Vector Valued Functions and Motion in Space

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Recall

Let $\mathbf{r}(t)$ be a smooth curve in space, and if s is the arc length parameter of the curve, then:

1. The unit tangent vector $\mathbf{T}$ is $d\mathbf{r}/ds = \underline{\mathbf{v}/|\mathbf{v}|}$. ✓

   $\because |T| = 1$

   $\frac{dT}{ds} \cdot T = 0$ ✓

2. The curvature in space is then defined to be

$$\kappa = \left|\frac{d\mathbf{T}}{ds}\right| = \frac{1}{|\mathbf{v}|}\left|\frac{d\mathbf{T}}{dt}\right|$$

3. The principal unit normal to be

$$\mathbf{N} = \frac{1}{\kappa}\frac{d\mathbf{T}}{ds} = \frac{d\mathbf{T}/dt}{|d\mathbf{T}/dt|}.$$

Application 1: Circle of curvature for plane curves

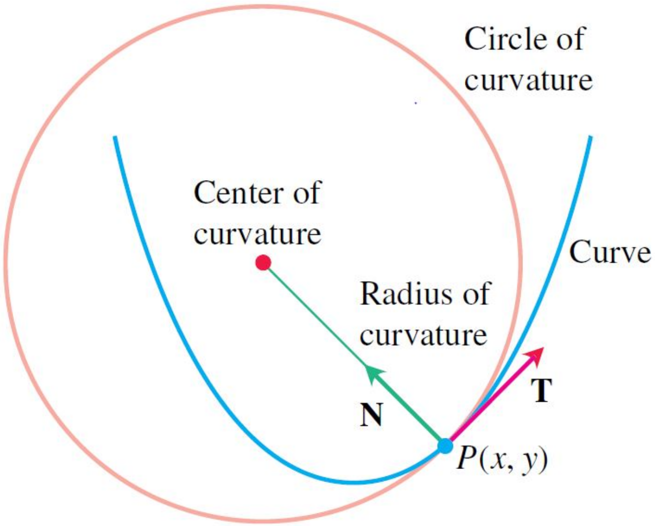
The **circle of curvature** or **osculating circle** at a point P on a plane curve where $\kappa \neq 0$ is the circle in the plane of the curve that

1. is tangent to the curve at P (has the same tangent line the curve has)
2. has the same curvature the curve has at P
3. has center that lies toward the concave or inner side of the curve

curvature of circle $= \frac{1}{\text{radius}}$
$=$ curvature of curve

$\Rightarrow \frac{1}{\text{radius}} =$ curvature of curve

$\Rightarrow$ radius of circle $= 1/k$



Radius and center of curvature

Definition 0.1.

The **radius of curvature** of the curve at P is the radius of the circle of curvature, which is

$$\text{Radius of curvature} = \rho = \frac{1}{\kappa}.$$

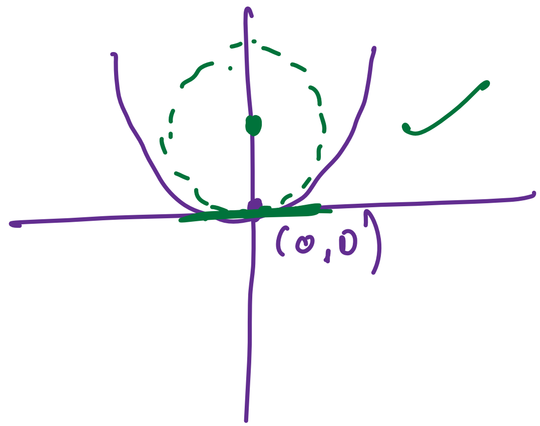
The **center of curvature** of the curve at P is the center of the circle of curvature.

Example

Find and graph the osculating circle of the parabola $y = x^2$ at the origin.

$r(t) = (t, t^2)$

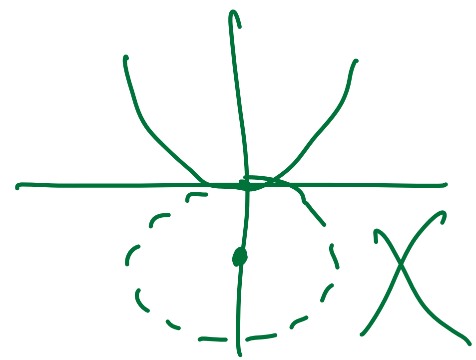
$t = 0, \quad r(0) = (0,0)$



$v(t) = (1, 2t)$

$|v(t)| = \sqrt{1+4t^2}$

$T = \frac{1}{\sqrt{1+4t^2}}(1, 2t)$

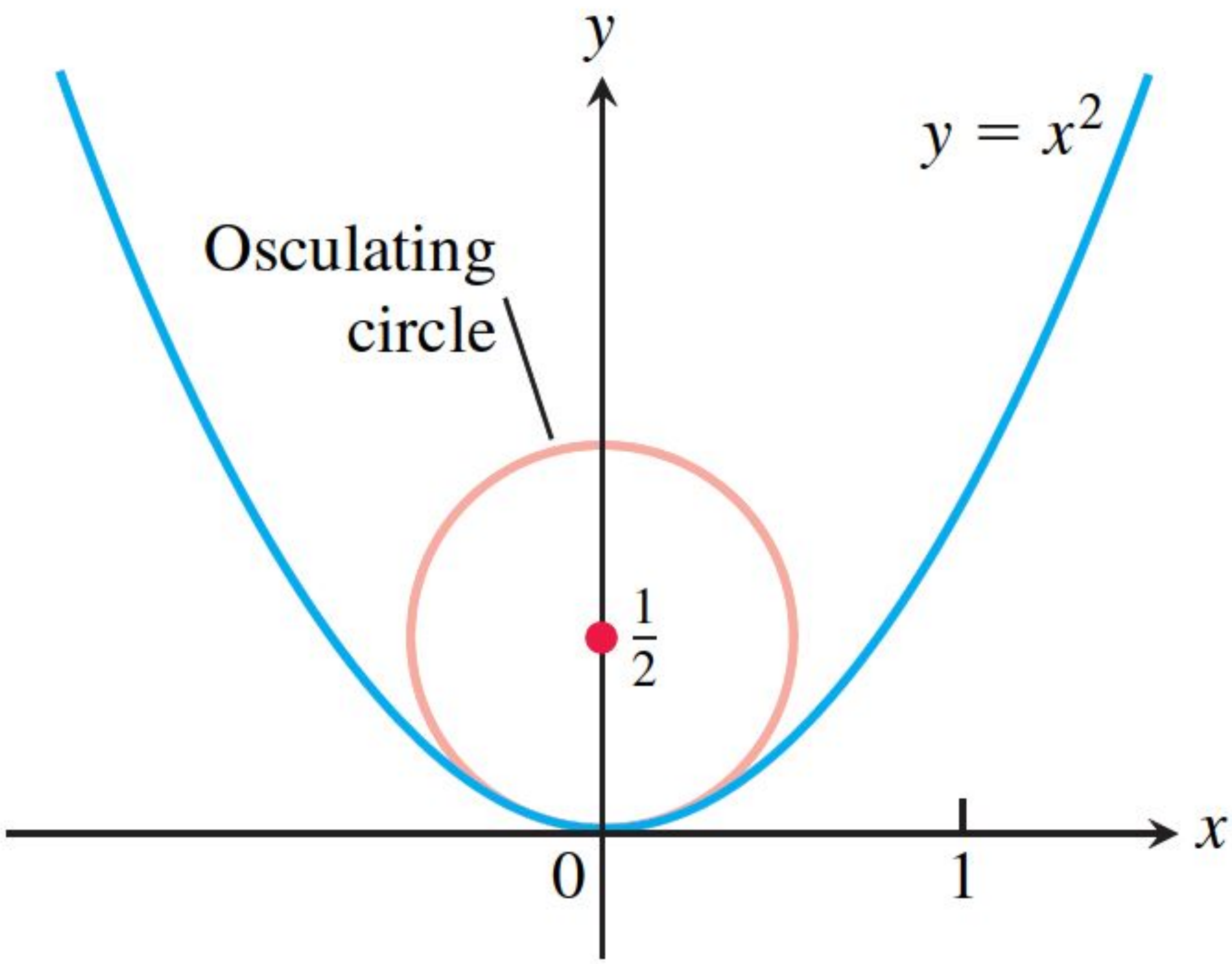


$$\kappa = \left.\frac{2}{1+4t^2}\right|_{(0,0)} = 2$$

$\Rightarrow$ radius of circle of curvature $= \frac{1}{2}$

Example

Find and graph the osculating circle of the parabola $y = x^2$ at the origin.



Binormal and TNB frame

Definition 0.2.

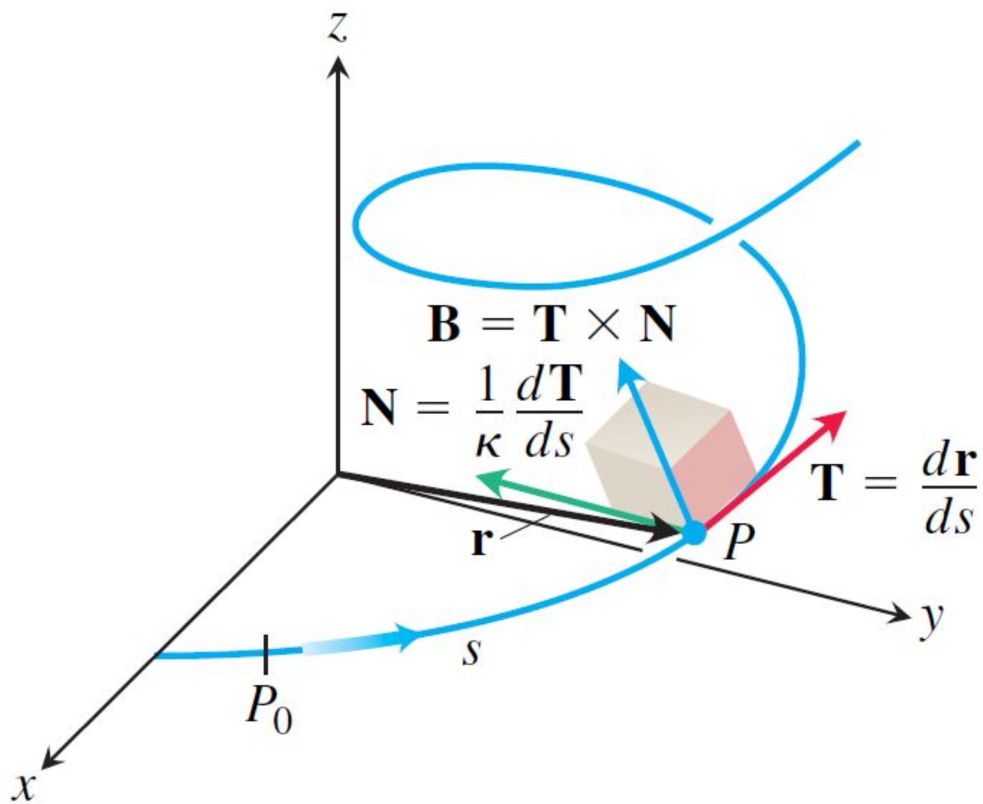
The **binormal vector** of a curve is define by

$$\mathbf{B} = \mathbf{T} \times \mathbf{N},$$

where $\mathbf{T}$ is the unit tangent and $\mathbf{N}$ is the unit normal vector of the curve.

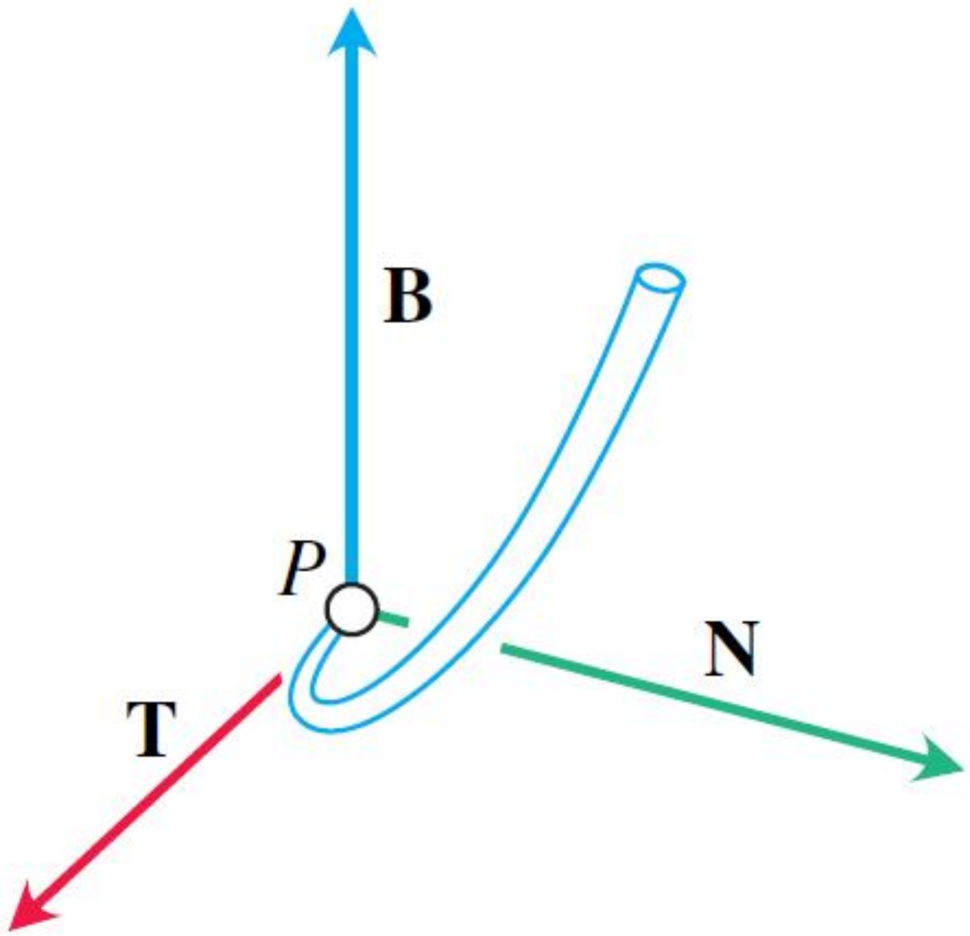
- Binormal vector $\mathbf{B}$ is a unit vector orthogonal to both $\mathbf{T}$ and $\mathbf{N}$.
- Together $\mathbf{T}$, $\mathbf{N}$ and $\mathbf{B}$ define a moving right-handed vector frame.

Binormal and TNB frame



- It is called **Frenet** ("fre-*nay*") **frame** (after Jean-Frederic Frenet) or the **TNB frame**.

Binormal and TNB frame



Example

Find the binormal **B** to the helix given by

$$\mathbf{r}(t) = (a\cos t)\mathbf{i} + (a\sin t)\mathbf{j} + (bt)\mathbf{k}, \;\; a, b > 0, \;\; a^2 + b^2 \neq 0.$$

Solution.

We already found that

$$\mathbf{T} = \frac{1}{\sqrt{a^2+b^2}}[-(a\sin t)\mathbf{i} + (a\cos t)\mathbf{j} + b\mathbf{k}] \;\; \text{and}$$

$$\mathbf{N} = -(\cos t)\mathbf{i} - (\sin t)\mathbf{j}.$$

$$B = T \times N = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{-a\sin t}{\sqrt{a^2+b^2}} & \frac{a\cos t}{\sqrt{a^2+b^2}} & \frac{b}{\sqrt{a^2+b^2}} \\ -\cos t & -\sin t & 0 \end{vmatrix} = \hat{i}\left(\frac{a\cos t\sin t}{\sqrt{a^2+b^2}}\right) - \hat{j}\,\frac{b\cos t}{\sqrt{a^2+b^2}} + \hat{k}\left(\_\_\_\_ \right)$$

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We already found that

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$$\mathbf{N} = -(\cos t)\mathbf{i} - (\sin t)\mathbf{j}.$$

Therefore,

$$\mathbf{B} = \mathbf{T} \times \mathbf{N} = \frac{1}{\sqrt{a^2+b^2}}[(b\sin t)\mathbf{i} - (b\cos t)\mathbf{j} + a\mathbf{k}].$$

Application 2: Tangent and Normal Components of Acceleration

Observe that,

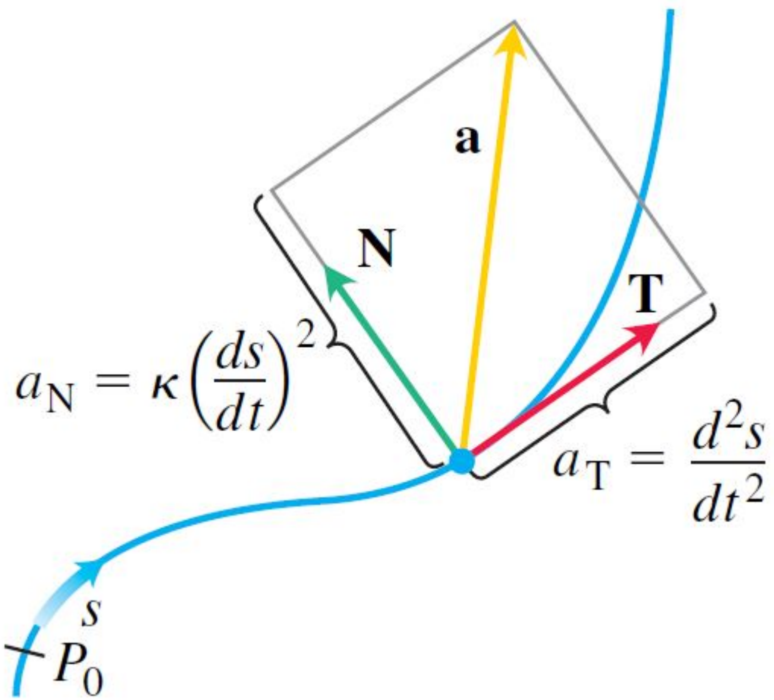
$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \frac{d\mathbf{r}}{ds}\frac{ds}{dt} = \mathbf{T}\frac{ds}{dt}.$$

Then we differentiate both ends of the above equation to get

$$\begin{aligned}
\mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{d}{dt}\left(\mathbf{T}\frac{ds}{dt}\right) &= \frac{d^2s}{dt^2}\mathbf{T} + \frac{ds}{dt}\frac{d\mathbf{T}}{dt} \\
&= \frac{d^2s}{dt^2}\mathbf{T} + \frac{ds}{dt}\left(\frac{d\mathbf{T}}{ds}\frac{ds}{dt}\right) \\
&= \frac{d^2s}{dt^2}\mathbf{T} + \frac{ds}{dt}\left(\kappa\mathbf{N}\frac{ds}{dt}\right), \quad \text{since } \mathbf{N} = \frac{1}{\kappa}\frac{d\mathbf{T}}{ds} \\
&= \left(\frac{d^2s}{dt^2}\right)\mathbf{T} + \kappa\left(\frac{ds}{dt}\right)^2\mathbf{N}.
\end{aligned}$$

Tangent and Normal Components of Acceleration

In practical situations, it is important to know that how much of acceleration acts in the direction of motion, in tangential direction **T**.



If the acceleration vector is written as

$$\mathbf{a} = a_{\mathbf{T}}\mathbf{T} + a_{\mathbf{N}}\mathbf{N},$$

then

$$\underline{a_{\mathbf{T}} = \frac{d^2s}{dt^2} = \frac{d}{dt}|\mathbf{v}|} \quad \text{and} \quad a_{\mathbf{N}} = \kappa\left(\frac{ds}{dt}\right)^2 = \kappa|\mathbf{v}|^2$$

are tangential and normal scalar components of acceleration.

$$a \cdot a = |a|^2 = (a_T T + a_N N)^2$$

$$= a_T^2 \, T \cdot T + a_N^2 \, N \cdot N + a_T a_N \, \underset{\overset{\shortparallel}{0}}{T \cdot N}$$

$$a^2 = a_T^2 + a_N^2$$

$$\Rightarrow \quad a_N = \sqrt{a^2 - a_T^2}$$

If the acceleration vector is written as

$$\mathbf{a} = a_{\mathbf{T}}\mathbf{T} + a_{\mathbf{N}}\mathbf{N},$$

then

$$a_{\mathbf{T}} = \frac{d^2s}{dt^2} = \frac{d}{dt}|\mathbf{v}| \quad \text{and} \quad a_{\mathbf{N}} = \kappa\left(\frac{ds}{dt}\right)^2 = \kappa|\mathbf{v}|^2$$

are tangential and normal scalar components of acceleration.

Formula for calculation the normal component of acceleration:

$$a_{\mathbf{N}} = \sqrt{|\mathbf{a}|^2 - a_{\mathbf{T}}^2},$$

where $a_{\mathbf{T}} = \frac{d^2s}{dt^2} = \frac{d}{dt}|\mathbf{v}|$, tangential component of acceleration.

Examples

Without finding **T** and **N**, write the acceleration of the motion $\mathbf{r}(t) = (\cos t + t\sin t)\mathbf{i} + (\sin t - t\cos t)\mathbf{j},\ t > 0$ in the form $\mathbf{a} = a_{\mathbf{T}}\mathbf{T} + a_{\mathbf{N}}\mathbf{N}$.

$$v(t) = (-\sin t + \sin t + t\cos t)\hat{i} + (\cos t - \cos t + t\sin t)\hat{j}$$

$$= t\cos t\,\hat{i} + t\sin t\,\hat{j}$$

$$a_T = \frac{d}{dt}|v|$$

$$|v| = t \quad , \quad a_T = 1 \Rightarrow a_T^2 = 1$$

$$a(t) = (-t\sin t + \cos t)\hat{i} + (\sin t + t\cos t)\hat{j}$$

$$a^2 = 1 + t^2$$

$$a_N = \sqrt{1 + t^2 - 1} = t$$

$$a = 1T + tN$$

Examples

Without finding $\mathbf{T}$ and $\mathbf{N}$, write the acceleration of the motion $\mathbf{r}(t) = (\cos t + t\sin t)\mathbf{i} + (\sin t - t\cos t)\mathbf{j},\ t > 0$ in the form $\mathbf{a} = a_{\mathbf{T}}\mathbf{T} + a_{\mathbf{N}}\mathbf{N}$.

Solution.

We first find $a_{\mathbf{T}}$ by using the formula $a_{\mathbf{T}} = \frac{d}{dt}|\mathbf{v}|$.

$\mathbf{v} = r'(t) = (t\cos t)\mathbf{i} + (t\sin t)\mathbf{j} \Rightarrow |\mathbf{v}| = t$ for $t > 0$.

$$a_{\mathbf{T}} = \frac{d}{dt}|\mathbf{v}| = \frac{d}{dt}(t) = 1.$$

Since $\mathbf{T}\cdot\mathbf{N} = \mathbf{0}$, $|\mathbf{a}|^2 = |a_{\mathbf{T}}|^2 + |a_{\mathbf{N}}|^2$ and hence

$$a_{\mathbf{N}} = \sqrt{|\mathbf{a}|^2 - a_{\mathbf{T}}^2}.$$

Examples

Now we need to find **a**.

$$\mathbf{a} = (\cos t - t\sin t)\mathbf{i} + (\sin t + t\cos t)\mathbf{j} \Rightarrow |\mathbf{a}|^2 = t^2 + 1.$$

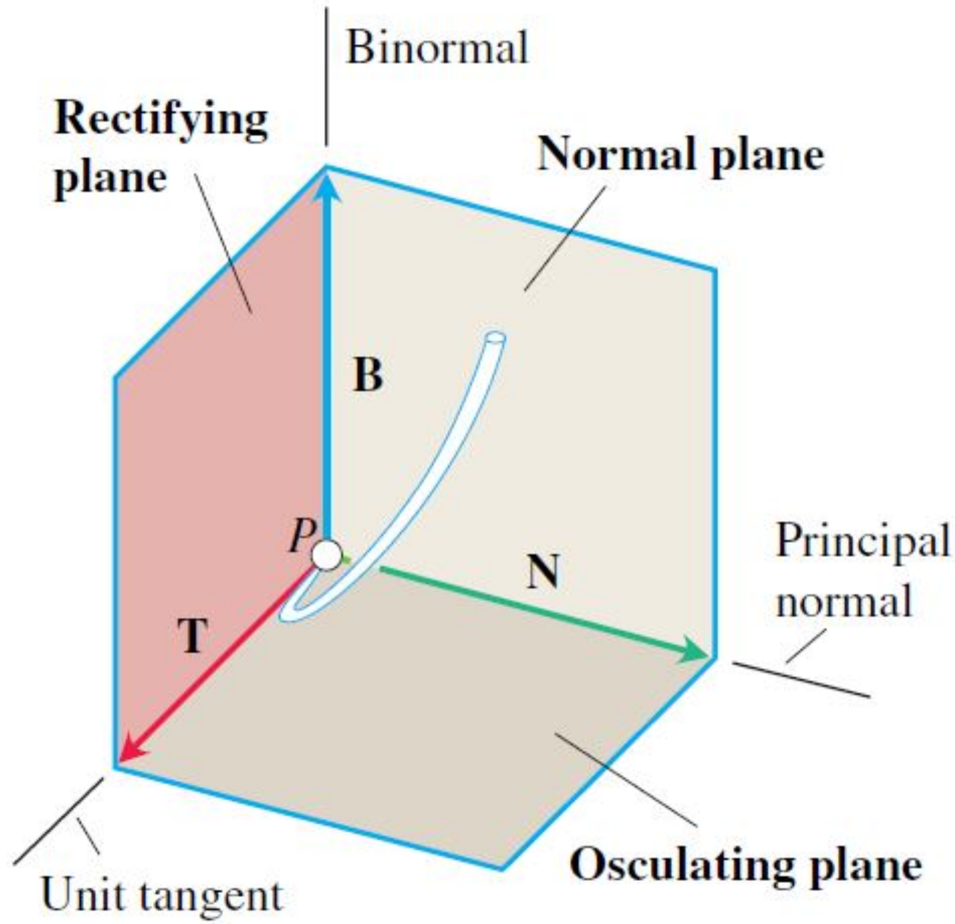
Therefore,

$$a_{\mathbf{N}} = \sqrt{|\mathbf{a}|^2 - a_{\mathbf{T}}^2} = \sqrt{(t^2+1) - 1} = t.$$

The required expression is

$$\mathbf{a} = a_{\mathbf{T}}\mathbf{T} + a_{\mathbf{N}}\mathbf{N} = \mathbf{T} + t\mathbf{N}.$$

Frame and planes determined by $\mathbf{T}, \mathbf{N}$ and $\mathbf{B}$



Frame and planes determined by $\mathbf{T}, \mathbf{N}$ and $\mathbf{B}$

Osculating Plane: The plane containing unit tangent $\mathbf{T}$ and principal normal $\mathbf{N}$.

Rectifying Plane: The plane containing unit tangent $\mathbf{T}$ and binormal $\mathbf{B}$.

Normal Plane: The plane containing principal normal $\mathbf{N}$ and binormal $\mathbf{B}$.

Lets find the relation of $\frac{d\mathbf{B}}{ds}$ with $\mathbf{T}$ and $\mathbf{N}$:

$$\frac{d\mathbf{B}}{ds} = \frac{d(\mathbf{T}\times\mathbf{N})}{ds} = \left(\frac{d\mathbf{T}}{ds}\times\mathbf{N}\right) + \left(\mathbf{T}\times\frac{d\mathbf{N}}{ds}\right)$$
$$= \mathbf{0} + \mathbf{T}\times\frac{d\mathbf{N}}{ds}$$
$$= \mathbf{T}\times\frac{d\mathbf{N}}{ds}.$$

Recall
if $r(t)$ is a curve with constant length then $\frac{dr}{dt}\cdot r = 0$

1) B is unit vector, $|B| = 1 \Rightarrow \frac{dB}{ds}\cdot B = 0$

$\Rightarrow \frac{dB}{ds} \perp B$

2) $\frac{dB}{ds} \perp T$

$\Rightarrow \frac{dB}{ds} \parallel N$

Torsion

Lets find the relation of $\frac{d\mathbf{B}}{ds}$ with $\mathbf{T}$ and $\mathbf{N}$:

$$\frac{d\mathbf{B}}{ds} = \frac{d(\mathbf{T} \times \mathbf{N})}{ds} = \left(\frac{d\mathbf{T}}{ds} \times \mathbf{N}\right) + \left(\mathbf{T} \times \frac{d\mathbf{N}}{ds}\right)$$
$$= \mathbf{0} + \mathbf{T} \times \frac{d\mathbf{N}}{ds}$$
$$= \mathbf{T} \times \frac{d\mathbf{N}}{ds}.$$

- It is clear that $\frac{d\mathbf{B}}{ds}$ is orthogonal to $\mathbf{T}$ and $\mathbf{B}$, so it is orthogonal to the plane containg $\mathbf{T}$ and $\mathbf{B}$.
- In other words, $\frac{d\mathbf{B}}{ds}$ is parallel to $\mathbf{N}$.
- We can write $\frac{d\mathbf{B}}{ds} = -\tau\mathbf{N}$ for a scalar function τ, called **torsion**.

Definition 0.3 (Torsion τ).

If $\mathbf{r}(t)$ is a smooth curve, then the **torsion** function is defined by

$$\tau = -\frac{d\mathbf{B}}{ds} \cdot \mathbf{N} = -\frac{1}{|\mathbf{v}|}\left(\frac{d\mathbf{B}}{dt} \cdot \mathbf{N}\right).$$

Torsion is the rate at which the osculating plane turns about $\mathbf{T}$ as the point moves along the curve. Torsion measures how the curve twists

Find τ for the helix

$$\mathbf{r}(t) = (a\cos t)\mathbf{i} + (a\sin t)\mathbf{j} + bt\mathbf{k} \quad a, b > 0.$$

Solution.
We already found that

$$\mathbf{T} = \frac{1}{\sqrt{a^2+b^2}}[-(a\sin t)\mathbf{i} + (a\cos t)\mathbf{j} + b\mathbf{k}]$$

$$\mathbf{N} = -(\cos t)\mathbf{i} - (\sin t)\mathbf{j}, \quad \text{and}$$

$$\mathbf{B} = \mathbf{T} \times \mathbf{N} = \frac{1}{\sqrt{a^2+b^2}}[(b\sin t)\mathbf{i} - (b\cos t)\mathbf{j} + a\mathbf{k}].$$

Example

$$|v| = \sqrt{a^2+b^2}$$

$$\frac{dB}{dt} = \frac{b}{\sqrt{a^2+b^2}}\cos t\,\hat{i} + \frac{b}{\sqrt{a^2+b^2}}\sin t\,\hat{j} + 0\hat{k}$$

$$\frac{dB}{dt}\cdot N = -\frac{b}{\sqrt{a^2+b^2}}\cos^2 t - \frac{b}{\sqrt{a^2+b^2}}\sin^2 t = \frac{-b}{\sqrt{a^2+b^2}}$$

$$\tau = -\frac{1}{|v|}\left(\frac{dB}{dt}\cdot N\right)$$

$$= \frac{-1}{\sqrt{a^2+b^2}} \times \left(-\frac{b}{\sqrt{a^2+b^2}}\right) = \frac{b}{a^2+b^2}$$

Example

$$\frac{d\mathbf{B}}{dt} = \frac{1}{\sqrt{a^2+b^2}}[(b\cos t)\mathbf{i} + (b\sin t)\mathbf{j}].$$

and $|\mathbf{v}| = \sqrt{a^2+b^2}$. Therefore,

$$\tau = -\frac{1}{|\mathbf{v}|}\left(\frac{d\mathbf{B}}{dt}\cdot\mathbf{N}\right) = \frac{b}{a^2+b^2}.$$

Formulas for κ and τ without finding $\mathbf{T}$ and $\mathbf{B}$

Remark 0.4 (Vector formula for Curvature).

$$\kappa = \frac{|\mathbf{v} \times \mathbf{a}|}{|\mathbf{v}|^3};$$

Proof. We know that $\mathbf{v} = \dfrac{ds}{dt}\mathbf{T}$ and

$$\mathbf{a} = \Big(\frac{d^2 s}{dt^2}\Big)\mathbf{T} + \kappa\Big(\frac{ds}{dt}\Big)^2\mathbf{N},$$

therefore we have that

$$\begin{aligned}\mathbf{v} \times \mathbf{a} &= \Big(\frac{ds}{dt}\mathbf{T}\Big) \times \Big[\Big(\frac{d^2 s}{dt^2}\Big)\mathbf{T} + \kappa\Big(\frac{ds}{dt}\Big)^2\mathbf{N}\Big] \\ &= \kappa\Big(\frac{ds}{dt}\Big)^3\mathbf{B}, \quad \text{since } \mathbf{T} \times \mathbf{T} = \mathbf{0}, \mathbf{T} \times \mathbf{N} = \mathbf{B}.\end{aligned}$$

Formulas for κ and τ without finding $\mathbf{T}$ and $\mathbf{B}$

It follows that

$$|\mathbf{v} \times \mathbf{a}| = \kappa \left| \frac{ds}{dt} \right|^3 |\mathbf{B}| = \kappa |\mathbf{v}|^3.$$

Hence, we get the formula,

$$\kappa = \frac{|\mathbf{v} \times \mathbf{a}|}{|\mathbf{v}|^3}.$$

Formulas for κ and τ without finding $\mathbf{T}$ and $\mathbf{B}$

Remark 0.5 (Formula for Torsion).

$$\tau = \frac{(\mathbf{v} \times \mathbf{a}) \cdot \mathbf{a}'(t)}{|\mathbf{v} \times \mathbf{a}|^2}; \quad (\text{if } \mathbf{v} \times \mathbf{a} \neq \mathbf{0}).$$

Problems: 1. Find the curvature and the torsion of the curve

$$\mathbf{r}(t) = (\cosh t)\mathbf{i} - (\sinh t)\mathbf{j} + t\mathbf{k}.$$

Solution $\mathbf{v}(t) = (\sinh t)\mathbf{i} - (\cosh t)\mathbf{j} + \mathbf{k}$;

$$\mathbf{a}(t) = (\cosh t)\mathbf{i} - (\sinh t)\mathbf{j}, \mathbf{a}'(t) = (\sinh t)\mathbf{i} - (\cosh t)\mathbf{j}$$

Problems

$$\mathbf{v} \times \mathbf{a} = (\sinh t)\mathbf{i} + (\cosh t)\mathbf{j} + \mathbf{k}$$

$$\kappa = \frac{|\mathbf{v} \times \mathbf{a}|}{|\mathbf{v}|^3} = \frac{1}{\cosh^2 t + \sinh^2 t + 1} = \frac{1}{2\cosh^2 t} \text{ and}$$

$$\tau = \frac{(\mathbf{v} \times \mathbf{a}) \cdot \mathbf{a}'(t)}{|\mathbf{v} \times \mathbf{a}|^2} = \frac{-1}{\cosh^2 t + \sinh^2 t + 1} = \frac{1}{2\cosh^2 t}$$

2. Find the the equations for the osculating, normal, and rectifying planes of the curve

$$\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + t\mathbf{k}, \ \boxed{\text{at } t = 0.}$$

Osculating plane – contains T and N

– it is orthogonal to B

A plane passing through a Point (x_0, y_0, z_0) and $\perp$ to a vector $\vec{n}$ then the eq. is $(x-x_0, y-y_0, z-z_0)\cdot\vec{n} = 0$

Find T, N, B

$T = v/|v|$, $N = \dfrac{dT/dt}{|dT/dt|}$

Normal plane – T is perpendicular vector

$v(t) = -\sin t\,\hat{i} + \cos t\,\hat{j} + \hat{k}$

$|v(t)| = \sqrt{1+1} = \sqrt{2}$

$T = -\frac{1}{\sqrt{2}}\sin t\,\hat{i} + \frac{1}{\sqrt{2}}\cos t\,\hat{j} + \frac{1}{\sqrt{2}}\hat{k}$

$P = (1, 0, 0)$

$T = \left(0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$

Eq. of normal plane $= ((x-1), y, z) \cdot T = 0$

$\Rightarrow \frac{1}{\sqrt{2}} y + \frac{1}{\sqrt{2}} z = 0 \Rightarrow \boxed{y + z = 0}$

3. What can be said about torsion of a smooth plane curve $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j}$?Give reasons for your answer.

4. For the curve $\mathbf{r}(t) = (\cos t + t\sin t)\mathbf{i} + (\sin t - t\cos t)\mathbf{j},\ t > 0$, we already found $a_{\mathbf{N}} = t$ and $|\mathbf{v}| = t$, from these values find the curvature.

Computation Formulas for Curves in Space

1. Unit tangent vector: $\mathbf{T} = \dfrac{\mathbf{v}}{|\mathbf{v}|}$

2. Principal unit normal vector: $\mathbf{N} = \dfrac{d\mathbf{T}/dt}{|d\mathbf{T}/dt|}$

3. Binormal vector: $\mathbf{B} = \mathbf{T} \times \mathbf{N}$

4. Curvature: $\kappa = \left|\dfrac{d\mathbf{T}}{ds}\right| = \dfrac{1}{|\mathbf{v}|}\left|\dfrac{d\mathbf{T}}{dt}\right| = \dfrac{|\mathbf{v} \times \mathbf{a}|}{|\mathbf{v}|^3}$

5. Torsion: $\tau = -\dfrac{d\mathbf{B}}{ds} \cdot \mathbf{N} = -\dfrac{1}{|\mathbf{v}|}\dfrac{d\mathbf{B}}{dt} \cdot \mathbf{N} = \dfrac{(\mathbf{v} \times \mathbf{a}) \cdot \mathbf{a}'(t)}{|\mathbf{v} \times \mathbf{a}|^2}$

(6). Tangential and normal scalar components of accelerations:

$$\begin{aligned} \mathbf{a} &= a_{\mathbf{T}}\mathbf{T} + a_{\mathbf{N}}\mathbf{N}, \\ a_{\mathbf{T}} &= \frac{d}{dt}|\mathbf{v}|, \\ a_{\mathbf{N}} &= \kappa|\mathbf{v}|^2 = \sqrt{|\mathbf{a}|^2 - a_{\mathbf{T}}^2}. \end{aligned}$$