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CS109 Handout #5

Introduction to Predicate Logic

Now that we have a good understanding of propositional logic, we need to generalize it to **predicate** or **first-order logic**. **Predicates** are simply functions of zero or more variables that return a Boolean. So, predicates can be true sometimes and false other times, depending on the values of the variables.

Using predicates as operands instead of propositional variables gives us a much more powerful language than propositional logic. In predicate logic, we can manipulate the variables in a way that is just not possible with propositional logic. In fact, predicate logic is expressive enough to form the basis of a number of useful programming languages including Prolog, and SQL (structured-query language). Predicate logic is also used in reasoning systems or "expert" systems, such as medical-diagnosis programs and theorem-proving programs.

As mentioned above, a statement that would be a proposition except for the fact that it includes variables whose values are not specified is called a predicate. The following statements are predicates:

1) $x+y = 4$
2) The sum of the first n odd integers is n^2
3) if $x < 3$, then the square of $x = -1$.
4) $x+y = 4$ if and only if $y = 4-x$

A predicate involving one variable is called a one-place predicate; one involving two variables is called a two-place predicate; one involving n variables is called (you guessed it) an n-place predicate. The collection of values that can replace a variable in a predicate is called the **universe** of the predicate. In a one-place predicate, if a value from the universe can be substituted for the variable to make it true, we say the value **satisfies** the predicate. If a set of n values exists which can be substituted for the variables in an n-place predicate, the set satisfies the predicate. The predicate is said to be **satisfiable**. If *all* sets of n values satisfy the predicate, it is said to be **valid.**

We generally denote predicates with uppercase letters, with the variables following in parentheses: $P(x)$ or $Q(x,y,z)$. This gives predicates their "functional" look. Two predicates are **equivalent** if they have the same truth value for all possible values of their variables.

Some of the terminology relating to predicates should seem familiar. Remember in algebra how one would say -1 *satisfies* the equation $x + 4 = 3$. This equation is a predicate, and solving it means we are trying to find all values of the variables which satisfy the predicate.

When all the variables in a predicate have been assigned values, the resulting statement has a truth value. However, there is another important way to change predicates into propositions called **quantification.**

Consider a predicate from algebra P(a,b): a + b = b + a. How can P(a,b) be regarded as a proposition? No matter what values we substitute for a and b, this predicate is always true. So the truth of the statement does not depend on the values of a or b at all, but on the fact that the statement is true for all values. We say that the variables a and b have been **quantified universally**, and we indicate this fact with the symbol meaning "for all": a b P(a,b). The predicate P is true for all values of a and b in the universe. Note that it is standard practice to *not* explicitly state universal quantifiers if they apply to the entire predicate, unless it helps to avoid confusion. So, in the previous example we could have said just P(a,b).

Consider another predicate from algebra S(a,b): For every *a* there is a *b* such that a+b = 0 and b+a = 0. In this case, a has been quantified but has b? Given the value of a, we can find one value of b that makes the predicate true. Again, we do not care what the actual value is, we just care that one exists. In this case, we say that b is **quantified existentially**, i.e., the predicate can be made true by proper selection of the value. We indicate this with the symbol , so we say a b S(a,b) which translates to "for all a, there exists b such that S(a,b)". Note that changing the order of the quantifiers changes things considerably: b a S(a,b) means there is some value of b which can be chosen so as to make S true no matter what value of a is chosen.

If all the variables in a predicate have been quantified, we can determine if it is true.

Example

> What is the truth value of x P(x) where P(x) is the statement $x + 1 > x$ and the universe of the predicate is all real numbers?
>
> Since P(x) is true for all real numbers, the quantification is true.

Example

> What is the truth value of x P(x) where P(x) is the statement $x = x + 1$ and the universe of the predicate is all real numbers?
>
> Since P(x) is false for all real numbers, the quantification is false.

Summary of Quantifiers in One-Place Predicates

Statement	When true?	When false?
x P(x)	P(x) is true for all x	an x exists where P(x) is false
x P(x)	There is an x where P(x) is true	P(x) is false for all x

A variable that has been either quantified or has been substituted into a predicate is called a **bound** variable. Variables that have not been bound are **free**. With the "binding" of a variable, it becomes possible to determine the truth of the predicate. Statements with free variables are predicates while statements with all bound variables are propositions.

Lewis Carroll (the author of <u>Alice in Wonderland</u>) wrote several books on symbolic logic. He gives many examples of reasoning using quantifiers:

Example

All lions are fierce.
Some lions do not drink coffee.
Some fierce creatures do not drink coffee.

If P(x) denotes "x is a lion"; Q(x) denotes "x is fierce"; and R(x) denote "x drinks coffee", we can express the 3 statements above using quantifiers and P(x), Q(x), and R(x) (assuming the universe is the set of all creatures).

x (P(x) -> Q(x))
x (P(x) ^ ~R(x))
x (Q(x) ^ ~R(x))

Rules for Negation of Predicates

1) ~ x P(x) <=> x ~P(x)
2) ~ x P(x) <=> x ~P(x)

Example

P(x): x is taller than 2 feet; the universe of x is all adult males so we have the universally qualified statement "all adult males are taller than 2 feet". For this statement to be false, we just have to find one man who is shorter than 2 feet. So, the negation is "there is an adult male who is not taller than 2 feet". The negated original statement is ~ x P(x) and its negation is x ~P(x).

Similarly, let Q(x): x is shorter than 2.5 feet. To quantify this statement existentially is to say that there exists an adult male who is shorter than 2.5 feet. If we negate this statement, we are saying that there is no adult male shorter than

2.5 feet, i.e., all adult males are taller than 2.5 feet. The negated original statement is ~ x Q(x) and its negation is x ~Q(x).

Example

Q(x,y): x + y = 0. What are the truth values of the quantifications y x Q(x,y) and x y Q(x,y) if the universe is all real numbers?

y x Q(x,y) means there is a real number y such that for every possible real number x, Q(x,y) is true. No matter what value of y is chosen, there is one and only one value of x which will make x+y=0 true, so this quantification is false.

x y Q(x,y) means for every possible real number x, there exists a real number y such that x+y = 0. In this case, y = -x, and the quantification is true.

Summary of Quantifiers in Two-Place Predicates

Statement	When true?	When false?
x y P(x,y)	P(x,y) is true for every pair (x,y)	a pair (x,y) exists where P(x,y) is false
x y P(x,y)	For every x, there is a y for which P(x,y) is true	There is an x such that P(x,y) is false for every y
x y P(x,y)	There is an x for which P(x,y) is true for every y	For every x, there is a y for which P(x,y) is false
x y P(x,y)	There is a pair (x,y) for which P(x,y) is true	P(x,y) is false for all pairs (x,y)

In working with quantifications of more than one variable it is sometimes useful to think in terms of nested loops. Of course, if the universe is infinite this won’t quite work, but it’s still useful.

To check:

1) x y P(x,y): loop through all the values of x and for each x, loop through all the values of y. If P(x,y) is true for all these iterations, we have determined x y P(x,y) is true. If we ever hit an x value for which we hit a y value for which P(x,y) is false, then we know x y P(x,y) is false.

2) x y P(x,y): loop through all the values of x and for each x, loop through all the values of y until we find one y value for which P(x,y) is true. If for all x, we find such a y value, then we have determined x y P(x,y) is true. If for some x, we never find a y value for which P(x,y) is true, then we know x y P(x,y) is false.

3) x y P(x,y): What is the analogy for this one?

4) x y P(x,y): loop through the values for x, where for each x, we loop through the values for y until we hit an x for which we hit a y for which P(x,y) is true. If we can never find an x for which we hit a such a y, then it's false.

I'm sure you will be excited to learn that there are quantifications with more then two variables too:

Let Q(x,y,z) be the statement x + y = z. What are the truth values of the statements: x y z Q(x,y,z) and z x y Q(x,y,z)?

Use quantifiers to express the statement: "There is a woman who has taken a flight on every airline in the world."

P(x): "X is a professor"
Q(x): "X is ignorant"
R(x): "X is vain"

Express the following statements using quantifiers, logical connectives and P(x), Q(x), and R(x), where the universe is the set of all people.

a) No professors are ignorant.

b) All ignorant people are vain.

c) No professors are vain.

d) Does (c) follow from (a) and (b)?

As you can see, predicate logic is very useful for capturing facts, and organizing them in such a way that conclusions can be drawn. This is exactly what is needed in certain reasoning systems found in AI research.

More Practice...

1) If x y P(x, y) is true, does it necessarily follow that x y P(x, y) is true?

2) Express the following statement using quantifiers: Every student in this class has taken some course in every department in the School of Engineering. (You have been busy!)

Bibliography

As shown above, an entertaining way to study logic is to read Lewis Carroll's book:

L. Carroll, *Symbolic Logic*, New York: Crown, 1978.

General references for logic include:

E. Mendelson, *Introduction to Mathematical Logic*, New York: Van Nostrand Reinhold, 1964.

P. Suppes, *Introduction to Logic*, Princeton, NJ: D. Van Nostrand, 1987.

Historical Notes

The formal concept of a quantifier was introduced into the symbolic logic of George Boole by the American philosopher, logician and engineer Charles Sanders Peirce (1839 - 1914). He was the founder of **pragmatism**, a branch of philosophy which attempts to bring philosophical and intellectual pursuits into the realm of the practical. Peirce made his living working for the US Coast Survey doing gravity research, but his primary interests were in logic and philosophy. He was a very gifted scientist and logician; he was also ambidextrous such that he could write a question with one hand, and write the answer simultaneously with the other.