MATH F111- Mathematics I

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Understanding limit of a function

Definition

Let $f(x)$ be defined on an open interval about x_0, except possibly at x_0 itself. We say that the limit of $f(x)$ as x approaches x_0 is the number L, and write

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if, for every number $\epsilon > 0$, there exists a corresponding number $\delta > 0$ such that for all x,

$$0 < |x - x_0| < \delta \implies |f(x) - L| < \epsilon.$$

Refer Chapter 2 in Thomas Calculus for a thorough understanding of definition of limit of a function.

Definition: Continuity

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Let $f : D \to \mathbb{R}$ be a function where $D \subseteq \mathbb{R}$. For x_0, we say that the function is **continuous at** x_0 if the following conditions hold:

1. $x_0 \in D$.
2. $\lim_{x \to x_0} f(x)$ exists.
3. $\lim_{x \to x_0} f(x) = f(x_0)$.

A function is **continuous** if it is continuous at all points of it's domain.

Continuous function theorem for Sequences

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- *If (a_n) is a sequence and if f is any function from $\mathbb{R} \to \mathbb{R}$, is $f(a_n)$ a sequence?*

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Theorem

Theorem 3: Let (a_n) be a sequence of real numbers. If

- *$a_n \to \ell$ and*
- *if f is a function that is continuous at ℓ and defined at all a_n, then*

$$f(a_n) \to f(\ell).$$

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- Show that $(2^{\frac{1}{n}}) \to 1$. $\frac{1}{n} \to 0$ and $f(x) = 2^x$ is continuous at $x = 0$. Thus $2^{\frac{1}{n}} \to 1$.

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Theorem

Sequential criteria for continuity *A function $f : D \to \mathbb{R}$ is continuous at $x_0 \in D$ iff for every sequence (x_n) in D such that $x_n \to x_0$, we have $f(x_n) \to f(x_0)$.*

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This theorem is particularly useful if you want to show that a function is not continuous at x_0. We only need to construct two sequences x_n and y_n both converging to x_0, but $f(x_n) \neq f(y_n)$.

Let

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$$

Show that f is discontinuous at every real number $x \in \mathbb{R}$.

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- Let x_0 be any real number. Let x_n be a rational sequence converging to x_0 and y_n be an irrational sequence converging to x_0.
- Thus $f(x_n) = 1$ for every n and $f(y_n) = 0$ for every n.

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- If $x_0 \in \mathbb{Q}$, then $f(x_0) = 1$. Thus $y_n \to x_0$, but $f(y_n) = 0 \nrightarrow f(x_0) = 1$.
- If $x_0 \in \mathbb{R} \setminus \mathbb{Q}$, then $f(x_0) = 0$. Thus $x_n \to x_0$, but $f(x_n) = 1 \nrightarrow f(x_0) = 0$.

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Therefore by sequential criteria for continuity f is not continuous at x_0.

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- Let x_0 be any real number. Let x_n be a rational sequence converging to x_0 and y_n be an irrational sequence converging to x_0.
- Thus $f(x_n) = 1$ for every n and $f(y_n) = 0$ for every n.
- If $x_0 \in \mathbb{Q}$, then $f(x_0) = 1$. Thus $y_n \to x_0$, but $f(y_n) = 0 \nrightarrow f(x_0) = 1$.
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Therefore by sequential criteria for continuity f is not continuous at x_0. Since x_0 is arbitrary, $f(x)$ is discontinuous at every real number.

Subsequences

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Let $(a_n) = \{a_1, a_2, a_3, \ldots\}$. Then
$\{a_1, a_5, a_6, a_{13}, \ldots\}$
$\{a_1, a_3, a_5, a_7, \ldots\}$
$\{a_{1001}, a_{100001}, a_{200001} \ldots\}$ are subsequences
$\{a_5, a_4, a_6, a_7, \ldots\}$ is not a subsequence.

Remark

Why are we interested in subsequences?

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$\{1, -1, 1, -1, \cdots\}$ doesn't converge as it has a subsequence $\{1, 1, 1, \cdots\}$ that converges to 1 and another subsequence $\{-1, -1, -1, \cdots\}$ that converges to -1. Since the limits of both subsequences are different we can conclude that original sequence doesn't converge.

Bounded Sequences

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A sequence (a_n) of real numbers is said to be **bounded below** if there is a real number β such that $\beta \leq a_n$ for every $n \in \mathbb{N}$. The number β is a lower bound for a_n. If β is a lower bound for a_n but no number greater than β is a lower bound for a_n, then β is **the greatest lower bound** for a_n.

A sequence (a_n) of real numbers is said to be **bounded** if there are real numbers α, β such that $\beta \leq a_n \leq \alpha$ for every $n \in \mathbb{N}$.

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If a sequence is not bounded, it is said to be **unbounded**.

Monotone sequences and convergence

- A sequence (x_n) is said to be **monotone increasing** or nondecreasing if $x_n \leq x_{n+1}$ for all $n \in \mathbb{N}$, that is, $x_1 \leq x_2 \leq x_3 \leq \cdots$.

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- A sequence is **monotone** if it is either monotone increasing or monotone decreasing.

A nondecreasing sequence that is bounded from above always has a least upper bound. Likewise, a nonincreasing sequence bounded from below always has a greatest lower bound. These results are based on the completeness property of the real numbers.

Monotone convergence theorem

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- Let a_n be a monotone increasing sequence that is bounded above.
- Let L be its least upperbound. By definition of least upper bound, $L - \epsilon$ is not an upperbound. i.e some term a_k from the sequence satisfies $a_k > L - \epsilon$.
- Since (a_n) is increasing $a_n \geq a_k, \forall n \geq k$.
- Thus we have $L - \epsilon < a_k \leq a_n \leq L < L + \epsilon, \forall n \geq k$. Thus $a_n \to L$.

Example:

$$\text{Let } a_1 := \frac{3}{2} \quad \text{and} \quad a_{n+1} := \frac{1}{2}\left(a_n + \frac{2}{a_n}\right) \quad \text{for } n \in \mathbb{N}.$$

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$$a_n - a_{n+1} = a_n - \frac{a_n^2 + 2}{2a_n} = \frac{a_n^2 - 2}{2\,a_n} \quad \text{for all } n \in \mathbb{N},$$

(a_n) is decreasing if and only if $a_n^2 - 2 \geq 0$ for all $n \in \mathbb{N}$. But

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$$a_1^2 \geq 2 \quad \text{and} \quad a_{n+1}^2 - 2 = \frac{\left(a_n^2 - 2\right)^2}{4a_n^2} \geq 0 \text{ for all } n \in \mathbb{N}.$$

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Hence the sequence (a_n) is decreasing.

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Since the limit of a sequence is unique, we see that $\frac{1}{2}\left(a + \frac{2}{a}\right) = a$, that is, $a^2 = 2$.

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Since the limit of a sequence is unique, we see that $\frac{1}{2}\left(a + \frac{2}{a}\right) = a$, that is, $a^2 = 2$.

Also, $a_n > 0$ for all $n \in \mathbb{N}$ and $a_n \to a$, so that $a \geq 0$. Thus a is the positive square root of 2, that is, $a = \sqrt{2}$.

Exercises

Determine if the sequences is monotonic and bounded.

- $a_n = \frac{n}{n+1}$
- $a_n = \frac{3n+1}{n+1}$
- $a_n = \frac{(2n+3)!}{(n+1)!}$

Theorem

Suppose that $f(x)$ is a function defined for all $x \geq n_0$ and that (a_n) is a sequence of real numbers such that $a_n = f(n)$ for $n \geq n_0$. Then $\lim_{n\to\infty} a_n = \ell$ whenever $\lim_{x\to\infty} f(x) = \ell$.

(i) Show that $\lim_{n\to\infty} \dfrac{\log n}{n} = 0$.

Functions and sequences

Theorem

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(i) Show that $\lim_{n\to\infty} \dfrac{\log n}{n} = 0$.

We take $f(x) = \frac{\log x}{x}$ and $f(x)$ is defined for $x \geq 1$. Therefore

$$\lim_{n\to\infty} \frac{\log n}{n} = \lim_{x\to\infty} \frac{\log x}{x} = 0.$$

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$$\begin{aligned}
\lim_{n\to\infty} \ln(a_n) &= \lim_{n\to\infty} n \ln(\frac{n+1}{n-1}) \\
&= \lim_{n\to\infty} \frac{\ln(\frac{n+1}{n-1})}{1/n} \\
&= \lim_{n\to\infty} \frac{-2/(n^2-1)}{-1/n^2} \\
&= \lim_{n\to\infty} \frac{2n^2}{n^2-1} = 2.
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\end{aligned}$$

Since $\ln(a_n) \to 2$ and $f(x) = e^x$ is continuous, $a_n \to e^2$.