Chi-squared distribution

The chi-squared distribution with m degrees of freedom is denoted $\chi^2(m)$.

By definition, if we have m variables A_i that are independent and follow $N(0,1)$, then

$$\sum_{i=1}^{m} A_i^2$$

follows

$$\chi^2(m)$$

Derivation of the pdf of the chi-squared distribution

Pdf of the standard normal

.

$$f_w(w) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}w^2\right)$$

$k = 1$ degrees of freedom

$$a = w^2$$

$$da = 2w\,dw$$

$$f_a(a) = 2\tilde{f}_a(a)$$

$$\tilde{f}_a(a)\,da = f_w(w)\,dw$$

$$(w, w + dw) \to (a, a + da)$$

$$\tilde{f}_a(a) = f_w(w)\frac{dw}{da} = f_w(w)\frac{1}{2w}$$

$$f_a(a) = f_w(|w|)\frac{1}{|w|}$$

$$f_a(a) = f_w\left(\sqrt{a}\right)\frac{1}{\sqrt{a}} = \frac{1}{\sqrt{2\pi}}\frac{1}{\sqrt{a}} \exp\left(-\frac{1}{2}a\right)$$

$k = 2$ degrees of freedom

$$w \equiv \sqrt{w_1^2 + w_2^2}$$

$$a = w^2$$

$$da = 2w\, dw$$

$$w_1 = w \cos\phi$$

$$w_2 = w \sin\phi$$

$$f_a(a)\, da = \int_\phi f_{w_1.w_2}(w_1, w_2)\, w\, dw\, d\phi$$

$$f_a(a)\, da = f_{w_1.w_2}(w_1, w_2)\, (2\pi w\, dw)$$

$$f_a(a)\, da = 2\pi \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2} w_1^2\right) \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2} w_2^2\right) w\, dw$$

$$f_a(a)\, da = \exp\left(-\frac{1}{2} w^2\right) w\, dw$$

$$f_a(a)\, da = \frac{1}{2} \exp\left(-\frac{1}{2} a\right) da$$

$$f_a(a) = \frac{1}{2} \exp\left(-\frac{1}{2} a\right)$$

Student's t-distribution

Student's t-distribution $t(m)$ with m degrees of freedom, defined using a normal distribution and a chi-squared distribution.

$$W \sim N(0,1)$$

$$S \sim Chi^2(m) = \chi^2(m)$$

S and W are independent.

$$\frac{W}{(S/m)^{1/2}} \sim t(m)$$