Mathematics I- MATH F111

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Curves in Space

Suppose a particle is moving in space during a time interval I. We think of the particle's coordinates as functions defined on I:

$$x = f(t), y = g(t), z = h(t); t \in I$$

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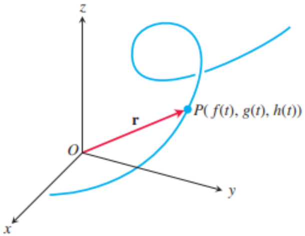
The points $(x, y, z) = (f(t), g(t), h(t))$, $t \in I$, make up the **curve** in space is called the particle's path.

A curve in space can also be represented in vector form. The vector

$$\overrightarrow{r}(t) = \overrightarrow{OP} = f(t)\hat{i} + g(t)\hat{j} + h(t)\hat{k} \tag{1}$$

from the origin to the particle's position $P(f(t), g(t), h(t))$ at time t is the particle's position vector.



The functions f, g and h are the **component functions** (or components) of the position vector.

Remark

We think of the particle's path as the curve traced by $\overrightarrow{r}$ during the time interval I.

Equation (1) defines $\vec{r}$ as a vector function of the real variable t on the interval I. More generally, a **vector-valued function** or **vector function** on a domain set D is a rule that assigns a vector in space to each element in D. The domains is the intervals of real numbers and the graph of the function represents a curve in space.

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Remark

Real-valued functions are called scalar functions to distinguish them from vector functions.

Example

Graph the vector function

$$\vec{r}(t) = (\cos t)\hat{i} + (\sin t)\hat{j} + t\hat{k}.$$

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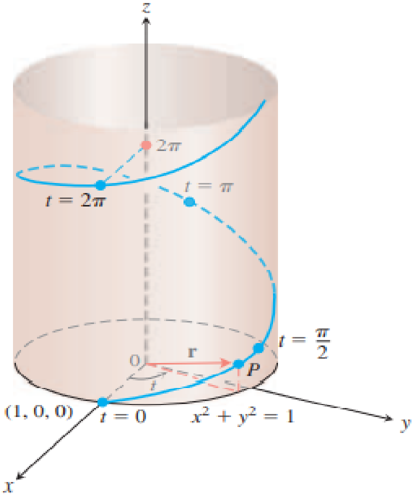
Solution: We first observe that

$$x^2 + y^2 = \cos^2 t + \sin^2 t = 1.$$

Thus the entire curve lies in the cylinder $x^2 + y^2 = 1$.
The curve rises as the k-component $z = t$ increases. Each time t increases by 2π, the curve completes one turn around the cylinder. The curve is called a helix (from an old Greek word for 'spiral').


z
2π
$t = 2\pi$
$t = \pi$
$t = \frac{\pi}{2}$
0
r
P
t
(1, 0, 0)
$t = 0$
$x^2 + y^2 = 1$
y
x

Limit of a vector valued function

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if, for every number $\epsilon > 0$, there exists a corresponding number $\delta > 0$ such that for all t with $|t - t_0| < \delta \implies |\vec{r}(t) - \vec{L}| < \epsilon$.

Properties of a Limit

Proposition

Let $\vec{r}(t) = f(t)\hat{i} + g(t)\hat{j} + h(t)\hat{k}$ *be a vector valued function and* $\vec{L} = \ell_1\hat{i} + \ell_2\hat{j} + \ell_3\hat{k}$. *Then* $\lim_{t \to t_0} \vec{r}(t) = \vec{L}$ *if and only if*

$$\lim_{t \to t_0} f(t) = \ell_1, \lim_{t \to t_0} g(t) = \ell_2, \lim_{t \to t_0} h(t) = \ell_3.$$

Example

If $\vec{r}(t) = (\cos t)\hat{i} + (\sin t)\hat{j} + t\hat{k}$*, then find* $\lim_{t \to \frac{\pi}{4}} \vec{r}(t)$.

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Solution: We have $\lim\limits_{t \to \frac{\pi}{4}} \cos t = \dfrac{1}{\sqrt{2}}$, $\lim\limits_{t \to \frac{\pi}{4}} \sin t = \dfrac{1}{\sqrt{2}}$, $\lim\limits_{t \to \frac{\pi}{4}} t = \dfrac{\pi}{4}$. Then by the above proposition, we have $\lim\limits_{t \to \frac{\pi}{4}} \vec{r}(t) = \dfrac{1}{\sqrt{2}}\hat{i} + \dfrac{1}{\sqrt{2}}\hat{j} + \dfrac{\pi}{4}\hat{k}$.

Continuity

Definition

Let $\vec{r}(t) = f(t)\hat{i} + g(t)\hat{j} + h(t)\hat{k}$ be a vector valued function with domain I, an interval. Then we say $\vec{r}(t)$ is **continuous** at a point $t = t_0$ in I, if

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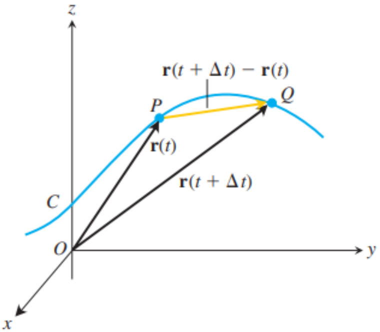
A vector function $\vec{r}(t)$ is **continuous** if it is continuous at every points in it's domain.

- The vector function $\vec{r}(t) = (\cos t)\hat{i} + (\sin t)\hat{j} + t\hat{k}$ is continuous everywhere.
- The vector function $\vec{r}(t) = (\cos t)\hat{i} + (\sin t)\hat{j} + \lfloor t \rfloor \hat{k}$ is discontinuous at every integer, where the $\lfloor t \rfloor$ is discontinuous.

Derivatives and Motion

Let a particle move along a vector function $\vec{r}(t) = f(t)\hat{i} + g(t)\hat{j} + h(t)\hat{k}$ where $f(t), g(t), h(t)$ are differentiable functions of t. The difference between the particle's positions at time t and time $t + \Delta t$ is

$$\Delta\vec{r} = \vec{r}(t + \Delta t) - \vec{r}(t)$$



$$
\begin{aligned}
\Delta\vec{r} =& \vec{r}(t+\Delta t) - \vec{r}(t) \\
=& f(t+\Delta t)\hat{i} + g(t+\Delta t)\hat{j} + h(t+\Delta t)\hat{k} - [f(t)\hat{i} + g(t)\hat{j} + h(t)\hat{k}] \\
=& [f(t+\Delta t) - f(t)]\hat{i} + [g(t+\Delta t) - g(t)]\hat{j} + [h(t+\Delta t) - h(t)]\hat{k}.
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\end{aligned}$$

Then

$$\begin{aligned}
\lim_{\Delta t\to 0}\frac{\Delta\vec{r}}{\Delta t} =& \Big[\lim_{\Delta t\to 0}\frac{f(t+\Delta t) - f(t)}{\Delta t}\Big]\hat{i} + \Big[\lim_{\Delta t\to 0}\frac{g(t+\Delta t) - g(t)}{\Delta t}\Big]\hat{j} \\
& + \Big[\lim_{\Delta t\to 0}\frac{h(t+\Delta t) - h(t)}{\Delta t}\Big]\hat{k}
\end{aligned}$$

Thus

$$\lim_{\Delta t\to 0}\frac{\Delta r}{\Delta t} = \frac{df}{dt}\hat{i} + \frac{dg}{dt}\hat{j} + \frac{dh}{dt}\hat{k}.$$

Differentiability of a Vector Function

The earlier observation leads us to the following definition:

Definition

Let $\vec{r}(t) = f(t)\hat{i} + g(t)\hat{j} + h(t)\hat{k}$ be a vector function with f, g, h are differentiable at t_0. Then we say that $\vec{r}$ is also **differentiable** at t_0. In notation, we have

$$\vec{r}\,'(t) = \frac{d\vec{r}}{dt} = \frac{df}{dt}\hat{i} + \frac{dg}{dt}\hat{j} + \frac{dh}{dt}\hat{k}.$$

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Remark

A vector function $\vec{r}$ is said to be **differentiable** *if it is differentiable at every point of it's domain.*

Differentiability of a Vector Function

The curve traced by $\vec{r}$ is smooth if $\frac{d\vec{r}}{dt}$ is continuous and never $\vec{0}$, that is, if f, g and h have continuous first derivatives that are not simultaneously 0.

Definition

Let $\vec{r}(t)$ be a differentiable function. The vector $\frac{d\vec{r}}{dt}$, when different from the zero vector $\vec{0}$, is defined to be the vector **tangent** to the curve at P.

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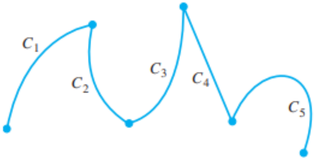
Definition

A curve that is made up of a finite number of smooth curves pieced together in a continuous fashion is called **piecewise smooth**.



A piecewise smooth curve made up of five smooth curves connected end to end in a continuous fashion. The curve here is not smooth at the points joining the five smooth curves.

1. Find parametric equations for the tangent line to the helix with parametric equations $x = 2\cos t, y = \sin t, z = t$ at the point $(0, 1, \frac{\pi}{2})$.

Determine whether the curves are smooth:

1. $\vec{r}(t) = (1 + t^3, t^2)$
2. $\vec{r}(t) = (t^3, t^4, t^5)$
3. $\vec{r}(t) = (t^3 + t, t^4, t^5)$

Motion of a Particle

Definition

If $\vec{r}$ is the position vector of a particle moving along a smooth curve in space, then

$$\vec{v}(t) = \frac{d\vec{r}}{dt}$$

is the particle's **velocity vector**, tangent to the curve. At any time t, the direction of $\vec{v}$ is the direction of motion, the magnitude of $\vec{v}$ is the particle's speed, and the derivative $\vec{a} = d\vec{v}/dt$, when it exists, is the particle's **acceleration vector**.

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Motion of a Particle

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1. *Velocity is the derivative of position:* $\vec{v} = \frac{d\vec{r}}{dt}$.
2. *Speed is the magnitude of velocity: speed*$= |\vec{v}|$.
3. *Acceleration is the derivative of velocity:* $\vec{a} = \frac{d\vec{v}}{dt} = \frac{d^2\vec{v}}{dt^2}$.
4. *The unit vector* $\vec{v}/|\vec{v}|$ *is the direction of motion at time* t.

Example

In this exercise, $\vec{r}(t)$ is the position of a particle in the xy-plane at time t. Then find the particle's velocity, speed and acceleration vectors at the given value of t.

$$\vec{r}(t) = (t+1)\hat{i} + (t^2-1)\hat{j},\ \ t = 1$$

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Solution: We have

$$\vec{v} = \frac{d\vec{r}}{dt} = \hat{i} + 2t\hat{j}.$$

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Therefore $|\vec{v}| = \sqrt{1+4t^2}$.
Thus $\vec{v}(1) = \hat{i} + 2\hat{j}$ and $\vec{a} = 2\hat{j}$ and $|\vec{v}(1)| = \sqrt{1+4} = \sqrt{5}$.

Properties

Let $\vec{u}$ and $\vec{v}$ be differentiable vector functions of t, C a constant vector, c any scalar, and f any differentiable scalar function.

- Constant function rule: $\frac{d}{dt}C = 0$.
- Scalar multiple rules: $\frac{d}{dt}c\vec{u}(t) = c\vec{u}\,'(t)$.
- $\frac{d}{dt}f(t)\vec{u}(t) = f'(t)\vec{u}(t) + f(t)\vec{u}\,'(t)$.

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- $\frac{d}{dt}f(t)\vec{u}(t) = f'(t)\vec{u}(t) + f(t)\vec{u}\,'(t)$.
- Sum rule: $\frac{d}{dt}(\vec{u}(t) + \vec{v}(t)) = \vec{u}\,'(t) + \vec{v}\,'(t)$.
- Difference rule: $\frac{d}{dt}(\vec{u}(t) - \vec{v}(t)) = \vec{u}\,'(t) - \vec{v}\,'(t)$.
- Dot product rule: $\frac{d}{dt}(\vec{u}(t) \cdot \vec{v}(t) = \vec{u}\,'(t) \cdot \vec{v}(t) + \vec{u}(t) \cdot \vec{v}\,'(t)$.
- Cross product rule: $\frac{d}{dt}(\vec{u}(t) \times \vec{v}(t) = \vec{u}\,'(t) \times \vec{v}(t) + \vec{u}(t) \times \vec{v}\,'(t)$.
- $\frac{d}{dt}[\vec{u}(f(t))] = f'(t)\vec{u}\,'(f(t))$.

Vector Functions of Constant Length

When we track a particle moving on a sphere centered at the origin, the position vector has a constant length equal to the radius of the sphere. The velocity vector $d\vec{r}/dt$, tangent to the path of motion, is tangent to the sphere and hence perpendicular to $\vec{r}$.

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$$\vec{r}(t) \cdot \vec{r}(t) = |r(t)|^2 = c^2$$

$$\frac{d}{dt}[\vec{r} \cdot \vec{r}] = \frac{d\vec{r}}{dt} \cdot \vec{r} + \vec{r} \cdot \frac{d\vec{r}}{dt} = 2\frac{d\vec{r}}{dt} \cdot \vec{r}.$$

$\frac{d}{dt}c^2 = 0 \implies \frac{d\vec{r}}{dt} \cdot \vec{r} = 0$

Vector Functions of Constant Length

Proposition

If $\vec{r}$ is a differentiable vector function of t and the length of $\vec{r}(t)$ is constant, then

$$\vec{r} \cdot \frac{d\vec{r}}{dt} = 0.$$

Exercise: If a curve has the property that the position vector is always perpendicular to the tangent vector, show that the curve lies on a sphere with center the origin.