MATH F111 - MATHEMATICS I
Tutorial Sheet 3

1. Investigate the convergence of the sequence (x_n) where

   (i) $x_n = \frac{1}{1+n^2} + \frac{2}{2+n^2} + \cdots + \frac{n}{n+n^2}$.

   (ii) $x_n = (a^n + b^n)^{1/n}$ where $0 < a < b$.

   (iii) $x_n = \left(\sqrt{2} - 2^{\frac{1}{3}}\right)\left(\sqrt{2} - 2^{\frac{1}{5}}\right)\cdots\left(\sqrt{2} - 2^{\frac{1}{2n+1}}\right)$.

   (iv) $x_n = n^\alpha - (n+1)^\alpha$ for some $\alpha \in (0,1)$.

   (v) $x_n = \frac{2^n}{n!}$.

   (vi) $x_n = \frac{1-2+3-4+\cdots+(-1)^{n-1}n}{n}$.

2. Let (x_n) be a sequence in $\mathbb{R}$. Prove or disprove the following statements:

   1. If $x_n \to 0$ and (y_n) is a bounded sequence, then $x_n y_n \to 0$.
   2. If $x_n \to \infty$ and (y_n) is a bounded sequence, then $x_n y_n \to \infty$.

3. Let (x_n) be a sequence in $\mathbb{R}$. Prove or disprove the following statements:

   1. If the sequence $\left(x_n + \frac{1}{n}x_n\right)$ converges, then (x_n) converges.
   2. If the sequence $\left(x_n^2 + \frac{1}{n}x_n\right)$ converges, then (x_n) converges.

4. Show that the sequence (x_n) is bounded and monotone, and find its limit where:

   1. $x_1 = 2$ and $x_{n+1} = 2 - \frac{1}{x_n}$ for $n \in \mathbb{N}$.
   2. $x_1 = \sqrt{2}$ and $x_{n+1} = \sqrt{2x_n}$ for $n \in \mathbb{N}$.
   3. $x_1 = 1$ and $x_{n+1} = \frac{4+3x_n}{3+2x_n}$ for $n \in \mathbb{N}$.

5. Let $0 < b_1 < a_1$ and define the sequences (a_n) and (b_n) by $a_{n+1} = \frac{a_n+b_n}{2}$ and $b_{n+1} = \sqrt{a_n b_n}$ for all $n \in \mathbb{N}$. Show that both (a_n) and (b_n) converge.

6. Let $a > 0$ and $x_1 > 0$. Define the sequence (x_n) by
$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{a}{x_n}\right)$$
for all $n \in \mathbb{N}$. Show that the sequence (x_n) converges to $\sqrt{a}$.

7. Let (x_n) be a sequence in $(0,1)$. Suppose $4x_n(1 - x_{n+1}) > 1$ for all $n \in \mathbb{N}$. Show that the sequence (x_n) is monotone and find its limit.

8. Show that the sequence (x_n) defined by
$$x_n = \left(1 + \frac{1}{n}\right)^n$$
is increasing and bounded above.