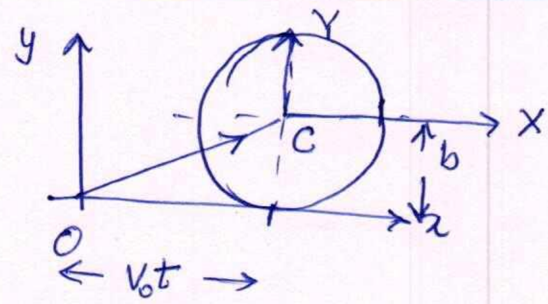
Tutorial 1 Vectors & Polar Co-ordinates 2nd Aug'2024

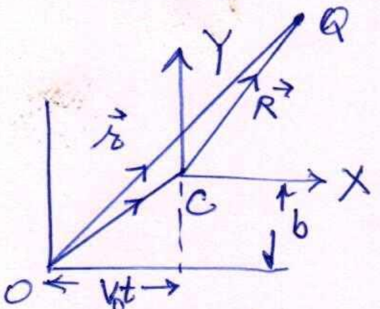
1.1



$\vec{OC} = V_0 t\,\hat{i} + b\hat{j}$

$\Rightarrow \vec{R}(C) = 0, \quad \vec{r}(\vec{OC}) = V_0 t\,\hat{i} + b\hat{j}$

1.2



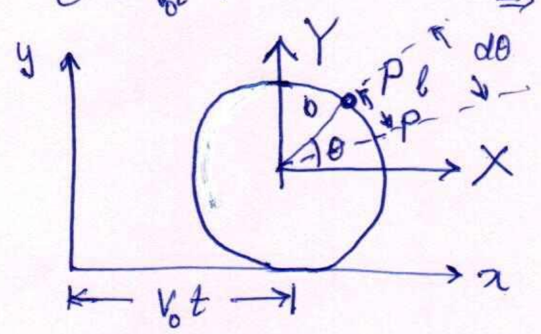
$\vec{r} = \vec{OC} + \vec{R}$

$\vec{R} = X\hat{i} + Y\hat{j}$

$\vec{OC} = V_0 t\,\hat{i} + b\hat{j}$

$\Rightarrow r(Q) = (X + V_0 t)\hat{i} + (b + Y)\hat{j}$

1.3



$X = b\cos\theta, \; Y = b\sin\theta$

as the wheel rotates, in the center of wheel reference frame the pebble moves from P to P' ie the angle changes from θ to $\theta - d\theta$; the length of the arc PP' is $l = |b\,d\theta|$. This length l must equal the distance that the wheel moves during the time $dt = V_0\,dt$. Therefore $|b\,d\theta| = V_0\,dt \Rightarrow \left|\frac{d\theta}{dt}\right| = \frac{V_0}{b} = \omega$. Note the angle θ is decreasing as time increases $\Rightarrow \frac{d\theta}{dt}$ must be negative. Let us put $\frac{d\theta}{dt} = -\frac{V_0}{b} = -\omega = \text{constant}$. Integrate both sides

$\theta = -\omega t + \text{const}$. Since $\theta = -\pi/2$ at $t = 0$, we must have $\theta = -(\omega t + \pi/2)$. The combination ωt appears often & it is better to give it some name such as α.
So $\alpha = \omega t$ & $\theta = -(\alpha + \pi/2)$

1.4 $X = b\cos[-\{\omega t + \frac{\pi}{2}\}] = b[\cos\omega t \cos\frac{\pi}{2} - \sin\omega t \underbrace{\sin\frac{\pi}{2}}_{1}]$, with $\cos\frac{\pi}{2} \to 0$

$\Rightarrow \quad X = -b\sin\omega t = -b\sin\alpha$

$Y = b\sin[-\{\omega t + \pi/2\}] = -b\sin(\omega t + \pi/2) = -b[\sin\omega t \cos\pi/2 + \cos\omega t \underbrace{\sin\pi/2}_{1}]$, with $\cos\pi/2 \to 0$

$= -b\cos\omega t = -b\cos\alpha$

$\vec{R} = X\hat{i} + Y\hat{j} = -b(\sin\alpha\,\hat{i} + \cos\alpha\,\hat{j})$

1.5 $\vec{r} = \vec{OC} + \vec{R} = (X + V_0 t)\hat{i} + (Y + b)\hat{j}$; $V_0 = \omega b$

$= [-b\sin\alpha + (\omega t) b]\hat{i} + [-b\cos\alpha + b]\hat{j}$

$= b[\alpha - \sin\alpha]\hat{i} + b(1 - \cos\alpha)\hat{j}$

Let us find $\vec{R}|_{t=0} = -b\hat{j}$ and $\vec{r}|_{t=0} = 0$ since the pebble is at the origin of the Ground based co-ordinate system. Aditionally, this result viz $\vec{r}|_{t=0} = 0$ makes sense, from the wheel center's origin C, the pebble lies directly below (–ve y-axis) at a distance = radius of the wheel = b. Hence $\vec{R}|_{t=0} = -b\hat{j}$ makes sense.

1.6 $\vec{V} = \frac{d\vec{R}}{dt} = \omega\frac{d\vec{R}}{d\alpha} = -\omega b[\cos\alpha\,\hat{i} - \sin\alpha\,\hat{j}]$

$\vec{A} = \frac{d\vec{V}}{dt} = -\omega^2 b[-\sin\alpha\,\hat{i} - \cos\alpha\,\hat{j}]$

$= -\omega^2\{-b(\sin\alpha\,\hat{i} + \cos\alpha\,\hat{j})\} = -\omega^2\vec{R}$

Next $\vec{v} = \frac{d\vec{r}}{dt} = \omega\frac{d\vec{r}}{d\alpha} = \omega b\{(1 - \cos\alpha)\hat{i} + \sin\alpha\,\hat{j}\}$

$\vec{a} = \frac{d\vec{v}}{dt} = \omega\frac{d\vec{v}}{d\alpha} = \omega^2 b\{\sin\alpha\,\hat{i} + \cos\alpha\,\hat{j}\}$

$= -\omega^2\{-b[\sin\alpha\,\hat{i} + \cos\alpha\,\hat{j}]\} = -\omega^2\vec{R}$

Thus $\vec{a} = \vec{A}$ because $\vec{OC}$ is linear in time.

1.7 $\vec{R} = -b(\sin\alpha\,\hat{i} + \cos\alpha\,\hat{j})$

$|\vec{R}| = b$ & therefore $\hat{R} = \dfrac{\vec{R}}{|\vec{R}|} = -[\sin\alpha\,\hat{i} + \cos\alpha\,\hat{j}]$

$\hat{T} = \hat{k}\times\hat{R} = -[\sin\alpha\,(\hat{k}\times\hat{i}) + \cos\alpha\,(\hat{k}\times\hat{j})]$

$= -[\sin\alpha\,\hat{j} + \cos\alpha\,(-\hat{i})] = \cos\alpha\,\hat{i} - \sin\alpha\,\hat{j}$

Check what happens at $t=0$ i.e $\alpha = 0$

$\hat{R}(0) = -\hat{j}$ and $\hat{T} = \hat{i}$ (Note $\hat{T}$ is positive along the direction of the increase of the angle θ.

1.8 $\vec{r} = b\{[\alpha - \sin\alpha]\hat{i} + [1-\cos\alpha]\hat{j}\}$

$|\vec{r}| = b[\alpha^2 + \sin^2\alpha - 2\alpha\sin\alpha + 1 + \cos^2\alpha - 2\cos\alpha]^{1/2}$

$= b[\alpha^2 + (\sin^2\alpha + \cos^2\alpha) + 1 - 2\cos\alpha - 2\alpha\sin\alpha]^{1/2}$

$= b[\alpha^2 + 2 - 2\cos\alpha - 2\alpha\sin\alpha]^{1/2}$

$= b[\alpha^2 + 2(1 - \cos\alpha - \alpha\sin\alpha)]^{1/2} = bD$

where $D = [\alpha^2 + 2(1-\cos\alpha - \alpha\sin\alpha)]^{1/2}$, $|\vec{r}| = bD$

& $\hat{r} = \dfrac{\vec{r}}{|\vec{r}|} = \dfrac{\cancel{b}\{(\alpha - \sin\alpha)\hat{i} + (1-\cos\alpha)\hat{j}\}}{\cancel{b}\,D}$

$= \frac{1}{D}[(\alpha - \sin\alpha)\hat{i} + (1-\cos\alpha)\hat{j}]$

$\hat{t} = \hat{k}\times\hat{r} = \frac{1}{D}[-(1-\cos\alpha)\hat{i} + (\alpha - \sin\alpha)\hat{j}]$

Thus we have found $\hat{r}$ & $\hat{t}$. Note that since the pebble is at the origin in the ground based system at $t=0$; neither $\hat{r}$ nor $\hat{t}$ are defined at $t=0$.

1.9 a) Consider the radial & tangential components of the velocity vector $\vec{V}$ in the X-Y co-ordinate system.

$$V_R = \hat{R}\cdot\vec{V} = -(\sin\alpha\,\hat{i}+\cos\alpha\,\hat{j})\cdot(-\omega b)[\cos\alpha\,\hat{i}-\sin\alpha\,\hat{j}]$$

$$= \omega b[\sin\alpha\cos\alpha - \cos\alpha\sin\alpha] = 0$$

This result makes sense because the pebble is always at a fixed distance from the center of the wheel. Note $|\vec{R}| = b \ \forall$ time.

Next consider $V_T = \hat{T}\cdot\vec{V}$

$$= (\cos\alpha\,\hat{i} - \sin\alpha\,\hat{j})\cdot(-\omega b)(\cos\alpha\,\hat{i} - \sin\alpha\,\hat{j})$$

$$= -\omega b(\cos^2\alpha + \sin^2\alpha) = -\omega b$$ (Also makes sense)

b) Consider radial & tangential components of the acceleration of the pebble in the wheel centered frame.

$$A_R = \hat{R}\cdot\vec{A} = -(\sin\alpha\,\hat{i}+\cos\alpha\,\hat{j})\cdot(-\omega^2)\vec{R}$$

$$= \hat{R}\cdot(-\omega^2)\bar{R} = -\omega^2 R = -\omega^2 b(-1)(\sin\alpha\,\hat{i}+\cos\alpha\,\hat{j})$$

$$= \omega^2 b(\sin\alpha\,\hat{i}+\cos\alpha\,\hat{j}) = -\omega^2 b\hat{R} \ \forall \text{time}$$

Note the acceleration is "centripetal" ie towards the center

$$A_T = \hat{T}\cdot\bar{A} = (\cos\alpha\,\hat{i} - \sin\alpha\,\hat{j})\cdot(\omega^2 b)(\sin\alpha\,\hat{i}+\cos\alpha\,\hat{j})$$

$$= \omega^2 b[\cos\alpha\sin\alpha - \sin\alpha\cos\alpha] = 0 \quad \forall \text{time}$$

1.10 Now consider the radial & tangential components of the velocity & acceleration vectors.

$$\hat{r} = \frac{1}{D}[(\alpha-\sin\alpha)\,\hat{i} + (1-\cos\alpha)\,\hat{j}]$$

$$\hat{t} = \frac{1}{D}[-(1-\cos\alpha)\hat{i} + (\alpha-\sin\alpha)\,\hat{j}]$$

a) The velocity vector

$$\vec{v} = \omega b\,[(1-\cos\alpha)\,\hat{i} + \sin\alpha\,\hat{j}]$$

$$v_r = \hat{r}\cdot\vec{v} = \frac{1}{D}[(\alpha-\sin\alpha)\,\hat{i} + (1-\cos\alpha)\,\hat{j}]\cdot(\omega b)[(1-\cos\alpha)\hat{i}+\sin\alpha\hat{j}]$$

$$= \frac{\omega b}{D}[(\alpha-\sin\alpha)(1-\cos\alpha) + \sin\alpha(1-\cos\alpha)]$$

$$= \frac{\omega b}{D}(1-\cos\alpha)(\alpha-\cancel{\sin\alpha}+\cancel{\sin\alpha}) = \frac{\omega b}{D}\alpha(1-\cos\alpha)$$

$$v_t = \hat{t}\cdot\vec{v} = \frac{1}{D}[-(1-\cos\alpha)\,\hat{i} + (\alpha-\sin\alpha)\,\hat{j}]$$
$$\cdot(\omega b)[(1-\cos\alpha)\,\hat{i}+\sin\alpha\,\hat{j}]$$

$$= \frac{1}{D}[-(1-\cos\alpha)^2 + \sin\alpha(\alpha-\sin\alpha)]$$

$$= \frac{1}{D}[-1-\cos^2\alpha + 2\cos\alpha + \alpha\sin\alpha - \sin^2\alpha]$$

$$= \frac{1}{D}[-1-(\cos^2\alpha+\sin^2\alpha) + 2\cos\alpha + \alpha\sin\alpha]$$

$$= \frac{1}{D}[2\cos\alpha - \alpha\sin\alpha - 2]$$

b) Next the acceleration vector

$$\vec{a} = \omega^2 b\,(\sin\alpha\,\hat{i} + \cos\alpha\,\hat{j})$$

$$a_r = \hat{r}\cdot\vec{a} = \frac{\omega^2 b}{D}[(\alpha-\sin\alpha)\,\hat{i} + (1-\cos\alpha)\,\hat{j}]\cdot(\sin\alpha\,\hat{i}+\cos\alpha\,\hat{j})$$

$$= \frac{\omega^2 b}{D}[\sin\alpha(\alpha-\sin\alpha) + \cos\alpha(1-\cos\alpha)]$$

$$= \frac{\omega^2 b}{D}[\alpha\sin\alpha - \sin^2\alpha + \cos\alpha - \cos^2\alpha]$$

$$= \frac{\omega^2 b}{D}[\alpha\sin\alpha + \cos\alpha - (\sin^2\alpha+\cos^2\alpha)]$$

$$= \frac{\omega^2 b}{D}[\alpha\sin\alpha + \cos\alpha - 1]$$

$$a_t = \hat{t}\cdot\vec{a} = \frac{1}{D}\left[-(1-\cos\alpha)\,\hat{i} + (\alpha-\sin\alpha)\,\hat{j}\right]\cdot (\omega^2 b)(-\sin\alpha\,\hat{i}+\cos\alpha\,\hat{j})$$

$$= \frac{\omega^2 b}{D}\left[-\sin\alpha(1-\cos\alpha) + \cos\alpha(\alpha-\sin\alpha)\right]$$

$$= \frac{\omega^2 b}{D}\left[-\sin\alpha + \cancel{\sin\alpha\cos\alpha} + \alpha\cos\alpha - \cancel{\sin\alpha\cos\alpha}\right]$$

$$= \frac{\omega^2 b}{D}\left[\alpha\cos\alpha - \sin\alpha\right]$$

<u>Exercise</u> Determine the limits of v_r, v_t, a_r & a_t as $t\to 0$.

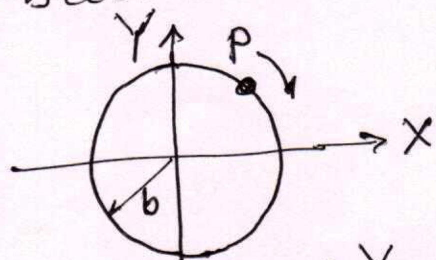
1.11 $\vec{r} = b\left[(\alpha-\sin\alpha)\,\hat{i} + (1-\cos\alpha)\,\hat{j}\right]$

$\vec{R} = -b\left[\sin\alpha\,\hat{i} + \cos\alpha\,\hat{j}\right]$

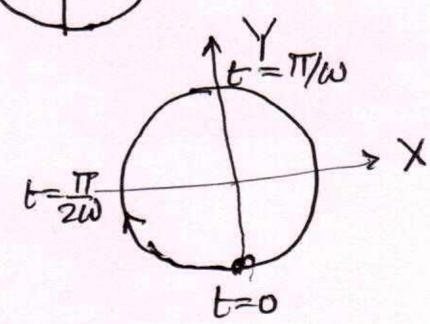
a) In the wheel center fixed frame

$\Rightarrow X(t) = -b\sin\omega t$, $Y(t) = -b\cos\omega t$ (Note $\alpha = \omega t$)

$\Rightarrow X^2 + Y^2 = b^2$



$\Rightarrow$ pebble lies on a circle of radius b and it travels clockwise



At $t=0$, $X(t)\big|_{t=0} = 0$, $Y(t)\big|_{t=0} = -b$

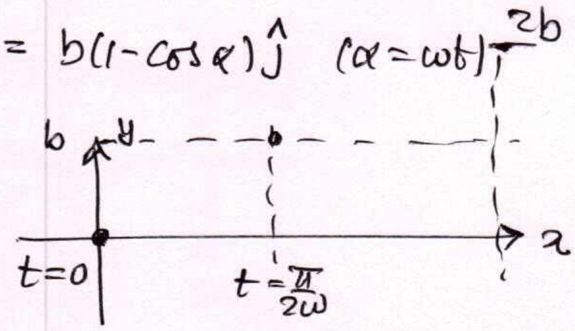
At $\alpha = \pi/2$ ie $t = \frac{\pi}{2\omega}$

$X = -b$, $Y = 0$

At $\alpha = \pi$, $t = \frac{\pi}{\omega}$, $X = 0$, $Y = b$

Thus one can find the direction of rotation as time passes

1.11 b) $x = b(\omega t - \sin \omega t)$, $y = b(1-\cos\alpha)\hat{j}$ $(\alpha = \omega t)$

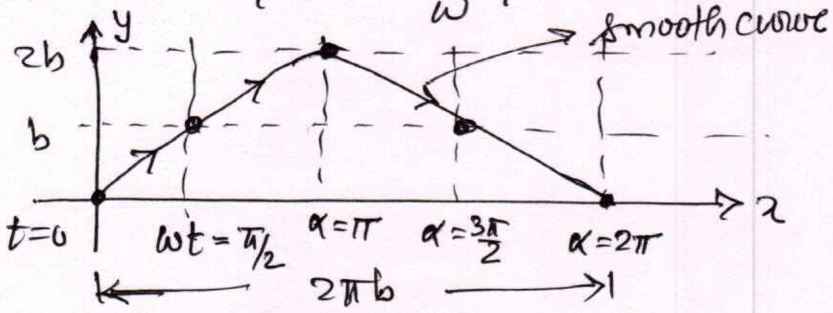


At $t=0$; $x=y=0$

At $\omega t = \frac{\pi}{2}$; $t = \frac{\pi}{2\omega}$

$x = b[\frac{\pi}{2} - 1]$, $y = b[1]$

At $\omega t = \pi$, ; $t = \frac{\pi}{\omega}$, $x = b(\pi - 0)$, $y = 2b$



The trajectory is a smooth curve called the cycloid

if $t > \frac{2\pi}{\omega}$, the graph repeats itself

if $\alpha = 2\pi + \alpha'$ $(\alpha' \in [0, 2\pi])$ then

$$x = b[(2\pi + \alpha') - \sin(2\pi + \alpha')]$$

$$= 2\pi b + b[\alpha' - \sin\alpha']$$

$$y = b[1 - \cos(2\pi + \alpha')] = b(1-\cos\alpha')$$

Thus the y versus x graph has the same shape but is shifted to the right by a distance $2\pi b$ which is the circumference of the wheel.