Vector Valued Functions and Motion in Space

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September 10, 2024

Interval $f: I \subseteq \mathbb{R} \longrightarrow \mathbb{R}^n$

$(x_1, x_2, \ldots, x_n)$ - vector in $\mathbb{R}^n$

f is vector valued function, In this course, we will look at $\boxed{n=3}$ i.e $f: I \longrightarrow \mathbb{R}^3$ $\quad I \subseteq \mathbb{R}$

$t \longmapsto (x(t), y(t), z(t))$

$x: I \longrightarrow \mathbb{R} \quad : t \longmapsto x(t)$

$y: I \longrightarrow \mathbb{R} \quad : t \longmapsto y(t)$

$z: I \longrightarrow \mathbb{R} \quad : t \longmapsto z(t)$

$f(I) \subseteq \mathbb{R}^3$

(x, y) - 2 tuple

(x, y, z) - 3 tuple

$(x_1, \ldots, x_n)$ - n tuple

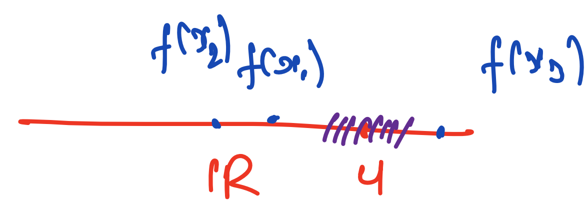
$f: I \longrightarrow \mathbb{R}$

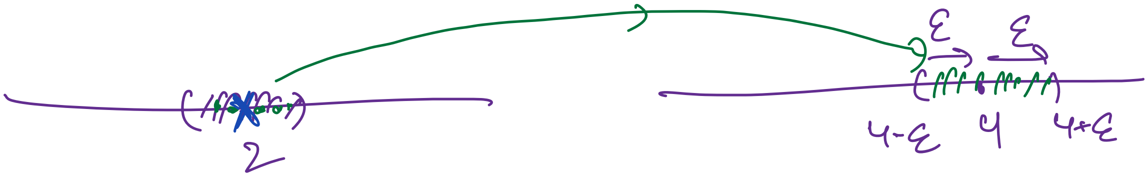
$f(x) = x^2$, $\quad x \in (-\infty, \infty) = I$

$\lim_{x \to 2} f(x) = 4$

or $x \to 2 \Rightarrow f(x) \to 4$

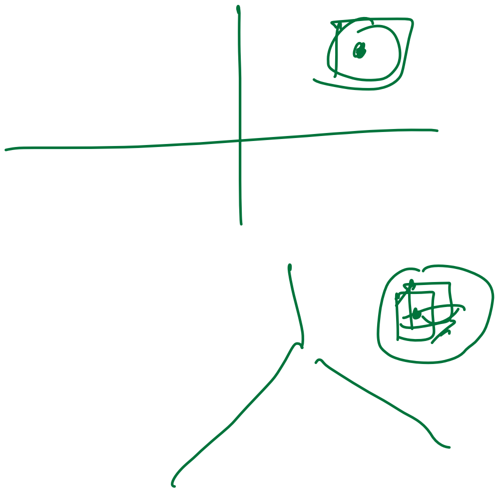
$I \subseteq \mathbb{R}$





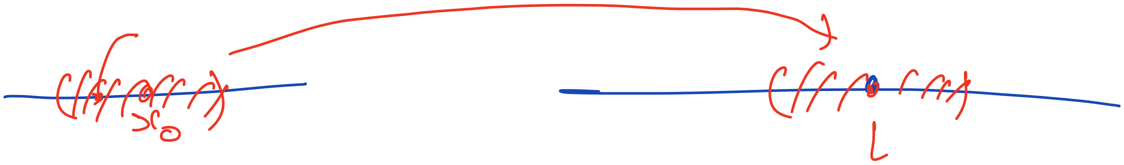
$x \to 2 \Rightarrow f(x) \to 4$

$\exists\, \delta > 0$ $\qquad$ $\varepsilon > 0$



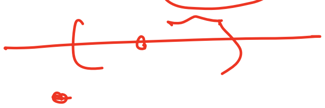
$\lim_{x \to x_0} f(x) = ?$ s.t $f(x_0)$ may or may not be defined

$= L$



$x \to x_0 \Rightarrow f(x) \to L$

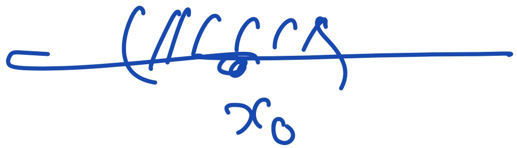
$\delta = 1$



ε

$\delta = \min\{1, \text{quantity on } \varepsilon\}$

Definition: Limit in Single Variable



Definition 1.

Let f be a single variable real-valued function with x_0, a point on the domain and $L \in \mathbb{R}$. Now for any $\epsilon > 0$ if there exists a $\delta > 0$ (depended on ϵ) such that

$\Rightarrow f(x) \in (L-\varepsilon, L+\varepsilon)$

$\Rightarrow x \in (x_0-\delta, x_0+\delta) - \{x_0\}$

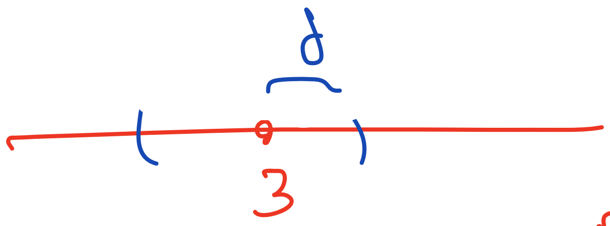
$$|f(x) - L| < \epsilon \text{ whenever } 0 < |x - x_0| < \delta$$

then we say that the limit of f at x_0 exists and equals to L.

We denote $\lim_{x \to x_0} f(x) = L$.

Example

Use ϵ, δ-definition to show that $\lim_{x\to 3} x^2 = 9$.



Given $\varepsilon > 0$, Find $\boxed{\delta = ?}$ s.t

$$0 < |x-3| < \delta \overset{?}{\Longrightarrow} |f(x) - 9| < \varepsilon$$

i.e $|x^2 - 9| < \varepsilon$

L.H.S $|x^2 - 9| = |x+3|\,|x-3| < |x+3|\,\delta$

take $|x+3|\,\delta = \varepsilon \Rightarrow \delta = \dfrac{\varepsilon}{|x+3|}$

take $\underline{\delta \leq 1}$

$|x-3|<1 \Rightarrow 2<x<4$

$\Rightarrow 5<|x+3|<7$

$\Rightarrow \frac{1}{5} > \frac{1}{|x+3|} > \frac{1}{7} \Rightarrow \frac{|x+3|}{7} < 1$

$\Rightarrow \frac{\varepsilon}{5} > \frac{\varepsilon}{|x+3|} > \frac{\varepsilon}{7} \Rightarrow \frac{\varepsilon}{7} < \frac{\varepsilon}{|x+3|}$

choose $\delta = \min\left\{1, \frac{\varepsilon}{7}\right\}$

<u>Claim</u> This is the correct choice of δ

i.e. Need to show $|x^2-9|<\varepsilon$

$|x^2-9| = |x+3|\,|x-3|$

$\leq |x+3|\,\delta \qquad \because \delta \leq \frac{\varepsilon}{7}$

$< |x+3| \times \frac{\varepsilon}{7} < 1 \cdot \varepsilon$

$< \varepsilon$

Example

Use ϵ, δ-definition to show that $\lim_{x \to 3} x^2 = 9$.

Solution: Let $\epsilon > 0$ be given. We want $|f(x) - 9| < \epsilon$ when $0 < |x - 3| < \delta$.

In other words, we expect $0 < |x - 3| < \delta$ gives us $|x^2 - 9| < \epsilon$.

Now $|x^2 - 9| < \epsilon \Rightarrow |(x + 3)(x - 3)| < \epsilon \Rightarrow |x - 3| < \frac{\epsilon}{|x+3|}$.

But we cannot choose $\delta = \frac{\epsilon}{|x+3|}$ as δ must be a real value depending upon ϵ (not a function of x).

We assume $\delta \leq 1 \Rightarrow 0 < |x - 3| < 1 \Rightarrow 2 < x < 4 \Rightarrow 5 < x + 3 < 7 \Rightarrow$ $5 < |x + 3| < 7 \Rightarrow \frac{1}{5} > \frac{1}{|x+3|} > \frac{1}{7}$.

Thus we choose $\delta = \min\{1, \frac{\epsilon}{7}\}$.

Example

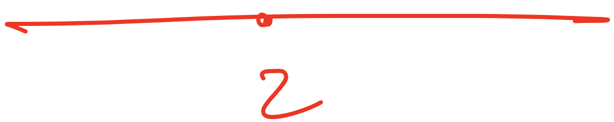
If we choose the above δ, then how $|f(x) - 9| < \epsilon$?

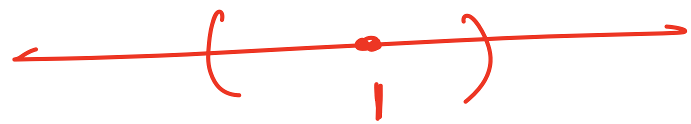
$0 < |x - 3| < \delta \leq \frac{\epsilon}{7} \Rightarrow |x - 3||x + 3| < \epsilon \times \frac{1}{7}|x + 3| < \epsilon \frac{1}{|x+3|}|x + 3| = \epsilon$ (as $\delta < 1$).

Then $|f(x) - 9| < \epsilon$, as required.

Example

Prove that $\lim_{x\to 2} x^3 - x^2 - 4x + 5 = 1$.





Given $\varepsilon > 0$, Find $\delta > 0$ s.t

$0 < |x-2| < \delta \implies |f(x) - 1| < \varepsilon$

i.e $|x^3 - x^2 - 4x + 4|$.

$= |x-2|\,|x^2 + x - 2|$

$|x-2|\,|x^2+x-2| < \delta\,|x^2+x-2| \stackrel{\text{if}}{=} \varepsilon \implies \delta = \dfrac{\varepsilon}{|x^2+x-2|}$

take $\delta \leq 1/2$

$\Rightarrow 0 < |x-2| < 1/2 \Rightarrow \frac{3}{2} < x < 5/2$

$\Rightarrow 9/4 < x^2 < 25/4 \Rightarrow \frac{7}{4} < x^2+x-2 < \frac{27}{4}$

$\frac{4|x^2+x-2|}{7} < 1$ ✓ $\Rightarrow \frac{4}{7} > \frac{1}{|x^2+x-2|} > \frac{4}{27}$?

choose $\delta = \min\left\{\frac{1}{2}, \frac{4}{27}\varepsilon\right\}$ $\delta \leq \frac{4}{27}\varepsilon$

Given. $|x-2| < \delta$

we need to prove

$|f(x)-1| < \varepsilon$

$|f(x)-1| = |x-2|\,|x^2+x-2|$

$< \delta\,|x^2+x-2| < \frac{4}{27}\varepsilon\,|x^2+x-2|$

$< 1 \cdot \varepsilon < \varepsilon$

Example

Prove that $\lim_{x \to 2} x^3 - x^2 - 4x + 5 = 1$.

Solution: Let $\epsilon > 0$ be given. Now $|x^3 - x^2 - 4x + 5 - 1| =$
$|x^3 - x^2 - 4x + 4| = |(x-2)(x^2+x-2)| < \epsilon \Rightarrow |x-2| < \frac{\epsilon}{|x^2+x-2|}$.
Let $0 < |x-2| < \frac{1}{2} \Rightarrow -\frac{1}{2} < x - 2 < \frac{1}{2} \Rightarrow \frac{3}{2} < x < \frac{5}{2} \Rightarrow \frac{9}{4} < x^2 < \frac{25}{4} \Rightarrow$
$\frac{15}{4} < x^2 + x < \frac{35}{4} \Rightarrow \frac{7}{4} < x^2 + x - 2 < \frac{27}{4} \Rightarrow \frac{4}{7} > \frac{1}{x^2+x-2} > \frac{4}{27}$.
Now we choose $\delta = \min\{\frac{1}{2}, \frac{4\epsilon}{27}\}$.
Then $0 < |x-2| < \delta \Rightarrow |x-2| < \frac{4}{27}\epsilon < \frac{1}{|x^2+x-2|}\epsilon$ (as $|x-2| < \frac{1}{2}$).
Then $|(x-2)(x^2+x-2)| < \epsilon$ and hence $|f(x) - 1| < \epsilon$.

Example: (Source Internet)

Prove that $\lim_{x\to 0} e^x = 1$.

Solution: Let $1 > \epsilon > 0$ be given.
Now $|e^x - 1| < \epsilon \Rightarrow -\epsilon < e^x - 1 < \epsilon \Rightarrow (1-\epsilon) < e^x < 1+\epsilon \Rightarrow$
$\ln(1-\epsilon) < x < \ln(1+\epsilon)$.
Let $\delta = \min\{|\ln(1-\epsilon)|, \ln(1+\epsilon)\} = \ln(1+\epsilon)$.
Now $0 < |x| < \delta = \ln(1+\epsilon) \Rightarrow -\ln(1+\epsilon) < x < \ln(1+\epsilon)$.
Now as $\ln(1-\epsilon) < -\ln(1+\epsilon)$, we have

$$\ln(1-\epsilon) < -\ln(1+\epsilon) < x < \ln(1+\epsilon).$$

Then $1-\epsilon < e^x < (1+\epsilon) \Rightarrow -\epsilon < e^x - 1 < \epsilon \Rightarrow |e^x - 1| < \epsilon$.

Example

Discuss $\lim_{x \to 0} f(x)$ where $f(x) = \frac{x \sin x}{x+1}$ if $x > -1$ and $x \neq 0$.

$x \to 0$ where $f(x) \to 0$
$= x_0$ $\qquad = L$

Given $\varepsilon > 0$, Find $\delta > 0$

s.t

NTP

$0 < |x - 0| < \delta \implies |f(x) - 0| < \varepsilon$

$\Rightarrow |x| < \delta$

i.e $\left| \frac{x \sin x}{x+1} \right| < \varepsilon$

L.H.S $= \left| \frac{x \sin x}{x+1} \right| \leq \frac{|x|}{|x+1|} < \frac{\delta}{|x+1|} = \varepsilon$

i.e $\delta = \varepsilon |x+1|$

take $\delta \le 1/2$ i.e $\delta \in (0, 1/2)$

$|x| < \delta \Rightarrow |x| < 1/2 \Rightarrow -1/2 < x < 1/2$

$\Rightarrow 1/2 < |x+1| < 3/2$

choose $\delta = \cancel{\varepsilon} \min\left\{ \frac{1}{2}, \varepsilon/2 \right\}$

<u>why</u> this is the correct choice

Example

Discuss $\lim_{x\to 0} f(x)$ where $f(x) = \frac{x\sin x}{x+1}$ if $x > -1$ and $x \neq 0$. **Solution**:

Let we choose $\delta_1 \in (0, \frac{1}{2})$. Then for $|x-0| < \delta_1 \Rightarrow |x-0| < \frac{1}{2} \Rightarrow -\frac{1}{2} < x < \frac{1}{2} \Rightarrow 1 - \frac{1}{2} < x+1 < 1+\frac{1}{2} \Rightarrow \frac{1}{2} < |x+1| < \frac{3}{2} \Rightarrow \frac{3}{2} < \frac{1}{|x+1|} < 2$. Now

$$\begin{aligned} |\frac{x\sin x}{x+1} - 0| &\leq \frac{|x|}{|x+1|} \text{ as } |\sin x| \leq 1 \\ &\leq 2|x| \text{ as } \frac{1}{|x+1|} < 2 \\ &\leq \epsilon \text{ if } \delta = \min\{\frac{\epsilon}{2}, \delta_1\} \end{aligned}$$