Mathematics I- MATH F111

Saranya G. Nair
Department of Mathematics

BITS Pilani

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In this section we study how a curve turns or bends. First we look at curves in the coordinate plane. Then we consider curves in space.

Rate of change of r with respect to arc length

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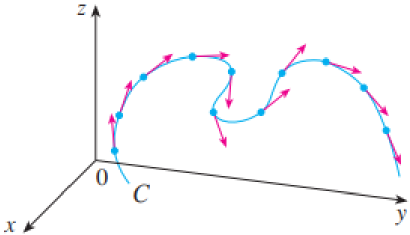

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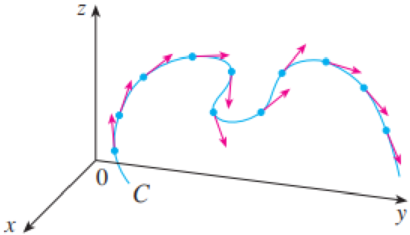

T changes direction very slowly when C is fairly straight, but it changes direction more quickly when bends or twists more sharply.

- The curvature of C at a given point is a measure of how quickly the curve changes direction at that point.
- Specifically, we define it to be the magnitude of the rate of change of the unit tangent vector with respect to arc length.
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Definition

The curvature of a curve is

$$\kappa = \left| \frac{d\mathbf{T}}{ds} \right|$$

where $\mathbf{T}$ is the unit tangent vector.

If a smooth curve $\vec{r}(t)$ is already given in terms of some parameter t other than the arc length parameter s, we can use the Chain Rule to compute it in terms of t.

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$$\kappa = \left|\frac{d\mathbf{T}}{ds}\right| = \left|\frac{d\mathbf{T}}{dt}\frac{dt}{ds}\right|$$
$$= \frac{1}{\left|\frac{ds}{dt}\right|}\left|\frac{d\mathbf{T}}{dt}\right|$$
$$= \frac{1}{|v|}\left|\frac{d\mathbf{T}}{dt}\right|.$$

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2. Find the curvature of a circle of radius a parametrized by $r(t) = a\cos t\vec{i} + a\sin t\vec{j}$.

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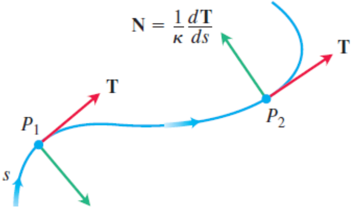
- Therefore, if we divide $\frac{d\mathbf{T}}{ds}$ by its length κ, we obtain a unit vector $\mathbf{N}$ orthogonal to $\mathbf{T}$.

$$\mathbf{N} = \frac{1}{\kappa}\frac{d\mathbf{T}}{ds}$$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T
T
P_1
P_2
s

If a smooth curve $r(t)$ is already given in terms of some parameter t other than the arc length parameter s, we can use the Chain rule to calculate $\mathbf{N}$ directly:

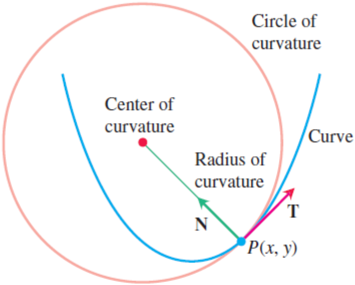
$$\mathbf{N} = \frac{d\mathbf{T}/ds}{|d\mathbf{T}/ds|} = \frac{(d\mathbf{T}/dt)(dt/ds)}{|d\mathbf{T}/dt||dt/ds|} = \frac{d\mathbf{T}/dt}{|d\mathbf{T}/dt|}$$

since $\frac{dt}{ds} = \frac{1}{ds/dt} > 0$. This formula helps us to calculate $\mathbf{N}$ without finding κ and s first.

Find $\vec{\mathbf{T}}$ and $\vec{\mathbf{N}}$ for the circular motion

$$r(t) = (\cos 2t)\vec{i} + (\sin 2t)\vec{j}.$$

Circle of Curvature



The center of the osculating circle at $P(x, y)$ lies toward the inner side of the curve.

Circle of Curvature

The circle of curvature or osculating circle at a point P on a plane curve where $\kappa \neq 0$ is the circle in the plane of the curve that

- is tangent to the curve at P (has the same tangent line the curve has)
- has the same curvature the curve has at P
- has center that lies toward the concave or inner side of the curve

The radius of curvature of the curve at P is the radius of the circle of curvature.

$$\text{Radius of curvature} = \rho = \frac{1}{\kappa}.$$

The center of curvature of the curve at P is the center of the circle of curvature.

Find and graph the circle of curvature of the parabola $y = x^2$ at the origin.

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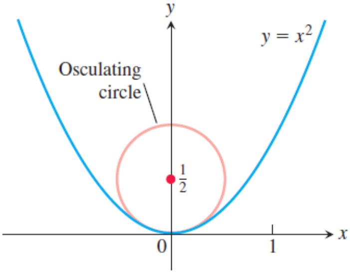

Curvature and Normal vectors for Space curves

If a smooth curve in space is specified by the position vector $\mathbf{r}(t)$ as a function of some parameter t, and if s is the arc length parameter of the curve, then the unit tangent vector $\mathbf{T}$ is $d\mathbf{r}/ds = v/|v|$. The curvature in space is defined to be

$$\kappa = \left|\frac{d\mathbf{T}}{ds}\right| = \frac{1}{|v|}\left|\frac{d\mathbf{T}}{dt}\right|$$

The vector $d\mathbf{T}/ds$ is orthogonal to $\mathbf{T}$, and we define the principal unit normal to be

$$\mathbf{N} = \frac{1}{\kappa}\frac{d\mathbf{T}}{ds} = \frac{d\mathbf{T}/dt}{|d\mathbf{T}/dt|}.$$

Find the curvature and Normal for the helix

$$r(t) = (a\cos t)\vec{i} + (a\sin t)\vec{j} + bt\vec{k},\ \ a, b \geq 0,\ \ a^2 + b^2 \neq 0.$$

What happens to the curvature when $a = 0$? or when $b = 0$?