Double Integrals - Area and Polar Form

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As we pass through the limit we get 1.

Examples

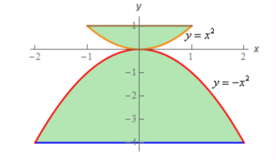
1. Sketch the region of integration, reverse the order of integration, and evaluate the integrals:

   1. $\displaystyle\int_0^3 \int_{\sqrt{x/3}}^1 e^{y^3}\, dy dx$ $(e - 1)$
   2. $\displaystyle\int_0^1 \int_y^1 x^2 e^{xy}\, dx dy$ $((e-2)/2)$

2. Sketch the region of integration and evaluate $\displaystyle\iint_R (y - 2x^2) dA$ where R is the region bounded by $|x| + |y| = 1$. (-2/3)

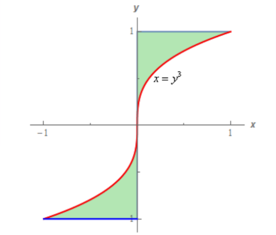
3. Compute the integral: $\displaystyle\int_0^\infty \int_0^\infty x e^{-(x+2y)} dx dy$ (1/2)

4. Evaluate $\iint\limits_R (3-6xy)dA$ where D is the region shown below:



(36 - Split the integral up and do the actual integration over separate sub regions.)

5. Evaluate $\iint\limits_D e^{y^4} dA$ where D is the region shown below:



((e-1)/2)

6. Find the volume of the solid bounded by the planes $x = 0, y = 0, z = 0$, and $2x + 3y + z = 6$. (6 cubic units - the solid is a tetrahedron with the base on the xy plane and a height $z = 6 - 2x - 3y$. The base is the region bounded by the lines, $x = 0, y = 0$, and $2x + 3y = 6$ where $z = 0$.)

7. Find the volume of the first octant part of the solid bounded by the cylinders $x^2 + y^2 = 1$ and $y^2 + z^2 = 1$.

$$\left(\int_0^1 \int_0^{\sqrt{1-y^2}} \sqrt{1 - y^2}\, dx dy = 2/3 \right)$$

8. Find the region R on the xy plane that maximizes the integral

$$\iint_R (4 - x^2 - y^2) dy dx.$$

(The region must be $x^2 + y^2 \leq 4$ - jusify using the monotonicity property of double integral).

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Average

Average value of f over R is

$$\frac{1}{\text{Area of } R} \iint\limits_R f \, dA.$$

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Example 1: Find the average value of the function $f(x, y) = 7xy^2$ on the region R bounded by the line $x = y$ and the curve $x = \sqrt{y}$.

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Example 2. Sketch the region bounded by the curve $y = e^x$, and the lines $y = 0$, $x = 0$, and $x = \ln 2$. Then express the region's area as an iterated double integral and evaluate the integral. (1)

Example 3. Find the average height of the paraboloid $z = x^2 + y^2$ over the square $0 \le x \le 2$, $0 \le y \le 2$. (8/3)

Integral of Function of Polar Coordinates

Suppose that a function $f(r, \theta)$ is defined in polar coordinates over the region R which lies between the angles $\alpha \leq \theta \leq \beta$ and the radius bounded as $g_1(\theta) \leq r \leq g_2(\theta)$ where $g_1(\theta), g_2(\theta)$ are continuous functions.

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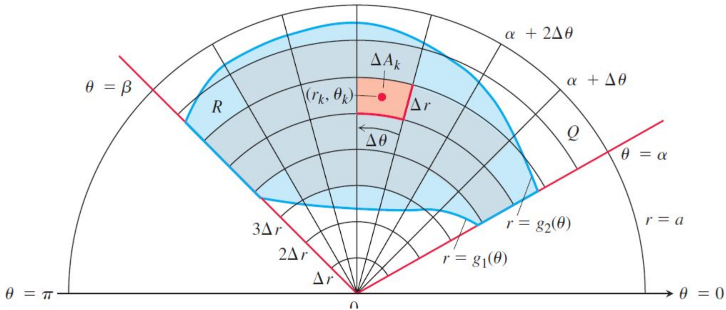
Suppose also that $0 \leq g_1(\theta) \leq g_2(\theta) \leq a$ for every value of θ between α and β. Then R lies in a fan shaped region Q defined by the inequality $0 \leq r \leq a$ and $\alpha \leq \theta \leq \beta$.



We cover Q by a grid of circular arcs and rays. The arcs are cut from circles centered at the origin, with radii Δr, $2\Delta r, \ldots, m\Delta r$, where $\Delta r = a/m$. The rays are given by

$$\theta = \alpha, \ \theta = \alpha + \Delta\theta, \ \ldots, \ \theta = \alpha + m'\Delta\theta = \beta,$$

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where $\Delta\theta = (\beta - \alpha)/m'$.
The arcs and rays partition Q into small patches called "polar rectangles". We number the polar rectangles that lies inside R, calling their areas $\Delta A_1, \Delta A_2, \ldots, \Delta A_n$. We let (r_k, θ_k) be any point in the polar rectangles whose area is ΔA_k. We then form the sum

$$S_n = \sum_{k=1}^{n} f(r_k, \theta_k)\Delta A_k.$$

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If f is continuous on R, this sum will approach a limit as we refine the grid to make Δr and $\Delta\theta$ go to zero. The limit is called the double integral of f over R.

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We note that $\Delta A_k = r_k \ \Delta r \ \Delta\theta$, therefore

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A version of Fubini's theorem says that

$$\iint_R f(r,\theta)dA = \int_{\theta=\alpha}^{\theta=\beta} \int_{r=g_1(\theta)}^{r=g_2(\theta)} f(r,\theta) r\ dr\ d\theta.$$

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$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \int_0^{\frac{6}{\sin\theta}} r^2 \cos\theta \, dr d\theta = 72 \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{\cos\theta d\theta}{\sin^3\theta} = 72 \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cot\theta \operatorname{cosec}^2\theta \, \mathrm{d}\theta$$

$$= -36 \left[\cot^2\theta \right]_{\frac{\pi}{4}}^{\frac{\pi}{2}} = 36.$$

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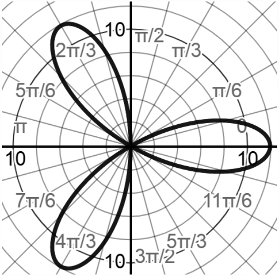

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