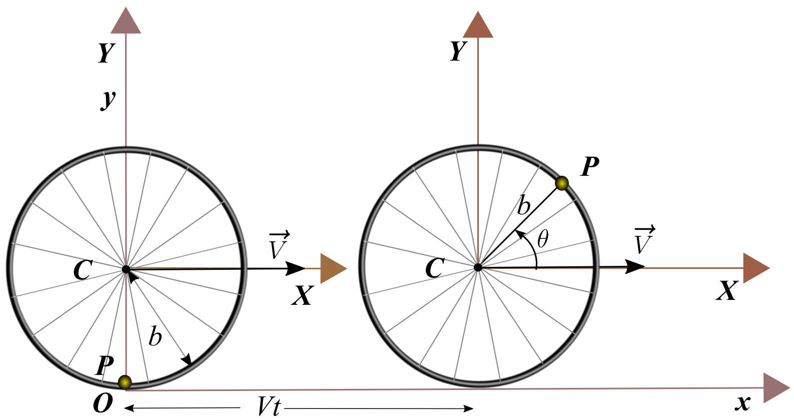
Tutorial 1 **Vectors and Polar coordinates** 6 August 2024

Problem statement:

The Cartesian unit vectors are constants, independent of location and time. The polar unit vectors depend on location. If the origin of the polar coordinate system is moving with time, then the polar unit vectors will depend on time as well.

1. A tyre of radius b is rolling without slipping such that the center moves at a constant velocity $V_0\hat{\boldsymbol{\imath}}$. A pebble on the ground gets lodged in the rim of the wheel at $t = 0$.



Consider two coordinate systems:

- Ground-fixed x-y system with its origin O at the location of the pebble at $t = 0$.
- Wheel Center-fixed non-rotating X-Y system with its origin C at the center of the wheel.
- The corresponding Cartesian axes of the two systems are parallel to each other, at all times.

(1.1) Find the coordinates of the center of the wheel as a function of time in both the coordinate systems.

(1.2) Find the coordinates (x, y) of an arbitrary point Q in terms of its Wheel-Center-fixed coordinates (X, Y) at some time t.

(1.3) If θ is the angle between the radius vector CP and the horizontal CX, at some time $t > 0$, determine the relation between the horizontal distance moved by the center of the wheel during an additional time dt and the change in the angle $d\theta$. Hence find a relation between $\omega = \dfrac{d\theta}{dt}$, V_0 and b. Integrate this relation to find $\theta(t)$ describing the position of the pebble at time t. Note that $\theta(0) = -\pi/2$.

(1.4) In terms of θ, find the Cartesian coordinates of the pebble in the Wheel-Center-fixed system and construct the vector $\vec{\boldsymbol{R}} = X\hat{\boldsymbol{\imath}} + Y\hat{\boldsymbol{\jmath}}$.

(1.5) Use the result of (1.1) above and find the position vector $\vec{\boldsymbol{r}} = x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}}$. in the Ground fixed system.

(1.6) From (1.4) find $\vec{\boldsymbol{V}} = d\vec{\boldsymbol{R}}/dt$ and $\vec{\boldsymbol{A}} = d\vec{\boldsymbol{V}}/dt$. From (1.5) determine $\vec{\boldsymbol{v}} = \dfrac{d\vec{\boldsymbol{r}}}{dt}$ and $\vec{\boldsymbol{a}} = \dfrac{d\vec{\boldsymbol{v}}}{dt}$.

(1.7) Construct the unit radial and unit tangential vectors $\hat{\boldsymbol{R}}$, $\hat{\boldsymbol{T}}$ at the location of the pebble P at time t, in the Wheel-Center-fixed frame in terms of the Cartesian unit vectors $\hat{\boldsymbol{\imath}}$ and $\hat{\boldsymbol{\jmath}}$.

(1.8) Likewise, construct the radial and tangential vectors $\hat{\boldsymbol{r}}$, $\hat{\boldsymbol{t}}$ at P at time t, in the Ground-fixed frame in terms of the Cartesian unit vectors $\hat{\boldsymbol{\imath}}$ and $\hat{\boldsymbol{\jmath}}$.

(1.9) Obtain the radial and tangential velocities and accelerations of the pebble in the Wheel-Center fixed frame.

(1.10) Obtain the radial and tangential velocities and accelerations of the pebble in the Ground fixed frame. **Note:** *the radial velocity and acceleration are defined as* $v_r = \hat{\boldsymbol{r}} \cdot \frac{d\vec{\boldsymbol{r}}}{dt} = \frac{d(|\vec{\boldsymbol{r}}|)}{dt}$ *and* $a_r = \hat{\boldsymbol{r}} \cdot \frac{d\vec{\boldsymbol{v}}}{dt}$. *The tangential velocities and accelerations are likewise* $\hat{\boldsymbol{t}} \cdot \vec{\boldsymbol{v}}$ and $\hat{\boldsymbol{t}} \cdot \vec{\boldsymbol{a}}$.

(1.11) Sketch the trajectory of the pebble in the x-y system. Describe how you would go about it.

Feedback Problem

2. A particle moves with constant angular velocity $\dot{\theta} = \omega$ along a path given by $r = r_0 e^{\beta t}$. You will need to express the velocity and acceleration of the particle in plane polar coordinates before you can answer the following questions.

(2.1) For what value of β is the radial acceleration zero?

(2.2) When the radial acceleration is zero, what is $\ddot{r}$?

(2.3) What does this mean?

(2.4) What is the radial velocity of the particle?

Problems for Practice

3. A bead is moving along the spoke of a wheel at uniform speed u. The wheel is rotating at constant angular speed ω about its center which is fixed. At the start of the experiment, the bead is at the center, and the spoke is horizontal.

(3.1) Is the bead accelerating?

(3.2) Express the velocity of the bead in polar and Cartesian coordinates

(3.3) Express the acceleration in polar coordinates

(3.4) Visualize the path of the bead. Can you find the equation for the trajectory.

(3.5) Plot the radial and tangential components of velocity and acceleration at a few select points on the trajectory.

(K-K) 1.5, 1.15, 1.16