Lecture 10

Angular Momentum

Radhika Vathsan, BITS-Goa, 2010-11

Lecture 10

ANGULAR MOMENTUM

1. **Angular Momentum: Point Particle**
   - Definition
   - Example 1: Linear Motion
   - Example 2: Conical Pendulum
2. **Torque**
   - Example: Conical Pendulum
   - Precession of L

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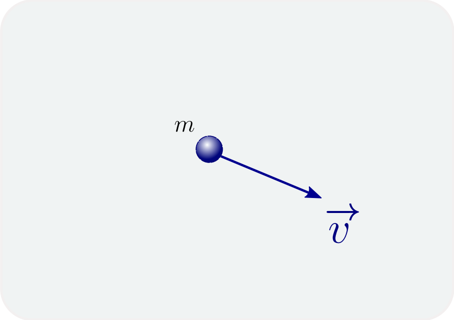
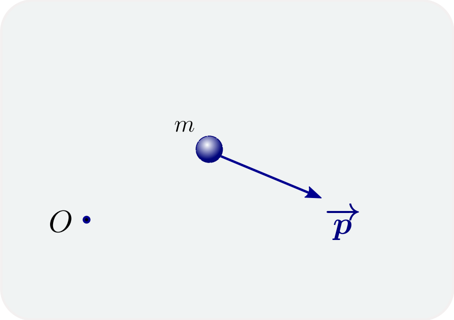
Angular Momentum of a point particle

Definition:



Angular Momentum of a point particle

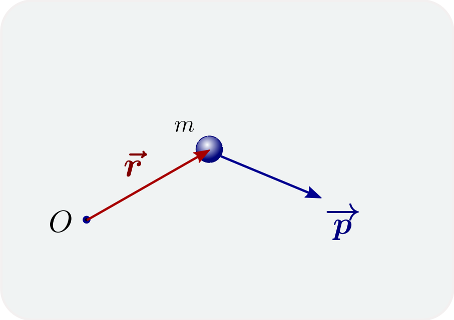
Definition: **Always w.r.t a specific point say O.**



Angular Momentum of a point particle

Definition: **Always w.r.t a specific point say O.**

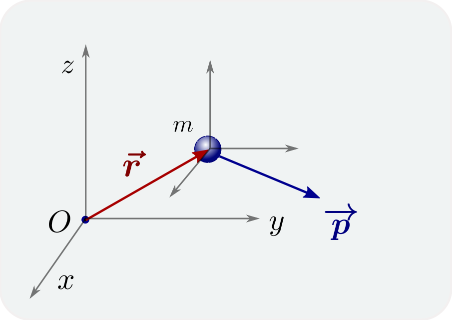
$$\overrightarrow{L}_o = \vec{r} \times \vec{p}$$



Angular Momentum of a point particle

Definition: **Always w.r.t a specific point say O.**

$$
\begin{aligned}
\overrightarrow{\boldsymbol{L}}_o &= \vec{\boldsymbol{r}} \times \vec{\boldsymbol{p}} \\
&= (yp_z - zp_y)\hat{\boldsymbol{i}} \\
&\quad + (zp_x - xp_z)\hat{\boldsymbol{j}} \\
&\quad + (xp_y - yp_x)\hat{\boldsymbol{k}}
\end{aligned}
$$



Angular Momentum of a point particle

Definition: **Always w.r.t a specific point say O.**

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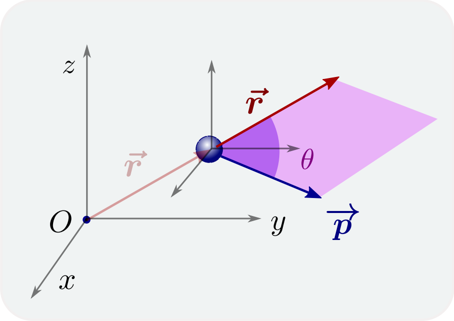
\end{aligned}
$$

$$
L = |\overrightarrow{\boldsymbol{L}}| = rp_\perp = r_\perp p
$$

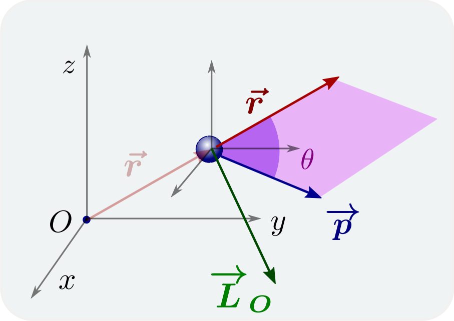


Angular Momentum of a point particle

Definition: **Always w.r.t a specific point say O.**

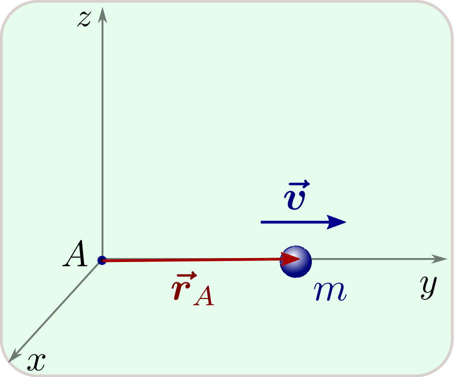
$$\vec{L}_o = \vec{r} \times \vec{p}$$
$$= (yp_z - zp_y)\hat{i} + (zp_x - xp_z)\hat{j} + (xp_y - yp_x)\hat{k}$$

$$L = |\vec{L}| = rp_\perp = r_\perp p$$
$$= rp\sin\theta = mvr\sin\theta$$



Example 1: Particle in linear motion

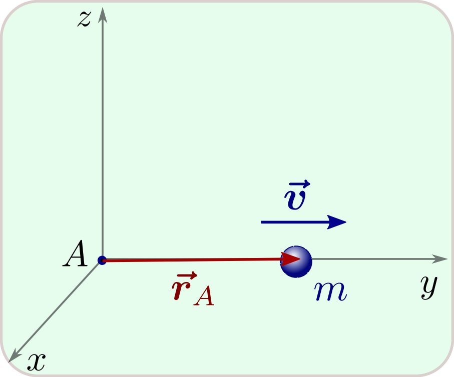
- $\vec{L}$ about A



Example 1: Particle in linear motion

- $\overrightarrow{\boldsymbol{L}}$ about A

$$\overrightarrow{\boldsymbol{L}}_A = \vec{\boldsymbol{r}}_A \times (m\vec{\boldsymbol{v}}) = 0$$

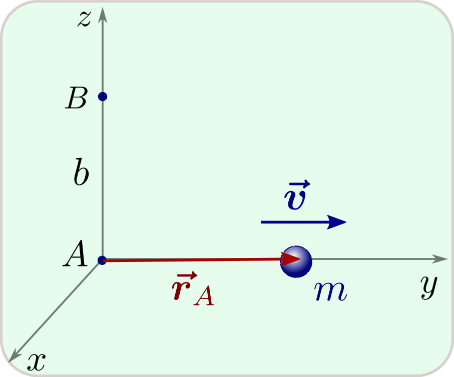


Example 1: Particle in linear motion

- $\vec{L}$ about A

$$\vec{L}_A = \vec{r}_A \times (m\vec{v}) = 0$$

- $\vec{L}$ about B



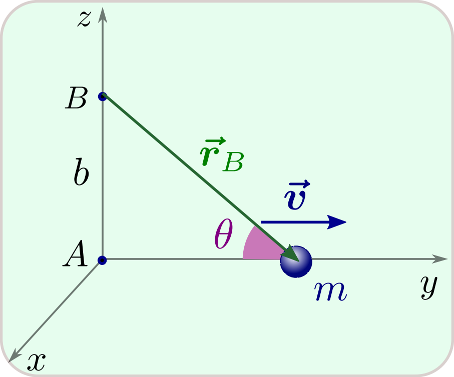
Example 1: Particle in linear motion

- $\vec{L}$ about A

$$\vec{L}_A = \vec{r}_A \times (m\vec{v}) = 0$$

- $\vec{L}$ about B

$$\vec{L}_B = \vec{r}_B \times (m\vec{v})$$



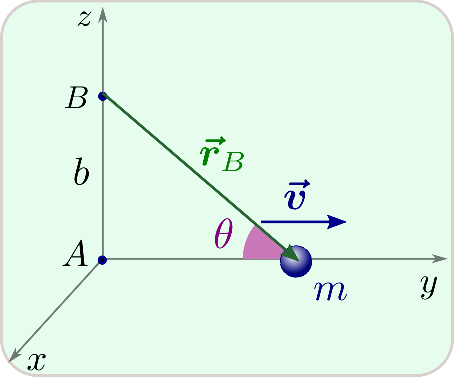
Example 1: Particle in linear motion

- $\vec{\boldsymbol{L}}$ about A

$$\vec{\boldsymbol{L}}_A = \vec{\boldsymbol{r}}_A \times (m\vec{\boldsymbol{v}}) = 0$$

- $\vec{\boldsymbol{L}}$ about B

$$\begin{aligned} \vec{\boldsymbol{L}}_B &= \vec{\boldsymbol{r}}_B \times (m\vec{\boldsymbol{v}}) \\ &= mvr_B \sin\theta \ \hat{\boldsymbol{i}} \end{aligned}$$



Example 1: Particle in linear motion

- $\overrightarrow{L}$ about A

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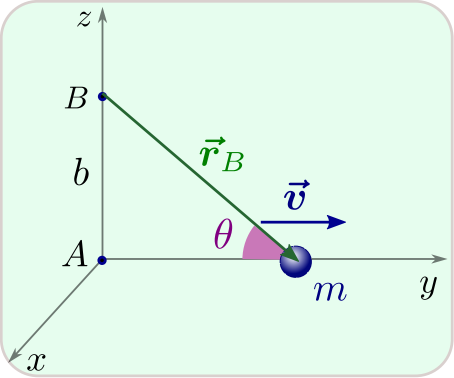
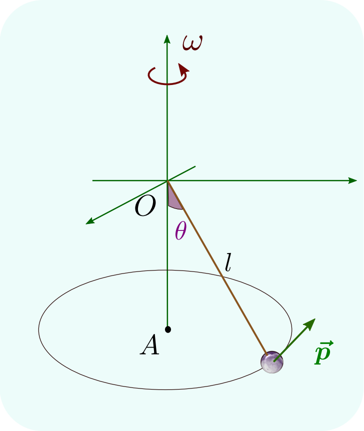
- $\overrightarrow{L}$ about B

$$\begin{aligned} \overrightarrow{\boldsymbol{L}}_B &= \vec{\boldsymbol{r}}_B \times (m\vec{\boldsymbol{v}}) \\ &= mvr_B \sin\theta \, \hat{\boldsymbol{i}} \\ &= mvb \, \hat{\boldsymbol{i}} \end{aligned}$$



Example 2: Conical Pendulum



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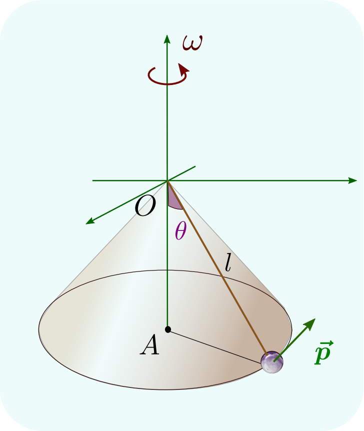
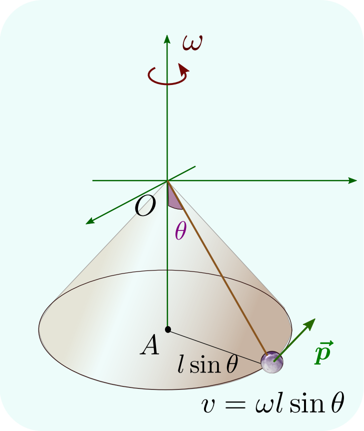


Example 2: Conical Pendulum

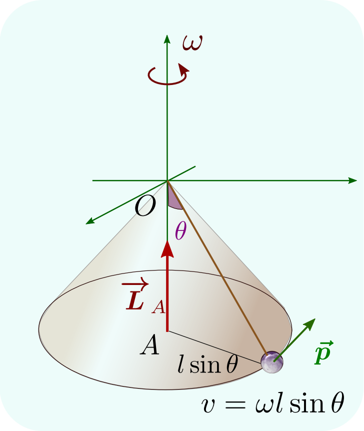
- $\vec{L}$ about A

$$\vec{L}_A \quad =$$



Example 2: Conical Pendulum

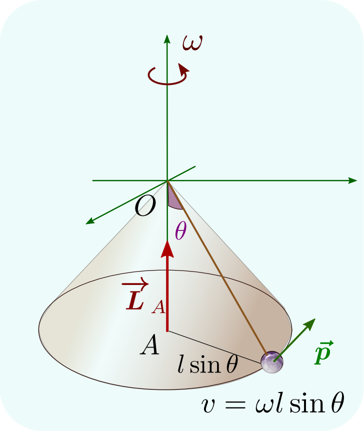
- $\vec{L}$ about A
$$\vec{L}_A = mvl\sin\theta\hat{k}$$



Example 2: Conical Pendulum

- $\vec{L}$ about A

$$\begin{aligned}\vec{L}_A &= mvl\sin\theta\hat{k}\\ &= m\omega l^2\sin^2\theta\hat{k}\end{aligned}$$



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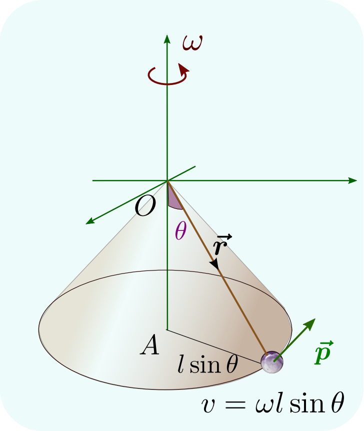
- $\vec{L}$ about O

$$\vec{L}_O =$$



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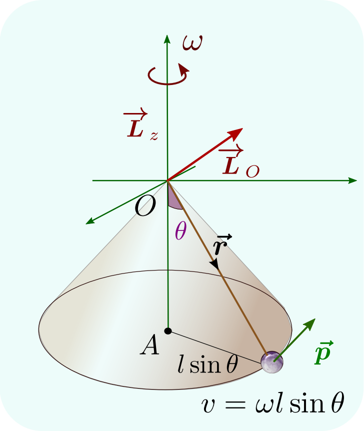
$$= m\omega l^2 \sin^2\theta\hat{k}$$
- $\vec{L}$ about O
$$\vec{L}_O = \vec{r}\times\vec{p}$$



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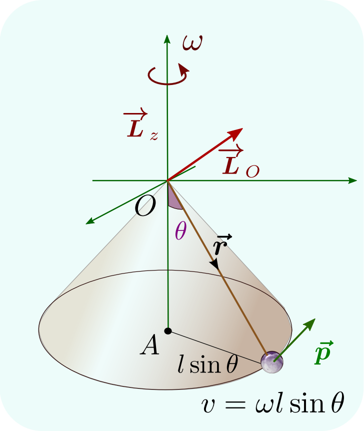
$$\begin{aligned} \vec{L}_A &= mvl\sin\theta\hat{k} \\ &= m\omega l^2\sin^2\theta\hat{k} \end{aligned}$$

- $\vec{L}$ about O

$$\begin{aligned} \vec{L}_O &= \vec{r}\times\vec{p} \\ |\vec{L}_O| &= mvl = m\omega l^2\sin\theta \end{aligned}$$



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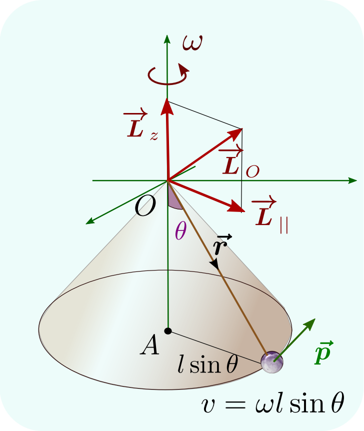
$$\begin{aligned}\vec{L}_O &= \vec{r}\times\vec{p}\\ |\vec{L}_O| &= mvl = m\omega l^2\sin\theta\\ L_z &= L\sin\theta = m\omega l^2\sin^2\theta\end{aligned}$$



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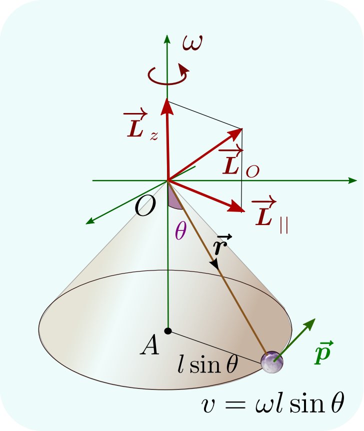

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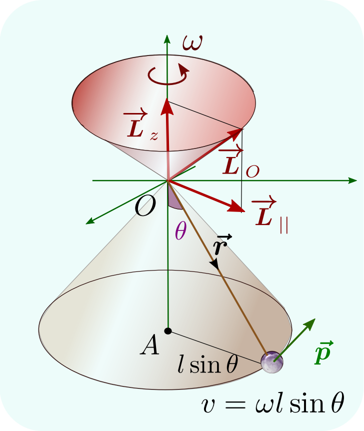
$\vec{L}_O$ precesses about z axis



Torque

$$\frac{d\overrightarrow{\boldsymbol{L}}}{dt} =$$

Torque

$$\frac{d\overrightarrow{\boldsymbol{L}}}{dt} = \frac{d}{dt}\left(\vec{\boldsymbol{r}} \times \vec{\boldsymbol{p}}\right)$$

Torque

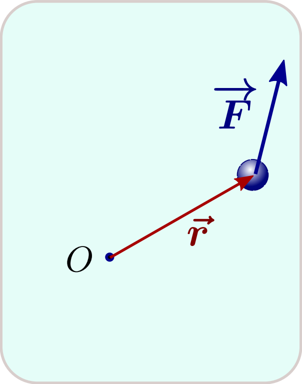
$$\begin{aligned}
\frac{d\overrightarrow{\boldsymbol{L}}}{dt} &= \frac{d}{dt}\left(\vec{\boldsymbol{r}} \times \vec{\boldsymbol{p}}\right) \\
&= \vec{\boldsymbol{v}} \times \vec{\boldsymbol{p}} + \vec{\boldsymbol{r}} \times \frac{d\vec{\boldsymbol{p}}}{dt}
\end{aligned}$$

Torque

$$
\begin{aligned}
\frac{d\overrightarrow{\boldsymbol{L}}}{dt} &= \frac{d}{dt}\left(\vec{\boldsymbol{r}} \times \vec{\boldsymbol{p}}\right) \\
&= \overset{=0}{\cancel{\vec{\boldsymbol{v}} \times \vec{\boldsymbol{p}}}} + \vec{\boldsymbol{r}} \times \frac{d\vec{\boldsymbol{p}}}{dt}
\end{aligned}
$$

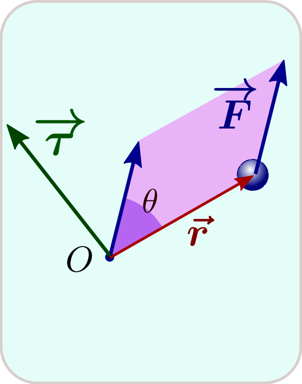
Torque

$$
\begin{aligned}
\frac{d\vec{\boldsymbol{L}}}{dt} &= \frac{d}{dt}(\vec{\boldsymbol{r}} \times \vec{\boldsymbol{p}}) \\
&= \underbrace{\vec{\boldsymbol{v}} \times \vec{\boldsymbol{p}}}_{=0} + \vec{\boldsymbol{r}} \times \frac{d\vec{\boldsymbol{p}}}{dt} \\
&= \vec{\boldsymbol{r}} \times \vec{\boldsymbol{F}}
\end{aligned}
$$



Torque

$$
\begin{aligned}
\frac{d\,\vec{L}}{dt} &= \frac{d}{dt}\left(\vec{r}\times\vec{p}\right) \\
&= \overset{=\,0}{\cancel{\vec{v}\times\vec{p}}} + \vec{r}\times\frac{d\vec{p}}{dt} \\
&= \vec{r}\times\vec{F} \\
&\equiv \vec{\tau}
\end{aligned}
$$



Torque

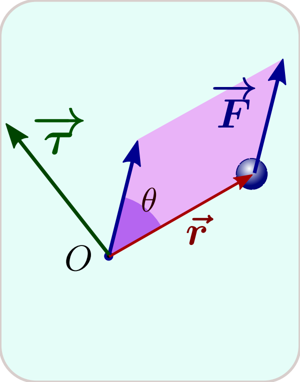
$$\frac{d\vec{L}}{dt} = \frac{d}{dt}(\vec{r} \times \vec{p})$$

$$= \underbrace{\cancel{\vec{v} \times \vec{p}}}_{=0} + \vec{r} \times \frac{d\vec{p}}{dt}$$

$$= \vec{r} \times \vec{F}$$

$$\equiv \vec{\tau}$$

$$\boxed{\frac{d\vec{L}}{dt} = \vec{\tau} = \vec{r} \times \vec{F}}$$



$$
\begin{aligned}
\frac{d\vec{L}}{dt} &= \frac{d}{dt}(\vec{r} \times \vec{p}) \\
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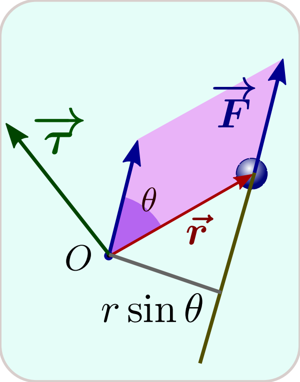
&= \vec{r} \times \vec{F} \\
&\equiv \vec{\tau}
\end{aligned}
$$

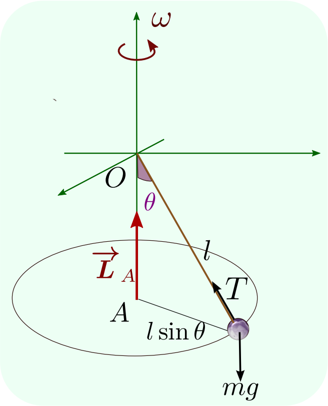


$$
\boxed{\frac{d\vec{L}}{dt} = \vec{\tau} = \vec{r} \times \vec{F}}
$$

$$
\begin{aligned}
\tau = &\ |\vec{\tau}| = rF\sin\theta \\
= &\ r_\perp F = rF_\perp
\end{aligned}
$$

Example: Conical Pendulum

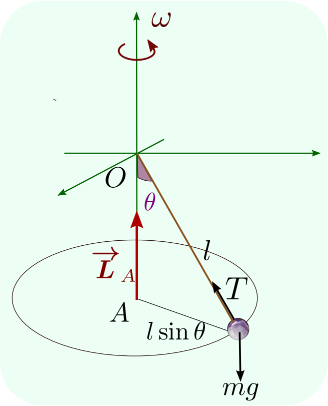
- Torque about A:



Example: Conical Pendulum

- Torque about A:

$$\vec{\tau}_A = 0$$

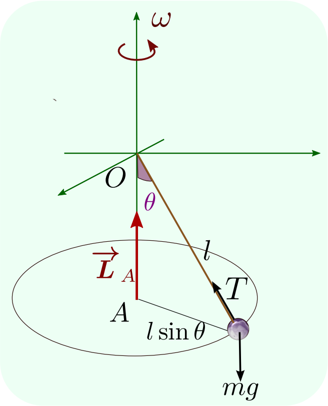


Example: Conical Pendulum

- Torque about A:

$$\vec{\tau}_A = 0 \text{ (Why??)}$$

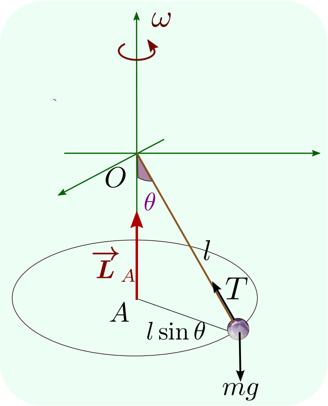
$$\Longrightarrow \frac{d\vec{L}_A}{dt} = 0$$



Example: Conical Pendulum

- Torque about A:

$$\begin{aligned} \vec{\tau}_A &= 0 \text{ (Why??)} \\ \Longrightarrow \frac{d\vec{\boldsymbol{L}}_A}{dt} &= 0 \\ \Longrightarrow \vec{\boldsymbol{L}}_A &= \text{constant} \end{aligned}$$



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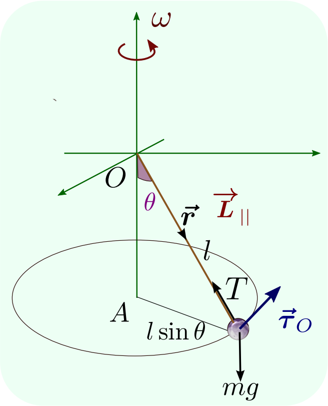
- Torque about O

$$\vec{\tau}_O = mgl\sin\theta \perp \vec{L}_O$$



Example: Conical Pendulum

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$$\vec{\tau}_A = 0 \text{ (Why??)}$$

$$\implies \frac{d\vec{\boldsymbol{L}}_A}{dt} = 0$$

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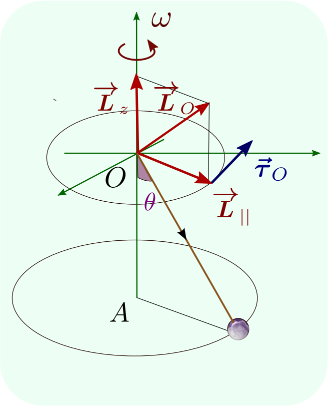
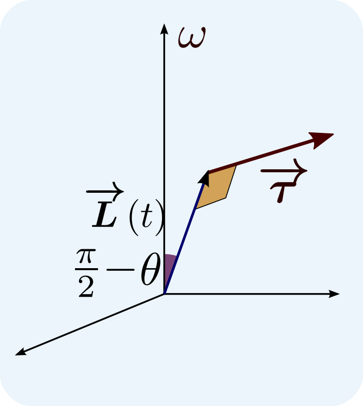


Conical Pendulum: Precession of Angular momentum

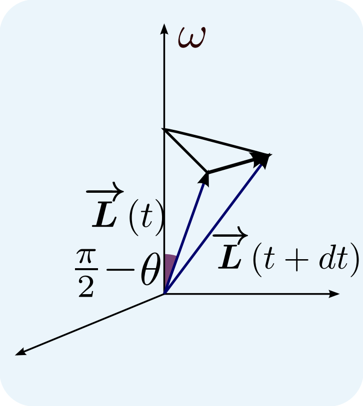
$$\frac{d\vec{\boldsymbol{L}}}{dt} = \vec{\boldsymbol{\tau}} \perp \vec{\boldsymbol{L}}$$



Conical Pendulum: Precession of Angular momentum

$$\frac{d\overrightarrow{\boldsymbol{L}}}{dt} = \vec{\boldsymbol{\tau}} \perp \overrightarrow{\boldsymbol{L}}$$

$$\Delta\overrightarrow{\boldsymbol{L}} \perp \overrightarrow{\boldsymbol{L}}$$

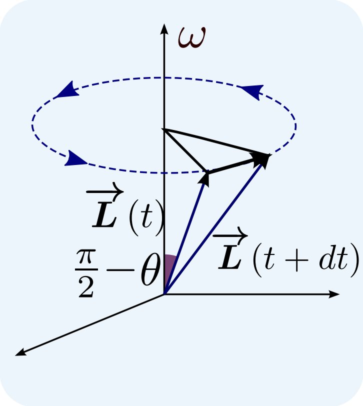


Conical Pendulum: Precession of Angular momentum

$$\frac{d\vec{\boldsymbol{L}}}{dt} = \vec{\boldsymbol{\tau}} \perp \vec{\boldsymbol{L}}$$

$$\Delta\vec{\boldsymbol{L}} \perp \vec{\boldsymbol{L}}$$

$\vec{\boldsymbol{L}}$ has constant *magnitude* but precesses about the $z-$axis.



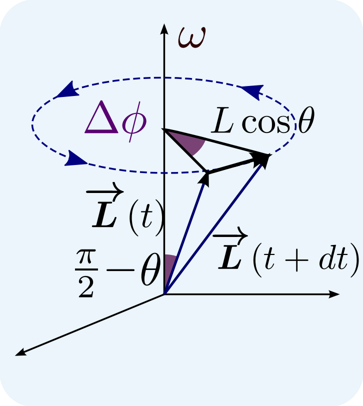
Conical Pendulum: Precession of Angular momentum

$$\frac{d\overrightarrow{\boldsymbol{L}}}{dt} = \vec{\boldsymbol{\tau}} \perp \overrightarrow{\boldsymbol{L}}$$

$$\Delta\overrightarrow{\boldsymbol{L}} \perp \overrightarrow{\boldsymbol{L}}$$

$\overrightarrow{\boldsymbol{L}}$ has constant *magnitude* but precesses about the $z-$axis.

$$\begin{aligned} \Delta L &= L\cos\theta\Delta\phi \\ &= \tau\Delta t \end{aligned}$$



Conical Pendulum: Precession of Angular momentum

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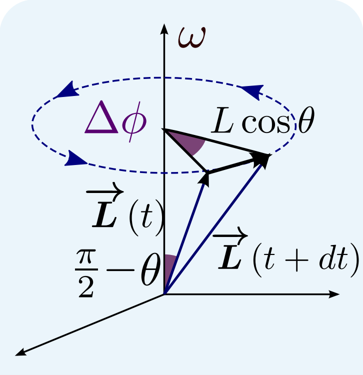


Precession frequency $\qquad \Omega_P = \frac{\Delta\phi}{\Delta t}$

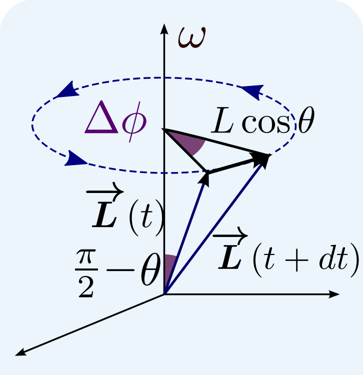
Conical Pendulum: Precession of Angular momentum

$$\frac{d\vec{L}}{dt} = \vec{\tau} \perp \vec{L}$$

$$\Delta\vec{L} \perp \vec{L}$$

$\vec{L}$ has constant *magnitude* but precesses about the $z-$axis.

$$\begin{aligned} \Delta L &= L\cos\theta\Delta\phi \\ &= \tau\Delta t \end{aligned}$$



Precession frequency $\Omega_P = \frac{\Delta\phi}{\Delta t} = \frac{\tau}{L\cos\theta} = \omega$