MATH F111- Mathematics I

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If we can find power series representation of a function, they provide useful polynomial approximations of the original functions. Because approximation by polynomials is extremely useful to both mathematicians and scientists, we are interested to see when a function can have power series representation.

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By repeated term-by-term differentiation $\forall x \in I$ we obtain

$$f'(x) = a_1 + 2a_2(x-a) + 2 \cdot (x-a)3(x-a)^2 + \cdots$$
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Since $x = a \in I$, we get $f^n(a) = n! a_n$. Therefore $a_n = \frac{f^n(a)}{n!}$. This answers Question (2). i.e

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots$$

Let us discuss what gives the motivation to think for a power series representation of a function f?

Suppose that we have a function like $f(x) = \cos(x)$ that needs a calculator to evaluate. Can you find a linear function close to $\cos x$ for x near 0?

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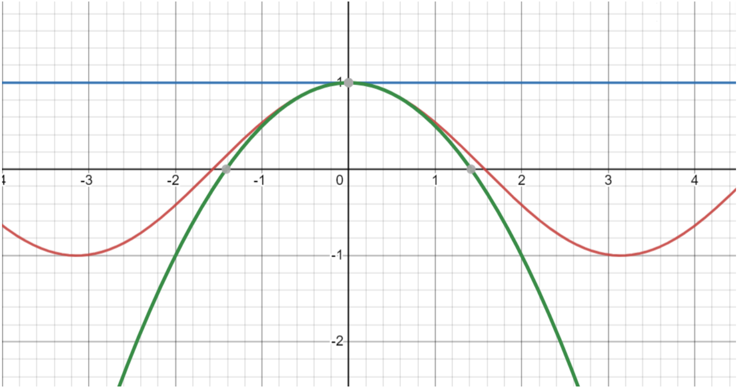
As a result, our quadratic approximation is $g(x) = 1 - \frac{1}{2}x^2$.



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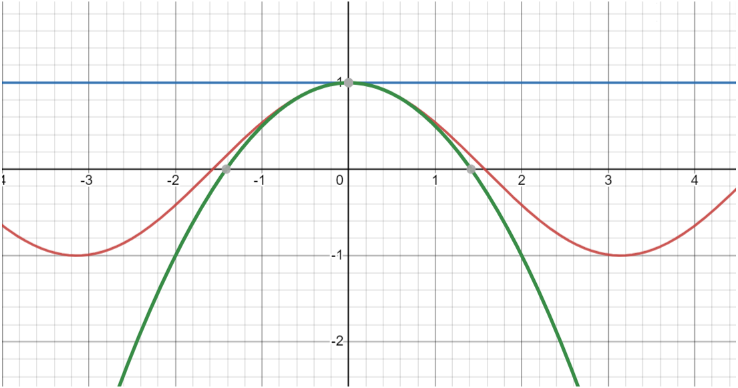

To get better approximations, we could continue approximating our function $f(x) = \cos(x)$ with polynomials of higher and higher degrees. Let

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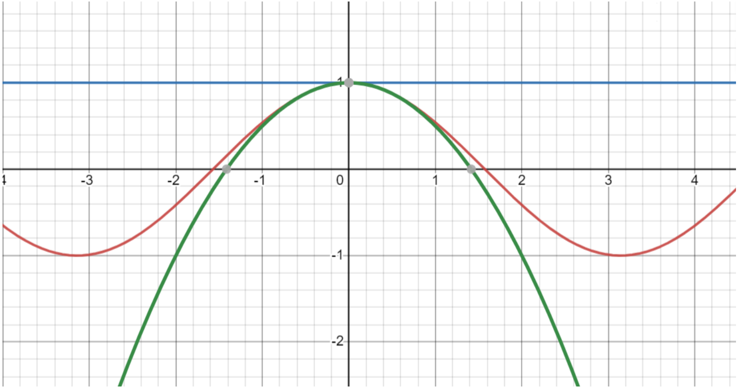

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Then we want $f^m(0) = g^m(0)$ and $g^m(0) = m! a_m$. Thus $a_m = \frac{f^m(0)}{m!}$.

Thus
$$g(x) = f(0) + f'(0)x + \frac{f^{''}(0)}{2!}x^2 + \cdots + \frac{f^{n}(0)}{n!}x^n$$
is a polynomial that can be used for approximating $f(x)$ for x in a neighbourhood of 0. This motivates the following definition.

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Taylor Polynomial of order n: Let f be a function with derivatives of order k for $k = 1, 2, \cdots, N$ in some interval containing a as an interior point. Then for any integer n from 0 through N, the Taylor polynomial of order n generated by f at $x = a$ is the polynomial

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$$P_n(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^n(a)}{n!}(x-a)^n$$

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We speak of a Taylor polynomial of order n rather than degree n because $f^n(a)$ may be zero.

Definition

Let f be a function with derivatives of all orders throughout some interval containing a as an interior point. Then the Taylor series generated by f at $x = a$ is $\sum_{k=0}^{\infty} \frac{f^k(a)}{k!}(x-a)^k$

$$= f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^n(a)}{n!}(x-a)^n + \cdots$$

The Maclaurin series generated by f is the Taylor series generated by f at $x = 0$ given by

$$\sum_{k=0}^{\infty} \frac{f^k(0)}{k!}x^k = f(0) + f'(0) + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^n(0)}{n!}x^n + \cdots$$

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- For what values of x can we normally expect a Taylor series to converge to its generating function?
- How accurately do a function's Taylor polynomials approximate the function on a given interval?

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where $R_n(x) = \dfrac{f^{n+1}(c)}{(n+1)!}(x-a)^{n+1}$ *for some c between a and x.*

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When we state Taylor's theorem this way, it says that for each $x \in I$, there exists $c \in (a, x)$ such that

$$f(x) = P_n(x) + R_n(x).$$

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Often we can estimate $R_n(x)$ without knowing the value of c, as the following example illustrates.

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for some c between 0 and x. Since e^x is an increasing function of x, $e^c < 1$ if $x \leq 0$ and $e^c \leq e^x$ for $x > 0$. Thus $|R_n(x)| \to 0$ as $n \to 0$. (Why?)

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Thus the series $\displaystyle\sum_{k=0}^{\infty} \frac{x^k}{k!}$ converges to e^x for every x.

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For every value of x, $R_{2k} \to 0$ as $k \to 0$. Thus $f(x) = \cos x$ converges for all x. Thus

$$\begin{aligned} \cos x &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!} \end{aligned}$$

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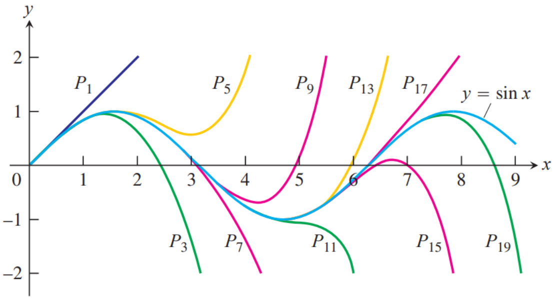

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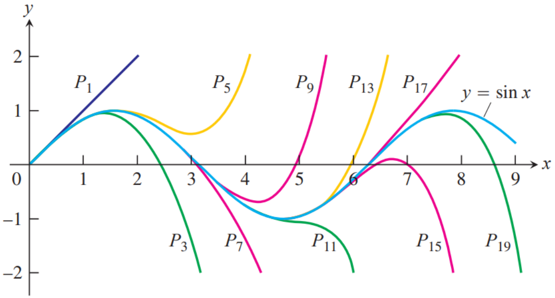

Use this to find Taylor series for $f(x) = x \sin x$ at $x = 0$.

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has derivatives of all orders at $x = 0$ and that $f^n(0) = 0$ for all n. What is the Taylor series generated by f at $x = 0$? Does the Taylor series converge? Does it converge to $f(x)$ for $x \in I$ where I is an interval containing 0?

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But $f(x_0) = e^{\frac{-1}{x_0^2}} \neq 0$ when $x_0 \neq 0$. Thus the series converges to $f(x)$ only at $x = 0$.