Partial Derivatives

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Mathematics 1 after Mid Semester Exam

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Consultation Hour in A EX 5: Wednesday 2:00 pm - 3:00 pm

For course materials, login : https://quantaaws.bits-goa.ac.in

Text Book:
Thomas' Calculus by George B. Thomas Jr., Joel Hass, Christopher Heil, Maurice D.Weir, Pearson education 12th edition, 2015.

Reference Books:

- R1 : Essential Calculus Early Transcendentals by J. Stewart, Thomson Learning, 2014.
- R2 : A First Course in Calculus by Serge Lang, Springer-Verlag 5th edition, 2009.
- R3 : Advanced Engineering Mathematics by Erwin Kreyszig Wiley 10th edition, 2015.
- R4 : Calculus Vol. 1 and Vol. 2, by T M Apostol, 2nd edition, 2007.

Recall

Definition (Limit of a function of two real variables)

We say that a function $f(x, y)$ approaches the limit L as (x, y) approaches (x_0, y_0) and write

$$\lim_{(x,y)\to(x_0,y_0)} f(x, y) = L$$

if, for every number $\epsilon > 0$, there exists a corresponding number $\delta > 0$ such that for all (x, y) in the domain of f,

$$|f(x, y) - L| < \epsilon \text{ whenever } 0 < \sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta.$$

Recall

Definition (Continuous function)

A function $f(x,y)$ is continuous at a point (x_0, y_0) if

- f is defined at (x_0, y_0),
- $\lim_{(x,y)\to(x_0,y_0)} f(x,y)$ exists,
- $\lim_{(x,y)\to(x_0,y_0)} f(x,y) = f(x_0, y_0)$.

A function is continuous if it is continuous at every point of its domain.

Partial Derivatives of a Function of Two Variables

Definition

The partial derivative of $f(x, y)$ with respect to x at a point (x_0, y_0) is

$$\left.\frac{\partial f}{\partial x}\right|_{(x_0,y_0)} := \lim_{h \to 0} \frac{f(x_0 + h, y_0) - f(x_0, y_0)}{h},$$

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Equivalent expression: $\left.\frac{\partial f}{\partial x}\right|_{(x_0,y_0)} = \left.\frac{d}{dx} f(x, y_0)\right|_{x=x_0} = f_x(x_0, y_0).$

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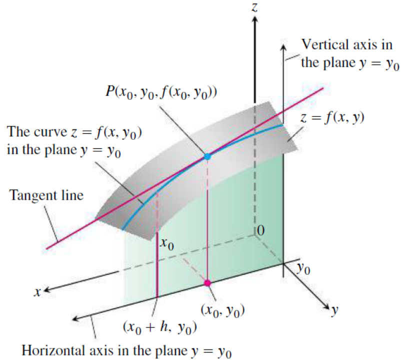
Similarly we have

$$\left.\frac{\partial f}{\partial y}\right|_{(x_0,y_0)} := \lim_{h\to 0} \frac{f(x_0, y_0+h) - f(x_0, y_0)}{h},$$

provided the limit exists.

The partial derivative $f_x(x_0, y_0)$ is the slope of the tangent line to the curve of intersection of surface $z = f(x, y)$ and plane $y = y_0$ at the point $P(x_0, y_0, f(x_0, y_0))$.



Continued

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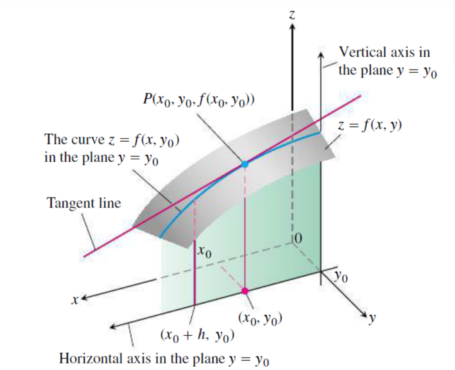

And the partial derivative $f_y(x_0, y_0)$ is the slope of the tangent line to the curve of the intersection of surface $z = f(x, y)$ and plane $x = x_0$ at the point $P(x_0, y_0, f(x_0, y_0))$.

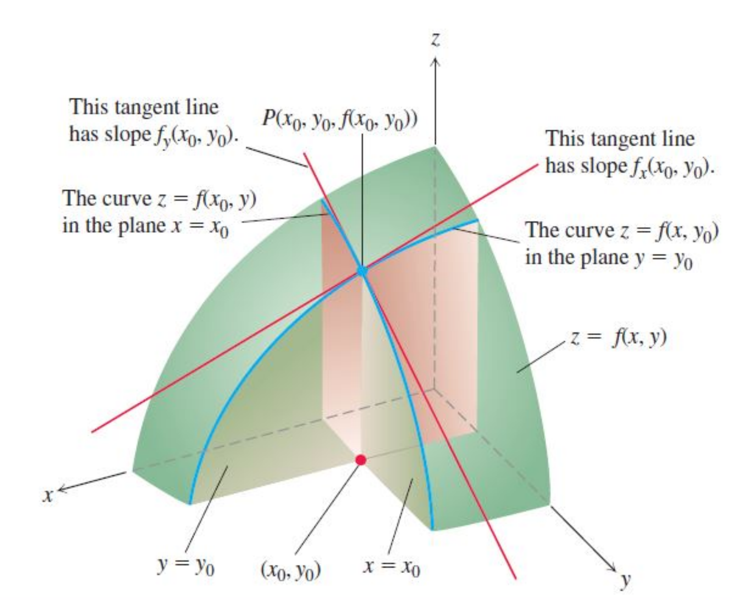

Figure: Partial derivative as slope

Functions of more than two variables

Definition

Let $f(x, y, z)$ be a function of three variable defined on a region $D \subset \mathbb{R}^3$. The partial derivative of $f(x, y, z)$ with respect to x at the point $(x_0, y_0, z_0) \in D$ is defined by

$$\left.\frac{\partial f}{\partial x}\right|_{(x_0,y_0,z_0)} = \lim_{h \to 0} \frac{f(x_0 + h, y_0, z_0) - f(x_0, y_0, z_0)}{h},$$

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provided the limit exists.

In the same way, we define the partial derivatives $\frac{\partial f}{\partial y}$ and $\frac{\partial f}{\partial z}$.

$$\left.\frac{\partial f}{\partial y}\right|_{(x_0,y_0,z_0)} = \lim_{h\to 0} \frac{f(x_0, y_0+h, z_0) - f(x_0, y_0, z_0)}{h},$$

$$\left.\frac{\partial f}{\partial z}\right|_{(x_0,y_0,z_0)} = \lim_{h\to 0} \frac{f(x_0, y_0, z_0+h) - f(x_0, y_0, z_0)}{h}.$$

Examples

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3. Find f_x and f_y for $f(x, y) = x/(x^2 + y^2)$
4. Let $f(x, y) = 2x + 3y - 4$. Find the slope of the line tangent to this surface at the point $(2, -1)$ and lying in the
   - plane $x = 2$
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   - plane $x = 2$
   - plane $y = -1$.
5. Find the value of $\partial x/\partial z$ at the point $(1, -1, -3)$ if the equation

$$xz + y \ln x - x^2 + 4 = 0$$

defines x as a function of the two independent variables y and z and the partial derivatives exist.

Examples

6. Find f_x and f_y for

$$f(x,y) = \begin{cases} \frac{xy}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{otherwise} \end{cases}.$$

Show that f_x and f_y exist at every point but not continuous at $(0,0)$.

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6. Find f_x and f_y for

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7. Compute f_x and f_y at $(0,0)$ of $f(x,y)$ where

$$f(x,y) = \begin{cases} \frac{\sin(x^3+y^4)}{x^2+y^2}, & (x,y) \neq (0,0), \\ 0 & (x,y) = (0,0). \end{cases}$$

Higher-Order Partial Derivatives

- $\dfrac{\partial^2 f}{\partial x^2} = f_{xx} := \dfrac{\partial}{\partial x}\left(\dfrac{\partial f}{\partial x}\right)$,
- $\dfrac{\partial^2 f}{\partial y^2} = f_{yy} := \dfrac{\partial}{\partial y}\left(\dfrac{\partial f}{\partial y}\right)$,
- $\dfrac{\partial^2 f}{\partial x \partial y} = f_{yx} := \dfrac{\partial}{\partial x}\left(\dfrac{\partial f}{\partial y}\right)$,
- $\dfrac{\partial^2 f}{\partial y \partial x} = f_{xy} := \dfrac{\partial}{\partial y}\left(\dfrac{\partial f}{\partial x}\right)$,
- $f_{yxzxx} := \dfrac{\partial^5 f}{\partial x^2 \partial z \partial x \partial y}$.

We can define all higher order partial derivatives in a similar manner.

1. Find all the second-order partial derivatives of
    - $g(x, y) = xe^y + y + 1$
    - $r(x, y) = \ln(x + y)$
    - $w(x, y) = ye^{x^2 - y}$

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2. **Laplace Equations:**

$$\text{Three dimensional} : \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0.$$

$$\text{Two dimensional} : \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0.$$

Check that the following functions satisfy Laplace equations.

- $f(x, y, z) = x^2 + y^2 - 2z^2.$
- $f(x, y) = e^{-2y} \cos 2x.$

A Counterexample

Example: Let

$$f(x,y) = \begin{cases} \frac{xy(x^2-y^2)}{x^2+y^2} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{else.} \end{cases}$$

1. Show that $\frac{\partial f}{\partial y}(x,0) = x$ for all x, and $\frac{\partial f}{\partial x}(0,y) = -y$ for all y.
2. Show that $\frac{\partial^2 f}{\partial y \partial x}(0,0) \neq \frac{\partial^2 f}{\partial x \partial y}(0,0)$.

Clairaut's Theorem

Theorem - The Mixed Derivative Theorem

If $f(x, y)$ and its partial derivatives f_x, f_y, f_{xy}, and f_{yx} are defined throughout an open region containing a point (a, b) and are all continuous at (a, b), then

$$f_{xy}(a, b) = f_{yx}(a, b).$$