Sequence and Series

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Series

Given a sequence of numbers (a_n), an expression of the form $a_1 + a_2 + a_3 + \cdots + a_n + \cdots$ is an infinite series. Then a_n is the nth term of the series. We write $\sum_{n=1}^{\infty} a_n$ or $\sum a_n$ to denote the series.

The sequence (s_n) defined by

$$
\begin{aligned}
s_1 &= a_1 \\
s_2 &= a_1 + a_2 \\
s_n &= a_1 + a_2 + \cdots + a_n = \sum_{k=1}^{n} a_k
\end{aligned}
$$

is the **sequence of partial sums** of the series, the number s_n being the nth partial sum.

Convergence of a series

If the **sequence of partial sums** (s_n) **converges to a limit** L**, we say that the series converges and that its sum is** L**.** That is,

$$a_1 + a_2 + \cdots + a_n + \cdots = \sum_{n=1}^{\infty} a_n = L.$$

If the sequence of partial sums of the series does not converge, we say that the series diverges.

Remarks

$$\boxed{1+\frac{1}{2}+\frac{1}{4}+\cdots = 2}$$

$$\underbrace{1+1+1+1}+\underbrace{\frac{1}{2}+\frac{1}{4}+\cdots}_{2} = 4$$

- Adding or deleting terms: We can add a finite number of terms to a series or delete a finite number of terms without altering the series convergence or divergence, although in the case of convergence this will usually change the sum.

- Re-indexing: As long as we preserve the order of its terms, we can reindex any series without altering its convergence.

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = \sum_{n=5}^{\infty} \frac{1}{2^{n-4}}$$

$$\sum_{n=-1}^{\infty} a_n = \sum_{n=1}^{\infty} a_{n-2}$$

Example: Geometric series

The series of the form $a + ar + ar^2 + \cdots + ar^{n-1} + \cdots = \sum_{n=1}^{\infty} ar^{n-1}$ in which a and r are fixed real numbers and $a \neq 0$.

$= \frac{a}{1-r}$ $\quad |r|<1$

Case 1 $\quad \boxed{r=1}$

$\sum_{n=1}^{\infty} a = a + a + a + \cdots$

$S_1 = a, \ S_2 = 2a, \ \cdots \ S_n = (na)$

$\lim_{n\to\infty} S_n = \pm\infty$ depends on sign of a

$\Rightarrow S_n$ diverges $\Rightarrow \sum a$ diverges

Case 2 $\quad \boxed{r=-1}$

$\sum_{n=1}^{\infty} a(-1)^{n-1} = a - a + a - \cdots$

$S_1 = a$

$S_2 = 0$

$S_3 = a$

$S_n = \{a, 0, a, 0, \cdots\}$ $\quad S_n$ diverges $\Rightarrow \sum a(-1)^{n-1}$ diverges

$\Rightarrow |r| \neq 1$ ✓

$\sum_{n=1}^{\infty} a r^{n-1}$

$S_n = a + ar + \cdots + ar^{n-1}$

$= a\dfrac{(1-r^n)}{1-r}$

$\lim_{n\to\infty} r^n = 0 \quad \text{if } |r|<1$

Case 3(a) $\quad |r|<1$

$\lim S_n = \dfrac{a}{1-r} - \lim_{n\to\infty} \dfrac{r^n}{1-r}$

$= \dfrac{a}{1-r}$

3(b) $\quad |r|>1$

$S_n \to \infty \Rightarrow \sum ar^{n-1}$ diverges

Example: Geometric series

The series of the form $a + ar + ar^2 + \cdots + ar^{n-1} + \cdots = \sum_{n=1}^{\infty} ar^{n-1}$ in which a and r are fixed real numbers and $a \neq 0$.

- If $r = 1$, then $s_n = na$ and the series diverges because $\lim_{n\to\infty} na = \pm\infty$, depending on the sign of a.
- If $r = -1$, then s_n alternate between a and 0 and never approach a single limit. Hence diverges.
- If $|r| \neq 1$, then $s_n = \frac{a(1-r^n)}{1-r}$.
- If $|r| < 1$, then $r^n \to 0$ and $s_n \to \frac{a}{1-r}$ and thus the series converges.
- If $|r| > 1$, then $|r^n| \to \infty$ and the series diverges.

Conclusion

Geometric Series

- The series $\sum_{n=1}^{\infty} ar^{n-1}$ converges to $\dfrac{a}{1-r}$, if $|r| < 1$.
- If $|r| \geq 1$, the series $\sum_{n=1}^{\infty} ar^{n-1}$ diverges.

Examples

Find the sum $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$. $= \sum_{n=1}^{\infty} \underbrace{\left(\frac{1}{n} - \frac{1}{n+1}\right)}_{a_n}$

$S_1 = a_1 = 1 - \frac{1}{2}$

$S_2 = a_1 + a_2 = 1 - \cancel{\frac{1}{2}} + \cancel{\frac{1}{2}} - \frac{1}{3} = 1 - \frac{1}{3}$

$a_n \to 0$

$\vdots$

$S_n = 1 - \frac{1}{n+1}$

$\lim_{n\to\infty} S_n = 1$

$\Rightarrow \sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$

Examples

Find the sum $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$.

- $a_n = \frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$.
- $s_n = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \cdots + \left(\frac{1}{n} - \frac{1}{n+1}\right) = 1 - \frac{1}{n+1}$.
- $s_n \to 1$ as $n \to \infty$.
- Thus the series converges and $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$.

Examples

The series

$$\sum_{n=1}^{\infty} \frac{n+1}{n} = \frac{2}{1} + \frac{3}{2} + \cdots + \frac{n+1}{n} + \cdots$$

$$\sum_{n=1}^{\infty} 1 + \frac{1}{n}$$

$$a_n = 1 + \frac{1}{n} \not\to 0$$

$$S_n = 1 + \cdots + 1 + 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$$

$$= n + \sum_{n=1}^{\infty} \frac{1}{n}$$

$$\lim_{n\to\infty} S_n = \infty \Rightarrow \sum \frac{n+1}{n} \text{ diverges}$$

Examples

The series

$$\sum_{n=1}^{\infty} \frac{n+1}{n} = \frac{2}{1} + \frac{3}{2} + \cdots + \frac{n+1}{n} + \cdots$$

- $a_n = \frac{n+1}{n} = 1 + \frac{1}{n}$.
- Thus $s_n = n + 1 + \frac{1}{2} + \cdots + \frac{1}{n} > n$.
- Thus s_n is not bounded above and hence s_n, the sequence of partial sums diverges.
- Thus the series diverges.

Theorem

If $\sum_{n=1}^{\infty} a_n$ *converges, then* $a_n \to 0$.

It implies,

Theorem

The n th term test for divergence: $\sum_{n=1}^{\infty} a_n$ *diverges if* $\lim_{n\to\infty} a_n$ *fails to exist or is different from* 0.

Examples.

- $$\sum_{n=1}^{\infty} \frac{n+1}{n} = \frac{2}{1} + \frac{3}{2} + \cdots + \frac{n+1}{n} + \cdots$$

  diverges as $a_n = \frac{n+1}{n} \to 1 \neq 0$.

- $\sum_{n=1}^{\infty} n^2$

$a_n = n^2$ $\lim_{n \to \infty} n^2$ does not exist.

$\Rightarrow$ Σn^2 diverges

Examples.

- $$\sum_{n=1}^{\infty} \frac{n+1}{n} = \frac{2}{1} + \frac{3}{2} + \cdots + \frac{n+1}{n} + \cdots$$

  diverges as $a_n = \frac{n+1}{n} \to 1 \neq 0$.
- $\sum_{n=1}^{\infty} n^2$ diverges as $n^2 \to \infty$
- $\sum_{n=1}^{\infty} (-1)^{n+1}$ $\lim_{n\to\infty} a_n$ fails to exist

  diverges

Examples.

- $$\sum_{n=1}^{\infty} \frac{n+1}{n} = \frac{2}{1} + \frac{3}{2} + \cdots + \frac{n+1}{n} + \cdots$$

  diverges as $a_n = \frac{n+1}{n} \to 1 \neq 0$.
- $\sum_{n=1}^{\infty} n^2$ diverges as $n^2 \to \infty$
- $\sum_{n=1}^{\infty} (-1)^{n+1}$ diverges as $\lim_{n\to\infty} (-1)^{n+1}$ does not exist.
- $\sum_{n=1}^{\infty} \frac{-n}{2n+5}$ diverges

$$a_n = \frac{-n}{2n+5} = \frac{-1}{2+5/n}$$

$$a_n \to -1/2 \neq 0$$

Examples.

- $$\sum_{n=1}^{\infty} \frac{n+1}{n} = \frac{2}{1} + \frac{3}{2} + \cdots + \frac{n+1}{n} + \cdots$$
  diverges as $a_n = \frac{n+1}{n} \to 1 \neq 0$.
- $\displaystyle\sum_{n=1}^{\infty} n^2$ diverges as $n^2 \to \infty$
- $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1}$ diverges as $\lim_{n\to\infty} (-1)^{n+1}$ does not exist.
- $\displaystyle\sum_{n=1}^{\infty} \frac{-n}{2n+5}$ diverges as $\displaystyle\lim_{n\to\infty} \frac{-n}{2n+5} = -\frac{1}{2} \neq 0$.

<u>What about converse?</u>

converse is not true

$\lim a_n = 0 \not\Rightarrow \sum a_n$ converges or diverges

A series whose $a_n \to 0$, but series diverges

$a_n = \frac{1}{2^n} + \cdots + \frac{1}{2^n}$

What about the series

$$1 + \underbrace{\frac{1}{2} + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \cdots + \underbrace{\frac{1}{2^n} + \frac{1}{2^n} + \cdots + \frac{1}{2^n}}_{2^n \text{ times}}} + \cdots ?$$

$\lim a_n = 0$

$\underbrace{\frac{1}{8} + \frac{1}{8} + \cdots + \frac{1}{8}}_{8 \text{ times}}$

$\sum_{n=1}^{\infty} \underbrace{\frac{1}{2^n} + \frac{1}{2^n} + \cdots \frac{1}{2^n}}_{2^n \text{ terms}}$

$S_n = n$

$S_n \to \infty \Rightarrow$ the above series diverges

A series whose $a_n \to 0$, but series diverges

What about the series

$$1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \cdots + \frac{1}{2^n} + \frac{1}{2^n} + \cdots + \frac{1}{2^n} + \cdots ?$$

- This series diverges because the terms can be grouped into infinitely many clusters each of which adds to 1, so the partial sums increase without bound.
- Thus s_n is not bounded hence does not converge.
- However, the terms of the series form a sequence that converges to 0.

Theorem

A series $\sum a_n$ with $a_n \geq 0$ converges if and only if its partial sums are bounded from above.

Let $\sum a_n$ be an infinite series with $a_n \geq 0$ for all n. Then each partial sum is greater than or equal to its predecessor because

$$s_n = s_{n-1} + a_n$$

and $a_n \geq 0$, so

$$s_1 \leq s_2 \leq s_3 \leq \cdots \leq s_n$$

Thus s_n is monotonic increasing and hence by Monotonic increasing theorem s_n converges if it is bounded above.

Harmonic series

What about the convergence of

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} + \cdots ?$$

$a_n = \frac{1}{n}$

$a_n \to 0$

$$\sum \frac{1}{n} = \overbrace{\left(1 + \frac{1}{2}\right)}^{> 1/2} + \overbrace{\left(\frac{1}{3} + \frac{1}{4}\right)}^{> 1/2} + \underbrace{\left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right)}_{} + \cdots$$

$> 1/2$

$$+ \left(\frac{1}{9} + \frac{1}{10} + \cdots + \frac{1}{16}\right) + \cdots$$

$||$

$$\sum \frac{1}{2^{n-1}+1} + \frac{1}{2^{n-1}+2} + \cdots + \frac{1}{2^n}$$

$$a_n = \frac{1}{2^{n-1}+1} + \cdots + \frac{1}{2^n}$$

$$> 1/2$$

$$S_1 = a_1 = 1 + \frac{1}{2} > \frac{1}{2}$$

$$S_2 = a_1 + a_2 > \frac{1}{2} + \frac{1}{2} = \frac{2}{2}$$

$$S_3 = a_1 + a_2 + a_3 > \frac{1}{2} + \frac{1}{2} + \frac{1}{2} = 3/2$$

$$\vdots$$

$$S_k > k/2$$

Clearly $\{S_k\}$ is not bounded above.

$\Rightarrow \sum \frac{1}{n}$ diverges

Harmonic series

What about the convergence of

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} + \cdots ?$$

- Although $a_n = \frac{1}{n}$ does go to 0, the series diverges because there is no upper bound for its partial sums. We can group the terms of the series in the following way:

$$1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) + \left(\frac{1}{9} + \cdots + \frac{1}{16}\right) + \cdots$$

- The sum of the first two terms is 1.5. The sum of the next two terms is $> \frac{1}{2}$.
- The sum of the next four terms is greater than $\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} > \frac{1}{2}$.

- The sum of the next eight terms is $> \frac{1}{2}$. The sum of the next 16 terms is greater than $1/2$ and so on.
- In general, the sum of 2^n terms ending with $\frac{1}{2^{n+1}}$ is $> \frac{1}{2}$.
- If $n = 2^k$, the partial sum $s_n > \frac{k}{2}$, so the sequence of partial sums is not bounded from above. Thus the harmonic series diverges.

Properties

Theorem

If $\sum a_n = A$ *and* $\sum b_n = B$ *are convergent series, then*

1. Sum Rule: $\sum(a_n + b_n) = \sum a_n + \sum b_n = A + B$

2. Difference Rule: $\sum(a_n - b_n) = \sum a_n - \sum b_n = A - B$

3. Constant Multiple Rule: $\sum ka_n = k\sum a_n = kA$. *(any number* k*).*

Corollary

- *Every nonzero constant multiple of a divergent series diverges.*
- *If* $\sum a_n$ *converges and* $\sum b_n$ *diverges, then* $\sum(a_n + b_n)$ *and* $\sum(a_n - b_n)$ *both diverge.*

$k \neq 0$ Σka_n diverges if Σa_n diverges

(i) Find $\sum_{n=1}^{\infty} \frac{3^{n-1} - 1}{6^{n-1}}$.

$$= \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^{n-1} - \sum_{n=1}^{\infty} \frac{1}{6^{n-1}}$$

$$= \frac{1}{1 - 1/2} - \frac{1}{1 - 1/6}$$

$$= 2 - \frac{6}{5}$$

$$= \frac{4}{5}$$

(i) Find $\sum_{n=1}^{\infty} \frac{3^{n-1}-1}{6^{n-1}}$.

$$\sum_{n=1}^{\infty} \frac{3^{n-1}-1}{6^{n-1}} = \sum_{n=1}^{\infty} \frac{1}{2^{n-1}} - \sum_{n=1}^{\infty} \frac{1}{6^{n-1}} = 2 - \frac{6}{5} = \frac{4}{5}$$

(ii) Find $\sum_{n=1}^{\infty} \frac{4}{2^n}$.

$$= 4 \sum_{n=1}^{\infty} \frac{1}{2^n}$$

$$= 4 \frac{(1/2)}{1 - 1/2} = 4$$

(i) Find $\sum_{n=1}^{\infty} \frac{3^{n-1}-1}{6^{n-1}}$.

$$\sum_{n=1}^{\infty} \frac{3^{n-1}-1}{6^{n-1}} = \sum_{n=1}^{\infty} \frac{1}{2^{n-1}} - \sum_{n=1}^{\infty} \frac{1}{6^{n-1}} = 2 - \frac{6}{5} = \frac{4}{5}$$

(ii) Find $\sum_{n=1}^{\infty} \frac{4}{2^n}$.

$$\sum_{n=1}^{\infty} \frac{4}{2^n} = 4 \sum_{n=1}^{\infty} \frac{1}{2^n} = 4.$$