Vector Valued Functions and Motion in Space

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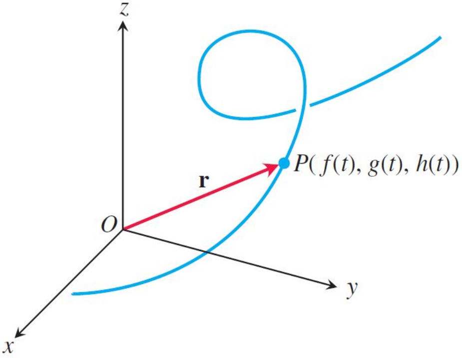
Lecture-15

Chapter-13.1 & 13.2

Motion of a Particle in Space:

Motion of a Particle in Space:

- Suppose that a particle is moving in the space. Let the particle is at the point $P(x(t), y(t), z(t))$ at time t units.



Introduction

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- If the time variable t is varying in an interval $I = [a, b]$, then the point $P(x(t), y(t), z(t))$ traces a curve (can be called **path of the particle**).
- We also use the following notation to denote the position of the particle at time t:

$$\mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j} + z(t)\,\mathbf{k} \tag{0.1}$$

- If the functions $x(t)$, $y(t)$ and $z(t)$ are continuous functions on the interval I, then the graph of $\mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j} + z(t)\,\mathbf{k}$, $t \in I$ gives a curve in the space.

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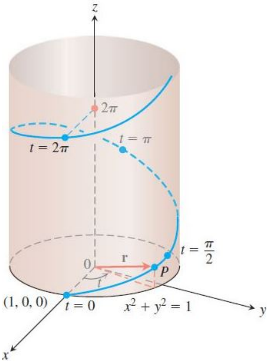
- Here we are interested to study the motion of the particle and its various properties which help us to understant more about the path of the particle.

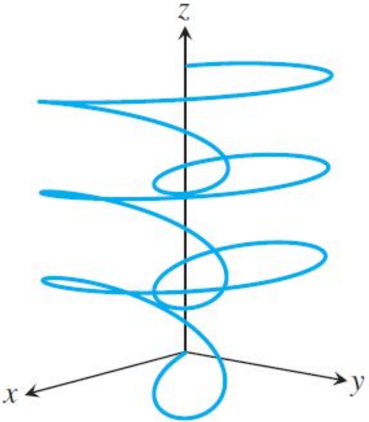
Examples for Curves



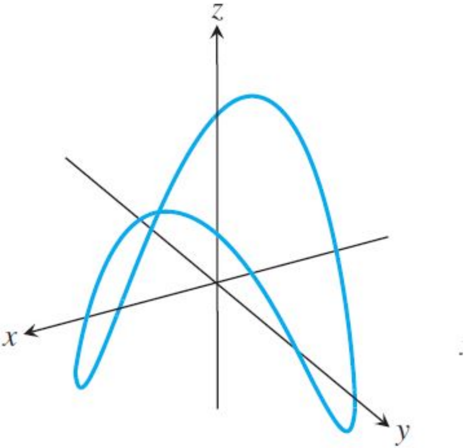
- The above curve is the graph of the following equation in upper half space

$$\mathbf{r}(t) = \cos t\ \mathbf{i} + \sin t\ \mathbf{j} + t\ \mathbf{k}.$$

Examples for Curves



$$\mathbf{r}(t) = (\sin 3t)(\cos t)\mathbf{i} + (\sin 3t)(\sin t)\mathbf{j} + t\mathbf{k}$$



$$\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + (\sin 2t)\mathbf{k}$$

Parametrization for Curves

A curve is a graph of a continuous vector functions $\mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j} + z(t)\,\mathbf{k}$, it has parametric equations:

$$x = x(t), \quad y = y(t), \quad z = z(t).$$

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Examples: Find the parametric equation of the curve of interesection of the cylinder $x^2 + y^2 = 1$ and the plane $z = 1/2$.

A curve can have different parametrizations. For instance $\mathbf{r}(t) = t\,\mathbf{i} + t^2\,\mathbf{j} + 1\,\mathbf{k}$ and $\mathbf{r}(t) = t^3\,\mathbf{i} + t^6\,\mathbf{j} + 1\,\mathbf{k}$ gives ths same curve.

- Motion of a particle (or the path of the particle) can be studied by the equation of its position vector, which is a vector valued function:

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$$\mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j} + z(t)\,\mathbf{k} \qquad (0.2)$$

- So, its important to study some properties of vector valued functions: Limits, Continuity, Differentiability etc.

Definition 0.1.

Let $\mathbf{r}(t)$ be as in (0.2), a vector-function which defined on an interval I, $t_0 \in I$ and Let $\mathbf{L} = x_0\mathbf{i} + y_0\mathbf{j} + z_0\mathbf{k}$. We say that $\mathbf{r}(t)$ has limit $\mathbf{L}$ as t approaches to t_0 and write

$$\lim_{t \to t_0} \mathbf{r}(t) = \mathbf{L},$$

if for every number $\varepsilon > 0$, there exists a corresponding number $\delta > 0$, such that

$$|\mathbf{r}(t) - \mathbf{L}| < \varepsilon \quad \text{whenever} \quad 0 < |t - t_0| < \delta.$$

Theorem 0.2.

Let $\mathbf{r}(t)$ *be as in the above. The vector-function* $\mathbf{r}(t)$ *has a limit at* $t = t_0$ *if and only if the component functions* $x(t), \; y(t)$ *and* $z(t)$ *have the limits at* $t = t_0$. *Moreover,*

$$\lim_{t \to t_0} \mathbf{r}(t) = \lim_{t \to t_0} x(t) \, \mathbf{i} + \lim_{t \to t_0} y(t) \, \mathbf{j} + \lim_{t \to t_0} z(t) \, \mathbf{k}.$$

Definition 0.3.

Vector function $\mathbf{r}(t)$ is said to be continuous at $t = t_0 \in I$ if

$$\lim_{t \to t_0} \mathbf{r}(t) = \mathbf{r}(t_0).$$

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Example: $\mathbf{r}(t) = \cos t\ \mathbf{i} + \sin t\ \mathbf{j} + t\ \mathbf{k}$ is a continuous vector-function.

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- Suppose that $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$ is the position vector of a particle moving along a curve in the space and that $x(t),\ y(t)$ and $z(t)$ are differentiable functions of t.

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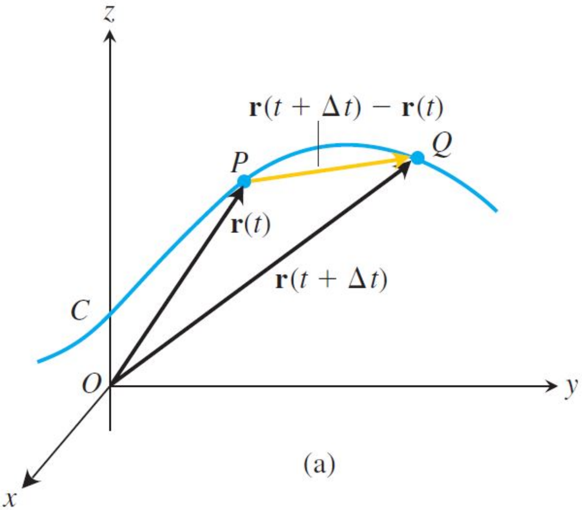
- In terms of components,

$$\Delta\mathbf{r} = [x(t + \Delta t) - x(t)]\,\mathbf{i} + [y(t + \Delta t) - y(t)]\,\mathbf{j} + [z(t + \Delta t) - z(t)]\,\mathbf{k}$$

Derivatives



(a)

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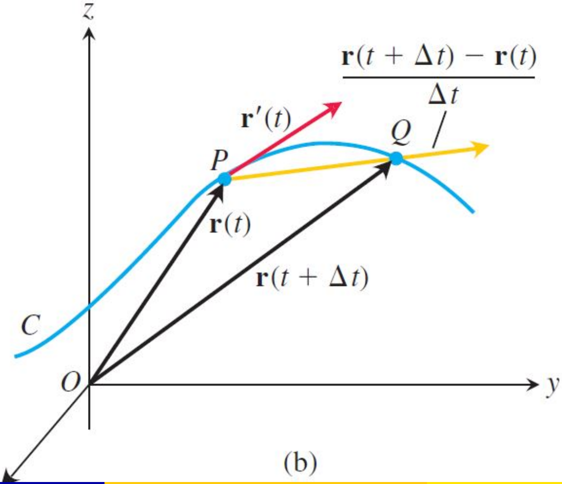
- Second, the secant line PQ seems to approach a limiting position tangent to the curve at P.
- Third, the quotient $\Delta \mathbf{r}/\Delta t$ approaches to the limit:

$$\lim_{\Delta t \to 0} \frac{\Delta \mathbf{r}}{\Delta t} = \left[\lim_{\Delta t \to 0} \frac{x(t+\Delta t) - x(t)}{\Delta t}\right] \mathbf{i} + \left[\lim_{\Delta t \to 0} \frac{y(t+\Delta t) - y(t)}{\Delta t}\right] \mathbf{j} + \left[\lim_{\Delta t \to 0} \frac{z(t+\Delta t) - z(t)}{\Delta t}\right] \mathbf{k}$$

Therefore, $\lim_{\Delta t \to 0} \dfrac{\Delta \mathbf{r}}{\Delta t} = \left[\dfrac{dx}{dt}\right] \mathbf{i} + \left[\dfrac{dy}{dt}\right] \mathbf{j} + \left[\dfrac{dy}{dt}\right] \mathbf{k}$



(b)

The above expression lead us to define:

Definition 0.4.

The vector function $\mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j} + z(t)\,\mathbf{k}$ has a derivateve (is differentiable) at t, if $x(t)$, $y(t)$ and $z(t)$ have derivatives at t. The derivative of $\mathbf{r}(t)$ is a vector function given by

$$\mathbf{r}'(t) = \lim_{\Delta t \to 0} \frac{r(t+\Delta t) - r(t)}{\Delta t} = \frac{dx}{dt}\,\mathbf{i} + \frac{dy}{dt}\,\mathbf{j} + \frac{dz}{dt}\,\mathbf{k}.$$

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- The tangent line to the curve

$$\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$$

at a point $(x(t_0), y(t_0), z(t_0))$ is defined to be the line through the point and parallel to the vector

$$\mathbf{r}'(t_0) = x'(t_0)\mathbf{i} + y'(t_0)\mathbf{j} + z'(t_0)\mathbf{k}.$$

Therefore, equation of the tangent line is given by

$$\gamma(\lambda) = \lambda\mathbf{r}'(t_0) + \mathbf{r}(t_0).$$

Smooth Curves

- A Curve is said to be smooth, if there exists a parametrization $\mathbf{r}(t) = (x(t), y(t), z(t))$ of the curve such that
    - $\mathbf{r}'(t)$ is continuous and
    - $\mathbf{r}'(t) \neq \mathbf{0}$ for any t in its domain, that is $x(t)$, $y(t)$ and $z(t)$ have continuous first derivates that are not simultaneously 0.

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  - $\mathbf{r}'(t) \neq \mathbf{0}$ for any t in its domain, that is $x(t)$, $y(t)$ and $z(t)$ have continuous first derivates that are not simultaneously 0.
- We require $d\mathbf{r}/dt \neq \mathbf{0}$ for a smooth curve to make sure that the curve has a continuously turning tangent at each point. On smooth curve there are no sharp corners.

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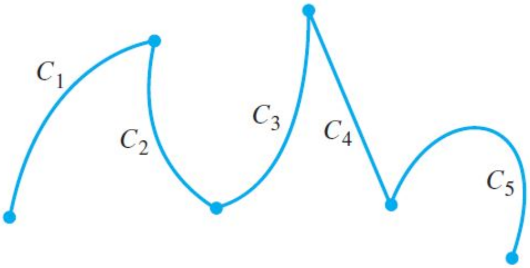
- $\mathbf{r}(t) = \cos t\ \mathbf{i} + \sin t\ \mathbf{j} + t\ \mathbf{k}$ gives a smooth curve.
- $y = x^2$ is a smooth plane curve. The parametric vector form is $r(t) = t\ \mathbf{i} + t^2\ \mathbf{j}$ and $r'(t) = \ \mathbf{i} + 2t\ \mathbf{j}$. Since $x^{'}(t) = 1$ for all t, $r^{'}(t) \neq 0$ for any $t \in (-\infty, \infty)$.

Piecewise smooth curve

A curve that is made up of a finite number of smooth curves pieced (joined) together in a continuous fashion.



Examples

Q. Determine the open intervals on which each of the following vector-valued functions is smooth:

1. $r(t) = (2t^3 - 3t^2)\ \mathbf{i} + (t^2 - 2t)\ \mathbf{j} + 2\ \mathbf{k}$
2. $r(t) = t^3\ \mathbf{i} + (\sqrt{t^2 - 1})\ \mathbf{j}$
3. $r(t) = t^4\ \mathbf{i} + \ln(t^2 + 1)\ \mathbf{j}$

Velocity, Speed, Acceleration and Unit Tangent

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2. At any time t, the direction of $\mathbf{v}$ is the **direction of motion**.
3. The magnitude of $\mathbf{v}$ is the particle's **speed**.
4. And the derivative $\mathbf{a} = d\mathbf{v}/dt$, when it exists, is the particle's **acceleration vector**.

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1. Velocity is the derivative of position: $\mathbf{v} = \mathbf{r}'(t)$.
2. Speed is the magnitude of velocity: Speed $= |\mathbf{v}(t)|$.
3. Acceleration is the derivative of velocity: $\mathbf{a} = \mathbf{v}'(t) = \mathbf{r}''(t)$.
4. The direction of motion at any time t is the unit vector $\mathbf{v}/|\mathbf{v}|$ (Unit tangent).

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4. The direction of motion at any time t is the unit vector $\mathbf{v}/|\mathbf{v}|$ (Unit tangent).
5. Velocity$=|\mathbf{v}|\left(\frac{\mathbf{v}}{|\mathbf{v}|}\right)=$(speed)(direction).

Example: Find the velocity, speed, and acceleration of a particle whose motion in space is given by the vector $\mathbf{r}(t) = (2\cos t)\,\mathbf{i} + (2\sin t)\,\mathbf{j} + (5\cos^2 t)\,\mathbf{k}$.

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Solution:

$$\mathbf{v} = \mathbf{r}'(t) = (-2\sin t)\,\mathbf{i} + (2\cos t)\,\mathbf{j} - (5\sin 2t)\,\mathbf{k}$$
$$\mathbf{a} = \mathbf{a}''(t) = (-2\cos t)\,\mathbf{i} - (2\sin t)\,\mathbf{j} - (10\cos 2t)\,\mathbf{k},$$

and the speed is given by

$$\begin{aligned} |\mathbf{v}(t)| &= \sqrt{(-2\sin t)^{+}(2\cos t)^2 + (-5\sin 2t)^2} \\ &= \sqrt{4 + 25\sin^2 2t}. \end{aligned}$$

When $t = 7\pi/4$, we have

$$\mathbf{v}\left(\frac{7\pi}{4}\right) = \sqrt{2}\,\mathbf{i} + \sqrt{2}\,\mathbf{j} + 5\mathbf{k}; \quad \left|\mathbf{v}\left(\frac{7\pi}{4}\right)\right| = \sqrt{29}.$$

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$$\mathbf{a}\left(\frac{7\pi}{4}\right) = -\sqrt{2}\,\mathbf{i} + \sqrt{2}\,\mathbf{j}.$$

Problems:

In the following, $\mathbf{r}(t)$ is the position of a particle in space at time t. Find velocity, speed, acceleration and the direction of the motion.

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3. $\mathbf{r}(t) = (2\ln(t+1))\,\mathbf{i} + t^2\,\mathbf{j} + \frac{t^2}{2}\,\mathbf{k}$, at $t = 1$

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2. $\mathbf{r}(t) = t^3\,\mathbf{i} + (2t+1)\,\mathbf{j} + t^2\,\mathbf{k}$, at $t_0 = 2$

Differentiation Rules for Vector Functions

Let $\mathbf{u}$ and $\mathbf{v}$ be differentiable vector functions of t, $\mathbf{C}$ a constant vector, c any scalar, and f any differentiable scalar function.

1. *Constant Function Rule:* $\dfrac{d}{dt}\mathbf{C} = \mathbf{0}$
2. *Scalar Multiple Rules:* $\dfrac{d}{dt}[c\mathbf{u}(t)] = c\mathbf{u}'(t)$

   $\dfrac{d}{dt}[f(t)\mathbf{u}(t)] = f'(t)\mathbf{u}(t) + f(t)\mathbf{u}'(t)$
3. *Sum Rule:* $\dfrac{d}{dt}[\mathbf{u}(t) + \mathbf{v}(t)] = \mathbf{u}'(t) + \mathbf{v}'(t)$
4. *Difference Rule:* $\dfrac{d}{dt}[\mathbf{u}(t) - \mathbf{v}(t)] = \mathbf{u}'(t) - \mathbf{v}'(t)$
5. *Dot Product Rule:* $\dfrac{d}{dt}[\mathbf{u}(t) \cdot \mathbf{v}(t)] = \mathbf{u}'(t) \cdot \mathbf{v}(t) + \mathbf{u}(t) \cdot \mathbf{v}'(t)$
6. *Cross Product Rule:* $\dfrac{d}{dt}[\mathbf{u}(t) \times \mathbf{v}(t)] = \mathbf{u}'(t) \times \mathbf{v}(t) + \mathbf{u}(t) \times \mathbf{v}'(t)$
7. *Chain Rule:* $\dfrac{d}{dt}[\mathbf{u}(f(t))] = f'(t)\mathbf{u}'(f(t))$

13.2 Integrals of Vector Functions

Integrals of Vector Functions:

Definition 0.6.

The indefinite integral of $\mathbf{r}$ with respect to t is the set of all antiderivates of $\mathbf{r}$, denoted by $\int \mathbf{r}(t)dt$. If $\mathbf{R}$ is any antiderivative of $\mathbf{r}$ (i.e. $\mathbf{R}'(t) = \mathbf{r}(t)$), then

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Note: Since $\mathbf{r}'(t) = x'(t)\mathbf{i} + y'(t)\mathbf{j} + z'(t)\mathbf{k}$, we have

$$\int \mathbf{r}(t)dt = \Big(\int x(t)dt\Big)\mathbf{i} + \Big(\int y(t)dt\Big)\mathbf{j} + \Big(\int z(t)dt\Big)\mathbf{k}$$

Example:

If $\mathbf{r}(t) = t^2\ \mathbf{i} + (2t-1)\ \mathbf{j} + t^3\ \mathbf{k}$, then

$$\int \mathbf{r}(t)dt = \frac{t^3}{3}\ \mathbf{i} + (t^2 - t)\ \mathbf{j} + \frac{t^4}{4}\ \mathbf{k} + \mathbf{C}.$$

Example:

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$$\int \mathbf{r}(t)dt = \frac{t^3}{3}\,\mathbf{i} + (t^2 - t)\,\mathbf{j} + \frac{t^4}{4}\,\mathbf{k} + \mathbf{C}.$$

Definition 0.7.

If the components of $\mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j} + z(t)\,\mathbf{k}$ are integrable over $[a,b]$, the so is $\mathbf{r}$ and the **definite integral** of $\mathbf{r}$ from a and b is

$$\int_a^b \mathbf{r}(t)dt = \Big(\int_a^b x(t)dt\Big)\mathbf{i} + \Big(\int_a^b y(t)dt\Big)\mathbf{j} + \Big(\int_a^b z(t)dt\Big)\mathbf{k}$$

Example:

If $\mathbf{r}(t) = t^2\ \mathbf{i} + (2t-1)\ \mathbf{j} + t^3\ \mathbf{k}$, then

Example:

If $\mathbf{r}(t) = t^2\ \mathbf{i} + (2t-1)\ \mathbf{j} + t^3\ \mathbf{k}$, then

$$\begin{aligned}
\int_0^1 \mathbf{r}(t)dt &= \left(\int_0^1 t^2 dt\right)\mathbf{i} + \left(\int_0^1 (2t-1)dt\right)\mathbf{j} + \left(\int_0^1 t^3 dt\right)\mathbf{k} \\
&= \left[\frac{t^3}{3}\right]_0^1 \mathbf{i} + \left[(t^2-t)\right]_0^1 \mathbf{j} + \left[\frac{t^4}{4}\right]_0^1 \mathbf{k} \\
&= \frac{1}{3}\ \mathbf{i} + \frac{1}{4}\ \mathbf{k}.
\end{aligned}$$

Theorem 0.8 (The Fundamental Theorem of Calculus).

Let $\mathbf{r}(t)$ *be a continuous vector function on* $[a, b]$ *and* $\mathbf{R}(t)$ *be any antiderivative of* $\mathbf{r}$, *then*

$$\int_a^b \mathbf{r}(t)dt = \mathbf{R}(t)\Big|_a^b = \mathbf{R}(b) - \mathbf{R}(a).$$

Problem: Suppose we do not know the path of a hang glider, but only its acceleration vector which is given by

$$\mathbf{a}(t) = -(3\cos t)\mathbf{i} - (3\sin t)\mathbf{j} + 2\mathbf{k}.$$

We also know that initially (at time $t = 0$) the glider departed from the point $(4, 0, 0)$ with velocity $\mathbf{v}(0) = 3\mathbf{j}$. Find the glider's position as a function of t.

Thank you