MATHEMATICS-I

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Lecture 11

Infinite series

Theorem 0.1 (Limit Comparison Test).

Suppose that $a_n > 0$ and $b_n > 0$ for all $n \geq N$ for some $N \in \mathbb{N}$.

1. *If $\lim_{n\to\infty} \frac{a_n}{b_n} = c > 0$, then $\sum a_n$ and $\sum b_n$ both converge or diverge.*
2. *If $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$ and $\sum b_n$ converges then $\sum a_n$ converges.*
3. *If $\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$ and $\sum b_n$ diverges then $\sum a_n$ diverges.*

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Examples: Test the convergence of the following:
(a). $\sum_{n=1}^{\infty} \frac{100}{10n+1}$, (b). $\sum_{n=1}^{\infty} \frac{1}{2^n+10}$., (c). $\sum_{n=1}^{\infty} \frac{\ln(n)}{n}$.
(d). $\sum_{n=1}^{\infty} \frac{1}{n^2 \ln(n)}$.

Theorem 0.2 (The Ratio Test).

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Then;

1. *the series converges if $\rho < 1$,*
2. *the series diverges if $\rho > 1$*
3. *the test is inconclusive if $\rho = 1$.*

Test the convergence of the following:

(a). $\sum_{n=1}^{\infty} \frac{x^n}{n!}$, (b). $\sum_{n=1}^{\infty} \frac{(2n)!}{n!n!}$, (c). $\sum_{n=1}^{\infty} \frac{1}{n^2}$

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Discuss the convergence of the following:

(a). $\sum_{n=1}^{\infty} \frac{1-n}{3n-n^2}$, (b). $\sum_{n=1}^{\infty} \frac{3^n}{n^{10}}$, (c). $\sum_{n=1}^{\infty} \left(\frac{n}{n+2}\right)^{n^2}$.

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- **Examples:** $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n}{n},\ \sum_{n=1}^{\infty}(-4/3)^n,\ \sum_{n=1}^{\infty}(-1)^n\sqrt{n}.$

Theorem 0.4 (The alternating series test (Leibniz Test)).

The series:

$$\sum_{n=1}^{\infty}(-1)^{n+1}a_n = a_1 - a_2 + a_3 - a_4 + \cdots$$

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2. *The positive a_n's are (eventually) non-increasing:* $a_n \geq a_{n+1}$ *for all* $n \geq N$, *for some integer* N.
3. $a_n \to 0$ *as* $n \to \infty$.

Examples:

1. If $p > 0$, then the alternating p-series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^p} = 1 - \frac{1}{2^p} + \frac{1}{3^p} - \cdots$$

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2. What can you say about the converges of $\displaystyle\sum_{n=1}^{\infty}(-1)^n \left(\frac{2+n}{8n}\right)$?

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Theorem 0.5.

If the series $\sum a_n$ is absolutely convergent then it is convergent.

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Examples

Discuss whether the following series absolutely convergence or conditionally convergence.

(a). $\sum_{n=1}^{\infty}(-1)^n \frac{1-n}{3n-n^2}$, (b). $\sum_{n=1}^{\infty}(-1)^n \frac{ln(n)}{n}$

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5. **Series with some negative terms**: Does $\sum |a_n|$ converge? If yes, so does $\sum a_n$, since absolute convergence implies convergence.
6. **Alternating series**: $\pm \sum (-1)^n a_n$; $(a_n \geq 0)$ converges if the series satisfies the conditions of the Alternating Series Test.

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Thank you