MATH F111- MATHEMATICS I
Tutorial sheet 12

1. Find all the local maxima, local minima, and saddle points of the functions:
   (a) $f(x,y) = x^2 - 4xy + y^2 + 6y + 2$
   (b) $f(x,y) = \frac{1}{x} + xy + \frac{1}{y}$

2. Find the absolute maxima and minima of the functions
   (a) $f(x,y) = 2x^2 - 4x + y^2 - 4y + 1$ on the closed triangular plate bounded by the lines $x = 0, y = 2, y = 2x$ in the first quadrant.
   (b) $f(x,y) = (4x - x^2)\cos y$ on the rectangular plate $1 \le x \le 3$ and $-\frac{\pi}{4} \le y \le \frac{\pi}{4}$.

3. Find two numbers a and b with $a < b$ such that $\displaystyle\int_a^b (6 - x - x^2)dx$ has its largest value.

4. A flat circular plate has the shape of the region $x^2 + y^2 \le 1$. The plate, including the boundary where $x^2 + y^2 = 1$, is heated so that the temperature at the point (x,y) is $T(x,y) = x^2 + 2y^2 - x$. Find the temperatures at the hottest and coldest points on the plate.

5. Find the maxima, minima, and saddle points of $f(x,y)$, if any, given that
   (a) $f_x = 2x - 4y$ and $f_y = 2y - 4x$
   (b) $f_x = 2x - 2$ and $f_y = 2y - 4$

   Describe your reasoning in each case.

6. The discriminant $f_{xx}f_{yy} - f_{xy}^2$ is zero at the origin for each of the following functions, so the Second Derivative Test fails there. Determine whether the function has a maximum, a minimum, or neither at the origin by imagining what the surface $z = f(x,y)$ looks like. Describe your reasoning in each case.
   (a) $f(x,y) = x^2y^2$
   (b) $f(x,y) = x^3y^2$

7. Show that $(0,0)$ is a critical point of $f(x,y) = x^2 + kxy + y^2$ no matter what value the constant k has. (Hint: Consider two cases: $k = 0$ and $k \neq 0$.)

8. For what values of the constant k does the Second Derivative Test guarantee that $f(x,y) = x^2 + kxy + y^2$ will have a saddle point at $(0,0)$? A local minimum at $(0,0)$? For what values of k is the Second Derivative Test inconclusive? Give reasons for your answers.

9. Find the maximum value of $s = xy + yz + xz$ where $x + y + z = 6$.

Extreme Values on Parametrized Curves: To find the extreme values of a function $f(x,y)$ on a curve $x = x(t), y = y(t)$, we treat f as a function of the single variable t and use the Chain Rule to find where $\frac{df}{dt}$ is zero. As in any other single-variable case, the extreme values of f are then found among the values at the

(a) critical points (points where $\frac{df}{dt}$ is zero or fails to exist), and
(b) endpoints of the parameter domain.

10. Find the absolute maximum and minimum values of the following functions on the given curves.

    (a) $f(x, y) = x + y$
    (b) $g(x, y) = xy$

    Curves:

    (a) The semicircle $x^2 + y^2 = 4, y \geq 0$
    (b) The quarter circle $x^2 + y^2 = 4, x \geq 0, y \geq 0$

    Use the parametric equations $x = 2\cos t, y = 2\sin t$.