MATHEMATICS-I

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Lecture 6

Infinite Sequences

Convergence and divergence

- The sequence $\{a_n\}$ **converges** to the number L if for every positive number ε there corresponds an integer N such that for all n,

$$n > N \qquad \Rightarrow \qquad |a_n - L| < \varepsilon.$$

- Here we call L as limit of the sequence. If no such number L exists, we say that $\{a_n\}$ **diverges**.
- If $\{a_n\}$ converges to L, we write $\lim_{n\to\infty} a_n = L$, or simply $a_n \to L$ as $n \to \infty$. We call L as the **limit** of the sequence $\{a_n\}$.

Examples

1. Show that (a) $\lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$; (b) $\lim_{n\to\infty} k = k$ (k is any real constant).

   Part (a): To show that $\lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$, for given $\varepsilon > 0$, we need to find an $N \in \mathbb{N}$ such that

   $$|1/\sqrt{n} - 0| < \varepsilon \text{ for every } n > N.$$

   It is clear that $|1/\sqrt{n} - 0| = 1/\sqrt{n} < \varepsilon$ if $n > 1/\varepsilon^2$. Hence we choose $N = [1/\varepsilon^2] + 1$.

Examples

1. Show that the sequence $\{(-1)^n\} = \{-1, 1, -1, \ldots (-1)^n, \ldots\}$ diverges.

   Proof: We will prove that the limit does not exist. Suppose the limit exists and let the limit be L.

   For $\varepsilon = 1/2 > 0$, by definition, there is a positive integer N such that for all $n > N$ we have that

   $$\left|(-1)^n - L\right| < \frac{1}{2} \iff (-1)^n \in (L - \frac{1}{2}, L + \frac{1}{2})$$

   which is impossible as the interval $(L - \frac{1}{2}, L + \frac{1}{2})$ contains at most one of -1 and 1.

Diverging to $+\infty$

What can we say about the convergence of $\{n^2\}$?

- We say that the sequence $\{a_n\}$ **diverges to infinity** if for every number M there corresponds a positive integer N such that

$$a_n > M \quad \text{for all} \quad n > N.$$

- If this case we write

$$\lim_{n\to\infty} a_n = \infty \quad \text{or simply} \quad a_n \to \infty.$$

Diverging to $-\infty$

What can we say about the convergence of $\{-n\}$?

- We say that the sequence $\{a_n\}$ **diverges to negative infinity** if for every number m there is a positive integer N such that

$$a_n < m \quad \text{for all} \quad n > N$$

and in this case we write

$$\lim_{n \to -\infty} a_n = \infty \quad \text{or simply} \quad a_n \to -\infty.$$

Examples

1. Show that (a) $\lim_{n \to \infty} \sqrt{n} = \infty$; (b) $\lim_{n \to \infty} \frac{1 - n^3}{n^2} = -\infty$.

Theorem 0.1.

A convergent sequence $\{a_n\}$ *has a unique limit.*

1. Find the limits:

(a) $\lim_{n\to\infty}(\frac{n-1}{n})$; **Ans :** 1

(b) $\lim_{n\to\infty}\left(1+\frac{5}{n^2}\right)\left(\frac{1}{2^n}+4\right)$. **Ans :** 4

(c) $\lim_{n\to\infty}\sin(n\pi/2)$

Theorem 0.2.

Let $\{a_n\}$ and $\{b_n\}$ be sequences of real numbers, and let A and B be real numbers. If $\lim_{n\to\infty} a_n = A$ and $\lim_{n\to\infty} b_n = B$, then the following rules holds:

1. $\lim_{n\to\infty}(a_n + b_n) = A + B$
2. $\lim_{n\to\infty}(a_n - b_n) = A - B$
3. $\lim_{n\to\infty}(k \cdot a_n) = k \cdot A$
4. $\lim_{n\to\infty}(a_n \cdot b_n) = A \cdot B$
5. $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{A}{B}$; *if* $B \neq 0$

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   (b) $\lim_{n\to\infty} \frac{10 + n^2}{5 - n^2}$;

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   (b) $\lim_{n\to\infty}\frac{10+n^2}{5-n^2}$;
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   (c) $\lim_{n\to\infty}\left(\sqrt{n^2+5n}-n\right)$.

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   (c) $\lim_{n\to\infty}\left(\sqrt{n^2+5n}-n\right)$.
   Ans : $5/2$

   (d) $\lim_{n\to\infty}\left(\frac{\cos n}{n}\right)$.

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Ans : $5/2$

(d) $\lim_{n\to\infty}\left(\frac{\cos n}{n}\right)$. **Ans :** 0

Theorem 0.3 (Squeeze theorem).

Let $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ be sequences of real numbers. If $a_n \leq b_n \leq c_n$ holds for all n beyond some index N and if $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$, then $\lim_{n\to\infty} b_n = L$.

Examples

1. Discuss the convergence of the following sequences: (a) $\left\{\frac{\cos n^2}{n}\right\}$, (b) $\left\{\frac{\sin(n\pi/2)}{n}\right\}$ (c) $\left\{\frac{n}{2^n}\right\}$.

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2. What about the sequence $\left\{\log\left(\frac{1+n}{n}\right)\right\}$?

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2. What about the sequence $\left\{\log\left(\dfrac{1+n}{n}\right)\right\}$? **Ans :** It converges to 0.

Theorem 0.4.

Let $\{a_n\}$ be a sequence of real numbers. If $a_n \to L$ and if f is a function that is continuous at L and defined at all a_n, then $f(a_n) \to f(L)$.

Examples

1. Discuss the convergence of the following sequences:

   (a) $\left\{\sin\left(\frac{1+n}{n^2}\right)\right\}$, (b) $\left\{\sqrt{\frac{n+1}{n}}\right\}$ (c) $\left\{e^{\left(\frac{2n^2+3}{n^3+5n+6}\right)}\right\}$.

Theorem 0.5.

Suppose that $f(x)$ is a continuous function defined for all $x \geq n_0$ and $\{a_n\}$ is a sequence of real numbers such that $a_n = f(n)$ for $n \geq n_0$. Then

$$\lim_{x\to\infty} f(x) = L \quad \Rightarrow \lim_{n\to\infty} a_n = L.$$

Examples

1. Discuss the convergence of the following sequences:
   (a) $\left\{\frac{\ln n}{n}\right\}$, (b) $\left\{\left(\frac{n-1}{n}\right)^n\right\}$.