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Handout #8a
CS109

Problem Set #2 Solutions

Methods of Proof

Rosen, Section 3.1:

#16: if n is even then n^2 is even.

a)

statement	reason
n is even	given
n = 2k for some int k	definition of even
$n^2 = (2k)^2$	square both sides (algebra)
$(2k)^2 = 4k^2 = 2(2k^2)$	algebra
$2k^2$ is an int	k is an int
$2(2k^2)$ is even	definition of even

Thus, if n is even then n^2 is even.

c) if n is even then n^2 is not even (n^2 must be odd)

statement	reason
n is even	given
n = 2k for some int k	definition of even
$n^2 = (2k)^2$	square both sides (algebra)
$(2k)^2 = 4k^2 = 2(2k^2)$	algebra
$2k^2$ is an int	k is an int
$2(2k^2)$ is even	definition of even - contradiction

Thus, if n is even then n^2 is even.

#24 Prove that the product of a non-zero rational number and an irrational number is irrational. This is true. We give a proof by contradiction:

If a/b is a nonzero rational number and that x is an irrational number, then xa/b is rational (proof by contradiction)

statement	reason
a/b != 0	given
b != 0	definition of rational number
a != 0	arithmetic
b/a is rational	definition of rational number
xa/b * b/a = x	multiplication
x is irrational	given
product of rational and rational is rational	see helper proof - contradiction
xa/b is irrational	

Thus, if a/b is a nonzero rational number and that x is an irrational number, then xa/b is irrational

helper proof: if a/b and c/d are rational numbers, then their product is rational.

statement	reason
a/b * c/d = (ac)/(bd)	multiplication
ac is an int	a and c are ints

bd is an int	b and d are ints
bd != 0	neither b or d = 0 (definition of rational number)
a/b * c/d is rational	definition of rational

Thus, if a/b and c/d are rational numbers, then their product is rational.

#30 For all integers n not divisible by 5, $n^2/5$ leaves a remainder of 1 or 4.

By the definition of remainder and division, there are four possible remainders when an integer is divided by 5: 1, 2, 3 and 4. We can represent these 4 cases as 5k+1, 5k+2, 5k+3 and 5k+4. We must then show that the remainder when n^2 is divided by 5 is 1 or 4.

<u>Proof by cases:</u>
n=5k+1: $n^2 = (5k+1)^2 = 25k^2 + 10k + 1 = 5(5k^2+2k) + 1$; remainder =1
n=5k+2: $n^2 = (5k+2)^2 = 25k^2 + 20k + 4 = 5(5k^2+4k) + 4$; remainder =4
n=5k+3: $n^2 = (5k+3)^2 = 25k^2 + 30k + 9 = 5(5k^2+6k) + 9$; remainder =4
n=5k+4: $n^2 = (5k+4)^2 = 25k^2 + 40k + 16 = 5(5k^2+8k) + 16$; remainder =1

Therefore, we have shown that in all cases : For all integers n not divisible by 5, $n^2/5$ leaves a remainder of 1 or 4.

#54 Prove or disprove: Given a positive integer n, there are n consecutive positive odd integers that are prime.

This is false: In fact, for n >= 4, there are never n consecutive odd positive integers that are all primes. To see this, note that among any three consecutive odd numbers, exactly one will be divisible by 3. As we count through the odd numbers, the remainders modulo 3 go 1, 0, 2, 1, 0, 2... No number divisible by 3 is prime except for 3 itself. So the only case in which even 3 consecutive odd integers can be all prime is 3, 5, 7 (which is not enough to prove this). It can never happen that four odd consecutive odd numbers are all prime.

Inductions

#6 We guess by looking at the first few sums, that the sum is n/(n+1). We prove this using induction.

P(n): 1/(1*2) + 1/(2*3) +...+ 1/(n(n+1)) = n/(n+1).

base case: When n = 1, we have 1/(1*2) = 1/2; n/(n+1) also = 1/2. The base case is proven.

inductive hypothesis: Assume that 1/(1*2) + 1/(2*3) +...+ 1/(n(n+1)) = n/(n+1), we must prove that [1/1*2 + 1/2*3 +...+ 1/(n(n+1))] + 1/(n+1)(n+2) = (n+1)/(n+2).

Proof:
[1/1*2 + 1/2*3 +...+ 1/(n(n+1))] + 1/(n+1)(n+2)
= n/(n+1) + 1/(n+1)(n+2) — inductive hypothesis

$n/(n+1) + 1/(n+1)(n+2) = (n^2 + 2n + 1)/(n+1)(n+2)$ — algebra

$(n^2 + 2n + 1)/(n+1)(n+2) = (n+1)/(n+2)$.

By the principle of mathematical induction, P(n) is true for all n.

#8 P(n): $1^3 + 2^3 + \ldots + n^3 = [n(n+1)/2]^2$

base case: n=1; $1^3 = 1$; $[1(1+1)/2]^2 = 1$. Base case is proven.

inductive hypothesis: Assume that $1^3 + 2^3 + \ldots + n^3 = [n(n+1)/2]^2$ and we must prove that $1^3 + 2^3 + \ldots + n^3 + (n+1)^3 = [(n+1)(n+2)/2]^2$

Proof:

$1^3 + 2^3 + \ldots + n^3 + (n+1)^3 = [n(n+1)/2]^2 + (n+1)^3$	inductive hypothesis
$[n(n+1)/2]^2 + (n+1)^3 = (n+1)^2 [(n^2 / 4) + n + 1] =$ $(n+1)^2 [(n^2 + 4n + 4) / 4] = [(n+1)(n+2)/2]^2$	algebra

By the principle of mathematical induction, P(n) is true for all n.

#12 P(n): $3^n < n!$ for $n > 6$.

base case: n = 7; $3^7 < 7!$ since 2187 < 5040. Base case is proven.

inductive hypothesis: Assume $3^n < n!$ for $n > 6$. We must prove that $3^{n+1} < (n+1)!$

Proof:

$(n+1)! = (n+1) * n!$	definition of factorial
$3^{n+1} = 3 * 3^n$	law of exponents
$3 * 3^n < (n+1) * 3^n$	3 must be less than n+1
$< (n+1) * n!$	inductive hypothesis
$< (n+1)!$	substitution

By the principle of mathematical induction, P(n) is true for all n.

#22 P(n): 6 divides $n^3 - n$ for $n >= 0$, i.e., $6 \mid n^3 - n$ for $n >= 0$

base case: n = 0; 6 divides 0 - 0 since $6 \mid 0$. base case is proven.

inductive hypothesis: Assume that $6 \mid n^3 - n$. We must prove $6 \mid (n+1)^3 - (n+1)$.

Proof:

$(n+1)^3 - (n+1) = n^3 + 3n^2 + 3n + 1 - n - 1$	algebra
$n^3 + 3n^2 + 3n + 1 - n - 1 = (n^3 - n) + 3n(n+1)$	algebra
$6 \mid n^3 - n$ (the first term above)	inductive hypothesis
3 divides the second term	algebra
the second term is even	one of n and n+1 must be even
$6 \mid 3n(n+1)$	algebra
$6 \mid (n^3 - n) + 3n(n+1) = (n+1)^3 - (n+1)$	

By the principle of mathematical induction, P(n) is true for all n.

#32 P(n): From dimes and quarters, we can form all multiples of 5 cents greater then or equal to 20 cents, as well as 10 cents.

base case: 20 cents can be formed from 2 dimes. base case is proven.

inductive hypothesis: We can form 5k cents (where k >= 4); we must prove that we can form 5(k+1) = 5 k + 5 cents from dimes and quarters.

Proof:
If a quarter is used to form 5k cents, replace it with 3 dimes to get 5k+5 cents.

If a quarter is not used, then at least two dimes were used ($k >= 4$, and $5k >= 20$). So, replace the 2 dimes with one quarter to get $5k+5$ cents.

By the principle of mathematical induction, P(n) is true for all n.

#44 P(n): ~(p1 v p2 v ... v pn) <=> ~p1 ^ ~p2 ^ ... ^ ~pn

base case: n = 1; ~(p1) <=> ~p1. base case is proven.

inductive hypothesis: Assume ~(p1 v p2 v ... v pn) <=> ~p1 ^ ~p2 ^ ... ^ ~pn and we must prove:
~(p1 v p2 v ... v pn v p(n+1)) <=> ~p1 ^ ~p2 ^ ... ^ ~pn ^ ~p(n+1)

Proof:

~(p1 v p2 v ... v pn v p(n+1)) <=> ~(p1 v p2 v ... v pn) ^ ~p(n+1)	DeMorgans
~(p1 v p2 v ... v pn) ^ ~p(n+1) <=> ~p1 ^ ~p2 ^ ... ^ ~pn ^ ~p(n+1)	inductive hypothesis

By the principle of mathematical induction, P(n) is true for all n.

#48 Where we invoke the inductive hypothesis, we have a flaw where we are implicitly assuming that $n >= 1$ in order to talk about a^{n-1} in the denominator (otherwise the exponent is not a nonnegative integer, so we cannot apply the inductive hypothesis). We checked the base case for $n = 0$ only so we are not justified in assuming that $n >= 1$ when we try to prove $n+1$ in the inductive step. Indeed, it is precisely at $n=1$ that the proposition breaks down.

#54 P(n) denotes: n lines separate a plane into $(n^2 + n + 2)/2$ regions if no two of these lines are parallel and no three pass through a common point.

i) base case: show that P(1) is true: one line separates a plane into two regions and $(1^2+1+2)/2 = 2$
ii) induction: assume P(k): k lines separate a plane into $(k^2 + k + 2)/2$ regions if no two of these lines are parallel and no three pass through a common point; show P(k+1): k+1 lines separate a plane into $((k+1)^2 + (k+1) + 2)/2$ regions if no two of these lines are parallel and no three pass through a common point.

<u>PROOF:</u>
Consider an arrangement of k+1 lines. Remove the last line. We know by the inductive hypothesis that there are $(k^2 + k + 2)/2$ regions created. Now we put the last line back in drawing it very slowly and watching what happens to the regions. As we come in "from infinity", the line separates one infinite region into two (one on each side of it). This separation is complete as soon as the line hits one of the first k lines. Then, as we continue drawing from this first line of intersection to the second, the line again separates one region into two. We continue in this way. Every time we come to another point of intersection between the line we are drawing and the figure already present, we lop off an additional region. Furthermore, once we leave the last point of intersection and draw our line off to infinity again, we separate another region into two. Therefore, the number of additional regions formed = the number of intersections + 1. There are k intersections because the new line must intersect each of the other lines in a distinct point, so this new arrangement has k+1 more points of intersection than the previous arrangement and therefore,

$$\begin{aligned}(k^2 + k + 2)/2 + (k + 1) &= (k^2 + k + 2 + 2k + 2)/2 \\ &= (k^2 + 2k + 1 + k+1 + 2)/2 \\ &= ((k + 1)^2 + (k+1) + 2)/2 \quad \text{regions}\end{aligned}$$

P(k+1) is true when P(k) is true, and therefore P(n) is true for any number n of lines.

#56 P(n): $21 \mid 4^{n+1} + 5^{2n-1}$ for $n > 0$

base case: n = 1; 21 divides 16 + 5 = 21

inductive hypothesis: Assume $21 \mid 4^{n+1} + 5^{2n-1}$ and we must prove $21 \mid 4^{n+2} + 5^{2n+1}$

Proof:

$$4^{n+2} + 5^{2n+1} = 4 * 4^{n+1} + 25 * 5^{2n-1}$$
$$= 4 * 4^{n+1} + (4+21) * 5^{2n-1}$$
$$= 4 * (4^{n+1} + 5^{2n-1}) + 21 * 5^{2n-1}$$

By the inductive hypothesis, the expression in the parenthesis is divisible by 21, and obviously the second term is divisible by 21, so the whole thing is divisible by 21.

By the principle of mathematical induction, P(n) is true for all n.

P(n) : We must prove that all B-string of length n have odd parity. Let's prove it by induction on the length of the string.

Basis: There are four base cases :
Case 1: n=1, there is no B-string of length 1 so P(1) holds.
Case 2: n=2, there is no B-string of length 2 so P(2) holds.
Case 3: n=3, there is only one B-string of length 3 , from rule 2, that string is 100, which has odd parity so P(3) holds.
Case 4: n=4, there is only one B-string of length 4, from rule 1, that string is 0111, which had odd parity so P(4) holds.
Inductive hypothesis: Assume that P(n) holds for all n<=k
Prove that P(k+1) is true.

Proof:
All B-strings of length greater than 4 must be generated using rule 3 and 4 since rule 1 and 2 generate string only of length 4 and of length 3, respectively. Therefore, since k>=4, k+1>=5, any B-string B of length k+1 must be of the form $1B_1010B_21$ or B_11B_2.
Case 1 : $B=1B_1010B_21$. From our inductive hypothesis, B_1 and B_2 must have odd parity since their length is less than k-4. Letting |B| signify the sum of the digits of B, $|B| = |B_1| + |B_2| + 3$ which is odd if |B1| and |B2| are odd, so the parity of B is odd.
Case 2 : $B=B_11B_2$. From our inductive hypothesis, B_1 and B_2 must have odd parity since their length is less than k. Letting |B| signify the sum of the digits of B, $|B| = |B_1| + |B_2| + 1$ which is odd if |B1| and |B2| are odd, so the parity of B is odd.
Therefore P(k+1) is true and, by the principle of mathematical induction, P(n) is true for all n>=1.