Functions of Several Variables

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Lecture 21

Definition: Suppose D is a set of n-tuples of real numbers $(x_1, x_2, \ldots, x_n)$. A **real-valued function** f on D is a rule that assigns a unique (single) real number

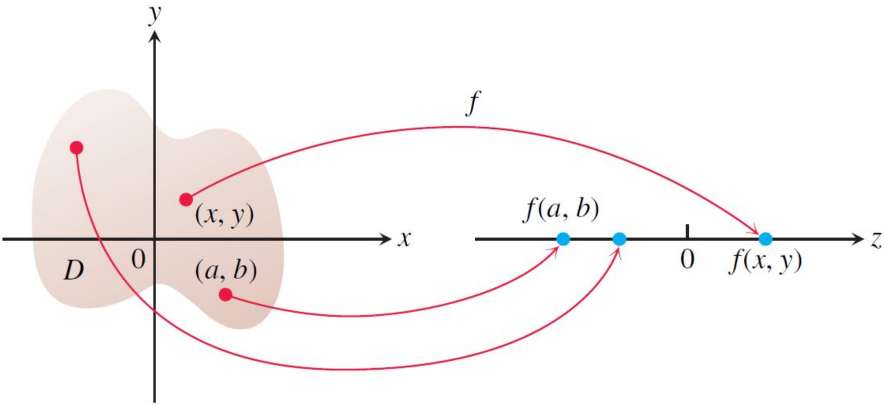
$$w = f(x_1, x_2, \ldots, x_n)$$

to each element in D. Here the symbol w is called **dependent variable** of f and f is said to be the function of several **independent variables** x_1 to x_n.

Domain and Range:

The set D is the function's **domain**. The set of w-values taken by f is called the **range** of the function f.

Examples



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3. $f(x,y) = \sqrt{1 - x^2 - y^2}$.
4. $f(x,y,z) = \sin(x+y) + |z|$.

Examples

- If f is a function of two independent variables, we write it as $z = f(x, y)$ and we usually call the independent variables x and y and the dependent variable z, and we picture the domain of f as a region in the xy-plane.

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- If f is a function of three independent variables, we write it as $w = f(x, y, z)$ and we call the independent variables x, y, and z and the dependent variable w, and we picture the domain as a region in space.

Examples

Function	Domain	Range
$z = \sqrt{y - x^2}$,	$\{(x, y) \ : \ y \geq x^2\}$	$[0, \infty)$
$z = \dfrac{1}{x + y}$	$\{(x, y) \ : \ x + y \neq 0\}$	$\mathbb{R} - \{0\}$
$z = \sin xy$	Entire plane $\mathbb{R}^2$	$[-1, 1]$
$w = \sqrt{x^2 + y^2 + z^2}$	Entire space $\mathbb{R}^3$	$[0, \infty)$
$w = \frac{1}{x^2+y^2+z^2}$	$\mathbb{R}^3 - \{(0, 0, 0)\}$	$(0, \infty)$
$w = xy \ln z$	$\{(x, y, z) \ : \ z > 0\}$	$(0, \infty)$

Functions of two variables

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- If the domain of real-valued function f is a subset of $\mathbb{R}^2$ (i.e., a some region in xy-plane), then f is called function of two variables.
- Here we define an interior point and boundary point of regions in xy-planes (subsets of $\mathbb{R}^2$).

Interior Point and Boundary Point

Interior Point: A point (x_0, y_0) in a region R in the xy-plane is an interior point of R if the region R contains a disk centered (x_0, y_0).

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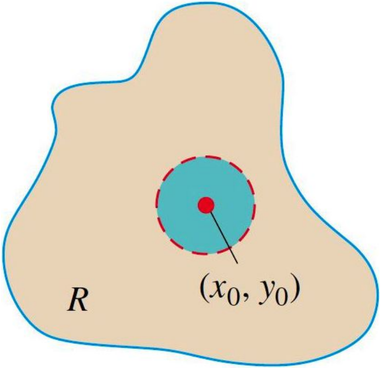

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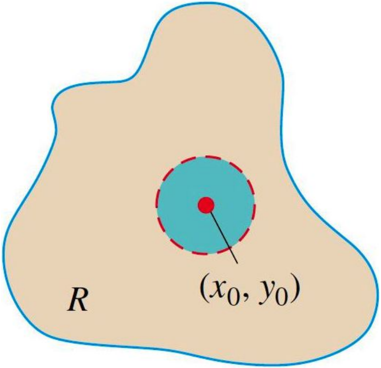

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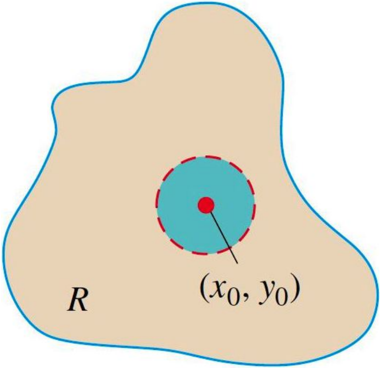

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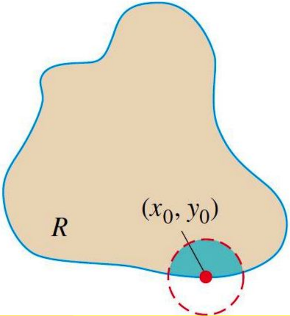

Interior and Boundary of Regions in xy-plane

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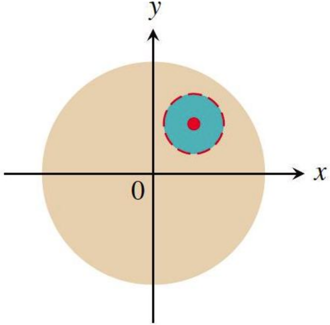

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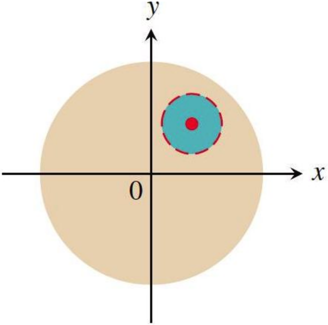

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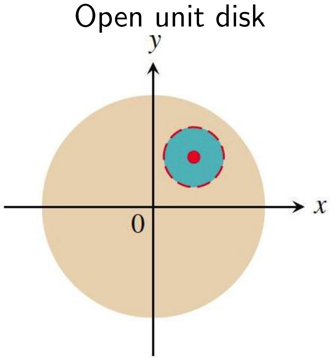

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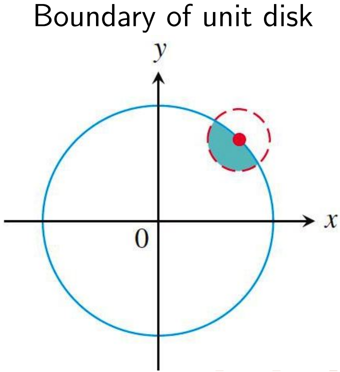

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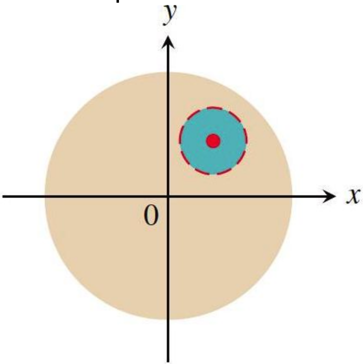

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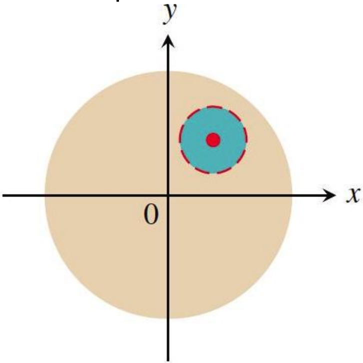

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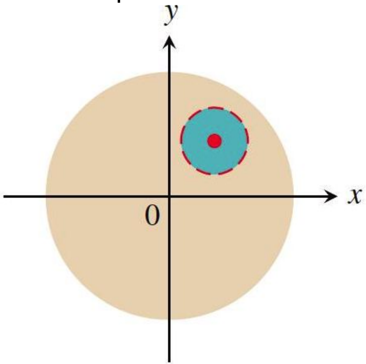
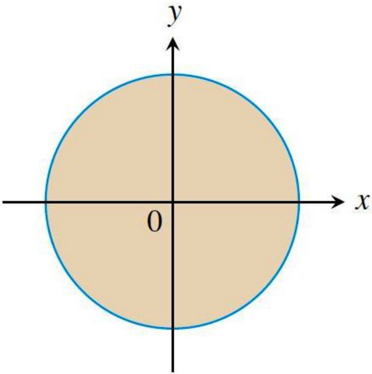


Closed unit disk



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A region in the xy-plane is **bounded** if it is lies inside a disk of fixed radius. A region is **unbounded** if it is not bounded.

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The lines, coordinate axes, quadrants, half-planes, and the full plane itself are unbounded regions.

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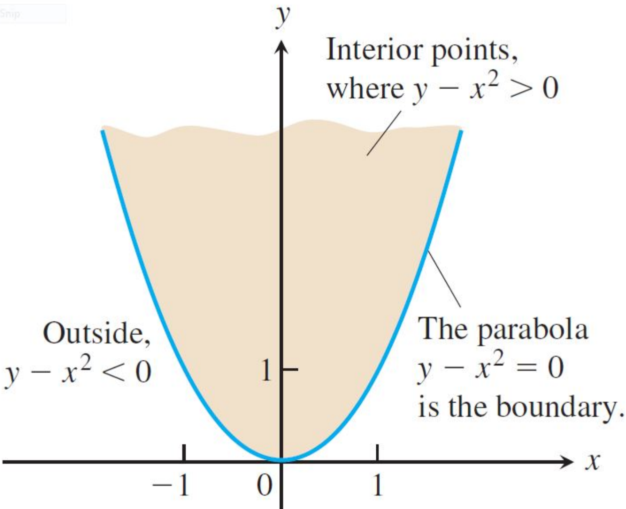

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Ans. The domain is given by $D = \{(x, y)\ :\ y - x^2 \geq 0\}$. It is closed, not open and it is unbonded.

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Describe the domain of the function

$$f(x, y) = \frac{1}{\ln(25 - x^2 - y^2)}.$$

Graphs, Level Curves and Contours of Functions of Two Variables

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- How to draw the graph two variable function $z = f(x, y)$?
- There are two standard ways, one is to draw and labels the curves in the domain on which f has a constant value.
- The other is to sketch the surface $z = f(x, y)$ in space.

Definition 0.2.

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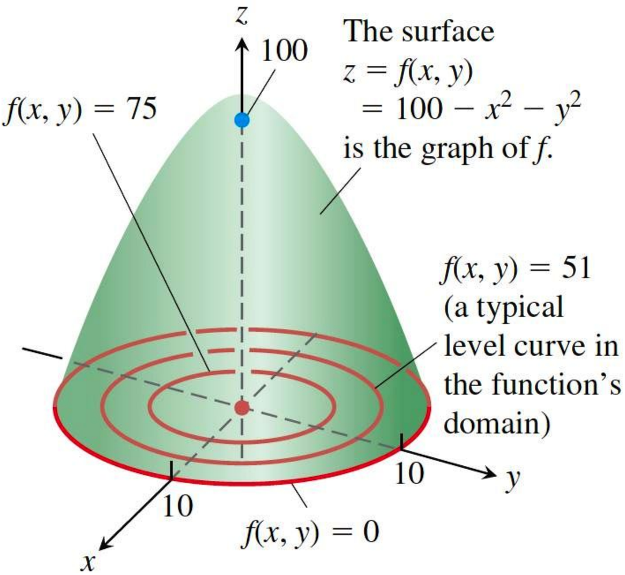
- The graph of f is also called surface $z = f(x, y)$.

Level Curves and Graph of Two Variable Function



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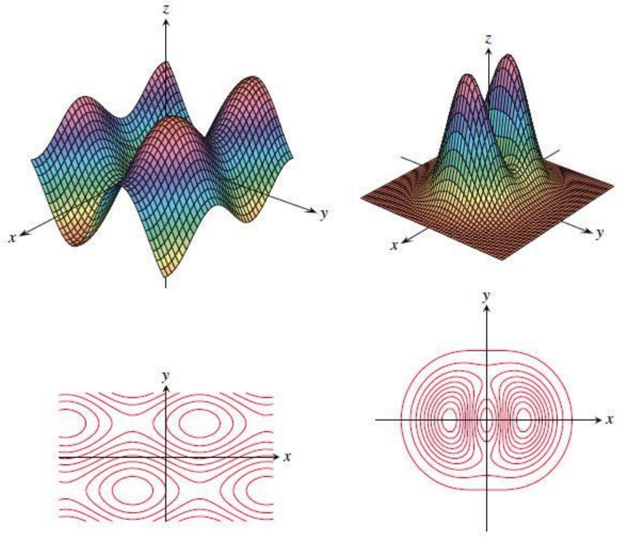

(a) $z = \sin x + 2\sin y$

(b) $z = (4x^2 + y^2)e^{-x^2-y^2}$

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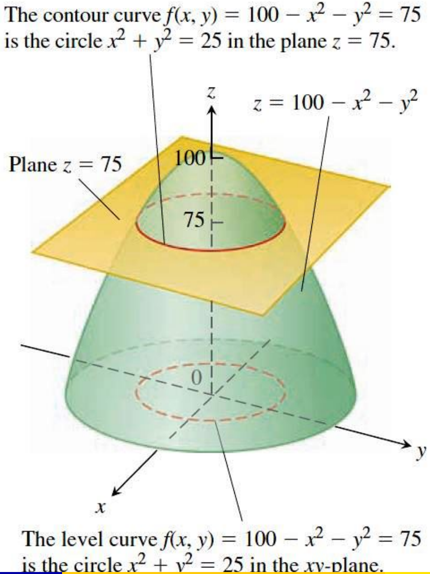
- It is called the contour curve $f(x, y) = c$ to the distinguish it from the level curve $f(x, y) = c$ in the domain of f.

Contour of Two Variable Function



Level curves of functions of two variables

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2. $f(x, y) = \sqrt{x^2 - 1}, \quad (1, 0)$.

Functions of Three Variable

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- Let $\varepsilon > 0$ and (x_0, y_0, z_0) be a point in space. We define **open ball** of radius $\varepsilon > 0$ centered at (x_0, y_0, z_0) by the set

$$\{(x, y, z) \in \mathbb{R}^3 \ : \ \sqrt{(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2} < \varepsilon\}$$

Interior Point and Boundary Point

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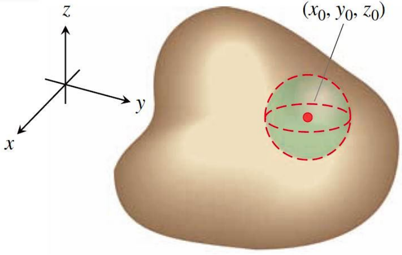

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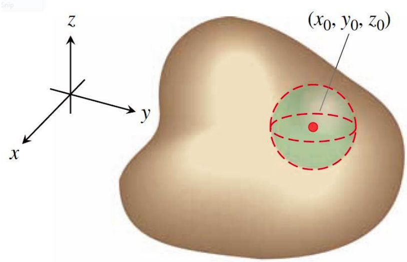

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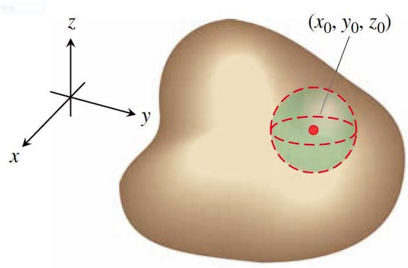

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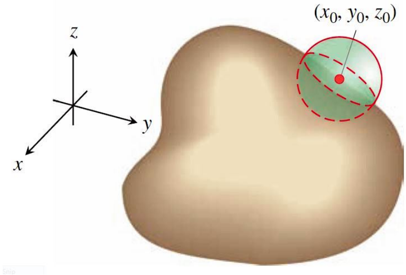

Interior and Boundary of Regions in space

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The lines, coordinate axes, octants, half-spaces, and the full space itself are unbounded regions.

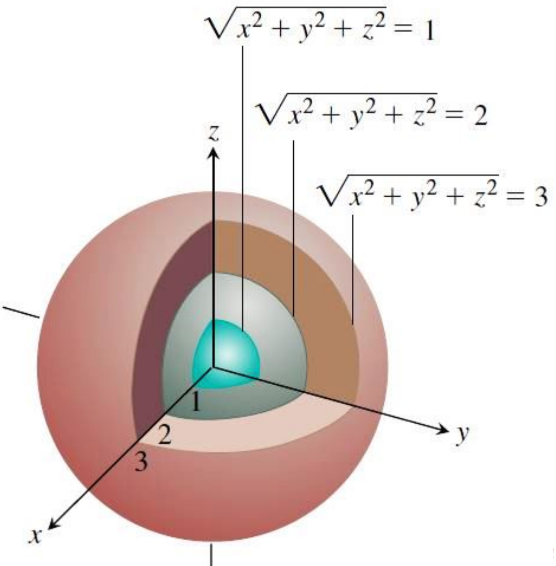
Definition 0.5.

The set of points (x, y, z) in space where a function $f(x, y, z)$ of three independent variables has a constant value $f(x, y, z) = c$ is called a level surface of f.

Example. Discribe the level surfaces of the function

$$f(x, y, z) = \sqrt{x^2 + y^2 + z^2}.$$

Level surfaces of functions of three variables



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