Lecture 13

ANGULAR MOMENTUM IV

MOMENT OF INERTIA TENSOR

Angular Momentum IV

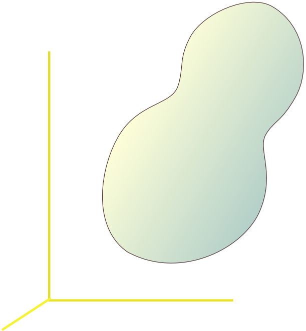
Moment of Inertia Tensor

Radhika Vathsan, BITS-Goa, 2010-11

Rigid Bodies

Rigid Bodies

Qn: How many degrees of freedom does a Rigid Body have?



Rigid Bodies

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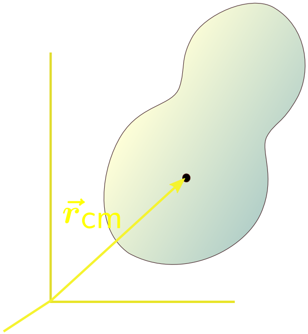
- 3 for $\vec{r}_{CM}$ (posn. of CM



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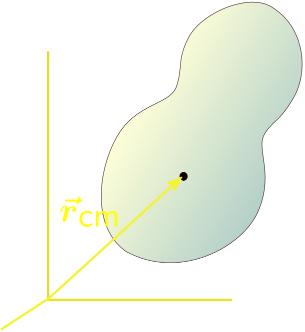
- 3 for $\vec{r}_{CM}$ (posn. of CM
- 3 more to fix the orientation of body



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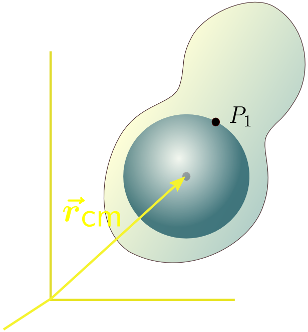
  - Fix P_1 on body. can rotate on sphere around CM $\rightarrow$ 2 angles



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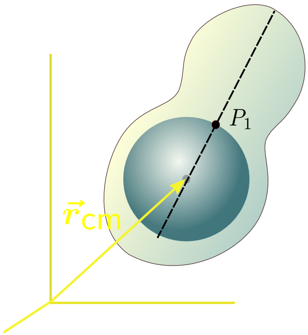

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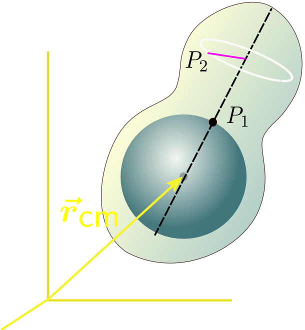

Space and Body Fixed Axes

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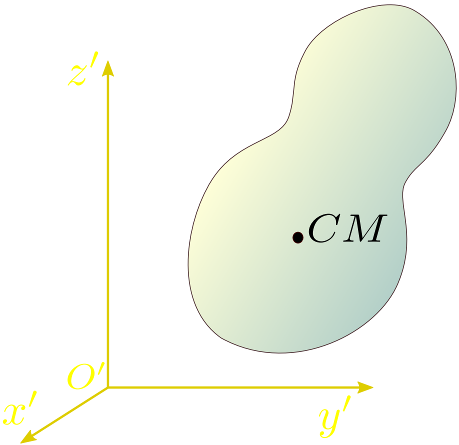
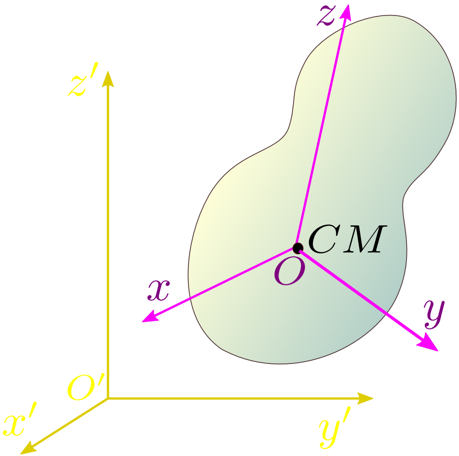


$(x', y', z'); O'$: Space (Lab) Fixed Axes

Space and Body Fixed Axes



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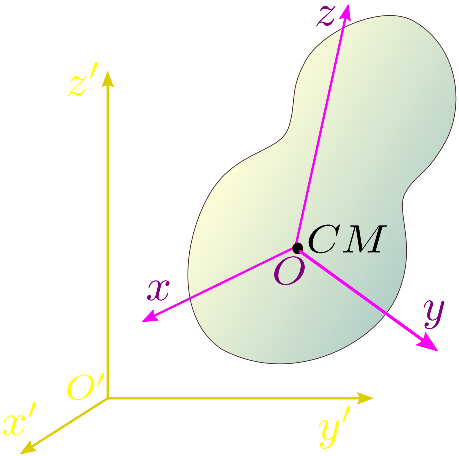
$(x, y, z); O = CM$: Body Fixed Axes

Space and Body Fixed Axes



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x, y, z frame moves/rotates with body

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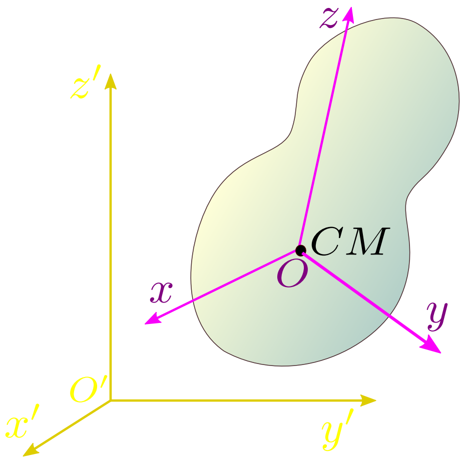

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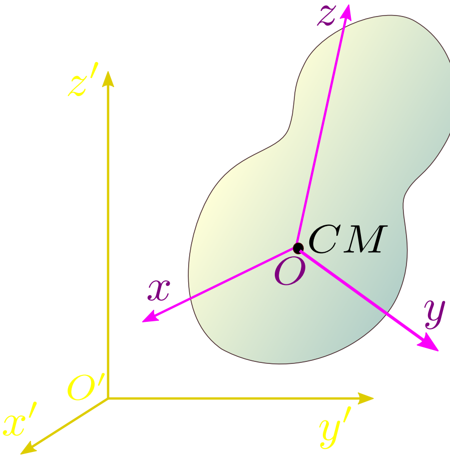


$(x', y', z'); O'$: Space (Lab) Fixed Axes

$(x, y, z); O = CM$: Body Fixed Axes

x, y, z frame moves/rotates with body

Orientation wrt body remains fixed

3 CM coords +

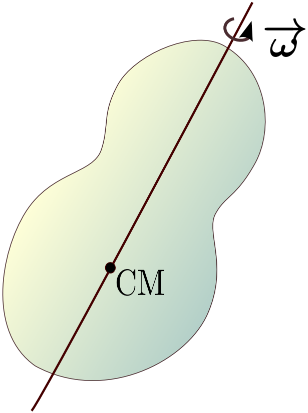
3 angles specify (x, y, z) frame wrt (x', y', z') axes

Angular Momentum in CM (Body) Frame

Rigid body rotating about axis thru CM

Angular Momentum in CM (Body) Frame

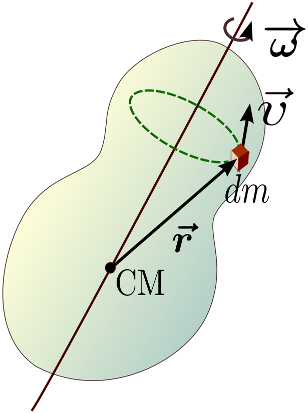
Rigid body rotating about axis thru CM



Angular Momentum in CM (Body) Frame

Rigid body rotating about axis thru CM

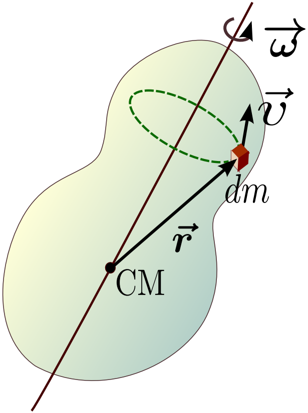
$$\vec{\boldsymbol{L}} = \int dm \;\; \vec{\boldsymbol{r}} \times \vec{\boldsymbol{v}}$$



Angular Momentum in CM (Body) Frame

Rigid body rotating about axis thru CM

$$
\begin{aligned}
\vec{\boldsymbol{L}} &= \int dm \;\; \vec{\boldsymbol{r}} \times \vec{\boldsymbol{v}} \\
&= \int dm \;\; \vec{\boldsymbol{r}} \times (\vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{r}})
\end{aligned}
$$



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&= \int dm \;\; [\vec{\boldsymbol{\omega}} r^2 - \vec{\boldsymbol{r}}(\vec{\boldsymbol{\omega}} \cdot \vec{\boldsymbol{r}})]
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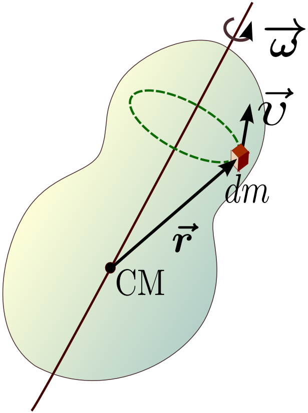

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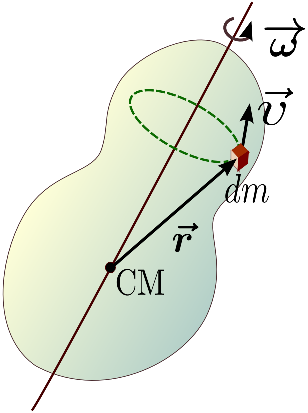

Moment of Inertia tensor

$$L_x = \int dm \left[r^2\omega_x - x(x\omega_x + y\omega_y + z\omega_z)\right]$$

Moment of Inertia tensor

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$$L_x = \int dm\left[\omega_x(y^2+z^2) + \omega_y(-xy) + \omega_z(-xz)\right]$$

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L_z &= \int dm\left[\omega_x(-zx) + \omega_y(-zy) + \omega_z(x^2+y^2)\right] \\
L_x &= I_{xx}\omega_x + I_{xy}\omega_y + I_{xz}\omega_z \\
L_y &= I_{yx}\omega_x + I_{yy}\omega_y + I_{yz}\omega_z \\
L_z &= I_{zx}\omega_x + I_{zy}\omega_y + I_{zz}\omega_z
\end{aligned}
$$

Moment of Inertia tensor

$$\begin{bmatrix} L_x \\ L_y \\ L_z \end{bmatrix} = \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{bmatrix} \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}$$

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$$\begin{aligned} \bar{I} &= [I] = \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{bmatrix} \\ &= \begin{bmatrix} \int dm(y^2+z^2) & -\int dmxy & -\int dmxz \\ -\int dmyx & \int dm(x^2+z^2) & -\int dmyz \\ -\int dmzx & -\int dmzy & \int dm(x^2+y^2) \end{bmatrix} \end{aligned}$$

$\bar{I} = [I]_{3\times 3}$: 3X3 Moment of Inertia Matrix

Moment of Inertia tensor

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Moment of Inertia tensor

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$\bar{I} = [I]_{3\times3}$: 3X3 Moment of Inertia Matrix or

(3X3 Moment of Inertia Tensor)

What is a tensor?

Scalar: a number (eg. distance)

Vector: direction & magnitude, 3 numbers

2nd rank tensor: a 3×3 matrix: Nine numbers!

formed by a ”product” (NOT scalar, NOT cross) of two vectors

$$T = \vec{A}\vec{B}$$

$$[T] = \begin{bmatrix} A_x B_x & A_x B_y & A_x B_z \\ A_y B_x & A_y B_y & A_y B_z \\ A_z B_x & A_z B_y & A_z B_z \end{bmatrix}$$

$\vec{C} \cdot T$ or $T \cdot \vec{D}$ are vectors. $\vec{C} \cdot T \cdot \vec{D}$ is a scalar!

scalars, vectors, tensors: quantities with well defined transformation under rotation

Moment of Inertia tensor

- $\bar{I}$ is a 3×3 symmetric matrix

Moment of Inertia tensor

- $\bar{I}$ is a 3×3 symmetric matrix with 6 independent components
- $$\left.\begin{aligned} I_{xx} &= \int dm(y^2+z^2) \\ I_{yy} &= \int dm(x^2+z^2) \\ I_{zz} &= \int dm(x^2+y^2) \end{aligned}\right\}$$
  Moments of inertia abt x, y & z
- $$\left.\begin{aligned} I_{xy}(=I_{yx}) &= -\int dmxy \\ I_{yz}(=I_{zy}) &= -\int dmzy \\ I_{xz}(=I_{zx}) &= -\int dmxz \end{aligned}\right\}$$
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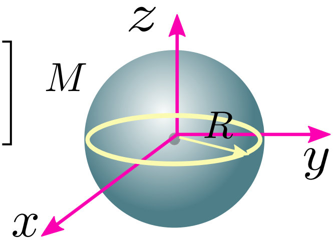
  Products of Inertia Not $+ve$ definite!

Example I: MI tensor for sphere

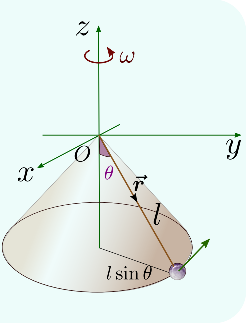
$$\bar{I} = \begin{bmatrix} \frac{2}{5}MR^2 & 0 & 0 \\ 0 & \frac{2}{5}MR^2 & 0 \\ 0 & 0 & \frac{2}{5}MR^2 \end{bmatrix}$$



Example II: Conical Pendulum

$$\vec{r} = (l\sin\theta\cos\omega t, l\sin\theta\sin\omega t, -l\cos\theta).$$

$$\begin{aligned}
I_{xx} &= ml^2(\cos^2\theta + \sin^2\theta\sin^2\omega t) \\
I_{yy} &= ml^2(\cos^2\theta + \sin^2\theta\cos^2\omega t) \\
I_{zz} &= ml^2\sin^2\theta \\
I_{xy} &= I_{yx} = -ml^2\sin^2\theta\cos\omega t\sin\omega t \\
I_{yz} &= I_{zy} = ml^2\cos\theta\sin\theta\sin\omega t \\
I_{xz} &= I_{zx} = ml^2\sin^2\theta\cos\theta\cos\omega t
\end{aligned}$$



Example II: Conical Pendulum

$$\vec{L} = \bar{I}\vec{\omega} \implies$$

$$\begin{bmatrix} L_x \\ L_y \\ L_z \end{bmatrix} =$$

$$ml^2 \begin{bmatrix} \cos^2\theta + \sin^2\theta\sin^2\omega t & -\sin^2\theta\cos\omega t\sin\omega t & \sin\theta\cos\theta\cos\omega t \\ -\sin^2\theta\cos\omega t\sin\omega t & \cos^2\theta + \sin^2\theta\cos^2\omega t & \cos\theta\sin\theta\sin\omega t \\ \sin\theta\cos\theta\cos\omega t & \cos\theta\sin\theta\sin\omega t & \sin^2\theta \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \omega \end{bmatrix}$$

$$\begin{aligned} \vec{L} &= (m\omega l^2\sin\theta\cos\theta\cos\omega t)\hat{\boldsymbol{i}} + (m\omega l^2\sin\theta\cos\theta\sin\omega t)\hat{\boldsymbol{j}} \\ &\quad +(m\omega l^2\sin^2\theta)\hat{\boldsymbol{k}} = \vec{L}(t) \end{aligned}$$

Calculate the torque $\vec{\tau} = d\vec{L}/dt$