Polar Coordinates

Gunja Sachdeva

Department of Mathematics
BITS Pilani, Goa Campus

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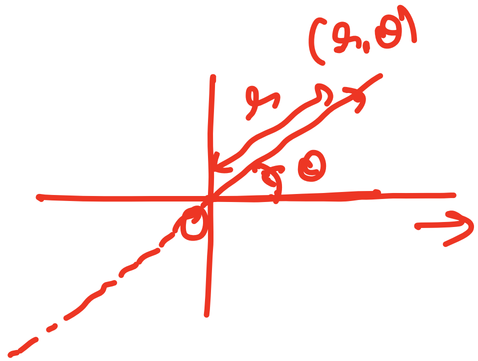
Recall

locate a point (an ordered pair) in the plane. using distance 'r' angle 'θ'

(r, θ)



(x, y) (r, θ) are related as follows

$(r, \theta) \longmapsto (x, y)$ Use $x = r\cos\theta$, $y = r\sin\theta$ ✓

$(x, y) \longmapsto (r, \theta)$ Use $r^2 = x^2 + y^2$, $\tan\theta = \frac{y}{x}$

Fix $r = +\sqrt{x^2+y^2}$, choose θ such that (x, y) and (r, θ) lie in the same quadrant.

Convert the following polar equations into Cartesian equations:

1. $r\cos\theta = 2$ $\Rightarrow x = 2$
2. $r\sin\theta = r\cos\theta$ $\Rightarrow y = x$
3. $r^2 = 4r\sin\theta$ $\Rightarrow x^2 + y^2 = 4y \Rightarrow x^2 + y^2 - 4y + 4 = 4$
   $\Rightarrow x^2 + (y-2)^2 = 2^2$
4. $r = \mathrm{cosec}\theta\, e^{r\cos\theta}$ $\Rightarrow y = e^x$
5. $r\sin\theta = \ln r + \ln(\cos\theta)$. $y = \ln x$

Convert the following Cartesian equations into polar equations:

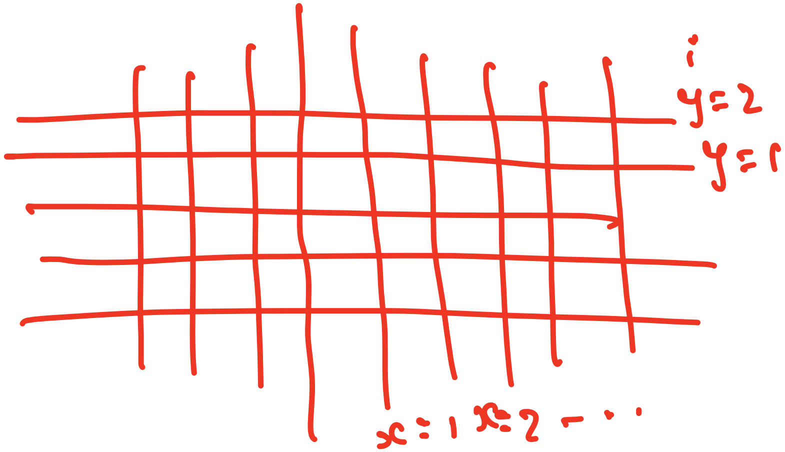
1. $x = 2$ $\Rightarrow r\cos\theta = 2$
2. $x^2 + y^2 = 4$ $\Rightarrow r^2 = 4 \Rightarrow r = \pm 2$
3. $\frac{x^2}{9} + \frac{y^2}{4} = 1$ $\Rightarrow \frac{r^2\cos^2\theta}{9} + \frac{r^2\sin^2\theta}{4} = 1 \Rightarrow \frac{r^2}{9} - \frac{r^2}{9}\sin^2\theta + \frac{r^2\sin^2\theta}{4}$
   $= 1$
4. $x^2 + xy + y^2 = 1$
5. $(x+2)^2 + (y-5)^2 = 16$

Graphing of Polar equations

xy- coordinate system

$y = f(x) \longrightarrow$ curve in plane

Plot : vary x gives different values of y



vertical and horizontal lines gives a graph paper

rθ- coordinate system

$r = f(\theta) \longrightarrow$ curve in the plane

vary θ gives values of r

θ=1, θ=2, - - -
r=1, r=2, - -

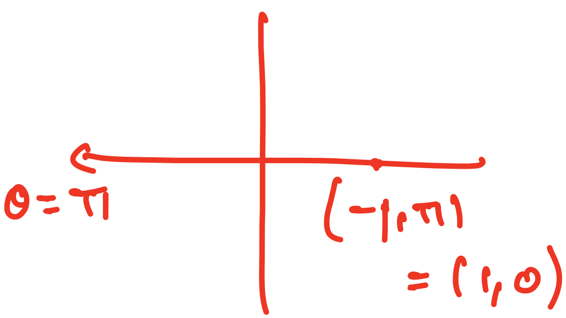
circles and the lines passing through the origin gives a graph paper.

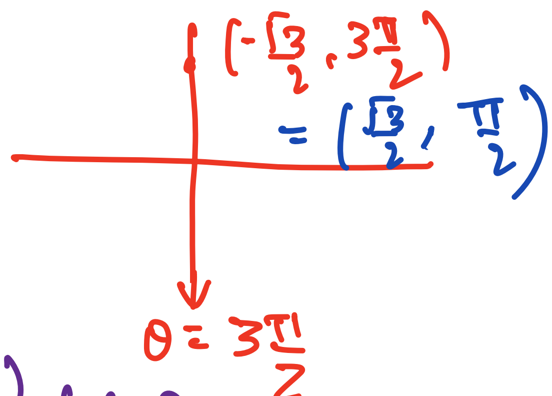
Polar curves

The graph of a polar equation $r = f(\theta)$ [or, more generally, $F(r, \theta) = 0$] consists of all points that have at least one polar representation (r, θ), whose coordinates satisfy the equation.

Example. Which of the following points lies on the polar curve $r = \cos(\theta/3)$

$$a)(r,\theta) = (-1,\pi) \checkmark \qquad b)(1,0) \checkmark \qquad \cancel{c)}\left(-\frac{\sqrt{3}}{2}, \frac{3\pi}{2}\right)$$





(r, θ), $(r, \theta - 2\pi)$, $(r, \theta + 2\pi)$

$(r, \theta + 2\pi n)$

$(-r, \theta + \pi)$

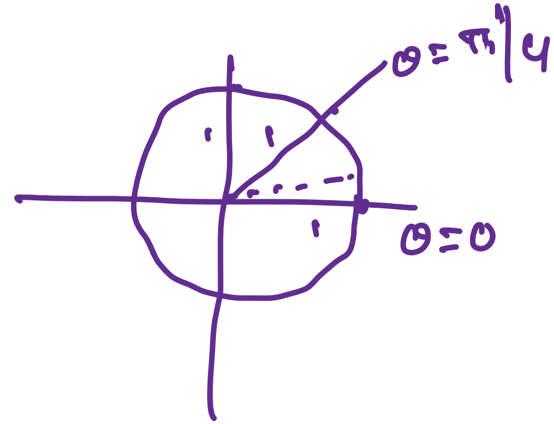
$= (-r, \theta + \pi + 2\pi n)$

$(-r, \theta + \pi)$, $(-r, \theta + 3\pi)$, $(-r, \theta - \pi)$

$(r, \theta) \longrightarrow (r, \theta + 2\pi)$, $(r, \theta - 2\pi)$

Example. What curve is represented by the polar equation $r = 2$?

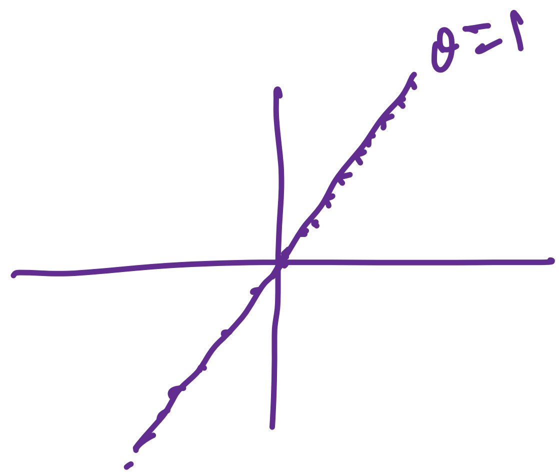
Example. Sketch the polar curve $\theta = 1$.



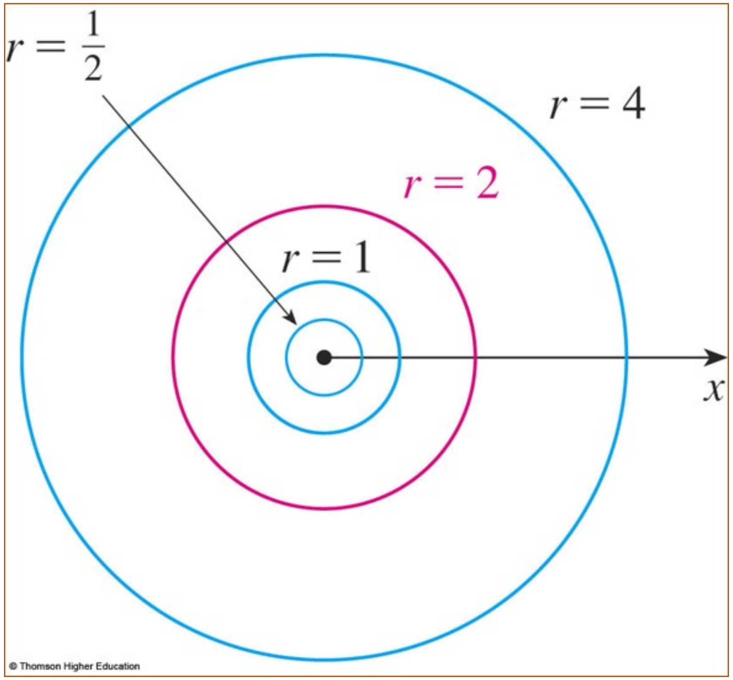
$r = 2 \Rightarrow r^2 = 4$

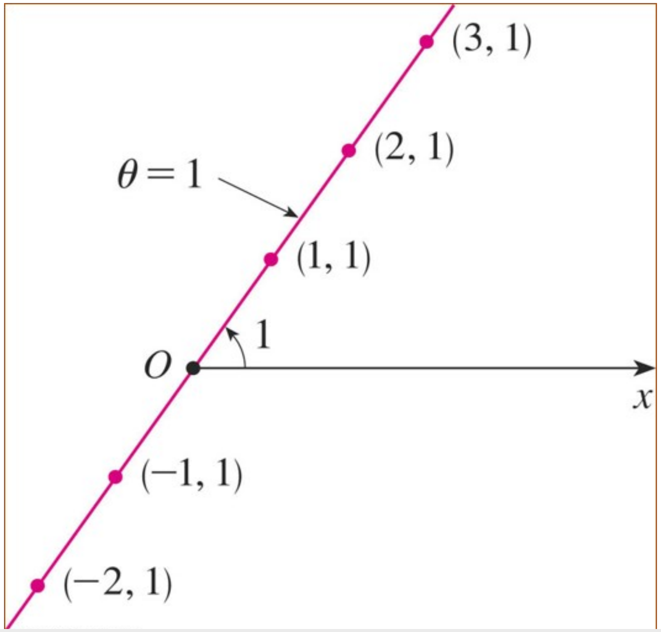
$x^2 + y^2 = 4$

$\theta = 1$

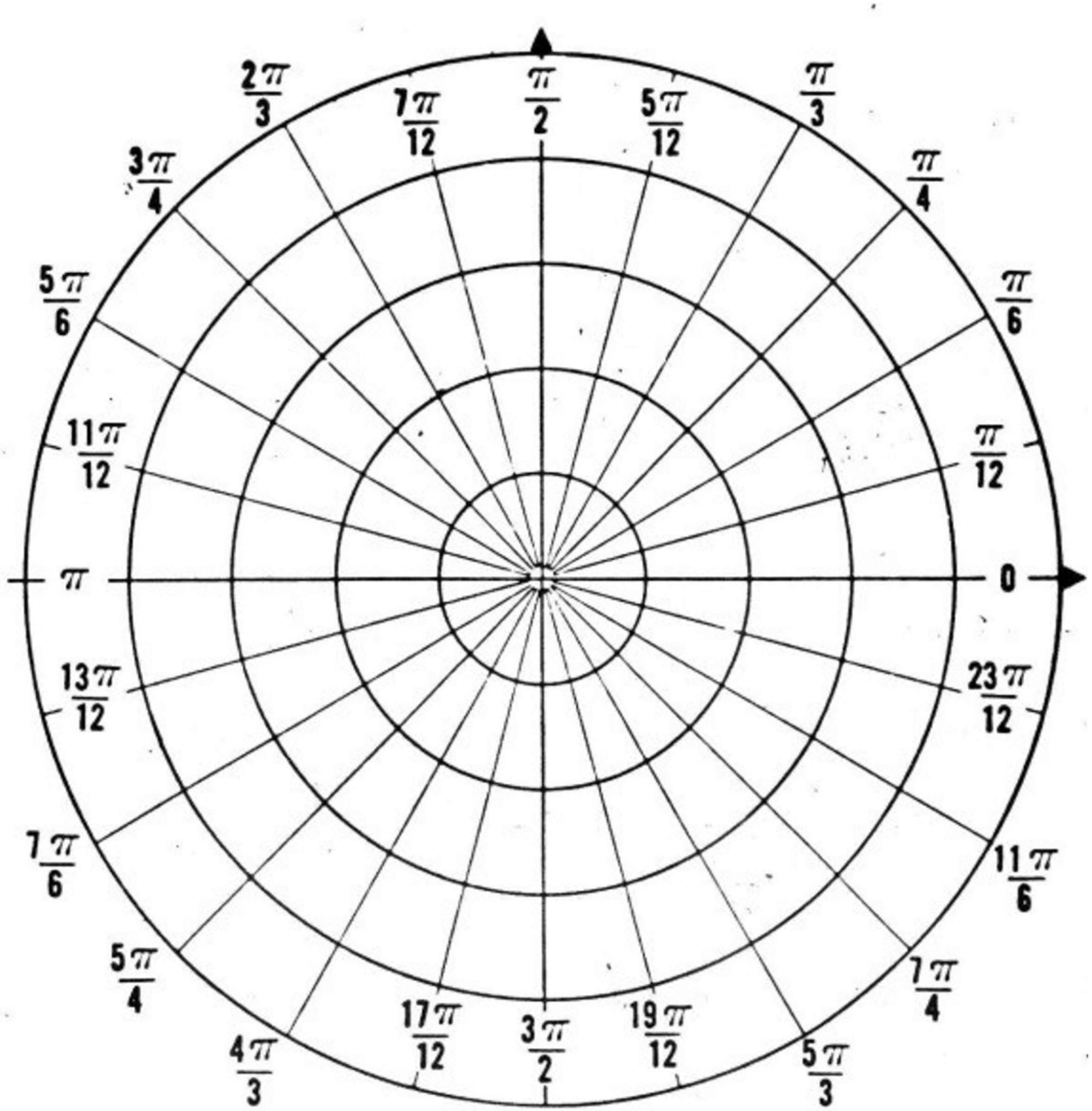


Circles and lines



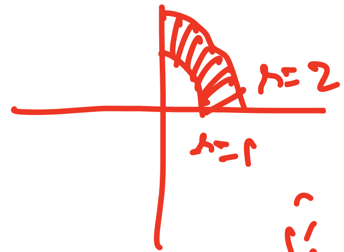


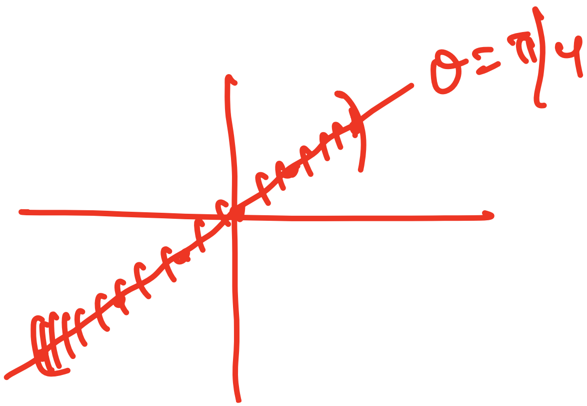
Graph paper of Polar coordinate system

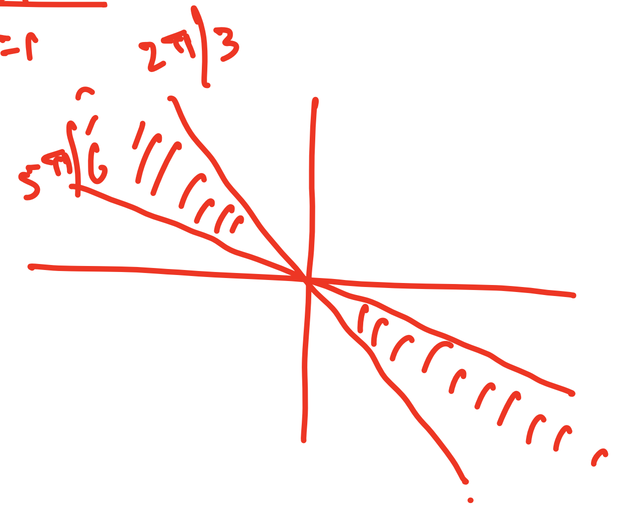


Graph the sets of points whose polar coordinates satisfy the following conditions:

1. $1 \leq r \leq 2, \quad 0 \leq \theta \leq \frac{\pi}{2}$
2. $-3 \leq r \leq 2, \quad \theta = \frac{\pi}{4}$
3. $\frac{2\pi}{3} \leq \theta \leq \frac{5\pi}{6}, \quad$ no restriction on r







Plotting the Polar curve

One way to graph a polar equation $r = f(\theta)$ is to make a table of (r, θ)-values, plot the corresponding points, and connect them in order of increasing θ.

Example. Plot $r = 2\cos\theta$ $\Rightarrow$

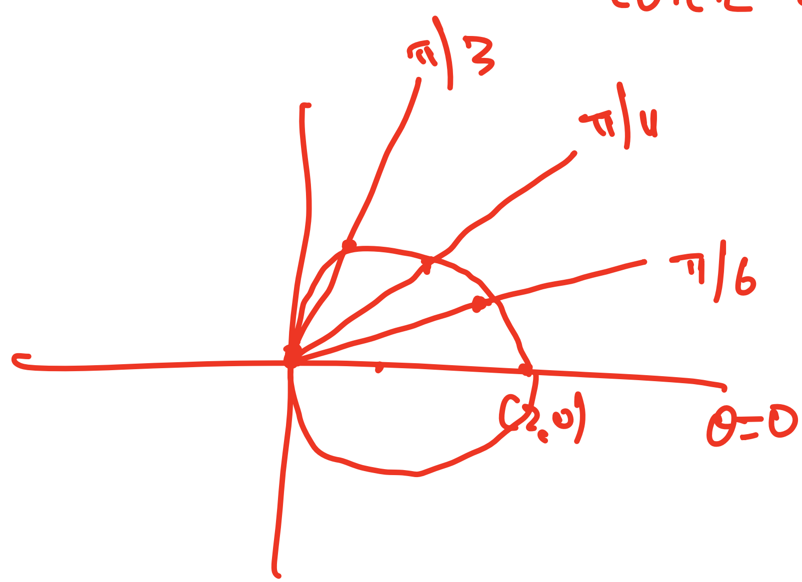
$$r^2 = 2r\cos\theta$$

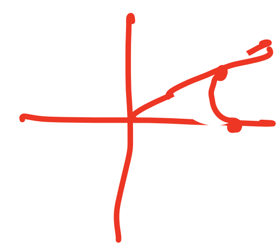
$$x^2 + y^2 = 2x \Rightarrow x^2 - 2x + 1 + y^2 = 1$$

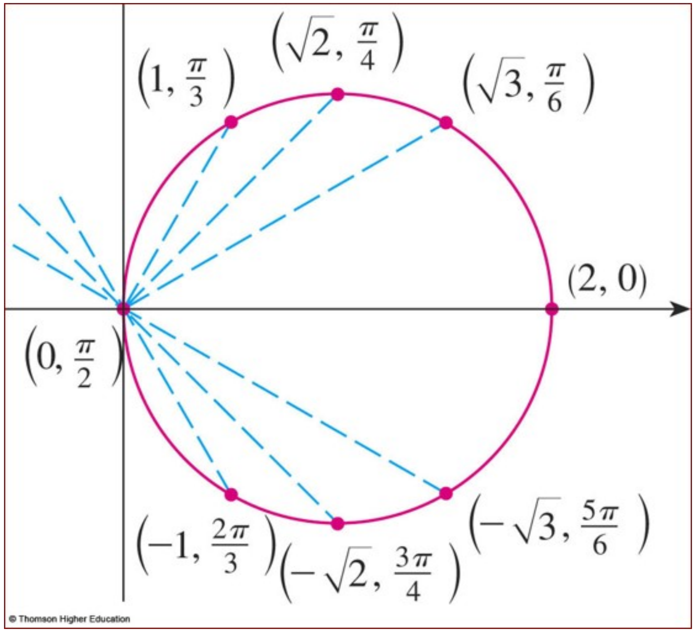
$$(x-1)^2 + y^2 = 1$$

circle of radius 1 centered at (1, 0)

θ	$r = 2\cos\theta$
0	2
$\pi/6$	$\sqrt{3}$
$\pi/4$	$\sqrt{2}$
$\pi/3$	1
$\pi/2$	0
$2\pi/3$	-1
$3\pi/4$	$-\sqrt{2}$
$5\pi/6$	$-\sqrt{3}$
π	-2







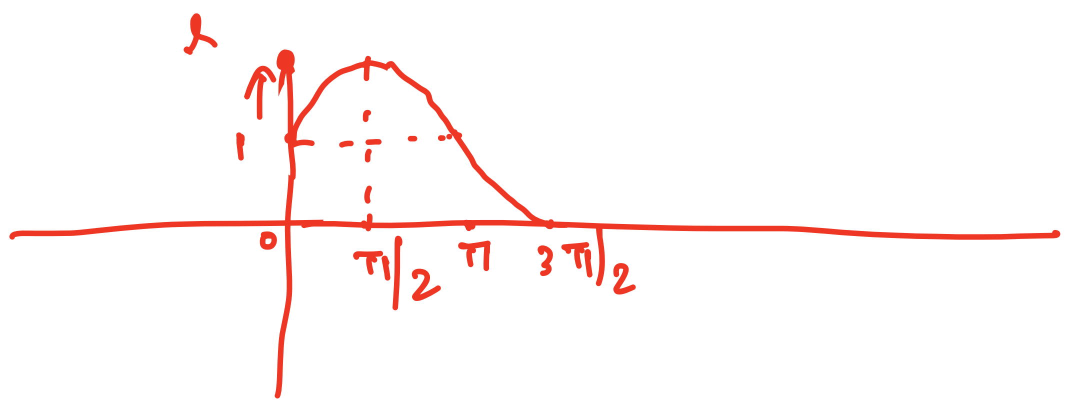
This method can work well if enough points have been plotted to reveal all the loops and dimples in the graph.

Another technique for graphing

- First graph $r = f(\theta)$ in the Cartesian $r\theta$-plane.
- then use the Cartesian graph as a "table" and guide to sketch the polar coordinate graph

Example. Sketch the curve $r = 1 + \sin\theta$.

Step 1. We first sketch the graph of $r = 1 + \sin\theta$ in Cartesian coordinates. This enables us to read at a glance the values of r that correspond to increasing values of θ



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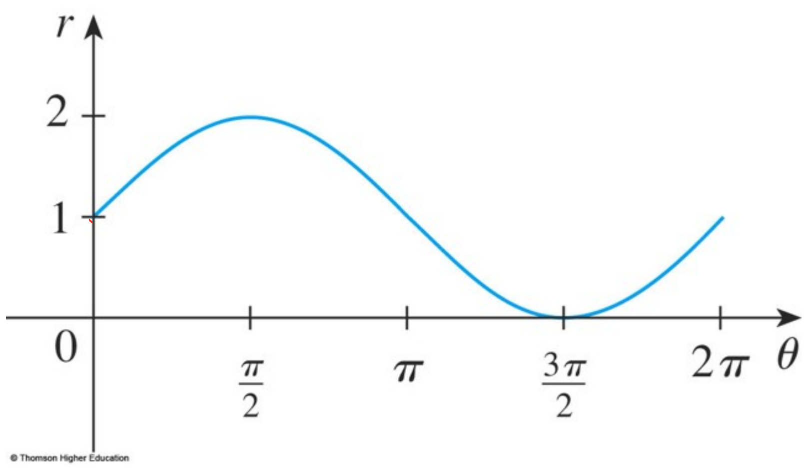
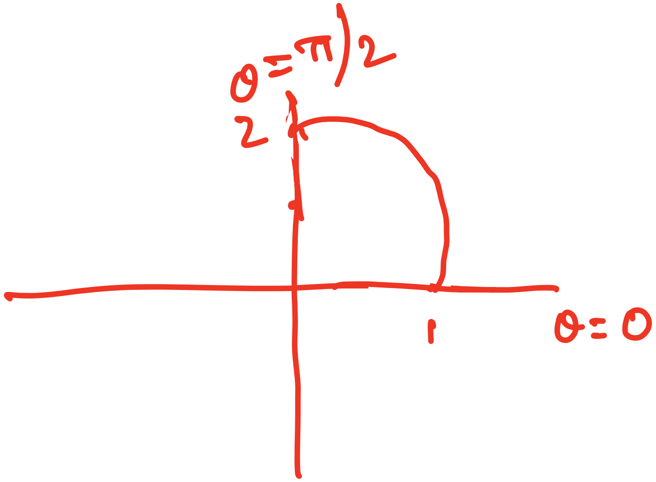


Step 2. We plot the curve in the polar graph

- We see that, as θ increases from 0 to $\frac{\pi}{2}$, r (the distance from O) increases from 1 to 2. So, we sketch the corresponding part of the polar curve.



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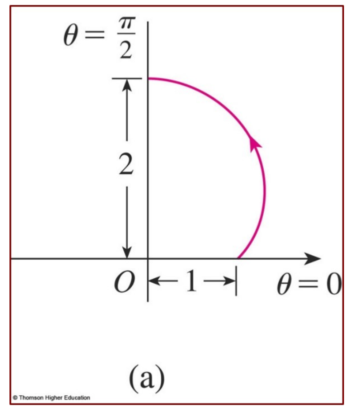
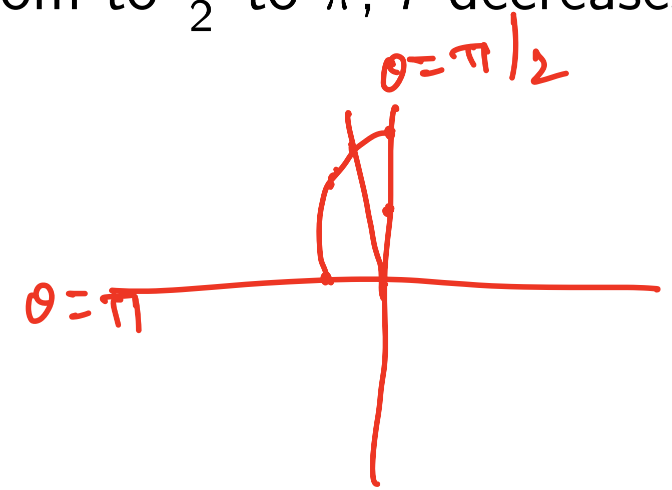


- As θ increases from to $\frac{\pi}{2}$ to π, r decreases from 2 to 1.



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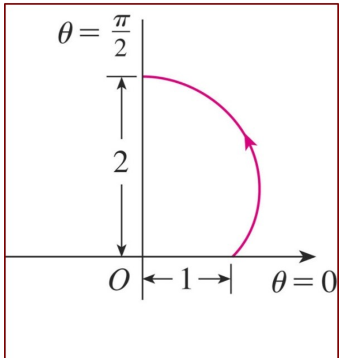
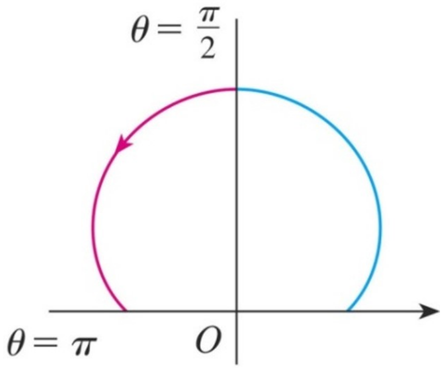


(a)

- As θ increases from to $\frac{\pi}{2}$ to π, r decreases from 2 to 1.

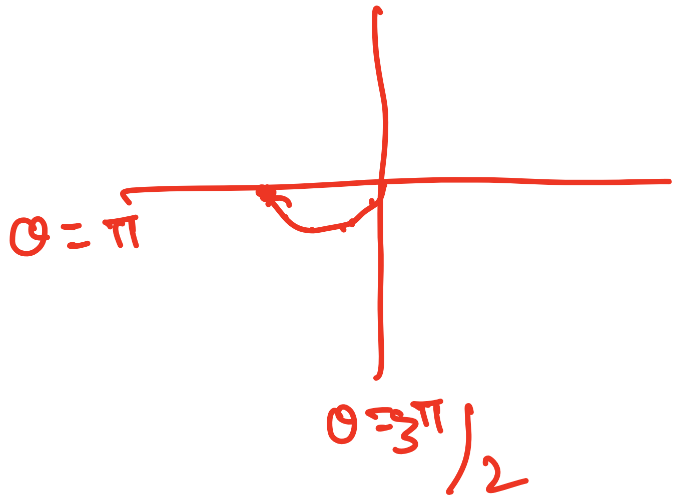


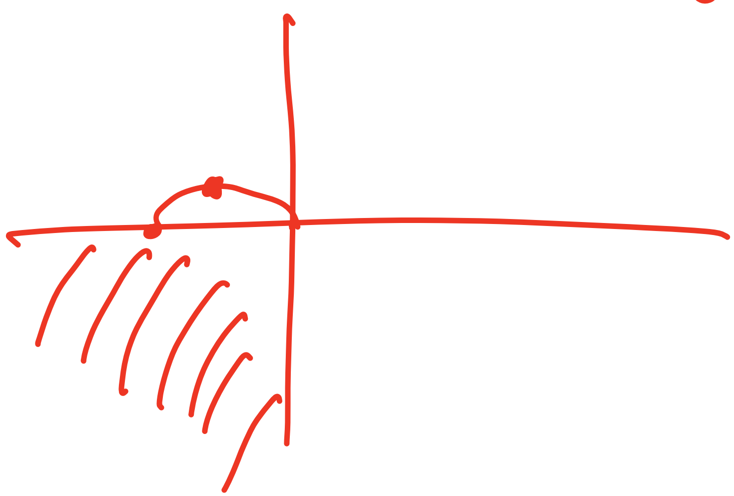
(b)

- As θ increases from to π to $\frac{3\pi}{2}$, r decreases from 1 to 0.

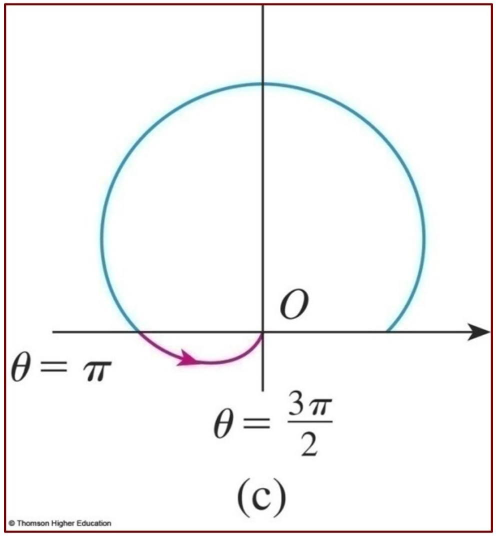
$\pi \leq \theta \leq 3\pi/2$

$r \downarrow 1 - 0$





- As θ increases from to π to $\frac{3\pi}{2}$, r decreases from 1 to 0.



- Finally, as θ increases from $\frac{3\pi}{2}$ to 2π, r increases from 0 to 1.