MATH F111- Mathematics I

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Series

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$a_1 + a_2 + a_3 + \cdots + a_n + \cdots$

- We plan to to understand the meaning of such an infinite sum and to develop methods to calculate it.

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  is an ordinary finite sum and can be calculated by normal addition. It is called the nth partial sum.
- Then s_n itself is a sequence and where does s_n converge to?

Definition

Given a sequence of numbers (a_n), an expression of the form $a_1 + a_2 + a_3 + \cdots + a_n + \cdots$ is an infinite series. Then a_n is the nth term of the series. The sequence (s_n) defined by

$$s_1 = a_1$$
$$s_2 = a_1 + a_2$$
$$s_n = a_1 + a_2 + \cdots + a_n = \sum_{k=1}^{n} a_k$$

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$$\begin{aligned} s_1 &= a_1 \\ s_2 &= a_1 + a_2 \\ s_n &= a_1 + a_2 + \cdots + a_n = \sum_{k=1}^{n} a_k \end{aligned}$$

is the **sequence of partial sums** of the series, the number s_n being the nth partial sum. If the **sequence of partial sums converges to a limit L, we say that the series converges and that its sum is L.** In this case, we also write

$$a_1 + a_2 + \cdots + a_n + \cdots = \sum^{\infty} a_n = L.$$

If the sequence of partial sums of the series does not converge, we say that the series diverges.

We write $\sum_{n=1}^{\infty} a_n$ or $\sum a_n$ to denote the series.

Example: Geometric series

The series of the form $a + ar + ar^2 + \cdots + ar^{n-1} + \cdots = \sum_{n=1}^{\infty} ar^{n-1}$ in which a and r are fixed real numbers and $a \neq 0$.

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- If $|r| \neq 1$, then $s_n = \frac{a(1-r^n)}{1-r}$.

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- If $|r| > 1$, then $|r^n| \to \infty$ and the series diverges.

Geometric Series

- The series $\sum_{n=1}^{\infty} ar^{n-1}$ converges to $\frac{a}{1-r}$, if $|r| < 1$.
- If $|r| \geq 1$, the series $\sum_{n=1}^{\infty} ar^{n-1}$ diverges.

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- $s_n = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \cdots + \left(\frac{1}{n} - \frac{1}{n+1}\right) = 1 - \frac{1}{n+1}$.

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- $s_n \to 1$ as $n \to \infty$.
- Thus the series converges and $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$.

Examples

The series

$$\sum_{n=1}^{\infty} \frac{n+1}{n} = \frac{2}{1} + \frac{3}{2} + \cdots + \frac{n+1}{n} + \cdots$$

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- Thus the series diverges.

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Question: What can you say about $\lim_{n\to\infty} a_n$ *if* $\sum_{n=1}^{\infty} a_n$ *converges?*

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$$\sum_{n=1}^{\infty} \frac{n+1}{n} = \frac{2}{1} + \frac{3}{2} + \cdots + \frac{n+1}{n} + \cdots$$

diverges as $a_n = \frac{n+1}{n} \to 1 \neq 0$.

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- $\sum_{n=1}^{\infty}(-1)^{n+1}$ diverges as $\lim_{n\to\infty}(-1)^{n+1}$ does not exist.

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- $\sum_{n=1}^{\infty} (-1)^{n+1}$ diverges as $\lim_{n\to\infty} (-1)^{n+1}$ does not exist.
- $\sum_{n=1}^{\infty} \frac{-n}{2n+5}$ diverges as $\lim_{n\to\infty} \frac{-n}{2n+5} = -\frac{1}{2} \neq 0$.

A series whose $a_n \to 0$, but series diverges

What about the series

$$1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \cdots + \frac{1}{2^n} + \frac{1}{2^n} + \cdots + \frac{1}{2^n} + \cdots ?$$

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- This series diverges because the terms can be grouped into infinitely many clusters each of which adds to 1, so the partial sums increase without bound.
- Thus s_n is not bounded hence does not converge.
- However, the terms of the series form a sequence that converges to 0.

Whenever we have two convergent series, we can add them term by term, subtract them term by term, or multiply them by constants to make new convergent series.

Theorem

If $\sum a_n = A$ *and* $\sum b_n = B$ *are convergent series, then*

1. *Sum Rule:* $\sum(a_n + b_n) = \sum a_n + \sum b_n = A + B$
2. *Difference Rule:* $\sum(a_n - b_n) = \sum a_n - \sum b_n = A - B$
3. *Constant Multiple Rule:* $\sum ka_n = k \sum a_n = kA$. *(any number* k*).*

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Corollary

- *Every nonzero constant multiple of a divergent series diverges.*
- *If* $\sum a_n$ *converges and* $\sum b_n$ *diverges, then* $\sum(a_n + b_n)$ *and* $\sum(a_n - b_n)$ *both diverge.*

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$$\sum_{n=1}^{\infty} \frac{4}{2^n} = 4\sum_{n=1}^{\infty} \frac{1}{2^n} = 4.$$

In this section, we consider the series that have $a_n \geq 0$.

Theorem

A series $\sum a_n$ with $a_n \geq 0$ converges if and only if its partial sums are bounded from above.

Let $\sum a_n$ be an infinite series with $a_n \geq 0$ for all n. Then each partial sum is greater than or equal to its predecessor because

$$s_n = s_{n-1} + a_n$$

and $a_n \geq 0$, so

$$s_1 \leq s_2 \leq s_3 \leq \cdots \leq s_n$$

Thus s_n is monotonic increasing and hence by Monotonic increasing theorem s_n converges if it is bounded above.

Harmonic series

What about the harmonic series

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- Although $a_n = \frac{1}{n}$ does go to 0, the series diverges because there is no upper bound for its partial sums. We can group the terms of the series in the following way:

$$1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) + \left(\frac{1}{9} + \cdots + \frac{1}{16}\right) + \cdots$$

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- The sum of the first two terms is 1.5. The sum of the next two terms is $> \frac{1}{2}$.
- The sum of the next four terms is greater than $\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} > \frac{1}{2}$.

- The sum of the next eight terms is $> \frac{1}{2}$. The sum of the next 16 terms is greater than $1/2$ and so on.
- In general, the sum of 2^n terms ending with $\frac{1}{2^{n+1}}$ is $> \frac{1}{2}$.
- If $n = 2^k$, the partial sum $s_n > \frac{k}{2}$, so the sequence of partial sums is not bounded from above. Thus the harmonic series diverges.