Extreme Values & Saddle Points

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Extreme Values

Let $f(x, y)$ be defined on a region R containing the point (a, b).

1. $f(a, b)$ is a local maximum value of f if $f(a, b) \geq f(x, y)$ for all domain points (x, y) in an open disk centered at (a, b).
2. $f(a, b)$ is a local minimum value of f if $f(a, b) \leq f(x, y)$ for all domain points (x, y) in an open disk centered at (a, b).
3. An interior point of the domain of a function f is called a critical point of the function if either both f_x, f_y vanish or at least one of f_x and f_y does not exist at the point.
4. A differentiable function $f(x, y)$ has a saddle point at a critical point (a, b) if in every open disk centered at (a, b) there are domain points (x, y) where $f(x, y) > f(a, b)$ and domain points (x, y) where $f(x, y) < f(a, b)$.

Theorem (First Derivative Test for Local Extreme Values)

If $f(x, y)$ has a local maximum or minimum value at an interior point (a, b) of its domain and if the first partial derivatives exist there, then

$$f_x(a, b) = 0 \text{ and } f_y(a, b) = 0.$$

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f is defined and differentiable for all$x$ and y and its domain has no boundary points. The extreme values can occur only at the points where f_x and f_y are simultaneously zero. Solving $2x = 0$ and $2y = 0$, $(0, 0)$ is the only point where f may take on an extreme value. CHECK!

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Hence the only critical point is the origin. But at this point we see that along y axis,
$f(0,y) = y^2 > 0$.
Along x axis, $f(x,0) = -x^2 < 0$.
Hence, the function has a saddle point at the origin and no local extreme values.

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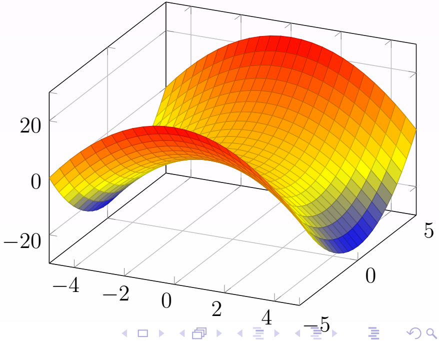

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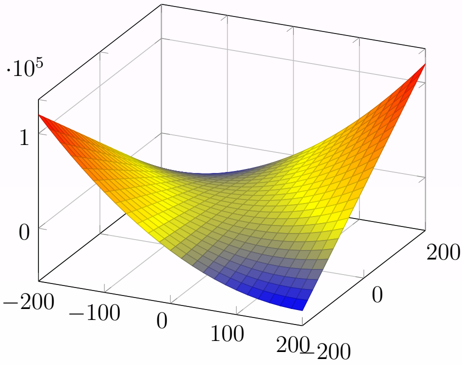

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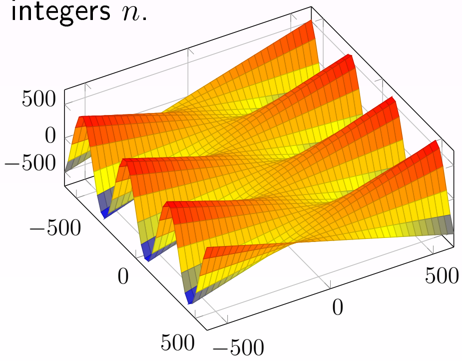

Second Derivative Test

Note that $f_x = f_y = 0$ at an interior point (a, b) of R does not guarantee f has a local extreme value.

Theorem (Second Derivative Test for Local Extreme Values)

Suppose that $f(x, y)$ and its first and second partial derivatives are continuous throughout a disk centered at (a, b) and that $f_x(a, b) = f_y(a, b) = 0$. Then

1. *f has a local maximum at (a, b) if $f_{xx} < 0$ and $f_{xx}f_{yy} - f_{xy}^2 > 0$ at (a, b).*
2. *f has a local minimum at (a, b) if $f_{xx} > 0$ and $f_{xx}f_{yy} - f_{xy}^2 > 0$ at (a, b).*
3. *f has a saddle point at (a, b) if $f_{xx}f_{yy} - f_{xy}^2 < 0$ at (a, b).*
4. *the test is inconclusive at (a, b) if $f_{xx}f_{yy} - f_{xy}^2 = 0$ at (a, b). In this case, we must find some other way to determine the behavior of f at (a, b).*

Examples

The expression $f_{xx}f_{yy} - f_{xy}^2$ is called the discriminant or Hessian of f. It is sometimes easier to remember it in determinant form,

$$f_{xx}f_{yy} - f_{xy}^2 = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{vmatrix}.$$

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Absolute Maxima and Minima on Closed Bounded Regions

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Steps to find absolute maxima and minima

1. *List the interior points* of R where f may have local maxima and minima and evaluate f at these points. These are the critical points of f.
2. *List the boundary points* of R where f has local maxima and minima and evaluate f at these points.
3. *Look through the lists* for the maximum and minimum values of f. These will be the absolute maximum and minimum values of f on R.

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Here the function takes the form $f(x, y) = y^2$ and the only local extremum is at the boundary points $y = 0, 2$ and the function values are $0, 4$ respectively.

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On the line segment $y + 2x = 2, 0 \leq x \leq 1$, the function becomes $f(x) = 5x^2 - 8x + 4$.

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Table: Points and corresponding $f(x, y)$

(0,0)	0
(0,2)	4
(1,0)	1
(4/5,2/5)	4/5

So the absolute maximum is 4 and the absolute minimum is 0.

Example 2. Find the absolute maximum and minimum of the function $f(x, y) = (4x - x^2)\cos y$ in the region $1 \leq x \leq 3$, $-\pi/4 \leq y \leq \pi/4$.

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Solution. First let us deal with the interior points.

$$f_x = (4 - 2x)\cos y = 0$$
$$f_y = -(4x - x^2)\sin y = 0$$

Solving these two gives us $(2, 0)$. The value at this point is 4.

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Boundary points:
Along the line $1 \le x \le 3, y = -\pi/4$. The function becomes $g(x) = \frac{4x - x^2}{\sqrt{2}}$ the critical point is $x = 2$ and the value is $2\sqrt{2}$.
Along the line $1 \le x \le 3, y = \pi/4$. $\cos y$ being an even function we have the same situation as above.

Examples

Along the line $-\pi/4 \le y \le \pi/4, x = 1$. The function becomes $h(y) = 3\cos y$ it has a critical point at $y = 0$ and the value is 3
Along the line $-\pi/4 \le y \le \pi/4, x = 3$. The function becomes $3\cos y$, so it is the same situation as above and the value is same 3.
Now the boundary of the lines $(1, -\pi/4), (1, \pi/4), (3, -\pi/4), (3, \pi/4)$ the values are all equal to $3/\sqrt{2}$. Let us now build the table:

Table: Points and corresponding $f(x, y)$

(2,0)	4
(2,$-\pi/4$)	$2\sqrt{2}$
(2,$\pi/4$)	$2\sqrt{2}$
(1,0)	3
(3,0)	3
$(1, -\pi/4), (1, \pi/4), (3, -\pi/4), (3, \pi/4)$	$3/\sqrt{2}$

So the absolute maximum is at (2,0) and the value is 4, and absolute minimum is $3/\sqrt{2}$ at $(1, -\pi/4), (1, \pi/4), (3, -\pi/4), (3, \pi/4)$.

3. Find the absolute maximum and minimum of the function $T(x, y) = x^2 + xy + y^2 - 6x + 2$ in the region given by $0 \leq x \leq 5$ and $-3 \leq y \leq 0$.

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3. Find the absolute maximum and minimum of the function $T(x, y) = x^2 + xy + y^2 - 6x + 2$ in the region given by $0 \leq x \leq 5$ and $-3 \leq y \leq 0$. (The absolute maximum is 11 attained at the point (0,-3) and the absolute minimum is -10 attained at the point (4,-2).)

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4. Among all closed rectangular boxes of volume 27 cm^3, what is the smallest surface area? (Volume=$xyz = 27$ and Surface area $S = 2(xy + yz + xz)$. Eliminate z and find the local minima for $S(x,y)$. Local minimum of $S(3,3,3) = 54$.)