Tutorial 9

MATH F111- Mathematics I

September 30, 2024

1. Describe level curve of $f(x, y) = e^{y-x^2-1}$ that passes through the point $(4, 17)$.
   Hint: Let the level curve be $f(x, y) = e^c$ for a constant c.

2. A thin metal plate, located in the xy plane, has temperature $T(x, y)$ at the point (x, y). The level curves of T are called isothermals because at all points on an isothermal the temperature is the same. Sketch some isothermals if the temperature function is given by

$$T(x, y) = 100/(1 + x^2 + 2y^2).$$

3. Let

$$f(x, y) = \begin{cases} x^2 & \text{if } x \geq 0 \\ x^3 & \text{if } x < 0 \end{cases}$$

   Find the following limits.

   - $\lim\limits_{(x,y)\to(3,-2)} f(x, y)$
   - $\lim\limits_{(x,y)\to(-2,1)} f(x, y)$
   - $\lim\limits_{(x,y)\to(0,0)} f(x, y)$.

4. The Sandwich Theorem for functions of two variables states that if $g(x, y) \leq f(x, y) \leq h(x, y)$ for all $(x, y) \neq (x_0, y_0)$ in a disk centered at (x_0, y_0) and if g and h have the same finite limit L as $(x, y) \to (x_0, y_0)$, then

$$\lim_{(x,y)\to(x_0,y_0)} f(x, y) = L.$$

   Use this result to support your answers to the following questions.

- Does knowing that

$$1 - \frac{x^2y^2}{3} < \frac{\tan^{-1} xy}{xy} < 1$$

tell you anything about

$$\lim_{(x,y)\to(0,0)} \frac{\tan^{-1} xy}{xy}?$$

- Does knowing that $|\sin(\frac{1}{x})| \leq 1$ tell you anything about

$$\lim_{(x,y)\to(0,0)} y \sin \frac{1}{x}?$$

- Find $\lim_{(x,y)\to(0,0)} \frac{x^2 \sin^2 y}{x^2+2y^2}$?
- Find $\lim_{(x,y)\to(0,0)} xy\frac{x^2-y^2}{x^2+y^2}$?

5. Define $f(0,0)$ in a way that f extends to be be continuous at the origin.

(i) $f(x,y) = xy\frac{x^2-y^2}{x^2+y^2}$

(ii) $f(x,y) = \ln\left(\frac{3x^2-x^2y^2+3y^2}{x^2+y^2}\right)$

(iii) $f(x,y) = \frac{3x^2y}{x^2+y^2}$

6. Find the limit, if it exists, or show that the limit does not exist.

- $\lim_{(x,y)\to(0,0)} \frac{5y^4\cos^2 x}{x^4+y^4}$
- $\lim_{(x,y,z)\to(0,0,0)} \frac{xy+yz}{x^2+y^2+z^2}$

7. Determine the set of points at which the function is continuous.

- $F(x,y) = \frac{1+x^2+y^2}{1-x^2-y^2}$
- $F(x,y) = \cos\sqrt{(1-x-y)}$
- $F(x,y) = \frac{1}{|xy|+|z|}$
- $F(x,y,z) = \frac{1}{4-\sqrt{x^2+y^2+z^2-1}}$

8. Changing variables to Polar Coordinates:

- How do you define $\lim_{(x,y)\to(0,0)} f(x,y)$ in polar co-ordinates?
- Let $f(x,y) = \frac{2x^2y}{x^4+y^2}$. Convert the function into polar co-ordinates. Does the limit exists at $(0,0)$? What happens to the limit for a fixed θ and $r \to 0$? What happens on the path $y = x^2$, i.e $r\sin\theta = r^2\cos\theta$?

9. Discuss limit at the given point.

   (a) $f(x,y) = \dfrac{1}{\sqrt{x^2+y^2}}(||x|-|y||-|x|-|y|)$ at $(0,0)$ Hint: Try $y = mx$ with $m = 3, 4, 5$.

   (b) $f(x,y) = \dfrac{x^2y^2}{x^2+y^2}$ at $(0,0)$ Hint: $\epsilon - \delta$

   (c) $f(x,y) = \dfrac{x^3+y^3}{x-y}$ at $(0,0)$ Hint: $y = x - mx^3$.

   (d) $f(x,y) = \dfrac{y^4\sqrt{xy}}{x^2+xy^3}$ at $(0,0)$ Hint: $y = my^3$.