [subject = SubID]	Variance	.238225	.044439

a. Dependent Variable: Rapport Student report of rapport with teacher.

Full Random Slope Model

Information Criteria[a]

-2 Restricted Log Likelihood	314.464
Akaike's Information Criterion (AIC)	322.464
Hurvich and Tsai's Criterion (AICC)	322.640
Bozdogan's Criterion (CAIC)	340.268
Schwarz's Bayesian Criterion (BIC)	336.268

The information criteria are displayed in smaller-is-better forms.

a. Dependent Variable: Rapport Student report of rapport with teacher.

Covariance Parameters

Estimates of Covariance Parameters[a]

Parameter		Estimate	Std. Error
Residual		.089937	.014417
Intercept + Time0 [subject = SubID]	UN (1,1)	.191068	.044357
	UN (2,1)	.016242	.017142
	UN (2,2)	.022339	.012976

a. Dependent Variable: Rapport Student report of rapport with teacher.

Centering and Missing Data

Centering Time (the Random Slope Variable)
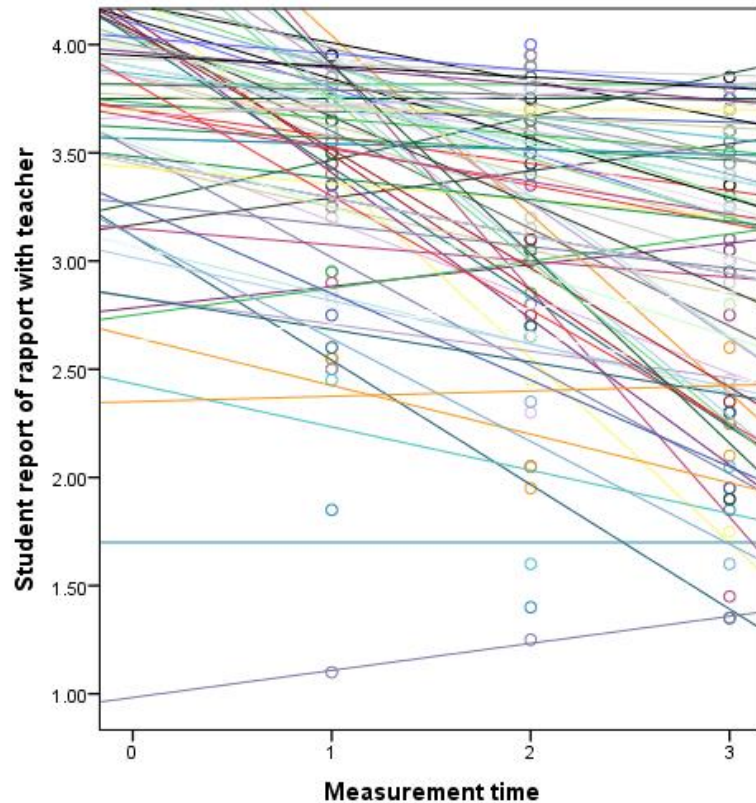
Centering Time
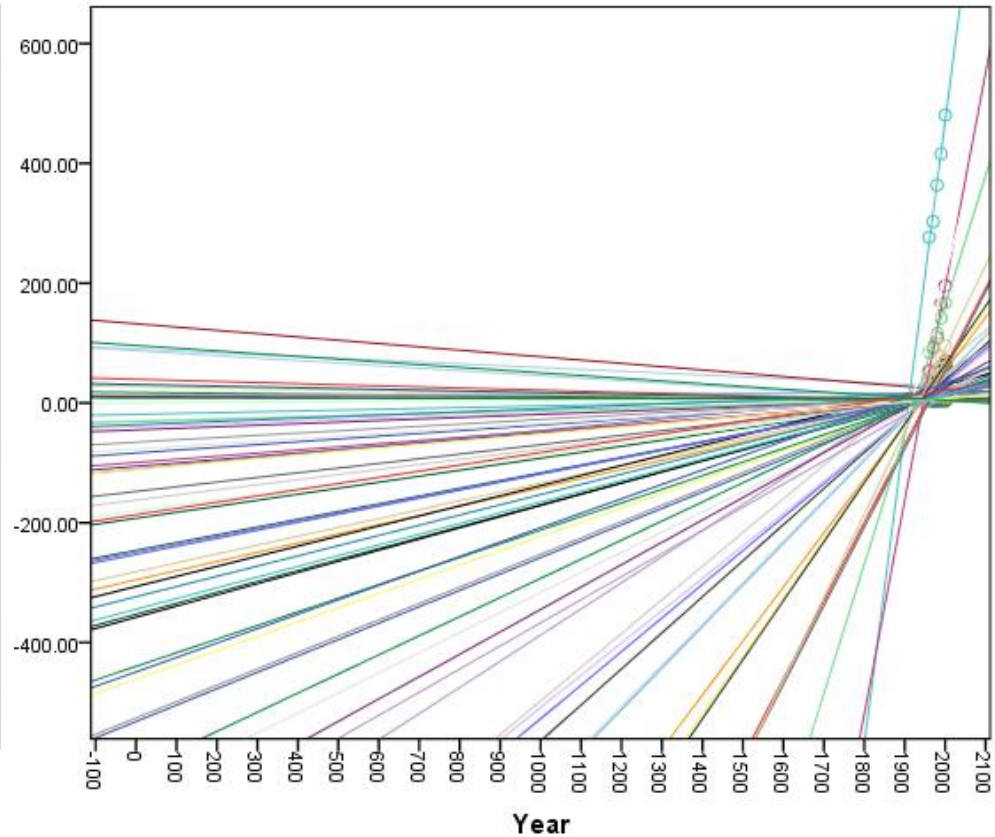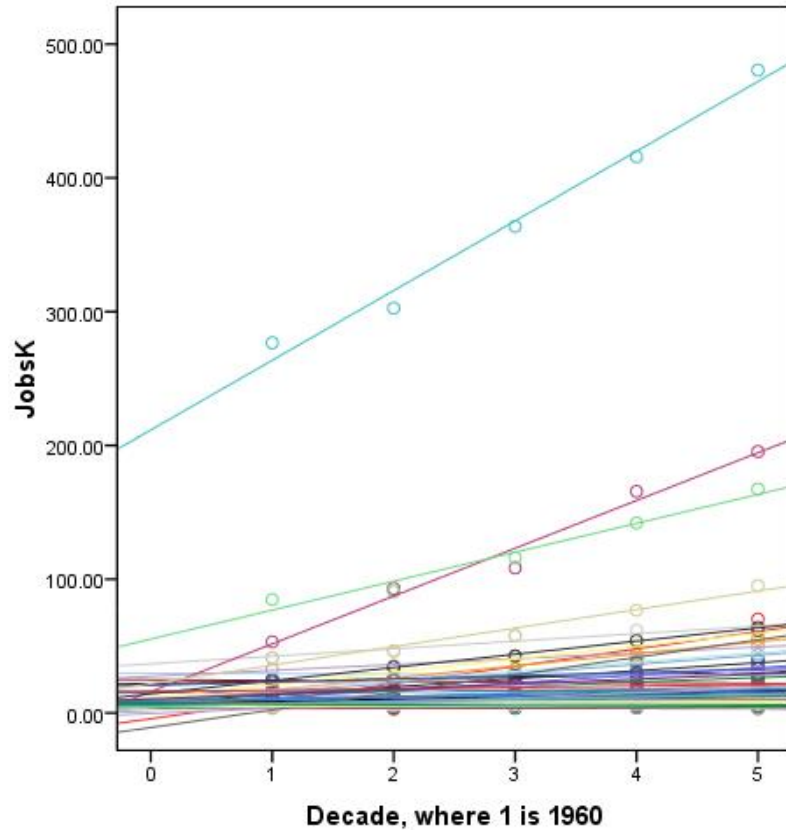
Centering Time
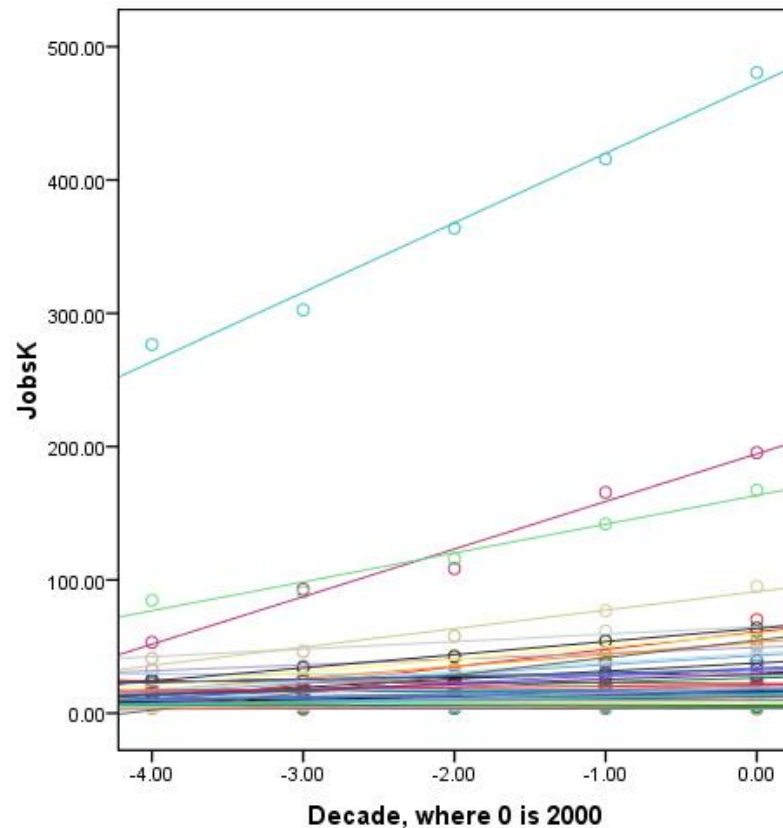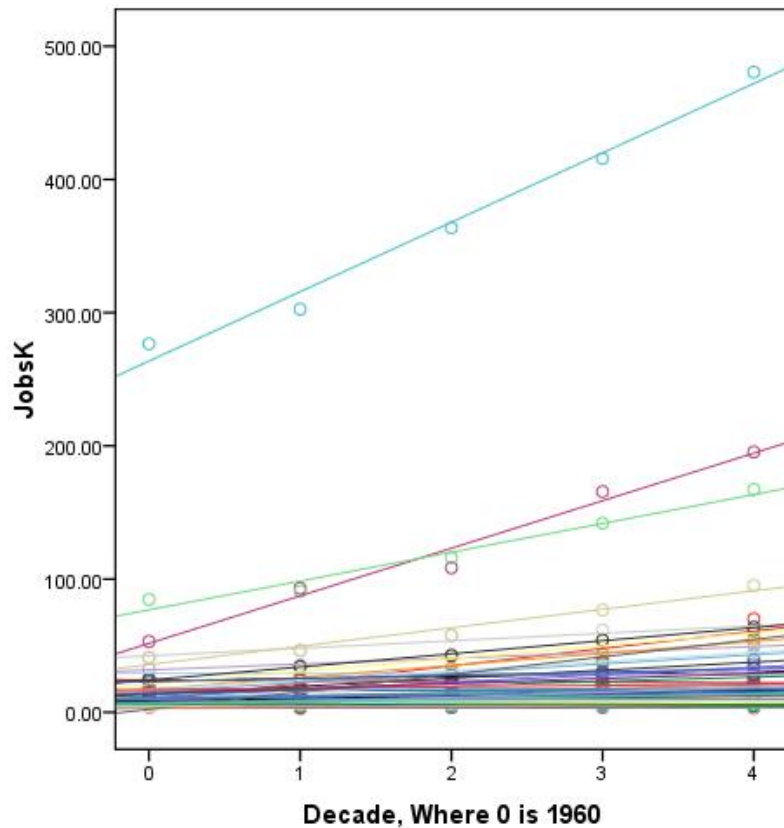
Missing Data
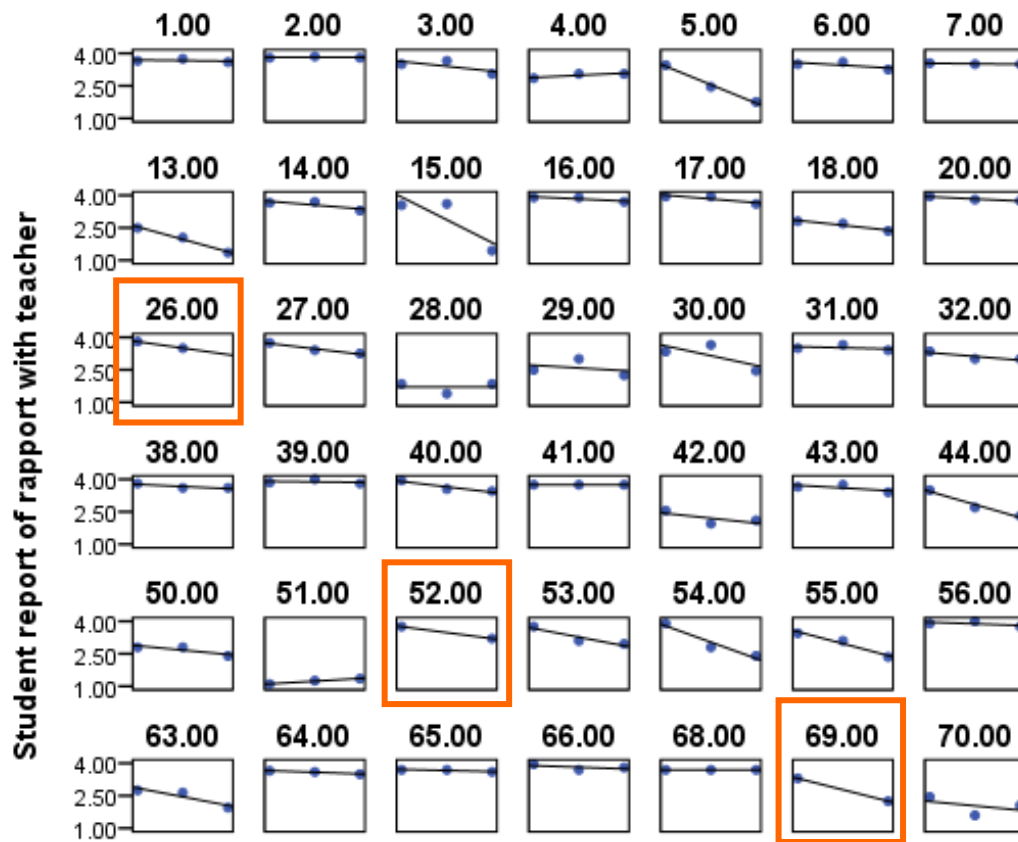
**Example: Physical Training Data
(When Random Slope Models Don't Work)**

Example: Physical Training Data (When Random Slope Models Don't Work)
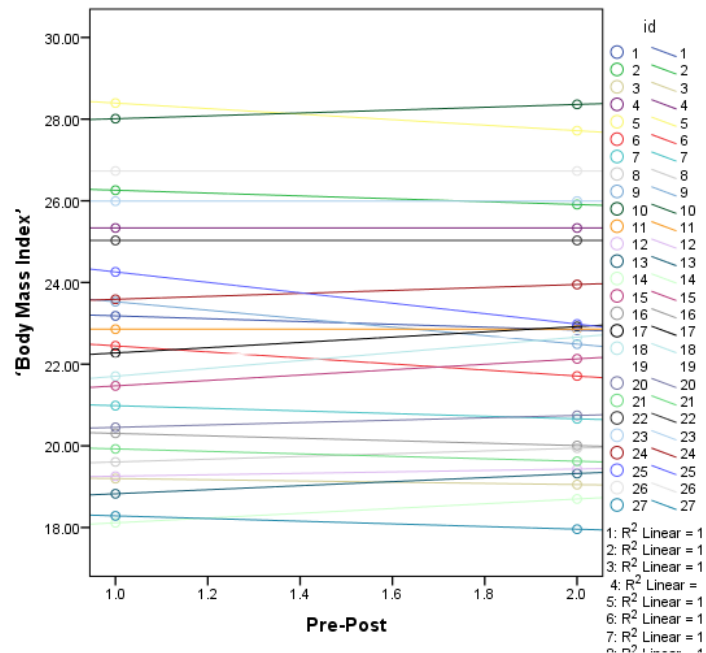
$$BMI_{ij} = \beta_0 + \beta_1 Time + \beta_2 Endurance + \beta_3 Strength$$
$$+ \beta_4 Time*Endurance + \beta_4 Time*Strength$$
$$+ u_{0i} + u_{1i} Time + \varepsilon_{ij}$$

Example: Physical Training Data (When Random Slope Models Don't Work)

Empty Random Intercept Model

Fit Statistics	
-2 Res Log Likelihood	175.4
AIC (smaller is better)	179.4
AICC (smaller is better)	179.7
BIC (smaller is better)	182.0

Empty Random Slope Model

Fit Statistics	
-2 Res Log Likelihood	178.1
AIC (smaller is better)	182.1
AICC (smaller is better)	182.3
BIC (smaller is better)	184.7

Example: Physical Training Data (When Random Slope Models Don't Work)

Empty Random Intercept Model

Convergence criteria met.

Covariance Parameter Estimates		
Cov Parm	Subject	Estimate
Intercept	id	8.4146
Residual		0.1425

Empty Random Slope Model

Convergence criteria met.

Covariance Parameter Estimates		
Cov Parm	Subject	Estimate
Intercept	id	8.4118
time	id	0
Residual		0.1480

```
NOTE: Convergence criteria met.
NOTE: Estimated G matrix is not positive definite.
NOTE: PROCEDURE MIXED used (Total process time):
      real time               0.17 seconds
      cpu time                0.04 seconds
```

Example: Swallowing Data
(Random Slope for Categorical Repeat)

Example: Swallowing Data Set (Random Intercept When the Repeat is Categorical)
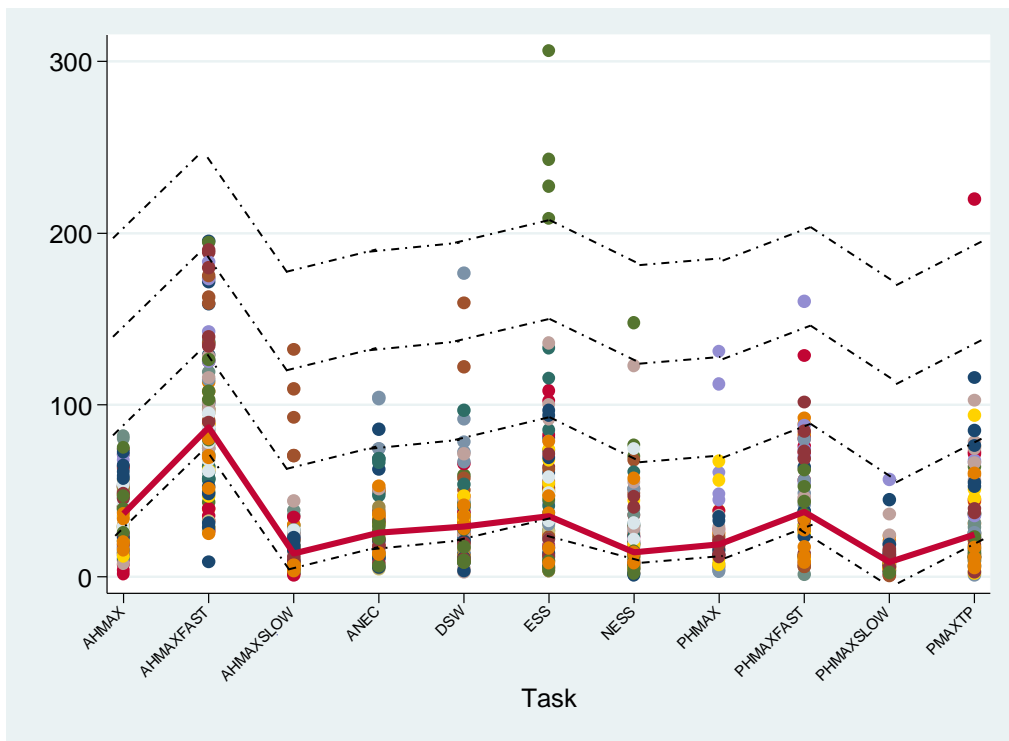
Example: Swallowing Data Set (Random Slope When the Repeat is Categorical)
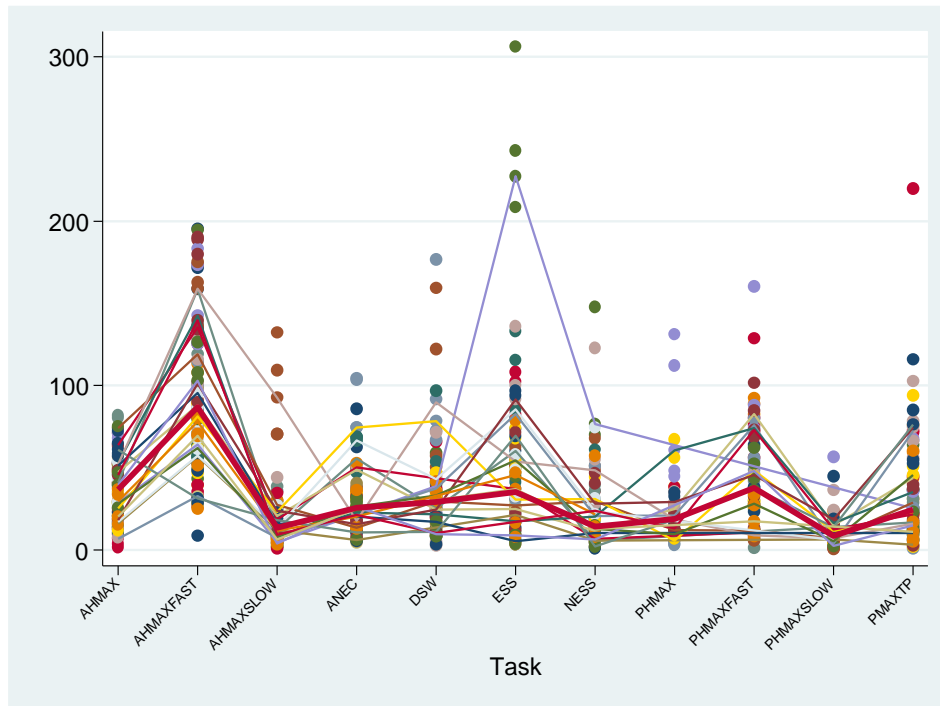
$$\text{RiseSlope}_{ij} = \beta_0 + \beta_1\text{Task1} + \beta_2\text{Task2} + \dots + \beta_{10}\text{Task10} + u_{0i} + u_{1i}\text{Task} + \varepsilon_{ij}$$

Example: Swallowing Data Set (Random Slope When the Repeat is Categorical)

Random Intercept Model

```
Akaike's information criterion and Bayesian information criterion

-------------------------------------------------------------------------
      Model |        Obs  ll(null)  ll(model)      df        AIC        BIC
------------+------------------------------------------------------------
          . |        845         .  -4091.984      13   8209.968   8271.579
-------------------------------------------------------------------------
```

Random Slope Model

```
Akaike's information criterion and Bayesian information criterion

-------------------------------------------------------------------------
      Model |        Obs  ll(null)  ll(model)      df        AIC        BIC
------------+------------------------------------------------------------
          . |        845         .  -3913.606      14   7855.212   7921.563
-------------------------------------------------------------------------
```

Example: Swallowing Data Set (Random Slope When the Repeat is Categorical)

Random Intercept Model

```
-----------------------------------------------------------------------
 Random-effects Parameters   |   Estimate   Std. Err.    [95% Conf. Interval]
-----------------------------+-----------------------------------------------
Participan~D: Identity       |
                 var(_cons)  |   98.735598  41.971577     42.917654    2.27e+02
-----------------------------+-----------------------------------------------
              var(Residual)  |   9.74e+02   48.276475     8.84e+02     1.07e+03
-----------------------------------------------------------------------
```

Random Slope Model

```
-----------------------------------------------------------------------
 Random-effects Parameters   |   Estimate   Std. Err.    [95% Conf. Interval]
-----------------------------+-----------------------------------------------
Participan~D: Identity       |
                 var(_cons)  |   35.329114  41.342096     3.564983     3.50e+02
-----------------------------+-----------------------------------------------
Task: Identity               |
                 var(_cons)  |   6.17e+02   82.206460     4.76e+02     8.01e+02
-----------------------------+-----------------------------------------------
              var(Residual)  |   4.33e+02   23.748250     3.89e+02     4.83e+02
-----------------------------------------------------------------------
```

Example: Swallowing Data Set (Random Slope When the Repeat is Categorical)

Random Intercept Model

```
----------------------------------------------------
            |       df       chi2      P>chi2
------------+---------------------------------------
RiseSlopecm |
       Task |       10      410.76     0.0000
----------------------------------------------------
```

Random Slope Model

```
----------------------------------------------------
            |       df       chi2      P>chi2
------------+---------------------------------------
RiseSlopecm |
       Task |       10      117.00     0.0000
----------------------------------------------------
```

```
-------------------------------------------------
           |              Delta-method
           |     Margin     Std. Err.
-----------+-------------------------------------
     Task  |
    AHMAX   |     37.386       3.931
 AHMAXFAST  |     93.351       3.931
 AHMAXSLOW  |     19.705       4.105
     ANEC   |     32.093       4.244
      DSW   |     35.848       4.361
      ESS   |     50.688       3.945
     NESS   |     24.384       4.005
    PHMAX   |     24.022       5.067
 PHMAXFAST  |     44.539       4.662
 PHMAXSLOW  |     10.558       4.932
    PMAXTP  |     35.982       4.321
-------------------------------------------------
```

```
-------------------------------------------------
           |              Delta-method
           |     Margin     Std. Err.
-----------+-------------------------------------
     Task  |
    AHMAX   |     37.386       6.238
 AHMAXFAST  |     93.351       6.238
 AHMAXSLOW  |     18.872       6.330
     ANEC   |     31.282       6.527
      DSW   |     35.853       6.851
      ESS   |     50.508       6.243
     NESS   |     25.145       6.264
    PHMAX   |     25.362       8.570
 PHMAXFAST  |     43.628       7.564
 PHMAXSLOW  |     11.181       7.774
    PMAXTP  |     36.141       6.873
-------------------------------------------------
```