Defining conditional probability

In the discrete case

$$P\left(A_2|A_1\right) = \frac{P\left(A_1 \cap A_2\right)}{P\left(A_1\right)}$$

In general, the definition is not so simple because we can't divide by zero. Instead, the definition is inspired by partitioning the outcome space:

$$\cup_{i=1}^{n} A_2^i = \Omega$$

$$A_2^i \cap A_2^j = 0 \text{ for } i \neq j$$

$$i = 1, ..., n$$

$$P\left(A_1\right) = \sum_{i=1}^{n} P\left(A_1 \cap A_2^i\right)$$

$$P\left(A_1\right) = \sum_{i=1}^{n} P\left(A_1|A_2^i\right) P\left(A_2^i\right)$$

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Definition of conditional probability for arbitrary distributions

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random variable X with distribution Q_X

collection of probability measures P_x – conditional distribution given X

requirements:

(1) $P_x\left(\{X = x\}\right) = 1$ except for a set N with $Q_X\left(N\right) = 0$

(2) $P_x\left(A\right)$ must be a Borel measurable function (we call such P_x a probability kernel)

(3) For all events $A \in \mathcal{A}$ ($\mathcal{A}$ is the collection of events in the probability space $(\Omega, \mathcal{A}, P)$):

$$P(A) = \int_{\mathbb{R}} P_x(A)\, dQ_X(x)$$

$P_x(A)$ is the same as $P(A|X = x)$

note: $\{X = x\}$ represents the event of the random variable X taking value x

$$P(A \cap \{x_1 < X < x_2\}) = \int_{(x_1,x_2)} P_x(A)\, dQ_X(x)$$

$$P(A|\{x_1 < X < x_2\}) = \frac{P(A \cap \{x_1 < X < x_2\})}{P(\{x_1 < X < x_2\})}$$

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$$P(A_2|A_1) = \frac{P(A_1 \cap A_2)}{P(A_1)}$$

$$f_{X,Y}(x, y) \text{ ... pdf}$$

$$f_Y(y_0) = \int_R f_{X,Y}(x, y_0)\, dx$$

$$f(x|y = y_0) = \frac{f_{X,Y}(x, y_0)}{f_Y(y_0)}$$

Review of integral notation

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Non-negative random variable X

$$E(X) = \int_\Omega X(\omega)\, dP(\omega) = \int X\, dP$$

General random variable X

$$E(X) = E(X_+) - E(X_-) = \int_\Omega X(\omega)\, dP(\omega) = \int X\, dP$$

Random variable defined as a function of another random variable: $Y = f(X)$

$$x \sim \omega$$

$$\text{set of } x \sim \Omega$$

$$E(Y) = \int_R f(x)\, dP_X(x) = \int_\Omega f(X(\omega))\, dP(\omega)$$

If X has cdf F_X, we can write this also as

$$E(Y) = \int_R f(x)\, dF_X(x)$$

$$\lim_{x \to \infty} F_X(x) = 1$$

$$\lim_{x \to -\infty} F_X(x) = 0$$