Vector Valued Functions and Motion in Space

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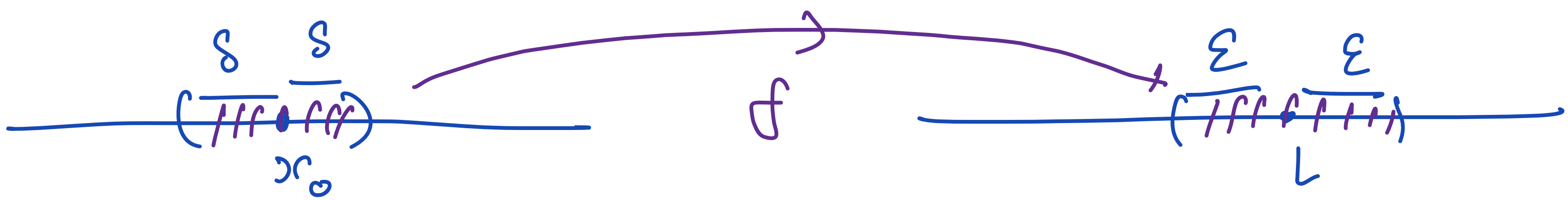
Recall

$f : \mathbb{R} \longrightarrow \mathbb{R}$

$\lim_{t \to t_0} f(t) = L$ if for every $\varepsilon > 0$, $\exists\, \delta > 0$ (depending on ε)

s.t

$$|x - x_0| < \delta \implies |f(x) - L| < \varepsilon$$



Recall an example

$$f(x) = \frac{(x+1)\sin x}{x}, \quad f(x) = \frac{x}{\log x}$$

$$\lim_{x \to 0} f(x)$$

Properties: Limit

Theorem 1.

Let $\lim_{x\to x_0} f(x) = L$, $\lim_{x\to x_0} g(x) = M$ *and* k *be a real number.*

- **Sum Rule** $\lim_{x\to x_0}[f(x) + g(x)] = L + M.$
- **Difference Rule** $\lim_{x\to x_0}[f(x) - g(x)] = L - M.$
- **Constant Multiple Rule** $\lim_{x\to x_0} kf(x) = kL.$
- **Product Rule** $\lim_{x\to x_0}[f(x) \cdot g(x)] = L \cdot M.$
- **Quotient Rule** $\lim_{x\to x_0} \frac{f(x)}{g(x)} = \frac{L}{M}$ *where* $M \neq 0.$
- **Power Rule** $\lim_{x\to x_0} f(x)^n = L^n$ *where* n *is a positive integer.*
- **Root Rule** $\lim_{x\to x_0} \sqrt[n]{f(x)} = \sqrt[n]{L} = L^{\frac{1}{n}}$, *where* n *is a positive integer.*

Definition: Continuity

Definition 2.

Let $f : D \to \mathbb{R}$ be a function where $D \subseteq \mathbb{R}$. For x_0, we say that the function is **continuous at** x_0 if the following conditions hold:

1. $f(x)$ should be defined at x_0.
2. $\lim_{x \to x_0} f(x)$ exists.
3. $\lim_{x \to x_0} f(x) = f(x_0)$.

A function is **continuous** if it is continuous at all points of it's domain.

Example

Consider the function

$$f(x) = \begin{cases} \frac{x \sin x}{x+1} & \text{if } x > -1 \text{ and } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

Is f continuous at the origin? $x_0 = 0$

1) Is $f(0)$ defined? ✓

2) Does $\lim_{x \to 0} f(x)$ exist? ✓ $= L$

3) $f(0) = L = 0$ ✓

Yes f is cont. at $x = 0$

Vector valued functions

$f : \mathbb{R} \longrightarrow \mathbb{R}^n$

$t \longmapsto f(t) \in \mathbb{R}^n$

$= (x_1(t), x_2(t), \ldots, x_n(t))$

(vector is an element of $\mathbb{R}^n$) i.e a pt. in $\mathbb{R}^n$

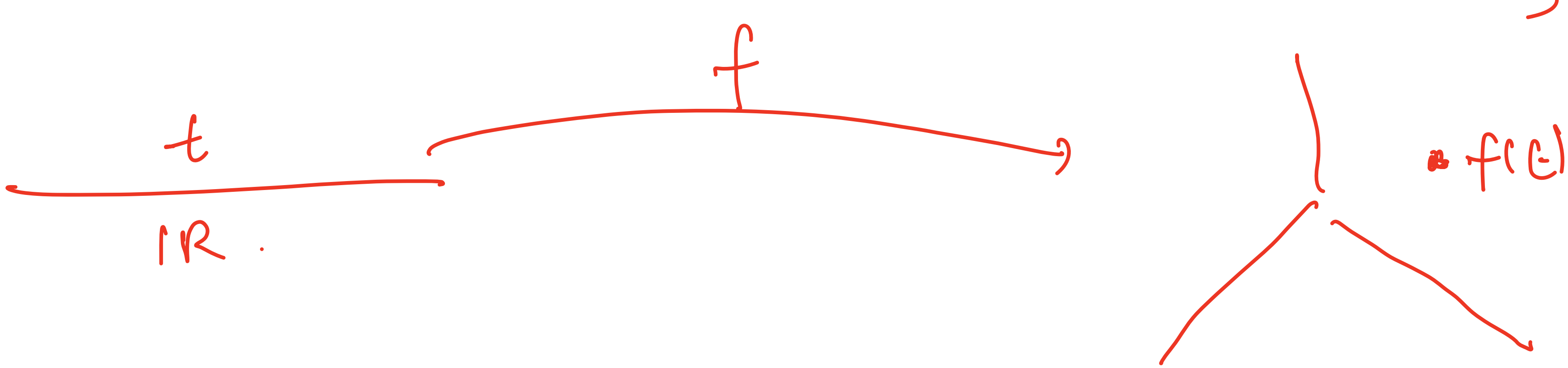
Ex.

$f : \mathbb{R} \longrightarrow \mathbb{R}^5$

$t \longmapsto (t, t^2, t^3, t^4, t-1)$

$f(1) = (1, 1, 1, 1, 0)$

$n = 3$



Curve in space

Suppose a particle is moving in space during a time interval I. We think of the particle's coordinates as functions defined on I :

$$x = f(t), \quad y = g(t), \quad z = h(t); \;\; t \in I$$

The points $(x, y, z) = (f(t), g(t), h(t)), t \in I$, make up the curve in space is called the particle's path.

$r : I \longrightarrow \mathbb{R}^3$ I is an interval

$t \longmapsto r(t) = (x, y, z) = (f(t), g(t), h(t))$

$\{ r(t) \mid t \in I \}$ = particle's path

i.e $\{ (x, y, z) = (f(t), g(t), h(t)) \mid t \in I \}$

Curve in space

Suppose a particle is moving in space during a time interval I. We think of the particle's coordinates as functions defined on I :

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The points $(x, y, z) = (f(t), g(t), h(t)), t \in I$, make up the curve in space is called the particle's path.

A curve in space can also be represented in vector form. The vector

$$r(t) = \overrightarrow{OP} = f(t)\hat{i} + g(t)\hat{j} + h(t)\hat{k}$$

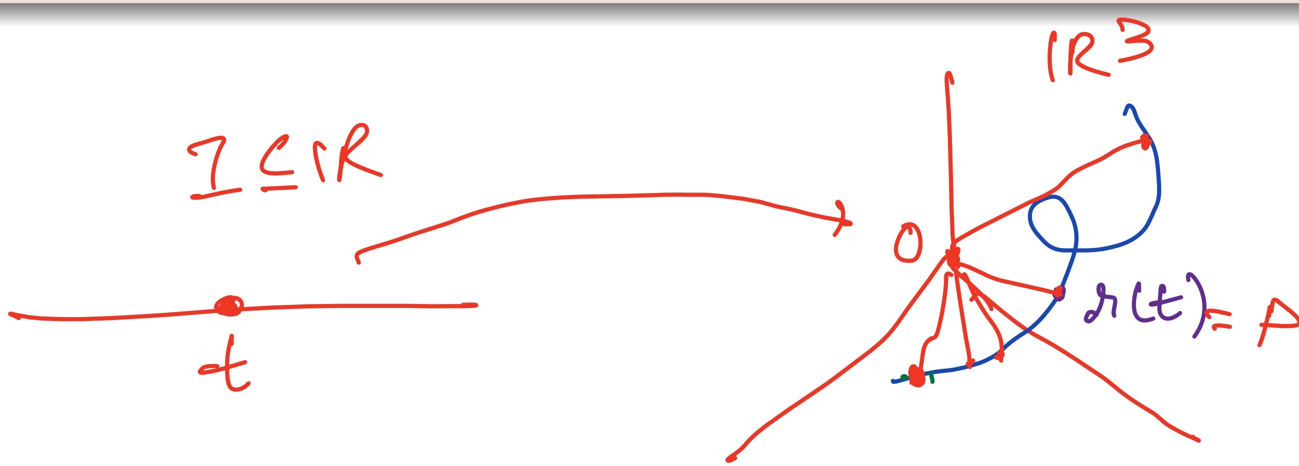
from the origin to the particle's position $P(f(t), g(t), h(t))$ at time t is the particle's position vector.

$$r : I \longrightarrow \mathbb{R}^3$$
$$t \longmapsto r(t) = \vec{OP}$$

The functions f, g and h are the component functions (or components) of the position vector.

Remark

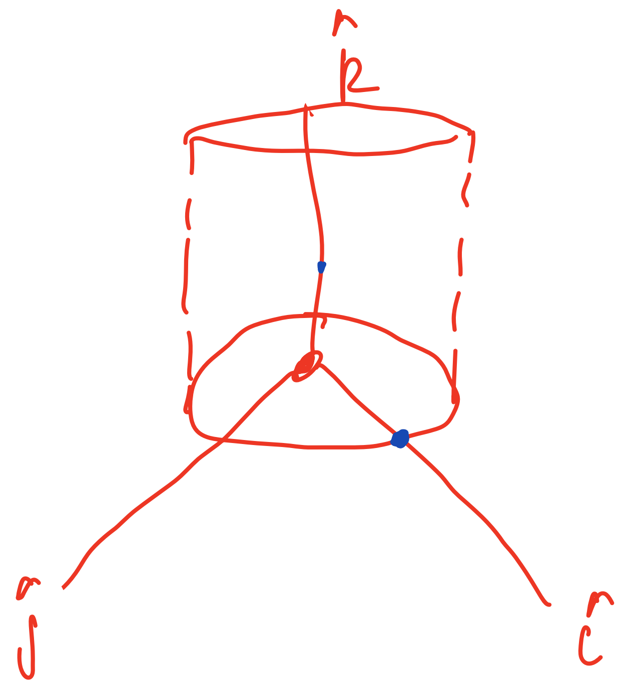
We think of the particle's path as the curve traced by r during the time interval I.



Examples

1. Plot the graph of the following equation in upper half space

$$\mathbf{r}(t) = \cos t\ \mathbf{i} + \sin t\ \mathbf{j} + t\ \mathbf{k}.$$



$r(t) = (\cos t, \sin t, t)$

$r(0) = (1, 0, 0)$

$r(\pi/2) = (0, 1, \pi/2)$

$t = 0$

Examples

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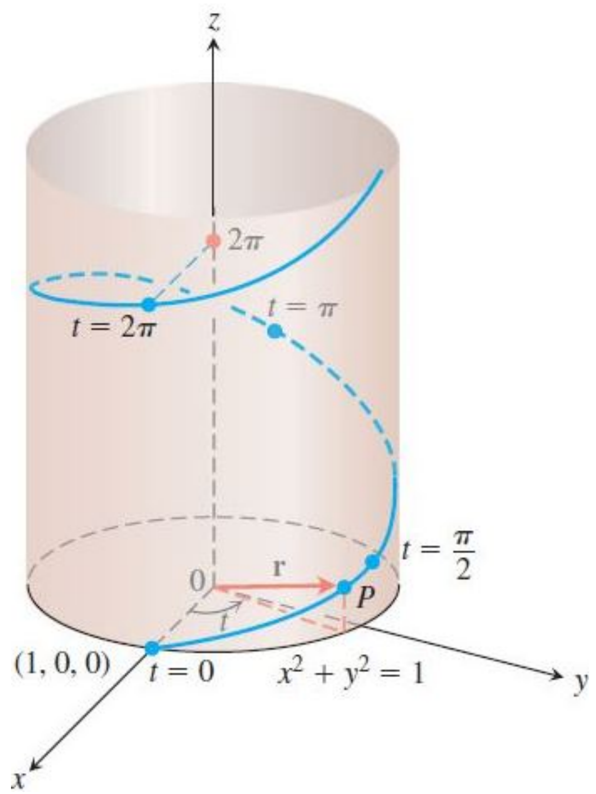
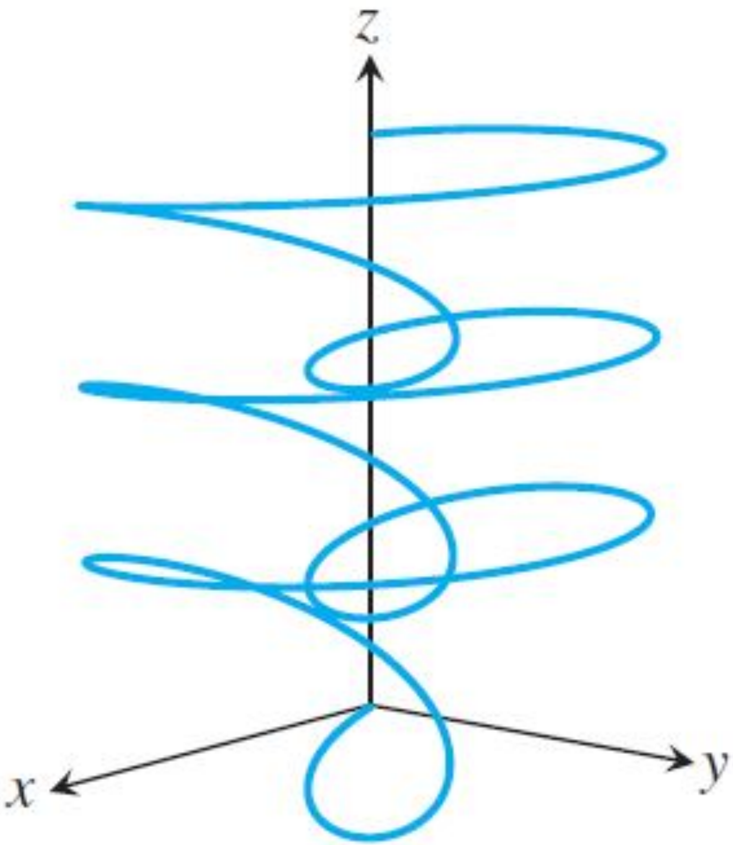
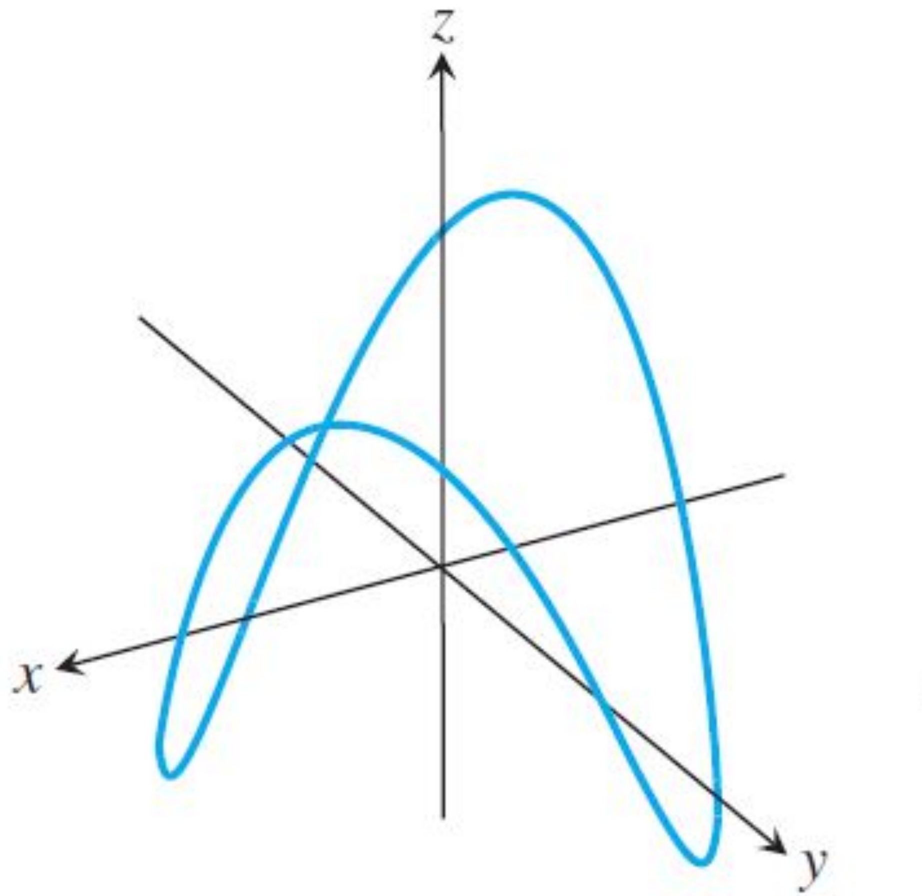


More Example of Curves



$$\mathbf{r}(t) = (\sin 3t)(\cos t)\mathbf{i} + (\sin 3t)(\sin t)\mathbf{j} + t\mathbf{k}$$



$$\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + (\sin 2t)\mathbf{k}$$

Limit of Vector Valued Functions

$$\lim_{t \to t_0} r(t) = ? = L \text{ (say)}$$

Definition 0.1.

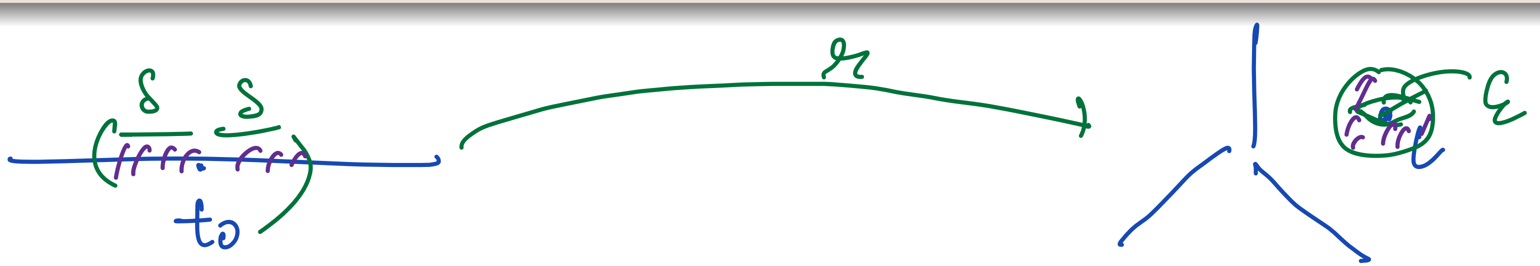
Let $\mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j} + z(t)\,\mathbf{k}$ be a vector-function which is defined on an interval I, and let $\mathbf{L} = x_0\mathbf{i} + y_0\mathbf{j} + z_0\mathbf{k}$. We say that $\mathbf{r}(t)$ has limit $\mathbf{L}$ as t approaches to t_0 and write

$$\lim_{t \to t_0} \mathbf{r}(t) = \mathbf{L},$$

if for every number $\varepsilon > 0$, there exists a corresponding number $\delta > 0$, such that

$$\sqrt{(x(t)-x_0)^2 + (y(t)-y_0)^2 + (z(t)-z_0)^2} < \varepsilon$$

$$|\mathbf{r}(t) - \mathbf{L}| < \varepsilon \quad \text{whenever} \quad 0 < |t - t_0| < \delta.$$



Theorem 0.2.

Let $\mathbf{r}(t)$ be as above. The vector-function $\mathbf{r}(t)$ has a limit at $t = t_0$ if and only if the component functions $x(t)$, $y(t)$ and $z(t)$ have the limits at $t = t_0$. Moreover,

$$\lim_{t \to t_0} \mathbf{r}(t) = \lim_{t \to t_0} x(t)\,\mathbf{i} + \lim_{t \to t_0} y(t)\,\mathbf{j} + \lim_{t \to t_0} z(t)\,\mathbf{k}.$$

$\underbrace{\lim_{t\to t_0} x(t), \quad \lim_{t\to t_0} y(t), \quad \lim_{t\to t_0} z(t)}$ $\longleftrightarrow$ $\lim_{t\to t_0} r(t)$

__Pf__ Given $\lim_{t\to t_0} x(t) = x_0$, $\lim_{t\to t_0} y(t) = y_0$, $\lim_{t\to t_0} z(t) = z_0$ exist ✓

__Claim__ $\lim_{t\to t_0} r(t) = (x_0, y_0, z_0)$

$\boxed{\text{Given } \underline{\varepsilon > 0}}$

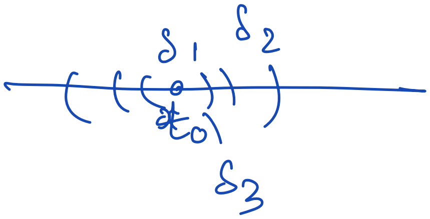
Consider $\varepsilon/\sqrt{3}$, $\exists\ \delta_1 > 0, \delta_2 > 0, \delta_3 > 0$ s.t.

$0 < |t - t_0| < \underline{\delta_1} \implies |x(t) - x_0| < \varepsilon/\sqrt{3}$

i.e. $(x(t) - x_0)^2 < \varepsilon^2/3$

$|t - t_0| < \delta_2 \implies |y(t) - y_0| < \varepsilon/\sqrt{3}$

$|t - t_0| < \delta_3 \implies |z(t) - z_0| < \varepsilon/\sqrt{3}$



take $\underline{\delta} = \min(\delta_1, \delta_2, \delta_3)$

$|t-t_0|<\delta \implies$

$$(x(t)-x_0)^2+(y(t)-y_0)^2+(z(t)-z_0)^2$$
$$\leq (x(t)-x_0)^2+(y(t)-y_0)^2+(z(t)-z_0)^2$$
$$< \varepsilon^2/3+\varepsilon^2/3+\varepsilon^2/3$$
$$< \varepsilon^2$$

$$\Rightarrow \sqrt{(x(t)-x_0)^2+(y(t)-y_0)^2+(z(t)-z_0)^2} < \varepsilon$$

Conversely Given $\lim_{t\to t_0} r(t)$ exist $= L = (x_0, y_0, z_0)$

Claim All $\lim_{t\to t_0} x(t)$, $\lim_{t\to t_0} y(t)$, $\lim_{t\to t_0} z(t)$ exist

It is enough to prove for $\lim_{t\to t_0} x(t) = x_0$

let $\varepsilon>0$

Use the same $\varepsilon>0$ for $r(t)$, $\exists\, \delta>0$

s.t $|t-t_0|<\delta \Rightarrow \sqrt{(x(t)-x_0)^2+(y(t)-y_0)^2+(z(t)-z_0)^2} < \varepsilon$

Clearly $(x(t)-x_0) \leq \sqrt{(x(t)-x_0)^2}$

$$< \sqrt{(x(t)-x_0)^2+(y(t)-y_0)^2+(z(t)-z_0)^2}$$

$$< \varepsilon$$

Ex. $\lim_{t\to\pi/2} r(t)$ $\quad r(t) = \sin t\,\hat{i} + \sin 2t\,\hat{j} + \cos t\,\hat{k}$

$$= \lim_{t\to\pi/2} \sin t\,\hat{i} + \lim_{t\to\pi/2} \sin 2t\,\hat{j} + \lim_{t\to\pi/2} \cos t\,\hat{k}$$

$$= 1.\hat{i} + 0\hat{j} + 0\hat{k}$$

$$= (1,0,0)$$

Continuity of Vector Valued Functions

Definition 0.3.

Vector function $\mathbf{r}(t)$ is said to be continuous at $t = t_0 \in I$ if $r(t_0)$ is defined and

$$\lim_{t \to t_0} \mathbf{r}(t) = \mathbf{r}(t_0).$$

The function $\mathbf{r}(t)$ is said to be continuous if the function is continuous at every point of its domain.

We note that $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$ is continuous at $t = t_0$ if and only if the component functions $x(t)$, $y(t)$ and $z(t)$ are continuous at $t = t_0$.

Example: 1) $\mathbf{r}(t) = \cos t\ \mathbf{i} + \sin t\ \mathbf{j} + t\ \mathbf{k}$ is a continuous vector-function.

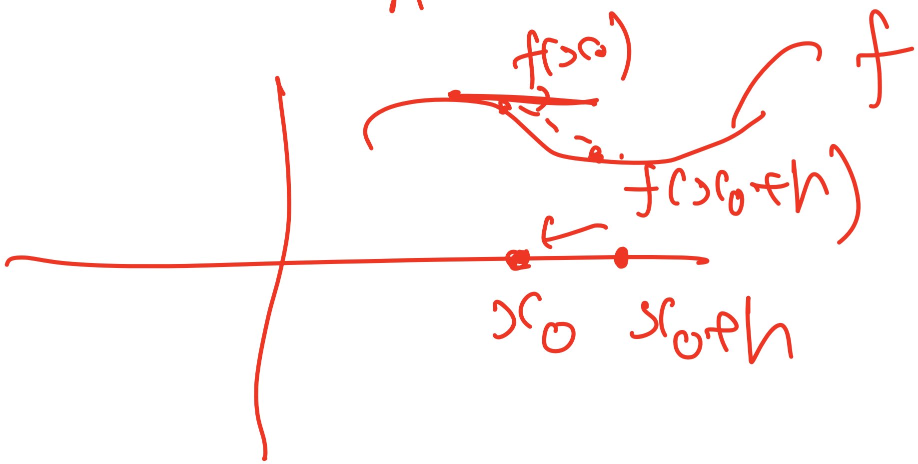
2) $r(t) = \cos t\,\hat{i} + \sin t\,\hat{j} + \lfloor t \rfloor\,\hat{k}$ — Not continuous

Differentiability

$f : \mathbb{R} \longrightarrow \mathbb{R}$

Is $f(x)$ differentiable at $x = x_0$?

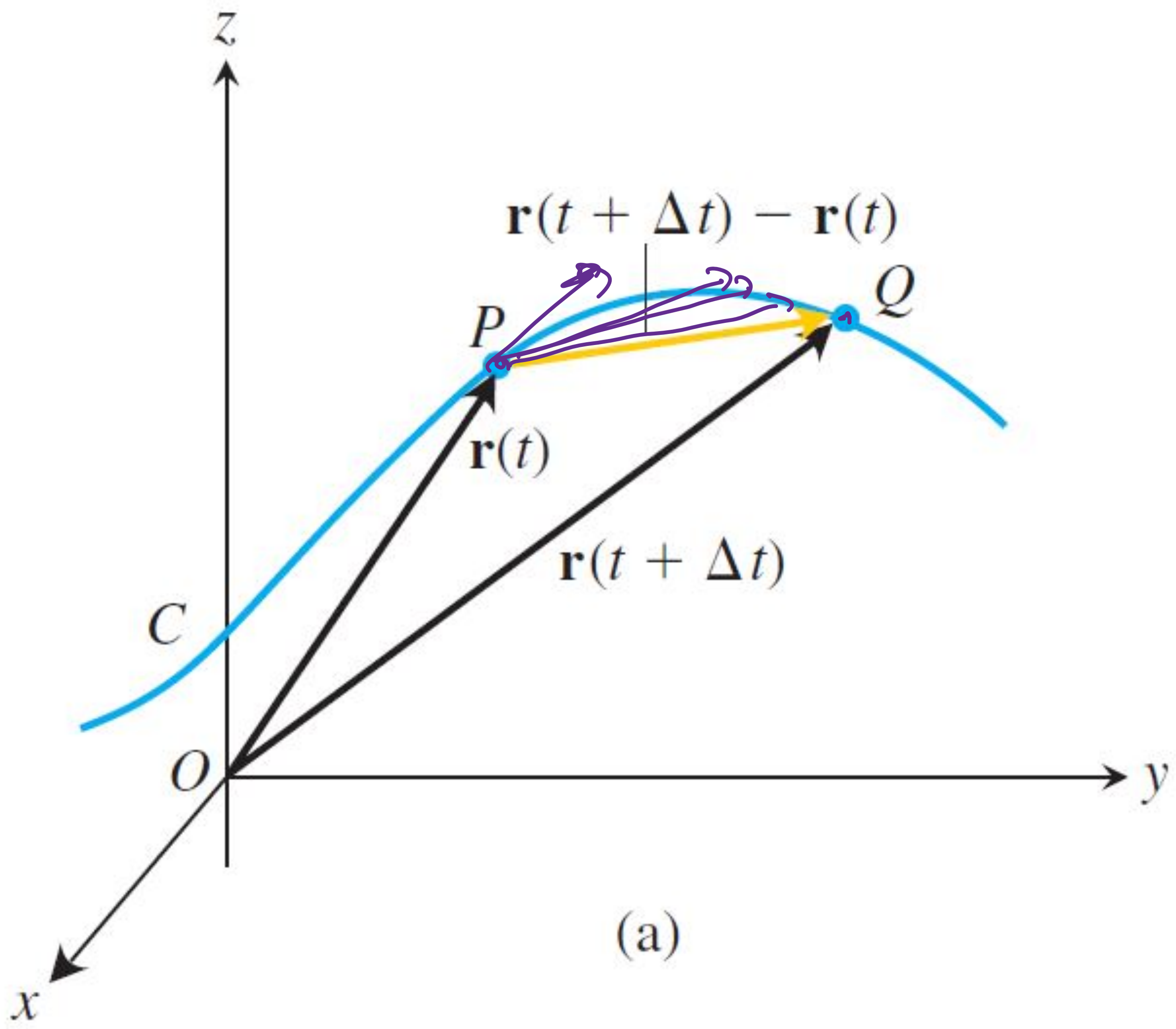
$$\lim_{h \to 0} \left(\frac{f(x_0+h) - f(x_0)}{h} \right) = f'(x_0)$$



Derivatives:

- Suppose that $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$ is the position vector of a particle moving along a curve in the space and that $x(t),\ y(t)$ and $z(t)$ are differentiable functions of t.
- Then the difference between the particle's position at time t and $t + \Delta t$ is
$$\Delta\mathbf{r} = \mathbf{r}(t + \Delta t) - \mathbf{r}(t)$$
- In terms of components,
$$\Delta\mathbf{r} = [x(t + \Delta t) - x(t)]\,\mathbf{i} + [y(t + \Delta t) - y(t)]\,\mathbf{j} + [z(t + \Delta t) - z(t)]\,\mathbf{k}$$

Derivatives



- As Δt approaches zero, three things seem to happen simultaneously. First the point Q approaches the point P along the curve.

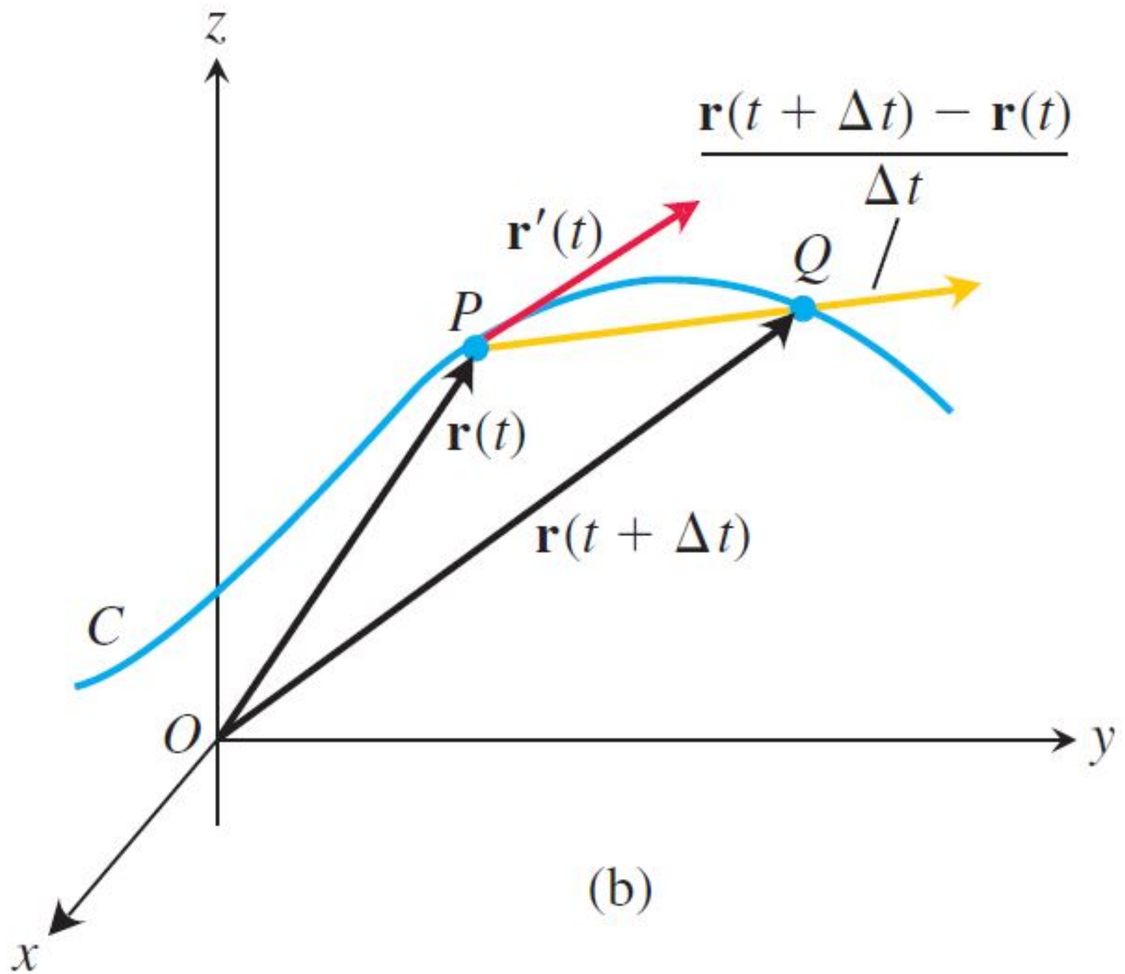
- Second, the secant line PQ seems to approach a limiting position tangent to the curve at P.

- Third, the quotient $\Delta \mathbf{r}/\Delta t$ approaches to the limit:

$$\lim_{\Delta t \to 0} \frac{\Delta \mathbf{r}}{\Delta t} = \left[\lim_{\Delta t \to 0} \frac{x(t+\Delta t) - x(t)}{\Delta t}\right] \mathbf{i} + \left[\lim_{\Delta t \to 0} \frac{y(t+\Delta t) - y(t)}{\Delta t}\right] \mathbf{j} + \left[\lim_{\Delta t \to 0} \frac{z(t+\Delta t) - z(t)}{\Delta t}\right] \mathbf{k}$$

Derivatives conti.

Therefore, $\displaystyle \lim_{\Delta t \to 0} \frac{\Delta \mathbf{r}}{\Delta t} = \left[\frac{dx}{dt}\right] \mathbf{i} + \left[\frac{dy}{dt}\right] \mathbf{j} + \left[\frac{d\cancel{y}z}{dt}\right] \mathbf{k}$



(b)

Derivatives conti.

The above expression lead us to define:

Definition 0.4.

The vector function $\mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j} + z(t)\,\mathbf{k}$ has a derivative (is differentiable) at t, if $x(t)$, $y(t)$ and $z(t)$ have derivatives at t. The derivative of $\mathbf{r}(t)$ is a vector function given by

$$\mathbf{r}'(t) = \lim_{\Delta t \to 0} \frac{r(t+\Delta t) - r(t)}{\Delta t} = \frac{dx}{dt}\,\mathbf{i} + \frac{dy}{dt}\,\mathbf{j} + \frac{dz}{dt}\,\mathbf{k}.$$

- A vector function $\mathbf{r}(t)$ is said to be differentiable if it is differentiable at every point of its domain.

- The vector $\mathbf{r}'(t)$, when different form $\mathbf{0}$, is defined to be the vector tangent to the curve at P.

- The tangent line to the curve

$$\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$$

at a point $(x(t_0), y(t_0), z(t_0))$ is defined to be the line through the point and parallel to the vector

$$\mathbf{r}'(t_0) = x'(t_0)\mathbf{i} + y'(t_0)\mathbf{j} + z'(t_0)\mathbf{k}.$$

Therefore, equation of the tangent line is given by

$$\boxed{\gamma(t) = \mathbf{r}'(t_0)t + \mathbf{r}(t_0).}$$