Sequence and Series

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August 13, 2024

Notations

$\mathbb{N}$- Set of Natural numbers
$\mathbb{Q}$- Set of rational numbers
$\mathbb{R}$- Set of real numbers
$\forall$ - For all
$\exists$- There exists

Definition of Interval

A subset I of $\mathbb{R}$ is said to be an interval if
$a, b \in I$ and $a < x < b \implies x \in I$.

Let $a, b \in \mathbb{R}$ and $a < b$.

- $(a, b) := \{x \in \mathbb{R} : a < x < b\}$ (open interval)
- $[a, b] := \{x \in \mathbb{R} : a \leq x \leq b\}$ (closed interval)
- $[a, b) := \{x \in \mathbb{R} : a \leq x < b\}$ and $(a, b] := \{x \in \mathbb{R} : a < x \leq b\}$ are half-open (or half-closed) intervals.
- $(a, \infty) := \{x \in \mathbb{R} : x > a\}$ and $(-\infty, a) := \{x \in \mathbb{R} : x < a\}$ are infinite open intervals.
- $[a, \infty) := \{x \in \mathbb{R} : x \geq a\}$ and $(-\infty, a] := \{x \in \mathbb{R} : x \leq a\}$ are infinite closed intervals.

Let $a \in \mathbb{R}$ and $\epsilon > 0$. Then $(a - \epsilon, a + \epsilon)$ is called the **ϵ-neighborhood of a.**

Sequences

Definition

A **sequence of real numbers** (or a **sequence in** $\mathbb{R}$) is a function $x : \mathbb{N} \to \mathbb{R}$.

If $x : \mathbb{N} \to \mathbb{R}$ is a sequence, we will usually denote the value of $x(n)$ by the symbol x_n. The values x_n are also called the terms or the elements of the sequence and x_n (that is, the value of x at n) is called the n-th term of the sequence. We will denote this sequence by the notations

$$(x_n), \quad \text{or} \quad (x_n : n \in \mathbb{N}).$$

Other commonly used notations are (a_n).

Examples

- $(n : n \in \mathbb{N}) =$
- $(1/n : n \in \mathbb{N}) =$
- $(n^2 : n \in \mathbb{N}) =$
- If $b \in \mathbb{R}$, the sequence $(b, b, b, \ldots)$, all of whose terms equal b, is called the constant sequence b.
- $(2^n : n \in \mathbb{N}) =$
- $((-1)^n : n \in \mathbb{N}) =$
- $x_1 := 1, x_2 := 1$ and $x_n := x_{n-1} + x_{n-2}$ for $n \geq 3$: $(1, 1, 2, 3, 5, 8, 13, 21, 34, 55, \ldots)$ This sequence is known as the **Fibonacci sequence**.

Bounded Sequences

A sequence (a_n) of real numbers is said to be **bounded above** if there is a real number α such that $a_n \leq \alpha$ for every $(\forall)$ $n \in \mathbb{N}$. eg. $(a_n) = -n$.

A sequence (a_n) of real numbers is said to be **bounded below** if there is a real number β such that $\beta \leq a_n$ for every $n \in \mathbb{N}$. eg. $(a_n) = n^2$

A sequence (a_n) of real numbers is said to be **bounded** if there are real numbers α, β such that $\beta \leq a_n \leq \alpha$ for every $n \in \mathbb{N}$. eg. $(a_n) = \frac{1}{n}$, $(a_n) = (-1)^n$

If a sequence is not bounded, it is said to be **unbounded**. eg. $(a_n) = (-1)^n n$

Convergence

- $(a_n) = \{1, \frac{1}{2}, \frac{1}{3}, \cdots, \frac{1}{n} \cdots\}$ approaches 0 as n gets large.
- $(a_n) = \{0, \frac{1}{2}, \frac{2}{3}, \cdots, 1 - \frac{1}{n} \cdots\}$ approaches 1 as n gets large.
- $(a_n) = \{\sqrt{1}, \sqrt{2}, \sqrt{3}, \cdots, \sqrt{n} \cdots\}$ have terms that get larger than any number as n increases.
- $(a_n) = \{1, -1, 1, -1, \cdots\}$ bounce back and forth between 1 and -1, never approaching to a single value.

Remark

Question: What do we mean by a sequence converges?

It says that if we go far enough out in the sequence, the difference between a_n and the limit of the sequence becomes less than any preselected number $\epsilon > 0$.

Let us see this with $(a_n) = \frac{1}{n}$.

- Can you make the distance between a_n and 0 as small as possible?
- Can you make $|a_n - 0| < \frac{1}{2}\ \forall n$? No
- Can you make $|a_n - 0| < \frac{1}{2}$, $\forall n \geq N$ for some integer N? Yes, choose $N = 3$.
- Given any $\epsilon > 0$, can you make $|a_n - 0| < \epsilon$, $\forall n \geq N$ for some integer N? Yes, choose $N > \frac{1}{\epsilon}$. In particular choose $N = \lfloor \frac{1}{\epsilon} \rfloor + 1$.

The Limit of a Sequence

Definition

A sequence (a_n) in $\mathbb{R}$ is said to converge to $\ell \in \mathbb{R}$, or ℓ is said to be a limit of (a_n), if for every $\epsilon > 0$, there exists an integer $N(\epsilon) \in \mathbb{N}$ such that

$$|a_n - \ell| < \epsilon \text{ for all } n \geq N(\epsilon).$$

ie, $a_n \in (\ell - \epsilon, \ell + \epsilon), \forall n \geq N(\epsilon)$.

Remark

The notation $N(\epsilon)$ is used to emphasize that the choice of N depends on the value of ϵ. However, it is often convenient to write N instead of $N(\epsilon)$, When a sequence (a_n) has limit ℓ, we will use the notation

$$\lim a_n = \ell.$$

We will sometimes use the symbolism $a_n \to \ell$, which indicates the intuitive idea that the values a_n “approach" the number ℓ as $n \to \infty$. If a sequence has a limit, we say that the sequence is **convergent**; if it has no limit, we say that the sequence is **divergent**.