Vector Valued Functions and Motion in Space

Gunja Sachdeva

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Recall

$r(t) = (f(t), g(t), h(t))$

$\vec{r} : \text{Domain} \subseteq \mathbb{R} \longrightarrow \mathbb{R}^3$

$t \longmapsto r(t) \in \mathbb{R}^3$

$\{r(t) \mid t \in \text{Domain}\}$ – curve in the space

Example : motion of a particle, projectile motion etc

$r : \mathbb{R} \longrightarrow \mathbb{R}^2$

$v(t) = \dfrac{dr(t)}{dt} = \dfrac{df}{dt}\hat{i} + \dfrac{dg}{dt}\hat{j} + \dfrac{dh}{dt}\hat{k}$ – tangent vector

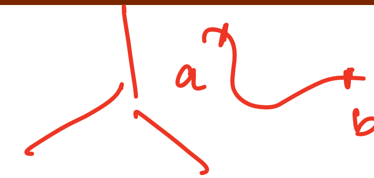
$|v(t)|$ – speed

Unit tangent vector $= \dfrac{v(t)}{|v(t)|}$

$\therefore \; v(t) = \underbrace{|v(t)|}_{\text{speed}} \; \underbrace{\dfrac{v(t)}{|v(t)|}}_{\text{Unit T.V.}}$

Smoothness of $r(t)$: $r(t)$ is smooth on the domain if $r'(t)$ is cont. and $r'(t) \neq 0$ $\forall$ $t \in$ Domain

Arc Length



Definition

The length of a smooth curve $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k},\ a \leq t \leq b$, that is traced exactly once as t increases from $t = a$ to $t = b$, is

$$\int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt.$$

The integrant in the above formula is $|\mathbf{v}(t)|$, therefore, the formula for length a shorter way.

$$L = \int_a^b |\mathbf{v}(t)|\ dt$$

Examples

- The length of the curve $\mathbf{r}(t) = (2+t)\mathbf{i} - (t+1)\mathbf{j} + t\mathbf{k}$, $0 \le t \le 3$ is

$$\text{Length} = \int_0^3 |v(t)|\, dt$$

at $t=0$

$r(0) = (2, -1, 0)$

$t = 3$

$r(3) = (5, -4, 3)$

$$v(t) = r'(t) = \hat{i} - \hat{j} + \hat{k}$$

$$|v(t)| = \sqrt{1+1+1} = \sqrt{3}$$

$$L = \int_0^3 \sqrt{3}\, dt = \sqrt{3} \times 3 = 3\sqrt{3}$$

Examples

- The length of the curve $\mathbf{r}(t) = (2+t)\mathbf{i} - (t+1)\mathbf{j} + t\mathbf{k}$, $0 \leq t \leq 3$ is

$$\int_0^3 |\mathbf{v}(t)|\ dt = \int_0^3 |\mathbf{i} - \mathbf{j} + \mathbf{k}|\ dt = \int_0^3 \sqrt{3}\ dt = 3\sqrt{3}.$$

- Find the length of the curve

$$\mathbf{r}(t) = \cos^3 t\mathbf{i} + \sin^3 t\mathbf{j} + \frac{3}{2}\sin^2 t\mathbf{k}, \quad 0 \leq t \leq \pi/2.$$

$$\mathbf{r}'(t) = -3\cos^2 t \sin t\mathbf{i} + 3\sin^2 t \cos t\mathbf{j} + 3\sin t \cos t\mathbf{k}.$$

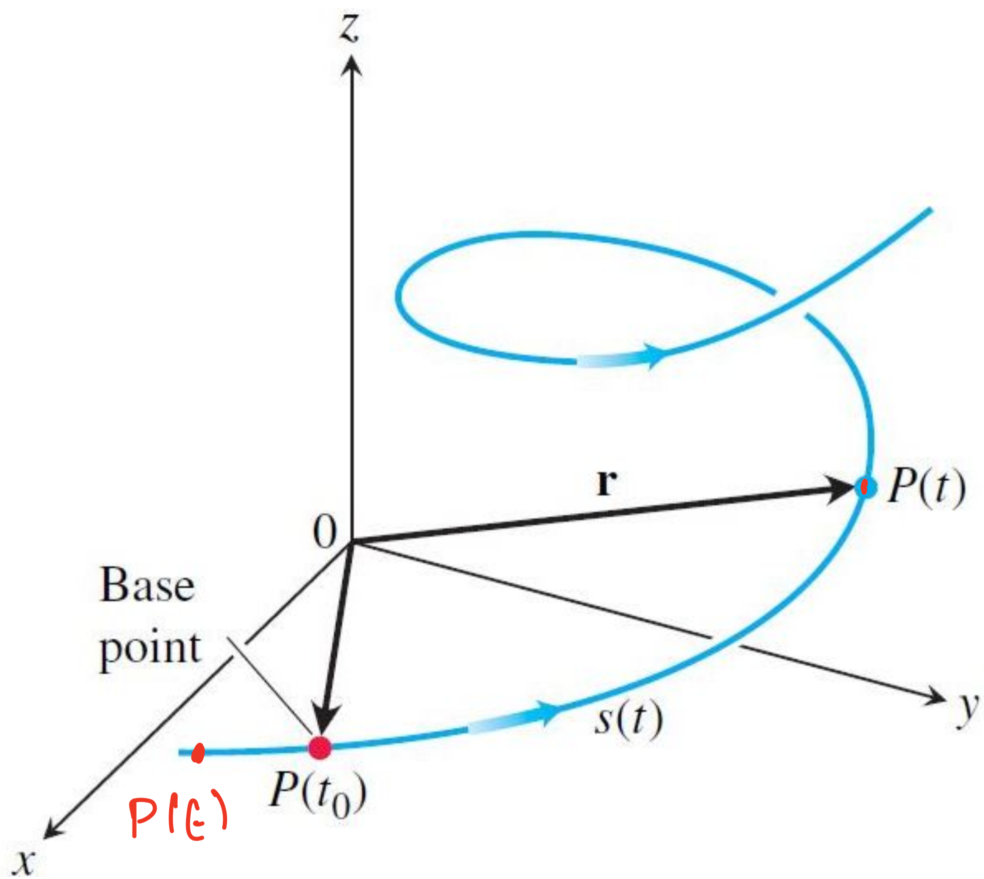
$$|\mathbf{v}(t)| = |\mathbf{r}'(t)| = 3\sqrt{2}|\sin t \cos t|$$

Arc Length is

$$\int_0^{\pi/2} |\mathbf{v}(t)| dt = 3\sqrt{2}\int_0^{\pi/2} \sin t \cos t dt = \frac{3}{\sqrt{2}}.$$

Arc Length Parameter

- Let C be a space curve with smooth parametric equation $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$.
- Now we are interested to find the length of the curve from a base point $P(t_0) = \mathbf{r}(t_0)$ on the curve C.



Arc Length Parameter

- The "directed" distance of any point $\mathbf{r}(t)$ from the base point $\mathbf{r}(t_0)$ along the curve C is defined by

$$s(t) = \int_{t_0}^{t} |\mathbf{v}(\tau)| \, d\tau = \int_{t_0}^{t} \sqrt{[x'(\tau)]^2 + [y'(\tau)]^2 + [z'(\tau)]^2} \, d\tau.$$

- Here $s(t)$ is called arc length function, if $t > t_0$, $s(t)$, the distance along the curve from $P(t_0)$ to $P(t)$ is positive. If $t < t_0$, $s(t)$ is negative of the distance.

- We call s is arc length parameter for the curve.

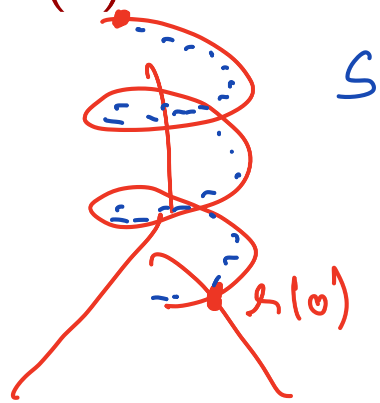
$S : \mathbb{R} \longrightarrow \mathbb{R}$

$t \longmapsto s(t)$

$\int_2^4 : \int_4^2 = -\int_2^4$

Example

Let C be the curve given by $\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + t\mathbf{k}$ and s is any real number. Find a point on C whose directed distance from $\mathbf{r}(0)$ is s.



$$s = \int_0^t |v(\tau)|\, d\tau$$

$$v(t) = -\sin t\,\hat{i} + \cos t\,\hat{j} + \hat{k}$$

$$|v(t)| = \sqrt{2}$$

$$s = \int_0^t \sqrt{2}\, d\tau \Rightarrow \boxed{t\sqrt{2} = s} \Rightarrow t = s/\sqrt{2}$$

at $t = 1/\sqrt{2}$, $s = 1$

Example

Let C be the curve given by $\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + t\mathbf{k}$ and s is any real number. Find a point on C whose directed distance from $\mathbf{r}(0)$ is s.

Solution: Velocity vector is given by $\mathbf{v}(t) = (-sint)\mathbf{i} + (\cos t)\mathbf{j} + \mathbf{k}$, hence $|\mathbf{v}(t)| = \sqrt{2}$.

Let $\mathbf{r}(t)$ be the required point, the distance from $\mathbf{r}(0)$ to this point along the curve is given by

$$s = \int_0^t |\mathbf{v}(\tau)| \, d\tau = \int_0^t \sqrt{2} \, d\tau = \sqrt{2}t.$$

Arc Length Parameter

We see that

$$t = s/\sqrt{2},$$

hence the required point is

$$\mathbf{r}(s/\sqrt{2}) = \cos(s/\sqrt{2})\mathbf{i} + \sin(s/\sqrt{2})\mathbf{j} + (s/\sqrt{2})\mathbf{k}.$$

Further, if we want a point on the curve which is at distance $\pi/\sqrt{2}$ from the base point $\mathbf{r}(0)$,

$s = \pi/\sqrt{2}$

$r(\pi/\sqrt{2} \times \sqrt{2}) = \cos \pi/2 \, \hat{i} + \sin \pi/2 \, \hat{j} + \pi/2 \, \hat{k}$

$r(\pi/2) = 0\hat{i} + \hat{j} + \pi/2 \, \hat{k}$

$= (0, 1, \pi/2)$

Arc Length Parameter

We see that

$$t = s/\sqrt{2},$$

hence the required point is

$$\mathbf{r}(s/\sqrt{2}) = \cos(s/\sqrt{2})\mathbf{i} + \sin(s/\sqrt{2})\mathbf{j} + (s/\sqrt{2})\mathbf{k}.$$

Further, if we want a point on the curve which is at distance $\pi/\sqrt{2}$ from the base point $\mathbf{r}(0)$,then the substitution of $s = \pi/\sqrt{2}$ in the above gives the point

$$0\ \mathbf{i} + \mathbf{j} + (\pi/2)\mathbf{k}.$$

Arc Length Parametrization

of position vector r(t)

- In the above example, we have expressed the parameter t in terms of arc lenght parameter s, (say $t = t(s)$), then the vector $\mathbf{r}(t(s))$ gives the point on the curve which is at distance (measured along the curve) s from the base point $P(t_0) = \mathbf{r}(t_0)$.
- Then $\mathbf{r}(t(s))$ gives another parametrization of the curve C, called arc length parametrization.

Example

Let $\mathbf{r}(t) = (e^t \cos t)\mathbf{i} + (e^t \sin t)\mathbf{j} + e^t\mathbf{k}$, then find the following:

1. The arc length parameter with base point $\mathbf{r}(0)$,
2. Arc length parametrization of the curve with the same base point.
3. The point on the curve which is at distance $\sqrt{3}(e^{\pi/2} - 1)$ from the base point.

$$s(t) = \int_0^t |v(z)|\,dz \qquad t \in \text{Domain}$$

$$r'(t) = (-\sin t \cdot e^t + \cos t \cdot e^t)\hat{i} + (e^t \sin t + \cos t \cdot e^t)\hat{j} + e^t\hat{k}$$

$$|r'(t)| = \sqrt{e^{2t} + e^{2t} + e^{2t}} = \sqrt{3}e^t$$

$$|v(t)| = \sqrt{3}e^t,$$

$$s(t) = \int_0^t |v(\tau)|\,d\tau = \int_0^t \sqrt{3}\, e^{\tau}\, d\tau$$

arc length parameter $\boxed{s = \sqrt{3}\,(e^t - 1)}$

$$e^t = \frac{s}{\sqrt{3}} + 1$$

$$\Rightarrow \quad t = \ln\left(\frac{s}{\sqrt{3}} + 1\right)$$

$$\Rightarrow \quad r\left(\ln \frac{s}{\sqrt{3}} + 1\right) = \left(\frac{s}{\sqrt{3}} + 1\right) \cos\left(\ln\left(\frac{s}{\sqrt{3}} + 1\right)\right)\hat{i} + \left(\frac{s}{\sqrt{3}} + 1\right) \sin\left(\ln\left(\frac{s}{\sqrt{3}} + 1\right)\right)\hat{j} + \left(\frac{s}{\sqrt{3}} + 1\right)\hat{k}$$

$$s = \sqrt{3}\left(e^{\pi/2} - 1\right)$$

$$r(\pi/2) = e^{\pi/2}\left(\hat{j} + \hat{k}\right)$$

Example

Let $\mathbf{r}(t) = (e^t \cos t)\mathbf{i} + (e^t \sin t)\mathbf{j} + e^t\mathbf{k}$, then find the following:

1. The arc length parameter with base point $\mathbf{r}(0)$,
2. Arc length parametrization of the curve with the same base point.
3. The point on the curve which is at distance $\sqrt{3}(e^{\pi/2} - 1)$ from the base point.

Answer: 1. $s(t) = \sqrt{3}[e^t - 1]$,
2. $\mathbf{r}(t(s)) = \left(\frac{s}{\sqrt{3}} + 1\right)[\cos\left(\ln\left(\frac{s}{\sqrt{3}} + 1\right)\right)\mathbf{i} + \sin\left(\ln\left(\frac{s}{\sqrt{3}} + 1\right)\right)\mathbf{j} + \mathbf{k}]$ and
3. $e^{\pi/2}[\mathbf{j} + \mathbf{k}]$

Speed

Remark

Let $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$ be a smooth parametrization of a curve C. Then the arc lenght parameter with base point $\mathbf{r}(t_0)$ is given by

$$s(t) = \int_{t_0}^{t} |\mathbf{v}(\tau)| \, d\tau.$$

Clearly, we have

$$\frac{ds}{dt} = |\mathbf{v}(t)| > 0,$$

which is speed of the particle with displacement $\mathbf{r}(t)$.

Unit Tangent Vector

$$T = \frac{v}{|v|} = \frac{dr/dt}{|dr/dt|}$$

$$= \frac{dr}{dt} \Big/ \frac{ds}{dt}$$

$$= \frac{dr}{dt} \times \frac{dt}{ds}$$

$$\boxed{T = \frac{dr}{ds}}$$

We already know that the velocity vector $\mathbf{v} = d\mathbf{r}/dt$ is tangent to the curve $\mathbf{r}(t)$ and the vector

$$\mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|}$$

is therefore a unit vector tangent to the curve, called the **unit tangent vector.**

Remark 0.1.

$$\mathbf{T} = \frac{d\mathbf{r}}{ds}. \tag{0.1}$$

Since, $\dfrac{d\mathbf{r}}{ds} = \dfrac{d\mathbf{r}}{dt}\,\dfrac{dt}{ds} = \dfrac{d\mathbf{r}}{dt}\,\dfrac{1}{ds/dt} = \dfrac{\mathbf{v}}{|\mathbf{v}|} = \mathbf{T}.$

Examples

1. Let $\mathbf{r}(t) = (e^t \cos t)\mathbf{i} + (e^t \sin t)\mathbf{j} + e^t\mathbf{k}$, then find the unit tangent of the curve.

$$r(t(s)) = \left(\frac{s}{\sqrt{3}}+1\right)\left[\cos\ln\left(\frac{s}{\sqrt{3}}+1\right)\hat{i} + \sin\ln\left(\frac{s}{\sqrt{3}}+1\right)\hat{j} + \hat{k}\right]$$

$$\frac{dr}{ds} =$$

Examples

1. Let $\mathbf{r}(t) = (e^t \cos t)\mathbf{i} + (e^t \sin t)\mathbf{j} + e^t\mathbf{k}$, then find the unit tangent of the curve.

2. Let $\mathbf{r}(t) = (t \sin t + \cos t)\mathbf{i} + (t \cos t - \sin t)\mathbf{j}$, then find the unit tangent of the curve at $t = 0$.

Curvature of A Plane Curve

- When a particle moves along a smooth curve in the plane, the unit tangent $\mathbf{T} = d\mathbf{r}/ds$ changes its direction (turns) wherever the curve bends.
- The rate at which $\mathbf{T}$ turns per unit of length along the curve is called the curvature.

Definition 0.2.

If $\mathbf{T}$ is the unit tangent of a smooth curve, the curvature of the curve is defined by

$$\kappa = \left| \frac{d\mathbf{T}}{ds} \right|.$$

Curvature formula

$$\kappa = \left|\frac{d\mathbf{T}}{ds}\right| = \left|\frac{d\mathbf{T}}{dt} \times \frac{dt}{ds}\right|$$

Remark 0.3 (Formula for Calculating Curvature).

If $\mathbf{r}(t)$ is smooth curve, then the curvature is the scalar function

$$\kappa = \frac{1}{|\mathbf{v}|}\left|\frac{d\mathbf{T}}{dt}\right|,$$

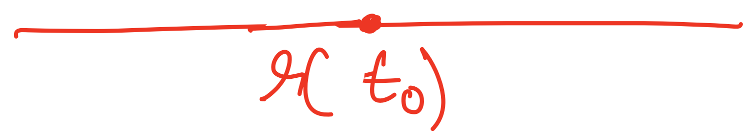
where $\mathbf{T} = \mathbf{v}/|\mathbf{v}|$ is the unit tangent vector.

Examples: Curvature of A Plane Curve

Curvature for straight lines and circles are constant.

Pf

$$r(t) = \underline{r'(t_0)}\, t + r(t_0)$$



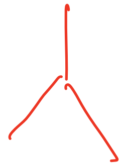
$$\kappa = \frac{1}{|v|} \left| \frac{dT}{dt} \right|$$

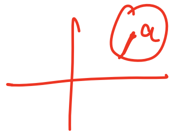
$$T = \frac{v(t)}{|v(t)|} \qquad v(t) = r'(t) = r'(t_0)$$

$$|v(t)| = |r'(t_0)|$$

$$T = \frac{r'(t_0)}{|r'(t_0)|} \qquad \frac{dT}{dt} = 0 \Rightarrow \kappa = 0$$

Curvature of a circle:





a - radius

$$r(t) = a\cos t\,\hat{i} + a\sin t\,\hat{j} + 0\hat{k}$$

$$v(t) = r'(t) = -a\sin t\,\hat{i} + a\cos t\,\hat{j} + 0\hat{k}$$

$$|v(t)| = a$$

$$T = \frac{-a\sin t\,\hat{i} + a\cos t\,\hat{j} + 0\hat{k}}{a}$$

$$\frac{dT}{dt} = \frac{-a\cos t\,\hat{i} - a\sin t\,\hat{j}}{a} = -\cos t\,\hat{i} - \sin t\,\hat{j}$$

$$\left|\frac{dT}{dt}\right| = \sqrt{1} = 1$$

$$\kappa = \frac{1}{a} = \frac{1}{\text{radius of circle}}$$

Examples: Curvature of A Plane Curve

Curvature for straight lines and circles are constant.

Any straight line can be parametrized by

$$\mathbf{r}(t) = t\mathbf{v}_0 + \mathbf{a}$$

where $\mathbf{v}_0$ and $\mathbf{a}$ constant vectors. Then $\mathbf{T} = \frac{\mathbf{v}_0}{|\mathbf{v}_0|}$, therefore $\kappa = 0$.

Now we will find the curvature of a circle.

A parametrization of the circle of radius 'a' with center at the origin O is given by

$$\mathbf{r}(t) = (a\cos t)\mathbf{i} + (a\sin t)\mathbf{j}.$$

Example

- We first find the velocity vector

$$\mathbf{v} = \mathbf{r}'(t) = -(a \sin t)\mathbf{i} + (a \cos t)\mathbf{j}.$$

- The speed is given by

$$|\mathbf{v}| = \sqrt{(-a \sin t)^2 + (a \cos t)^2} = |a| = a.$$

- Therefore, the unit tangent

$$\mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|} = -(\sin t)\mathbf{i} + (\cos t)\mathbf{j}.$$

- $\dfrac{d\mathbf{T}}{dt} = -(\cos t)\mathbf{i} - (\sin t)\mathbf{j} \Rightarrow \left|\dfrac{d\mathbf{T}}{dt}\right| = 1$

Example

- Hence, for any value of the parameter t, the curvature of the circle is

$$\kappa = \frac{1}{|\mathbf{v}|}\left|\frac{d\mathbf{T}}{dt}\right| = \frac{1}{a}(1) = \frac{1}{a} = \frac{1}{\text{radius}}$$

- Find the curvature of the parabola $y = x^2$ at the points $(0,0),\ (1,1),\ (2,4)$.

$r(t) = (t, t^2)$

$(0,0) \Rightarrow t=0$
$(1,1) \Rightarrow t=1$
$(2,4) \Rightarrow t=2$

Example

- Hence, for any value of the parameter t, the curvature of the circle is

$$\kappa = \frac{1}{|\mathbf{v}|}\left|\frac{d\mathbf{T}}{dt}\right| = \frac{1}{a}(1) = \frac{1}{a} = \frac{1}{\text{radius}}$$

- Find the curvature of the parabola $y = x^2$ at the points $(0,0),\ (1,1),\ (2,4)$.
  Ans.

$$\frac{dT}{dt} = \frac{2}{(1+4t^2)^{3/2}}[-2t\mathbf{i}+\mathbf{j}],\quad \kappa(t) = \frac{2}{(1+4t^2)^{3/2}}.$$

$\kappa(0) = 2$, $\kappa(1) = \frac{2}{5^{3/2}}$ $\kappa(2) = \frac{2}{17^{3/2}}$