Friday 10-11 am
Tuesday 12-1 pm

Sequence and Series

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$\sum \frac{(-1)^{n+1}}{n}$ $\sum \frac{1}{n}$ $\sum \frac{1}{n^2}$ September 3, 2024 $-|a_n| \le a_n \le |a_n|$

$$\sum_{n=1}^{\infty} a r^{n-1}, \quad a - \text{constant}$$

$$r = \frac{1}{2}, \quad \sum_{n=1}^{\infty} a\left(\frac{1}{2}\right)^{n-1}, \quad r = 2, \quad \sum a(2)^{n-1}$$

$$\sum x^n = f(x) \qquad \mathbb{R} \xrightarrow{f} \mathbb{R}$$

$$x \longmapsto \sum x^n$$

Does $f(1)$ exist? No!

Does $f(2)$ exist? No!

$$(-1, 1) \xrightarrow{f} \mathbb{R}$$

Does $f\left(\frac{1}{2}\right)$ exist? Yes

Power series

$\sum(-1)^{n+1}x^n \quad \sum 2x^n$

Definition

A power series about $x = 0$ is a series of the form

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n + \cdots = f(x)$$

Domain $\to \mathbb{R}$

??

A power series about $x = a$ is a series of the form

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots + c_n(x-a)^n + \cdots$$

in which the center a and the coefficients $c_0, c_1, c_2, \cdots, c_n, \cdots$ are constants.

When we fix a value for x, say $x = 1$, the power series

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 + c_2 + \cdots + c_n + \cdots$$

is an infinite series whose convergence or divergence can be investigated.

For $x = 2$, the power series $\sum\limits_{n=0}^{\infty} c_n x^n = c_0 + 2c_1 + 4c_2 + \cdots + 2^n c_n + \cdots$.

If for an x, the series $\sum\limits_{n=0}^{\infty} c_n x^n$ converges we can use the limit of partial sequences to define a function f at x.

- We will see that a power series defines a function $f(x)$ on a certain interval where it converges.
- Finding this interval of convergence is important. Moreover, this function will be shown to be continuous and differentiable inside the interval.

Geometric Power series

Let us consider some familiar power series.

$$f(x) = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + x^4 + \cdots + x^n + \cdots .$$

$x=1$, $\sum_{n=0}^{\infty} 1^n$ diverges

$x=-1$, $\sum (-1)^n$ diverges

$x \neq 1$, $S_n(x) = 1 + x + x^2 + \cdots + x^n$

$= \frac{1-x^{n+1}}{1-x}$

$|x|>1$, $S_n(x) \to \infty \Rightarrow \sum x^n$ diverges $|x|>1$

$|x|<1$, $x^{n+1} \to 0$ $\therefore \lim_{n\to\infty} S_n(x) = \frac{1}{1-x}$

$\boxed{R=1}$

$\sum x^n = \frac{1}{1-x}$ $\boxed{-1<x<1}$

$\sum x^n : (-1,1) \to \mathbb{R}$

$= 1/1-x$

Geometric Power series

Let us consider some familiar power series.

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + x^4 + \cdots + x^n + \cdots .$$

This is the geometric series with first term 1 and common ratio x. It converges to $\frac{1}{(1-x)}$ for $|x| < 1$. We express this fact by writing

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \cdots + x^n + \cdots , -1 < x < 1.$$

We think of the partial sums of the series on the right as polynomials $P_n(x)$ that approximate the function on the left.

$$\begin{aligned} P_1(x) &= 1 \\ P_2(x) &= 1 + x \\ P_3(x) &= 1 + x + x^2 \\ P_n(x) &= 1 + x + x^2 + \cdots + x^n \end{aligned}$$

Example

Consider the power series

$$f(x) = 1 - \frac{1}{2}(x-2) + \frac{1}{4}(x-2)^2 + \cdots + (-\frac{1}{2})^n(x-2)^n + \cdots$$

$a = 1$

$r = -\frac{1}{2}(x-2)$

$$\frac{a(1-r^n)}{1-r}$$

$f(x)$ converges on $|r| < 1$

$$\Rightarrow \frac{|x-2|}{2} < 1 \Rightarrow \underline{\underline{|x-2| < 2}}$$

$$\Rightarrow x \in (0,4)$$

$\therefore f(x) : (0,4) \longrightarrow \mathbb{R}$

$$f(x) = \sum \left(-\frac{1}{2}\right)^n (x-2)^n$$

$$= \frac{1}{1 + \frac{(x-2)}{2}} = \frac{2}{2+x-2} = \frac{2}{x}$$

$\boxed{R = 2}$

Example

Consider the power series

$$1 - \frac{1}{2}(x-2) + \frac{1}{4}(x-2)^2 + \cdots + (-\frac{1}{2})^n (x-2)^n + \cdots$$

This is a geometric series with first term 1 and ratio $r = -\frac{(x-2)}{2}$. The series converges for

$$\left|\frac{x-2}{2}\right| < 1$$

which simplifies to $0 < x < 4$. The sum is $\frac{1}{1-r} = \frac{2}{x}$. Thus

$$\frac{2}{x} = 1 - \frac{1}{2}(x-2) + \frac{1}{4}(x-2)^2 + \cdots + (-\frac{1}{2})^n (x-2)^n + \cdots, 0 < x < 4.$$

The Convergence Theorem for Power Series

✓ $\sum x^n$ ✓ $\sum (-1)^{n+1} \frac{x^n}{n}$ ✓

$|x| > 1$ $|x| < 1$

Theorem

If the power series

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots$$

converges at $x = c \neq 0$*, then it converges absolutely for all* x *with* $|x| < |c|$*. If the series diverges at* $x = d$*, then it diverges for all* x *with* $|x| > |d|$*.*

Radius of convergence

$x \in (a-R, a+R)$

$\Leftarrow$ converges $|x-a| < R$

$x \in (-\infty, a-R) \cup (a+R, \infty)$ diverges $|x-a| > R$

Definition

The radius of convergence of the series $\sum_{n=0}^{\infty} a_n(x-a)^n$ is a positive number R such that the series diverges for x with $|x-a| > R$ but converges absolutely for x with $|x-a| < R$. The series may or may not converge at either of the endpoints $x = a - R$ and $x = a + R$.

Remarks.

1. If the series converges absolutely for every x, then $R = \infty$.
2. If the series converges at $x = a$ and diverges elsewhere, then $R = 0$.

The interval of radius R centered at is called the interval of convergence.
$(a-R, a+R)$ or $[a-R, a+R)$ or $(a-R, a+R]$, $[a-R, a+R]$

How to calculate R?

- Use ratio test or root test to find R such that in $|x-a| < R$, the power series $\sum_{n=0}^{\infty} a_n(x-a)^n$, converges.
- Check for the convergence at $|x-a| = R$ to conclude if R also a part of interval of convergence.

Example

For what values of x the power series $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}$ converge?

$$\lim_{n\to\infty} \left| \frac{a_{n+1}(x)}{a_n(x)} \right| = \lim_{n\to\infty} \left| \frac{(-1)^n x^{n+1}}{n+1} \times (-1)^{n-1} \frac{n}{x^n} \right|$$

$$= \lim_{n\to\infty} \left| (-1)\, x \times \frac{1}{1+\frac{1}{n}} \right|$$

$$= |-x| = |x| = l$$

converges when $|x| < 1$

diverges when $|x| > 1$

$x = 1$, $x = -1$

$$\sum \frac{(-1)^{n-1}}{n} \qquad \sum \frac{(-1)^{2n-1}}{n}$$

Example

converge $= \sum \frac{1}{n}$

diverges

For what values of x the power series $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{n}$ converge?

- $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{x^{n+1}}{n+1}\cdot\frac{n}{x^n}\right| = \lim_{n\to\infty}\frac{n}{n+1}|x| = |x|.$
- By the Ratio Test, the series converges absolutely for $|x| < 1$ and diverges for $|x| > 1$.
- At $x = 1$, we get the alternating harmonic series $1 - \frac{1}{2} + \frac{1}{3} - \cdots$ which converges.
- At $x = -1$, we get the harmonic series $1 + \frac{1}{2} + \frac{1}{3} + \cdots$ which diverges. Thus $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{n}$ converges for $-1 < x \leq 1$ and diverges elsewhere.

$\boxed{R=1}$

Interval of convergence $= (-1, 1]$

Example

For what values of x the power series $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^{2n-1}}{2n-1}$ converges?

$$\lim_{n\to\infty}\left|\frac{x^{2(n+1)-1}}{2n+1}\times\frac{2n-1}{x^{2n-1}}\right|$$

$$= |x|^2 = r$$

converges $|x|^2<1 \Rightarrow -1<x<1$

diverges $|x|^2>1 \Rightarrow x\in(-\infty,-1)\cup(1,\infty)$

$x=1$, $\sum (-1)^{n-1}\frac{1}{2n-1}$ converges

$x=-1$ $\sum \frac{(-1)^{n-1+2n-1}}{2n-1} = \sum\frac{(-1)^{3n-2}}{2n-1}$

Example

converges

For what values of x the power series $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^{2n-1}}{2n-1}$ converges?

- By the Ratio Test, the series converges for $x^2 < 1$ and diverges for $x^2 > 1$.
- At $x = \pm 1$, the alternating series converges.
- Thus the series converges for $-1 \leq x \leq 1$ and diverges elsewhere.

Interval of convergence : $[-1, 1]$

$\boxed{R = 1}$

Example

For what values of x the power series $\sum_{n=1}^{\infty} \frac{x^n}{n!}$ converges?

$$\lim_{n\to\infty} \left| \frac{x^{n+1}}{(n+1)!} \times \frac{n!}{x^n} \right|$$

$$= |x| \lim_{n\to\infty} \frac{1}{n+1} = 0 \quad \forall x$$

converges $\forall x$

Interval of convergence

$= (-\infty, \infty)$

$$\boxed{R = \infty}$$

Example

For what values of x the power series $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{n!}$ converges?

- By ratio test, $\left|\dfrac{a_{n+1}}{a_n}\right| = \left|\dfrac{x^{n+1}}{(n+1)!} \cdot \dfrac{n!}{x^n}\right| = \dfrac{|x|}{n+1} \to 0$ for every x.
- The series converges absolutely and hence converges for all $x \in \mathbb{R}$.

Example

For what values of x the power series $\sum_{n=1}^{\infty} n! x^n$ converges?

$$\lim_{n\to\infty} \left| \frac{(n+1)!\, x^{n+1}}{n!\, x^n} \right| = |x| \lim_{n\to\infty} n+1$$

$$= \begin{cases} 0 & x = 0 \\ \infty & x \neq 0 \end{cases}$$

converges at $x = 0$

otherwise divergence

Interval of convergence $= \{0\}$

$R = 0$

Example

For what values of x the power series $\sum_{n=1}^{\infty} n!x^n$ converges?

By ratio test, $\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(n+1)!x^{n+1}}{n!x^n}\right| = (n+1)|x| \to \infty$ except for $x = 0$. The series diverges for all values of x except at $x = 0$.