Sequence and Series

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Recall

$\sum_{n=1}^{\infty} a_n$ where a_n is the n^{th} term of the series

Construct a seq. of its partial sums $S_n = a_1 + a_2 + \cdots + a_n$

if $S_n \to l$ then $\sum a_n$ converges and $\sum a_n = l$

Remark

if $\sum a_n$ converges then $a_n \to 0$ as $n \to \infty$

Its contrapositive gives the n^{th} divergent test : if a_n fails to exist or $a_n \to \pm\infty$ as $n \to \infty$, then $\sum a_n$ diverges

Result : If $a_n > 0$ then

$\sum a_n$ converges $\Longleftrightarrow$ $\{S_n\}$ is bdd above

Theorem

The Integral Test: *Let a_n be a sequence of positive terms ($a_n > 0$). Suppose that $a_n = f(n)$, where f is a continuous, positive, decreasing function of x for all $x \geq N$ (N a positive integer). Then the series $\sum_{n=N}^{\infty} a_n$ and the integral $\int_N^{\infty} f(x)dx$ both converge or both diverge.*

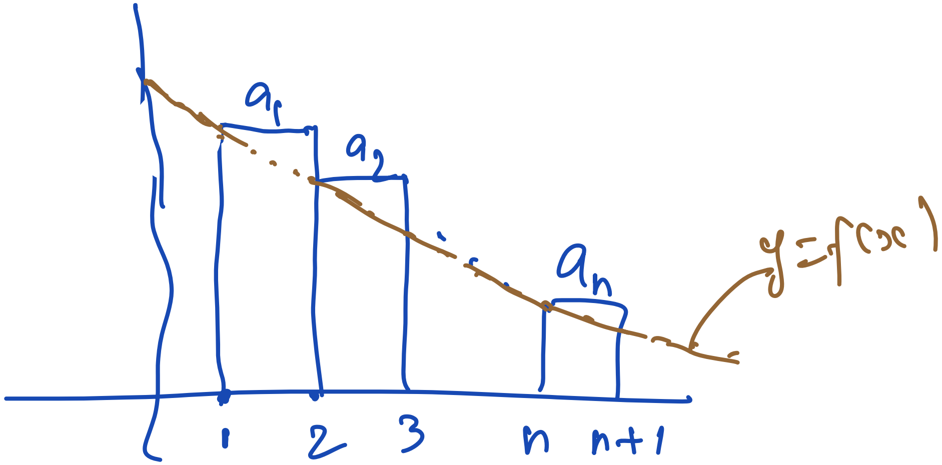
[Given f cont., decreasing s.t $f(n) = a_n \; \forall \; n \geq N$
is defined on $(N-1, \infty)$

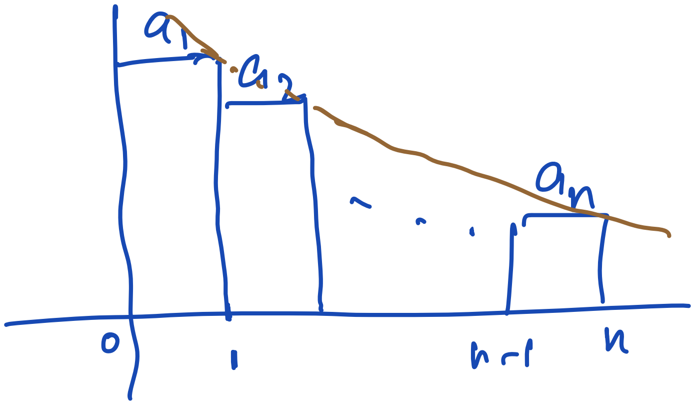
WLOG, take $N = 1$

$\Rightarrow$ f is decreasing in $(1, \infty)$ and $f(n) = a_n$ $\boxed{\sum_{n=1}^{\infty} a_n}$

i.e $a_1 = f(1)$
$a_2 = f(2)$
$\vdots$

$a_2 < a_1$





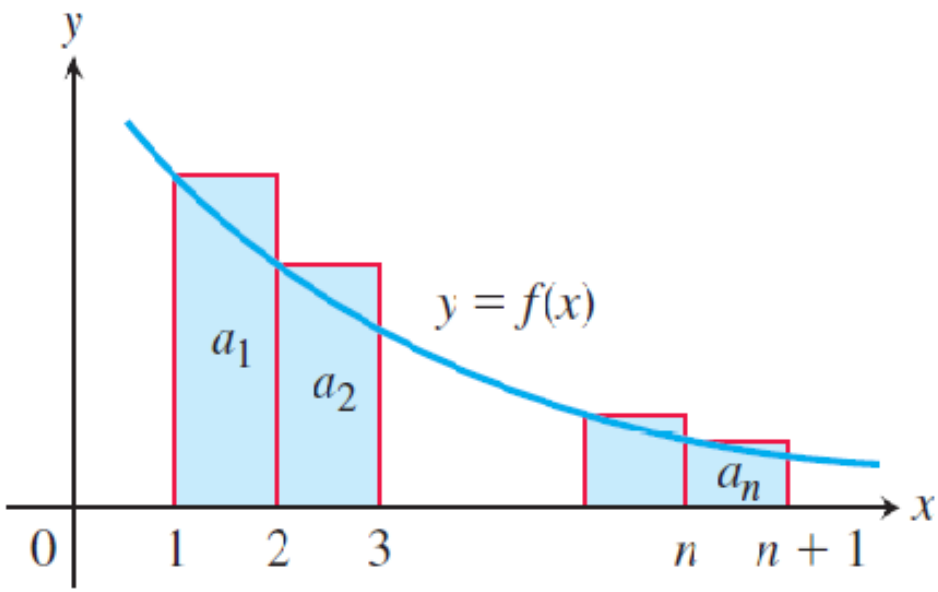
Area from 1 to n+1 bdd by f

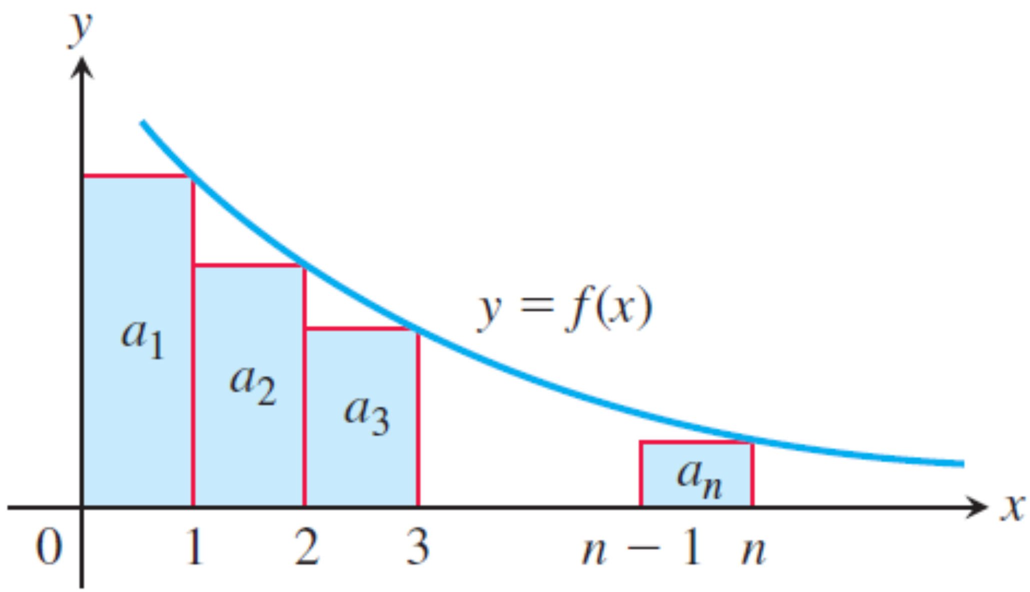
$$\int_1^{n+1} f(x)\,dx \le a_1 + a_2 + \cdots + a_n \le a_1 + \int_1^n f(x)\,dx$$

let $n \to \infty$

$$\underbrace{\lim_{n\to\infty} \int_1^{n+1} f(x)\,dx}_{\text{if } \int \text{ diverges then } \Sigma \text{ diverges}} \le \underbrace{\sum a_n \le a_1 + \lim_{n\to\infty} \int_1^n f(x)\,dx}_{\text{if } \int \text{ converges then } \Sigma \text{ converges}}$$

Let us prove this result when $N = 1$. The proof for general N is similar. Given that f is decreasing and $f(n) = a_n, \forall n$.





$$\int_1^{n+1} f(x)dx \leq a_1 + a_2 + a_3 + \cdots + a_n \leq a_1 + \int_1^n f(x)dx$$

- If $\int_1^n f(x)dx$ is finite as $n \to \infty$, the RHS inequality shows that $\sum_{n=1}^{\infty} a_n$ is finite.
- If $\int_1^n f(x)dx$ is infinite as $n \to \infty$, the LHS inequality shows that $\sum_{n=1}^{\infty} a_n$ is infinite.

Hence, existence of one assures existence of other one.

Example

Does the series $\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} + \cdots$ converge? $= \pi^2/6$

$f(x) = \frac{1}{x^2}$ $x \geq 1$ $f(n) = a_n$ - decreasing, cont.

$a_n = \frac{1}{n^2}$

$$\int_1^{\infty} \frac{1}{x^2}\,dx = -\frac{1}{x}\Big|_1^{\infty} = 1$$

$\Rightarrow$ integral converges $\Rightarrow$ $\sum \frac{1}{n^2}$ converges

Example

Does the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} + \cdots$ converge?

Remark

The series and integral need not have the same value in the convergent case as $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2} \neq \int_1^{\infty} \frac{1}{x^2} dx = 1.$

$$\frac{\pi^2}{6}$$

$$1 + \frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{4^3} + \cdots$$

Example

Using integral test discuss the convergence or divergence of the series $\sum_{n=1}^{\infty} \frac{1}{n^p} = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \cdots + \frac{1}{n^p} + \cdots +$ for any fixed real number p.

Case 1   if $p \leq 0 \Rightarrow -p \geq 0$

$\sum n^{-p}$ by n^{th} divergent test as $a_n \to \infty$ as $n \to \infty$

$\Rightarrow \sum n^{-p}$ diverges

Case 2 $p > 0$

2(i)   if $p = 1$ $\sum \frac{1}{n}$ diverges left over cases are $\quad p > 0 \;, \; p \neq 1$

$f(x) = \frac{1}{x^p} \quad x \geq 1 \quad$ decreasing, continuous.

$$\int_1^\infty f(x)\,dx = \lim_{b\to\infty} \int_1^b \frac{1}{x^p}\,dx$$

$$= \lim_{b\to\infty} \left[\frac{x^{-p+1}}{-p+1} \Big|_1^b \right]$$

$$= \frac{1}{1-p} \lim_{b\to\infty} \left[b^{-p+1} - 1 \right]$$

$0 < p < 1$, $\lim_{b\to\infty} b^{-p+1} \longrightarrow \infty$ $\quad$ and $\quad p > 1$ $\quad \lim_{b\to\infty} b^{-p+1} \longrightarrow 0$

$\int_1^\infty f(x)\,dx \longrightarrow \infty$ $\quad \int_1^\infty f(x)\,dx = \frac{1}{p-1}$

Example

Using integral test discuss the convergence or divergence of the series $\sum_{n=1}^{\infty} \frac{1}{n^p} = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \cdots + \frac{1}{n^p} + \cdots +$ for any fixed real number p.

Case I: Let $p > 1$.

- $f(x) = \frac{1}{x^p}$ is continuous and decreasing for $x \geq 1$.

$$
\begin{aligned}
\int_1^{\infty} \frac{1}{x^p} dx = \lim_{b \to \infty} \int_1^b x^{-p} dx &= \lim_{b \to \infty} \frac{x^{-p+1}}{-p+1} \bigg|_1^b \\
&= \frac{1}{1-p} \lim_{b \to \infty} \left(\frac{1}{b^{p-1}} - 1 \right) \\
&= \frac{1}{p-1}
\end{aligned}
$$

- Thus the series converges by the Integral test when $p > 1$.

Case II: Let $p \leq 0$.

By n th term test, the series diverges.

Case III: Let $0 < p < 1$.

Then $1 - p > 0$ and

$$\int_1^\infty \frac{1}{x^p} dx = \lim_{b\to\infty} \int_1^b x^{-p} dx = \lim_{b\to\infty} \frac{x^{1-p}}{1-p}\bigg|_1^b$$
$$= \infty$$

- Thus the series diverges by the Integral test when $0 < p < 1$.

Case IV: If $p = 1$, we have the (divergent) harmonic series
$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} + \cdots +$

More Examples

2. Discuss the convergence of the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+1}$.

$\displaystyle\int_1^{\infty} \frac{1}{x^2+1}dx = \frac{\pi}{4}$ and thus the series converges by the integral test.

3. Determine the convergence or divergence of the series. (Use integral test)

(i) $\displaystyle\sum_{n=1}^{\infty} ne^{-n^2}$

(ii) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^{\ln n}}$

Comparison tests

Theorem

Direct Comparison test: *Let $\sum a_n$, $\sum c_n$ and $\sum d_n$ be three series of non-negative terms such that $c_n \leq a_n \leq d_n$ for all $n \geq N$. Then*

- *If $\sum d_n$ converges, then $\sum a_n$ also converges.*
- *If $\sum c_n$ diverges, then $\sum a_n$ also diverges.*

Exercise

Determine the convergence or divergence of the series.

$(i) \sum_{n=1}^{\infty} \frac{5}{5n-1}$

$$\frac{1}{n - 1/5} > \frac{1}{n}$$

$\sum \frac{1}{n}$ diverges

$\sum \frac{1}{n} < \sum \frac{5}{5n-1} \Rightarrow \sum \frac{5}{5n-1}$ diverges

Exercise

Determine the convergence or divergence of the series.

$$(i) \sum_{n=1}^{\infty} \frac{5}{5n-1}$$

$$\frac{5}{5n-1} = \frac{1}{\frac{5n-1}{5}} = \frac{1}{n-\frac{1}{5}} > \frac{1}{n}.$$

Thus $\sum \frac{5}{5n-1}$ diverges as $\sum \frac{1}{n}$ diverges.

$$(ii) \sum_{n=1}^{\infty} \frac{1}{n!}$$

$1! = 1$

$2! = 2$

$3! = 3\times 2 > 2\times 2$

$4! = 4\times 3\times 2 > 2\times 2\times 2$

$n! > 2^{n-1}$

$\sum \frac{1}{n!} < \sum \frac{1}{2^{n-1}}$ $\quad n \geq 1$

Exercise

converges for $n>1$

$\sum_{n=1} \frac{1}{2^{n-1}}$

Determine the convergence or divergence of the series.

$\sum_{n=2}^{\infty} \frac{1}{n!} < \sum_{n=2}^{\infty} \frac{1}{2^{n-1}}$

$$(i) \sum_{n=1}^{\infty} \frac{5}{5n-1}$$

$$\frac{5}{5n-1} = \frac{1}{\frac{5n-1}{5}} = \frac{1}{n-\frac{1}{5}} > \frac{1}{n}.$$

Thus $\sum \frac{5}{5n-1}$ diverges as $\sum \frac{1}{n}$ diverges.

$$(ii) \sum_{n=1}^{\infty} \frac{1}{n!}$$

$1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots + \frac{1}{n!} < 1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots + \frac{1}{2^{n-1}}$ Thus $\sum \frac{1}{n!}$ converges as $\sum \frac{1}{2^n}$ converges.

Limit comparison test

$$\frac{\ln n}{n^{3/2}} n^{1/2} = \frac{\ln n}{n}$$

Suppose that $a_n > 0$ and $b_n > 0$ for all $n \geq N$ for some $N \in \mathbb{N}$.

(i) If $\lim_{n\to\infty} \frac{a_n}{b_n} = c > 0$, then $\sum a_n$ and $\sum b_n$ both converge or diverge. ✓

(ii) If $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$ and $\sum b_n$ converges then $\sum a_n$ converges. ✓

(iii) If $\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$ and $\sum b_n$ diverges then $\sum a_n$ diverges.

Which of the following series converge and, which diverge?

$(i) \sum_{n=1}^{\infty} \frac{2n+1}{(n+1)^2}$ ✓

$(ii) \sum_{n=1}^{\infty} \frac{1}{2^n - 1}$

$(iii) \sum_{n=1}^{\infty} \frac{1 + n \ln n}{n^2 + 5}$

$(iv) \sum_{n=1}^{\infty} \frac{\ln n}{n^{\frac{3}{2}}}$

$a_n = \frac{2n+1}{(n+1)^2}$ $\qquad b_n = 1/n$

$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{2n+1}{(n+1)^2} \times n$

$= \lim_{n\to\infty} \frac{2n^2+n}{n^2+1+2n}$

$= \lim_{n\to\infty} \frac{2+1/n}{1+1/n^2+2/n} = 2$

$\sum b_n = \sum \frac{1}{n}$ diverges $\Rightarrow \sum \frac{2n+1}{(n+1)^2}$ diverges

2) $\sum_{n=1}^{\infty} \frac{1}{2^n-1}$ $\quad a_n = \frac{1}{2^n-1}$ $\quad b_n = \frac{1}{2^n}$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{2^n}{2^n-1} = \lim_{n\to\infty} \frac{1}{1-1/2^n} = 1$$

$\sum \frac{1}{2^n}$ converges $\Rightarrow$ $\sum \frac{1}{2^n-1}$ converges

3. $a_n = \frac{1+n\ln n}{n^2+5}$ $\quad b_n = \frac{1}{n}$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{n+n^2\ln n}{n^2+5} = \lim_{n\to\infty} \frac{1/n + \ln n}{1+5/n^2}$$

$$= \infty$$

$\sum b_n = \sum \frac{1}{n}$ diverges

$\Rightarrow$ $\sum a_n$ diverges

4. $a_n = \frac{\ln n}{n^{3/2}}$ $\quad b_n = \frac{1}{n^{5/4}}$

$\lim_{n\to\infty} \frac{a_n}{b_n} = 0$ $\quad \sum \frac{1}{n^{5/4}}$ converges

$\Rightarrow$ $\sum a_n$ converges

Alternating Series test

Theorem

The series

$$\sum_{n=0}^{\infty}(-1)^{n+1}u_n = u_1 - u_2 + u_3 - u_4 + \cdots$$

converges if the following conditions are satisfied:

1. The $u_n > 0$.

2. $u_n \geq u_{n+1}$ *for all* $n \geq N$*, for some integer* N.

3. $u_n \to 0$. as $n \to \infty$

$\sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{n}$ converges by alternating series test.

$u_n = \frac{1}{n}$ $\quad u_n > 0$ $\quad u_n \to 0$

$u_n > u_{n+1}$

Alternating Series test

Theorem

The series

$$\sum_{n=0}^{\infty}(-1)^{n+1}u_n = u_1 - u_2 + u_3 - u_4 + \cdots$$

converges if the following conditions are satisfied:

1. The $u_n > 0$.

2. $u_n \geq u_{n+1}$ *for all* $n \geq N$, *for some integer* N.

3. $u_n \to 0$.

$\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{n}$ converges by alternating series test.

$\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{n^p}$, $p > 0$ converges by alternating series test.