Binormal and Torsion of a Curve

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Chapter 13.5

Binormal and TNB frame

Definition 0.1.
The **binormal vector** of a curve is define by

$$\mathbf{B} = \mathbf{T} \times \mathbf{N},$$

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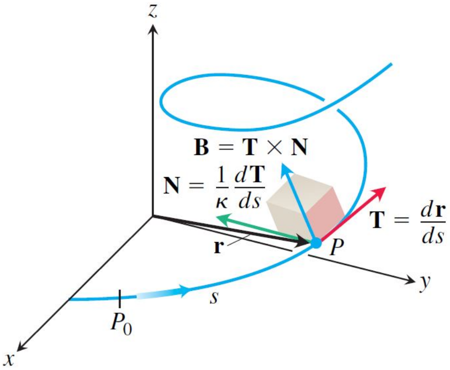
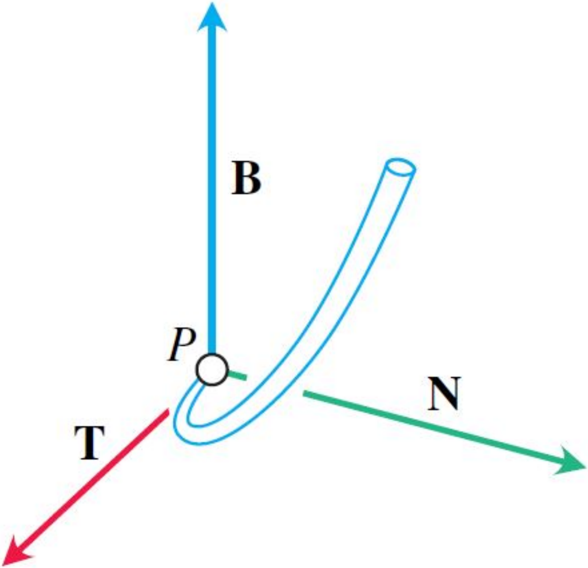
- Binormal vector $\mathbf{B}$ is a unit vector orthogonal to both $\mathbf{T}$ and $\mathbf{N}$.
- Together $\mathbf{T}$, $\mathbf{N}$ and $\mathbf{B}$ define a moving right-handed vector frame.



- It is called **Frenet** ("fre-*nay*") **frame** (after Jean-Frederic Frenet) or the **TNB frame**.

Binormal and TNB frame



Example

Find the binormal $\mathbf{B}$ to the helix given by

$\mathbf{r}(t) = (a\cos t)\mathbf{i} + (a\sin t)\mathbf{j} + (bt)\mathbf{k},\ \ a, b > 0,\ \ a^2 + b^2 \neq 0.$

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Solution.

We already found that

$$\mathbf{T} = \frac{1}{\sqrt{a^2+b^2}}[-(a\sin t)\mathbf{i} + (a\cos t)\mathbf{j} + b\mathbf{k}] \quad \text{and}$$

$$\mathbf{N} = -(\cos t)\mathbf{i} - (\sin t)\mathbf{j}.$$

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Therefore,

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Tangent and Normal Components of Acceleration

In practical situations, it is important to know that how much of acceleration acts in the direction of motion, in tangential direction $\mathbf{T}$.

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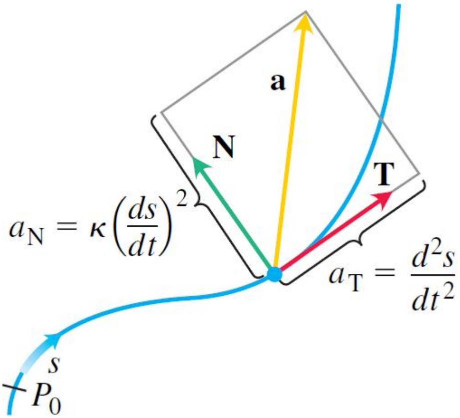

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- Then we differentiate both ends of the above equation to get

$$
\begin{aligned}
\mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{d}{dt}\left(\mathbf{T}\frac{ds}{dt}\right) &= \frac{d^2s}{dt^2}\mathbf{T} + \frac{ds}{dt}\frac{d\mathbf{T}}{dt} \\
&= \frac{d^2s}{dt^2}\mathbf{T} + \frac{ds}{dt}\left(\frac{d\mathbf{T}}{ds}\frac{ds}{dt}\right) \\
&= \frac{d^2s}{dt^2}\mathbf{T} + \frac{ds}{dt}\left(\kappa\mathbf{N}\frac{ds}{dt}\right), \quad \text{since } \mathbf{N} = \frac{1}{\kappa}\frac{d\mathbf{T}}{ds} \\
&= \left(\frac{d^2s}{dt^2}\right)\mathbf{T} + \kappa\left(\frac{ds}{dt}\right)^2\mathbf{N}.
\end{aligned}
$$

Remark 0.2.

If the acceleration vector is written as

$$\mathbf{a} = a_{\mathbf{T}}\mathbf{T} + a_{\mathbf{N}}\mathbf{N},$$

then

$$a_{\mathbf{T}} = \frac{d^2s}{dt^2} = \frac{d}{dt}|\mathbf{v}| \quad \text{and} \quad a_{\mathbf{N}} = \kappa\left(\frac{ds}{dt}\right)^2 = \kappa|\mathbf{v}|^2$$

are tangential and normal scalar components of acceleration.

Examples

Without finding $\mathbf{T}$ and $\mathbf{N}$, write the acceleration of the motion $\mathbf{r}(t) = (\cos t + t\sin t)\mathbf{i} + (\sin t - t\cos t)\mathbf{j},\ \ t > 0$ in the form $\mathbf{a} = a_{\mathbf{T}}\mathbf{T} + a_{\mathbf{N}}\mathbf{N}$.

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Since $\mathbf{T} \cdot \mathbf{N} = \mathbf{0}$, $|\mathbf{a}|^2 = |a_{\mathbf{T}}|^2 + |a_{\mathbf{N}}|^2$ and hence

$$a_{\mathbf{N}} = \sqrt{|\mathbf{a}|^2 - a_{\mathbf{T}}^2}.$$

Examples

Now we need to find $\mathbf{a}$.

$$\mathbf{a} = (\cos t - t\sin t)\mathbf{i} + (\sin t + t\cos t)\mathbf{j} \Rightarrow |\mathbf{a}|^2 = t^2 + 1.$$

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The required expression is

$$\mathbf{a} = a_{\mathbf{T}}\mathbf{T} + a_{\mathbf{N}}\mathbf{N} = \mathbf{T} + t\mathbf{N}.$$

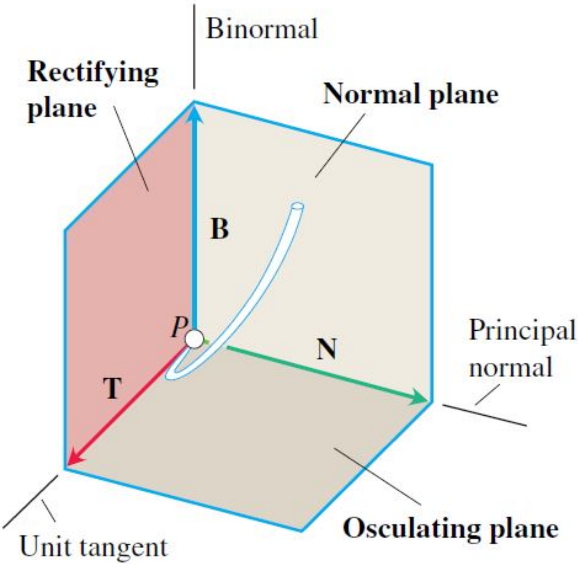
Remark 0.3.

Formula for calculation the normal component of acceleration:

$$a_{\mathbf{N}} = \sqrt{|\mathbf{a}|^2 - a_{\mathbf{T}}^2},$$

where $a_{\mathbf{T}} = \frac{d^2 s}{dt^2} = \frac{d}{dt}|\mathbf{v}|$, tangential component of acceleration.

Frame and planes determined by **T**, **N** and **B**



Frame and planes determined by $\mathbf{T}, \mathbf{N}$ and $\mathbf{B}$

Osculating Plane: The plane containg unit tangent $\mathbf{T}$ and principal normal $\mathbf{N}$.

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Osculating Plane: The plane containing unit tangent $\mathbf{T}$ and principal normal $\mathbf{N}$.

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Normal Plane: The plane containing principal normal $\mathbf{N}$ and binormal $\mathbf{B}$.

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To give mathematical expression for torsion, we find the relation of $\frac{d\mathbf{B}}{ds}$ with $\mathbf{T}$ and $\mathbf{N}$:

$$\begin{aligned}\frac{d\mathbf{B}}{ds} = \frac{d(\mathbf{T} \times \mathbf{N})}{ds} &= \left(\frac{d\mathbf{T}}{ds} \times \mathbf{N}\right) + \left(\mathbf{T} \times \frac{d\mathbf{N}}{ds}\right) \\ &= \mathbf{0} + \mathbf{T} \times \frac{d\mathbf{N}}{ds} \\ &= \mathbf{T} \times \frac{d\mathbf{N}}{ds}.\end{aligned}$$

Torsion

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- We can write $\dfrac{d\mathbf{B}}{ds} = -\tau\mathbf{N}$ for a scalar function τ, called **torsion**.

Definition 0.4 (Torsion τ).

If $\mathbf{r}(t)$ is a smooth curve, then the **torsion** function is defined by

$$\tau = -\frac{d\mathbf{B}}{ds} \cdot \mathbf{N} = -\frac{1}{|\mathbf{v}|}\left(\frac{d\mathbf{B}}{dt} \cdot \mathbf{N}\right).$$

Example

Find $\mathbf{B}$ and τ for the helix

$$\mathbf{r}(t) = (a\cos t)\mathbf{i} + (a\sin t)\mathbf{j} + bt\mathbf{k} \quad a, b > 0.$$

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Solution.

We already found that

$$\mathbf{T} = \frac{1}{\sqrt{a^2+b^2}}[-(a\sin t)\mathbf{i} + (a\cos t)\mathbf{j} + b\mathbf{k}]$$

$$\mathbf{N} = -(\cos t)\mathbf{i} - (\sin t)\mathbf{j}, \quad \text{and}$$

$$\mathbf{B} = \mathbf{T} \times \mathbf{N} = \frac{1}{\sqrt{a^2+b^2}}[(b\sin t)\mathbf{i} - (b\cos t)\mathbf{j} + a\mathbf{k}].$$

Example

$$\frac{d\mathbf{B}}{dt} = \mathbf{T} \times \mathbf{N} = \frac{1}{\sqrt{a^2 + b^2}}[(b\cos t)\mathbf{i} + (b\sin t)\mathbf{j}.$$

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and $|\mathbf{v}| = \sqrt{a^2+b^2}$. Therefore,

$$\tau = -\frac{1}{|\mathbf{v}|}\left(\frac{d\mathbf{B}}{dt}\cdot\mathbf{N}\right) = \frac{b}{a^2+b^2}.$$

Formulas for κ and τ without finding $\mathbf{T}$ and $\mathbf{B}$

Remark 0.5 (Vector formula for Curvature).

$$\kappa = \frac{|\mathbf{v} \times \mathbf{a}|}{|\mathbf{v}|^3};$$

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Proof. We know that $\mathbf{v} = \dfrac{ds}{dt}\mathbf{T}$ and

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therefore we have that

$$\mathbf{v} \times \mathbf{a} = \left(\frac{ds}{dt}\mathbf{T}\right) \times \left[\left(\frac{d^2s}{dt^2}\right)\mathbf{T} + \kappa\left(\frac{ds}{dt}\right)^2\mathbf{N}\right]$$
$$= \kappa\left(\frac{ds}{dt}\right)^3\mathbf{B}, \quad \text{since } \mathbf{T} \times \mathbf{T} = \mathbf{0}, \mathbf{T} \times \mathbf{N} = \mathbf{B}.$$

Formulas for κ and τ without finding $\mathbf{T}$ and $\mathbf{B}$

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Hence, we get the formula,

$$\kappa = \frac{|\mathbf{v} \times \mathbf{a}|}{|\mathbf{v}|^3}.$$

Remark 0.6 (Formula for Torsion).

$$\tau = \frac{(\mathbf{v} \times \mathbf{a}) \cdot \mathbf{a}'(t)}{|\mathbf{v} \times \mathbf{a}|^2}; \quad (\text{if } \mathbf{v} \times \mathbf{a} \neq \mathbf{0}).$$

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5. Torsion:
$$\tau = -\frac{d\mathbf{B}}{ds} \cdot \mathbf{N} = -\frac{1}{|\mathbf{v}|}\frac{d\mathbf{B}}{dt} \cdot \mathbf{N} = \frac{(\mathbf{v} \times \mathbf{a}) \cdot \mathbf{a}'(t)}{|\mathbf{v} \times \mathbf{a}|^2}$$

(6). Tangential and normal scalar components of accelerations:

$$\mathbf{a} = a_{\mathbf{T}}\mathbf{T} + a_{\mathbf{N}}\mathbf{N},$$

$$a_{\mathbf{T}} = \frac{d}{dt}|\mathbf{v}|,$$

$$a_{\mathbf{N}} = \kappa|\mathbf{v}|^2 = \sqrt{|\mathbf{a}|^2 - a_{\mathbf{T}}^2}.$$

Questions

1. Suppose that B and N are unit binormal and unit normal vectors. Then prove that $\frac{dN}{ds} = \tau B - \kappa T$.
2. Find the curvature and the torsion of the curve

$$\mathbf{r}(t) = (3t - t^3)\mathbf{i} + (3t^2)\mathbf{j} + (3t + t^3)\mathbf{k}.$$

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$$\mathbf{r}(t) = (3t - t^3)\mathbf{i} + (3t^2)\mathbf{j} + (3t + t^3)\mathbf{k}.$$

3. Find the the equations for the osculating, normal, and rectifying planes of the curve

$$\mathbf{r}(t) = (2\sin 3t)\mathbf{i} + (t)\mathbf{j} + 2\cos(3t)\mathbf{k}.$$

at $(0, \pi, -2)$