Sequence and Series

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Recall

$I \subseteq \mathbb{R}$

$\sum x^n$

$\sum x^n n!$

$\sum \left(\frac{x-2}{2}\right)^n (-1)^{n+1}$

$f : I \longrightarrow \mathbb{R}$

s.t. f is cont.

and all order

derivative

of f exist

$\sum_{n=0}^{\infty} c_n (x-a)^n$ – power series

x - variable

a, c_n are constants

$\sum x^n \longrightarrow$ Interval of convergence 'I'

define $f : I \longrightarrow \mathbb{R}$

$f(x) = \sum x^n$

Operations on Power series

On the **intersection of their intervals of convergence**, two power series can be added and subtracted term by term just like series of constants.

$\sum a_n x^n$ with I_1 as interval of convergence $\rightsquigarrow$ $f_1 : I_1 \to \mathbb{R}$

$\sum b_n x^n$ with I_2 as interval of convergence $\rightsquigarrow$ $f_2 : I_2 \to \mathbb{R}$

$\sum (a_n \pm b_n) x^n$ with $I_1 \cap I_2$ as interval of convergence $\longrightarrow$ $f_1 \pm f_2 : I_1 \cap I_2 \to \mathbb{R}$

Operations on Power series

On the **intersection of their intervals of convergence**, two power series can be added and subtracted term by term just like series of constants.

Also, they can be multiplied just as we multiply polynomials as follows:

Theorem

If $A(x) = \sum_{n=0}^{\infty} a_n x^n$ *and* $B(x) = \sum_{n=0}^{\infty} b_n x^n$ *converge absolutely for* $|x| < R$,

and $c_n = a_0 b_n + a_1 b_{n-1} + a_2 b_{n-2} + \cdots + a_n b_0 = \sum_{k=0}^{n} a_k b_{n-k}$, *then*

$\sum_{n=0}^{\infty} c_n x^n$ *converges absolutely to* $A(x)B(x)$ *for* $|x| < R$.

$$\left(\sum_{n=0}^{\infty} a_n x^n\right)\left(\sum_{n=0}^{\infty} b_n x^n\right) = \sum_{n=0}^{\infty} c_n x^n$$

Example

$$\sum_{n=1}^{\infty} x^n \cdot \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n} \quad - \quad I_2 = [-1, 1]$$

$\downarrow$ "$f(x)$" $\qquad g(x)$

$I_1 = (-1, 1)$

$I_1 \cap I_2 = (-1, 1)$

$\Sigma x^n = 1 + x + x^2 + x^3 + \cdots$

$\Sigma (-1)^{n-1} \frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} \cdots$

$f(x)\, g(x)$ is function defined on $(-1, 1)$

$$= 1 \cdot \left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots\right) + x\left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots\right)$$

$$+ x^2\left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots\right) + \cdots$$

$$= x - \frac{x^2}{2} + x^2 + \frac{x^3}{3} - \frac{x^3}{2} + x^3 + \cdots$$

$2 + 3 + 6$

$$= x + \frac{1}{2} x^2 + \frac{5}{6} x^3 + \cdots$$

From variable to functions

$\mathbb{R} \xrightarrow{f} \mathbb{R}$

$(x-R, x+R) \longrightarrow \mathbb{R}$

$\sum a_n x^n$

$I_1 \quad (x-R, x+R)$

$I_2 \quad (f(x)+R, f(x)-R)$

Theorem

If $\sum_{n=0}^{\infty} a_n x^n$ *converges absolutely for* $|x| < R$ *and* f *is a continuous function, then* $\sum_{n=0}^{\infty} a_n f(x)^n$ *converges absolutely on the set of points* x *where* $|f(x)| < R$.

$(x-R, x+R) \xrightarrow{f} (\tilde{x}-R, \tilde{x}+R) \longrightarrow \mathbb{R}$

$\sum a_n x^n$

$\sum a_n f(x)^n$ converges

$(f(x)-R, f(x)+R)$

$x-R=-1$

$x+R=1$

1. Find the interval of convergence for $\sum_{n=0}^{\infty}(4x^2)^n$. To which function this series will converge in this interval?

$\sum x^n$ converges on $\boxed{(-1,1)}$

$(-1,1) \xrightarrow{f} (-1,1) \longrightarrow \mathbb{R}$

$f(x)=4x^2$

$\sum x^n = g(x)$

$|f(x)| < R$

$|4x^2| < 1$

$|x^2| < \frac{1}{4}$

$|x| < \frac{1}{2}$

$g(f(x)) = g(4x^2) = \sum (4x^2)^n$

converges on

$\left(-\frac{1}{2}, \frac{1}{2}\right)$

1. Find the interval of convergence for $\sum_{n=0}^{\infty}(4x^2)^n$. To which function this series will converge in this interval?

Since $\sum_{n=0}^{\infty} x^n$ converges absolutely to the function $\frac{1}{1-x}$ for $|x| < 1$,
it follows from above theorem (with $f(x) = 4x^2$) that

$\sum_{n=0}^{\infty}(4x^2)^n$ converges absolutely to $\frac{1}{1-4x^2}$ when x satisfies $|4x^2| < 1$ or
equivalently when $|x| < \frac{1}{2}$.

Term by term differentiation

Theorem

If $\sum_{n=0}^{\infty} a_n(x-a)^n$ *has radius of convergence* $R > 0$, *it defines a function* $f(x) = \sum_{n=0}^{\infty} a_n(x-a)^n$ *on the interval* $a - R < x < a + R$. *This function* f *has derivatives of all orders inside the interval, and the derivatives are obtained by differentiating the original series term by term*

$$f'(x) = \sum_{n=1}^{\infty} n a_n (x-a)^{n-1}$$

$$f''(x) = \sum_{n=2}^{\infty} n(n-1) a_n (x-a)^{n-2}$$

and so on. Each of these derived series converges at every point of the interval $a - R < x < a + R$.

Example

Find series for $f'(x)$ and $f''(x)$ if

$$f(x) = \frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots = \sum_{n=1}^{\infty} x^n. \quad -1 < x < 1$$

$f : (-1,1) \to \mathbb{R}$

$$f(x) = \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

$$f'(x) = \left(\sum x^n\right)' = 1 + 2x + 3x^2 + \cdots$$

converges $-1 < x < 1$

$$f''(x) = \left(1 + 2x + 3x^2 + \cdots\right)'$$

$$= 2 + 6x + 12x^2 + \cdots$$

converges $-1 < x < 1$

Example

Find series for $f'(x)$ and $f''(x)$ if

$$f(x) = \frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots = \sum_{n=1}^{\infty} x^n.$$

We differentiate the power series on the right term by term:

$$f'(x) = \tfrac{1}{(1-x)^2} = 1 + 2x + 3x^2 + \cdots = \sum_{n=1}^{\infty} nx^{n-1}, -1 < x < 1$$

$$f''(x) = \tfrac{2}{(1-x)^3} = 2 + 6x + 12x^2 \cdots = \sum_{n=1}^{\infty} n(n-1)x^{n-2}, -1 < x < 1$$

Now look at $\sum_{n=1}^{\infty} \frac{\sin(n!x)}{n^2}$

converges $\forall x$

Using direct comparison test

with $\sum \frac{1}{n^2}$

$$\sum_{n=1}^{\infty} \frac{n!\cos(n!x)}{n^2}$$

n^{th} divergent test. as $n \to \infty$ $a_n \not\to 0$

diverges

$\frac{d}{dx} \sum \frac{\sin(n!x)}{n^2} = \sum \frac{n!\cos(n!x)}{n^2}$

Remark

Term-by-term differentiation might not work for other kinds of series.

For example, the trigonometric series $\displaystyle\sum_{n=1}^{\infty} \frac{\sin(n!x)}{n^2}$ converges for all x. But if we differentiate term by term we get the series $\displaystyle\sum_{n=1}^{\infty} \frac{n!\cos(n!x)}{n^2}$ which diverges for all x. Note that this is not a power series since it is not a sum of positive integer powers of x.

Term by term integration

Theorem

Suppose that $\sum_{n=0}^{\infty} a_n(x-a)^n$ *converges for* $a-R<x<a+R$ *and is equal to* $f(x)$. *Then*

$$\sum_{n=0}^{\infty} a_n \frac{(x-a)^{n+1}}{n+1} \text{ converges for } a-R<x<a+R$$

and

$$\int f(x)dx = \sum_{n=0}^{\infty} a_n \frac{(x-a)^{n+1}}{n+1} + C \text{ for } a-R<x<a+R.$$

Example

Identify the function

$$f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots, -1 \le x \le 1.$$

$=$

$x = 0,$

$f(x) = 0$

$$= x - \frac{x^3}{3} + \frac{x^5}{5} - - - -$$

$$f'(x) = 1 - x^2 + x^4 - x^6 + - - -$$ converges $(-1, 1)$

$$= \frac{1}{1+x^2}$$

$$\int f'(x)\,dx = f(x) + C$$ for $(-1, 1)$ ✓

$$\int \frac{1}{1+x^2}\,dx = f(x) + C$$

$$\Rightarrow \tan^{-1} x = f(x) + C$$

Example

$0 = 0 + C \Rightarrow C = 0$

$f(x) = \tan^{-1} x$ on $(-1, 1)$

Identify the function

$$f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots, -1 \leq x \leq 1.$$

Differentiating the original series term by term, we get

$$f'(x) = 1 - x^2 + x^4 - \cdots, -1 < x < 1.$$

This is a geometric series with first term 1 and ratio $-x^2$, so $f'(x) = \frac{1}{1+x^2}$. We can now integrate $f'(x) = \frac{1}{1+x^2}$ to get

$$\int f'(x)dx = \int \frac{dx}{1+x^2} = tan^{-1}x + C.$$

The series for $f(x)$ is 0 when $x = 0$, so $C = 0$. Thus

$$f(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots = tan^{-1}x, -1 < x < 1.$$

Remark

Note that the original series $x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots$, converges at both endpoints of the original interval of convergence, but our theorem can only guarantee the convergence of the differentiated series inside the interval.

Example

The series

$$\frac{1}{1+t} = 1 - t + t^2 - t^3 + \cdots$$

converges on the open interval $-1 < t < 1$.

Therefore,

$$\ln(1+x) = \int_0^x \frac{1}{1+t}\,dt = \underbrace{x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots}_{f(x)}, -1 < x < 1.$$

$\int f'(x)\,dx = f(x) + c$

$\int \frac{1}{1+t}\,dt = f(x) + c$

$\ln(1+x) = f(x) + c$, $c = 0$

$f'(x) = 1 - \cancel{x} + x^2 - x^3 + \cdots \quad -1 < x < 1$

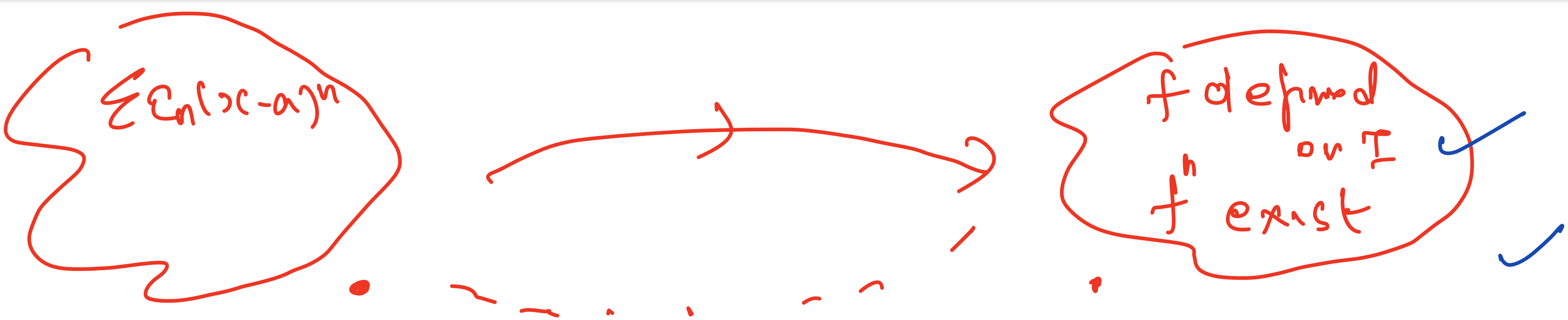
$= \frac{1}{1+x}$

We have seen that within its interval of convergence I, the power series is a continuous function with derivatives of all orders. Now we ask the reverse question.

Remark

(1) *If a function $f(x)$ has derivatives of all orders on an interval, can it be expressed as a power series on at least part of that interval?*
i.e Can you find a power series that converges to $f(x)$ for each point x in an interval?

(2) *And if it can, what are its coefficients?*



$$f(x) = \sum_{n=1}^{n} a_n (x-a)^n$$ a center of interval

$$= a_1(x-a) + a_2(x-a)^2 + \cdots \quad \text{I}$$

$$f'(x) = a_1 + 2a_2(x-a) + 3a_3(x-a)^2 + \cdots$$

at $x = a$, $f'(a) = a_1$

$$f''(x) = 2a_2 + 3\times 2\, a_3 (x-a) + 4\times 3\, a_4 (x-a)^2 + \cdots$$

at $x = a$

$$f''(a) = 2a_2$$

$$\vdots$$

$$n!\, a_n = f^n(a) \Rightarrow a_n = \frac{f^n(a)}{n!}$$

$$\sum \frac{f^n(a)}{n!} (x-a)^n$$

Suppose we have an answer to Question (1), i.e $f(x)$ has a power series representation in an interval of convergence I. i.e

$$f(x) = \sum_{n=0}^{\infty} a_n (x-a)^n, x \in I$$

By repeated term-by-term differentiation $\forall x \in I$ we obtain

$$\begin{aligned} f'(x) &= a_1 + 2a_2(x-a) + 2 \cdot (x-a)3(x-a)^2 + \cdots \\ f^n(x) &= n!a_n + \text{ sum of terms with } (x-a) \text{ as a factor} \end{aligned}$$

Since $x = a \in I$, we get $f^n(a) = n!a_n$. Therefore $a_n = \frac{f^n(a)}{n!}$. This answers Question (2). i.e

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots$$

Definition

Let f be a function with derivatives of all orders throughout some interval containing a as an interior point. Then the Taylor series generated by f at $x = a$ is $\sum_{k=0}^{\infty} \frac{f^k(a)}{k!}(x-a)^k$

$$= f(a) + f'(a)(x-a) + \frac{f^{''}(a)}{2!}(x-a)^2 + \cdots + \frac{f^n(a)}{n!}(x-a)^n + \cdots$$

The Maclaurin series generated by f is the Taylor series generated by f at $x = 0$ given by

$$\sum_{k=0}^{\infty} \frac{f^k(0)}{k!}x^k = f(0) + f'(0) + \frac{f^{''}(0)}{2!}x^2 + \cdots + \frac{f^n(0)}{n!}x^n + \cdots$$