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On the power generalized Weibull family: model for cancer censored data

Summary - In this paper we present the Power Generalized Weibull family. This family obtains by adding a second shape parameter to the Weibull family and consists of regular distributions with *bathtub shaped, unimodal* and monotone hazard rates. The Power Generalized Weibull family is suitable for modeling data that indicate nonmonotone hazard rates and can be used in survival analysis and reliability studies. Usefulness and flexibility of the family is illustrated by reanalyzing Efron's data pertaining to a head-and-neck cancer clinical trial. These data involve censoring and indicate unimodal hazard rate.

Key Words - Weibull family; Unimodal hazard rate; Cancer data.

1. Introduction

The Weibull distribution is commonly used for analyzing lifetime data. It has increasing and decreasing failure rates depending on the shape parameter. In presence of censoring, Weibull is much easier to handle compared to different monotone hazard rate models such as Gamma. But the Weibull distribution does not allow for nonmonotone failure rates, which is common in survival analysis and reliability. Several models, such as the Generalized Weibull distribution and the Exponentiated Weibull distribution have been introduced based on the Weibull distribution for modeling nonmonotone failure rate data. The Generalized Weibull family was suggested by Mudholkar *et al.* (1994). It is shown that this family contains distributions with increasing, decreasing, bathtub shaped and unimodal failure rates. The Exponentiated Weibull family was introduced by Mudholkar and Srivastava (1995). This family also accommodates unimodal, Bathtub shaped and monotone failure rates. Its applications in reliability and survival studies are illustrated using Davis's data on the bus-motor-failures and Efron's data on the head-and-neck cancer. The purpose of this paper is to study another Weibull extension termed the Power Generalized Weibull (PGW)

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family. The family was introduced by Bagdonavičius and Nikulin (2002) as a baseline distribution for the Accelerated Failure Time model. It not only contains distributions with unimodal and bathtub hazard shape, but also allows for a broader class of monotone hazard rate. The PGW family can be used as a possible alternative to the Exponentiated Weibull family for modeling lifetime data.

2. Power generalized Weibull distribution

The PGW family is most conveniently specified in terms of survival function,

$$S(t;\sigma,\nu,\gamma)=\exp\left\{1-\left(1+\left(\frac{t}{\sigma}\right)^{\nu}\right)^{\frac{1}{\gamma}}\right\},\quad (\sigma,\nu,\gamma>0),\quad t>0,\qquad (1)$$

The corresponding probability density function is

$$f(t)=\frac{\nu}{\gamma\sigma^{\nu}}t^{\nu-1}\left\{1+\left(\frac{t}{\sigma}\right)^{\nu}\right\}^{(\frac{1}{\gamma}-1)}\exp\{1-\left(1+\left(\frac{t}{\sigma}\right)^{\nu}\right)^{\frac{1}{\gamma}}\},\qquad (2)$$

and its quantile function is

$$Q(p)=\sigma\{(1-\log(1-p))^{\gamma}-1\}^{\frac{1}{\nu}},\quad 0<p<1,$$

The median is

$$Med(T)=\sigma\{(1-\log(0.5))^{\gamma}-1\}.$$

The hazard function of PGW family is

$$\alpha(t,\sigma,\nu,\gamma)=\frac{\nu}{\gamma\sigma^{\nu}}t^{\nu-1}\left\{1+\left(\frac{t}{\sigma}\right)^{\nu}\right\}^{\left(\frac{1}{\gamma}-1\right)}.$$

Particular cases of the Power Generalized Weibull distribution are:

- $\gamma=1$: the family of Weibull distribution;
- $\gamma=1$ and $\nu=1$: the family of Exponential distribution.

2.1. Density shapes

Differentiating $f(t)$ with respect to t, (2) gives

$$f'(t) = \left[(\nu-1)t^{-1} + \frac{\nu}{\gamma\sigma}\left(\frac{t}{\sigma}\right)^{\nu-1} \frac{1-\gamma-\left(1+(\frac{t}{\sigma})^{\nu}\right)^{\frac{1}{\gamma}}}{1+(\frac{t}{\sigma})^{\nu}} \right] f(t). \qquad (3)$$

If $\nu \leq 1$ then the $\frac{df(t))}{dt}$ is nonpositive implying that the p.d.f's of all PGW distributions with $\nu \leq 1$ are monotone decreasing. On the other hand, if $\nu > 1$ then $\frac{d^2 f(t))}{dt^2} \leq 0$. Hence for $\nu > 1$ the Power Generalized Weibull p.d.f's are unimodal. The shapes of the tails of the densities of the Power Generalized Weibull family can be understood by examining the limit of $f(t)$ as $t \to 0$ and $t \to \infty$ (Figure 1). We have the followings:

(i) If $\nu > 1$, then $f(t) \to 0$ as $t \to 0$ and $t \to \infty$.
(ii) If $\nu = 1$, then $f(t) \to \frac{1}{\sigma\gamma}$ as $t \to 0$, and $f(t) \to 0$ as $t \to \infty$, hence the p.d.f. is hight-tailed at the left end, similar to that of the Exponential distribution.
(iii) If $\nu < 1$, then $f(t) \to \infty$ as $t \to 0$, and $f(t) \to 0$ as $t \to \infty$, hence the family has a hight left tail asymptote at $t = 0$.

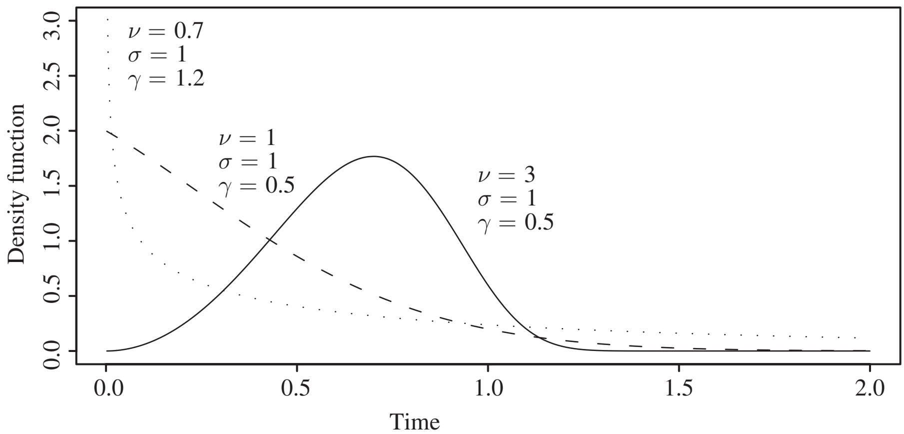

Figure 1. Typical Density Function of the Power Generalized Weibull family.

2.2. Hazard function shapes

The hazard rates of the Power Generalized Weibull family can be constant, monotone, unimodal, bathtub-shaped (Figure 2).

(a) constant for $\nu = \gamma = 1$;
(b) monotone increasing for $\nu > 1, : : \nu \geq \gamma$ and for $\nu = 1, : : 0 < \gamma < 1$;
(c) monotone decreasing for $0 < \nu < 1, : : \nu \leq \gamma$ and for $\nu = 1, : : \gamma > 1$;
(d) unimodal for $1 < \nu < \gamma$;
(e) bathtub-shaped for $0 < \gamma < \nu < 1$.

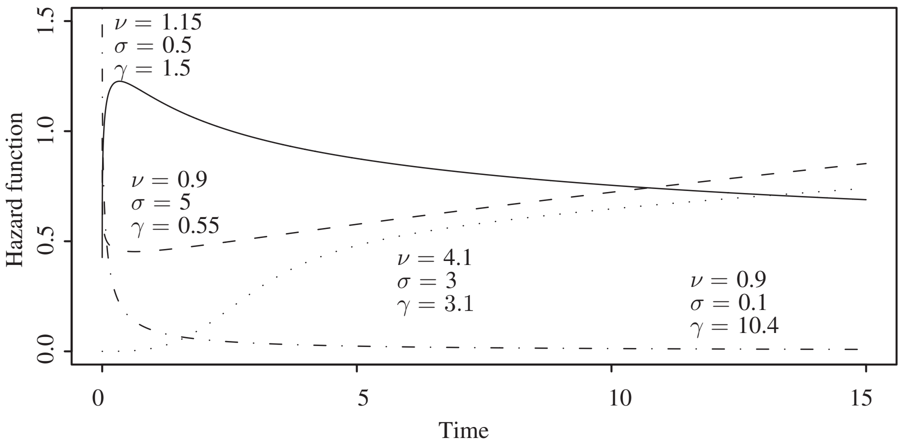

Figure 2. Typical Hazard Function of the Power Generalized Weibull family.

2.3. Moments

The following theorem gives a series representation for $E(T^k)$ for integer value of k/ν.

Theorem 1. *If T is a random variable having the pdf* (2) *for integer value of k/ν,*

$$E(T^k) = \gamma\sigma^k e(-1)^{k/\nu} \sum_{i=0}^{k/\nu} i \binom{\frac{k}{\nu}}{i} (-1)^i \Gamma(i\gamma, 1).$$

Proof. The moments of PGW distribution function given by

$$E(T^k) = \frac{\nu}{\gamma\sigma^\nu} \int_0^\infty t^{k+\nu-1} \left\{1 + \left(\frac{t}{\sigma}\right)^\nu\right\}^{\frac{1}{\gamma}-1} \exp\left\{1 - \left(1 + \left(\frac{t}{\sigma}\right)^\nu\right)^{\frac{1}{\gamma}}\right\} dt, \quad (4)$$

Substituting $u^\gamma = 1 + (\frac{t}{\sigma})^\nu$ and using the binomial expansion the (3) can be expanded to yield

$$E(T^k) = \sigma^k e \int_1^\infty (u^\gamma - 1)^{\frac{k}{\nu}} \exp\{-u\} du = \quad (5)$$

$$= \sigma^k e \sum_{i=0}^{k/\nu} \binom{\frac{k}{\nu}}{i} (-1)^{\frac{k}{\nu}-i} \int_1^\infty u^{i\gamma} \exp\{-u\} du \quad (6)$$

the last term is $\Gamma(i\gamma + 1, 1)$. So (5) can be reduced to

$$E(T^k) = \sigma^k e \sum_{i=0}^{k/\nu} \binom{\frac{k}{\nu}}{i} (-1)^{\frac{k}{\nu}-i} \Gamma(i\gamma + 1, 1)$$

By integration by parts we find that,

$$\begin{aligned} E(T^k) &= e\sigma^k \sum_{i=0}^{k/\nu} \binom{\frac{k}{\nu}}{i} (-1)^{\frac{k}{\nu}-i} \{i\gamma\Gamma(i\gamma, 1) + e^{-1}\} = \\ &= e\sigma^k \gamma (-1)^{\frac{k}{\nu}} \sum_{i=0}^{k/\nu} i \binom{\frac{k}{\nu}}{i} (-1)^i \Gamma(i\gamma, 1) + \sigma^k (-1)^{\frac{k}{\nu}} \sum_{i=0}^{k/\nu} \binom{\frac{k}{\nu}}{i} (-1)^i = \\ &= e\sigma^k \gamma (-1)^{\frac{k}{\nu}} \sum_{i=0}^{k/\nu} i \binom{\frac{k}{\nu}}{i} (-1)^i \Gamma(i\gamma, 1). \end{aligned}$$

Corollary 1. *Suppose T is a random variable with the pdf given by* (2)*. If γ and k/ν are any integer then*

$$E(T^k) = \sigma^k \gamma (-1)^{\frac{k}{\nu}} \sum_{i=1}^{k/\nu} i \binom{\frac{k}{\nu}}{i} (-1)^i (\gamma i - 1)! e_{\gamma i - 1}(1)$$

Proof. See Lemma 5.1 in Appendix. In particular, for Weibull distribution, if $\frac{k}{\nu}$ is any integer the moments are given by

$$E(T^k) = \sigma^k (-1)^{\frac{k}{\nu}} \sum_{i=1}^{k/\nu} (-1)^i \sum_{j=0}^{i-1} \binom{\frac{k}{\nu}}{i} \frac{\Gamma(i+1)}{\Gamma(j+1)}.$$

Corollary 2. *Suppose T is a random variable with the pdf given by* (2)*. If $\gamma = 1$, $\nu = 1$ (Exponential distribution) and k/ν is any integer then*

$$E(T^k) = \sigma^k (-1)^k \sum_{i=1}^{k} (-1)^i \frac{\Gamma(k+1)}{\Gamma(k-i+1)} e_{i-1}(1). \tag{7}$$

In particular, the first three moments are given by

$$\begin{aligned} E(T) &= \sigma, \\ E(T^2) &= 2\sigma^2, \\ E(T^3) &= 6\sigma^3. \end{aligned}$$

Proof. Set $\nu = 1$ and $k = 1, 2, 3$ into (6).

Corollary 3. *Suppose T is a random variable with the pdf given by* (2). *If $\nu = 1$ then*

$$E(T^k) = e\gamma\sigma^k(-1)^k \sum_{i=0}^{k} i \binom{k}{i} (-1)^i \Gamma(i\gamma, 1).$$

In particular, the first four moments are given by

$$\begin{aligned}
E(T) &= e\gamma\sigma\Gamma(\gamma, 1),\\
E(T^2) &= 2e\gamma\sigma^2[\Gamma(2\gamma, 1) - \Gamma(\gamma, 1)],\\
E(T^3) &= 3e\gamma\sigma^3[\Gamma(3\gamma, 1) - 2\Gamma(2\gamma, 1) + \Gamma(\gamma, 1)],\\
E(T^4) &= 4e\gamma\sigma^4[\Gamma(4\gamma, 1) - 3\Gamma(3\gamma, 1) + 3\Gamma(2\gamma, 1) - \Gamma(\gamma, 1)].
\end{aligned}$$

Table 1 gives the values of $\Gamma(i\gamma, 1)$ for $i = 1, 2, 3, 4$, computed using GAMIC in IMSL. Table 2 gives the values of the first moments for a random variable with pdf given by (2), where $\nu = 1$.

TABLE 1: Value of $\Gamma(i\gamma, 1)$ for $i = 1, 2, 3, 4$,$\gamma = 0.5, 1, 1.5$.

i	$\gamma = 0.5$	$\gamma = 1$	$\gamma = 1.5$
1	0.28	0.37	0.51
2	0.37	0.74	1.84
3	0.51	1.84	11.53
4	0.74	5.89	119.93

TABLE 2: Value of $E(T^k)$ for $k = 1, 2, 3, 4$, $\nu = 1$ and $\gamma = 0.5, 1, 1.5$.

	$\gamma = 0.5$ $\nu = 1$	$\gamma = 1$ $\nu = 1$	$\gamma = 1.5$ $\nu = 1$
$E(T)$	0.38σ	σ	2.07σ
$E(T^2)$	$0.24\sigma^2$	$2\sigma^2$	$10.86\sigma^2$
$E(T^3)$	$0.21\sigma^3$	$6\sigma^3$	$102.27\sigma^3$
$E(T^4)$	$0.21\sigma^4$	$24\sigma^4$	$1473.46\sigma^4$

2.4. Mode

To derive the mode we equate (7) with zero. Since $f(t)$ is always positive for $t > 0$, then we have

$$(\nu - 1)t^{-1} + \frac{\nu}{\gamma\sigma}\left(\frac{t}{\sigma}\right)^{\nu-1} \frac{1 - \gamma - \left(1 + \left(\frac{t}{\sigma}\right)^{\nu}\right)^{\frac{1}{\gamma}}}{1 + \left(\frac{t}{\sigma}\right)^{\nu}} = 0 \tag{8}$$

We discuss the following cases:

1. $\nu > 1, \gamma = 1$: We get the form of the mode from (8)

$$Mod(T) = \sigma(\frac{\nu - 1}{\nu})^{\frac{1}{\nu}}.$$

   which is the mode of the Weibull distribution for $\nu > 1$.
2. $\nu \leq 1$: This implies that $f(t)$ is decreasing and hence there is no mode.

3. Maximum likelihood estimators

In this section we discuss the maximum likelihood estimators of PGW distribution. Let $t_1, t_2, ..., t_n$ be a random sample from PGW distribution, then the log likelihood function can be written as:

$$L(\nu, \sigma, \gamma) = n(\log \nu - \log \gamma - \nu \log \sigma + 1) + (\nu - 1) \sum_{i=1}^{n} \log t_i + \left(\frac{1}{\gamma} - 1\right) \sum_{i=1}^{n} \log \left\{1 + \left(\frac{t_i}{\sigma}\right)^{\nu}\right\} - \sum_{i=1}^{n} \left\{1 + \left(\frac{t_i}{\sigma}\right)^{\nu}\right\}^{\frac{1}{\gamma}}.$$

Therefore, to obtain the MLE's of ν,σ and γ, either we can maximize the log-likelihood directly with respect to ν, σ and γ or we can solve the non-linear equations which are as follows:

$$\frac{\partial L}{\partial \nu} = \frac{n}{\nu} - n \log \sigma + \sum_{i=1}^{n} \log t_i + \left(\frac{1}{\gamma} - 1\right) \sum_{i=1}^{n} \frac{\left(\frac{t_i}{\sigma}\right)^{\nu} \log \frac{t_i}{\sigma}}{1 + \left(\frac{t_i}{\sigma}\right)^{\nu}} - \frac{1}{\gamma} \sum_{i=1}^{n} \left(\frac{t_i}{\sigma}\right)^{\nu} \log \left(\frac{t_i}{\sigma}\right) \left\{1 + \left(\frac{t_i}{\sigma}\right)^{\nu}\right\}^{\frac{1}{\gamma}} = 0,$$

$$\frac{\partial L}{\partial \gamma} = \frac{n}{\gamma} - \frac{1}{\gamma^2} \sum_{i=1}^{n} \log \left(1 + \left(\frac{t_i}{\sigma}\right)^{\nu}\right) + \frac{1}{\gamma^2} \sum_{i=1}^{n} \log \left\{1 + \left(\frac{t_i}{\sigma}\right)^{\nu}\right\} \left\{1 + \left(\frac{t_i}{\sigma}\right)^{\nu}\right\}^{\frac{1}{\gamma}} = 0,$$

$$\frac{\partial L}{\partial \sigma} = \frac{n\nu}{\sigma} + \left(1 - \frac{1}{\gamma}\right) \frac{\nu}{\sigma} \sum_{i=1}^{n} \frac{\left(\frac{t_i}{\sigma}\right)^{\nu}}{1 + \left(\frac{t_i}{\sigma}\right)^{\nu}} + \frac{\nu}{\gamma\sigma} \sum_{i=1}^{n} \left(\frac{t_i}{\sigma}\right)^{\nu} \left(1 + \left(\frac{t_i}{\sigma}\right)^{\nu}\right)^{\frac{1}{\gamma}-1} = 0.$$

We use the asymptotic normality results as follows:

$$\sqrt{n}(\hat{\theta} - \theta) \to N_3(0, I^{-1}(\theta)), \qquad \theta = (\nu, \sigma, \gamma)$$

where $I(\theta)$ is the Fisher information matrix. This asymptotic behavior remains valid if $I(\theta)$ replaced by consistent estimator $I(\hat{\theta})$ or more simply by the simple information matrix given by

$$i(\hat{\theta}) = \left(\frac{-\partial^2 L}{\partial\theta_i \partial\theta_j} \right) \Big|_{\theta=\hat{\theta}}, \qquad \theta = (\nu, \sigma, \gamma).$$

The asymptotic distribution of $\hat{\theta}$ is generally used to construct approximate confidence interval. We perform some numerical experiments to see how the MLE's and their asymptotic results work for large sample. For each sample size $n = 250, 500$, we compute the MLE's of $\theta = (\nu, \sigma, \gamma)$. We repeat this process 1000 times and compute the average estimators $AV(\hat{\theta})$, the square root of the mean squared errors SMSE($\hat{\theta}$) and the average numbers of iterations to convergence $AV(t)$ with the standard errors SE(t). The results are reported in Table 3. It is observed that the convergence has been achieved in all cases. For all the parametric values the MSE's and the biases decrease as the sample size increase. It verifies the consistency properties of the MLE's.

TABLE 3: The average and the square root of the mean squared errors of the $\theta = (\nu, \sigma, \gamma)$ and iterations to convergence from 1000 samples of size 250 and 500 generated from PGW.

n	θ $\nu = 1$	AV($\hat{\theta}$) $\nu = 1$	SMSE($\hat{\theta}$) $\nu = 1$	AV(t)	SE(t)
200	(2.50, 10.00, 0.75)	(2.37, 11.95, 0.67)	(1.33, 7.12, 0.67)	28.29	13.32
	(2.50, 5.00, 0.75)	(2.48, 5.67, 0.73)	(0.84, 2.72, 0.46)	25.42	11.12
	(0.75, 10.00, 2.50)	(0.76, 12.48, 2.52)	(0.10, 9.22, 0.58)	23.21	3.45
	(0.75, 5.00, 2.50)	(0.76, 5.87, 2.54)	(0.11, 3.97, 0.54)	21.17	3.20
	(2.00, 10.00, 2.50)	(2.02, 10.52, 2.51)	(0.32, 2.33, 0.60)	17.27	2.92
	(2.00, 5.00, 2.50)	(2.03, 5.17, 2.53)	(0.33, 1.01, 0.56)	15.32	2.74
500	(2.50, 10.00, 0.75)	(2.44, 10.62, 0.71)	(0.97, 2.47, 0.46)	23.90	6.15
	(2.50, 5.00, 0.75)	(2.49, 5.31, 0.73)	(0.44, 1.21, 0.28)	22.09	5.95
	(0.75, 10.00, 2.50)	(0.75, 11.17, 2.50)	(0.10, 7.45, 0.42)	22.61	3.31
	(0.75, 5.00, 2.50)	(0.75, 5.43, 2.52)	(0.08, 2.62, 0.40)	21.31	3.09
	(2.00, 10.00, 2.50)	(2.01, 10.18, 2.51)	(0.28, 1.56, 0.43)	15.60	2.41
	(2.00, 5.00, 2.50)	(2.01, 5.12, 2.50)	(0.24, 0.01, 0.41)	13.76	2.43

4. DATA ANALYSIS

The flexibility of Power Generalized Weibull family is shown by using it to model Efron's data given in Table 4. Efron's data from a head-and-neck cancer clinical trial consist of survival times (in days) of 51 patients in arm A who were given radiation therapy. Nine patients in arm A were lost to follow-up and were regarded as censored. Table 4 shows the data for arm A of the head-and-neck cancer study, discretized by one-month intervals. For each value of i from 1 to 47, the Table 5 shows:

n_i : the number of patients at risk at the beginning of month i,
s_i : the number of patients who died during month i.

TABLE 4: Survival Times (in days) for the patients in Arm A of the Head-and-Neck Cancer Trial.

7	34	42	63	64	74*	83	84	91	108	112	129
133	133	139	140	140	146	149	154	157	160	160	165
173	176	185*	218	225	241	248	273	277	279*	297	319*
405	417	420	440	523	523*	583	594	1101	1116*	1146	1226*
1349*	1412*	1417									

* indicates observations lost to follow-up.

TABLE 5: Data for Arm A of the Head-and-Neck Cancer Study Discretized by months.

month	n_i	s_i	month	n_i	s_i	month	n_i	s_i	month	n_i	s_i
1	51	1	16	11	0	31	7	0	46	2	0
2	50	2	17	11	0	32	7	0	47	2	1
3	48	5	18	11	1	33	7	0			
4	42	2	19	9	0	34	7	0			
5	40	8	20	9	2	35	7	0			
6	32	7	21	7	0	36	7	0			
7	25	0	22	7	0	37	7	1			
8	24	3	23	7	0	38	5	1			
9	21	2	24	7	0	39	4	0			
10	19	2	25	7	0	40	4	0			
11	16	0	26	7	0	41	4	0			
12	15	0	27	7	0	42	3	0			
13	15	0	28	7	0	43	3	0			
14	15	3	29	7	0	44	3	0			
15	12	1	30	7	0	45	3	0			

* indicates observations lost to follow-up.

Efron (1988) estimated the hazard function for each discretized interval using the logistic regression and showed that the estimated hazard rate for head-and-neck cancer data in arm A is unimodal. Later, Mudholkar *et al.* (1995) reanalyzed these data using the Exponentiated Weibull (EW) family,

$$S(t) = 1 - \left\{1 - \exp\left(-\left(\frac{t}{\sigma}\right)^{\nu}\right)\right\}^{\gamma}, \qquad \nu, \sigma, \gamma, t > 0.$$

They showed that the Exponentiated Weibull distribution provides a very good fit to the data and the estimated hazard function, in agreement with Efron's analysis, is unimodal.The MLE's of the parameters and log-likelihood for the Exponentiated Weibull distribution are as follows:

$$\hat{\nu} = 0.2944, \quad \hat{\sigma} = 0.144, \quad \hat{\gamma} = 18.0357, \quad LL = -149.6007 \tag{9}$$

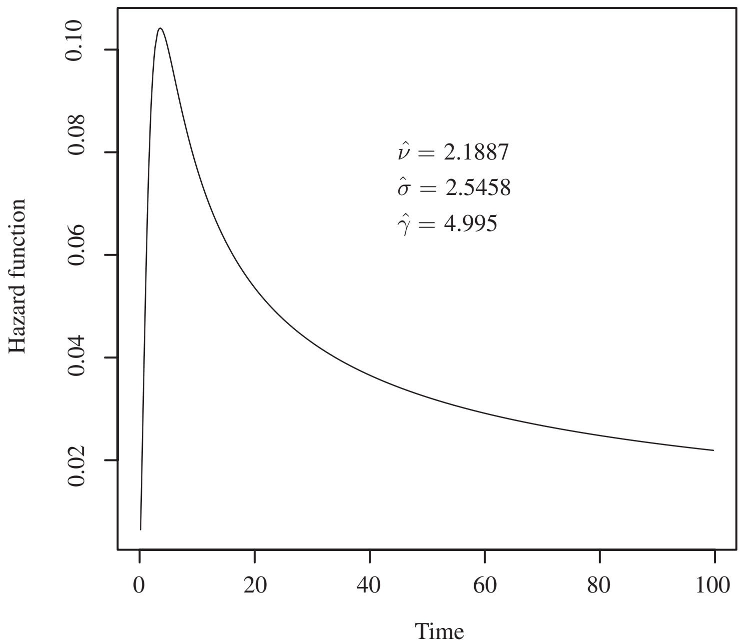

Figure 3. Estimated Hazard Function for Efron's data using Power Generalized Weibull distribution.

We propose the Power Generalized Weibull distribution for fitting to Efron's data and compare the results with the results obtained by using the Exponentiated Weibull distribution. The MLE's of the parameters and log-likelihood for the Power Generalized Weibull distribution are obtained as follows:

$$\hat{\nu} = 2.1887, \quad \hat{\sigma} = 2.5458, \quad \hat{\gamma} = 4.995, \quad LL = -148.8288 \tag{10}$$

It is observed that the estimated hazard function is unimodal ($\hat{\gamma} > \hat{\nu} > 1$) and the PGW distribution provides a better fit compared to the EW distribution in terms of log-likelihood. The goodness of fit of the models corresponding to the estimates given in (9) and (10) for arm A can be ascertained by the signed deviance residuals,

$$R_j = \sqrt{2}\,\mathrm{Sign}\,(S_j - E_j)\left[S_j \log\frac{S_j}{E_j} + (N_j - S_j)\log\frac{N_j - S_j}{N_j - E_j}\right]^{1/2},$$

discussed by McCullagh and Nelder (1983) and used by Efron(1988) and Mudholkar *et al.* (1995). For regrouped data in Table 5, N_j represent the number of patients at risk at beginning of the jth period, $j = 1, ..., 13$ and S_j represent the number of observed deaths in the jth period. For example, $N_7 = n_9 + n_{10} + n_{11} = 21 + 19 + 16 = 56$, and $S_j = s_9 + s_{10} + s_{11} = 2 + 2 + 0 = 4$.

The expected deaths for each model, $E_j = \sum_{j\text{th}:\ \text{time}:\ \text{period}} N_j \hat{h}_j$, obtained by integrating the estimated hazard function $\hat{h}$ over the given interval. If the model is correct – in the sense that it contains the true hazard function – then the $\sum_{i=1}^{13} R_i^2$ has the chi-squared distribution with (13−number of model parameters) df. The results in Table 6 show values of $\sum_{i=1}^{13} R_i^2 = 15.6053$ for the Power Generalized Weibull and $\sum_{i=1}^{13} R_i^2 = 17.3248$ for the Exponentiated Weibull with 10 df. The two distributions appear to provide similar fits. However, the PGW yields larger LL value and smaller value for signed deviance residuals. Hence, the Power Generalized Weibull offers an alternative to the Exponentiated Weibull for data set of above kind.

TABLE 6: Reanalysis of Head-and-Neck Cancer Data using Exponentiated Weibull distribution and Power Generalized Weibull distribution.

Class interval (in months)	Death N	S	Expected deaths EW	PGW	Signed deviance residuals PGW	EW
0-1	51	1	1.26	1.78	-0.24	-0.65
1-2	50	2	3.63	4.20	-0.97	-1.24
2-3	48	5	4.68	4.68	0.15	0.16
3-4	42	2	4.36	4.17	-1.32	-1.23
4-6	72	15	7.20	6.88	2.72	2.86
6-8	49	3	4.40	4.32	-0.74	-0.70
8-11	56	4	4.43	4.46	-0.22	-0.23
11-14	45	3	3.10	3.18	-0.06	-0.10
14-18	45	2	2.73	2.83	-0.48	-0.53
18-24	46	2	2.42	2.51	-0.28	-0.34
24-31	49	0	2.21	2.28	-2.13	-2.16
31-38	47	2	1.87	1.91	0.09	0.07
38-47	28	1	1.00	1.00	0.002	0.003
					$\sum_{i=1}^{13} R_i^2$	
					15.61	17.32

5. Appendix

5.1. Incomplete gamma function

The calculations of the paper make use of the incomplete gamma function defined by

$$\Gamma(a, x) = \int_x^{\infty} t^{a-1} e^{-t} dt$$

The properties of this function are:

Lemma 1. *If $a \geq 1$ is any integer,*

$$\Gamma(a, x) = (a-1)! \exp(-x) e_{a-1}(x)$$

where $e_{a-1}(x) = \sum_{k=0}^{a-1} \frac{x^k}{k!}$ denotes the exponential sum function. If $a = 0$

$$\Gamma(0, x) = \begin{cases} -Ei(-x) - i\pi, & \text{if } x < 0, \\ -Ei(-x), & \text{if } x > 0. \end{cases}$$

where $Ei(-x) = -\int_{-x}^{\infty} \frac{e^t dt}{t}$ is exponential integral.

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