MATH F111- Mathematics I

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For a general series with both positive and negative terms, we can apply the tests for convergence studied before to the series of absolute values of its terms.

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Now $a_n = (a_n + |a_n| - |a_n|)$ and $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty}(a_n + |a_n|) - \sum_{n=1}^{\infty} |a_n|$.

Thus $\sum a_n$ converges.

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Remark

Series is absolute convergent $\implies$ series is convergent

Converse not true: $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ *converges, but* $\sum_{n=1}^{\infty} \frac{1}{n}$ *doesn't converge, hence* $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ *doesn't converge absolutely.*

Ratio test

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For a geometric series ar^n, this rate is a constant $\left|\frac{ar^{n+1}}{ar^n}\right| = |r|$ and we know that the series converges if and only if $|r| < 1$. The Ratio Test is a powerful rule extending that result.

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- (c) *the test is inconclusive if* $r = 1$.

If we apply ratio test for $\displaystyle\sum_{n=0}^{\infty} \frac{1}{n}$ (diverges) and $\displaystyle\sum_{n=0}^{\infty} \frac{1}{n^2}$ (converges), both cases $r = 1$. This shows that some other test for convergence must be used when $r = 1$.

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- $\sum_{n=0}^{\infty} \frac{2^n}{n^3}$, $\lim_{n\to\infty} \sqrt[n]{|a_n|} = 2$, so diverges.

Alternating Series test

Theorem

The series

$$\sum_{n=0}^{\infty}(-1)^{n+1}u_n = u_1 - u_2 + u_3 - u_4 + \cdots$$

converges if the following conditions are satisfied:

1. *The* $u_n > 0$.
2. $u_n \geq u_{n+1}$ *for all* $n \geq N$, *for some integer* N.
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Rearranging terms in a series

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\begin{aligned}
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=& 2 - 1 + \frac{2}{3} - \frac{1}{2} + \frac{2}{5} - \frac{1}{3} + \frac{2}{7} - \frac{1}{4} + \frac{2}{9} - \frac{1}{5} + \frac{2}{11} \cdots \\
=& (2-1) - \frac{1}{2} + \left(\frac{2}{3} - \frac{1}{3}\right) - \frac{1}{4} + \left(\frac{2}{5} - \frac{1}{5}\right) - \frac{1}{6} + \left(\frac{2}{7} - \frac{1}{7}\right) - \frac{1}{8} + \cdots \\
=& 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots \\
=& L.
\end{aligned}
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Theorem

The Rearrangement Theorem for Absolutely Convergent Series: *If* $\sum_{n=0}^{\infty} a_n$ *converges absolutely, and* $b_1, b_2, \cdots, b_n, \cdots$ *is any arrangement of the sequence* (a_n)*, then* $\sum_{n=0}^{\infty} b_n$ *converges absolutely and* $\sum_{n=0}^{\infty} a_n = \sum_{n=0}^{\infty} b_n$*.*