Lagrange Multipliers

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To deal with the situation when we have to find maximum or minimum of a function $f(x, y, z)$ subject to a constraint $g(x, y, z) = 0$ (or more than one constraints, say, $g_1(x, y, z) = 0$ and $g_2(x, y, z) = 0$), we use Lagrange multiplier method.

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Observation

Theorem (The Orthogonal Gradient Theorem)

Suppose that $f(x, y, z)$ is differentiable in a region whose interior contains a smooth curve $C\colon \boldsymbol{r}(t) = g(t)\boldsymbol{i} + h(t)\boldsymbol{j} + k(t)\boldsymbol{k}$.
If P_0 is a point on C where f has a local maximum or minimum relative to its values on C, then ∇f is orthogonal to C at P_0.

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Proof: We want to show that $\nabla f \cdot \boldsymbol{v} = 0$ where $\boldsymbol{v} = \frac{d\boldsymbol{r}}{dt}$.
The f on C is given by $f(g(t), h(t), k(t))$. Hence,

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dg}{dt} + \frac{\partial f}{\partial y}\frac{dh}{dt} + \frac{\partial f}{\partial z}\frac{dk}{dt} = \nabla f \cdot \boldsymbol{v}.$$

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At any point P_0, where f has a local maxima or minima relative to its values on the curve, $\frac{df}{dt} = 0$. This completes the proof.

Corollary (Side remark: Two variable case)

At the points on a smooth curve $\boldsymbol{r}(t) = g(t)\boldsymbol{i} + h(t)\boldsymbol{j}$ where a differentiable function $f(x, y)$ takes on its local maxima and minima relative to its values on the curve, $\nabla f \cdot \boldsymbol{v} = 0$, where $\boldsymbol{v} = d\boldsymbol{r}/dt$.

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Theorem (The Method of Lagrange Multipliers)

Suppose that $f(x, y, z)$ and $g(x, y, z)$ are differentiable and $\nabla g \neq 0$ when $g(x, y, z) = 0$. To find the local maximum and minimum values of f subject to the constraint $g(x, y, z) = 0$ (if these exist), find the values of x, y, z, and λ that simultaneously satisfy the equations

$$\nabla f = \lambda \nabla g \qquad \text{and} \qquad g(x, y, z) = 0.$$

For functions of two independent variables, the condition is similar, but without the variable z.

Here, the scalar λ is called a Lagrange multiplier.

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- And the velocity vector of any curve at the point is orthogonal to the gradient of the defining equation of the surface g.
- Putting these two together we see that at the point of the surface where the function f take on extreme values, the gradient of the function f must be a scalar multiple of the gradient of the defining equation of the surface g.

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The problem is essentially to minimize the function $f(r,h)$ on the surface $g(r,h) = r^2 h - 16 = 0$.

- We set up the Lagrange multiplier method first.

$$\nabla f(r,h) = \lambda \nabla g(r,h), \quad g(r,h) = 0.$$

Examples

- Since

$$\nabla f(r,h) = (2\pi h + 4\pi r)\boldsymbol{i} + 2\pi r\boldsymbol{j}, \quad \nabla g(r,h) = 2rh\boldsymbol{i} + r^2\boldsymbol{j}.$$

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- Solving, we get $r = 2, h = 4$.

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$$2x\boldsymbol{i} + 2y\boldsymbol{j} = (2x - 2)\lambda\boldsymbol{i} + (2y - 4)\lambda\boldsymbol{j}.$$

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- Now $g(x, y, z) = x^2 - 2x + y^2 - 4y = 0$, substituting the above values into this equation gives: $5\lambda^2 - 10\lambda = 0$.
- $\lambda = 0, 2$ or the points $(0, 0)$ and $(2, 4)$, where the values of the functions are 0 and 20 respectively. So 0 is minimum and 20 is maximum.

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- $\boldsymbol{i} - 2\boldsymbol{j} + 5\boldsymbol{k} = 2x\lambda\boldsymbol{i} + 2y\lambda\boldsymbol{j} + 2z\lambda\boldsymbol{k}$.
- $2x\lambda = 1$, $2y\lambda = -2$, $2z\lambda = 5$.
- This implies $x = \frac{1}{2\lambda}$, $y = \frac{-1}{\lambda}$, $z = \frac{5}{2\lambda}$, $\lambda \neq 0$.

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- Substituting these into $g(x,y,z) = 0$ gives: $\lambda = \dfrac{1}{2}, -\dfrac{1}{2}$.
- The corresponding points are $(1,-2,5), (-1,2,-5)$ respectively and the values are 30 and -30 respectively which are the maximum and the minimum value in that order.

Lagrange Multipliers with Two Constraints

To find the extreme values of a differentiable function $f(x, y, z)$ whose variables are subject to two constraints. If the constraints are

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and g_1 and g_2 are differentiable, with ∇g_1 not parallel to ∇g_2, we find the constrained local maxima and minima of f by introducing two Lagrange multipliers λ and μ . That is, locate the points $P(x, y, z)$ where f takes on its constrained extreme values by finding the values of x, y, z, λ, and μ that simultaneously satisfy the equations

$$\nabla f = \lambda \nabla g_1 + \mu \nabla g_2, \qquad g_1(x, y, z) = 0, \qquad g_2(x, y, z) = 0.$$

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- Hence we have λ_1, λ_2 such that $\nabla f = \lambda_1 \nabla g_1 + \lambda_2 \nabla g_2$

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- If $x \neq 0$, then the second equation with $z = 1$ leads to $x^2 = 2y^2$. Substituting this in the equation of the sphere, we get:

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- If $x \neq 0$, then the second equation with $z = 1$ leads to $x^2 = 2y^2$. Substituting this in the equation of the sphere, we get: $x = \sqrt{6}, -\sqrt{6}$ and $y = \sqrt{3}, -\sqrt{3}$. The critical points would be $(\pm\sqrt{6}, \pm\sqrt{3}, 1)$. The function value would be $1 \pm 6\sqrt{3}$.

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- When $x = 0$, $y = \pm 3$ (use the equation of the sphere) or the points are $(0, \pm 3, 1)$ and the function values are 1.

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- From the first equation, either $x = 0$ or $yz = \lambda_2$.
- If $x \neq 0$, then the second equation with $z = 1$ leads to $x^2 = 2y^2$. Substituting this in the equation of the sphere, we get: $x = \sqrt{6}, -\sqrt{6}$ and $y = \sqrt{3}, -\sqrt{3}$. The critical points would be $(\pm\sqrt{6}, \pm\sqrt{3}, 1)$. The function value would be $1 \pm 6\sqrt{3}$.
- When $x = 0$, $y = \pm 3$ (use the equation of the sphere) or the points are $(0, \pm 3, 1)$ and the function values are 1.
- The extreme values are $1 + 6\sqrt{3}, 1 - 6\sqrt{3}$ at the points $(\pm\sqrt{6}, \sqrt{3}, 1)$, $(\pm\sqrt{6}, -\sqrt{3}, 1)$ respectively.

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Solving for x, y, z, we get $x = y = z$. Now, utilize the equation $g = 0$ to obtain $x = y = z = \sqrt{\frac{c}{6}}$. So the maximum volume is $(\frac{c}{6})^{\frac{3}{2}}$.

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- The Lagrange multiplier method is only a necessary condition need not be sufficient.

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