Mathematics I- MATH F111

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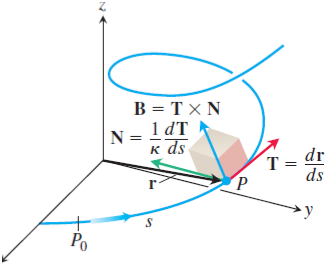
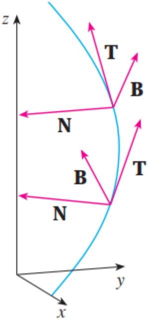
Expressing the acceleration vector along the curve as a linear combination of this **TNB** frame of mutually orthogonal unit vectors traveling with the motion can reveal much about the nature of the path and motion along it.

Definition

The binormal vector of a curve in space is $\mathbf{B} = \mathbf{T} \times \mathbf{N}$ which is a unit vector that is orthogonal to both $\mathbf{T}$ and $\mathbf{N}$. Together $\mathbf{T}, \mathbf{N}$, and $\mathbf{B}$ define a moving righthanded vector frame that plays a significant role in calculating the paths of particles moving through space. It is called the Frenet ("fre-nay") frame (after Jean-Frédéric Frenet, 1816–1900), or the **TNB** frame.





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Then we differentiate both ends of this string of equalities to get

$$\begin{aligned}\vec{a} = \frac{d\vec{v}}{dt} &= \frac{d}{dt}(\mathbf{T}\frac{ds}{dt}) = \frac{d^2s}{dt^2}\mathbf{T} + \frac{ds}{dt}\frac{d\mathbf{T}}{dt} \\ &= \frac{d^2s}{dt^2}\mathbf{T} + \frac{ds}{dt}\left(\frac{d\mathbf{T}}{ds}\frac{ds}{dt}\right) = \frac{d^2s}{dt^2}\mathbf{T} + \frac{ds}{dt}\left(\kappa\mathbf{N}\frac{ds}{dt}\right) \\ &= \frac{d^2s}{dt^2}\mathbf{T} + \kappa\left(\frac{ds}{dt}\right)^2\mathbf{N}.\end{aligned}$$

Definition

If the acceleration vector is written as $\vec{a} = a_T\mathbf{T} + a_N\mathbf{N}$, then $a_T = \frac{d^2s}{dt^2} = \frac{d}{dt}|\vec{v}|$ and $a_N = \kappa\left(\frac{ds}{dt}\right)^2 = \kappa|\vec{v}|^2$ are the tangential and normal scalar components of acceleration.

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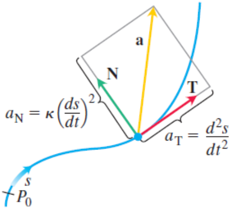
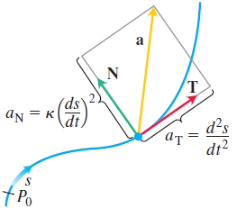
- The equation also tells us exactly how much of the acceleration takes place tangent to the motion and how much takes place normal to the motion.



By definition, acceleration $\vec{a}$ is the rate of change of velocity $\vec{v}$, and in general, both the length and direction of $\vec{v}$ change as an object moves along its path.



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- The tangential component of acceleration a_T measures the rate of change of the length of $\vec{v}$ (that is, the change in the speed).
- The normal component of acceleration a_N measures the rate of change of the direction of $\vec{v}$.

The normal scalar component of the acceleration is the curvature times the square of the speed. This explains why you have to hold on when your car makes a sharp (large κ), high-speed (large $|\vec{v}|$) turn. If you double the speed of your car, you will experience four times the normal component of acceleration for the same curvature.

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If an object moves in a circle at a constant speed, $\frac{d^2s}{dt^2}$ is zero and all the acceleration points along $\mathbf{N}$ toward the circle's center. If the object is speeding up or slowing down, $\vec{a}$ has a nonzero tangential component.

To calculate a_N, we usually use the formula $a_N = \sqrt{|\vec{a}^2| - a_T^2}$, which comes from solving the equation $|\vec{a}^2| = \vec{a} \cdot \vec{a} = a_T^2 + a_N^2$. Using this we can find a_N without having to calculate κ first.

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Problem 1: Without finding $\vec{T}$ and $\vec{N}$, write the acceleration of the motion

$$r(t) = (\cos t + t \sin t)\vec{i} + (\sin t - t \cos t)\vec{j}, t > 0$$

in the form

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Sol: $a_T = 1, a_N = t$.

Torsion

How does $\frac{d\mathbf{B}}{ds}$ behave in relation to $\mathbf{T}, \mathbf{N}$, and $\mathbf{B}$?

$$\frac{d\mathbf{B}}{ds} = \frac{d}{ds}(\mathbf{T} \times \mathbf{N}) = \frac{d\mathbf{T}}{ds} \times \mathbf{N} + \mathbf{T} \times \frac{d\mathbf{N}}{ds} = \mathbf{T} \times \frac{d\mathbf{N}}{ds}.$$

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$$\frac{d\mathbf{B}}{ds} = -\tau\mathbf{N}.$$

The scalar τ is called the torsion along the curve. Notice that

$$\frac{d\mathbf{B}}{ds} \cdot \mathbf{N} = -\tau \mathbf{N} \cdot \mathbf{N} = -\tau.$$

Definition

Let $\mathbf{B} = \mathbf{T} \times \mathbf{N}$. The torsion function of a smooth curve is

$$\tau = -\frac{d\mathbf{B}}{ds} \cdot \mathbf{N}.$$

Unlike the curvature κ, which is never negative, the torsion τ may be positive, negative, or zero.

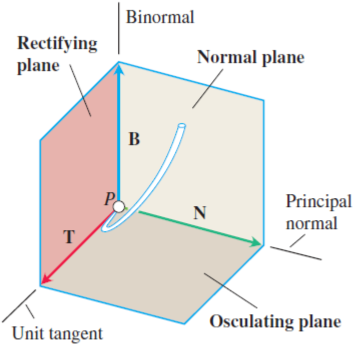

FIGURE 13.28 The names of the three planes determined by **T**, **N**, and **B**.

The curvature $\kappa = |\frac{d\mathbf{T}}{ds}|$ can be thought of as the rate at which the normal plane turns as the point P moves along its path. Similarly, the torsion τ is the rate at which the osculating plane turns about $\mathbf{T}$ as P moves along the curve. Torsion measures how the curve twists.

Vector Formula for Curvature

$$\kappa = \frac{|\mathbf{v} \times \mathbf{a}|}{|\mathbf{v}|^3}$$

Formula for Torsion

$$\tau = \frac{\begin{vmatrix} \dot{x} & \dot{y} & \dot{z} \\ \ddot{x} & \ddot{y} & \ddot{z} \\ \dddot{x} & \dddot{y} & \dddot{z} \end{vmatrix}}{|\mathbf{v} \times \mathbf{a}|^2} \qquad (\text{if } \mathbf{v} \times \mathbf{a} \neq \mathbf{0})$$

Find the curvature and torsion for the helix

$$r(t) = (a\cos t)\vec{i} + (a\sin t)\vec{j} + bt\vec{k}, \;\; a, b \geq 0, \;\; a^2 + b^2 \neq 0$$

using the above formula.

Derive the vector formula for curvature

$$\kappa = \frac{|v \times a|}{|v|^3}.$$

Derive

$$\tau = -\frac{1}{|v|}\left(\frac{d\mathbf{B}}{dt} \cdot \mathbf{N}\right).$$