Sequence and Series

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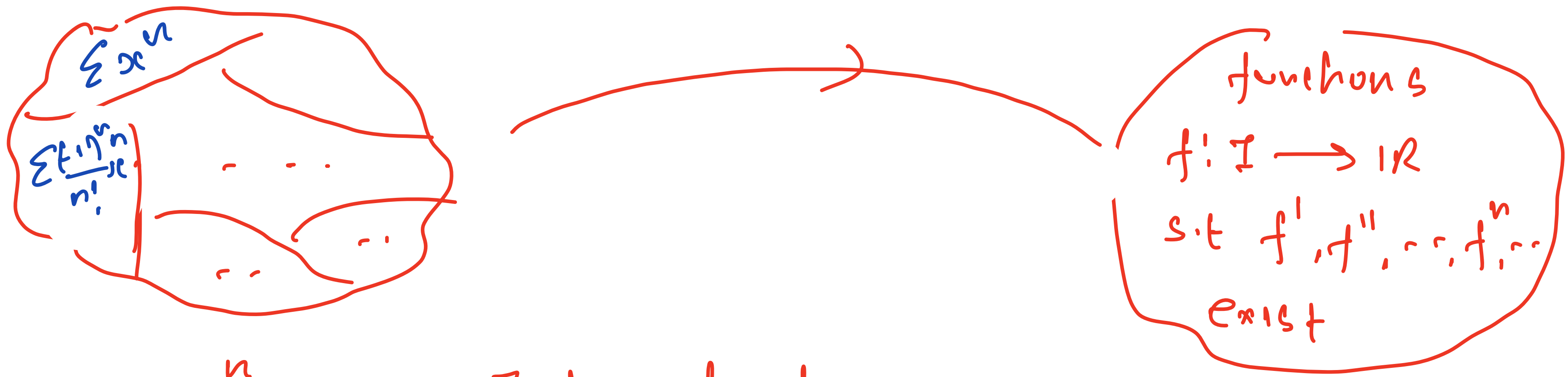
Recall



$\sum c_n (x-a)^n \implies$ Interval of convergence I

$\longrightarrow f: I \to \mathbb{R}$

Infact

$$\sum c_n (x-a)^n = f(x)$$

$$\sum_{n=0}^{\infty} \frac{f^n(a)}{n!}(x-a)^n \quad \longleftarrow \cdots\cdots\cdots \quad \text{Given } f: I \to \mathbb{R}$$

s.t derivatives f^n exist.

Definition

Let f be a function with derivatives of all orders throughout some interval containing a as an interior point. Then the Taylor series generated by f at $x = a$ is $\displaystyle\sum_{k=0}^{\infty} \frac{f^k(a)}{k!}(x-a)^k$

$$= f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^n(a)}{n!}(x-a)^n + \cdots$$

The Maclaurin series generated by f is the Taylor series generated by f at $x = 0$ given by

$$\sum_{k=0}^{\infty} \frac{f^k(0)}{k!}x^k = f(0) + f'(0) + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^n(0)}{n!}x^n + \cdots$$

Example

Find the Taylor series generated by $f(x) = \frac{1}{x}$ at $a = 2$.

$f : \mathbb{R} - \{0\} \longrightarrow \mathbb{R}$

$f(x) = \frac{1}{x}$ $\qquad f(2) = \frac{1}{2}$

$f'''(x) = -2\times3\frac{1}{x^4}\Big|_{x=2}$

$= \frac{-2\times3}{2^4}$

$\Rightarrow$

$f : (2-\varepsilon, 2+\varepsilon) \longrightarrow \mathbb{R}$ s.t. f^n exist

$f(x) = \frac{1}{x}$

$f'(2) = -\frac{1}{x^2}\Big|_{x=2} = -\frac{1}{4}$

$f''(2) = +\frac{2}{x^3}\Big|_{x=2} = \frac{1}{4}$

$\vdots$

Taylor series about $x=2$

$$= \sum_{n=0}^{\infty} \frac{f^n(2)}{n!}(x-2)^n = f(2) + f'(2)(x-a) + \frac{f''(2)}{2!}(x-a)^2 + \cdots$$

$$= \frac{1}{2} + \left(-\frac{1}{4}\right)(x-a) + \left(\frac{1}{4}\right)\frac{1}{2!}(x-a)^2 + \cdots$$

Example

Find the Taylor series generated by $f(x) = \frac{1}{x}$ at $a = 2$. Where does the series converge to?

$$\frac{1}{2} + \left(-\frac{1}{4}\right)(x-2)^1 + \frac{1}{4\times 2}(x-2)^2 - \frac{1}{2^4}(x-2)^3 + - \cdots$$

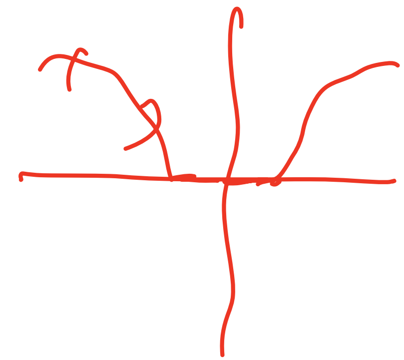
$\sum_{n=1} x^n$ converges $|x|<1$ and it is equal to $\frac{1}{1-x}$

This converges $\left|\frac{x-2}{2}\right| < 1$ and it is equal to $\frac{1/2}{1-\frac{x-2}{2}} = \frac{1}{x}$

$$\Updownarrow$$

$0 < x < 4$

$I = (-1,1)$ at $x = 1/2$



Consider

$$f(x) = \begin{cases} 0 & \text{if } x = 0 \\ e^{\frac{-1}{x^2}} & \text{if } x \neq 0 \end{cases}$$

Find the Taylor series generated by f at $x = 0$. Also where does it converge?

$f'(x)\Big|_{x=0} = 0$ Infact $f^n(x)\Big|_{x=0} = 0$

$f(x) \text{ - - - - } \to \sum \frac{f^n(0)}{n!}(x)^n = f(0) + f'(0)x^1 + \frac{f''(0)}{2!}x^2 + \cdots$

$= 0$

This series converges everywhere and converges to 0

at $x = 1/2$, $f(1/2) = e^{-4} \neq 0$

Consider

$$f(x) = \begin{cases} 0 & \text{if } x = 0 \\ e^{\frac{-1}{x^2}} & \text{if } x \neq 0 \end{cases}$$

Find the Taylor series generated by f at $x = 0$. Also where does it converge?

Clearly f has derivatives of all orders at $x = 0$ and that $f^n(0) = 0$ for all n.

The Taylor series generated by f at $x = 0$ is 0 and thus Taylor series converges for all values of x.

But $f(x_0) = e^{\frac{-1}{x_0^2}} \neq 0$ when $x_0 \neq 0$. Thus the series converges to $f(x)$ only at $x = 0$.

Hence, if we start with an arbitrary function f that is infinitely differentiable on an interval I centered at $x = a$ and use it to generate the series, will the series then converge to $f(x)$ at each x in the interior of I? The answer is
maybe for some functions it will but for other functions it will not.

$$f(x) \rightsquigarrow \sum \frac{f^{n}(a)}{n!}(x-a)^{n}$$

converges

$$= f(x)$$

$$f(x) = f(a) + \cdots + \frac{f^{n}(a)}{n!}(x-a)^{n} + \text{error term}$$

(depends)

$\lim_{n \to \infty}$ $\cdot \lim_{n \to \infty}$

Definition

Taylor Polynomial of order n: Let f be a function with derivatives of order k for $k = 1, 2, \cdots, N$ in some interval containing a as an interior point. Then for any integer n from 0 to N, the Taylor polynomial of order n generated by f at $x = a$ is the polynomial

$$P_n(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^n(a)}{n!}(x-a)^n$$

We speak of a Taylor polynomial of order n rather than degree n because $f^n(a)$ may be zero.

Ex. $f(x) = \sin x$ Find Taylor Poly of order 3 at $x = 0$

$\sin(0) = 0$

$f'(x) = \cos x$, $f'(0) = 1$, $f''(x) = -\sin x$, $f''(0) = 0$, $f'''(x) = -\cos x$

degree 4

$$P_n(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \frac{f^{iv}(a)}{4!}(x-a)^4$$

$$= 0 + 1(x) + \frac{0}{2!}(x)^2 \; \frac{-1}{3!}x^3 + \frac{0}{4!}x^4$$

$$= x - \frac{x^3}{3!}$$ ✓

Taylor poly of order 4

$$P_4(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}$$

Taylor's Formula

Theorem

Taylor's Theorem: If f has derivatives of all orders in an open interval I containing a, then for each positive integer n and for each $x \in I$,

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^n(a)}{n!}(x-a)^n + R_n(x)$$

where $R_n(x) = \dfrac{f^{n+1}(c)}{(n+1)!}(x-a)^{n+1}$ *for some c between a and x.*

That is, the Taylor's theorem says that for each $x \in I$, there exists $c \in (a, x)$ such that

$$f(x) = P_n(x) + R_n(x).$$

The function $R_n(x)$ is called the remainder of order n or the error term for the approximation of f by $P_n(x)$ over I.

Taylor series converging to f

If $R_n(x) \to 0$ as $n \to \infty$, for all $x \in I$, we say that the Taylor series generated by f at $x = a$ converges to f on I, and we write

$$f(x) = \sum_{k=0}^{\infty} \frac{f^k(a)}{k!}(x-a)^k.$$

Often we can estimate $R_n(x)$ without knowing the value of c, as the following example illustrates.

1. Show that the Taylor series generated by $f(x) = e^x$ at $x = 0$ converges to $f(x)$ for every real value of x.

$f'(x)\big|_{x=0} = e^x = 1 \quad \cdots \qquad f^n(x)\big|_{x=0} = e^x = 1$

$f(x) = \text{Taylor poly of order } n + R_n(x)$

$= 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + R_n(x)$

$R_n(x) = \frac{e^c}{(n+1)!} x^{n+1}$ where $c \in (0, x)$

e^x is an increasing function, when $x \geq 0$

$e^c \leq e^x$

when $x \leq 0$, $e^c \leq 1$

$$|R_n(x)| = \left| \frac{e^c x^{n+1}}{(n+1)!} \right| \leq \begin{cases} \dfrac{x^{n+1}}{(n+1)!} & x \leq 0 \\[2ex] \dfrac{e^x x^{n+1}}{(n+1)!} & x \geq 0 \end{cases}$$

✓

Apply ratio test

$x \leq 0$, $a_n = \dfrac{x^{n+1}}{(n+1)!}$ $\quad \lim_{n\to\infty} \left| \dfrac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \dfrac{|x|}{(n+2)}$

$\Rightarrow 0 < 1$

$x \geq 0$, $a_n = \dfrac{e^x x^{n+1}}{(n+1)!}$, $\lim_{n\to\infty} \left| \dfrac{a_{n+1}}{a_n} \right| \longrightarrow 0$

$\Rightarrow$ as $n \to \infty$, $R_n(x) \to 0$

$$\therefore e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots$$

$\forall x \in \mathbb{R}$

$= (-\infty, \infty)$

1. Show that the Taylor series generated by $f(x) = e^x$ at $x = 0$ converges to $f(x)$ for every real value of x.

The function has derivatives of all orders throughout the interval $(-\infty, \infty)$. We get

$$f(x) = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots + \frac{1}{n!}x^n + R_n(x)$$

and

$$R_n(x) = \frac{f^{n+1}(c)}{(n+1)!}x^{n+1} = \frac{e^c}{(n+1)!}x^{n+1}$$

for some c between 0 and x. Since e^x is an increasing function of x, $e^c < 1$ if $x \leq 0$ and $e^c \leq e^x$ for $x > 0$. Thus $|R_n(x)| \to 0$ as $n \to 0$.

Thus the series $\displaystyle\sum_{k=0}^{\infty} \frac{x^k}{k!}$ converges to e^x for every x.

2. Show that the Taylor series for $f(x) = \cos x$ at $x = 0$ converges to $\cos x$ for every value of x.

$$f(x) = P_n(x) + R_n(x)$$

$$P_n(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots (-1)^n \frac{x^{2n}}{(2n)!}$$

$$R_n(x) = f^{n+1}(c) \frac{x^{2n+1}}{(2n+1)!}, \qquad c \in (0, x)$$

$$\because f(x) = \sin(x), \qquad |f^{n+1}(c)| \leq 1 \quad \forall\, c \in (0, x)$$

$$|R_n(x)| \leq \frac{|x|^{2n+1}}{(2n+1)!} \longrightarrow 0 \quad \text{as } n \to \infty$$

Using Ratio test

2. Show that the Taylor series for $f(x) = \cos x$ at $x = 0$ converges to $\cos x$ for every value of x.

The Taylor series for $f(x) = \cos x$ around $x = 0$ given by

$$1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + (-1)^k \frac{x^{2k}}{(2k)!} + R_{2k}(x)$$

and

$$|R_{2k}(x)| \leq 1 \cdot \frac{|x|^{2k+1}}{(2k+1)!}.$$

For every value of x, $R_{2k} \to 0$ as $k \to 0$. Thus $f(x) = \cos x$ converges for all x. Thus

$$\begin{aligned}\cos x &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!}\end{aligned}$$

3. Show that the Taylor series for $f(x) = \sin x$ at $x = 0$ converges to $\sin x$ for every value of x.

$$P_n(x) = x - \frac{x^3}{3!} + \cdots (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

$$R_{2n+2}(x) = \frac{f^{2n+2}(c)\, x^{2n+2}}{(2n+2)!}$$

$$|R_{2n+2}(x)| \leq 1 \cdot \frac{|x|^{2n+2}}{(2n+2)!} \longrightarrow 0$$