Vector Valued Functions and Motion in Space

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Recall: $r : \mathbb{R} \longrightarrow \mathbb{R}^3$
$t \longmapsto r(t) \in \mathbb{R}^3$

$= (x, y, z) = (f(t), g(t), h(t))$

$= x\hat{i} + y\hat{j} + z\hat{k}$

curve in space $= \{ r(t) \mid t \in \text{Domain of } r \}$ ✓

$= f(t)\hat{i} + g(t)\hat{j} + h(t)\hat{k}$

Given a vector valued function $\vec{r}$, one can assign a curve in the space

Exs $r(t) = \sin t\,\hat{i} + \cos t\,\hat{j} + t\hat{k} = (\sin t, \cos t, t)$, $t \in \mathbb{R}$

$x = \sin t$, $z = t$ $\Rightarrow$ $x^2 + y^2 = 1$

$y = \cos t$

$t = \sin^{-1} x$

$z = \sin^{-1} x$ $\Rightarrow$ $x = \sin z$

$r(t)$ is obtained by taking the intersection of $x^2 + y^2 = 1$ and $x = \sin z$

2) $r(t) = (t, t^2, t^3)$ $\Rightarrow$ $x = t$, $y = t^2$, $z = t^3$ $\Rightarrow$ Intersection $\begin{cases} y = x^2 \\ z = x^3 \end{cases}$ ✓

Given a curve in the space, how do you find a function?

Ex Intersection of the cylinder $x^2+y^2=1$ and the plane $z = 1/2$

$\longrightarrow$ $r(t) = \sin t\,\hat{i} + \cos t\,\hat{j} + 1/2\,\hat{k} = (\sin t, \cos t, 1/2)$
$t \in \mathbb{R}$

Remark: A curve in space can have more than one parametrization.

Ex: (t, t^2, t^3) and (t^2, t^4, t^6) gives the same curve in the space.

Given a vector valued function $\vec{r}: \mathbb{R} \to \mathbb{R}^3$

$r(t) = (f(t), g(t), h(t)) \quad t \in \mathbb{R}$

Limit $\quad \lim_{t\to t_0} r(t) = \left(\lim_{t\to t_0} f(t), \lim_{t\to t_0} g(t), \lim_{t\to t_0} h(t)\right)$

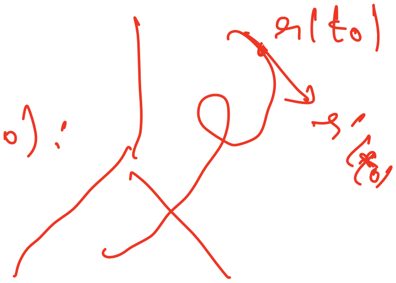
Continuity: If $f(t), g(t), h(t)$ are cont. at $t=t_0$, then $r(t)$ is also continuous at $t=t_0$

Differentiability: If $f(t), g(t), h(t)$ are differentiable at $t=t_0$, then $r(t)$ is also differentiable. and

$$r'(t)\Big|_{t=t_0} = \frac{df}{dt}\Big|_{t=t_0}\hat{i} + \frac{dg}{dt}\Big|_{t=t_0}\hat{j} + \frac{dh}{dt}\Big|_{t=t_0}\hat{k}$$

$$= (x'(t_0), y'(t_0), z'(t_0))$$

Remarks $\quad r'(t_0)$



eq. of the tangent line at $r(t_0)$:

$$\boxed{r(t) = r'(t_0)t + r(t_0)}$$

Examples

Find equation of the tangent line to the following curves at the value $t = t_0$.

1. $\mathbf{r}(t) = (\cos t)\,\mathbf{i} + (\sin t)\,\mathbf{j} + (\sin 2t)\,\mathbf{k}$, at $t_0 = \pi/2$
2. $\mathbf{r}(t) = t^2\,\mathbf{i} + (2t-1)\,\mathbf{j} + t^3\,\mathbf{k}$, at $t_0 = 2$

1)

$r(t_0) = r(\pi/2) = (0, 1, 0)$ $\qquad r'(t) = (-\sin t, \cos t, 2\cos 2t)$

$r'(t_0) = r'(\pi/2) = (-1, 0, -2)$

$v(t) = (-1, 0, -2)\,t + (0, 1, 0)$

$= (-t, 1, -2t)$

Smooth Curves

- A Curve $\mathbf{r}(t) = (x(t), y(t), z(t))$ is said to be smooth, if $\mathbf{r}'(t)$ is continuous and $\mathbf{r}'(t) \neq \mathbf{0}$ for any t in its domain, that is $x(t)$, $y(t)$ and $z(t)$ have continuous first derivatives that are not simultaneously 0.

- We require $d\mathbf{r}/dt \neq \mathbf{0}$ for a smooth curve to make sure that the curve has a continuously turning tangent at each point. On smooth curve there are no sharp carners.

Smooth Curves: Examples

- Is the curve of intersection of $x^2 + y^2 = 1$ and $x + z = 1$ smooth?

'Yes'

$x = \sin t$
$y = \cos t.$ $\quad z = 1 - x = 1 - \sin t$

$r(t) = (\sin t, \cos t, 1 - \sin t) \quad t \in \mathbb{R}$

$r'(t) = (\cos t, -\sin t, -\cos t)$ - continuous $\forall\, t \in \mathbb{R}$

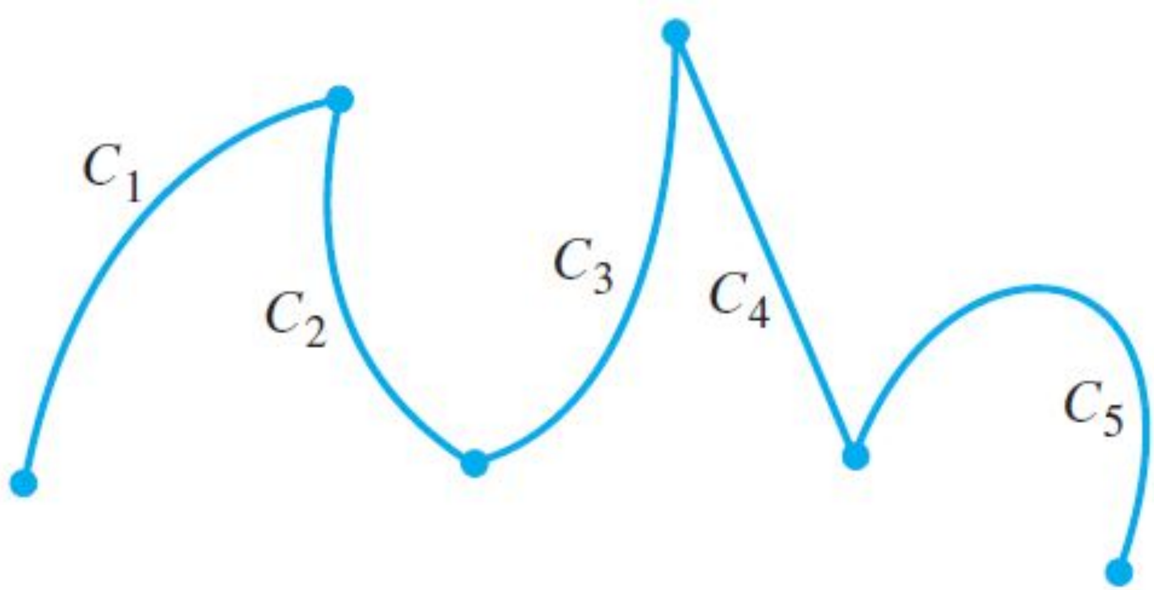
$r'(t) = 0 \iff \cos t = 0 \;\&\; \sin t = 0$

$\iff \underbrace{t = (2n+1)\pi/2 \text{ and } t = n'\pi}_{\text{for any } n, n' \text{ it is true simultaneously}}$

$\therefore r'(t) \neq 0$

Smooth Curves: Examples

- Is the curve of intersection of $x^2 + y^2 = 1$ and $x + z = 1$ smooth?
- $\mathbf{r}(t) = \cos t\ \mathbf{i} + \sin t\ \mathbf{j} + t\ \mathbf{k}$ gives a smooth curve.
- Piecewise smooth curve: A curve that is made up of a finite number of smooth curves pieced (joined) together in a continuous fashion.



Velocity, Speed, Acceleration and Unit Tangent

Definition 0.1.

If $\mathbf{r}(t)$ is the position vector of a particle moving along a smooth curve in space then:

1. The **velocity** of the particle is defined by

$$\mathbf{v}(t) = \frac{d\mathbf{r}}{dt} = r'(t)$$

   which is tangent to the curve.

2. At any time t, the direction of $\mathbf{v}$ is the **direction of motion**.
3. The magnitude of $\mathbf{v}$ is the particle's **speed**.
4. And the derivative $\mathbf{a} = d\mathbf{v}/dt$, when it exists, is the particle's **acceleration vector**.

Velocity, Speed, Acceleration and Unit Tangent

Summary,

1. Velocity is the derivative of position: $\mathbf{v} = \mathbf{r}'(t)$.
2. Speed is the magnitude of velocity: Speed $=|\mathbf{v}(t)|$.
3. Acceleration is the derivative of velocity: $\mathbf{a} = \mathbf{v}'(t) = \mathbf{r}''(t)$.
4. The direction of motion at any time t is the unit vector $\dfrac{\mathbf{v}}{|\mathbf{v}|}$ (Unit tangent).
5. Velocity$=|\mathbf{v}|\left(\dfrac{\mathbf{v}}{|\mathbf{v}|}\right)=$(speed)(direction).

unit tangent vector

Example

Find the velocity, speed, and acceleration of a particle whose motion in space is given by the vector $\mathbf{r}(t) = (2\cos t)\,\mathbf{i} + (2\sin t)\,\mathbf{j} + (5\cos^2 t)\,\mathbf{k}$. Sketch the velocity vector $\mathbf{v}(7\pi/4)$.

$$v(t) = r'(t) = -2\sin t\,\hat{i} + 2\cos t\,\hat{j} + (-10\cos t\sin t)\,\hat{k}$$

$$v(7\pi/4) = (\sqrt{2}, \sqrt{2}, 5) \quad , \quad |v(7\pi/4)| = \sqrt{2+2+25} = \sqrt{29}$$

$$|v| = \sqrt{4\sin^2 t + 4\cos^2 t + 100\cos^2 t\sin^2 t}$$

$$= \sqrt{4 + 100\cos^2 t\sin^2 t}$$

$$\frac{v}{|v|} = \frac{(\sqrt{2}, \sqrt{2}, 5)}{\sqrt{29}}$$

Example

Find the velocity, speed, and acceleration of a particle whose motion in space is given by the vector $\mathbf{r}(t) = (2\cos t)\,\mathbf{i} + (2\sin t)\,\mathbf{j} + (5\cos^2 t)\,\mathbf{k}$. Sketch the velocity vector $\mathbf{v}(7\pi/4)$.

Solution:

$$\mathbf{v} = \mathbf{r}'(t) = (-2\sin t)\,\mathbf{i} + (2\cos t)\,\mathbf{j} - (5\sin 2t)\,\mathbf{k}$$
$$\mathbf{a} = \mathbf{a}''(t) = (-2\cos t)\,\mathbf{i} - (2\sin t)\,\mathbf{j} - (10\cos 2t)\,\mathbf{k},$$

and the speed is given by

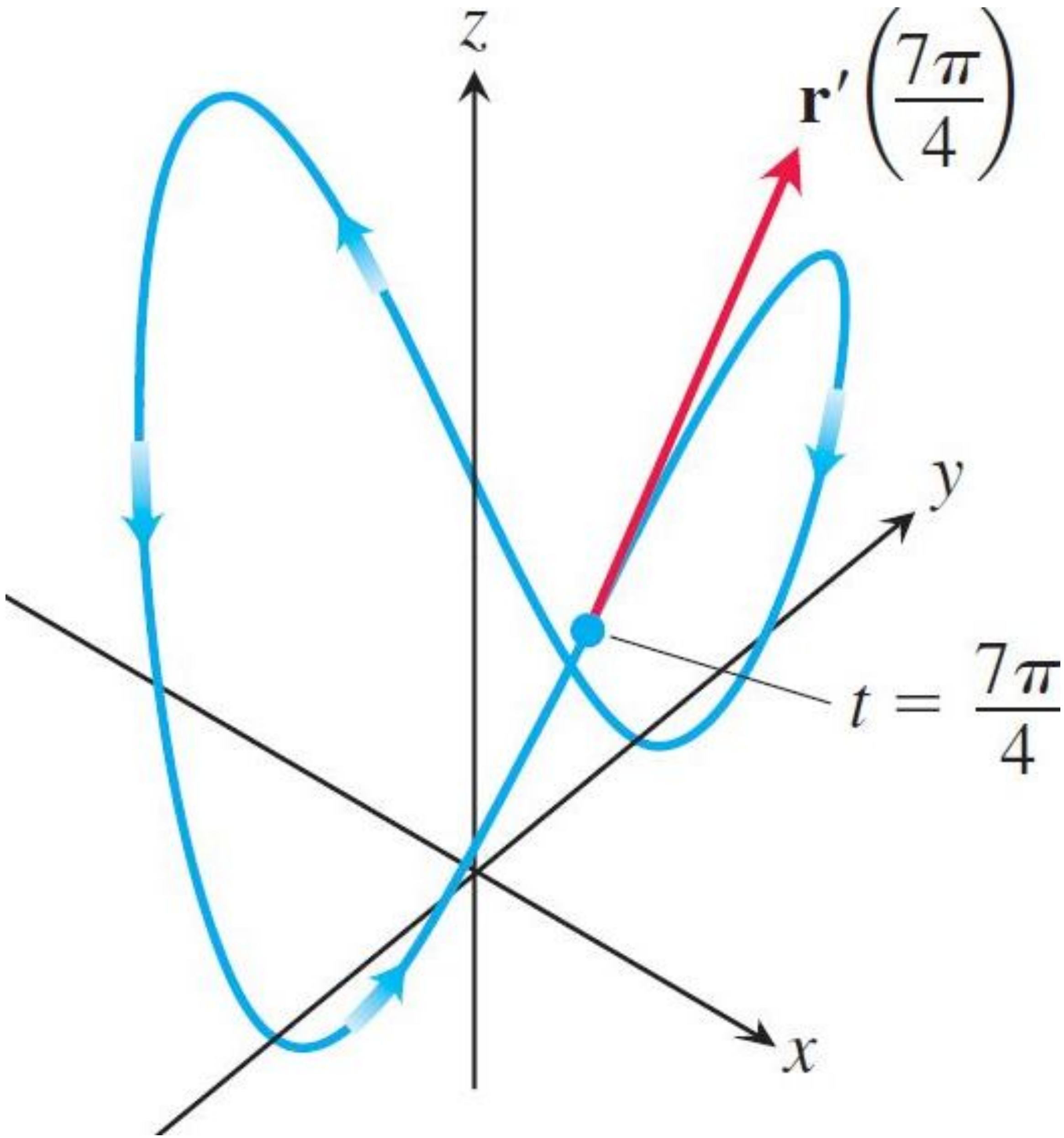
$$\begin{aligned}|\mathbf{v}(t)| &= \sqrt{(-2\sin t)^2 + (2\cos t)^2 + (-5\sin 2t)^2} \\ &= \sqrt{4 + 25\sin^2 2t}.\end{aligned}$$

When $t = 7\pi/4$, we have

$$\mathbf{v}\left(\frac{7\pi}{4}\right) = \sqrt{2}\,\mathbf{i} + \sqrt{2}\,\mathbf{j} + 5\mathbf{k}; \quad \left|\mathbf{v}\left(\frac{7\pi}{4}\right)\right| = \sqrt{29}.$$

$$\mathbf{a}\left(\frac{7\pi}{4}\right) = -\sqrt{2}\,\mathbf{i} + \sqrt{2}\,\mathbf{j}.$$

Sketch of the path with velocity



Problems

In the following, $\mathbf{r}(t)$ is the position of a particle in space at time t. Find velocity, speed, acceleration and the direction of the motion.

1. $\mathbf{r}(t) = (2\cos t)\,\mathbf{i} + (3\sin t)\,\mathbf{j} + 4t\,\mathbf{k}$, at $t = \pi/2$
2. $\mathbf{r}(t) = (e^{-t})\,\mathbf{i} + (2\cos 3t)\,\mathbf{j} + (2\sin 3t)\,\mathbf{k}$, at $t = 0$
3. $\mathbf{r}(t) = (2\ln(t+1))\,\mathbf{i} + t^2\,\mathbf{j} + \frac{t^2}{2}\,\mathbf{k}$, at $t = 1$

Differentiation Rules for Vector Functions

Let $\mathbf{u}$ and $\mathbf{v}$ be differentiable vector functions of t, $\mathbf{C}$ a constant vector, c any scalar, and f any differentiable scalar function.

1. *Constant Function Rule:* $\dfrac{d}{dt}\mathbf{C} = \mathbf{0}$
2. *Scalar Multiple Rules:* $\dfrac{d}{dt}[c\mathbf{u}(t)] = c\mathbf{u}'(t)$

   $\dfrac{d}{dt}[f(t)\mathbf{u}(t)] = f'(t)\mathbf{u}(t) + f(t)\mathbf{u}'(t)$
3. *Sum Rule:* $\dfrac{d}{dt}[\mathbf{u}(t) + \mathbf{v}(t)] = \mathbf{u}'(t) + \mathbf{v}'(t)$
4. *Difference Rule:* $\dfrac{d}{dt}[\mathbf{u}(t) - \mathbf{v}(t)] = \mathbf{u}'(t) - \mathbf{v}'(t)$
5. *Dot Product Rule:* $\dfrac{d}{dt}[\mathbf{u}(t) \cdot \mathbf{v}(t)] = \mathbf{u}'(t) \cdot \mathbf{v}(t) + \mathbf{u}(t) \cdot \mathbf{v}'(t)$
6. *Cross Product Rule:* $\dfrac{d}{dt}[\mathbf{u}(t) \times \mathbf{v}(t)] = \mathbf{u}'(t) \times \mathbf{v}(t) + \mathbf{u}(t) \times \mathbf{v}'(t)$
7. *Chain Rule:* $\dfrac{d}{dt}[\mathbf{u}(f(t))] = f'(t)\mathbf{u}'(f(t))$

Vector Function of Constant Length:

If $\mathbf{r}(t)$ is a differentiable vector function of t of $\underbrace{\text{constant length}}_{c}$, then prove that

$$\mathbf{r} \cdot \frac{d\mathbf{r}}{dt} = 0.$$

$\Rightarrow$ position vector is $\perp$ to the velocity vector.

Pf

$|r(t)| = c$

$\Rightarrow |r(t)|^2 = c^2$

$\Rightarrow r(t) \cdot r(t) = c^2$

Differentiate w.r.to t,

$\boxed{\text{Using rule (5)}}$

$r'(t) \cdot r(t) + r(t) \cdot r'(t) = 0$

$\Rightarrow 2\, r(t) \cdot r'(t) = 0$

$\Rightarrow r \cdot v = 0$

Definition 0.2.

The indefinite integral of $\mathbf{r}$ with respect to t is the set of all antiderivates of $\mathbf{r}$, denoted by $\int \mathbf{r}(t)dt$. If $\mathbf{R}$ is any antiderivative of $\mathbf{r}$ (i.e. $\mathbf{R}'(t) = \mathbf{r}(t)$), then

$$\int \mathbf{r}(t)\ dt = \mathbf{R}(t) + \mathbf{C}.$$

Note: Since $\mathbf{r}'(t) = x'(t)\mathbf{i} + y'(t)\mathbf{j} + z'(t)\mathbf{k}$, we have

$$\int \mathbf{r}(t)dt = \left(\int x(t)dt\right)\mathbf{i} + \left(\int y(t)dt\right)\mathbf{j} + \left(\int z(t)dt\right)\mathbf{k}$$

$$+ C_1\hat{i} + C_2\hat{j} + C_3\hat{k}$$

Example:

If $\mathbf{r}(t) = t^2\ \mathbf{i} + (2t-1)\ \mathbf{j} + t^3\ \mathbf{k}$,

$$\int r(t)\,dt = \int t^2\,dt\ \hat{i} + \int (2t-1)\,dt\ \hat{j} + \int t^3\,dt\ \hat{k}$$

$$= \frac{t^3}{3}\hat{i} + (t^2 - t)\hat{j} + \frac{t^4}{4}\hat{k}$$

$$+ C$$

Example:

If $\mathbf{r}(t) = t^2\,\mathbf{i} + (2t-1)\,\mathbf{j} + t^3\,\mathbf{k}$, then

$$\int \mathbf{r}(t)dt = \frac{t^3}{3}\,\mathbf{i} + (t^2 - t)\,\mathbf{j} + \frac{t^4}{4}\,\mathbf{k} + \mathbf{C}.$$

Definition 0.3.

If the components of $\mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j} + z(t)\,\mathbf{k}$ are integrable over $[a, b]$, the so is $\mathbf{r}$ and the **definite integral** of $\mathbf{r}$ from a to b is

$$\int_a^b \mathbf{r}(t)dt = \Big(\int_a^b x(t)dt\Big)\mathbf{i} + \Big(\int_a^b y(t)dt\Big)\mathbf{j} + \Big(\int_a^b z(t)dt\Big)\mathbf{k}$$

Example:

If $\mathbf{r}(t) = t^2\,\mathbf{i} + (2t-1)\,\mathbf{j} + t^3\,\mathbf{k}$, then

$$\begin{aligned}
\int_0^1 \mathbf{r}(t)dt &= \left(\int_0^1 t^2 dt\right)\mathbf{i} + \left(\int_0^1 (2t-1)dt\right)\mathbf{j} + \left(\int_0^1 t^3 dt\right)\mathbf{k} \\
&= \left[\frac{t^3}{3}\right]_0^1 \mathbf{i} + \left[(t^2-t)\right]_0^1 \mathbf{j} + \left[\frac{t^4}{4}\right]_0^1 \mathbf{k} \\
&= \frac{1}{3}\,\mathbf{i} + \frac{1}{4}\,\mathbf{k}.
\end{aligned}$$

Theorem 0.4 (The Fundamental Theorem of Calculus).

Let $\mathbf{r}(t)$ *be a continuous vector function on* $[a, b]$ *and* $\mathbf{R}(t)$ *be any antiderivative of* $\mathbf{r}$*, then*

$$\int_a^b \mathbf{r}(t)dt = \mathbf{R}(t)\Big|_a^b = \mathbf{R}(b) - \mathbf{R}(a).$$

Question

Suppose we do not know the path of a hang glider, but only its acceleration vector which is given by

$$\mathbf{a}(t) = -(3\cos t)\mathbf{i} - (3\sin t)\mathbf{j} + 2\mathbf{k}.$$

We also know that initially (at time $t = 0$) the glider departed from the point $(3, 0, 0)$ with velocity $\mathbf{v}(0) = 3\mathbf{j}$. Find the glider's position as a function of t.

$v'(t) = a(t) \Rightarrow \int a(t) = v(t) + C_1$

$\int(-3\cos t\,\hat{i} - 3\sin t\,\hat{j} + 2\hat{k})\,dt = v(t) + C_1$

$-3\sin t\,\hat{i} + 3\cos t\,\hat{j} + 2t\,\hat{k} = v(t) + C_1$

at $t=0$, $v(0) = 3\hat{j} \Rightarrow C_1 + 3\hat{j} = 3\hat{j} \Rightarrow C_1 = 0$

$$\Rightarrow \quad v(t) = -3\sin t\,\hat{i} + 3\cos t\,\hat{j} + 2t\,\hat{k}$$

$$r'(t) = v(t) \quad \Rightarrow \quad \int v(t) = r(t) + C_2$$

$$\int (-3\sin t\,\hat{i} + 3\cos t\,\hat{j} + 2t\,\hat{k})\,dt = r(t) + C_2$$

$$3\cos t\,\hat{i} + 3\sin t\,\hat{j} + t^2\,\hat{k} = r(t) + C_2$$

$$t = 0, \quad r(0) = (3, 0, 0) = 3\hat{i}$$

$$\Rightarrow C_2 = 0$$

$$\Rightarrow r(t) = 3\cos t\,\hat{i} + 3\sin t\,\hat{j} + t^2\,\hat{k}$$

Question

Suppose we do not know the path of a hang glider, but only its acceleration vector which is given by

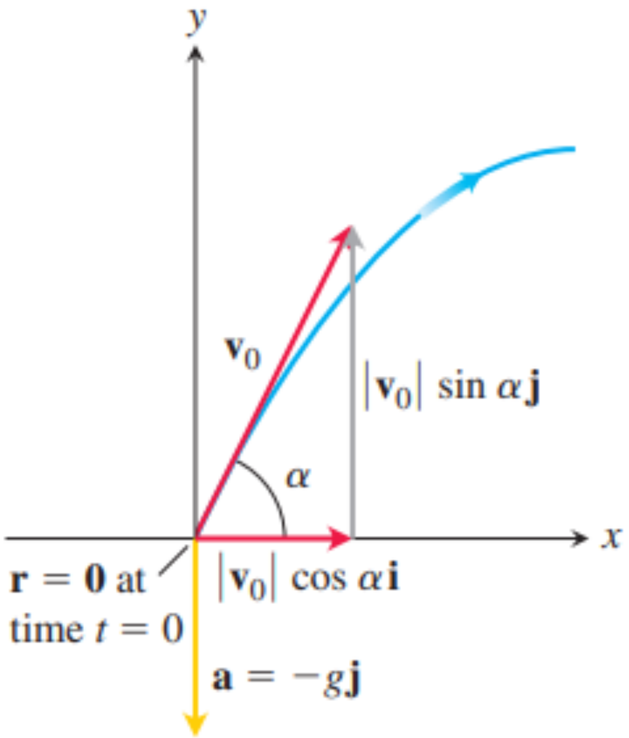
$$\mathbf{a}(t) = -(3\cos t)\mathbf{i} - (3\sin t)\mathbf{j} + 2\mathbf{k}.$$

We also know that initially (at time $t = 0$) the glider departed from the point $(3, 0, 0)$ with velocity $\mathbf{v}(0) = 3\mathbf{j}$. Find the glider's position as a function of t.

Answer: $\mathbf{r}(t) = 3\cos t\mathbf{i} + 3\sin t\mathbf{j} + t^2\mathbf{k}$.

Projectile motion

We assume that the projectile is launched from the origin at time $t = 0$ into the first quadrant with an initial velocity v_0.



If v_0 makes an angle α with the horizontal, then

$$v_0 = (|v_0| \cos\alpha)\vec{i} + (|v_0| \sin\alpha)\vec{j}$$

The projectile's initial position is $\vec{r} = 0\vec{i} + 0\vec{j} = \vec{0}$.

Newton's second law of motion says that the force acting on the projectile is equal to the projectile's mass m times its acceleration. If the force is solely the gravitational force, $-mg\hat{j}$, then

$$m\overrightarrow{a}(t) = -mg\hat{j}, \text{ where } \overrightarrow{a}(t) = \frac{d^2\vec{r}}{dt^2}.$$

$$\implies \frac{d^2\vec{r}}{dt^2} = -g\hat{j} \text{ with initial conditions } \vec{r} = \vec{r}_0$$

and $\frac{d\vec{r}}{dt} = \vec{v}_0$ when $t = 0$.

$$\Rightarrow \overrightarrow{v}(t) = -gt\hat{j} + c_1\hat{i} + c_2\hat{j}$$

Now by putting $t = 0$, we have

$$(|v_0|\cos\alpha)\hat{i} + (|v_0|\sin\alpha)\hat{j} = \overrightarrow{v}(0) = c_1\hat{i} + c_2\hat{j}$$

$$c_1 = |v_0|\cos\alpha, \; c_2 = |v_0|\sin\alpha.$$

Hence $\overrightarrow{v}(t) = |v_0| \cos\alpha \hat{i} + (|v_0| \sin\alpha - gt)\hat{j}$.
Now by integrating once more, we have

$$\overrightarrow{r}(t) = (|v_0| \cos\alpha) t \hat{i} + \left(|v_0| \sin\alpha t - g\frac{t^2}{2}\right)\hat{j} + d_1\hat{i} + d_2\hat{j}$$

Now $0\hat{i} + 0\hat{j} = \overrightarrow{r}(0) = d_1\hat{i} + d_2\hat{j}$ implies $d_1 = d_2 = 0$. Thus we have
Ideal Projectile Motion Equation

$$\boxed{\overrightarrow{r}(t) = (|v_0| \cos\alpha) t \hat{i} + \left(|v_0| \sin\alpha t - \tfrac{1}{2}gt^2\right)\hat{j}.}$$

The components of $\vec{r}$ gives the parametric equations

$$x = (|v_0| \cos\alpha)t, \ \ y = (|v_0| \sin\alpha)t - \frac{1}{2}gt^2.$$

Example 1.

A projectile is fired from the origin over horizontal ground at an initial speed of $500 m/sec$ *and a launch angle of* 60°*. Where will the projectile be* 10 *seconds later?*

Solution: We use $\alpha = 60^\circ, |v_0| = 500, g = 9.8$ and $t = 10$ in the above equation, we get

$$\overrightarrow{r}(t) = (500 \times \cos 60^\circ) \times 10\hat{i} + \left(500 \sin 60^\circ \times 10 - \frac{9.8 \times 100}{2}\right)\hat{j}$$

Remark 2.

The ideal projectiles move along parabolas.

Substitute $t = x/(|v_0|\cos\alpha)$ to get

$$y = -\left(\frac{g}{2|v_0|^2\cos^2\alpha}\right)x^2 + (\tan\alpha)x.$$

This has the form $y = ax^2 + bx$, so the graph is a parabola.

Height, Flight time and Range of Projectile motion

A projectile reaches highest point when its vertical component of velocity vector is 0.

$$\frac{dr}{dt} = |v_0|\cos\alpha\hat{i} + (|v_0|\sin\alpha - gt)\hat{j}.$$

Thus at maximum height, $t = \frac{|v_0|\sin\alpha}{g}$, which implies maximum height, $y_{\max} = \frac{(|v_0|\sin\alpha)^2}{2g}$.
Once the object reaches ground, the y co-ordinate of position vector equals 0 which gives flight time, $t = \frac{2|v_0|\sin\alpha}{g}$. At this time, the x co-ordinate is given by

$$x = \frac{|v_0|^2\sin 2\alpha}{g}$$

will give the range of projectile.

Properties

Height, Flight Time, and Range for Ideal Projectile Motion

For ideal projectile motion when an object is launched from the origin over a horizontal surface with initial speed v_0 and launch angle α:

$$\text{Maximum height}: \quad y_{\max} = \frac{(v_0 \sin \alpha)^2}{2g}$$

$$\text{Flight time}: \quad t = \frac{2v_0 \sin \alpha}{g}$$

$$\text{Range}: \quad R = \frac{v_0^2}{g} \sin 2\alpha.$$

Projectile with shifted origin

Let the projectile be fired from the point (x_0, y_0) instead of the origin. Then $m\overrightarrow{a}(t) = -mg\hat{j}$ with the initial condition $\overrightarrow{v}(0) = (v_0 \cos\alpha)\hat{i} + (v_0 \sin\alpha)\hat{j}$ and $\overrightarrow{r}(0) = x_0\hat{i} + y_0\hat{j}$.
Now

$$m\overrightarrow{a}(t) = -mg\hat{j}$$

$$\Rightarrow \overrightarrow{a}(t) = -g\hat{j}$$

$$\Rightarrow \overrightarrow{v}(t) = -gt\hat{j} + c_1\hat{i} + c_2\hat{j}$$

Now by putting $t = 0$, we have

$$(|v_0| \cos\alpha)\hat{i} + (|v_0| \sin\alpha)\hat{j} = \overrightarrow{v}(0) = c_1\hat{i} + c_2\hat{j}$$

$$c_1 = |v_0| \cos\alpha, c_2 = |v_0| \sin\alpha.$$

Hence $\overrightarrow{v}(t) = |v_0|\cos\alpha\hat{i} + (|v_0|\sin\alpha - gt)\hat{j}$.
Now by integrating once more, we have

$$\overrightarrow{r}(t) = (|v_0|\cos\alpha)t\hat{i} + \left(|v_0|\sin\alpha t - g\frac{t^2}{2}\right)\hat{j} + d_1\hat{i} + d_2\hat{j}$$

Now $x_0\hat{i} + y_0\hat{j} = \overrightarrow{r}(0) = d_1\hat{i} + d_2\hat{j}$ implies $d_1 = x_0$ and $d_2 = y_0$.
Thus we have

Projectile Motion Equation

$$\overrightarrow{r}(t) = [(|v_0|\cos\alpha)t + x_0]\hat{i} + \left[\left(|v_0|\sin\alpha t - \frac{gt^2}{2}\right) + y_0\right]\hat{j}.$$

Projectile motion with Wind gusts

A baseball is hit when it is 3 ft above the ground. It leaves the bat with initial speed of 152 ft/sec, making an angle of 20 degree with the horizontal. At the instant the ball is hit, an instantaneous gust of wind blows in the horizontal direction directly opposite the direction the ball is taking toward the outfield, adding a component of $-8.8\hat{i}$ (ft/sec) to the ball's initial velocity(8.8 ft/sec = 6 mph).

1. Find a vector equation (position vector) for the path of the baseball.
2. How high does the baseball go, and when does it reach maximum height?
3. Assuming that the ball is not caught, find its range and flight time.

Adding the effect of wind gust, the initial velocity of baseball

$$v_0 = |v_0|\cos\alpha\hat{i} + |v_0|\sin\alpha\hat{j} - 8.8\vec{i}$$
$$= (152\cos 20 - 8.8)\hat{i} + 152\sin 20\hat{j}.$$

The initial position is $r_0 = 3\hat{j}$. Integration of $\frac{d^2r}{dt^2} = -g\hat{j}$ gives

$$\frac{dr}{dt} = -(gt)\hat{j} + v_0 \text{ and}$$
$$r = -\frac{1}{2}gt^2\hat{j} + v_0t + r_0$$

Substituting the values of v_0 and r_0,

$$r = (152\cos 20 - 8.8)\hat{i} + (3 + (152\sin 20)t - 16t^2)\hat{j}.$$

(b). The baseball reaches its highest point when the vertical component of velocity is zero, or $\frac{dy}{dt} = 0$ that $\implies t = \frac{152 sin20}{32} \equiv 1.62$sec.
Substituting this time into the vertical component for $\vec{r}$ gives the maximum height

$$y_{max} = 3 + (152 \sin 20)(1.62) - 16(1.62)^2 \equiv 45.2 \text{ feet}$$

(c). To find when the baseball lands, we set the vertical component for r equal to 0 and solve for t:

$$3 + (152 \sin 20)t - 16t^2 = 0.$$

The values are at $t = 3.3$sec and $t = -0.06$ sec. Substituting the positive time into the horizontal component for $\vec{r}$, we find the range

$$R = (152 \cos 20 - 8.8)(3.3) \equiv 442 \text{ feet.}$$