Sequence and Series

Gunja Sachdeva

August 22, 2024

Recall $\lim_{n \to \infty} a_n = l$ meaning the sequence $\{a_n\}$ converges and it converges to l.

$\Longleftrightarrow$ For each $\varepsilon > 0$, $\exists$ N (depends on ε) s.t

$a_{N+1}, a_{N+2}, \ldots \in (l-\varepsilon, l+\varepsilon)$

Results 1. the limit l of a sequence $\{a_n\}$ is unique

2. Every convergent sequence is bdd

Equivalently, if $\{a_n\}$ is not bounded, then $\{a_n\}$ is not convergent.

Example. The sequence $\{(-1)^n n : n \in \mathbb{N}\}$ is divergent because it is not bounded.

- A bounded sequence need not be convergent. For example, the sequence $\{(-1)^n : n \in \mathbb{N}\}$ is bounded but not convergent.

Limit theorems

Let $\{a_n\}$ and $\{b_n\}$ be two convergent sequences that converge to A and B respectively. Then:

- $\lim(a_n \pm b_n) = A \pm B$.
- $\lim(a_n b_n) = AB$.
- $\lim(ca_n) = cA$ for $c \in \mathbb{R}$.
- $\lim \frac{a_n}{b_n} = \frac{A}{B}$, provided (b_n) is a sequence of non-zero real numbers and $B \neq 0$.

Find the limits.

- $\lim_{n\to\infty} \frac{-1}{n}$ $= (-1) \lim_{n\to\infty} \frac{1}{n} = (-1) \times 0 = 0$

Limit theorems

Let $\{a_n\}$ and $\{b_n\}$ be two convergent sequences that converge to A and B respectively. Then:

- $\lim(a_n \pm b_n) = A \pm B$.
- $\lim(a_n b_n) = AB$.
- $\lim(ca_n) = cA$ for $c \in \mathbb{R}$.
- $\lim \frac{a_n}{b_n} = \frac{A}{B}$, provided (b_n) is a sequence of non-zero real numbers and $B \neq 0$.

Find the limits.

- $\lim_{n\to\infty} \frac{-1}{n} = -\lim_{n\to\infty} \frac{1}{n} = 0.$
- $\lim_{n\to\infty} \frac{n+1}{n}$ $= \lim_{n\to\infty}\left(1+\frac{1}{n}\right) = \lim_{n\to\infty} 1 + \lim_{n\to\infty} \frac{1}{n}$

  $= 1 + 0 = 1$

Limit theorems

Let $\{a_n\}$ and $\{b_n\}$ be two convergent sequences that converge to A and B respectively. Then:

- $\lim(a_n \pm b_n) = A \pm B$.
- $\lim(a_n b_n) = AB$.
- $\lim(ca_n) = cA$ for $c \in \mathbb{R}$.
- $\lim \frac{a_n}{b_n} = \frac{A}{B}$, provided (b_n) is a sequence of non-zero real numbers and $B \neq 0$.

Find the limits.

- $\lim_{n\to\infty} \frac{-1}{n} = -\lim_{n\to\infty} \frac{1}{n} = 0$.
- $\lim_{n\to\infty} \frac{n+1}{n} = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right) = 1$.

- $\lim_{n\to\infty} \frac{4-7n^6}{n^6+3}$

$$= \lim_{n\to\infty} \frac{4/n^6 - 7}{1 + 3/n^6}$$

$$= \frac{\lim_{n\to\infty} \frac{4}{n^6} - 7}{\lim_{n\to\infty} 1 + \frac{3}{n^6}}$$

$$= -7$$

- $\lim_{n\to\infty} \frac{4-7n^6}{n^6+3} = \lim_{n\to\infty} \frac{\left(\frac{4}{n^6}-7\right)}{1+\frac{3}{n^6}} = \frac{0-7}{1+0} = -7.$
- $\lim_{n\to\infty} \frac{n^7+2n-1}{n^6+n^2+1} =$ $\lim_{n\to\infty} \frac{n+2/n^5-\frac{1}{n^6}}{1+1/n^4+\frac{1}{n^6}}$

$= \frac{\infty}{1} = \infty$

- $\lim_{n\to\infty} \frac{4-7n^6}{n^6+3} = \lim_{n\to\infty} \frac{\left(\frac{4}{n^6}-7\right)}{1+\frac{3}{n^6}} = \frac{0-7}{1+0} = -7.$
- $\lim_{n\to\infty} \frac{n^7+2n-1}{n^6+n^2+1} =$

Sandwich Theorem.

Let $\{a_n\}, \{b_n\}, \{c_n\}$ be three sequences of real numbers and there is a natural number m such that

$$a_n \leq b_n \leq c_n \text{ for all } n \geq m.$$

If $\lim a_n = \lim c_n = \ell$, then (b_n) is convergent and $\lim b_n = \ell$.

Examples:

(i) $\lim\limits_{n\to\infty} \dfrac{\cos n}{n}$

$$\frac{-1}{n} \leq \frac{\cos n}{n} \leq \frac{1}{n}$$

$\because$ $\left\{\frac{-1}{n}\right\}$ and $\left\{\frac{1}{n}\right\} \longrightarrow 0$

$\Rightarrow$ $\frac{\cos n}{n}$ converges and $\lim\limits_{n\to\infty} \frac{\cos n}{n} = 0$

Sandwich Theorem.

Let $\{a_n\}, \{b_n\}, \{c_n\}$ be three sequences of real numbers and there is a natural number m such that

$$a_n \leq b_n \leq c_n \text{ for all } n \geq m.$$

If $\lim a_n = \lim c_n = \ell$, then (b_n) is convergent and $\lim b_n = \ell$.

Examples:

(i) $\lim_{n\to\infty} \frac{\cos n}{n}$

$-1 \leq \cos n \leq 1$. Therefore $-\frac{1}{n} \leq \frac{\cos n}{n} \leq \frac{1}{n}$ and $\lim_{n\to\infty} \frac{\cos n}{n} = 0$.

(ii) $\lim_{n\to\infty} \frac{1}{2^n}$

$\because 2^n > n$

$0 < \frac{1}{2^n} < \frac{1}{n}$

$\{0\}$ and $\{\frac{1}{n}\} \to 0$

$\therefore \{\frac{1}{2^n}\} \to 0$

Sandwich Theorem.

Let $\{a_n\}, \{b_n\}, \{c_n\}$ be three sequences of real numbers and there is a natural number m such that

$$a_n \leq b_n \leq c_n \text{ for all } n \geq m.$$

If $\lim a_n = \lim c_n = \ell$, then (b_n) is convergent and $\lim b_n = \ell$.

Examples:

(i) $\lim\limits_{n\to\infty} \dfrac{\cos n}{n}$

$-1 \leq \cos n \leq 1$. Therefore $-\frac{1}{n} \leq \frac{\cos n}{n} \leq \frac{1}{n}$ and $\lim\limits_{n\to\infty} \dfrac{\cos n}{n} = 0.$

(ii) $\lim\limits_{n\to\infty} \dfrac{1}{2^n} = 0$

as $0 \leq \frac{1}{2^n} \leq \frac{1}{n}$

(iii) $\lim\limits_{n\to\infty} (-1)^n \dfrac{1}{n} = 0.$

Examples

(iv) Let $a_n := \dfrac{n^3 + 3n^2 + 1}{n^4 + 8n^2 + 2}$ for $n \in \mathbb{N}$.

$$0 < \frac{n^3+3n^2+1}{n^4+8n^2+2} = \frac{n^3}{n^4+8n^2+2} + \frac{3n^2}{n^4+8n^2+2} + \frac{1}{n^4+8n^2+2}$$

$$< \frac{n^3}{n^4} + \frac{3n^2}{n^4} + \frac{1}{n^4}$$

$$= \frac{1}{n} + \frac{3}{n^2} + \frac{1}{n^4}$$

$\because \{0\}$ and $\left\{ \frac{1}{n} + \frac{3}{n^2} + \frac{1}{n^4} \right\} \to 0$

$\Rightarrow \{a_n\} \to 0$

Examples

(iv) Let $a_n := \dfrac{n^3 + 3n^2 + 1}{n^4 + 8n^2 + 2}$ for $n \in \mathbb{N}$. Then $a_n \to 0$,

$$\text{since } 0 \leq a_n \leq \frac{1}{n} + \frac{3}{n^2} + \frac{1}{n^4} \to 0.$$

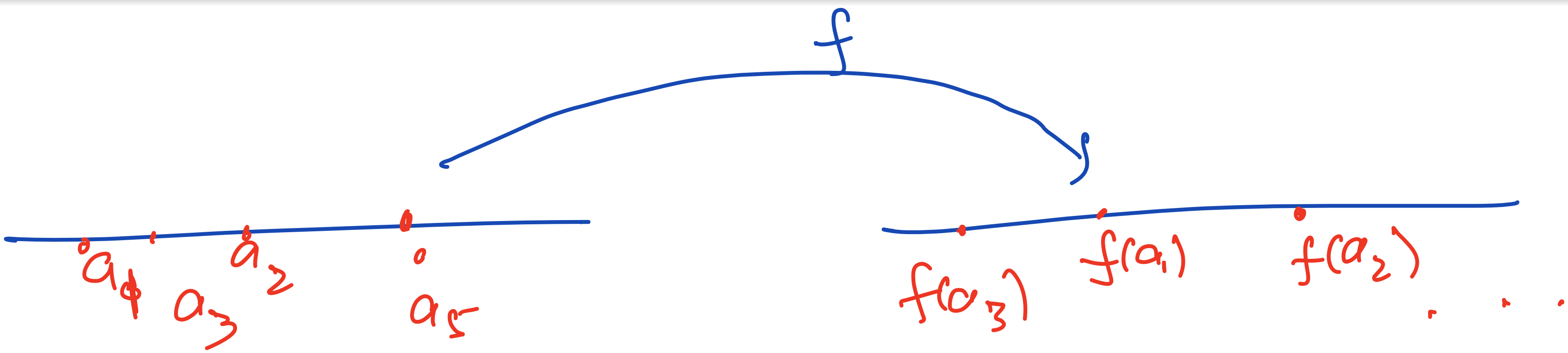
(iii) Let $a_n := \dfrac{1}{n} \sin\left(\dfrac{1}{n}\right)$ for $n \in \mathbb{N}$. Then $a_n \to 0$,

$$\text{since } |a_n| \leq \frac{1}{n} \to 0.$$

Continuous function theorem for Sequences

Remark

- *If* (a_n) *is a sequence and if* f *is any function from* $\mathbb{R} \to \mathbb{R}$, *is* $f(a_n)$ *a sequence? Yes* ✓
- *What can we say about convergence of* $f(a_n)$ *if we know about convergence of* a_n*?*



$a_n \longrightarrow l$

$\{f(a_n)\}$

$f(a_n) \xrightarrow{?}$

Continuous function theorem for Sequences

Remark

- *If (a_n) is a sequence and if f is any function from $\mathbb{R} \to \mathbb{R}$, is $f(a_n)$ a sequence? Yes*
- *What can we say about convergence of $f(a_n)$ if we know about convergence of a_n?*

Theorem

Theorem 3: Let (a_n) be a sequence of real numbers. If

- *$a_n \to \ell$ and*
- *if f is a function that is continuous at ℓ and defined at all a_n, then*

$$\boxed{f(a_n) \to f(\ell).}$$

- Show that $\sqrt{\frac{(n+1)}{n}} \to 1$.

$a_n = \frac{n+1}{n}$

$\downarrow$

1

defined $f : \mathbb{R} \to \mathbb{R}$

$x \mapsto \sqrt{x}$

$a_n \mapsto \sqrt{a_n}$

$1 \mapsto 1$

clearly $f(x) = \sqrt{x}$ is cont. at $x = 1$

$\Rightarrow \quad f(\{a_n\}) \longrightarrow f(l)$

$= 1$

- Show that $\sqrt{\frac{(n+1)}{n}} \to 1$. We know that $\frac{n+1}{n} \to 1$ and $f(x) = \sqrt{x}$ is continuous at $\ell = 1$. So by Theorem 3, $(\sqrt{\frac{(n+1)}{n}}) \to 1$.
- Show that $(2^{\frac{1}{n}}) \to 1$.

$\{a_n\} = \{\frac{1}{n}\} \to 0 = \ell$

$f : \mathbb{R} \to \mathbb{R}$

$x \mapsto 2^x$

$0 \mapsto 2^0 = 1$

$\because f$ is cont. at $x = 0$

$\therefore f\{a_n\} = f(\{\frac{1}{n}\}) = 2^{1/n} \to 2^0 = 1$

- Show that $\sqrt{\frac{(n+1)}{n}} \to 1$. We know that $\frac{n+1}{n} \to 1$ and $f(x) = \sqrt{x}$ is continuous at $\ell = 1$. So by Theorem 3, $(\sqrt{\frac{(n+1)}{n}}) \to 1$.
- Show that $(2^{\frac{1}{n}}) \to 1$. $\frac{1}{n} \to 0$ and $f(x) = 2^x$ is continuous at $x = 0$. Thus $2^{\frac{1}{n}} \to 1$.
- Find the limit of the sequence $\sin\left(\frac{1+n}{n^2}\right) \to 0$

  $a_n = \frac{1+n}{n^2} \to 0$

  $f : \mathbb{R} \to \mathbb{R}$

  $x \mapsto \sin x$

  $0 \mapsto 0$

- Find the limit of the sequence $e^{\underbrace{\frac{2n^2+3}{n^3+5n+6}}_{\downarrow\, 0}}$

  $e^0 = 1$

Functions and sequences

Theorem

Suppose that $f(x)$ is a function defined for all $x \geq n_0$ and that (a_n) is a sequence of real numbers such that $a_n = f(n)$ for $n \geq n_0$. Then

$$\lim_{x \to \infty} f(x) = \ell \quad \Longrightarrow \quad \lim_{n \to \infty} a_n = \ell.$$

(i) Show that $\displaystyle\lim_{n \to \infty} \frac{\log n}{n} = 0.$

$f : \mathbb{R} \to \mathbb{R}$

$x \mapsto \dfrac{\log x}{x}, \quad x \geq 1$

$\displaystyle\lim_{x \to \infty} f(x) = \lim_{x \to \infty} \frac{\log x}{x}$ Apply L'Hopital

$\displaystyle\Rightarrow \lim_{x \to \infty} \frac{1/x}{1} = 0$

Theorem

Suppose that $f(x)$ is a function defined for all $x \geq n_0$ and that (a_n) is a sequence of real numbers such that $a_n = f(n)$ for $n \geq n_0$. Then

$$\lim_{x \to \infty} f(x) = \ell \implies \lim_{n \to \infty} a_n = \ell.$$

(i) Show that $\displaystyle\lim_{n \to \infty} \frac{\log n}{n} = 0$.

We take $f(x) = \frac{\log x}{x}$ and $f(x)$ is defined for $x \geq 1$. Therefore

$$\lim_{n \to \infty} \frac{\log n}{n} = \lim_{x \to \infty} \frac{\log x}{x} = 0.$$

(ii) Let $a_n = (\frac{n+1}{n-1})^n$. Does a_n converge? Where?

$$f(x) = \left(\frac{x+1}{x-1}\right)^x = \left(\frac{1+1/x}{1-1/x}\right)^x, \quad x \geq 1$$

$$\lim_{x\to\infty} f(x) = 1^\infty \text{ form}$$

$$\log f(x) = \infty \times \log 1$$
$$= \infty \times 0$$
$$= \frac{0}{0}$$

$$\lim_{x\to\infty} \log\left(\frac{1+1/x}{1-1/x}\right)^x = x \log\frac{(x+1)}{x-1} = \lim_{x\to\infty}\frac{\log\left(\frac{x+1}{x-1}\right)}{1/x}$$

$$= \lim_{x\to\infty}\frac{2x^2}{x^2-1} = 2$$

(ii) Let $a_n = (\frac{n+1}{n-1})^n$. Does a_n converge? Where?
The limit leads to the indeterminate form 1^∞. We can apply l'Hopital's rule if we first change the form by taking the natural logarithm of a_n.

$$\begin{aligned}
\lim_{n\to\infty} \ln(a_n) &= \lim_{n\to\infty} n\ln(\frac{n+1}{n-1}) \\
&= \lim_{n\to\infty} \frac{\ln(\frac{n+1}{n-1})}{1/n} \\
&= \lim_{n\to\infty} \frac{-2/(n^2-1)}{-1/n^2} \\
&= \lim_{n\to\infty} \frac{2n^2}{n^2-1} = 2.
\end{aligned}$$

Since $\ln(a_n) \to 2$ and $f(x) = e^x$ is continuous, $a_n \to e^2$.

Bounded Sequences

A sequence (a_n) of real numbers is said to be **bounded above** if there is a real number α such that $a_n \leq \alpha$ for every $(\forall)$ $n \in \mathbb{N}$. The number α is an upper bound for (a_n). If α is an upper bound for a_n but no number less than α is an upper bound for a_n, then α is **the least upper bound** for a_n.

A sequence (a_n) of real numbers is said to be **bounded below** if there is a real number β such that $\beta \leq a_n$ for every $n \in \mathbb{N}$. The number β is a lower bound for a_n. If β is a lower bound for a_n but no number greater than β is a lower bound for a_n, then β is **the greatest lower bound** for a_n.

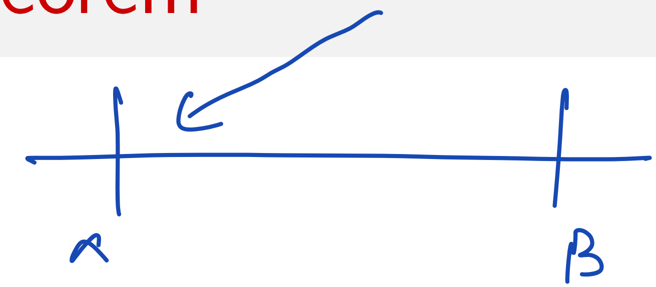
A sequence (a_n) of real numbers is said to be **bounded** if there are real numbers α, β such that $\beta \leq a_n \leq \alpha$ for every $n \in \mathbb{N}$.

If a sequence is not bounded, it is said to be **unbounded**.

Monotone sequences and convergence

- A sequence (x_n) is said to be **monotone increasing** or nondecreasing if $x_n \leq x_{n+1}$ for all $n \in \mathbb{N}$, that is, $x_1 \leq x_2 \leq x_3 \leq \cdots$.
- $\frac{1}{2}, \frac{2}{3}, \ldots, \frac{n}{n+1}, \ldots$ is monotone increasing
- A sequence (x_n) is said to be **monotone decreasing** or nonincreasing if $x_n \geq x_{n+1}$ for all $n \in \mathbb{N}$, that is, $x_1 \geq x_2 \geq x_3 \geq \cdots$.
- $1, \frac{1}{2}, \frac{1}{4}, \ldots, \frac{1}{2^n}, \ldots$ is monotone decreasing.
- A sequence is **monotone** if it is either monotone increasing or monotone decreasing.

Monotone convergence theorem



If a sequence (a_n) is both bounded and monotone, then the sequence converges.

In other words,

- A monotone increasing sequence that is bounded above, is convergent and it converges to the least upper bound.
- A monotone decreasing sequence that is bounded below, is convergent and it converges to the greatest lower bound.

Example:

$$\text{Let } a_1 := \frac{3}{2} \quad \text{and} \quad a_{n+1} := \frac{1}{2}\left(a_n + \frac{2}{a_n}\right) \quad \text{for } n \in \mathbb{N}.$$

$a_1 = \frac{3}{2} = 1.5 > 0$, $a_2 = \frac{1}{2}\left[\frac{3}{2} + \frac{4}{3}\right]$

$= \frac{1}{12} \times 9+8 = \frac{17}{12} = 1.4 > 0$

$a_2 < a_1$

$\boxed{a_{n+1} > 0} \Rightarrow \{a_n\}$ is bdd below by 0

claim $a_n - a_{n+1} \geq 0 \Rightarrow a_n - \frac{a_n}{2} - \frac{1}{a_n}$

$= \frac{a_n}{2} - \frac{1}{a_n} = \frac{a_n^2 - 2}{2a_n}$

$\{a_n\}$ is decreasing iff $a_n^2 - 2 \geq 0$

To show $a_n^2 - 2 \geq 0$. We show it by induction

$$a_1^2 = \frac{9}{4} = 2.25 \geq 2 \quad \Rightarrow a_1^2 - 2 \geq 0$$

Assume $a_n^2 - 2 \geq 0 \ \forall$ upto n

$$a_{n+1}^2 - 2 = \frac{(a_n^2 - 2)^2}{4a_n^2} \geq 0$$

$\{a_n\}$ is bdd below and decreasing ✓

$\Rightarrow \{a_n\}$ converges

$\{a_n\} \to a$, $\{a_{n+1}\} \to a$ ✓

$$\{a_{n+1}\} = \left\{\frac{a_n}{2} + \frac{1}{a_n}\right\} \downarrow \left\{\frac{a}{2} + \frac{1}{a}\right\} ✓$$

$$a = \frac{a}{2} + \frac{1}{a}$$

$$a^2 = 2 \quad \Rightarrow \quad a = \pm\sqrt{2}$$

$$\Rightarrow a = \sqrt{2} \quad \because \{a_n\} > 0$$

Example:

$$\text{Let } a_1 := \frac{3}{2} \quad \text{and} \quad a_{n+1} := \frac{1}{2}\left(a_n + \frac{2}{a_n}\right) \quad \text{for } n \in \mathbb{N}.$$

Then $a_n > 0$ for all $n \in \mathbb{N}$. Hence (a_n) is bounded below by 0.

Let us check whether the sequence (a_n) is decreasing. Since

$$a_n - a_{n+1} = a_n - \frac{a_n^2 + 2}{2a_n} = \frac{a_n^2 - 2}{2\,a_n} \quad \text{for all } n \in \mathbb{N},$$

(a_n) is decreasing if and only if $a_n^2 - 2 \geq 0$ for all $n \in \mathbb{N}$. But

$$a_1^2 \geq 2 \quad \text{and} \quad a_{n+1}^2 - 2 = \frac{\left(a_n^2 - 2\right)^2}{4a_n^2} \geq 0 \text{ for all } n \in \mathbb{N}.$$

Hence the sequence (a_n) is decreasing.

It follows that (a_n) is convergent. Let $a_n \to a$. Then $a_{n+1} \to a$ also. But

$$a_{n+1} = \frac{1}{2}\left(a_n + \frac{2}{a_n}\right) \to \frac{1}{2}\left(a + \frac{2}{a}\right).$$

Since the limit of a sequence is unique, we see that $\frac{1}{2}\left(a + \frac{2}{a}\right) = a$, that is, $a^2 = 2$.

Also, $a_n > 0$ for all $n \in \mathbb{N}$ and $a_n \to a$, so that $a \geq 0$. Thus a is the positive square root of 2, that is, $a = \sqrt{2}$.

Exercises

Determine if the sequences is monotonic and bounded.

- $a_n = \frac{n}{n+1}$
- $a_n = \frac{3n+1}{n+1}$
- $a_n = \frac{(2n+3)!}{(n+1)!}$