Mathematics I

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August 24, 2024

Lecture 9

Infinite series

Infinite Series:

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$$a_1 + a_2 + a_3 + \cdots$$

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- In this section we want to understand the meaning of such an infinite sum and to develop methods to calculate it.

- In order to give meaning for the infinite sum, we just consider the sum of the first n terms

$$s_n = a_1 + a_2 + \cdots + a_n = \sum_{k=1}^{n} a_k.$$

Infinite Series

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- For example consider the series $\sum_{k=1}^{\infty} 1/2^{k-1}$.

- $s_n = \displaystyle\sum_{k=1}^{n} 1/2^{k-1} = 2 - \frac{1}{2^{n-1}}$, therefore we can say that

$$\sum_{k=1}^{\infty} 1/2^{k-1} = \lim_{n\to\infty}\left(2 - \frac{1}{2^{n-1}}\right) = 2$$

Infinite Series Conti.

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is called an **infinite series.** The number a_n is called nth term of the series.

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- The sequence $\{s_n\}$ defined by

$$s_n = \sum_{k=1}^{n} a_k = a_1 + a_2 + \cdots a_n$$

is called the sequence of partial sums of the series and s_n is called ***n*th partial sum**.

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- If the sequence $\{s_n\}$ of partial sums converges to a limit L, we say the series converges and its **sum** is L. In this case we also write

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- If the sequence $\{s_n\}$ of partial sums does not converge, we say the the series **diverges.**

Infinite Series Conti.

Geometric Series: Geometric series are series of the form (for $a, r \in \mathbb{R}$)

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Theorem 0.1.
If $|r| < 1$ then the above geometric series converges and

$$a + ar + ar^2 + \cdots + ar^{n-1} + \cdots = \sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}.$$

if $|r| \geq 1$, the series diverges.

Infinite Series Conti.

Theorem 0.2 (The *n*-th term test).

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Examples:

(a). $\displaystyle\sum_{n=1}^{\infty} \frac{n}{\sqrt{n^2+10}}$. (b). $\displaystyle\sum_{n=1}^{\infty} \cos n\pi$.

(c). $\displaystyle\sum_{n=2}^{\infty} \frac{1}{4^n}$. (d). $\displaystyle\sum_{n=1}^{\infty} \left(\frac{1}{n(n+1)}\right)$.

Infinite Series Conti.

Part (a): Consider the n-th term $a_n = \sum_{n=1}^{\infty} \frac{n}{\sqrt{n^2+10}}$ which converges to 1 not equal to 0, hence by the n-th term test the series diverges.

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Part (c): Given series is geometric series with common ration $r = 1/4$ whose modulus value is less then 1, hence the series converges.

Part (d): $a_n = \left(\frac{1}{n(n+1)}\right) = \frac{1}{n} - \frac{1}{n+1}$, $s_n = 1 - \frac{1}{n+1}$ converges to 1 hence the series is convergent.

Theorem 0.3 (Algebra of Series).

Let $\sum_{n=1}^{\infty} a_n = A$ *and* $\sum_{n=1}^{\infty} b_n = B$. *Then*

1. $\sum_{n=1}^{\infty}(a_n \pm b_n) = \sum_{n=1}^{\infty} a_n \pm \sum_{n=1}^{\infty} b_n = A \pm B$
2. $\sum_{n=1}^{\infty} ka_n = k\sum_{n=1}^{\infty} a_n = kA.$

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Example: (a). $\sum_{n=1}^{\infty} \frac{2^n + 3^n}{4^n}$; (b). $\sum_{n=1}^{\infty} \frac{5^n - 3^n}{4^n}$

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Part (b): Since $\displaystyle\sum_{n=1}^{\infty} \frac{5^n}{4^n}$ is divergent and $\displaystyle\sum_{n=1}^{\infty} \frac{3^n}{4^n}$ is convergent, the given series is divergent.

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Theorem 0.4.

A series $\sum_{n=1}^{\infty} a_n$ *of non-negative terms converges if and only if the sequence* $\{s_n\}$ *of its partial sums are bounded from above.*

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Example:

1. Is the series $\sum_{n=1}^{\infty} \frac{1}{n}$ (**harmonic series**) convergent? **NO.**

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What can you say about the convergence of the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$? It is convergent which follows from the following theorem.

Integral Test:

Theorem 0.5.

Let $\{a_n\}$ *be a sequence of positive terms. Suppose that* $a_n = f(n)$, *where* $f(x)$ *is a positive, continuous, decreasing function of* x *for all* $x \geq N$ *(*N *is a positive integer). Then the series* $\sum_{n=N}^{\infty} a_n$ *and the integral* $\int_N^{\infty} f(x)dx$ *both converge or diverge.*

Examples:

(a). Find the values of p for which the following series converges

$$\sum_{n=1}^{\infty} \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \cdots + \frac{1}{n^p} + \cdots .$$

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Ans: The above series is convergent if and only if $p > 1$. Just apply integral test with the function $f(x) = 1/x^p$.

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(b). Test the convergence of $\displaystyle\sum_{n=1}^{\infty} \frac{1}{1+n^2}$?

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(b). Test the convergence of $\displaystyle\sum_{n=1}^{\infty} \frac{1}{1+n^2}$?

Ans: it is convergent. Just apply integral test with the function $f(x) = 1/(1+x^2)$.

Theorem 0.6 (The Comparison Test).

Let $\sum a_n$, $\sum c_n$ and $\sum d_n$ be series with non-negative terms. Suppose that for some integer N

$$d_n \leq a_n \leq c_n \qquad \text{for all} \qquad n > N.$$

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1. *If $\sum c_n$ converges, then $\sum a_n$ also converges.*

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1. *If $\sum c_n$ converges, then $\sum a_n$ also converges.*
2. *If $\sum d_n$ diverges, then $\sum a_n$ also diverges.*

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Let $\sum a_n$, $\sum c_n$ and $\sum d_n$ be series with non-negative terms. Suppose that for some integer N

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1. *If $\sum c_n$ converges, then $\sum a_n$ also converges.*
2. *If $\sum d_n$ diverges, then $\sum a_n$ also diverges.*

Examples:

(a) Test the convergence of $\displaystyle\sum_{n=1}^{\infty} \frac{5}{5n-1}$

(b) Test the convergence of $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n!}$