Maggie Johnson
CS109

Problem Set #5 Solutions

1) There are at most 2 real solutions of each quadratic equation so the number of solutions is countable as long as the number of triples (a,b,c) with a, b, and c as integers is countable. This follows from Exercise 37 (see Student Solutions Guide): There are a countable number of pairs (b,c) since for each b (and there are countably many b's), there are only a countable number of pairs with b as its first coordinate. Now, for each a (and there are countably many a's), there are only a countable number of triples with that a as its first coordinate (since we just showed there are a countable number of pairs (b,c)). So, using the result from Exercise 37, there are only countably many triples.

2) Disprove: Suppose there were such a book. If such a book did not refer to itself, then it would belong to the set of all books that do not refer to themselves. But it is supposed to refer to all books in this set, and so it would refer to itself. On the other hand, if such a book referred to itself, then it would belong to the set of books to which it refers and this set contains only books that do not refer to themselves. Thus, it would not refer to itself. The assumption that such a book exists leads to a contradiction so it cannot exist.

3) For the union, take the union of the directed graphs, i.e., create a new graph with vertices representing the union of the two vertex sets of the graphs of the relations. Then, put an edge from vertex i to j whenever there is an edge from i to j in either of the two directed graphs representing the two relations.

For intersection, put an edge from i to j if there is an edge from i to j in both of the graphs; To form the difference, put an edge from i to j whenever there is an edge from i to j in the first but not the second; and for composition, (S o R of relations R and S), put an edge from i to j whenever there is a vertex k such that there is an edge from i to k in R and from k to j in S.

4) In each case, we construct simplest possible such relation:
a) {(a,a), (b,b), (c,c), (a,b),(b,a), (b,c),(c,b),(d,d)}
b)
c) {(a,b), (b,c)}
d) {(a,a), (b,b), (c,c), (a,b),(b,a), (c,a), (c,b), (d,d)}
e) {(c,c), (a,b),(b,a), (c,a)}

5) This follows from Exercise 5 in section 6.5, where f is the function from the set of pairs of positive integers to the set of positive rational numbers that takes (a,b) to a/b, since ad = bc if and only if a/b = c/d.

6) We necessarily get an equivalence relation when we form the symmetric closure of the reflexive closure of the transitive closure of a relation.

This is not true: the result is not necessarily transitive. Here is a counterexample: the original relation is {(1,3), (2,3)}. Transitive closure adds nothing; reflexive adds {(1,1) (2,2) (3,3)}; symmetric closure adds {(3,1), (3,2)} for a final set of {(1,1), (1,3), (2,2), (2,3), (3,1), (3,2), (3,3)} which is not an equivalence relation; ((2,3) and (3,1) give (2,1) which is not in the set).

7) Any subset of the “diagonal line” of A x A in which (a,b) is an element of R implies a = b is both symmetric and anti-symmetric.

8) R is not reflexive because a line is never perpendicular to itself (in fact it is parallel to itself)
R is symmetric because if A^B, B^A.
R is not transitive because if A^B and B^C then A//C not A^C.

9a) chains (a subset of a chain is also a chain so we list only maximal chains here):

a) {a,b,c} and {a,b,d}
b) {a,b,e}, {a,b,d}, {a,c,d}
c) there are 9 maximal chains each consisting of one element from the top row, the element in the middle and one element from the bottom row.

antichains:

a) {c,d}
b) {b,c}, {c,e}, {d,e}
c) {a,b}, {a,c}, {b,c}, {a,b,c}, {d,e}, {d,f}, {e,f}, {d,e,f}

9b) The vertices are arranged in 3 columns. Each pair of vertices in the same column are comparable. Therefore the largest antichain can have at most 3 elements. One such antichain is {a, b, c}.

10) P(n): A set with n elements has n(n-1)(n-2)/6 subsets containing exactly three elements whenever n is an integer greater than or equal to 3.

Base case: n = 3: The set has exactly one subset and 3(2)(1) / 6 = 1.
Inductive hypothesis: Assume P(k): A set with k elements has k(k-1)(k-2)/6 subsets with exactly three elements.
Prove P(k+1): A set with k+1 elements has (k+1)(k)(k-1)/6 subsets with exactly three elements.

Proof:
Fix an element a in some set S. Let T be the set S without the element a. There are two varieties of subsets of S containing exactly three elements. First, there are those that do not contain a. These are precisely the three-element subsets of T, and by the inductive hypothesis, we have k(k-1)(k-2)/6 such subsets. Second, there are those that contain a together with two other elements. There are exactly k(k-1)/2 such subsets (see helper proof below). Therefore, we have k(k-1)/2 three-element subsets of S containing a. The total number of subsets = k(k-1)(k-2)/6 + k(k-1)/2:

$$\begin{aligned} k(k-1)(k-2)/6 + k(k-1)/2 &= [k(k-1)(k-2) + 3k(k-1)] / 6 \\ &= [(k^3 - 3k^2 + 2k) + 3k^2 - 3k]/6 \\ &= [k^3 - k] / 6 \\ &= (k+1)(k)(k-1)/6 \end{aligned}$$

By the principle of mathematical induction, P(n) is true for all n >= 3.

Helper proof:

P(n) denotes a set with n elements has n(n-1)/2 subsets containing exactly two elements whenever n is an integer greater than or equal to 2.

Base case: prove that $P(2)$ is true: a set with two elements has one subset with two elements (itself) and $2(2-1)/2 = 1$
Inductive hypothesis: assume $P(k)$: a set with k elements has $k(k-1)/2$ subsets containing exactly two elements whenever k is an integer greater than or equal to 2;
Prove: $P(k+1)$ is true: a set with k+1 elements has $(k+1)k / 2$ subsets containing exactly two elements whenever k is an integer greater than or equal to 2.

Proof:
Let S be a set with k+1 elements. Choose an element a in set S and let set $T = S - \{a\}$. A two-element subset of S either contains a or it does not. Those subsets not containing a are exactly the subsets of 2 elements of set T which we know by the inductive hypothesis that there are $k(k-1)/2$ of these. There are k subsets of 2 elements of S that contain a, since such a subset contains a and one of the k elements in T. Thus, there are

$$\begin{aligned} k(k-1)/2 + k &= (k^2 - k)/2 + k \\ &= (k^2 + 2k - k) / 2 \\ &= k(k+1)/2 \\ &= (k+1)k / 2 \text{ subsets of 2 elements in S} \end{aligned}$$

$P(k+1)$ is true when $P(k)$ is true, and therefore $P(n)$ is true for sets with 2 or more elements.