Functions of Several variables

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Limits for functions of two variables

We start with the definition of limit for two variable real valued functions.

Definition

Let f be a function of two variables whose domain D includes points arbitrarily close to (x_0, y_0). Then we say that the limit of $f(x, y)$ as (x, y) approaches (x_0, y_0), and write

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if for every $\epsilon > 0$, there exists a corresponding number $\delta > 0$ such that if (x, y) is in D and $0 < \sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta$, then $|f(x, y) - L| < \epsilon$.

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- For functions of two variables the situation is not as simple because we can let (x, y) approach (x_0, y_0) from an infinite number of directions in any manner whatsoever as long as (x, y) stays within the domain of f.
- Therefore, if the limit exists, then $f(x, y)$ must approach the same limit no matter how (x, y) approaches (x_0, y_0).
- Thus, if we can find two different paths of approach along which the function $f(x, y)$ has different limits, then it follows that $\lim_{(x,y)\to(x_0,y_0)} f(x, y)$ does not exist.

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Consider $f(x, y) = x$. *Show that* $\lim_{(x,y)\to(x_0,y_0)} f(x, y) = x_0$.

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Given any $\epsilon > 0$, choose $\delta = \epsilon$. Then for all $(x, y) \in B_\epsilon(x_0, y_0)$, we have

$$0 < \sqrt{(x - x_0)^2 + (y - y_0)^2} < \epsilon.$$

For all (x, y), we have $|x - x_0| \leq \sqrt{(x - x_0)^2 + (y - y_0)^2}$.
Therefore,
for all $(x, y) \in B_\epsilon(x_0, y_0)$,

$$|f(x, y) - f(x_0, y_0)| = |x - x_0| \leq \sqrt{(x - x_0)^2 + (y - y_0)^2} < \epsilon$$

Example

1. *Consider* $f(x,y) = y$. *Show that* $\lim_{(x,y)\to(x_0,y_0)} f(x,y) = y_0$.
2. *Consider* $f(x,y) = k$. *Show that* $\lim_{(x,y)\to(x_0,y_0)} f(x,y) = k$.

Algebra of limits

Let x_0, y_0, L and K be real numbers, let n be a positive integer, and let f and g be functions with the following limits:

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L \text{ and } \lim_{(x,y)\to(x_0,y_0)} g(x,y) = K.$$

Then

1. Sums/Differences: $\lim_{(x,y)\to(x_0,y_0)}(f(x,y) \pm g(x,y)) = L \pm K$
2. Products: $\lim_{(x,y)\to(x_0,y_0)} f(x,y)g(x,y) = LK$
3. Quotients: $\lim_{(x,y)\to(x_0,y_0)} \frac{f(x,y)}{g(x,y)} = \frac{L}{K}, (K \neq 0)$
4. Powers: $\lim_{(x,y)\to(x_0,y_0)} f(x,y)^n = L^n$
5. Root: $\lim_{(x,y)\to(x_0,y_0)} \sqrt[n]{f(x,y)} = \sqrt[n]{L}$

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$$\begin{aligned}
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&= \frac{\lim\limits_{(x,y)\to(1,\pi)} y}{\lim\limits_{(x,y)\to(1,\pi)} x} + \lim_{(x,y)\to(1,\pi)} \cos(xy) \\
&= \pi + \cos(\pi) = \pi - 1
\end{aligned}$$

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- The case where the limit does not exist is often easier to deal with, for we can often pick two paths along which the limit is different.

Example

Show that $\lim_{(x,y)\to(0,0)} \frac{3xy}{x^2+y^2}$ *does not exist.*

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Let $y = mx, x \neq 0$. Then

$$\lim_{\substack{(x,y)\to(0,0)\\ \text{along } y=mx}} f(x,y) = \lim_{x\to 0} \frac{3mx^2}{x^2+m^2x^2}$$
$$= \frac{3m}{1+m^2}$$

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The value of the limit is different for different values of m. i,e, if we choose different straight line paths to reach $(0,0)$ each path is giving a different value as limit. Hence limit does not exist in this case.

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Let $y = mx, x \neq 0$, then $\lim_{\substack{(x,y)\to(0,0) \\ \text{along } y=mx}} \frac{mx^3}{x^2+m^4x^4} = 0$.

Thus f has the same limiting value along every nonvertical line through the origin.

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Now let $(x,y) \to (0,0)$ along the parabola $x = y^2, y \neq 0$. Then

$$\lim_{\substack{(x,y)\to(0,0)\\ \text{along } x=y^2}} f(x,y) = \lim_{y\to(0)} \frac{y^4}{y^4+y^4} = \frac{1}{2}$$

Even though all the straight line paths give the same limit (which is 0 here), the limits were different for path $y = mx$ and $x = y^2$ and hence limit does not exist in this case.

Example

Let $f(x,y) = \frac{5x^2y^2}{x^2+y^2}$. *Find* $\lim\limits_{(x,y)\to(0,0)} f(x,y)$.

If we take the limit along $y = mx, x \neq 0$ line we get

$$\lim_{\substack{(x,y)\to(0,0)\\ \text{along } y=mx}} \frac{5x^2y^2}{x^2+y^2} = 0.$$

This is not enough to prove that the limit exists, as demonstrated in the previous example, but it tells us that if the limit does exist then it must be 0.

Remark

When indeterminate forms arise, if limit exists, it can be difficult to prove the existence of limit as we need to show the same limiting value is obtained regardless of the path chosen(i.e we need to apply the definition of limit.)

To prove the limit is 0, we apply definition. Let $\epsilon > 0$ be given. We need to find a $\delta > 0$ such that

$$\forall\ (x, y) \in B_\delta(0, 0) \implies |f(x, y) - 0| < \epsilon.$$

Now $x^2 \leq x^2 + y^2$ and $y^2 \leq x^2 + y^2$. Therefore

$$|f(x, y) - 0| = |\frac{5x^2y^2}{x^2 + y^2}| = \frac{5x^2y^2}{x^2 + y^2} \leq 5x^2 \leq 5(x^2 + y^2)$$

Remember we need to find a $\delta > 0$ such that

$$\text{if } 0 < \sqrt{(x-0)^2 + (y-0)^2} < \delta, \text{ then } |f(x,y) - 0| < \epsilon. \tag{1}$$

We know that

$$|f(x,y) - 0| \leq 5(x^2 + y^2) < \epsilon \text{ if } x^2 + y^2 < \frac{\epsilon}{5} \tag{2}$$

$$i.e \text{ if } \sqrt{x^2 + y^2} < \sqrt{\frac{\epsilon}{5}}. \tag{3}$$

From (1) and (3), choose $\delta = \sqrt{\frac{\epsilon}{5}}$, then $\forall (x,y)$ with $x^2 + y^2 < \delta^2 = \frac{\epsilon}{5}$ we have

$$|f(x,y) - 0| \leq 5(x^2 + y^2) < 5 \times \frac{\epsilon}{5} < \epsilon.$$

Thus we proved limit exists and equals to 0.

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$$\lim_{(x,y)\to(0,0)} \frac{4xy^2}{x^2+y^2} = \lim_{x\to 0} \frac{4x^3}{2x^2} = \lim_{x\to 0} 2x = 0.$$

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Thus if the limit exists, it must be 0.
But it remains to prove that the limit is 0. Now let $\epsilon > 0$ be given. Now

$$\left|\frac{4xy^2}{x^2+y^2} - 0\right| = \left|\frac{4xy^2}{x^2+y^2}\right| \leq |4x| \leq 4|x| \leq 4\sqrt{x^2+y^2}.$$

Choose $\delta = \frac{\epsilon}{4}$. Then whenever $0 < \sqrt{x^2+y^2} < \delta = \frac{\epsilon}{4}$,

$$|f(x,y) - 0| < 4 \times \frac{\epsilon}{4} = \epsilon.$$

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Thus for different values of m, we get different values for limit. Thus the limit does not exists.

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Now for different values of m, we get different values for $\frac{2m}{1+m^2}$. Thus the limit does not exists.