Polar Coordinates

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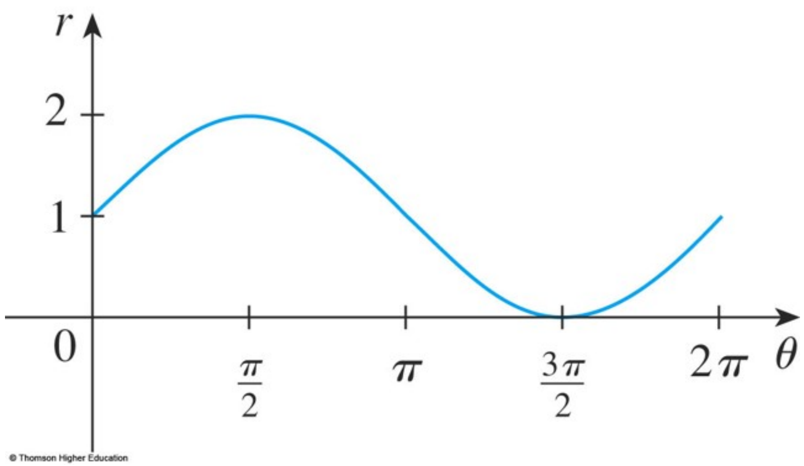
Recall

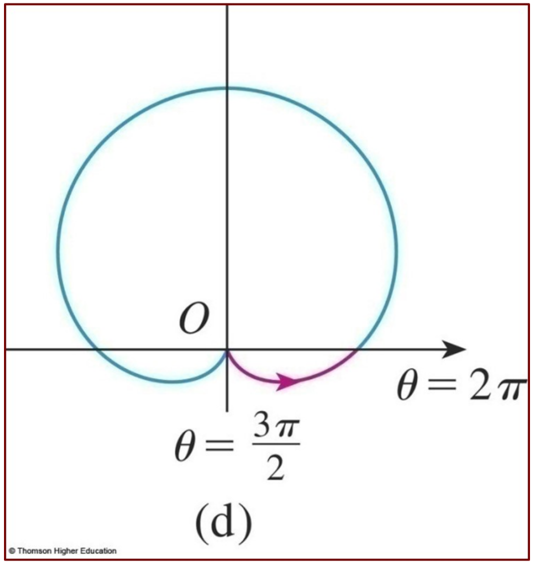
Algorithm to draw a polar curve:

- First graph $r = f(\theta)$ in the Cartesian $r\theta$-plane.
- then use the Cartesian graph as a "table" and guide to sketch the polar coordinate graph

Example. $r = 1 + \sin\theta$

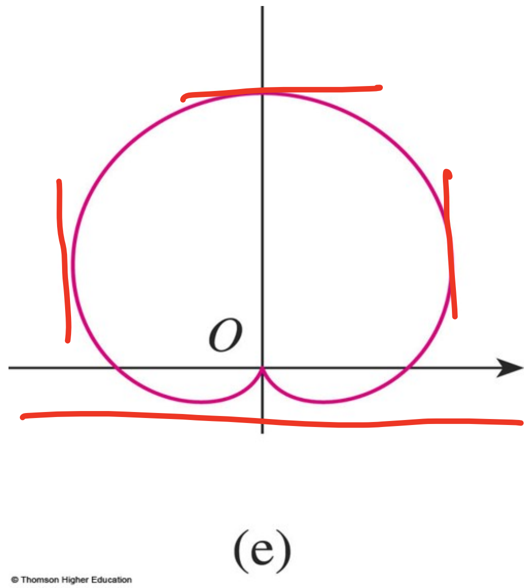


$y = 1 + \sin x$



Cardioids

The polar curve of $r = 1 + \sin\theta$ called a cardioid, because it's shaped like a heart.



(e)

$\frac{7\pi}{6}$

Cardioids are graphs of polar equations having the following form

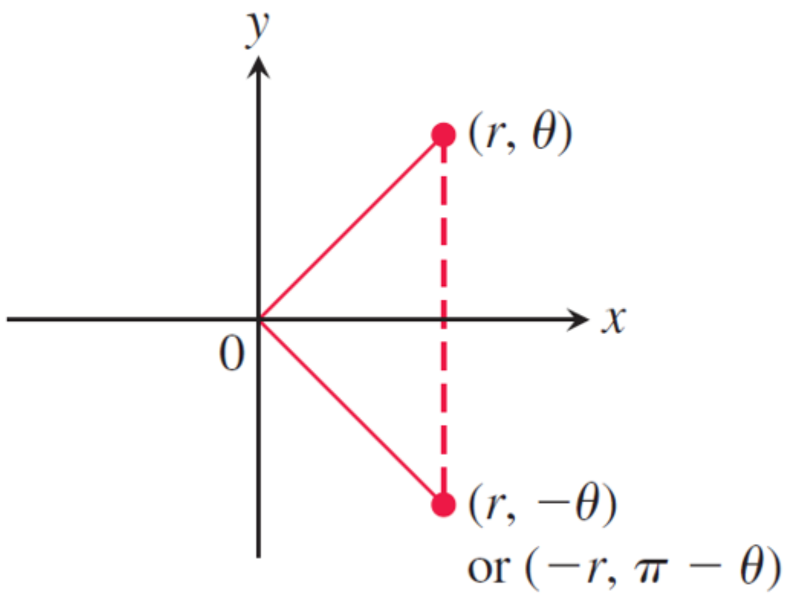
$$r = a \pm a\cos\theta, \quad r = a \pm \sin\theta$$

$= a \pm a\sin\theta$

where a is a positive real number.

Symmetry tests for Polar graphs

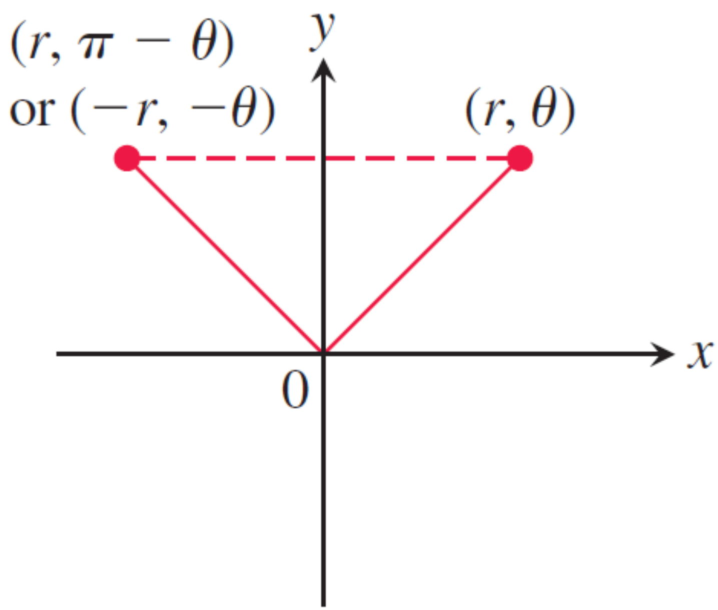
Symmetry about the x-axis: If the point (r, θ) lies on the graph, then the point $(r, -\theta)$ or $(-r, \pi - \theta)$ lies on the graph.



(a) About the x-axis

Example. $r = 1 + \cos\theta$

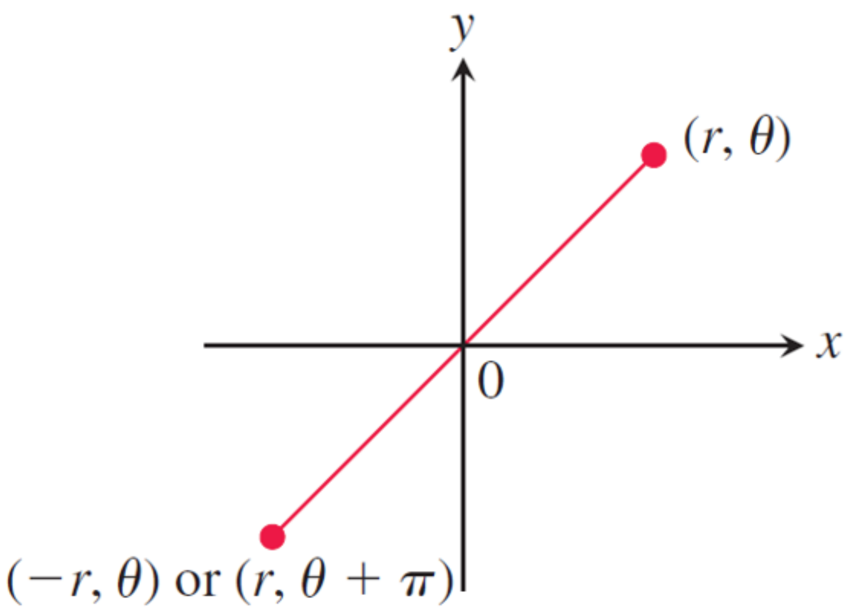
Symmetry about the y-axis: If the point (r, θ) lies on the graph, then the point $(r, \pi - \theta)$ or $(-r, -\theta)$ lies on the graph.



(b) About the y-axis

Example. $r = 1 + \sin\theta$

Symmetry about the origin: If the point (r, θ) lies on the graph, then the point $(-r, \theta)$ or $(r, \theta + \pi)$ lies on the graph.

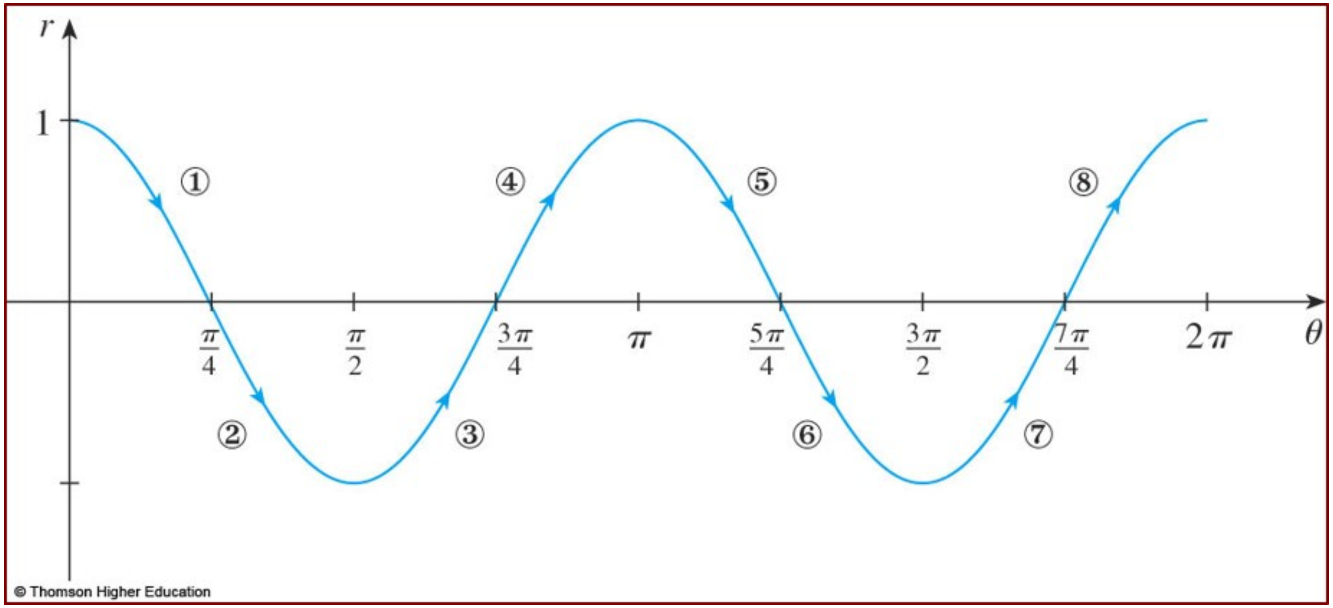


(c) About the origin

Example. $r^2 = \sin(2\theta)$

Plot $r = \cos(2\theta)$

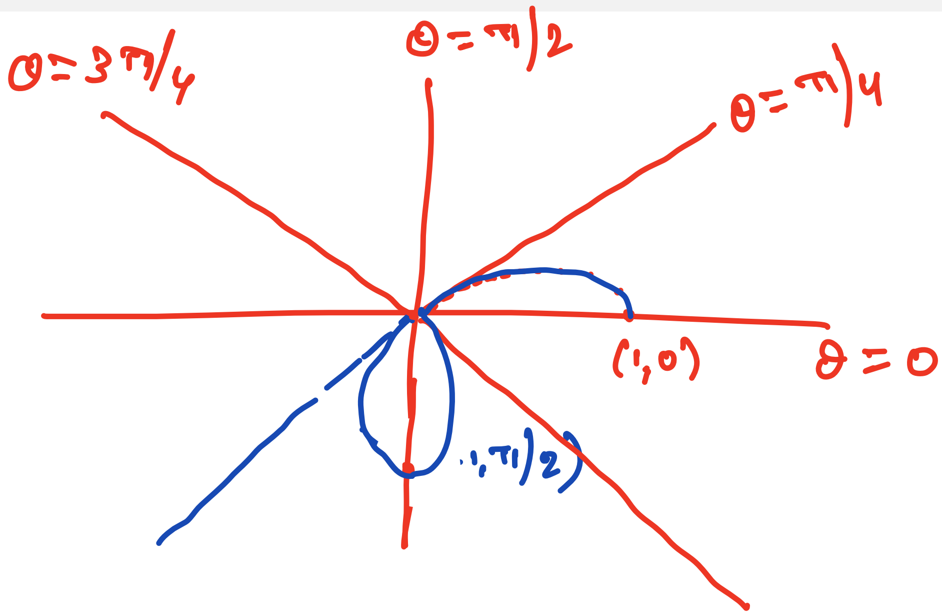
We first sketch $r = \cos(2\theta)$ for $0 \le \theta \le 2\pi$, in Cartesian coordinates.



$y = \cos(2x)$

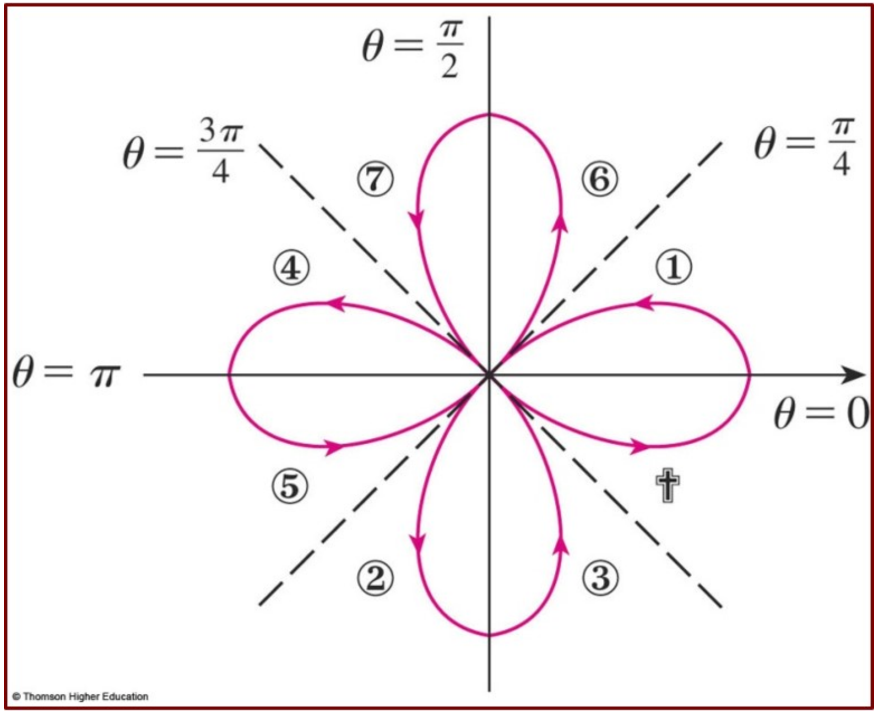
- $\theta \uparrow 0 - \frac{\pi}{4}, r \downarrow 1 - 0.$
- $\theta \uparrow \frac{\pi}{4} - \frac{\pi}{2}, r \downarrow 0 - (-1).$
- and so on....

Graph of the curve $r = \cos(2\theta)$



Graph of the curve $r = \cos(2\theta)$

$n = 2$



The resulting curve has four loops and is called a four-leaved rose.

Roses

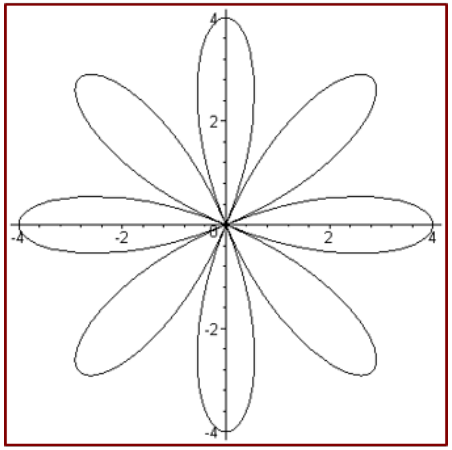
Roses are graphs of polar equations of the forms,

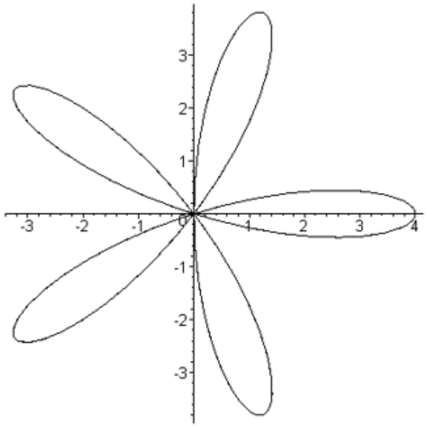
$n \in \mathbb{N}$
set of natural numbers

$$r = a\cos(n\theta), \quad r = a\sin(n\theta)$$

The roses given by above polar equations has n petals, if n is odd and $2n$ petals if n is even. **Examples.**

- $r = 4\cos(4\theta)$. Since n is even, number of petals are $2 \times 4 = 8$.
- $r = 4\cos(5\theta)$. Since n is odd, number of petals are 5.





$r = \cos 3\theta$

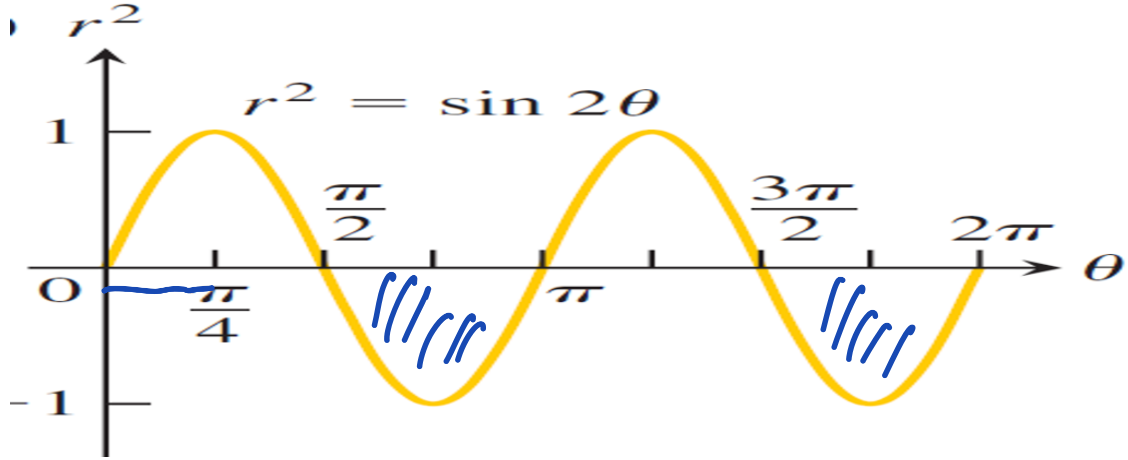
Sketch the curve $r^2 = \sin(2\theta)$

We begin by plotting r^2 (not r) as a function of θ in the Cartesian $r^2\theta$-plane.

$y^2 = \sin 2x$

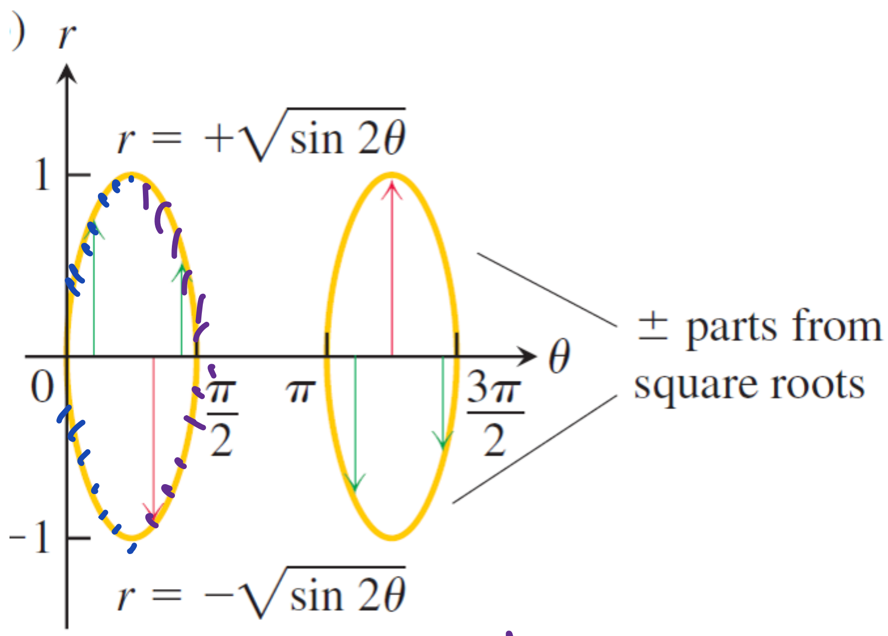
$\Rightarrow \; y = \pm\sqrt{\sin 2x}$

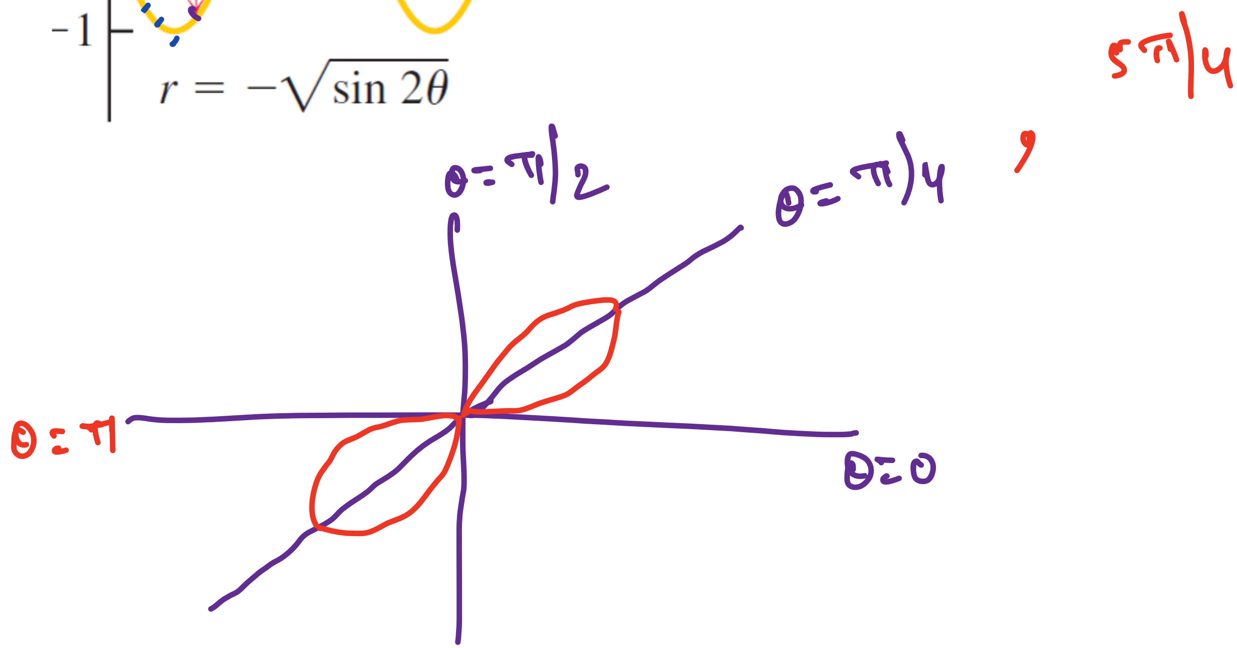
$\cancel{y = \pm\sqrt{-\sin 2x}}$

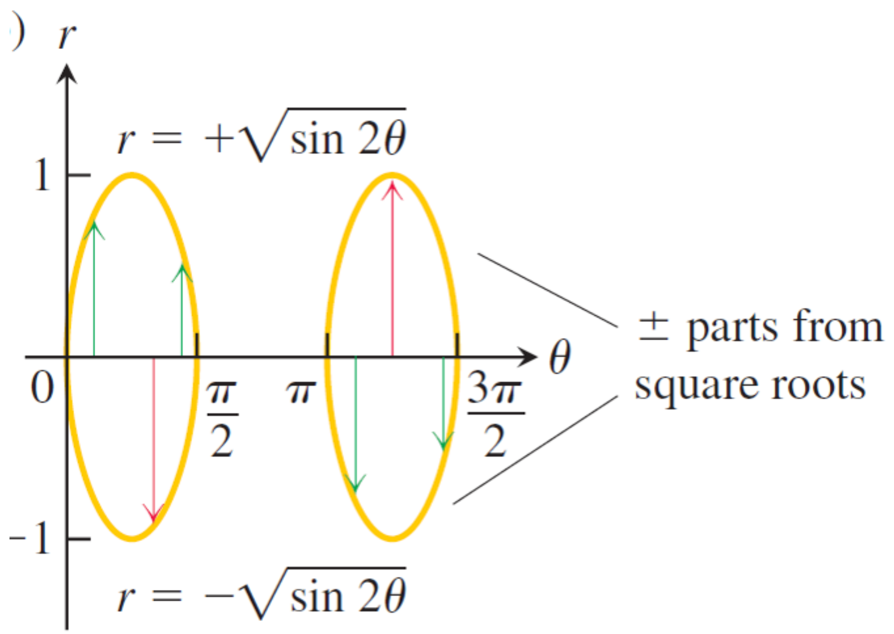


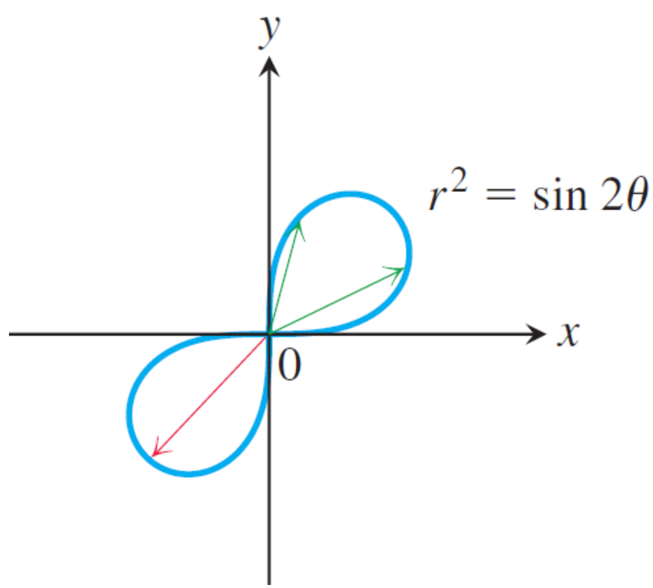
We pass from there to the graph of $r = \pm\sqrt{\sin(2\theta)}$ in $r\theta$-plane.

$r = \pm\sqrt{\sin 2\theta}$


r
1
-1
0
$\frac{\pi}{2}$
π
$\frac{3\pi}{2}$
θ
$r = +\sqrt{\sin 2\theta}$
$r = -\sqrt{\sin 2\theta}$
± parts from square roots


$5\pi/4$
$\theta = \pi/2$
$\theta = \pi/4$
$\theta = \pi$
$\theta = 0$


)
r
$r = +\sqrt{\sin 2\theta}$
1
-1
0
$\frac{\pi}{2}$
π
$\frac{3\pi}{2}$
θ
$\pm$ parts from
square roots
$r = -\sqrt{\sin 2\theta}$


y
$r^2 = \sin 2\theta$
x
0

Limacons

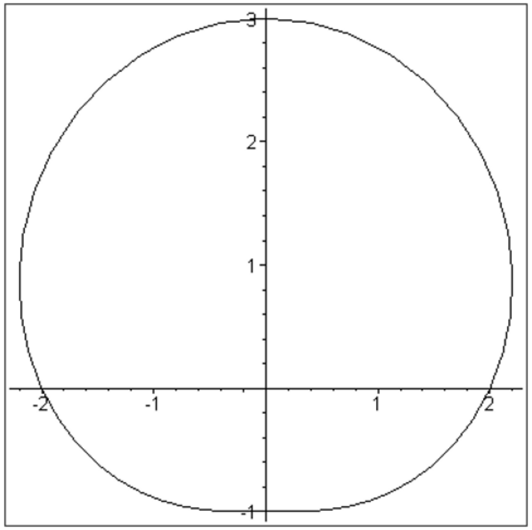
- The word limaçon comes from the old French for snail. There are two types of limaçons: ones without inner loop and ones with inner loop.
- Limaçons without inner loop are graphs of polar equations of the form:

$$r = a + b\cos\theta \quad \text{or} \quad r = a + b\sin\theta$$

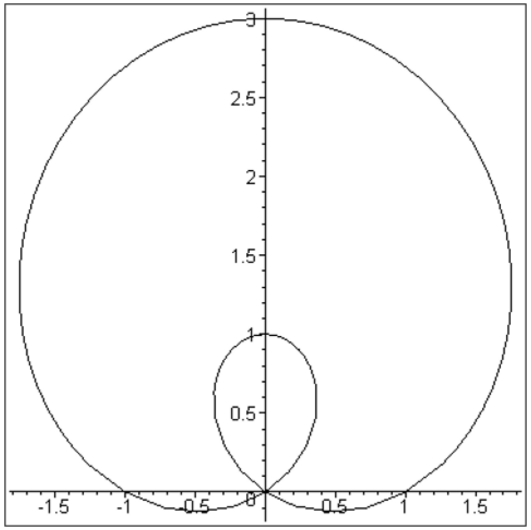
  where a and b are positive real numbers with $a > b$.
- When $a < b$ one gets limaçons with inner loops.

Examples

Limacon without inner loop: $r = 2 + \sin\theta$.



Limacon with inner loop: $r = 1 + 2\sin\theta$.



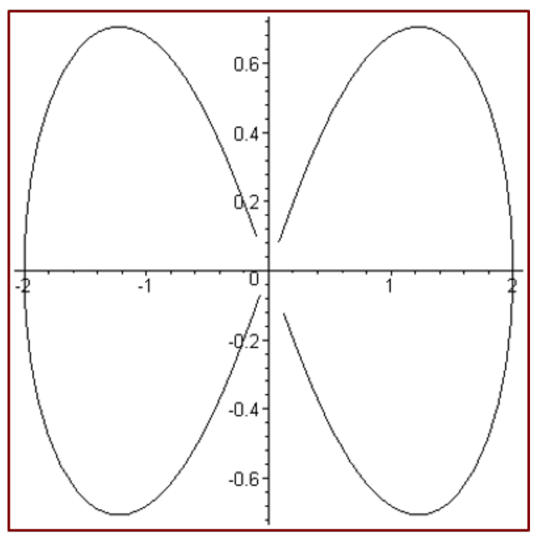
Lemniscates

The word lemniscates comes from the Latin word for ribbon. These are the graphs of polar equations of the form

$$r^2 = a^2\cos 2\theta, \quad r^2 = a^2\sin 2\theta.$$

Example. $\cancel{} = 4\cos 2\theta$

r^2



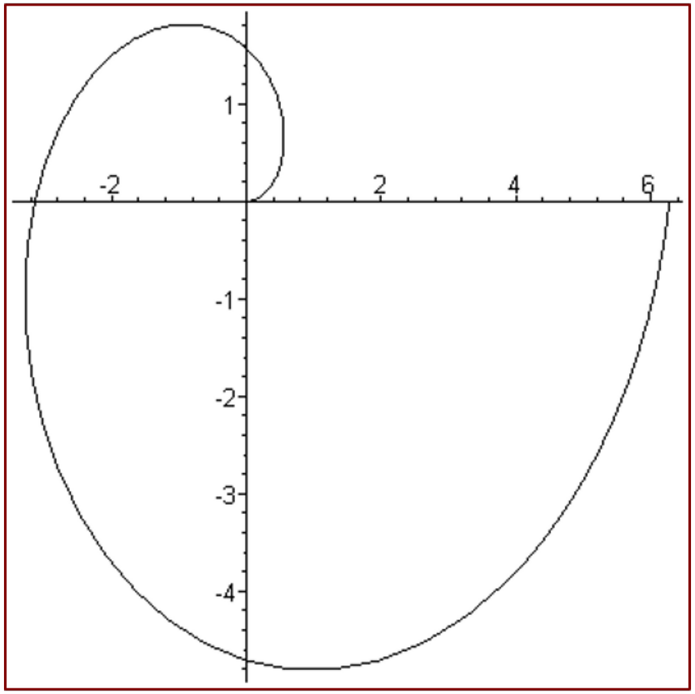
Spirals

There are many kinds of interesting spirals.

$$r = \pm\theta, \quad r = e^{\theta}, \quad r = \frac{a}{\theta}.$$

$r = a\theta$

Example. $r = \theta,\ 0 \le \theta \le 2\pi.$



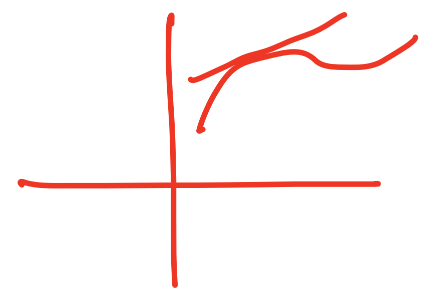
Slope of Tangents to Polar curve

To find the slope of a polar curve $r = f(\theta)$, think of the graph of f as the graph of the parametric equations

$$x = r\cos\theta = f(\theta)\cos\theta, \quad \underset{y}{\cancel{r}} = r\sin\theta = f(\theta)\sin\theta.$$

If f is differentiable function of θ, then so are x and y and when $\frac{dx}{d\theta} \neq 0$, then

$$\left.\frac{dy}{dx}\right|_{(r,\theta)} = \frac{dy/d\theta}{dx/d\theta} = \frac{\frac{d}{d\theta}(f(\theta)\sin\theta)}{\frac{d}{d\theta}(f(\theta)\cos\theta)} = \frac{f'(\theta)\sin\theta + f(\theta)\cos\theta}{f'(\theta)\cos\theta - f(\theta)\sin\theta}$$



Remark. $\frac{dy}{dx} \neq \frac{dr}{d\theta}$.

1. **Horizontal tangents.** We locale horizontal tangents by finding the points where $\frac{dy}{d\theta} = 0$, provided $\frac{dx}{d\theta} \neq 0$.

2. **Vertical Tangents.** We locale horizontal tangents by finding the points where $\frac{dx}{d\theta} = 0$, provided $\frac{dy}{d\theta} \neq 0$.

3. **Tangent through the origin.** If the curve $r = f(\theta)$ passes through the origin at $\theta = \theta_0$, then $f(\theta_0) = 0$, and the slope is

$$\left.\frac{dy}{dx}\right|_{(0,\theta_0)} = \frac{f'(\theta_0)\sin\theta_0}{f'(\theta_0)\cos\theta_0} = \tan\theta_0.$$

Note that, there can be many tangents with different slope at origin.

Example. $r = 1 + \sin\theta$ $= f(\theta)$

$$\frac{dy}{d\theta} = \cos\theta(1 + 2\sin\theta) = 0 \Rightarrow \theta = \frac{\pi}{2}, \cancel{\frac{3\pi}{2}}, \frac{7\pi}{6}, \frac{11\pi}{6}$$

$\Rightarrow$ Horizontal tangents

$$\theta = \frac{\pi}{2}, \frac{7\pi}{6}, \frac{11\pi}{6}$$

$$\frac{dx}{d\theta} = (1 + \sin\theta)(1 - 2\sin\theta) = 0 \Rightarrow \theta = \cancel{\frac{3\pi}{2}}, \frac{\pi}{6}, \frac{5\pi}{6}$$

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{\cos\theta(1+2\sin\theta)}{(1+\sin\theta)(1-2\sin\theta)}$$

$$\left.\frac{dy}{dx}\right|_{\text{at origin}}$$

$$\text{origin} = (0, \theta)$$

Vertical tangents

$$\theta = \frac{\pi}{6}, \frac{5\pi}{6}$$