MATH F111- Mathematics I

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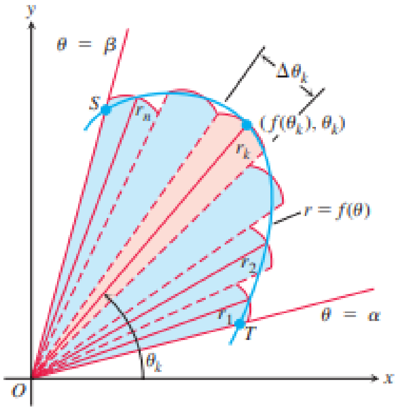
Area in the Plane



The region OTS in figure is bounded by the rays $\theta = \alpha$ and $\theta = \beta$ and the curve $r = f(\theta)$. We approximate the region with n nonoverlapping fan-shaped circular sectors based on a partition P of angle TOS. The typical sector has radius $r_k = f(\theta_k)$ and central angle of radian measure $\Delta\theta_k$. Its area is $\Delta\theta_k/2\pi$ times the area of a circle of radius r_k,

$$A_k = \frac{1}{2} r_k^2 \Delta\theta_k = \frac{1}{2} (f(\theta_k))^2 \Delta\theta_k.$$

The area of region OTS is approximately,

$$\sum_{k=1}^{n} A_k = \sum_{k=1}^{n} \frac{1}{2}(f(\theta_k))^2 \Delta\theta_k.$$

If f is continuous, we expect the approximations to improve as the norm of the partition P goes to zero, where the norm of P is the largest value of $\Delta\theta_k$. Thus

$$\begin{aligned} A &= \lim_{||P|| \to 0} \sum_{k=1}^{n} \frac{1}{2}(f(\theta_k))^2 \Delta\theta_k \\ &= \int_{\alpha}^{\beta} \frac{1}{2}(f(\theta))^2 d\theta \end{aligned}$$

Definition

Area of the Fan-Shaped Region Between the Origin and the Curve $r = f(\theta)$, $\alpha \leq \theta \leq \beta, r \geq 0$ and $\beta - \alpha \leq 2\pi$, is given by

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   Area$= \int_0^{2\pi} \frac{1}{2} r^2 d\theta = \int_0^{2\pi} \frac{1}{2}(1 - \cos\theta)^2 d\theta = \frac{3}{2}\pi$.

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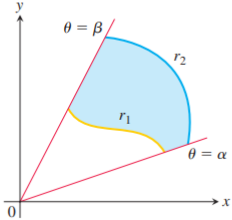
2. Find the area of the region in the xy-plane enclosed by the cardioid $r = 2(1 + \cos\theta)$.
   Ans: 6π.



Definition

Area of the Region $0 \leq r_1(\theta) \leq r \leq r_2(\theta)$, $\alpha \leq \theta \leq \beta$ and $\beta - \alpha \leq 2\pi$,

$$A = \int_\alpha^\beta \frac{1}{2} r_2^2 d\theta - \int_\alpha^\beta \frac{1}{2} r_1^2 d\theta = \int_\alpha^\beta \frac{1}{2}(r_2^2 - r_1^2) d\theta.$$

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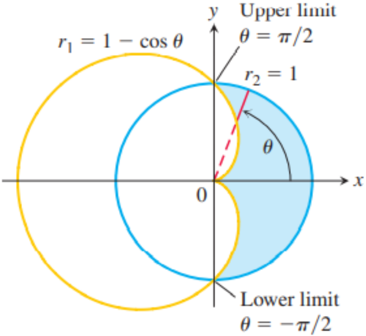

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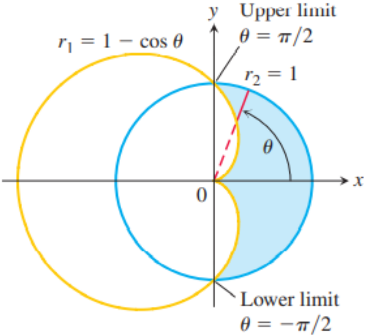

Required area$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{2}(r_2^2 - r_1^2)\,d\theta = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{2}(1 - (1 - \cos\theta)^2)\,d\theta = 2 - \frac{\pi}{4}$.

Example

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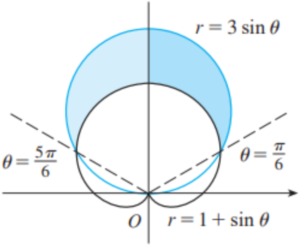


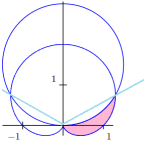
Required area$= \displaystyle\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \frac{1}{2}(3\sin\theta)^2 - (1+\sin\theta)^2 d\theta = \pi.$

Remark

The fact that a single point has many representations in polar coordinates sometimes makes it difficult to find all the points of intersection of two polar curves. For instance, it is obvious from figure of earlier example that the circle and the cardioid have three points of intersection; however, when we solved the equations we found only two such points. The origin is also a point of intersection, but we can't find it by solving the equations of the curves because the origin has no single representation in polar coordinates that satisfies both equations. When represented as $(0,0)$ *or* $(0,\pi)$ *the origin satisfies* $r = 3\sin\theta$ *and so it lies on the circle; when represented as* $(0, 3\frac{\pi}{2})$*, it satisfies* $r = 1 + \sin\theta$ *and so it lies on the cardioid.*

Thus, to find all points of intersection of two polar curves, it is recommended that you draw the graphs of both curves.

4. Find the area of the shaded region between the circle $r = 3\sin\theta$ and the cardioid $r = 1 + \sin\theta$.



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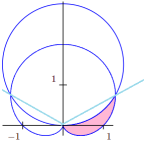

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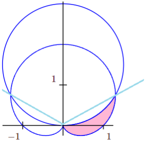
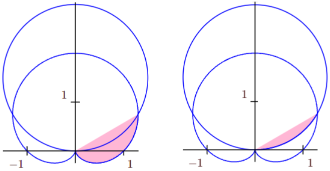


Note that $r < 0$ for the curve $r = 3\sin\theta$ in $-\pi/2 \leq \theta < 0$. So we cannot apply the formula directly. The shaded area from the graph can be found as the difference other two shaded areas given by the following figures.



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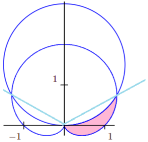

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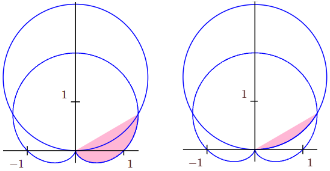

Required area$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{6}} \frac{1}{2}(1+\sin\theta)^2 d\theta$ - $\int_{0}^{\frac{\pi}{6}} \frac{1}{2}(3\sin\theta)^2 d\theta$

Example

Find all points of intersection of the curves $r = \cos 2\theta$ *and* $r = \frac{1}{2}$.

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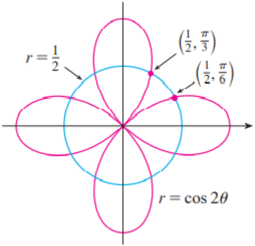
Solving the two equations, we get $\cos 2\theta = \frac{1}{2}$ which gives $\theta = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}$.



The curves have four other points of intersection—namely, $(\frac{1}{2}, \frac{\pi}{3}), (\frac{1}{2}, \frac{2\pi}{3}), (\frac{1}{2}, \frac{4\pi}{3}), (\frac{1}{2}, \frac{5\pi}{3})$.

Length of a Polar Curve

If $r = f(\theta)$ has a continuous first derivative for $\alpha \leq \theta \leq \beta$ and if the point $P(r, \theta)$ traces the curve $r = f(\theta)$ exactly once as θ runs from α to β, then the length of the curve is

$$L = \int_{\alpha}^{\beta} \sqrt{r^2 + (\frac{dr}{d\theta})^2} d\theta.$$

Example

Find the length of the cardioid $r = 1 - \cos\theta$.

$$L = \int_{0}^{2\pi} \sqrt{(1 - \cos\theta)^2 + (\sin\theta)^2} d\theta$$