

# Lecture 5

## Epipolar Geometry



1891

Professor Silvio Savarese

*Computational Vision and Geometry Lab*

# Lecture 5

## Epipolar Geometry

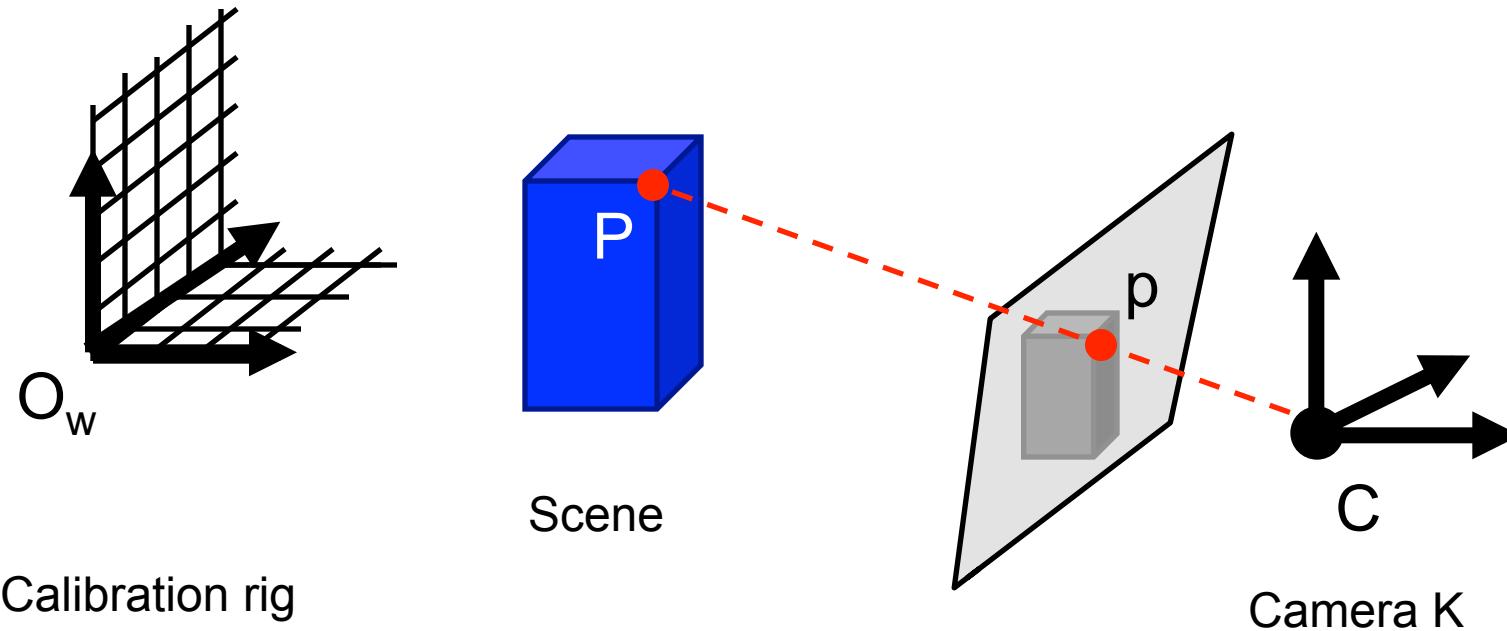


- Why is stereo useful?
- Epipolar constraints
- Essential and fundamental matrix
- Estimating  $F$
- Examples

**Reading:** [AZ] Chapter: 4 “Estimation – 2D perspective transformations  
Chapter: 9 “Epipolar Geometry and the Fundamental Matrix Transformation”  
Chapter: 11 “Computation of the Fundamental Matrix  $F$ ”

[FP] Chapter: 7 “Stereopsis”  
Chapter: 8 “Structure from Motion”

# Recovering structure from a single view



From calibration rig

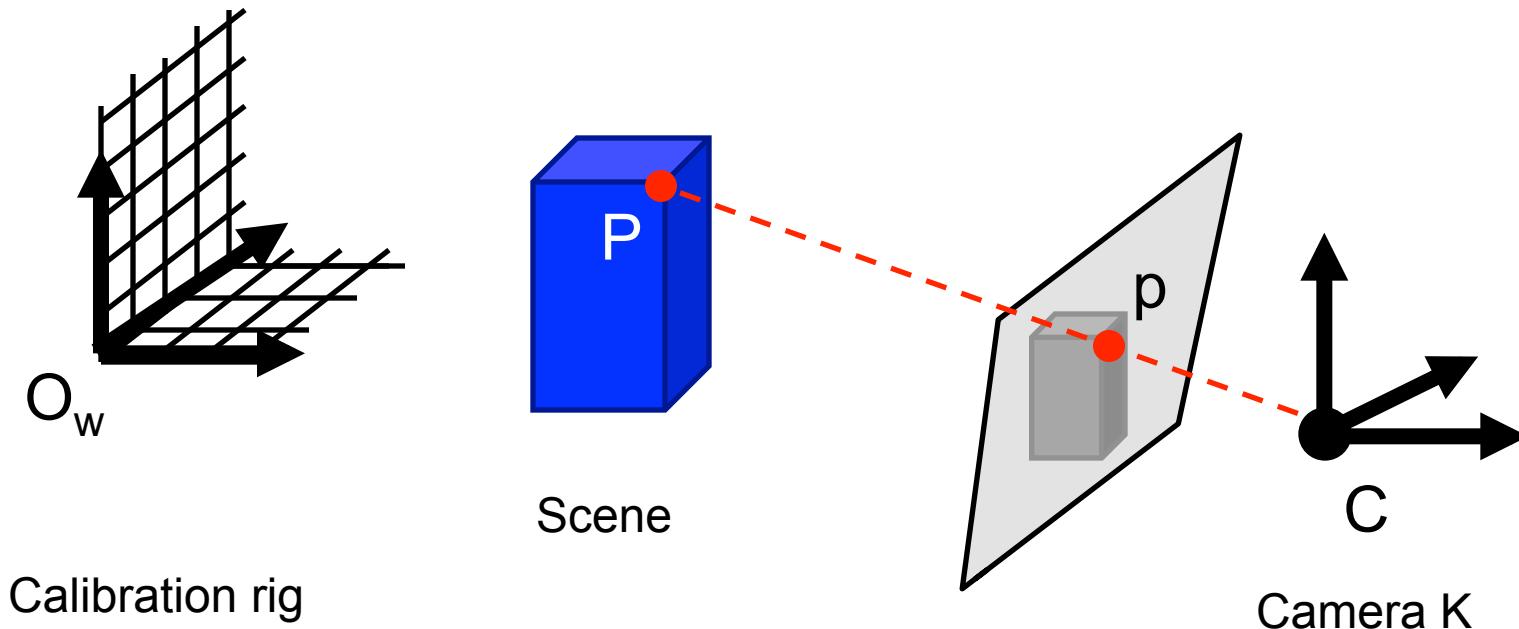
→ location/pose of the rig,  $K$

From points and lines at infinity  
+ orthogonal lines and planes

→ structure of the scene,  $K$

Knowledge about scene (point correspondences, geometry of lines & planes, etc...)

# Recovering structure from a single view



Why is it so difficult?

Intrinsic ambiguity of the mapping from 3D to image (2D)

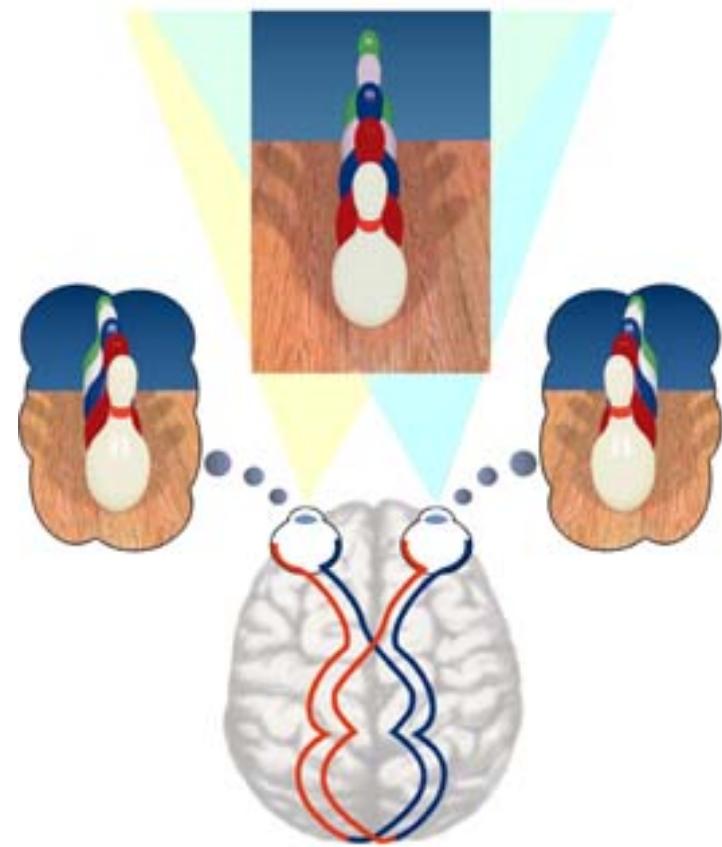
# Recovering structure from a single view

Intrinsic ambiguity of the mapping from 3D to image (2D)



Courtesy slide S. Lazebnik

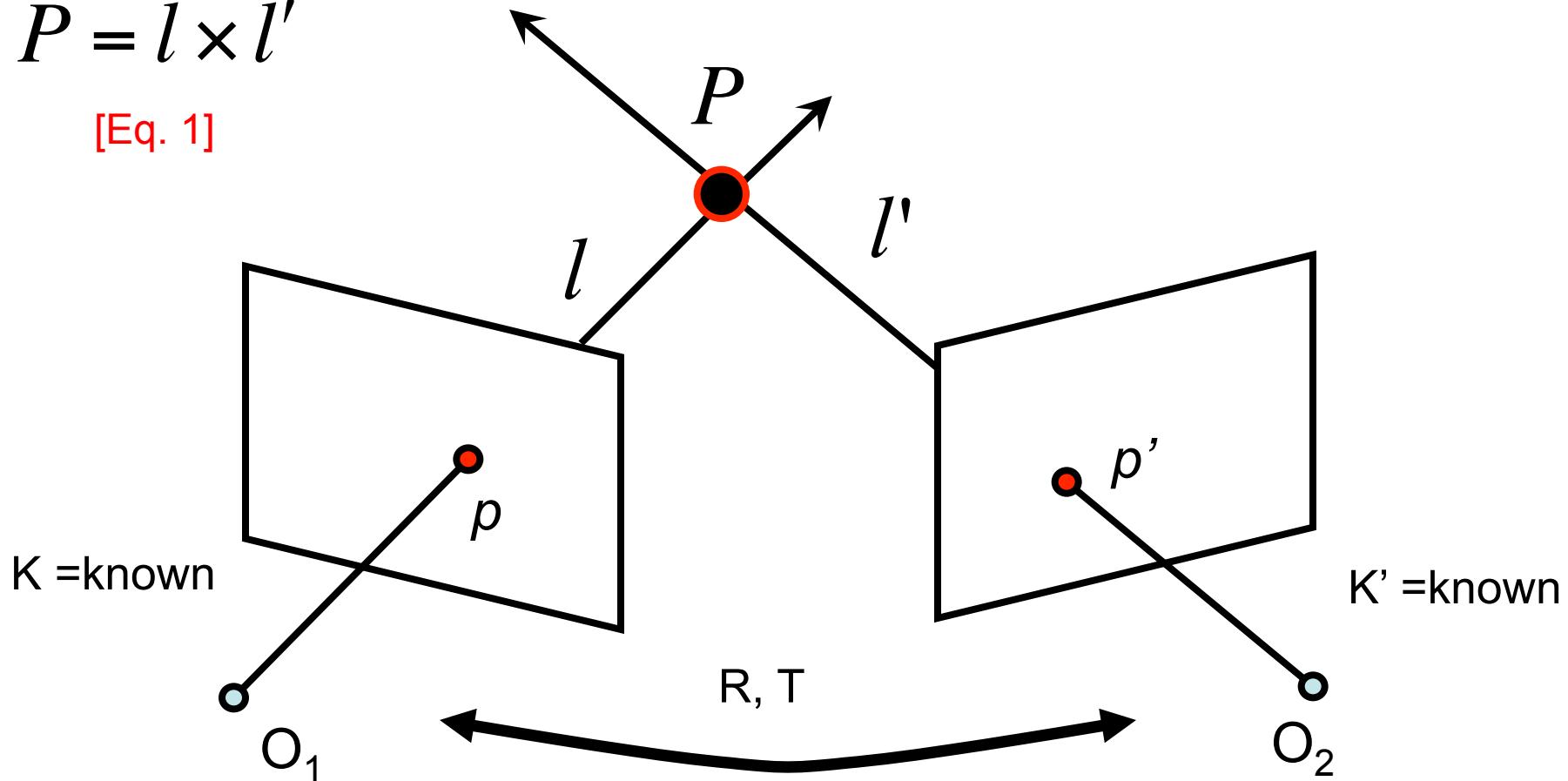
# Two eyes help!



# Two eyes help!

$$P = l \times l'$$

[Eq. 1]

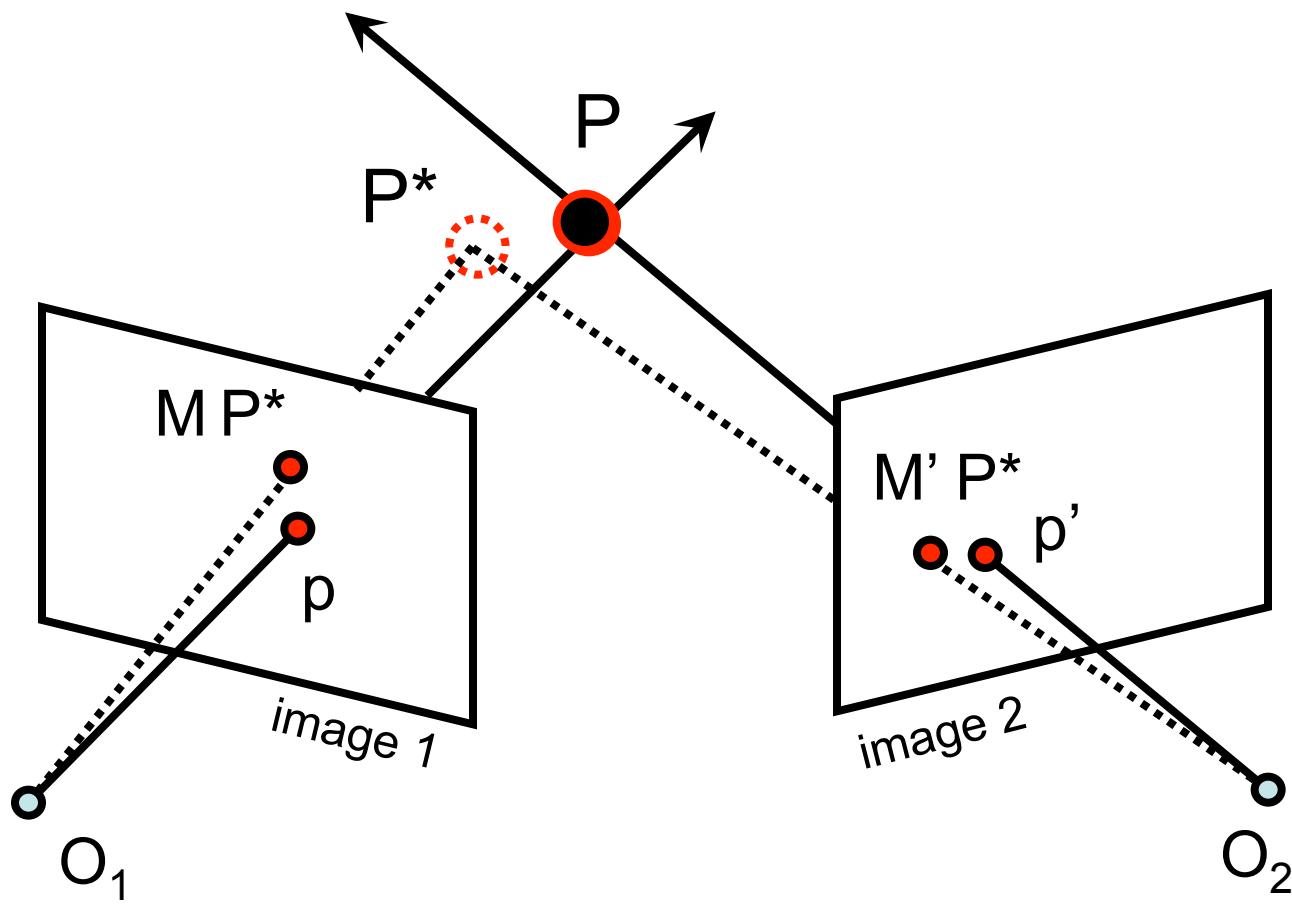


This is called **triangulation**

# Triangulation

- Find  $P'$  that minimizes

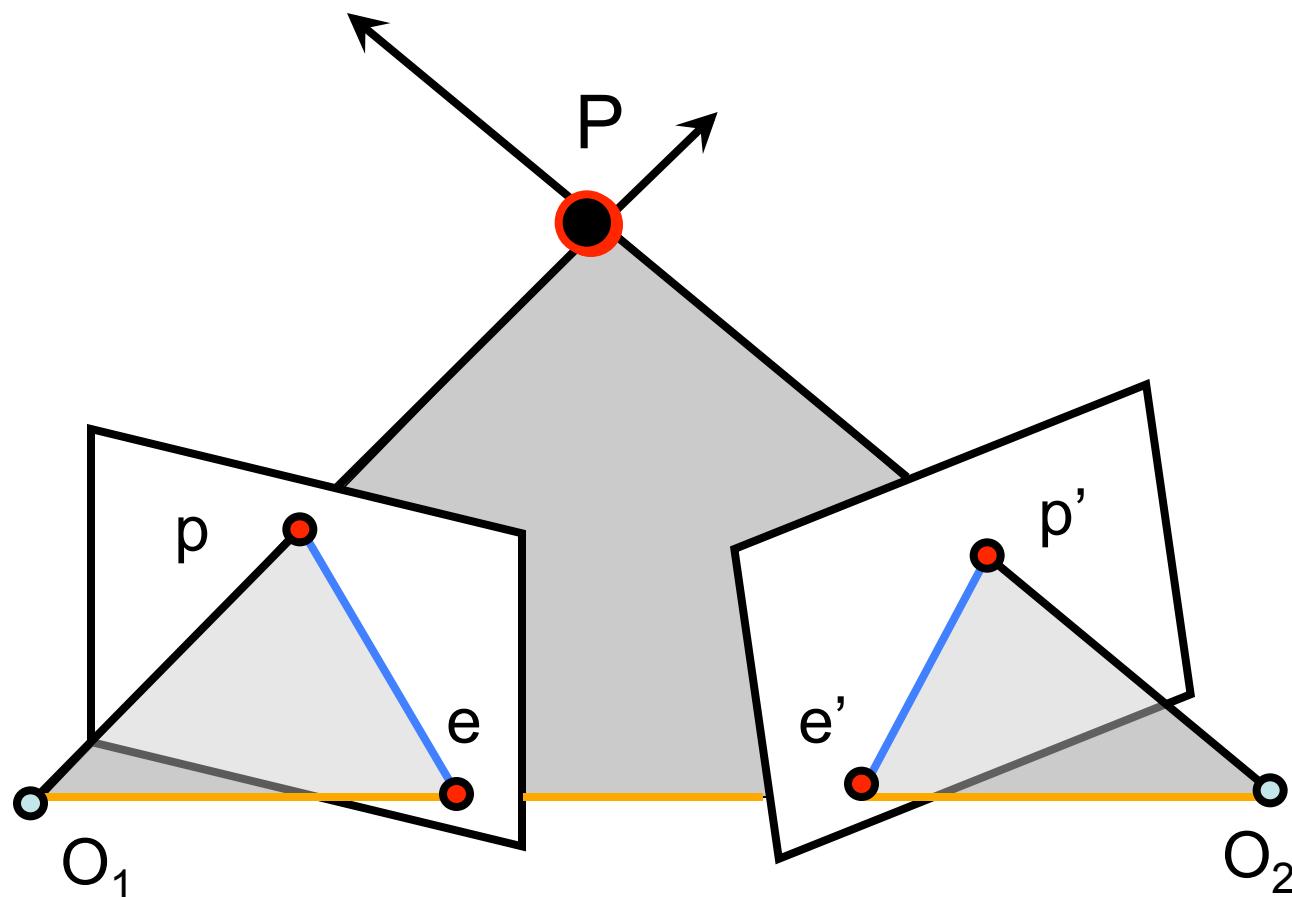
$$d(p, M P^*) + d(p', M' P^*) \quad [\text{Eq. 2}]$$



# Stereo-view geometry

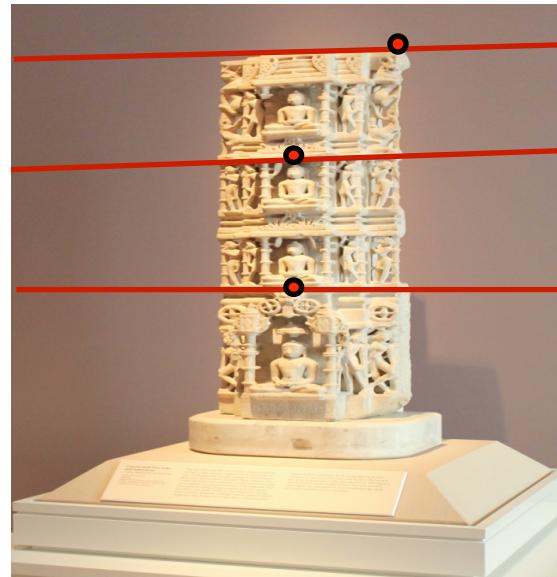
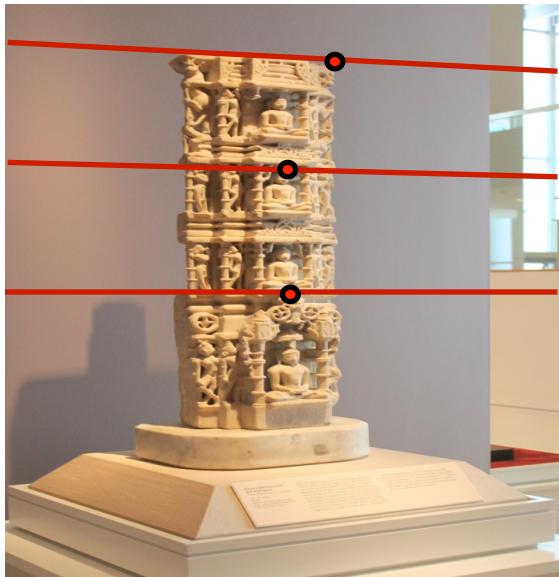
- **Correspondence:** Given a point  $p$  in one image, how can I find the corresponding point  $p'$  in another one?
- **Camera geometry:** Given corresponding points in two images, find camera matrices, position and pose.
- **Scene geometry:** Find coordinates of 3D point from its projection into 2 or multiple images.

# Epipolar geometry

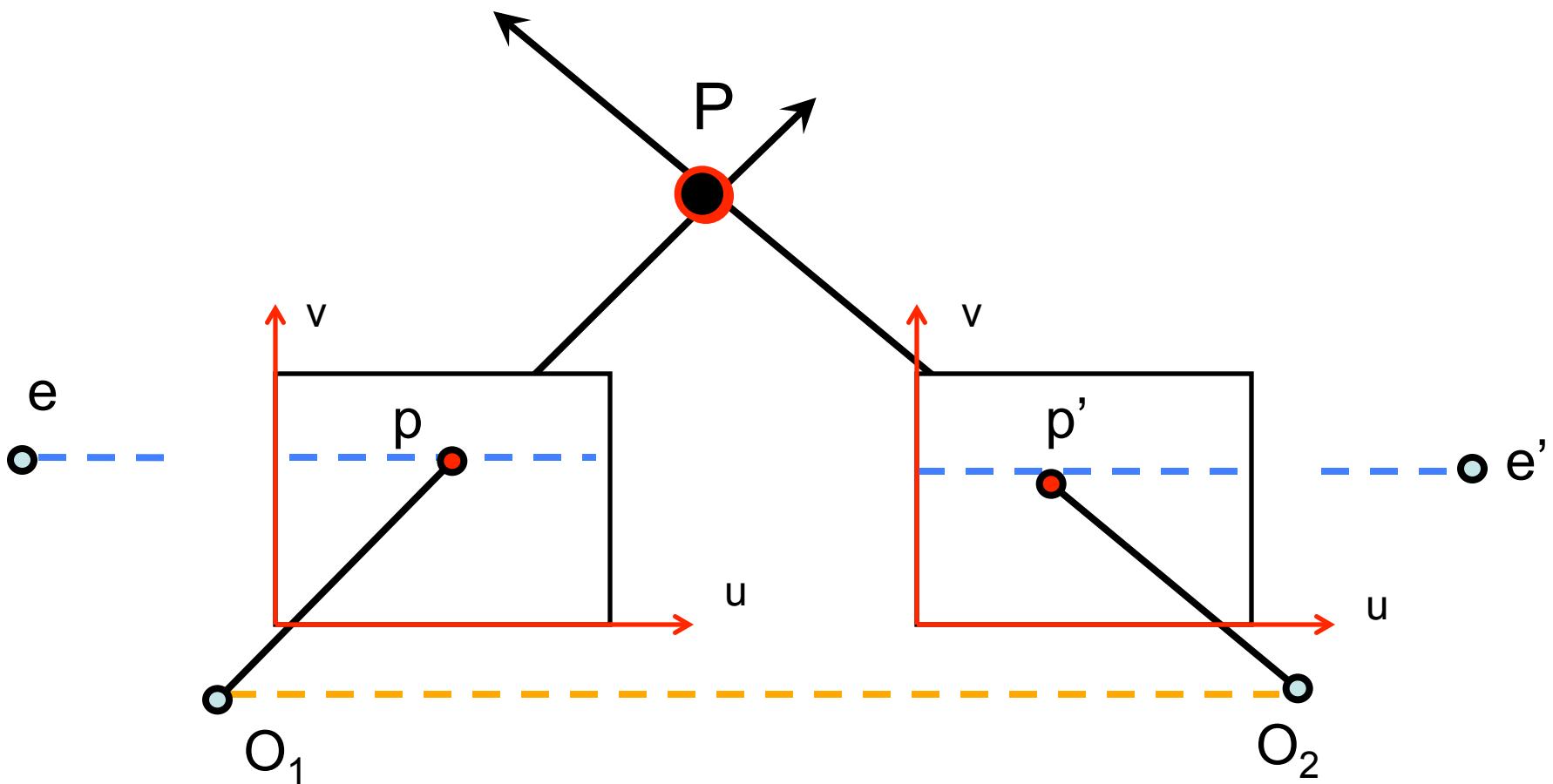


- Epipolar Plane
- Baseline
- Epipolar Lines
- Epipoles  $e, e'$ 
  - = intersections of baseline with image planes
  - = projections of the other camera center

# Example of epipolar lines



# Example: Parallel image planes

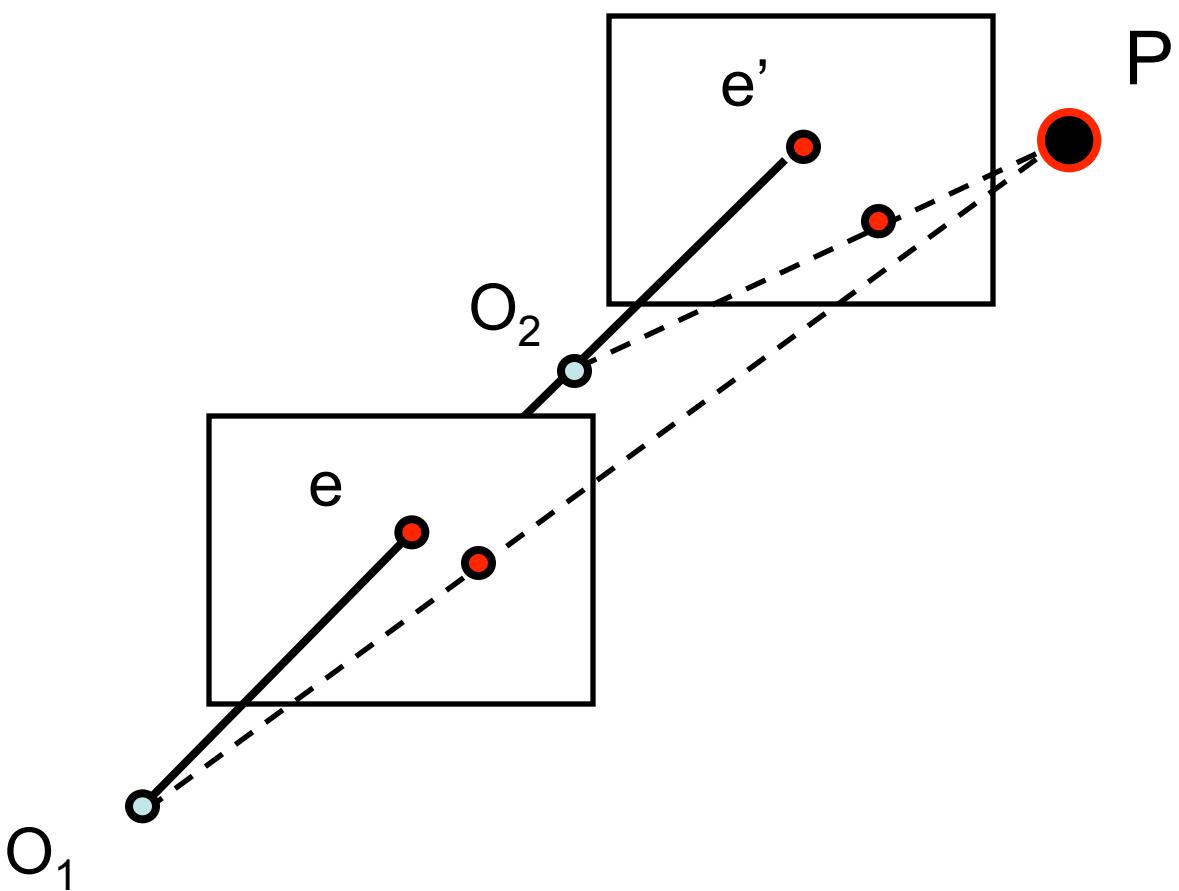


- Baseline intersects the image plane at infinity
- Epipoles are at infinity
- Epipolar lines are parallel to  $u$  axis

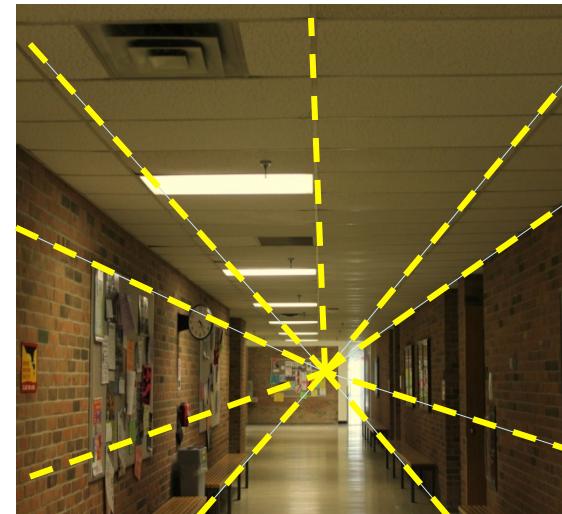
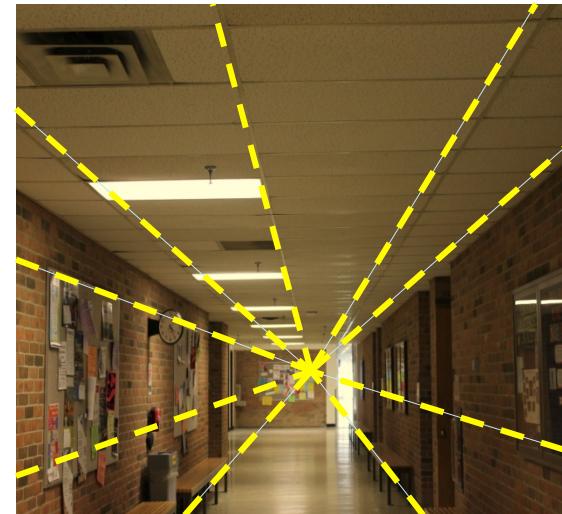
# Example: Parallel Image Planes



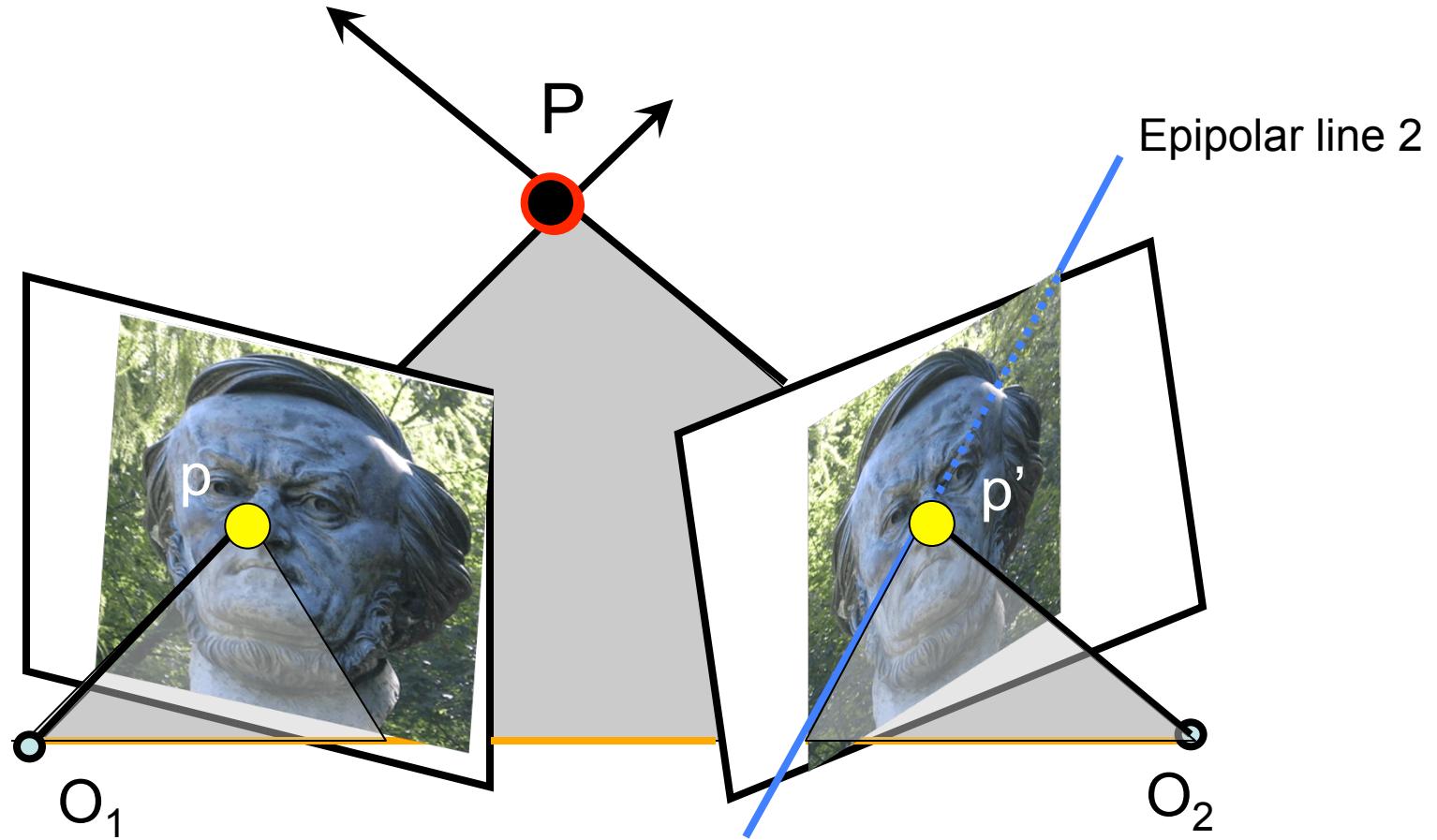
# Example: Forward translation



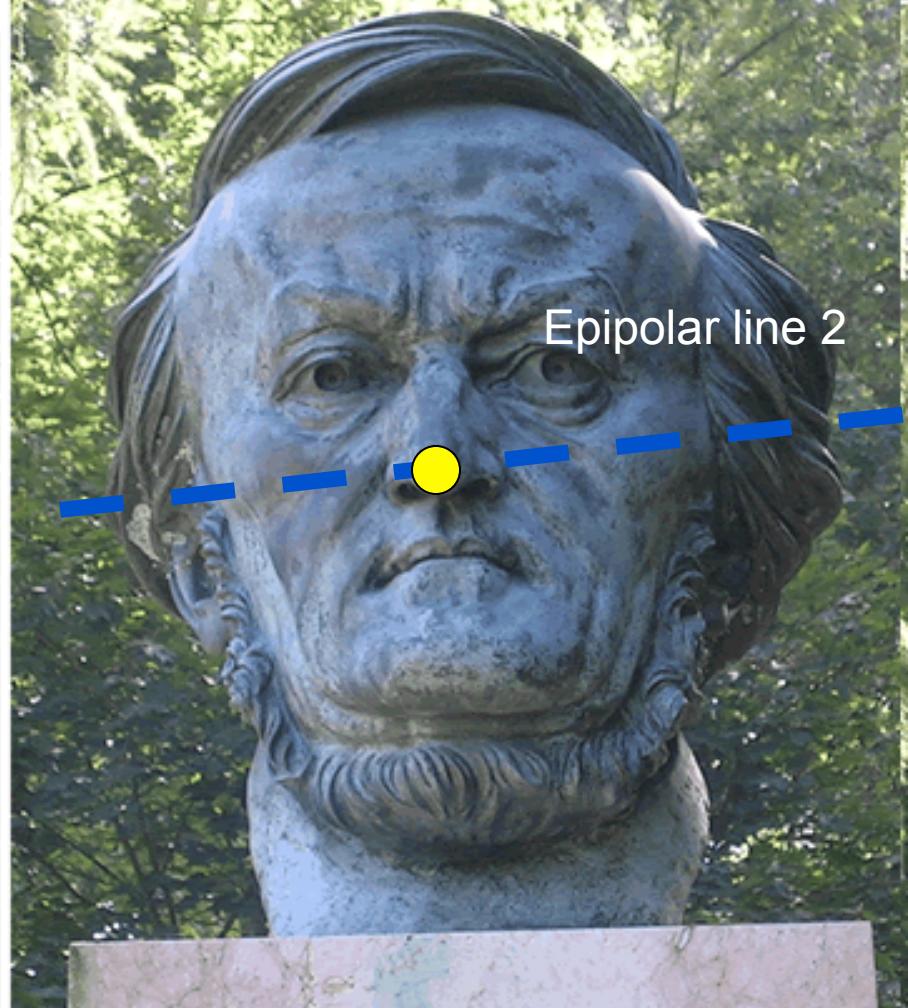
- The epipoles have same position in both images
- Epipole called FOE (focus of expansion)



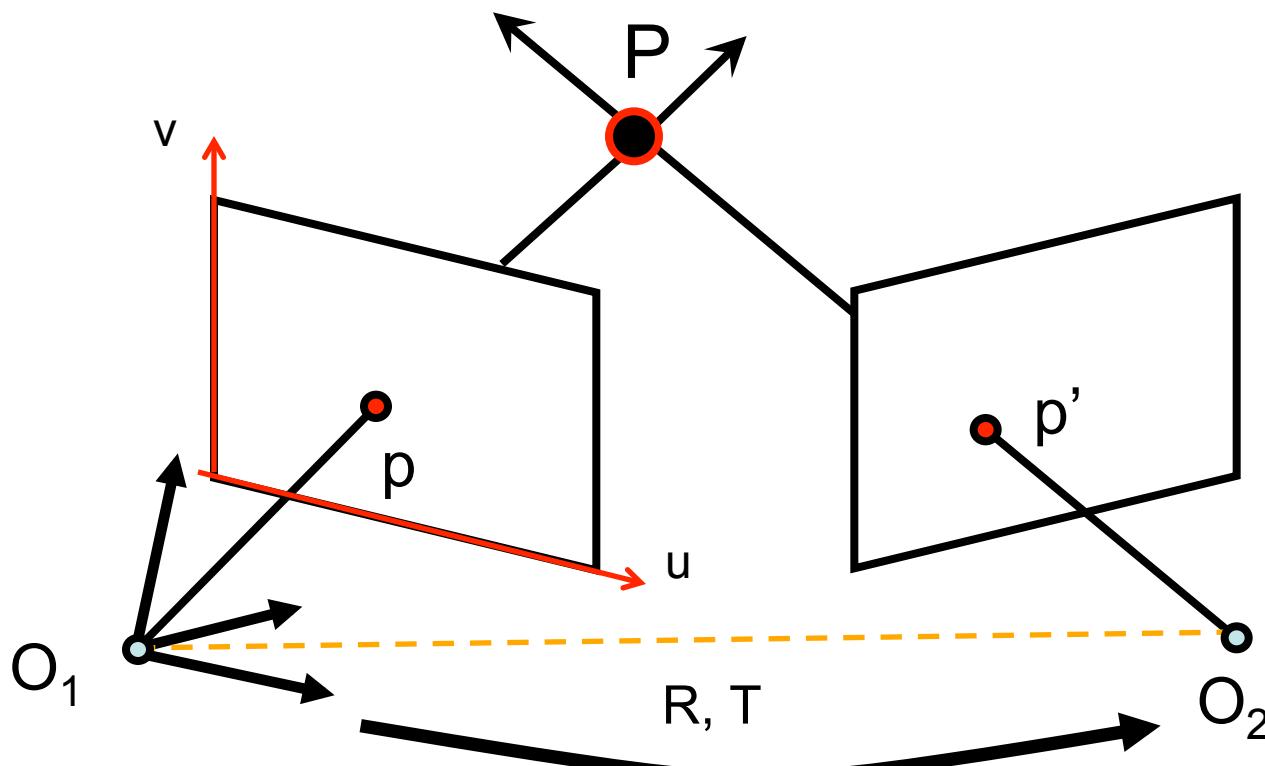
# Epipolar geometry



# Epipolar Constraint



# Epipolar Constraint



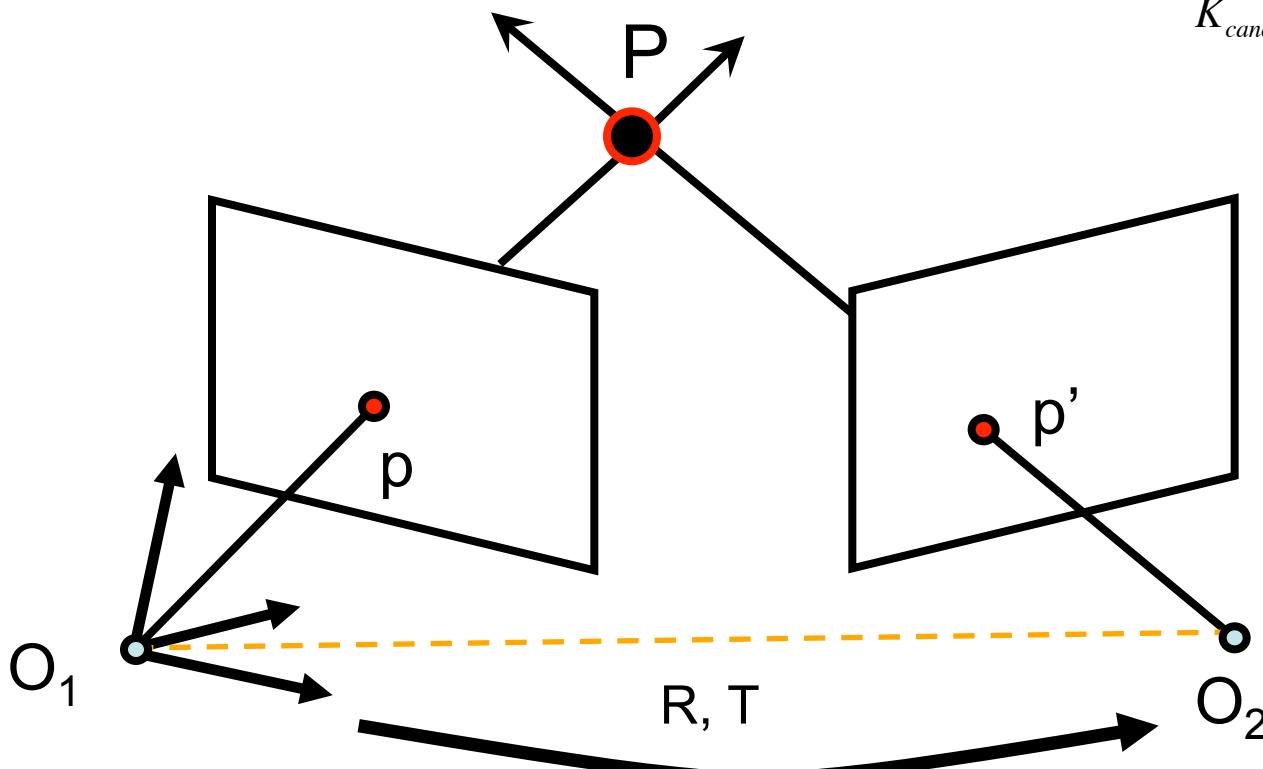
$$M = K \begin{bmatrix} I & 0 \end{bmatrix}$$

$$M P = \begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = p \quad [\text{Eq. 3}]$$

$$M' = K' \begin{bmatrix} R^T & -R^T T \end{bmatrix}$$

$$M' P = \begin{bmatrix} u' \\ v' \\ 1 \end{bmatrix} = p' \quad [\text{Eq. 4}]$$

# Epipolar Constraint



$$K_{\text{canonical}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$M = K \begin{bmatrix} I & 0 \end{bmatrix}$$

$$\downarrow$$

$$M = [I \quad 0] \quad [\text{Eq. 5}]$$

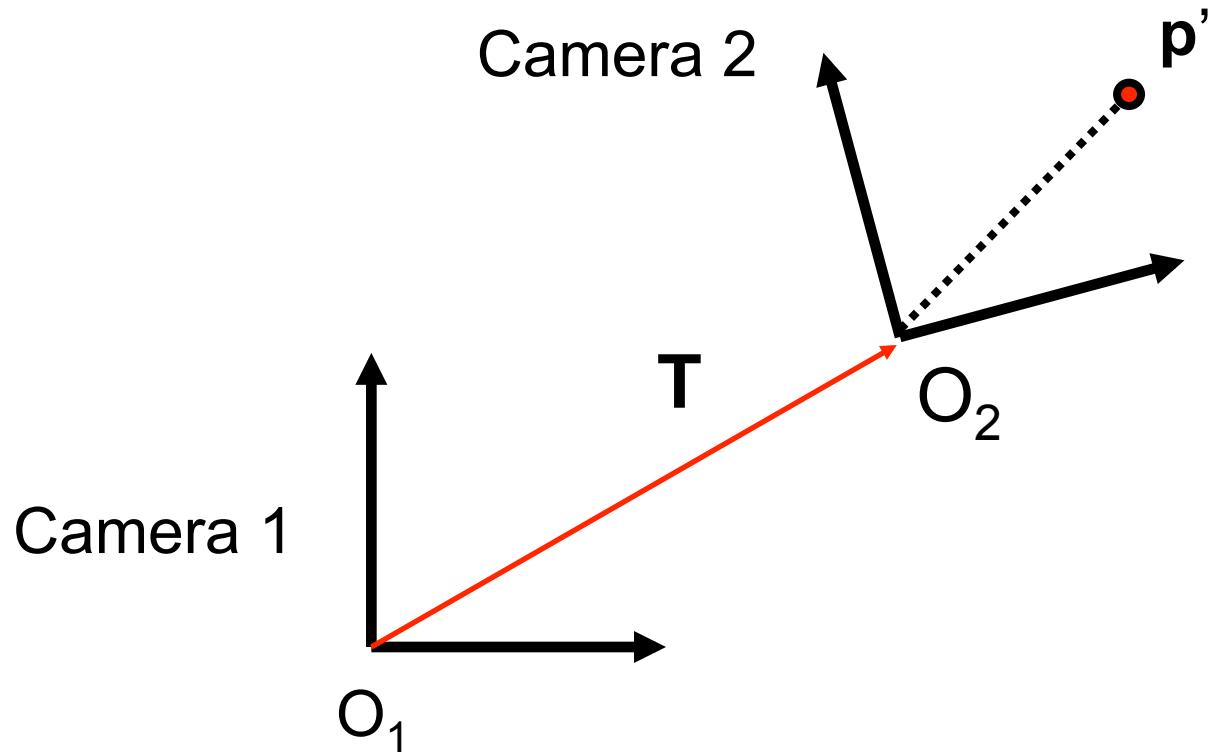
$K = K'$  are known  
(canonical cameras)

$$M' = K' \begin{bmatrix} R^T & -R^T T \end{bmatrix}$$

$$\downarrow$$

$$M' = \begin{bmatrix} R^T & -R^T T \end{bmatrix} \quad [\text{Eq. 6}]$$

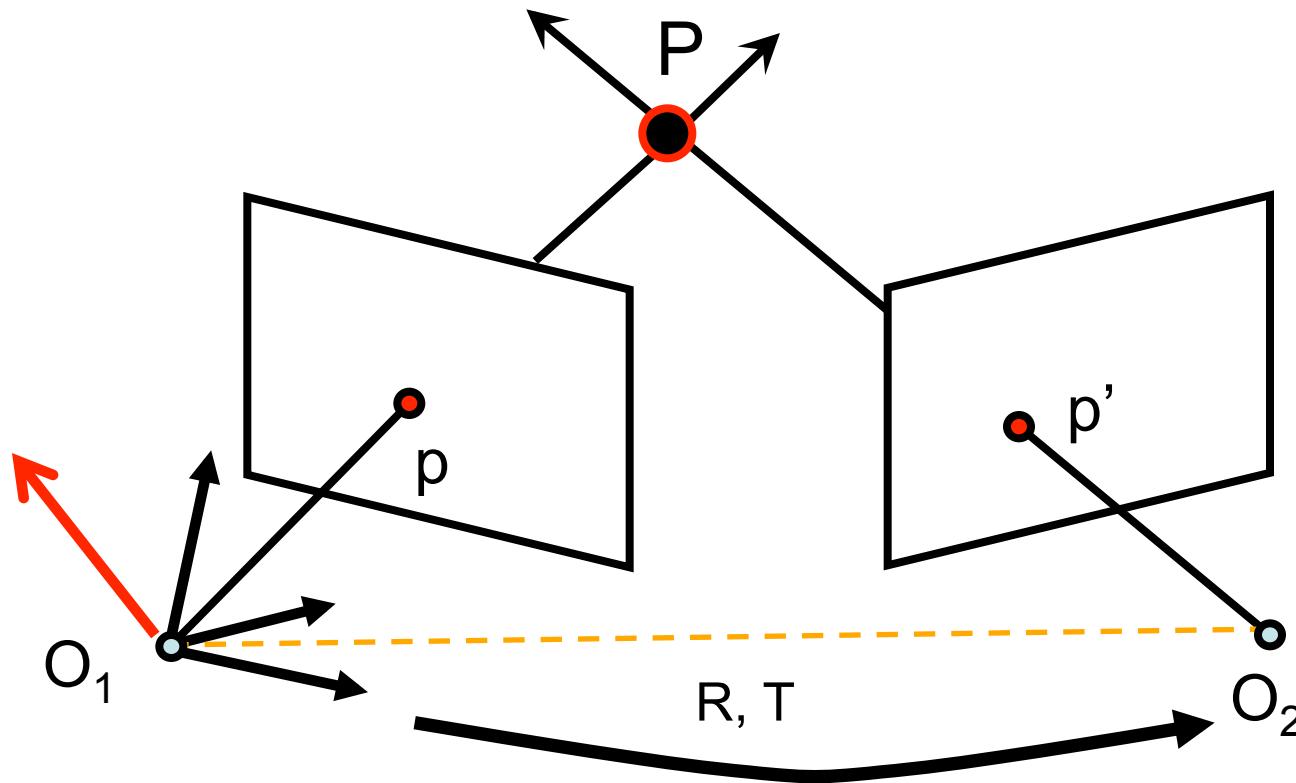
# The cameras are related by $R$ , $T$



$T = O_2$  in the camera 1 reference system

$R$  is the rotation matrix such that a vector  $p'$  in the camera 2 is equal to  $R p'$  in camera 1.

# Epipolar Constraint



$p'$  in first camera reference system is  $= R p'$

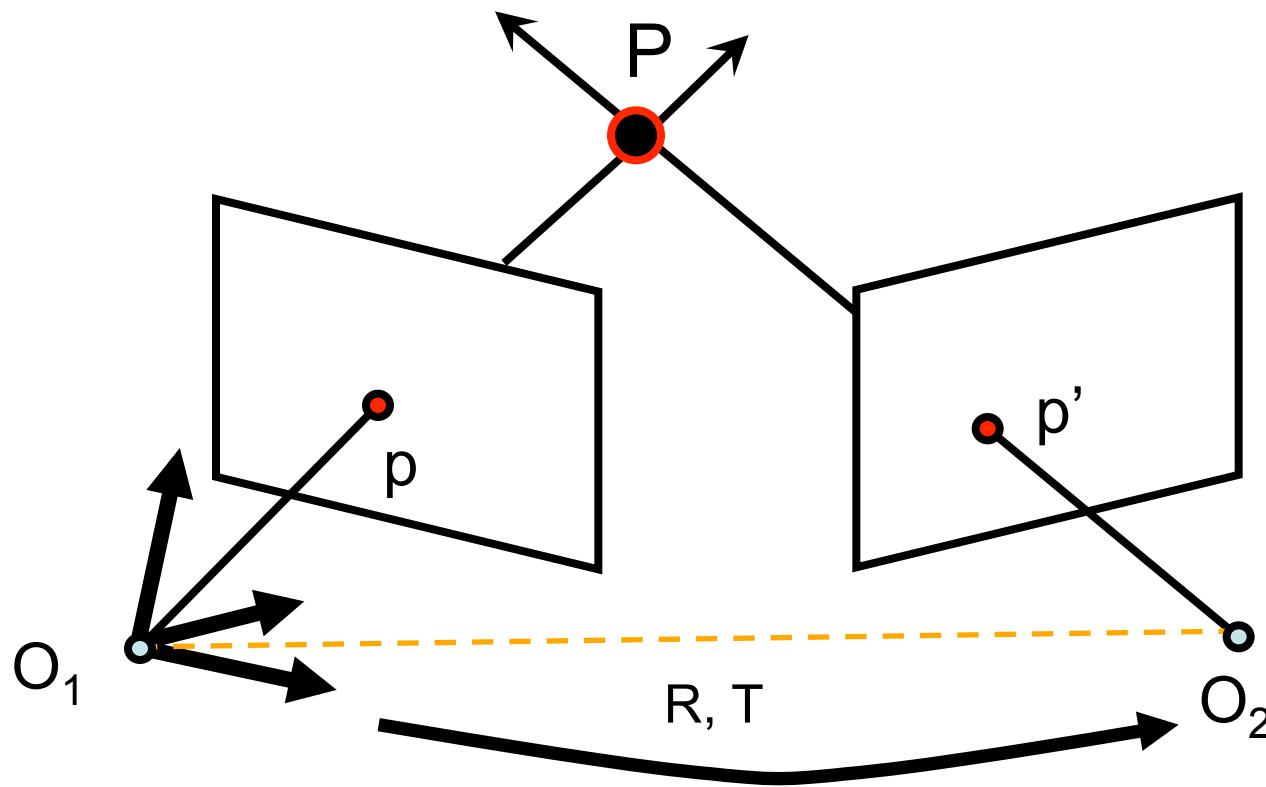
$T \times (R p')$  is perpendicular to epipolar plane

$$\rightarrow p^T \cdot [T \times (R p')] = 0 \quad [\text{Eq. 7}]$$

# Cross product as matrix multiplication

$$\mathbf{a} \times \mathbf{b} = \begin{bmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{bmatrix} \begin{bmatrix} b_x \\ b_y \\ b_z \end{bmatrix} = [\mathbf{a}_\times] \mathbf{b}$$

# Epipolar Constraint



$$p^T \cdot [T \times (R p')] = 0 \rightarrow p^T \cdot [T_x] \cdot R p' = 0 \quad [Eq. 9]$$

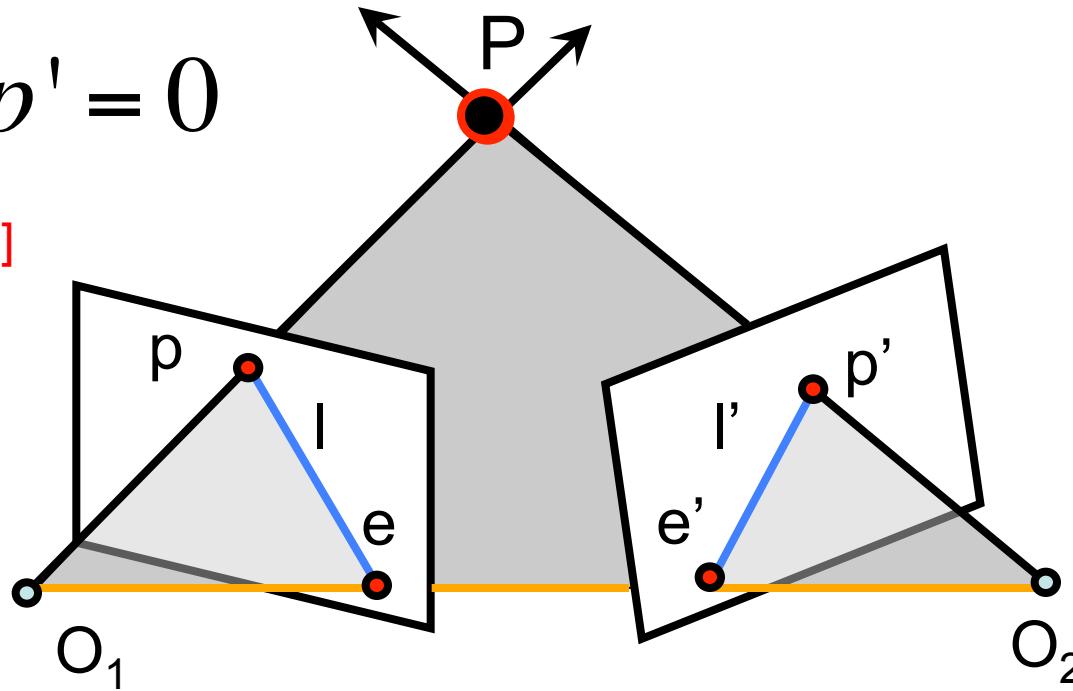
$E = \text{Essential matrix}$   
(Longuet-Higgins, 1981)

[Eq. 8]

# Epipolar Constraint

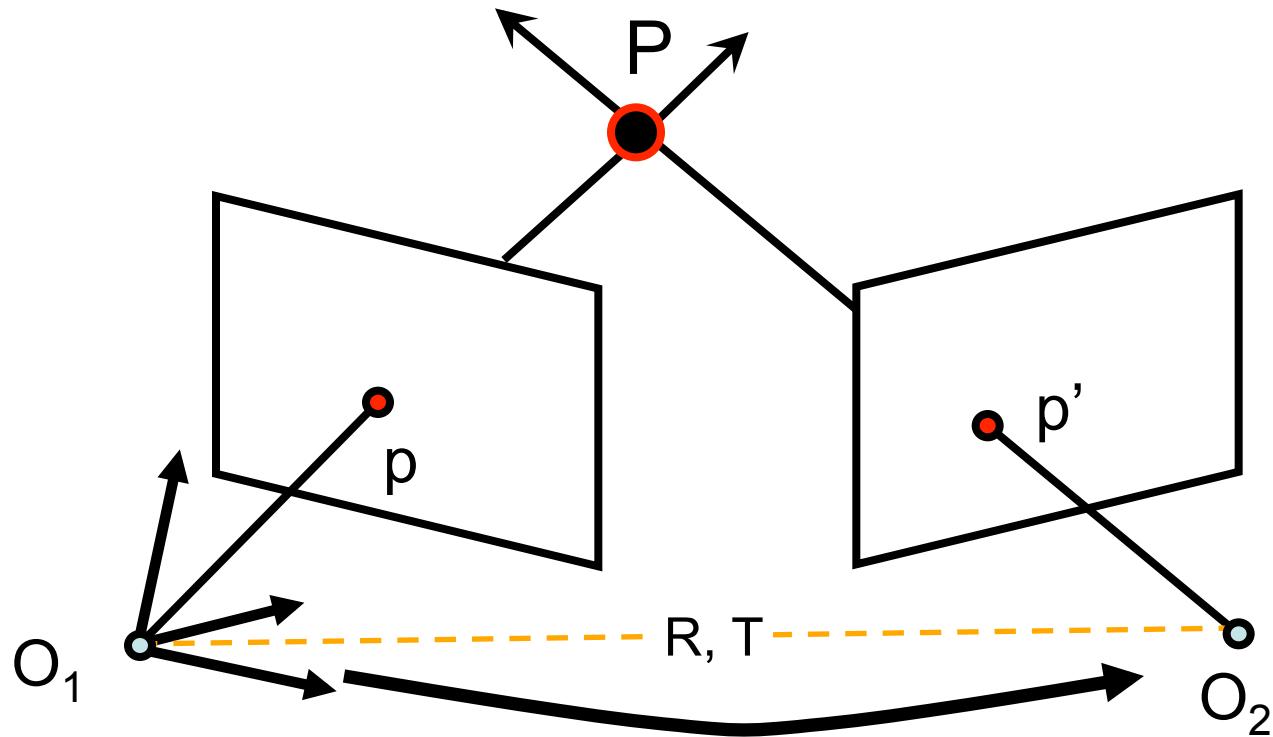
$$p^T \cdot E p' = 0$$

[Eq. 10]



- $l = E p'$  is the epipolar line associated with  $p'$
- $l' = E^T p$  is the epipolar line associated with  $p$
- $E e' = 0$  and  $E^T e = 0$
- $E$  is  $3 \times 3$  matrix; 5 DOF
- $E$  is singular (rank two)

# Epipolar Constraint



$$M = K \begin{bmatrix} I & 0 \end{bmatrix}$$

$$p_c = K^{-1} p \quad [\text{Eq. 11}]$$

$$M' = K' \begin{bmatrix} R^T & -R^T T \end{bmatrix}$$

$$p'_c = K'^{-1} p' \quad [\text{Eq. 12}]$$

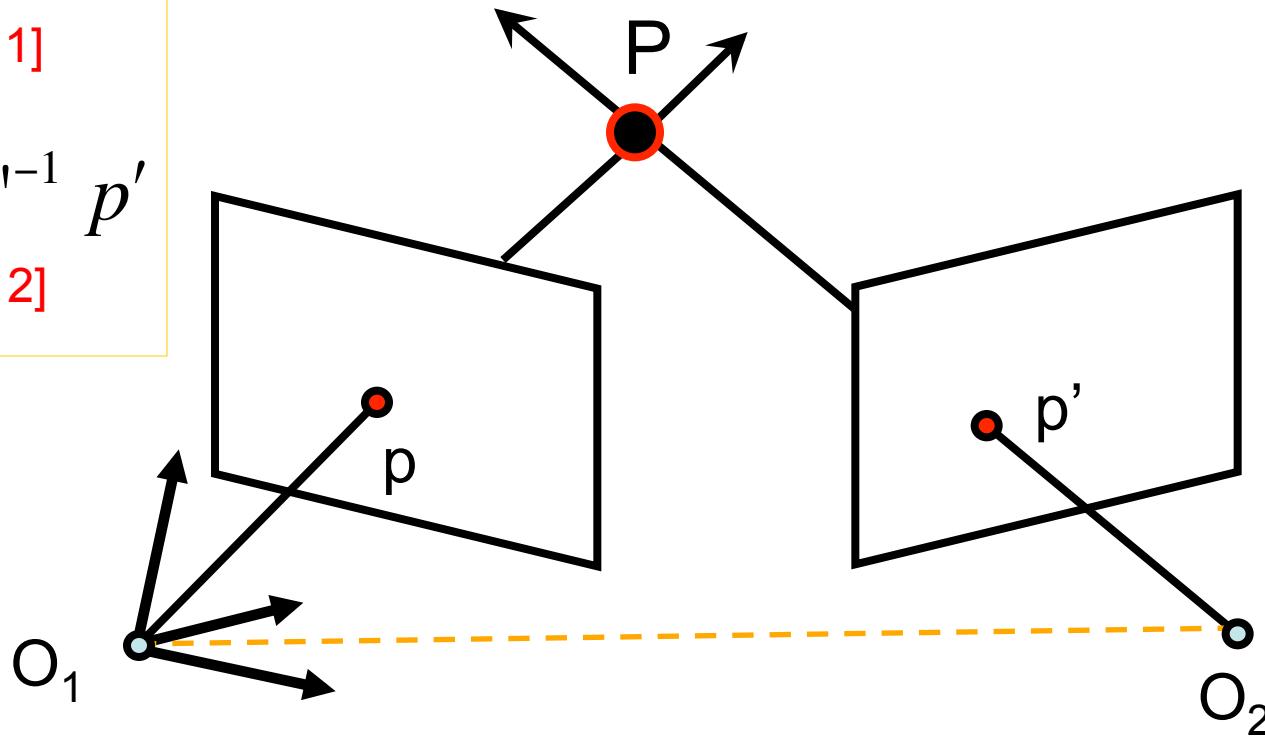
# Epipolar Constraint

$$p_c = K^{-1} p$$

[Eq. 11]

$$p'_c = K'^{-1} p'$$

[Eq. 12]



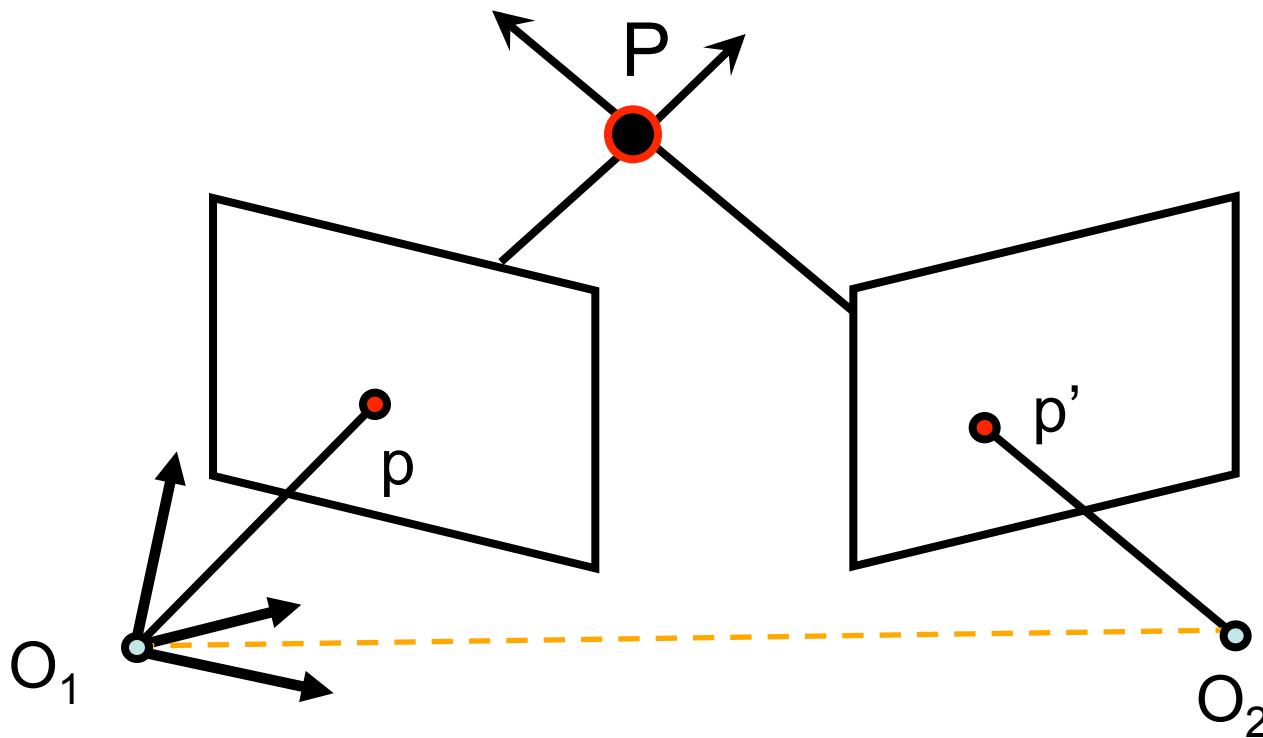
[Eq. 9]

$$p_c^T \cdot [T_x] \cdot R p'_c = 0 \rightarrow (K^{-1} p)^T \cdot [T_x] \cdot R K'^{-1} p' = 0$$

$$p^T [K^{-T} \cdot [T_x] \cdot R K'^{-1}] p' = 0 \rightarrow p^T [F] p' = 0$$

[Eq. 13]

# Epipolar Constraint



[Eq. 13]

$$p^T F p' = 0$$

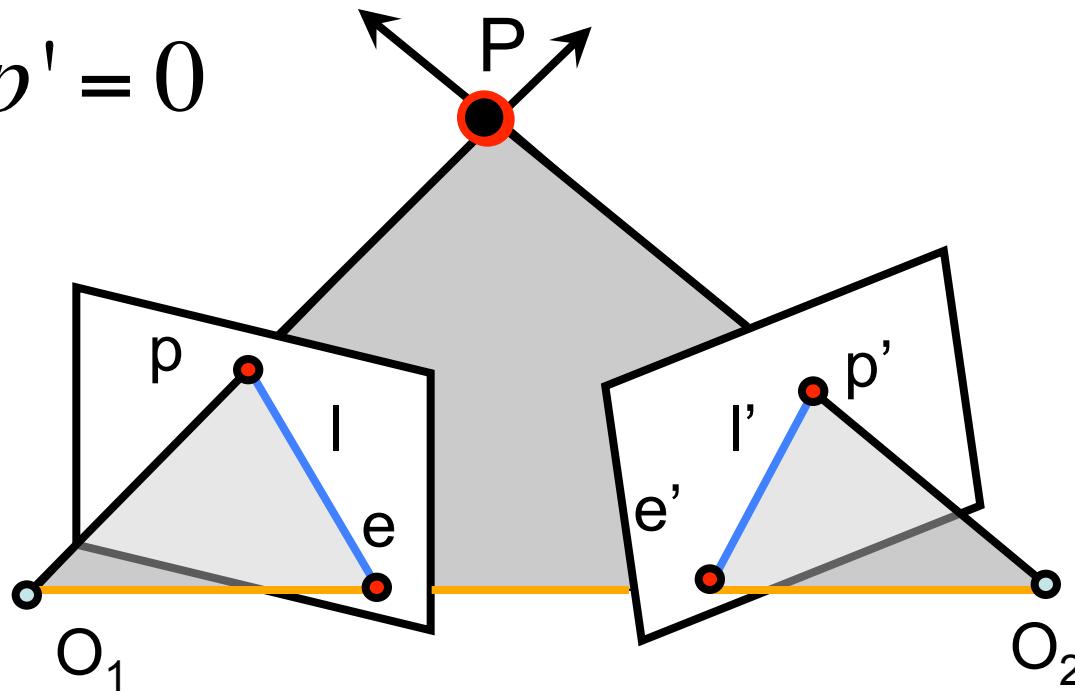
$$F = K^{-T} \cdot [T_x] \cdot R \cdot K'^{-1}$$

**F = Fundamental Matrix**  
(Faugeras and Luong, 1992)

[Eq. 14]

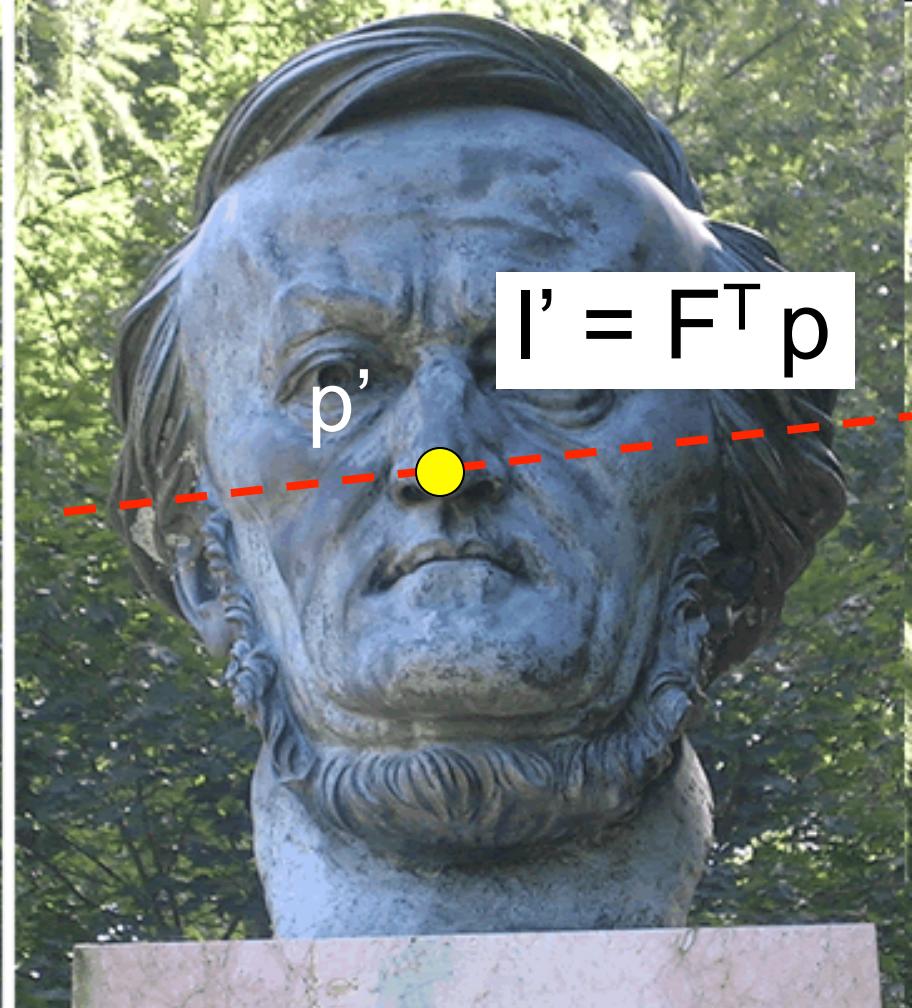
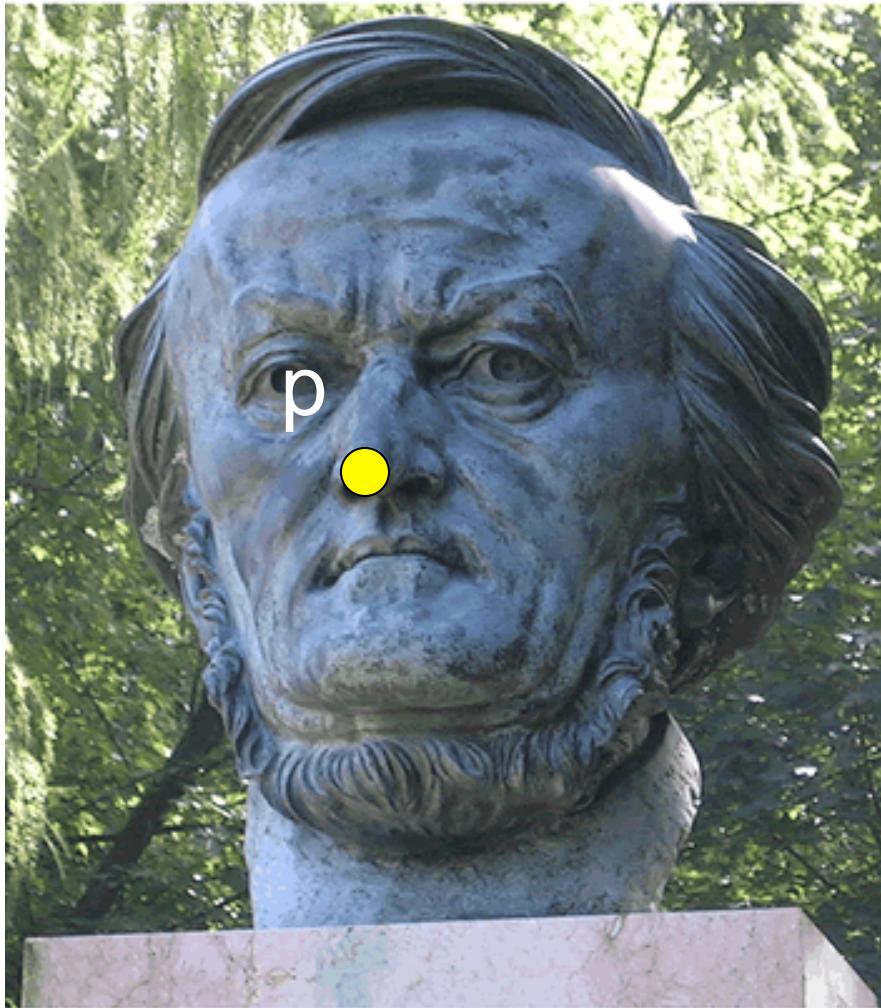
# Epipolar Constraint

$$p^T \cdot F p' = 0$$



- $l = F p'$  is the epipolar line associated with  $p'$
- $l' = F^T p$  is the epipolar line associated with  $p$
- $F e' = 0$  and  $F^T e = 0$
- $F$  is  $3 \times 3$  matrix; 7 DOF
- $F$  is singular (rank two)

# Why F is useful?



- Suppose  $F$  is known
- No additional information about the scene and camera is given
- Given a point on left image, we can compute the corresponding epipolar line in the second image

# Why F is useful?

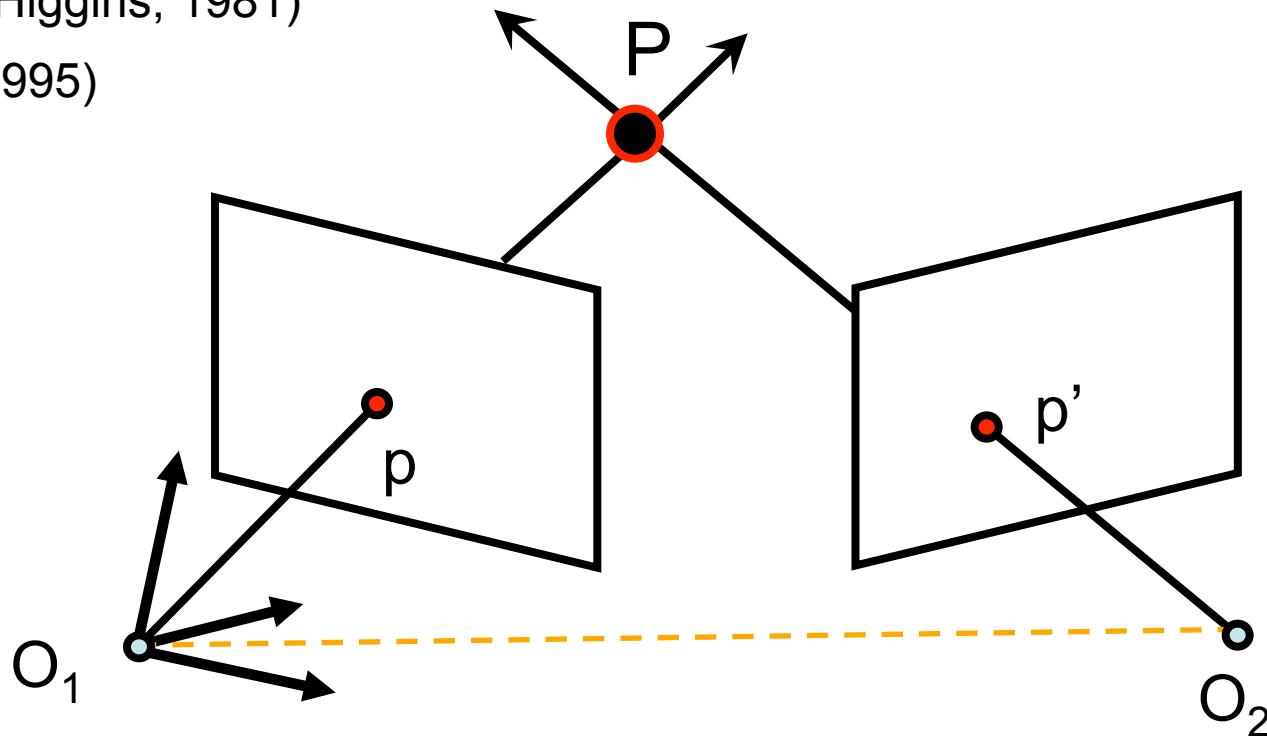
- F captures information about the epipolar geometry of 2 views + camera parameters
- **MORE IMPORTANTLY:** F gives constraints on how the scene changes under view point transformation (without reconstructing the scene!)
- Powerful tool in:
  - 3D reconstruction
  - Multi-view object/scene matching

# Estimating F

## The Eight-Point Algorithm

(Longuet-Higgins, 1981)

(Hartley, 1995)



$$p^T F p' = 0$$

# Estimating F

[Eq. 13]  $p^T F p' = 0 \quad \rightarrow$

$$p = \begin{bmatrix} u \\ v \\ 1 \end{bmatrix} \quad p' = \begin{bmatrix} u' \\ v' \\ 1 \end{bmatrix}$$

$$(u, v, 1) \begin{pmatrix} F_{11} & F_{12} & F_{13} \\ F_{21} & F_{22} & F_{23} \\ F_{31} & F_{32} & F_{33} \end{pmatrix} \begin{pmatrix} u' \\ v' \\ 1 \end{pmatrix} = 0$$



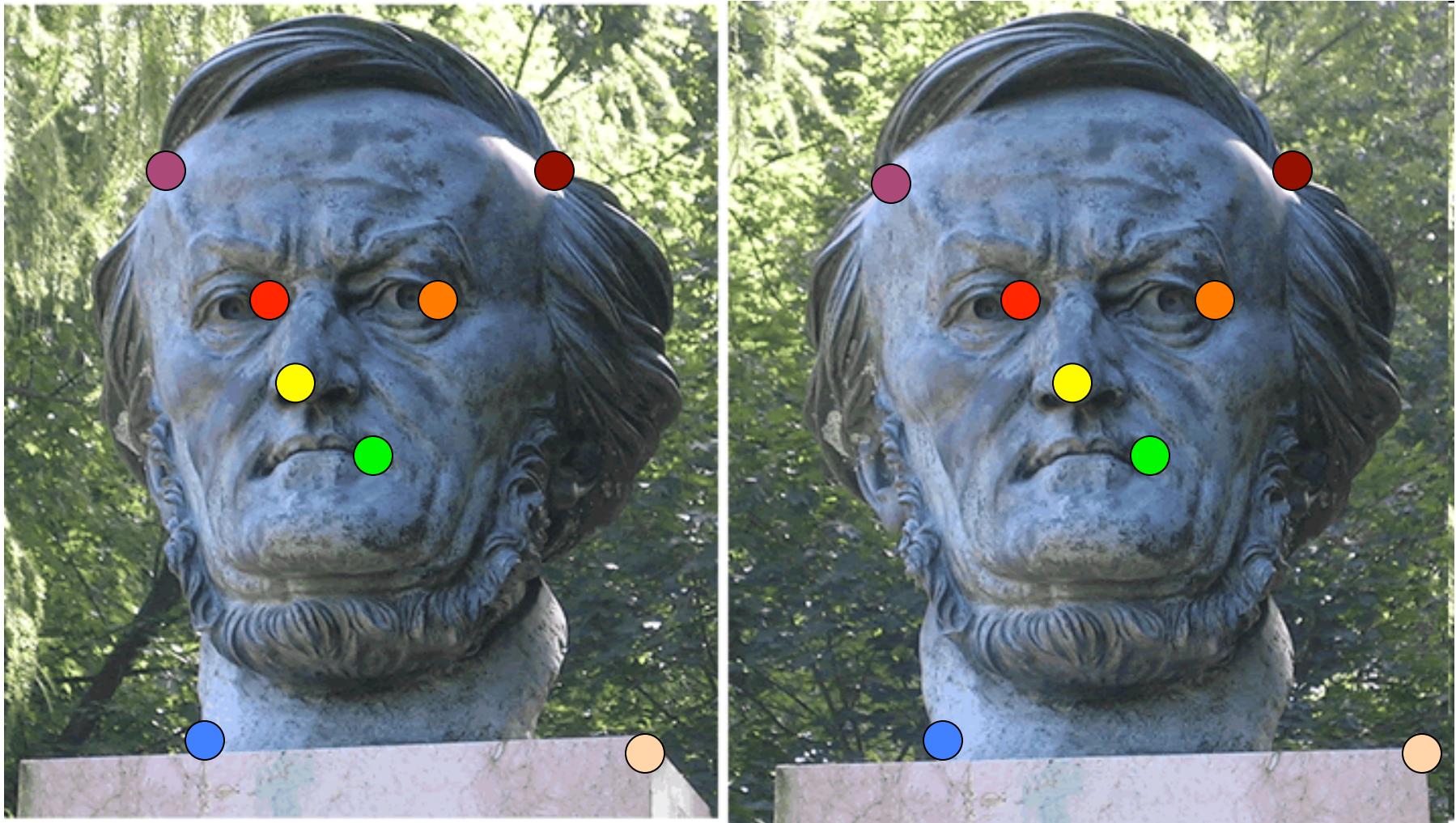
$$(uu', uv', u, vu', vv', v, u', v', 1)$$

$$\begin{pmatrix} F_{11} \\ F_{12} \\ F_{13} \\ F_{21} \\ F_{22} \\ F_{23} \\ F_{31} \\ F_{32} \\ F_{33} \end{pmatrix} = 0$$

[Eq. 14]

Let's take 8 corresponding points

# Estimating F



# Estimating F

$$\begin{pmatrix} u_i u'_i, u_i v'_i, u_i, v_i u'_i, v_i v'_i, v_i, u'_i, v'_i, 1 \end{pmatrix} \begin{pmatrix} F_{11} \\ F_{12} \\ F_{13} \\ F_{21} \\ F_{22} \\ F_{23} \\ F_{31} \\ F_{32} \\ F_{33} \end{pmatrix} = 0 \quad [\text{Eq. 14}]$$

# Estimating F

$$\mathbf{W} \begin{pmatrix} u_1 u'_1 & u_1 v'_1 & u_1 & v_1 u'_1 & v_1 v'_1 & v_1 & u'_1 & v'_1 & 1 \\ u_2 u'_2 & u_2 v'_2 & u_2 & v_2 u'_2 & v_2 v'_2 & v_2 & u'_2 & v'_2 & 1 \\ u_3 u'_3 & u_3 v'_3 & u_3 & v_3 u'_3 & v_3 v'_3 & v_3 & u'_3 & v'_3 & 1 \\ u_4 u'_4 & u_4 v'_4 & u_4 & v_4 u'_4 & v_4 v'_4 & v_4 & u'_4 & v'_4 & 1 \\ u_5 u'_5 & u_5 v'_5 & u_5 & v_5 u'_5 & v_5 v'_5 & v_5 & u'_5 & v'_5 & 1 \\ u_6 u'_6 & u_6 v'_6 & u_6 & v_6 u'_6 & v_6 v'_6 & v_6 & u'_6 & v'_6 & 1 \\ u_7 u'_7 & u_7 v'_7 & u_7 & v_7 u'_7 & v_7 v'_7 & v_7 & u'_7 & v'_7 & 1 \\ u_8 u'_8 & u_8 v'_8 & u_8 & v_8 u'_8 & v_8 v'_8 & v_8 & u'_8 & v'_8 & 1 \end{pmatrix} \begin{pmatrix} F_{11} \\ F_{12} \\ F_{13} \\ F_{21} \\ F_{22} \\ F_{23} \\ F_{31} \\ F_{32} \\ F_{33} \end{pmatrix} = 0 \quad [\text{Eqs. 15}]$$

- Homogeneous system  $\mathbf{W} \mathbf{f} = 0$
- Rank 8  $\rightarrow$  A non-zero solution exists (unique)
- If  $N > 8$   $\rightarrow$  Lsq. solution by SVD!  $\rightarrow \hat{\mathbf{F}}$   
 $\|\mathbf{f}\| = 1$

$\hat{F}$  satisfies:  $p^T \hat{F} p' = 0$

and estimated  $\hat{F}$  may have full rank ( $\det(\hat{F}) \neq 0$ )

But remember: fundamental matrix is Rank2

Find  $F$  that minimizes  $\|F - \hat{F}\| = 0$

Frobenius norm (\*)

Subject to  $\det(F) = 0$

SVD (again!) can be used to solve this problem

(\*) Sq. root of the sum of squares of all entries

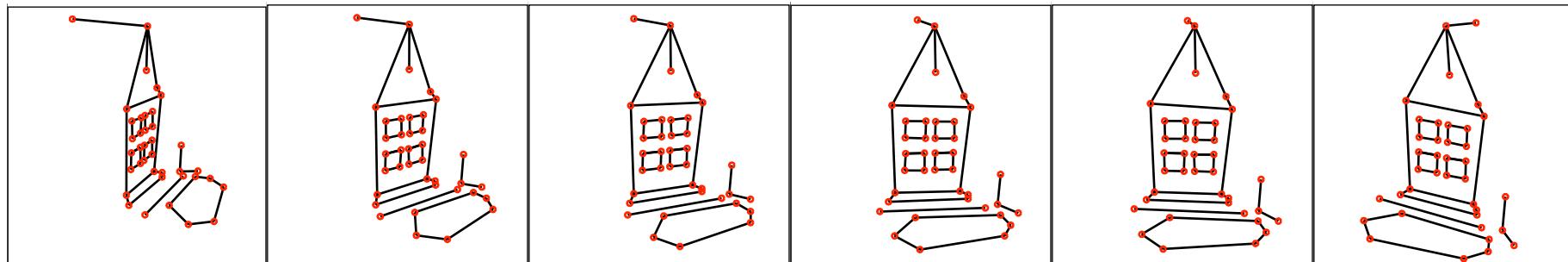
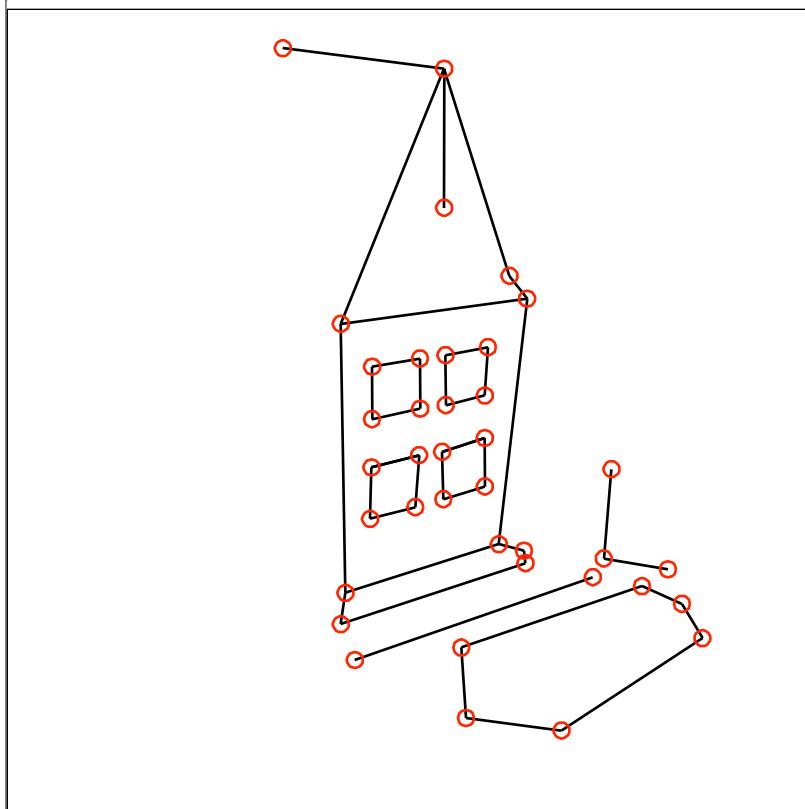
Find  $F$  that minimizes  $\|F - \hat{F}\| = 0$   
Frobenius norm (\*)

Subject to  $\det(F) = 0$

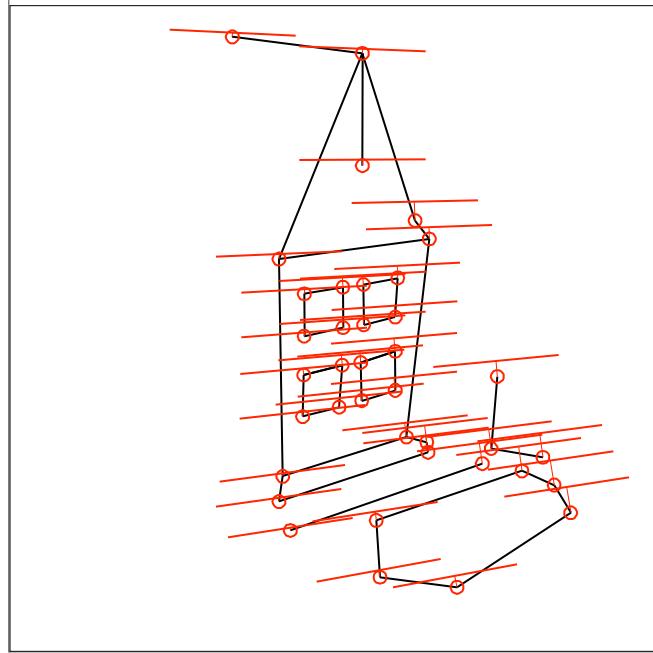
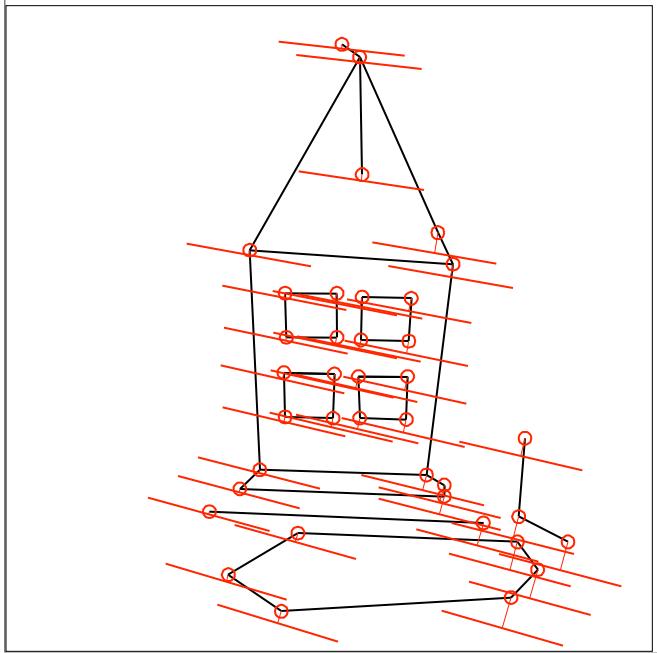
$$F = U \begin{bmatrix} s_1 & 0 & 0 \\ 0 & s_2 & 0 \\ 0 & 0 & 0 \end{bmatrix} V^T$$

Where:

$$U \begin{bmatrix} s_1 & 0 & 0 \\ 0 & s_2 & 0 \\ 0 & 0 & s_3 \end{bmatrix} V^T = SVD(\hat{F})$$

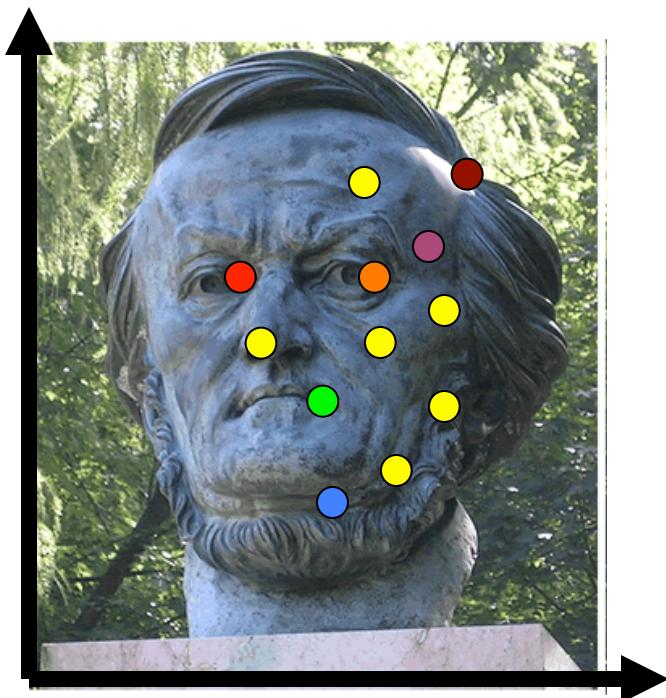


Data courtesy of R. Mohr and B. Boufama.



Mean errors:  
10.0pixel  
9.1pixel

# Problems with the 8-Point Algorithm



$$W f = 0,$$

$$\|f\| = 1$$

Lsq solution  
by SVD

$$F$$

- Recall the structure of  $W$ :
  - do we see any potential (numerical) issue?

# Problems with the 8-Point Algorithm

$$\mathbf{Wf} = 0$$

$$\begin{pmatrix} u_1 u'_1 & u_1 v'_1 & u_1 & v_1 u'_1 & v_1 v'_1 & v_1 & u'_1 & v'_1 & 1 \\ u_2 u'_2 & u_2 v'_2 & u_2 & v_2 u'_2 & v_2 v'_2 & v_2 & u'_2 & v'_2 & 1 \\ u_3 u'_3 & u_3 v'_3 & u_3 & v_3 u'_3 & v_3 v'_3 & v_3 & u'_3 & v'_3 & 1 \\ u_4 u'_4 & u_4 v'_4 & u_4 & v_4 u'_4 & v_4 v'_4 & v_4 & u'_4 & v'_4 & 1 \\ u_5 u'_5 & u_5 v'_5 & u_5 & v_5 u'_5 & v_5 v'_5 & v_5 & u'_5 & v'_5 & 1 \\ u_6 u'_6 & u_6 v'_6 & u_6 & v_6 u'_6 & v_6 v'_6 & v_6 & u'_6 & v'_6 & 1 \\ u_7 u'_7 & u_7 v'_7 & u_7 & v_7 u'_7 & v_7 v'_7 & v_7 & u'_7 & v'_7 & 1 \\ u_8 u'_8 & u_8 v'_8 & u_8 & v_8 u'_8 & v_8 v'_8 & v_8 & u'_8 & v'_8 & 1 \end{pmatrix} \begin{pmatrix} F_{11} \\ F_{12} \\ F_{13} \\ F_{21} \\ F_{22} \\ F_{23} \\ F_{31} \\ F_{32} \\ F_{33} \end{pmatrix} = 0$$

- Highly un-balanced (not well conditioned)
- Values of W must have similar magnitude
- This creates problems during the SVD decomposition

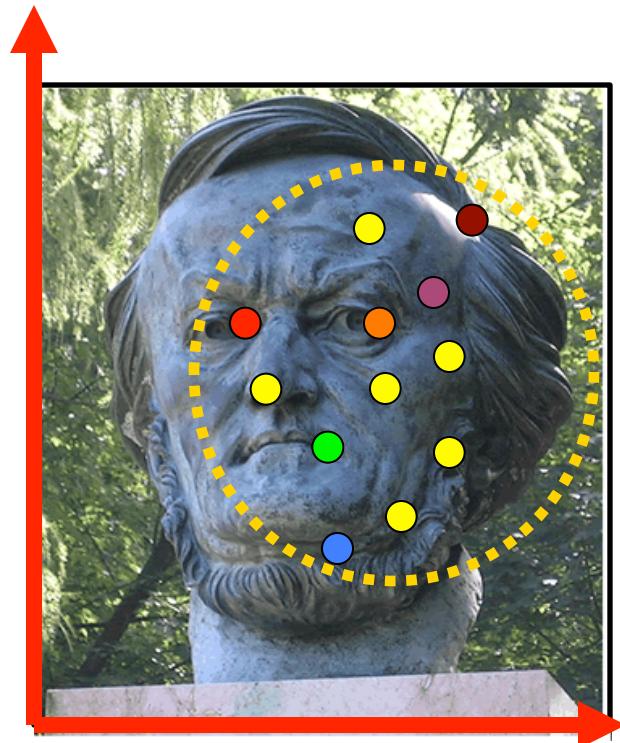
# Normalization

IDEA: Transform image coordinates such that the matrix  $\mathbf{W}$  becomes better conditioned (**pre-conditioning**)

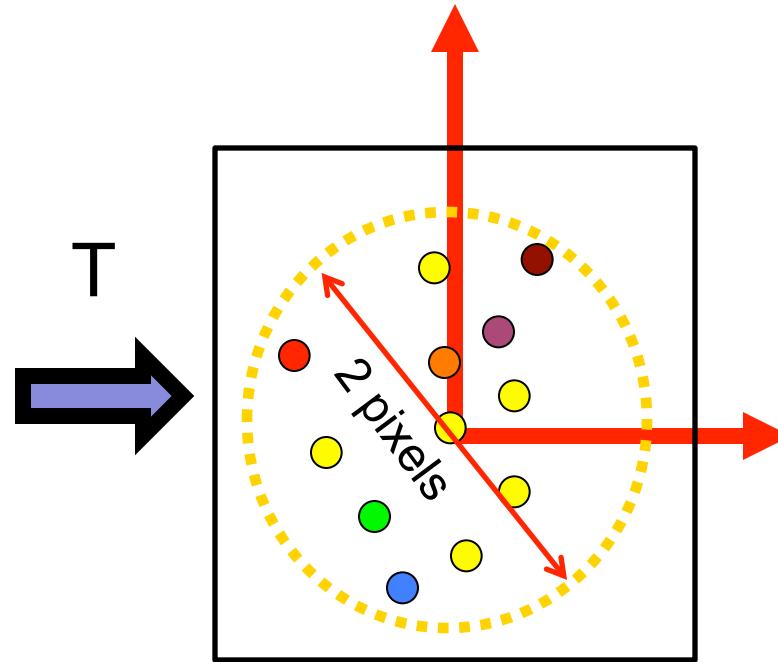
For each image, apply a following transformation  $T$  (translation and scaling) acting on image coordinates such that:

- Origin = centroid of image points
- Mean square distance of the image points from origin is  $\sim 2$  pixels

# Example of normalization



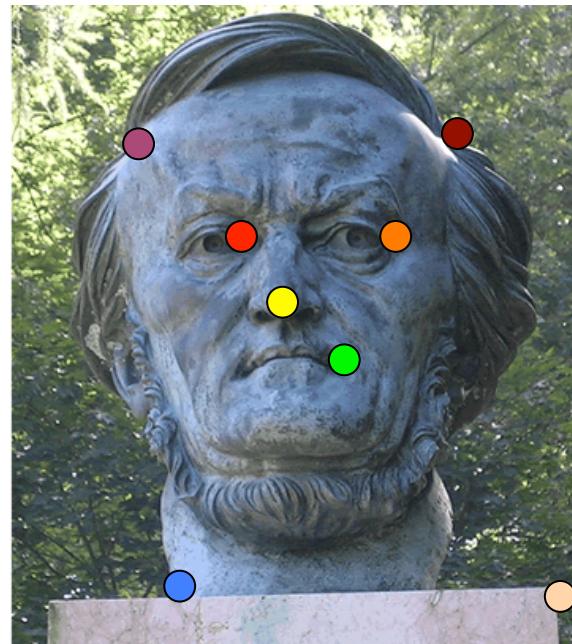
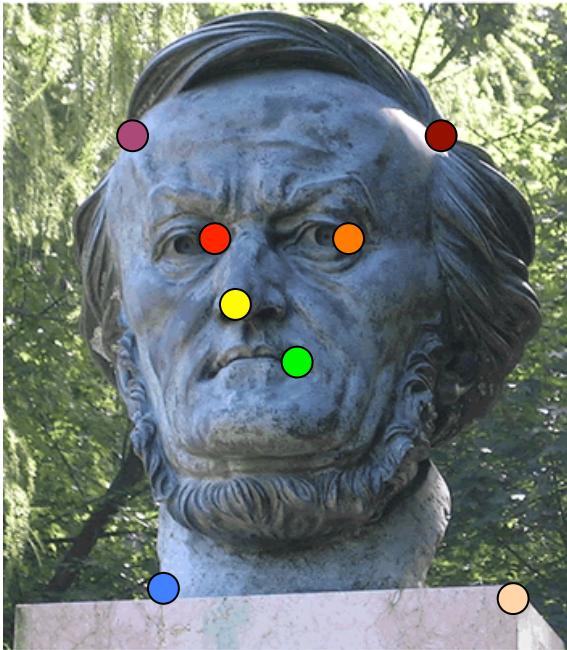
Coordinate system of the image before applying  $T$



Coordinate system of the image after applying  $T$

- Origin = centroid of image points
- Mean square distance of the image points from origin is  $\sim 2$  pixels

# Normalization



$$q_i = T \ p_i$$

$$q'_i = T' \ p'_i$$

# The Normalized Eight-Point Algorithm

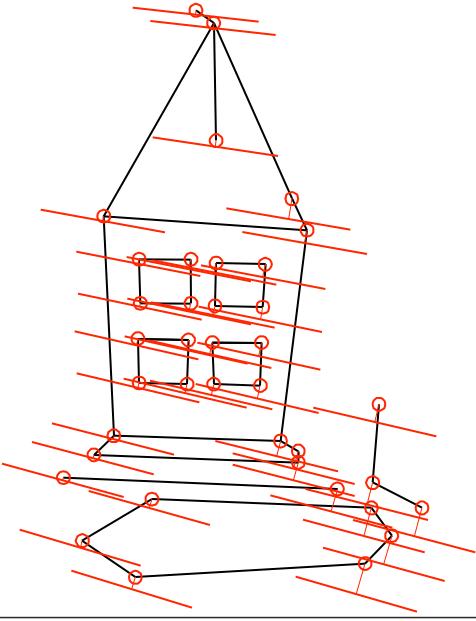
0. Compute  $T$  and  $T'$  for image 1 and 2, respectively
1. Normalize coordinates in images 1 and 2:

$$q_i = T p_i \quad q'_i = T' p'_i$$

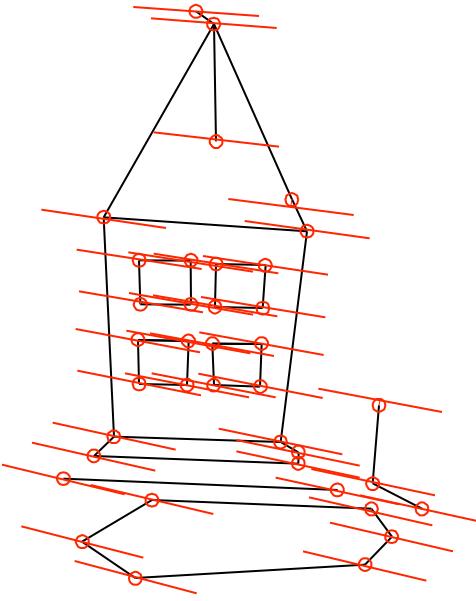
2. Use the eight-point algorithm to compute  $\hat{F}_q$  from the corresponding points  $q$  and  $q'$ .

1. Enforce the rank-2 constraint.  $\rightarrow F_q$  such that:  
$$\begin{cases} q^T F_q q' = 0 \\ \det(F_q) = 0 \end{cases}$$
2. De-normalize  $F_q$ :  $F = T^T F_q T'$

Without normalization



With normalization



Mean errors:  
10.0pixel  
9.1pixel

Mean errors:  
1.0pixel  
0.9pixel

# Lecture 5

## Epipolar Geometry

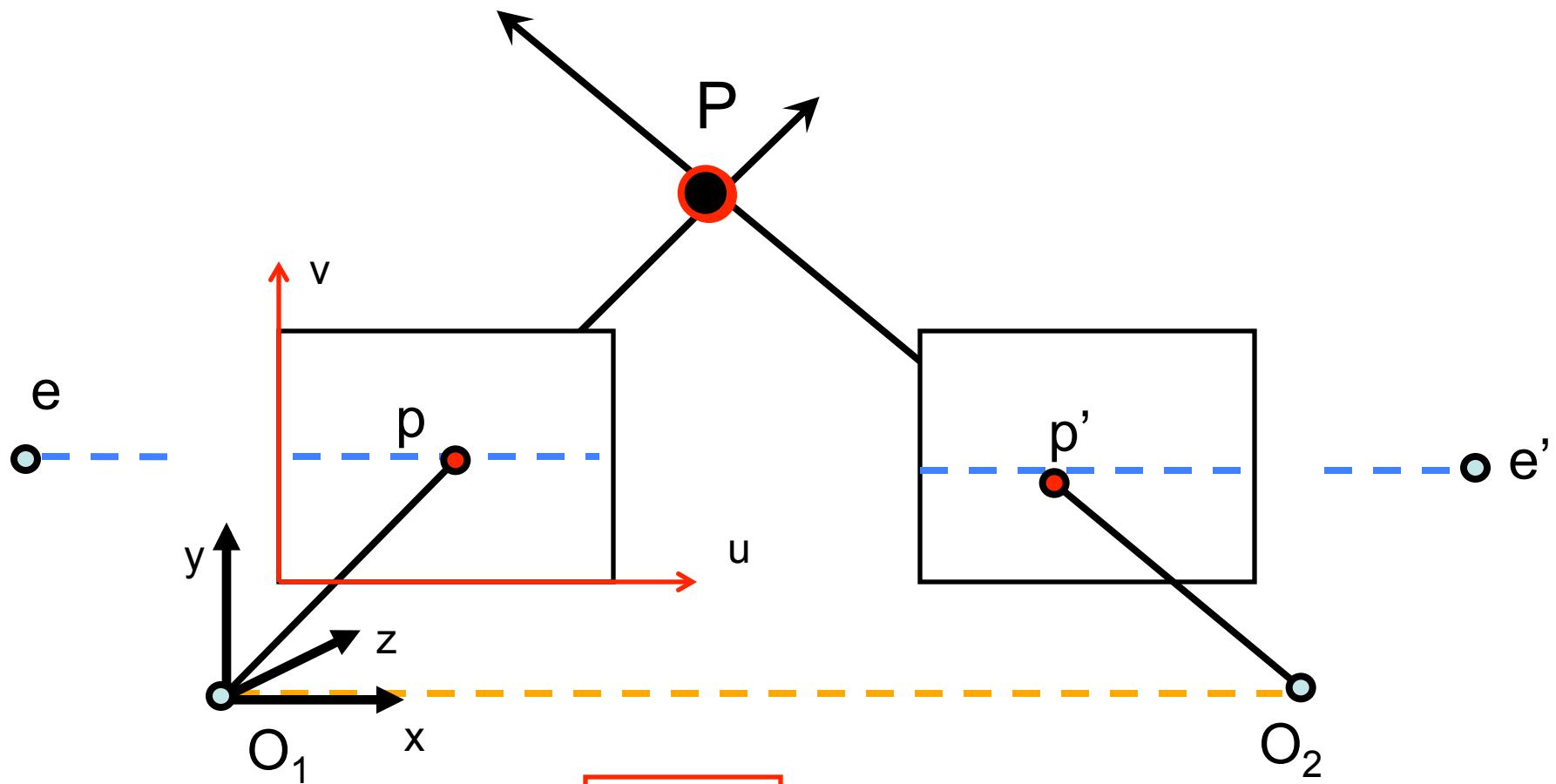


- Why is stereo useful?
- Epipolar constraints
- Essential and fundamental matrix
- Estimating  $F$
- Examples

**Reading:** [AZ] Chapter: 4 “Estimation – 2D perspective transformations  
Chapter: 9 “Epipolar Geometry and the Fundamental Matrix Transformation”  
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[FP] Chapter: 7 “Stereopsis”  
Chapter: 8 “Structure from Motion”

# Example: Parallel image planes



$K_1 = K_2 = \text{known}$

$x$  parallel to  $O_1O_2$

$$E=?$$

Hint :  
 $R = I$        $T = (T, 0, 0)$

# Essential matrix for parallel images

$$\mathbf{E} = [\mathbf{T}_\times] \cdot \mathbf{R}$$

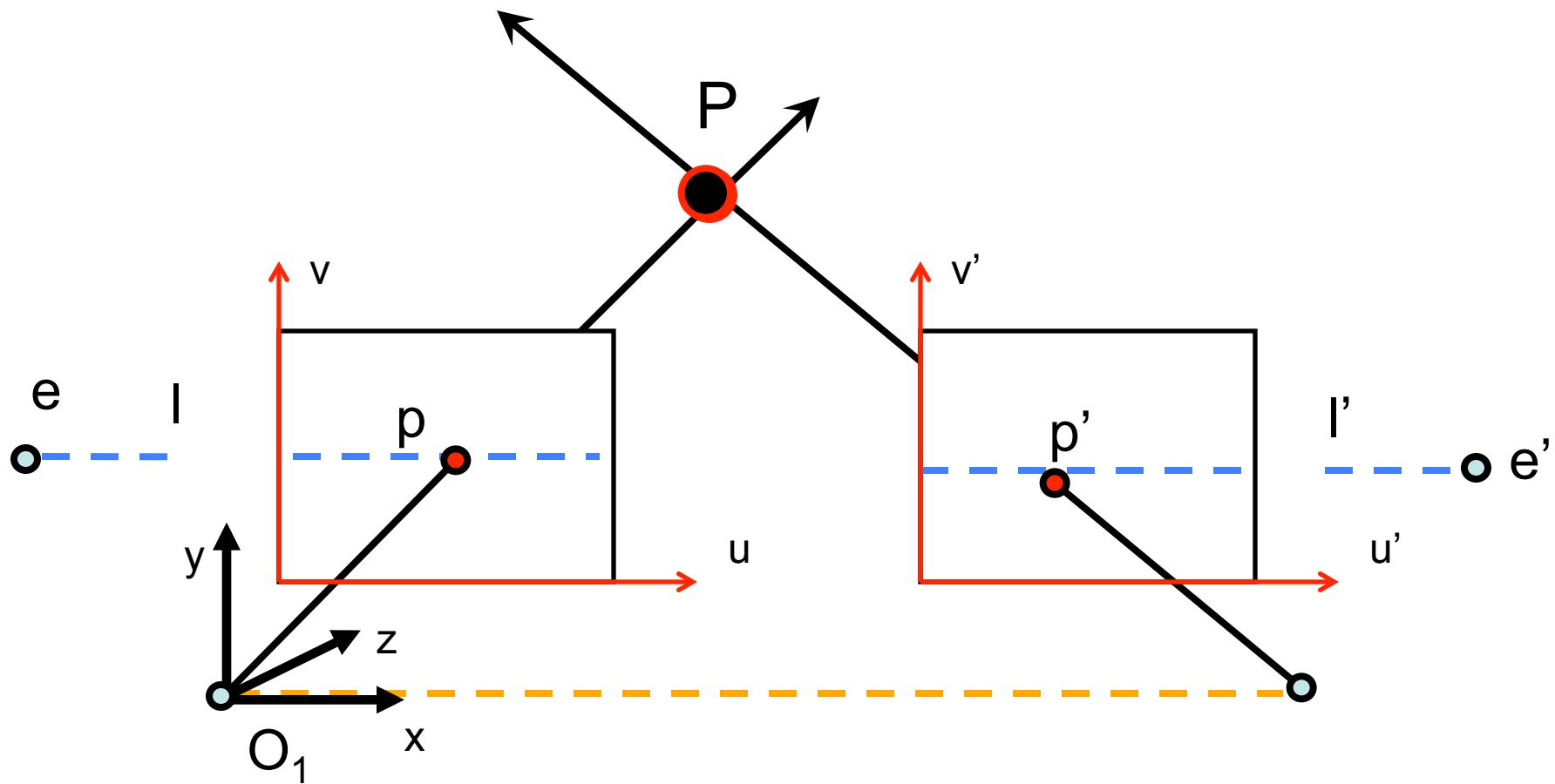
$$\mathbf{E} = \begin{bmatrix} 0 & -T_z & T_y \\ T_z & 0 & -T_x \\ -T_y & T_x & 0 \end{bmatrix} \quad \mathbf{R} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -T \\ 0 & T & 0 \end{bmatrix}$$

[Eq. 20]

$$\mathbf{T} = [ T \ 0 \ 0 ]$$

$$\mathbf{R} = \mathbf{I}$$

# Example: Parallel image planes

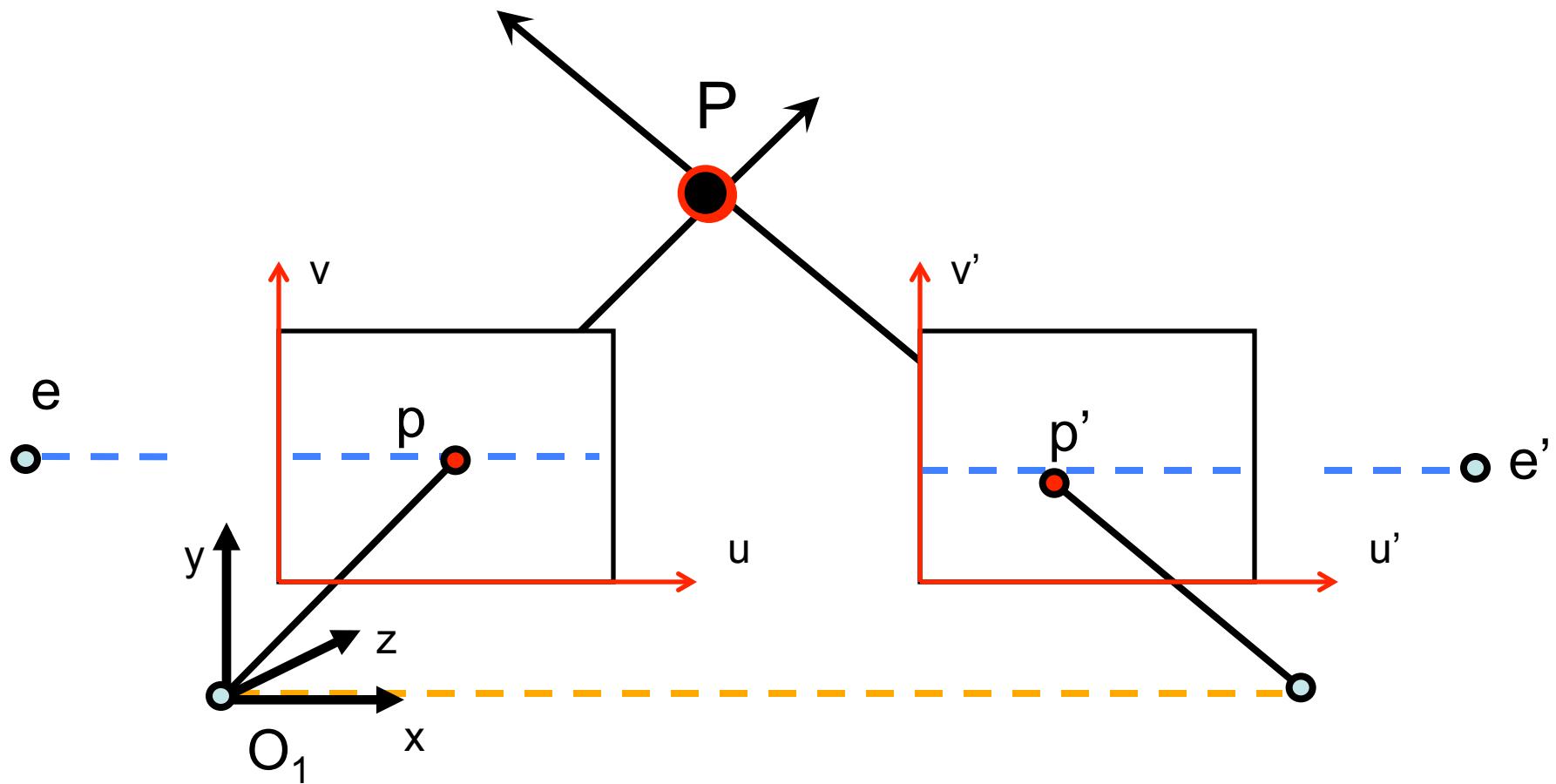


What are the directions of epipolar lines?

$$l = E p' = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -T \\ 0 & T & 0 \end{bmatrix} \begin{bmatrix} u' \\ v' \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -T \\ T v' \end{bmatrix}$$

horizontal!

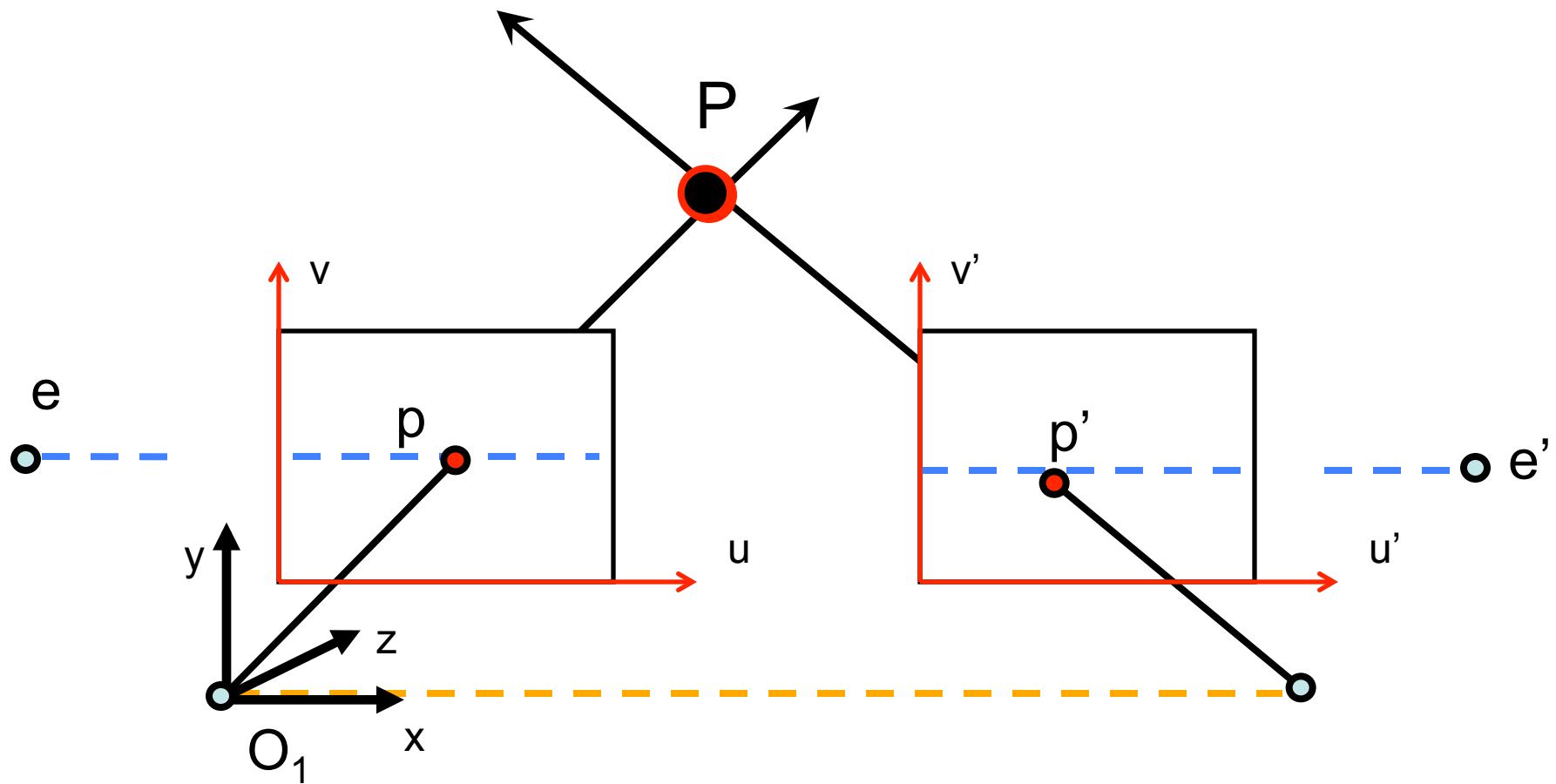
# Example: Parallel image planes



How are  $p$   
and  $p'$   
related?

$$p^T \cdot E \ p' = 0$$

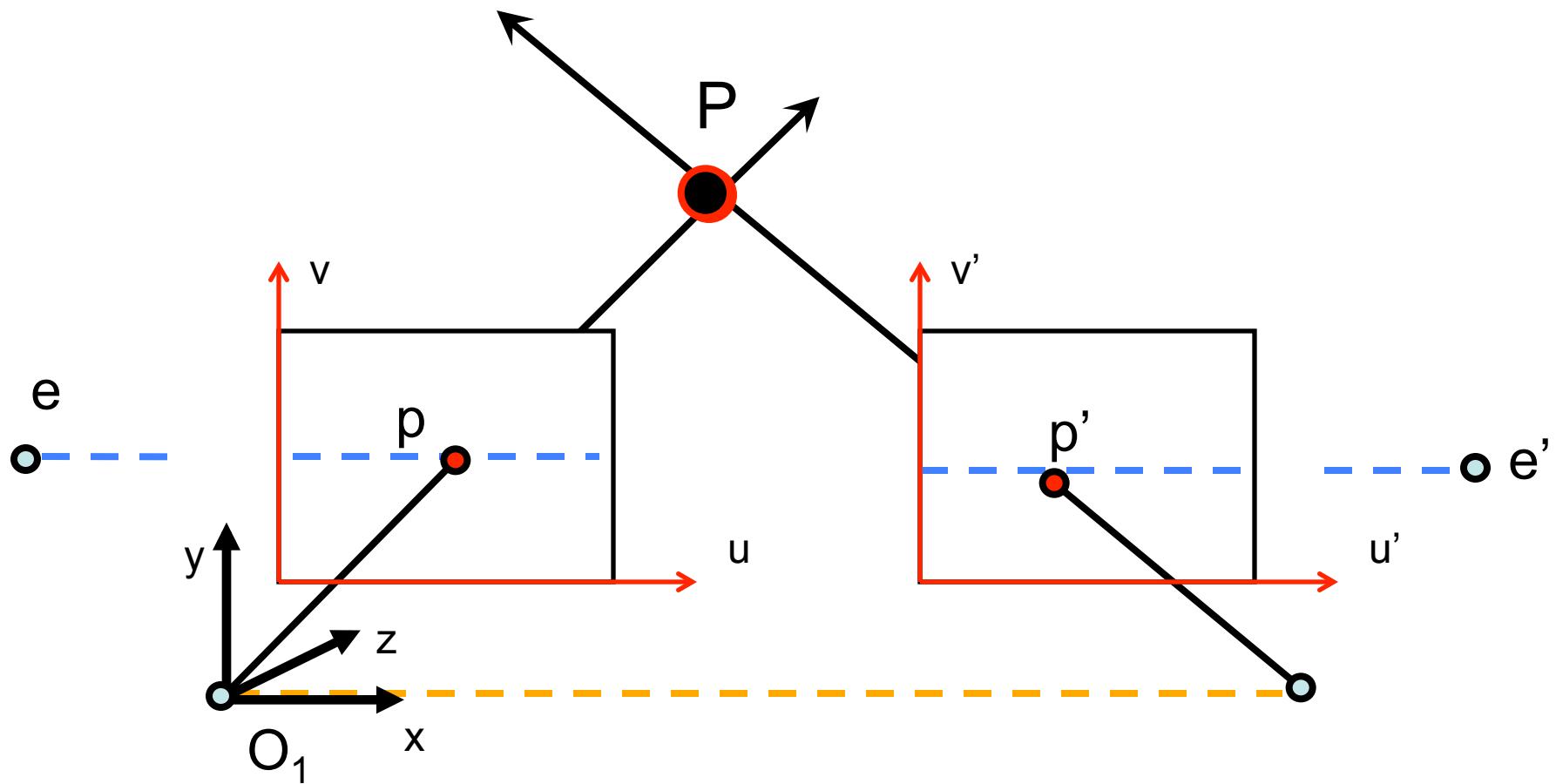
# Example: Parallel image planes



How are  $p$   
and  $p'$   
related?

$$\Rightarrow \begin{pmatrix} u & v & 1 \end{pmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -T \\ 0 & T & 0 \end{bmatrix} \begin{pmatrix} u' \\ v' \\ 1 \end{pmatrix} = 0 \Rightarrow \begin{pmatrix} u & v & 1 \end{pmatrix} \begin{pmatrix} 0 \\ -T \\ Tv' \end{pmatrix} = 0 \Rightarrow Tv = Tv' \Rightarrow v = v'$$

# Example: Parallel image planes



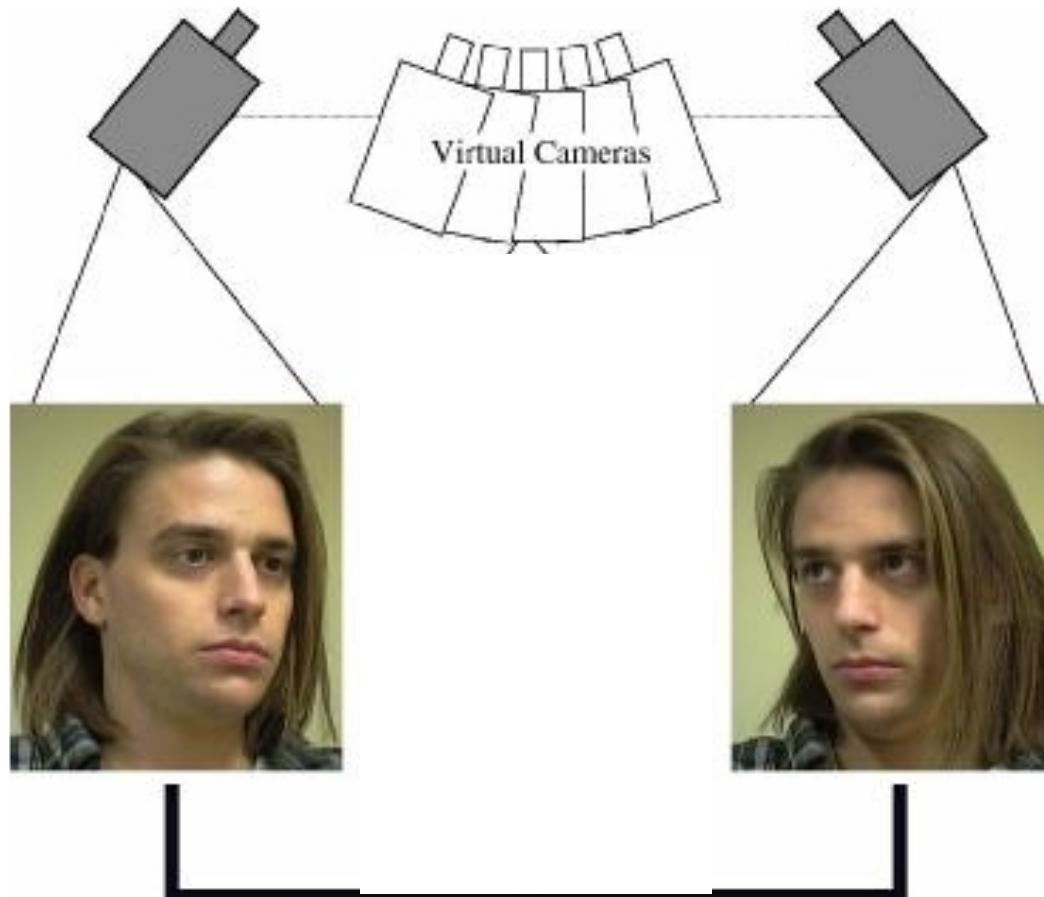
Rectification: making two images “parallel”

Why it is useful?

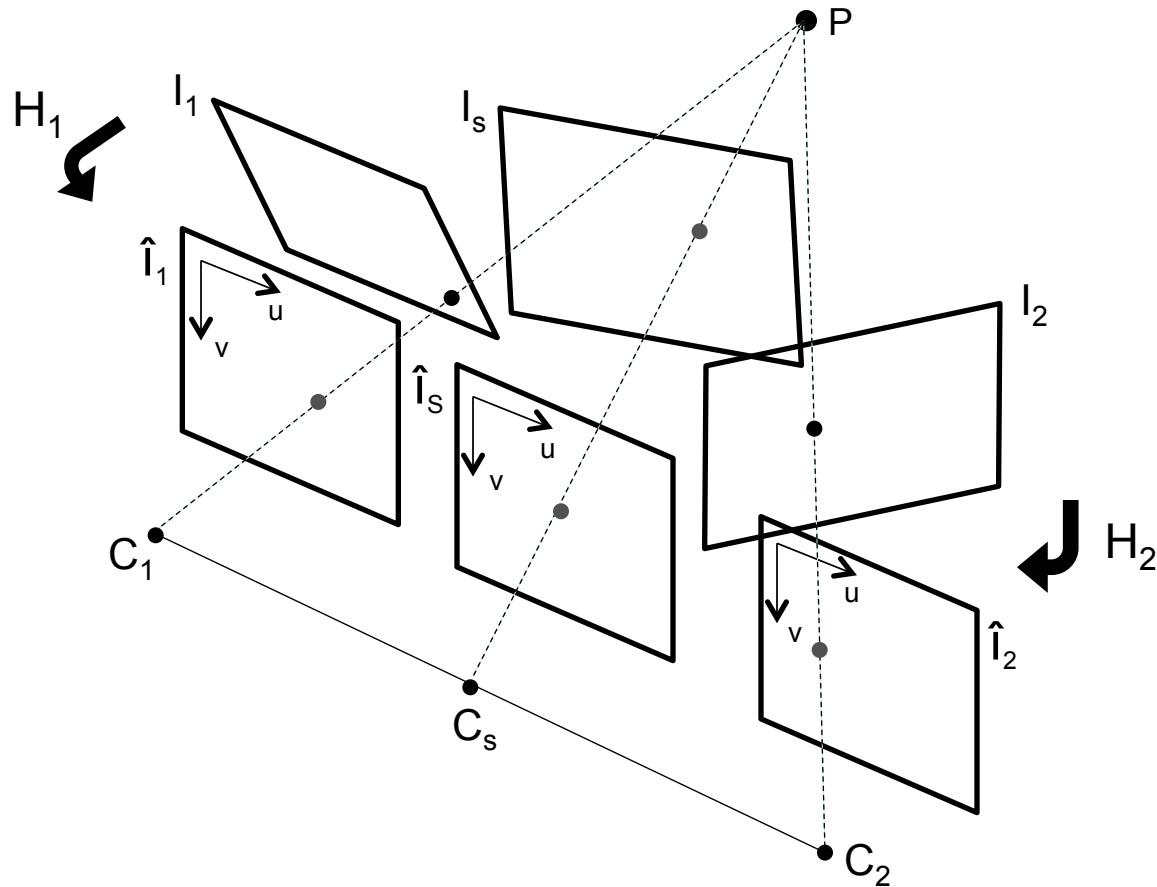
- Epipolar constraint  $\rightarrow v = v'$
- New views can be synthesized by linear interpolation

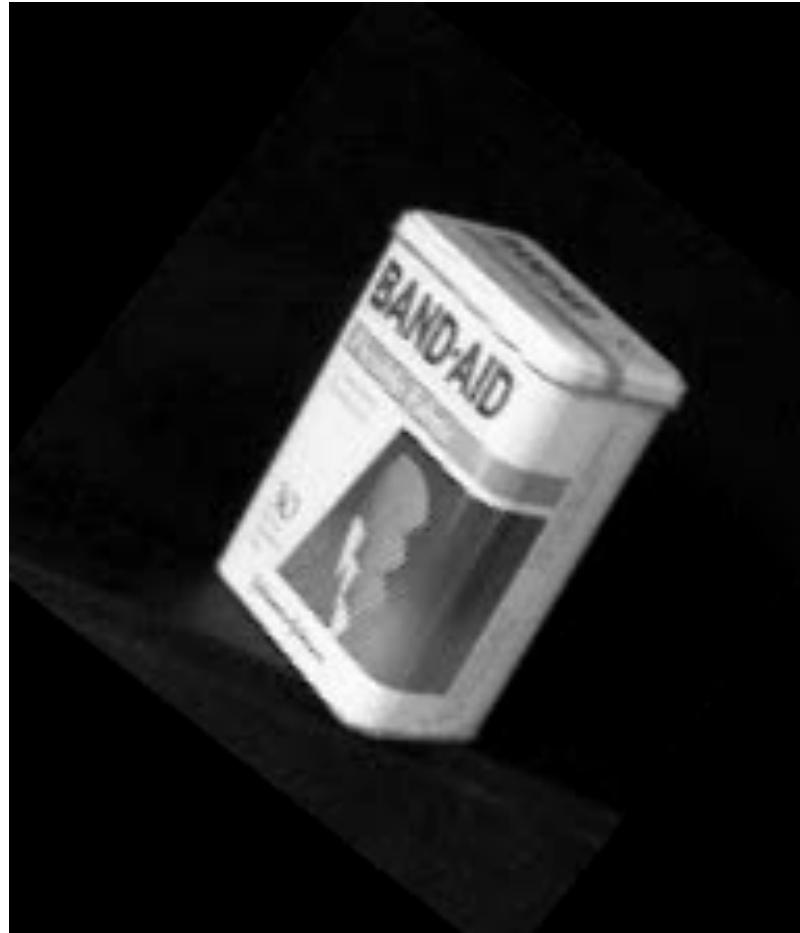
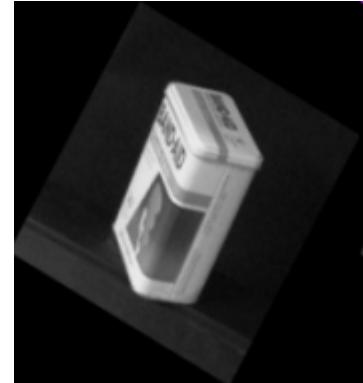
# Application: view morphing

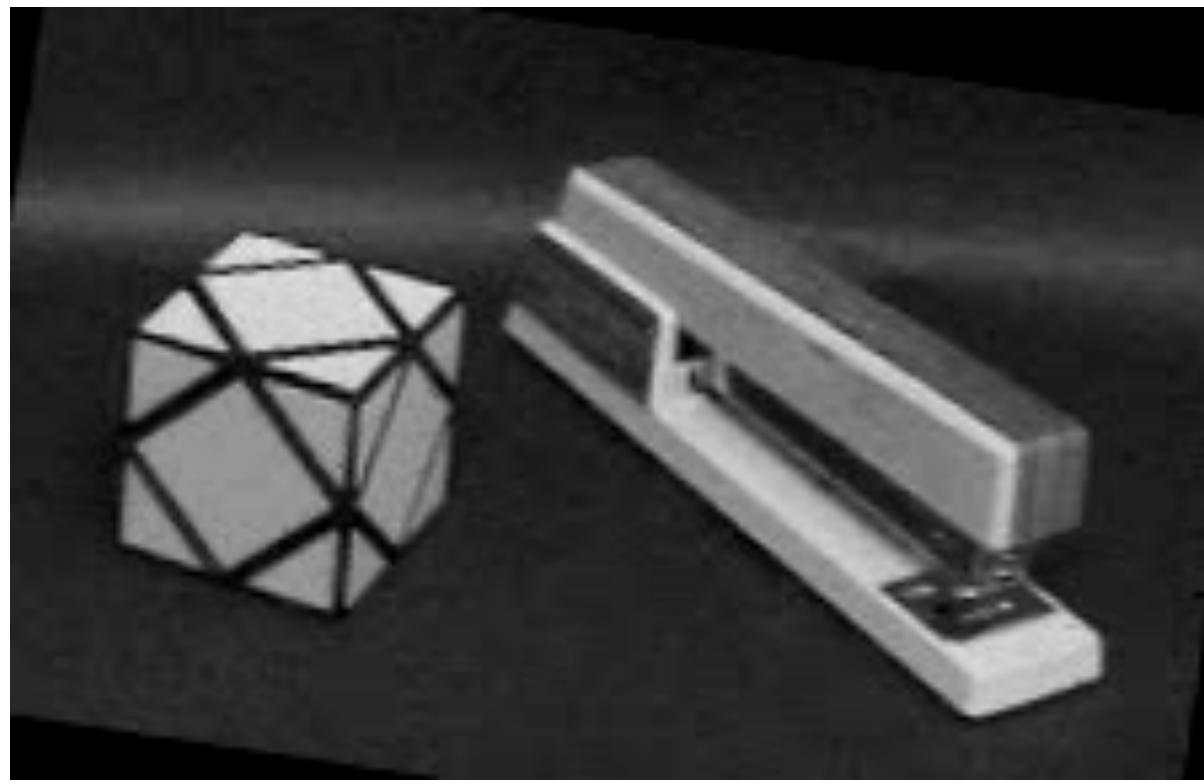
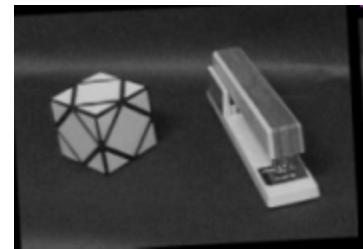
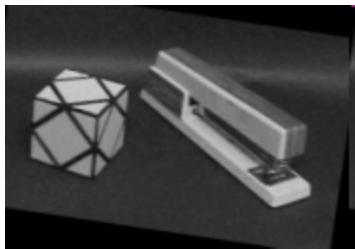
S. M. Seitz and C. R. Dyer, *Proc. SIGGRAPH 96*, 1996, 21-30



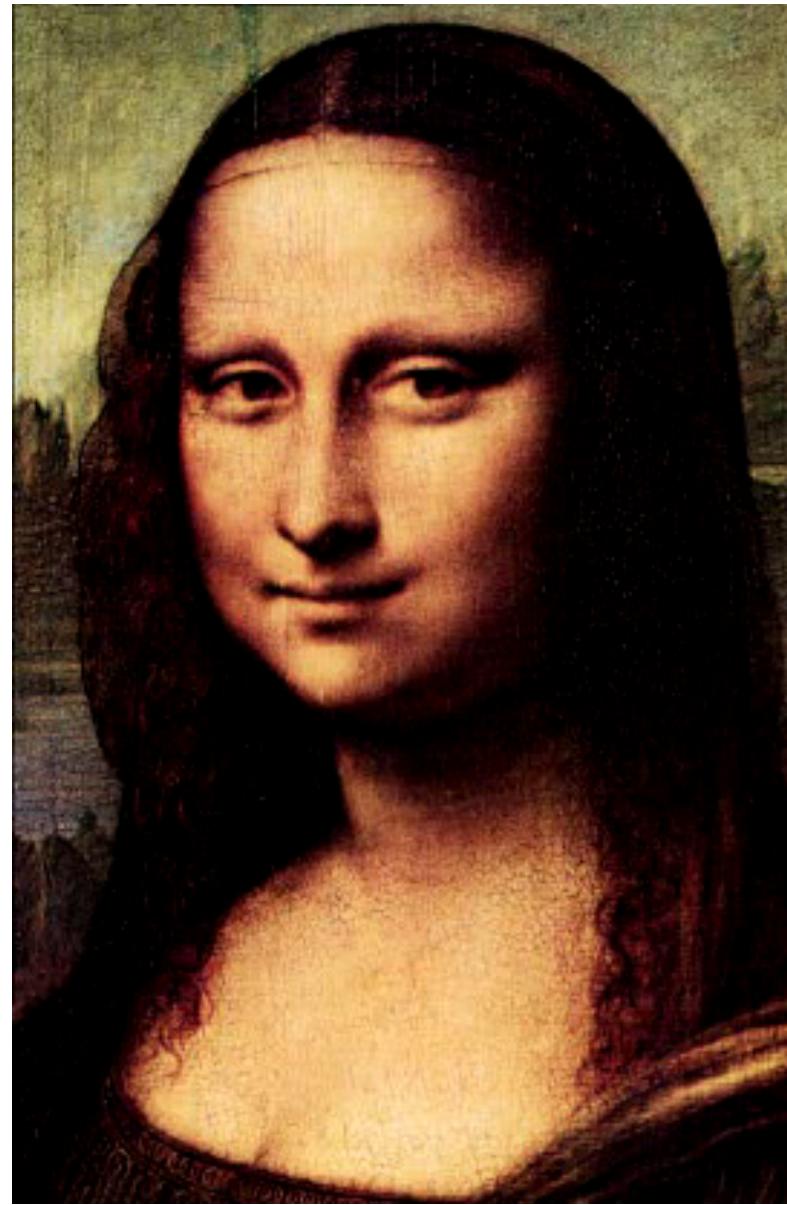
# Rectification



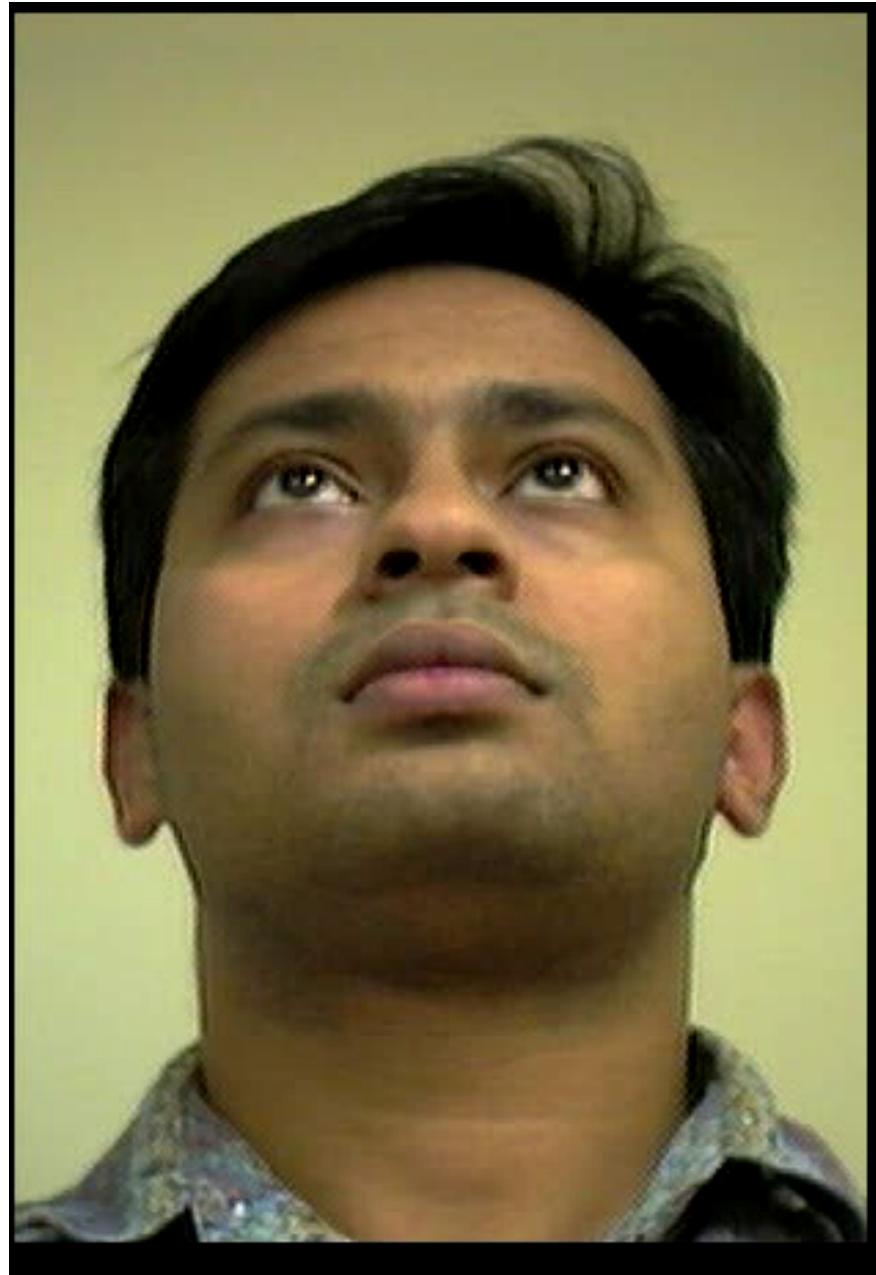








From its reflection!



# The Fundamental Matrix Song

