14. Partial Derivatives

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Domain and range for functions of several variables

$$\mathbb{R}^n \xrightarrow{f} \mathbb{R}$$
$$(x_1, \ldots, x_n) \longmapsto f(x_1, \ldots, x_n) = w$$

Definition (Function of several variables)

Suppose D is a set of n-tuples of real numbers $(x_1, x_2, \ldots, x_n)$. A **real-valued function** f on D is a rule that assigns a unique (single) real number

$$w = f(x_1, x_2, \ldots, x_n)$$

to each element in D. The set D is the function's **domain**. The set of w-values taken by f is called the **range** of the function f. Here the symbol w is called **dependent variable** of f and f is said to be the function of several **independent variables** x_1 to x_n.

Particular cases

$f: \text{Domain} \subseteq \mathbb{R}^2 \longrightarrow \mathbb{R}$
$(x,y) \longmapsto f(x,y) = z$

- If f is a function of two independent variables, we write it as $z = f(x,y)$ and we usually call the independent variables x and y and the dependent variable z, and we picture the domain of f as a region in the xy-plane.

$f: \text{Domain} \subseteq \mathbb{R}^3 \longrightarrow \mathbb{R}$
$(x,y,z) \longmapsto f(x,y,z) = w$

- If f is a function of three independent variables, we write it as $w = f(x,y,z)$ and we call the independent variables x, y, and z and the dependent variable w, and we picture the domain as a region in space.

Examples

1. The temperature at each point of an object.

2. Distance of a point in the space from the origin. $f: \mathbb{R}^3 \longrightarrow \mathbb{R}$

3. $f(x, y) = \sqrt{1 - x^2 - y^2}$. $f:$ $\longrightarrow \mathbb{R}$

4. $f(x, y, z) = \sin(x + y) + |z|$.

   $f: \mathbb{R}^3 \longrightarrow \mathbb{R}$

   Range $= [-1, \infty)$

Examples

Function	Domain	Range
$z = \sqrt{y - x^2},$	$\{(x, y) \ : \ y \geq x^2\}$	$[0, \infty)$
$z = \dfrac{1}{x + y}$	$\{(x, y) \ : \ x + y \neq 0\}$	$\mathbb{R} - \{0\}$
$z = \sin xy$	Entire plane $\mathbb{R}^2$	$[-1, 1]$
$w = \sqrt{x^2 + y^2 + z^2}$	Entire space $\mathbb{R}^3$	$[0, \infty)$
$w = \frac{1}{x^2+y^2+z^2}$	$\mathbb{R}^3 - \{(0, 0, 0)\}$	$(0, \infty)$
$w = xy \ln z$	$\{(x, y, z) \ : \ z > 0\}$	$(-\infty, \infty)$

Properties of Domain in $\mathbb{R}^2$ and $\mathbb{R}^3$

- Let (x_0, y_0) be a point in $\mathbb{R}^2$. The open disk centered (x_0, y_0) of radius $r > 0$ is defined as the set:

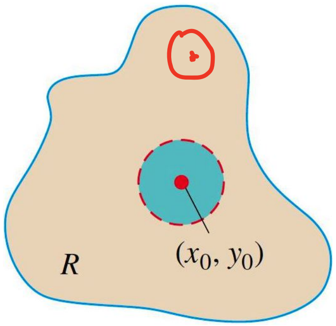
$$\{(x, y) \in \mathbb{R}^2 : (x - x_0)^2 + (y - y_0)^2 < r^2\}.$$

- Let $\varepsilon > 0$ and (x_0, y_0, z_0) be a point in space. We define open ball of radius $\varepsilon > 0$ centered at (x_0, y_0, z_0) by the set

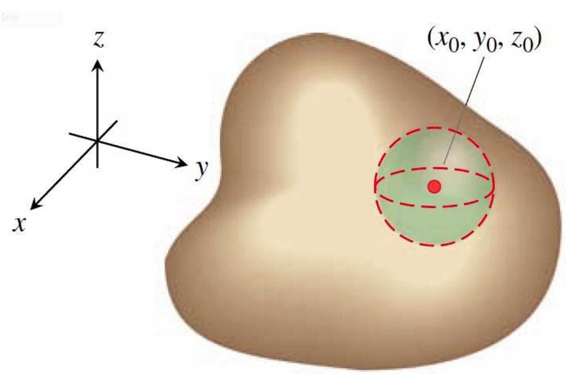
$$\{(x, y, z) \in \mathbb{R}^3 \ : \ \sqrt{(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2} < \varepsilon\}$$

Interior Point

A point (x_0, y_0) in a region R in the xy-plane is an interior point of R if the region R contains a disk centered (x_0, y_0).

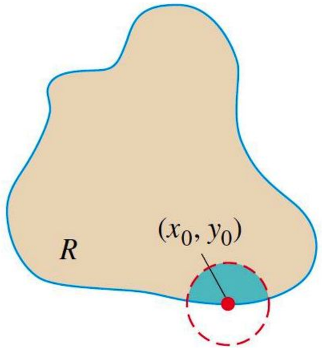


A point (x_0, y_0, z_0) in a region R in space is an interior point of R if the region R contains an open ball centered at (x_0, y_0, z_0) of some positive radius.

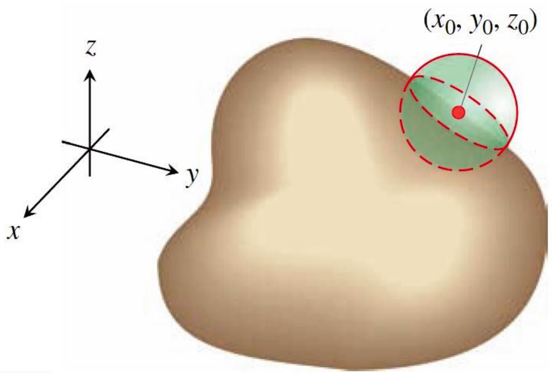


Boundary Point

A point (x_0, y_0) is a boundary point of R if every disk centered at (x_0, y_0) contains points that lies outside of R as well as points that lies in R.

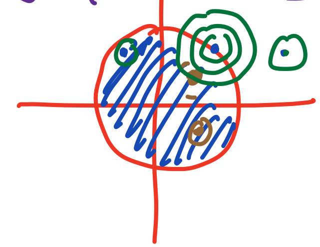


A point (x_0, y_0, z_0) is a boundary point of R if every open ball centered at (x_0, y_0, z_0) contains points that lies outside of R as well as points that lies in R.



Examples

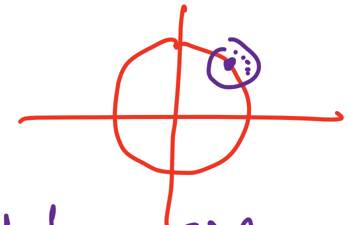
$\mathbb{R}^2$ $\quad D = \{(x,y)\in\mathbb{R}^2 \mid x^2+y^2<1\}$ — Not closed, open



Interior pts = D

Bdd pt = $(x,y) \mid x^2+y^2=1$

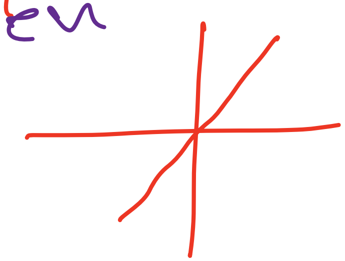
$D=\{(x,y)\in\mathbb{R}^2 \mid x^2+y^2=1\}$



No pt. is an interior pt.

Bd pts = D — closed, Not open

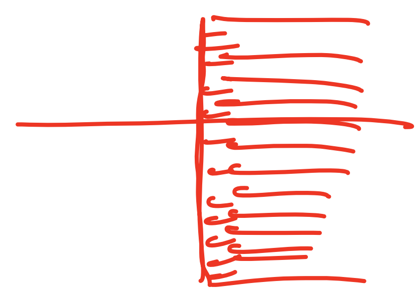
$D=\{(x,y)\in\mathbb{R}^2 \mid x=y\}$



Bd points = D

Interior pts = None

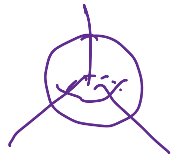
$D=\{(x,y)\in\mathbb{R}^2 \mid x\geq 0\}$



Interior pt = $\{(x,y)\in\mathbb{R}^2 \mid x>0\}$

Bd pts = $\{(0,y)\in\mathbb{R}^2\}$ — closed, Not open

$D = \{ (x,y,z) \in \mathbb{R}^3 \mid x^2+y^2+z^2 \leq 1 \}$



Interior pts $= \{ (x,y,z) \in \mathbb{R}^3 \mid x^2+y^2+z^2 < 1 \}$

Bdd pts $= \{ (x,y,z) \in \mathbb{R}^3 \mid x^2+y^2+z^2 = 1 \}$

- Not open, closed

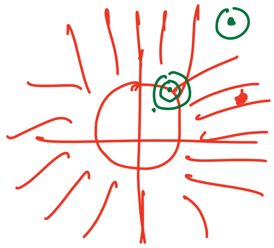
$D = \{ (x,y,z) \in \mathbb{R}^3 \mid z = 1 \}$

Bdd pts $= D$

No interior pt.

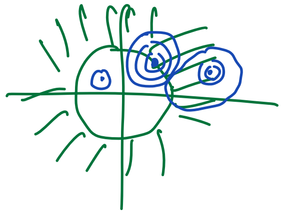
- closed, Not open.

$D = \{ (x,y) \in \mathbb{R}^2 \mid x^2+y^2 > 1 \}$

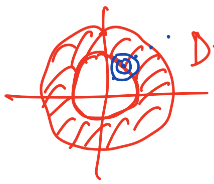


Interior pt $= D$

Bdd pts $= x^2+y^2 = 1$



- Not closed, open



$D = 1 \leq x^2+y^2 < 2$

— Not open

Int $D = \{ (x,y) \in \mathbb{R}^2 \mid 1 < x^2+y^2 < 2 \}$

Interior pts $= \{ (x,y) \in \mathbb{R}^2 \mid 1 < x^2+y^2 < 2 \}$

Bdd pt $= \{ (x,y) \in \mathbb{R}^2 \mid x^2+y^2 = 1,\ x^2+y^2 = 2 \}$

— Not closed

Open Set: A region R is said to be **open** if every point in it is an interior point of the region R. $Int(R) = R$

Closed Set: A region R is said to be **closed** if it contains all its boundary points. $Bd\,R \subseteq R$

Ex. $D = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2=1\}$ – Not open

$D = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2<1\}$ – open

$D = \{(x,y) \in \mathbb{R}^2 \mid x=y\}$ – Not open