Arc Length and Unit Tangent of a Curve

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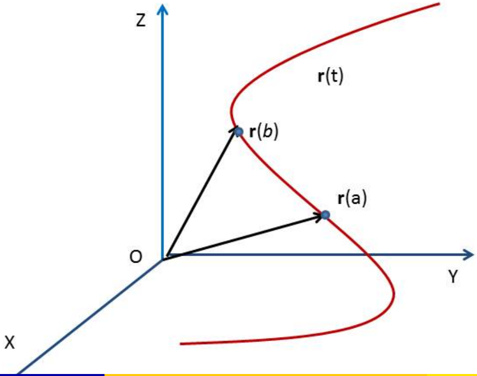
Lecture 16

Arc Length Along a Space Curve

In the case of polar curves how we calculate the length of the curve, we can also calculate the length of a space curve from any smooth parametrization of the curve.



Arc Length

Definition 0.1.

The length of a smooth curve $\mathbf{r}(t) = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k},\ a \leq t \leq b$, that is traced exactly once as t increases from $t = a$ to $t = b$, is

$$\int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt.$$

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The integrant in the above formula is $|\mathbf{v}(t)|$, therefore, the formula for length a shorter way.

$$L = \int_a^b |\mathbf{v}(t)|\ dt$$

Example

- The length of the curve $\mathbf{r}(t) = (2+t)\hat{i} - (t+1)\hat{j} + t\hat{k}$, $0 \leq t \leq 3$ is

$$\int_0^3 |\mathbf{v}(t)|\, dt = \int_0^3 |\hat{i} - \hat{j} + \hat{k}|\, dt = \int_0^3 \sqrt{3}\, dt = 3\sqrt{3}.$$

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- Find the length of the curve

$$\mathbf{r}(t) = \cos^3 t\hat{i} + \sin^3 t\hat{j} + \frac{3}{2}\sin^2 t\hat{k}, \quad 0 \le t \le \pi/2.$$

Arc Length Parameter

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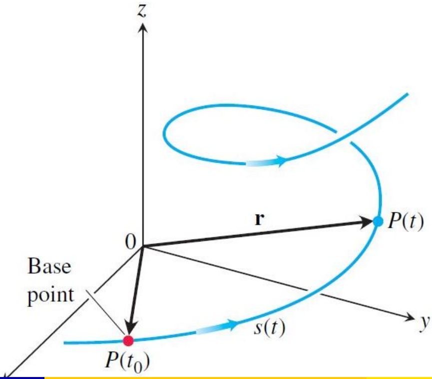
- Now we are interested to find the lenght of the curve from a base point $P(t_0) = \mathbf{r}(t_0)$ on the curve C.



Arc Length Parameter

- The "directed" distance of any point $\mathbf{r}(t)$ from the base point $\mathbf{r}(t_0)$ along the curve is defined by

$$s(t) = \int_{t_0}^{t} |\mathbf{v}(\tau)| \, d\tau.$$

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- Here $s(t)$ is called arc length function, if $t > t_0$, $s(t)$ is positive, the distance along the curve from $P(t_0)$ to $P(t)$. If $t < t_0$, $s(t)$ is negative of the distance.

Arc Length Parameter

- Different values of s (directed distance from base point) determines different points on C and vice versa, hence we can parametrize the curve C with respect to s.

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- Arc Length Function(Parameter) with Base Point $P(t_0)$:

$$s(t) = \int_{t_0}^{t} |\mathbf{v}(\tau)| \, d\tau = \int_{t_0}^{t} \sqrt{[x'(\tau)]^2 + [y'(\tau)]^2 + [z'(\tau)]^2} \, d\tau$$

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- We will see that the arc length parameter is particularly effective for investigating the turning and twisting nature of a space curve.

Arc Length Parameter

Let C be the curve given by $\mathbf{r}(t) = (\cos t)\hat{i} + (\sin t)\hat{j} + t\hat{k}$ and s is any real number. Find a point on C whose directed distance from $\mathbf{r}(0)$ is s.

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Let $\mathbf{r}(t)$ be the required point, the distance from $\mathbf{r}(0)$ to this point along the curve is given by

$$s = \int_0^t |\mathbf{v}(\tau)|\, d\tau = \int_0^t \sqrt{2}\, d\tau = \sqrt{2}t.$$

Arc Length Parameter

We see that

$$t = s/\sqrt{2},$$

hence the required point is

$$\mathbf{r}(s/\sqrt{2}) = \cos\left(s/\sqrt{2}\right)\hat{i} + \sin\left(s/\sqrt{2}\right)\hat{j} + \left(s/\sqrt{2}\right)\hat{k}.$$

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$$\mathbf{r}(s/\sqrt{2}) = \cos(s/\sqrt{2})\hat{i} + \sin(s/\sqrt{2})\hat{j} + (s/\sqrt{2})\hat{k}.$$

If we want a point on the curve which is at distance $\pi/\sqrt{2}$ from the base point $\mathbf{r}(0)$, then the substitution of $s = \pi/\sqrt{2}$ in the above gives the point

$$0\ \hat{i} + \hat{j} + (\pi/2)\hat{k}.$$

Examples

Let $\mathbf{r}(t) = (e^t \cos t)\hat{i} + (e^t \sin t)\hat{j} + e^t \hat{k}$, then find the following:

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2. Arc length parametrization of the curve with the same base point.

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Let $\mathbf{r}(t) = (e^t \cos t)\hat{i} + (e^t \sin t)\hat{j} + e^t\hat{k}$, then find the following:

1. The arc length parameter with base point $\mathbf{r}(0)$,
2. Arc length parametrization of the curve with the same base point.
3. The point on the curve which is at distance $\sqrt{3}(e^{\pi/2} - 1)$ from the base point.

Remark 0.2.

Let $\mathbf{r}(t) = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}$ be a smooth parametrization of a curve C. Then the arc lenght parameter with base point $\mathbf{r}(t_0)$ is given by

$$s(t) = \int_{t_0}^{t} |\mathbf{v}(\tau)| \, d\tau.$$

Speed

Remark 0.2.

Let $\mathbf{r}(t) = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}$ be a smooth parametrization of a curve C. Then the arc lenght parameter with base point $\mathbf{r}(t_0)$ is given by

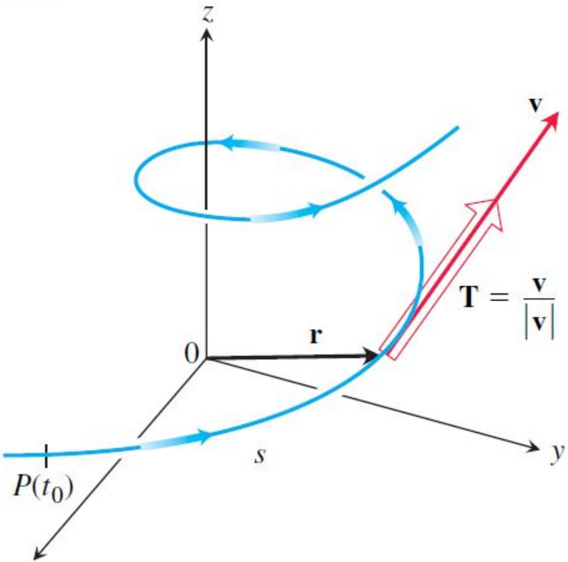
$$s(t) = \int_{t_0}^{t} |\mathbf{v}(\tau)| \, d\tau.$$

Clearly, we have

$$\frac{ds}{dt} = |\mathbf{v}(t)| > 0,$$

which is speed of the particle with displacement $\mathbf{r}(t)$.

Unit Tangent Vector



Unit Tangent Vector

We already know that the velocity vector $\mathbf{v} = d\mathbf{r}/dt$ is tangent to the curve $\mathbf{r}(t)$ and the vector

$$\mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|}$$

is therefore a unit vector tangent to the curve, called the **unit tangent vector.**

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Remark 0.3.

$$\mathbf{T} = \frac{d\mathbf{r}}{ds}. \tag{0.1}$$

Since, $\dfrac{d\mathbf{r}}{ds} = \dfrac{d\mathbf{r}}{dt}\,\dfrac{dt}{ds} = \dfrac{d\mathbf{r}}{dt}\,\dfrac{1}{ds/dt} = \dfrac{\mathbf{v}}{|\mathbf{v}|} = \mathbf{T}$.

Examples

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Let $\mathbf{r}(t) = (e^t \cos t)\hat{i} + (e^t \sin t)\hat{j} + e^t\hat{k}$, then find the following:

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2. Arc length parametrization of the curve with the same base point.

Examples

Let $\mathbf{r}(t) = (e^t \cos t)\hat{i} + (e^t \sin t)\hat{j} + e^t\hat{k}$, then find the following:

1. The arc length parameter with base point $\mathbf{r}(0)$ and speed,
2. Arc length parametrization of the curve with the same base point.
3. The unit tangent of the curve.