MATHEMATICS-I

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Lecture 12

Power series

- A power series about $x = a$ is a series of the form

$$\sum_{n=0}^{\infty} c_n(x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots. \tag{0.1}$$

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  The series converges when $|x-2| < 2$. In that case, the series converges to $f(x) = \dfrac{1}{1 - \frac{-1}{2}(x-2)} = \dfrac{2}{x}$.

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- If $|x| > 1$, then $\{(-1)^{n-1}\frac{x^n}{n}\}$ is divergent and hence the series diverges.
- For $x = 1$, the series is an alternating harmonic series and hence it converges.
- For $x = -1$ the series becomes negative of harmonic series hence it diverges.

Examples

Find the values of x where the following series converges and diverges.

- (a) $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^{2n-1}}{2n-1}$, (b) $\sum_{n=0}^{\infty}\frac{(n+1)(x-2)^n}{(2n+1)!}$

  (c) $\sum_{n=0}^{\infty} n!x^n$.

Theorem 0.1.

If the power series $\sum_{n=0}^{\infty} a_n x^n$ *converges at* $x = c \neq 0$, *then it converges absolutely for all* x *with* $|x| < |c|$. *If the series diverges at* $x = d$, *then it diverges for all* x *with* $|x| > |d|$.

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Proof: Suppose that the series converges for $x = c$ that is $\sum_{n=0}^{\infty} a_n c^n$ converges.

It implies $\lim_{n\to\infty} a_n c^n = 0$ by n-th term test. Hence there exists a positive integer N such that $|a_n c^n| < 1$ for all $n > N$.

We have that $|a_n| < \dfrac{1}{|c|^n}$ for $n > N$.

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Now for any $|x| < |c|$, we see that

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Therefore the $\sum_{n=0}^{\infty} a_n x^n$ converges absolutely for $|x| < |c|$.

Now suppose that the series $\sum_{n=0}^{\infty} a_n x^n$ diverges for $x = d$. We have to prove that the series $\sum_n a_n x^n$ diverges for $|x| > |d|$.

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Therefore, the series $\sum_n a_n x^n$ is not convergent (diverges) for any $|x| > |d|$.

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3. If the power series $\sum_{n=0}^{\infty} a_n (x-a)^n$ diverges for some $x = d$ then it diverges for all x such that $|x-a| > |x-d|$.