Lecture 4

Area under polar curves

Anushaya Mohapatra

Text book chapter: 11.4

Points of intersection

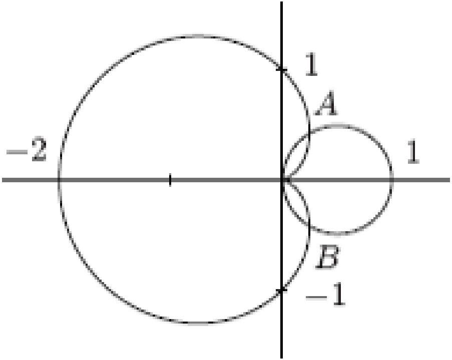
Given two polar equations $r = f(\theta)$ and $r = g(\theta)$, solving the two equations simultaneously by equating $f(\theta) = g(\theta)$ only gives some of the intersection points.

Example-5:
Find the points of intersection for the curves: $r = \cos\theta$ and $r = 1 - \cos\theta$

Example 5

For finding the point of intersection we have

$$\cos\theta = 1 - \cos\theta$$
$$\Rightarrow \cos\theta = \frac{1}{2}$$
$$\Rightarrow \theta = \frac{\pi}{3},\ r = \frac{1}{2} \ or\ \theta = \frac{5\pi}{3},\ r = \frac{1}{2}.$$



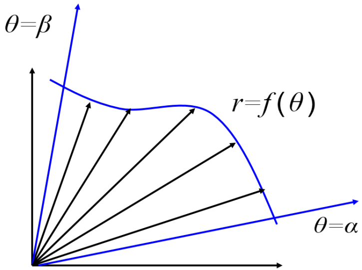
Example 5

The intersection points are $(1/2, \pi/3), (1/2, 5\pi/3)$ and $(0, 0)$.

Hence the solution obtained by solving $\cos(\theta) = 1 - \cos(\theta)$ is incomplete. The origin is also a point of intersection, but we can't find it by solving the equations of the curves because the origin has no single representation in polar coordinates that satisfies both equations.

Area between polar curves

We are interested in finding area bounded by polar equations $\theta = \alpha, \theta = \beta$, and $r = f(\theta)$.



Area between polar curves

We approximate the region with n non-overlapping fan-shaped circular sectors based on a partition P. The typical sector has radius $r_k = f(\theta_k)$ and central angle of radian measure $\Delta\theta_k$. The area of the region is approximately same as the sum

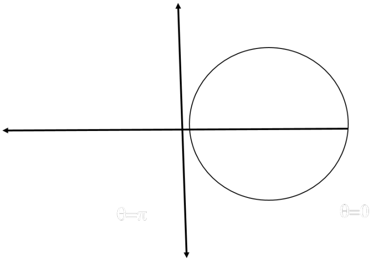
$$A = \sum_{k=1}^{n} \frac{1}{2} f(\theta_k)^2 \Delta\theta_k .$$

When the norm of the partition goes to zero, we get the following integration as the area of the region.

$$A = \int_{\alpha}^{\beta} \frac{1}{2} (f(\theta))^2 d\theta = \int_{\alpha}^{\beta} \frac{1}{2} r^2 d\theta .$$

Example-5

Find the area of the circle $r = 2\cos\theta$



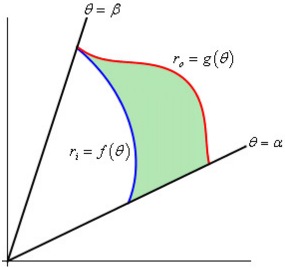
The area of the circle $A = \frac{1}{2}\int_0^{\pi}(2\cos(\theta))^2 d\theta$

Area bounded by two polar curve

Definition 1.

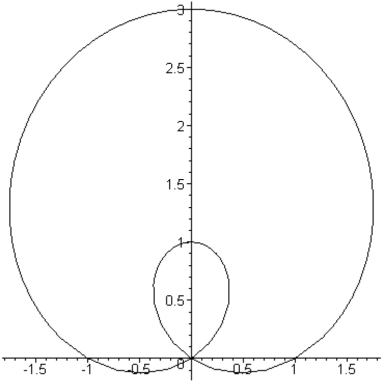
The area bounded by two polar curve $r_0 = g(\theta)$ and $r_1(\theta) = f(\theta)$ with $0 \leq r_1 \leq r \leq r_0, \alpha \leq \beta$ is given by

$$A = \int_{\alpha}^{\beta} \frac{1}{2}(r_0^2 - r_1^2)d\theta$$



Example-6

(a)Find the area of the inner loop $r = 1 + 2\sin(\theta)$. (b) Express the area of the region inside the outer loop and outside the inner loop in terms of integrals.



Example-6

(a)The right half of the inner loop is formed when θ varies from $7\pi/6$ to $3\pi/2$ and the graph is symmetric around y-axis. So the area is given by the following integral.

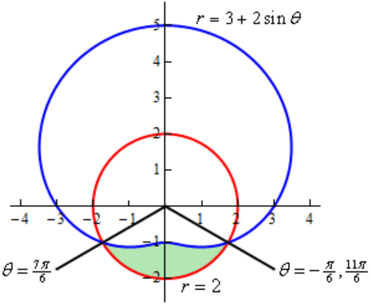
$$\begin{aligned} A &= \int_{7\pi/6}^{3\pi/2} r^2 d\theta \\ &= \int_{7\pi/6}^{3\pi/2} (1+4\sin^2\theta+4\sin\theta)\, d\theta \\ &= [3\theta-\sin 2\theta-4\cos\theta]_{7\pi/6}^{3\pi/2} \\ &= \pi-\frac{3\sqrt{3}}{2}. \end{aligned}$$

(b) Ans:

$$A = 2\left(\int_{\pi/2}^{7\pi/6} \frac{1}{2} r^2 d\theta - \int_{7\pi/6}^{3\pi/2} \frac{1}{2} r^2 d\theta\right)$$

Example-7

Find the area of the region outside $r = 3 + 2\sin\theta$ and inside $r = 2$.



Example 7

$$
\begin{aligned}
A &= \int_{\frac{7\pi}{6}}^{\frac{11\pi}{6}} \frac{1}{2}\left((2)^2 - (3+2\sin\theta)^2\right)\, d\theta \\
&= \int_{\frac{7\pi}{6}}^{\frac{11\pi}{6}} \frac{1}{2}\left(-5 - 12\sin\theta - 4\sin^2\theta\right)\, d\theta \\
&= \int_{\frac{7\pi}{6}}^{\frac{11\pi}{6}} \frac{1}{2}\left(-7 - 12\sin\theta + 2\cos(2\theta)\right)\, d\theta \\
&= \frac{1}{2}\left(-7\theta + 12\cos\theta + \sin(2\theta)\right)\Big|_{\frac{7\pi}{6}}^{\frac{11\pi}{6}} \\
&= \frac{11\sqrt{3}}{2} - \frac{7\pi}{3} = 2.196
\end{aligned}
$$