Maggie Johnson Handout #24
CS109

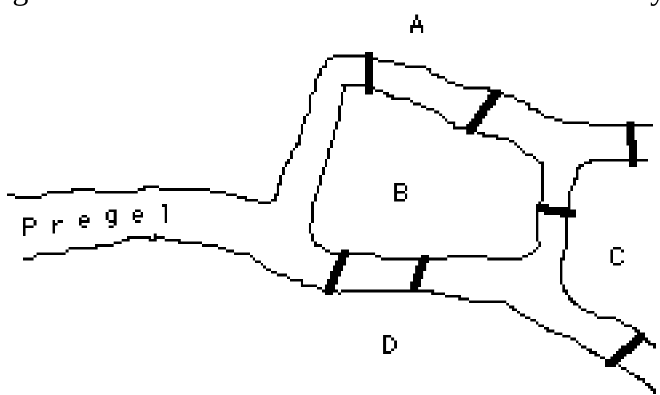
Paths and Circuits

Key Topics

* The Bridges of Königsberg
* Some Definitions
* Euler Circuits
* Hamilton Circuits
* The Travelling Salesperson problem

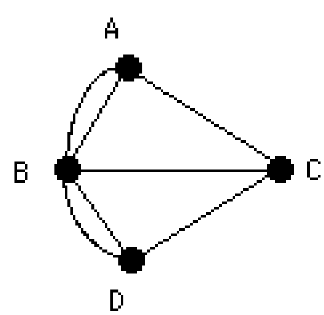
In the introductory graph handout, we talked about how the subject of graph theory began in 1736 when the great mathematician Leonhard Euler published a paper that contained the solution to the following puzzle:

> The town of Königsberg in Prussia (now Kaliningrad in Russia) is built at a point where two branches of the Pregel River join. The town consists of an island and some land along the river banks. All land masses are connected by seven bridges:



> Given this geographical situation, is it possible for a person to take a walk around town, starting and ending at the same location, and crossing each of the seven bridges exactly once?

To solve this problem, Euler modeled it using what we now call a graph. He noticed that all the points of a given land mass can be identified with each other because the person can travel from any point on the land mass to any other point without crossing a bridge. So, his map of Konigsberg looked like this:



So, in terms of the graph, the question is: *Is it possible to find a route through the graph that starts and ends at some one vertex, and traverses each edge exactly once? i.e.: Is it possible to trace this graph, starting and ending at the same point, without ever lifting your pencil from the paper, and without drawing over the same line twice?* ____

Any way you try, you end up on a vertex that does not have an unused edge on which to leave. If we start at A, for example, each time we pass through B, C and D, we use up two edges, one for arrival and one for departure. Therefore, the degrees of B, C and D must be even (a multiple of 2) in order for us to start at A and pass through B, C and D using each edge only once. The deg(B) = 5 and deg(C, D) = 3, so it is not possible to travel all around the city crossing each bridge only once.

Some Definitions

Travel around the edges of a graph is accomplished by an alternating sequence of vertex, edge, vertex, edge... Certain types of sequences of adjacent vertices along connecting edges are of special importance in graph theory.

Let G be a graph and v and w be vertices in G.

A **walk** from v to w is a finite alternating sequence of adjacent vertices and edges of G. The **trivial walk** from v to v consists of a single vertex v.

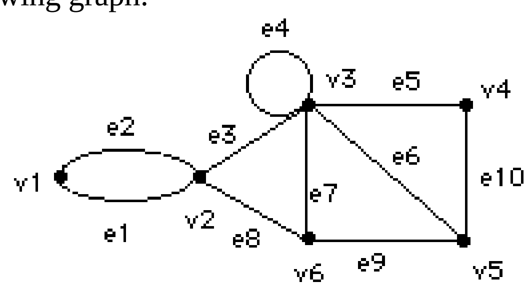
A **path** from v to w is a walk from v to w that does not contain a repeated edge. A **simple path** from v to w is a path that does not contain a repeated vertex.

A **circuit** is a path that starts and ends at the same vertex and does not contain a repeated edge. A **simple circuit** starts and ends at the same vertex and does not have any other repeated vertex.

The following table summarizes these definitions:

	repeated edge?	repeated vertex?	starts/ends same point
walk	allowed	allowed	possibly
path	no	allowed	possibly
simple path	no	no	no
circuit	no	allowed	yes
simple circuit	no	1st/last only	yes

Given the following graph:



Which of the following walks are paths, simple paths, circuits and simple circuits?

a. $v_1\ e_1\ v_2\ e_3\ v_3\ e_4\ v_3\ e_5\ v_4$

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b. $e_1\ e_3\ e_5\ e_5\ e_6$

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c. $v_2\ v_3\ v_4\ v_5\ v_3\ v_6\ v_2$

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Let G be a graph. Two vertices v nd w of G are **connected** iff there is a walk from v to w. The graph G is **connected** iff, given *any* two vertices v and w in G, there is a walk from v to w.

Euler Circuits

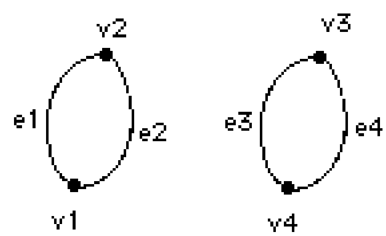
Let G be a graph. An **Euler circuit** for G is a circuit that contains every vertex and every edge of G. That is, an Euler circuit is a sequence of adjacent vertices and edges that starts and ends at the same vertex, uses every vertex at least once (maybe more), and uses every edge exactly once.

The analysis given above on the Königsberg puzzle can be generalized as follows:

If a graph has an Euler circuit, then every vertex of the graph has even degree.

A proof of this statement is trivial: we know that we start and end at the same vertex, and use each edge only once. Since we cannot use an edge more than once and we must visit each vertex, there must be an entry edge and exit edge from each vertex. If there is an additional edge, we either won't be able to use it or it will leave us stranded at a particular vertex.

This theorem can be used to show if a graph has an Euler circuit, i.e., if there is a vertex of odd degree, there cannot be an Euler circuit. What about if every vertex of a graph is of even degree, can we assume it has an Euler circuit? Only if the graph is connected. The following unconnected graph has all vertices of even degree, but no Euler circuit.

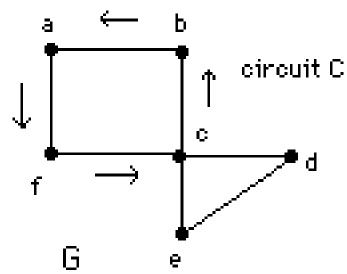


If, however, every vertex of a nonempty graph has even degree and if the graph is connected, then the graph has an Euler circuit. We will give a constructive proof of this consisting of an algorithm to find an Euler circuit for any connected graph in which every vertex has even degree.

Proof: Suppose G is a nonempty, connected graph, and all its vertices are of even degree, Construct a circuit C using the following algorithm:

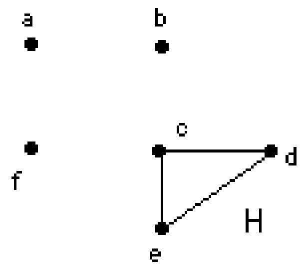
1) Pick any vertex v of G to start; we can do this because the graph is nonempty.

2) Pick any sequence of adjacent vertices and edges, starting and ending at v. Call this circuit C. We can perform this step because the degree of each vertex is even, so there is always a way out of any vertex via an unused edge. This path may or may not use all the edges in the graph. In the following graph G, the starting point v = a, so we start at a, go to f, c, and b.



3) Check if C contains all the edges and vertices of G. If so, we have found an Euler circuit. If not, do the following:

a) Remove all the edges of circuit C, and also any vertices that become isolated when the edges of C are removed. Call the resulting subgraph H (which may or may not be connected, but every vertex of H still has even degree since removing the edges of C removes an even number of edges from each vertex, and the difference of two evens is even).



b) Pick any vertex w, common to H and C - there must be at least one since G is a connected graph. (In the graph above, w = c.)

c) Pick any sequence of adjacent vertices and edges, starting and ending at w. Call this circuit C'.

d) Patch C and C' together to create C" by doing the following: Start at v and and follow C until you reach w. Then, follow C' back around to w. After that, continue along the untravelled portion of C back to v.

e) Let C = C" and goto step 3.

Since the graph G is finite, these steps will eventually terminate. At that point, we have an Euler circuit for the graph. Since there are lots of choices in this algorithm, we can come up with several Euler circuits for any particular graph.

The characteristics of an Euler walk can be applied to a path as well:

Let G be a graph and let v and w be two vertices of G. An **Euler path** from v to w is a sequence of adjacent edges and vertices that starts at v and ends at w, passes through every vertex of G at least once, and traverses each edge of G exactly once. An Euler path between v and w exists iff G is connected and v and w have odd degree and all other vertices of G have even degree.

Hamilton Circuits

We have handled all the possibilities of repeated edges in our discussion of Euler circuits and paths. What about repeated vertices? Is it possible to find a circuit for G in which all the vertices appear exactly once (except the first and last)? This idea comes from a puzzle invented in 1857 by the Irish mathematician Sir William Rowan Hamilton. Hamilton's puzzle consisted of a wooden dodecahedron (a solid figure with 12 regular pentagons as faces). The twenty vertices of the figure were labeled with the names of large cities around the world. The object was to start at one city and travel to each of the other cities exactly once, and return to the starting city. You don't have to use every edge, but you cannot repeat edges either. The graph representation of such a solid figure is given below along with one possible solution.

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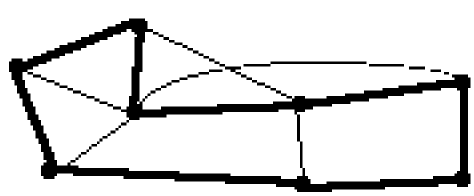
Given a graph G, a **Hamilton Circuit** for G is a simple circuit that includes every vertex of G. That is, a Hamilton circuit for G is a sequence of adjacent vertices and distinct edges in which every vertex of G appears exactly once.

So, an Euler circuit must include every vertex of G, but it may visit some vertices more than once in hitting every edge exactly once. On the other hand, a Hamilton circuit for G does not need to include all the edges of G, but must hit all the vertices exactly once.

Despite the analogies between the two, the mathematics of Euler vs Hamilton circuits are quite different. Since it was so easy to find an Euler circuit for a graph with particular characteristics, one might think that it is just as easy to find a Hamilton circuit. Surprisingly, there is no known simple, necessary criteria for the existence of Hamilton circuits. However, there is a simple technique for showing that a particular graph *cannot* have a Hamilton circuit. This comes from the following observations:



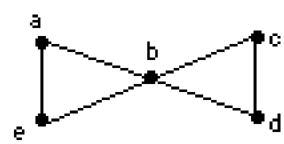
Suppose a graph G has a Hamilton circuit C; we know all the edges of the circuit are distinct and each vertex of G is visited exactly once. Let H be the subgraph of G that is formed by taking only the vertices and edges of C.

Note that H has the same number of edges as vertices since each of its edges are distinct and so are its vertices. We can also note that H has to be connected since it is a circuit. Every vertex of H is of degree 2 because there must be exactly two edges incident on each vertex.

If graph G has a Hamilton circuit, then G has a subgraph H with the following properties:

1) H contains every vertex of G
2) H is connected
3) H has the same number of edges as vertices
4) every vertex of H has degree 2.

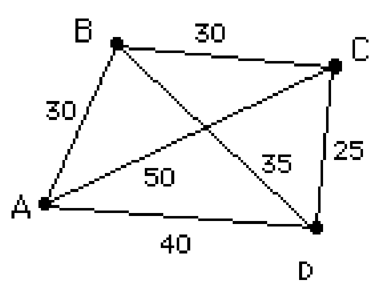
We can use this proposition to show that a particular graph does not have a Hamilton circuit.
Does the following graph have a Hamilton circuit? ________________________________



If this graph has a Hamilton circuit, there is a subgraph H with the properties described above. Suppose such a subgraph exists: there is a subgraph that has five vertices (a, b, c, d, e) and five edges such that each node of H has degree 2. The degree of b in G is 4, so we have to remove 2 edges from b in order to create H. But the removal of any 2 edges from b reduces the total number of edges in H to 4, (which must have 5 edges). So, G cannot have any such subgraph H. (Note that this graph does have an Euler circuit: a, b, c, d, b, e, a).

The Travelling Salesperson problem

The Travelling Salesperson problem concerns Hamilton circuits: Suppose a travelling salesperson must travel to a series of cities exactly once, starting and ending in city A. *Which route will minimize the number of miles travelled?* _______________________________________



Recall from 109A that this is an NP-Complete problem**:** no polynomial solution has been found, although exponential solutions exist. The exponential solution is to find all possible Hamilton circuits starting and ending at the given city, and calculating the total distance for each circuit:

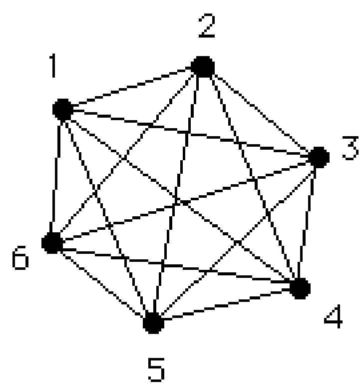
ABCDA	125 miles
ABDCA	140 miles
ACBDA	155 miles
ACDBA	140 miles
ABDCA	155 miles
ADCBA	125 miles

So, either route ABCDA or ADCBA is best. This is a manageable solution when the number of vertices is small. If, however, you have a complete graph on 30 vertices, there would be 29! = 8.84 * 10^{30} different Hamilton circuits starting and ending at a particular vertex. If each circuit could be found and its total distance computed in 1 microsecond, it would require 2.8 * 10^{17} years to finish the computation. At present, there is no known algorithm for solving this problem more efficiently, although there are algorithms that find "pretty good" solutions, i.e., it may not be the absolute minimal Hamilton circuit but it will be pretty small. One such algorithm for a "pretty good" solution is described below:

The Quick Travelling Salesperson Tour Construction

1) Pick any vertex as a starting circuit T_1 with one vertex and zero edges.
2) Given the k-vertex circuit T_k, find the vertex z_k not on T_k that is closest to a vertex, call it y_k, on T_k.
3) Let T_{k+1} be the k+1 circuit obtained by inserting z_k immediately before y_k in Tk.
4) Repeat steps 2 and 3 until a Hamilton circuit (containing all vertices) is formed.

We will trace through this construction on K_6 defined by the following cost matrix:



To From	1	2	3	4	5	6
1	x	3	3	2	7	3
2	3	x	3	4	5	5
3	3	3	x	1	4	4
4	2	4	1	x	5	5
5	7	5	4	5	x	4
6	3	5	4	5	4	x

Name the vertices x1, x2, x3, x4, x5, x6. We will start with x1 as T_1. Vertex x4 is closest to x1 according to the cost matrix, so T_2 consists of x1-x4-x1. Vertex x3 is closest to T_2 at x4 so T_3 = x1-x3-x4-x1. There are now two vertices each 3 units from T_3 at x2 and x6. Suppose we pick x2 and insert it before x3 to obtain T_4 of x1-x2-x3-x4-x1. Vertex x6 is still 3 units away from x1, so we insert x6 before x1 to get T_5: x1-x2-x3-x4-x6-x1. Finally, x5 is within 4 units of x3 and x6 so we choose to insert it before x6. This gives us a near-minimal tour T_6 of x1-x2-x3-x4-x5-x6-x1 with a cost of 19.

Note that the length of the tour generally depends on the starting vertex. Thus, by applying the algorithm to each starting vertex and taking the shortest of the six tours obtained, we would get an improved estimate of the true minimal tour.

Bibliography

* A translation of Euler's original paper on the Königsberg bridges can be found in the following. It's interesting to read this; his writing is very awkward because he lacked the modern terminology of graph theory.

L. Euler, "The Koenigsberg Bridges," *Scientific American*, 189, no. 1, (July, 1953), 66-70.

* Hamilton's instructions for his game are reprinted in:

N. Biggs, E. Lloyd, R.J. Wilson, *Graph Theory 1736-1936*, Oxford: Clarendon, 1986.

* For more on the Travelling Salesperson problem:

M. Bellmore, G. Nemhauser, "The Travelling Salesman Problem," *Operations Research*, 16 (1968) 538-558.

G. Dantzig, R. Fulkerson and S. Johnson, "Solution of a Large-Scale Travelling Salesman Problem," *Operations Research*, 2 (1954), 393-410.

E. Lawler, J. Lenstra, A. Kan, D. Shmoys, editors, *The Travelling Salesman Problem*, Chichester: Wiley, 1985.

* The Quick Travelling Salesperson Tour Construction is from:

Tucker, A., *Applied Combinatorics, 2nd edition*, New York: Wiley, 1984.

Historical Notes

Leonhard Euler (1707-1783) was a great Swiss mathematician, and an incredibly prolific writer contributing to many areas of mathematics including number theory, combinatorics, and analysis. He wrote over 700 books and papers, and left so much unpublished work that it took 47 years after he died for all his work to be published. The process of publishing his collected works (undertaken by the Swiss Society of Natural Science) is still going on and will require more than 75 (!) volumes.

The first use of the concept of a Hamilton circuit occurred in a 1771 paper by A. Vandermonde that presented a sequence of moves by which a knight could tour all the positions of a chessboard (without repeating a position). As mentioned above, Sir William Rowan Hamilton (1805-1865) formalized the concept. He made important contributions to optics, abstract algebra and dynamics. Hamilton invented algebraic objects called quarternions as an example of a noncommutative system. The story goes that he discovered the appropriate way to multiply quaternions while walking along a canal in Dublin. In his excitement, he carved the formula in the stone of a bridge crossing the canal, a spot marked today by a plaque.

The roots of the Travelling Salesperson problem come from studies of knight's tours (as mentioned above, this is a sequence of moves by a knight on a chessboard that begins and ends on a fixed square and visits all other squares exactly once). The first formal description of the problem appears to have occurred during the late 1930's in a seminar by Hassler Whitney. The problem was popularized by Merrill Flood of RAND Corporation during the next two decades. The first important paper on the subject was published in 1954 by G. Dantzig, R. Fulkerson and S. Johnson (see above). They found a tour of 49 cities, one in each state and Washington, DC, that has the shortest possible road distance. The work was carried out using an ingenious combination of linear programming, graph theory and map analysis. The study of this problem has grown enormously. The 1985 reference given above summarizes the subject in 465 pages.