Sequence and Series

Gunja Sachdeva

August 20, 2024

Ex 1) $\left\{\frac{1}{n}\right\}$
$\downarrow$
0

2) $\left\{1-\frac{1}{n}\right\} \longrightarrow 1$

$X : \mathbb{N} \longrightarrow \mathbb{R}$

$n \longmapsto X(n) \equiv x_n$

$\{x_n\}$ or $\{a_n\}$

$\{a_1, a_2, a_3, - - \}$

$a : \mathbb{N} \longrightarrow \mathbb{R}$

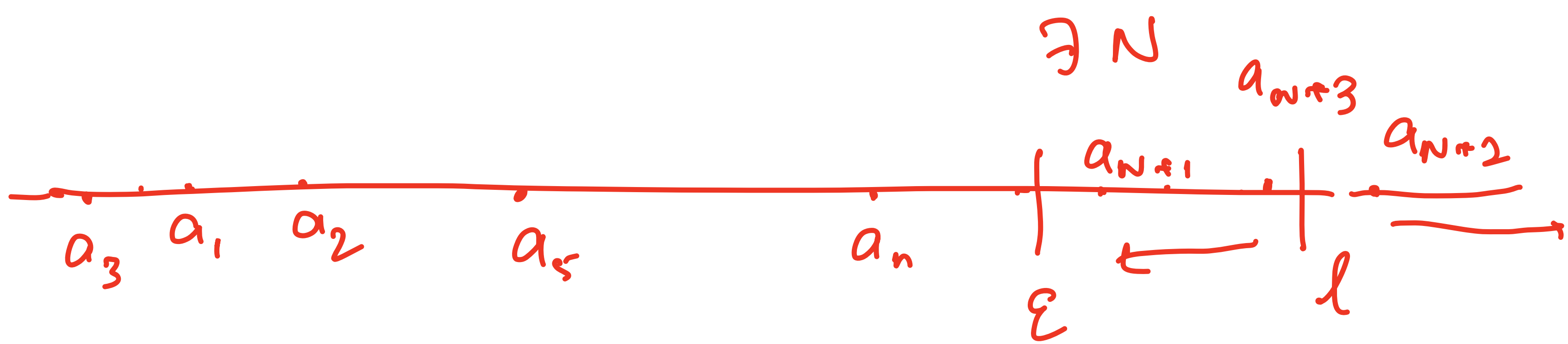
$n \longmapsto a(n) \equiv a_n$

Recall

A sequence (a_n) in $\mathbb{R}$ is said to converge to $\ell \in \mathbb{R}$, or ℓ is said to be a limit of (a_n), if for every $\epsilon > 0$, there exists an integer $N(\epsilon) \in \mathbb{N}$ such that

$$|a_n - \ell| < \epsilon \text{ for all } n \geq N(\epsilon).$$

ie, $a_n \in (\ell - \epsilon, \ell + \epsilon)$, $\forall n \geq \underline{N(\epsilon)}$.



Recall

A sequence (a_n) in $\mathbb{R}$ is said to converge to $\ell \in \mathbb{R}$, or ℓ is said to be a limit of (a_n), if for every $\epsilon > 0$, there exists an integer $N(\epsilon) \in \mathbb{N}$ such that

$$|a_n - \ell| < \epsilon \text{ for all } n \geq N(\epsilon).$$

ie, $a_n \in (\ell - \epsilon, \ell + \epsilon), \forall n \geq N(\epsilon)$.

Remarks.

- The notation is $\lim a_n = \ell$. or $a_n \to \ell$, as $n \to \infty$.
- If a sequence has a limit, we say that the sequence is **convergent**; if it has no limit, we say that the sequence is **divergent**.
- The convergence of a sequence is unaltered if a finite number of its terms are replaced by some other terms.

$\{1, 2, 3, \ldots\}$ $\{1, 1, 3, 1, 2, 3, \ldots\}$

Examples

(i) Let $a \in \mathbb{R}$ and $a_n := a$ for all $n \in \mathbb{N}$. Then $a_n \to a = l$.

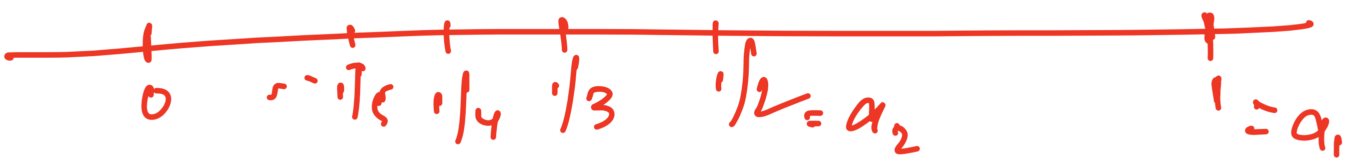
$\{a, a, a, \ldots\}$. $\varepsilon = 1/2$ Choose $N = [\frac{1}{2}] + 1 = 1$

s.t $|a_n - l| = |a - a| = 0 < \frac{1}{2} \quad \forall\, n \geq 1$

In general choose $N = [\varepsilon] + 1$

(ii) $a_n := 1/n$ for all $n \in \mathbb{N}$. Then $a_n \to 0$.

$\{\frac{1}{n}\} = \{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\}$ $\quad l = 0$



let $\varepsilon = 1/2$. Choose $N = 3$ s.t

$a_n \in (-\frac{1}{2}, \frac{1}{2}) \quad \forall\, n \geq 3$

let $\varepsilon = \frac{1}{5}$. choose $N = 6$

s.t

$a_n \in (-\frac{1}{5}, \frac{1}{5}) \quad \forall\, n \geq 6$

Examples

(i) Let $a \in \mathbb{R}$ and $a_n := a$ for all $n \in \mathbb{N}$. Then $a_n \to a$.

(ii) $a_n := 1/n$ for all $n \in \mathbb{N}$. Then $a_n \to 0$.

Let $\epsilon > 0$ be given. We want to find $N \in \mathbb{N}$ such that $|(1/n) - 0| < \epsilon$ for all $n \geq N$.

Choose any $N \in \mathbb{N}$ which is greater than $1/\epsilon$. (This is possible because of the Archimedean property of $\mathbb{R}$.)
For example, we can let $N := [1/\epsilon] + 1$.

$// \quad N = \left[\frac{1}{\varepsilon}\right] + 2$

$N > \frac{1}{\varepsilon} \Rightarrow \varepsilon > \frac{1}{N}$

$|a_n - \ell|$
$= |\frac{1}{n} - 0|$
$= \frac{1}{n} < \varepsilon \quad \forall n \geq N$

(iii) $a_n := 2/(n^2+1)$ for $n \in \mathbb{N}$. Then $a_n \to 0$.

$\left\{1, \frac{2}{5}, \frac{2}{10}, \frac{2}{17}, \ldots\right\}$ $\qquad l = 0$

$\varepsilon = 1/2$, choose $N = 2$

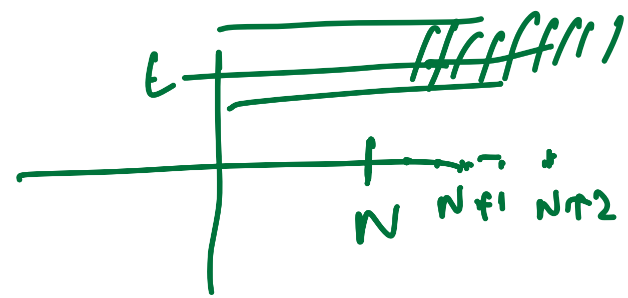
$\left(-\frac{1}{2}, \frac{1}{2}\right) = (-0.5, 0.5)$

s.t. $a_n \in (-0.5, 0.5)$ $\forall\ n \geq 2$

$\varepsilon = \frac{1}{5}$, choose $N = 4$

$a_n \in (-0.2, 0.2)$ $\forall\ n \geq 4$

$N = \left[\sqrt{\frac{2}{\varepsilon}}\right] + 1$



(iii) $a_n := 2/(n^2+1)$ for $n \in \mathbb{N}$. Then $a_n \to 0$. Now

$$\left|\frac{2}{n^2+1} - 0\right| = \frac{2}{n^2+1} < \frac{2}{n^2} \quad \text{for all } n \in \mathbb{N}.$$

$\frac{2}{n^2} = \varepsilon$

$n^2 = \frac{2}{\varepsilon}$

$n = \sqrt{\frac{2}{\varepsilon}}$

Choose $N \in \mathbb{N}$ such that $N > \sqrt{2}/\sqrt{\epsilon}$. For example, let $N := [\sqrt{2}/\sqrt{\epsilon}] + 1$. Then $|a_n - 0| < \frac{2}{n^2} < \epsilon$ for all $n \geq N$.

(iv) $a_n := 5/(3n+1)$ for $n \in \mathbb{N}$. Then $a_n \to \underline{0}$.

$\ell = 0$

$\left|\frac{5}{3n+1} - 0\right| = \frac{5}{3n+1} < \frac{5}{3n}$

$\frac{5}{3n} = \varepsilon$

$n = \frac{5}{3\varepsilon}$

$N = \left[\frac{5}{3\varepsilon}\right] + 1$

Given $\varepsilon > 0$, choose $N = \left[\frac{5}{3\varepsilon}\right] + 1$ s.t

$|a_n - \ell| = \left|\frac{5}{3n+1}\right| < \frac{5}{3n} < \varepsilon \quad \forall\, n \geq N$

(iii) $a_n := 2/(n^2+1)$ for $n \in \mathbb{N}$. Then $a_n \to 0$. Now

$$\left|\frac{2}{n^2+1} - 0\right| = \frac{2}{n^2+1} < \frac{2}{n^2} \quad \text{for all } n \in \mathbb{N}.$$

Choose $N \in \mathbb{N}$ such that $N > \sqrt{2}/\sqrt{\epsilon}$. For example, let $N := [\sqrt{2}/\sqrt{\epsilon}] + 1$. Then $|a_n - 0| < \frac{2}{n^2} < \epsilon$ for all $n \geq N$.

(iv) $a_n := 5/(3n+1)$ for $n \in \mathbb{N}$. Then $a_n \to 0$. Now

$$\frac{5}{3n+1} < \frac{5}{3n} \quad \text{for all } n \in \mathbb{N}.$$

Choose $N \in \mathbb{N}$ such that $N > 5/3\epsilon$. For example, let $N := [5/3\epsilon] + 1$. Then $|a_n - 0| < \frac{5}{3n} < \epsilon$ for all $n \geq N$.

Example

$(1+a)^n = 1 + na + \cdots$

$\frac{1}{(1+a)^n} = \frac{1}{1+na+\cdots}$

Prove that $\lim_{n\to\infty} r^n = 0$ for $|r| < 1$. i.e. $-1 < r < 1$

Case 1 $\quad r = 0$

$(0, 0, \ldots) \to 0$

Case 2 $\quad r \neq 0$ and $-1 < r < 1$

$\frac{1}{na} = \varepsilon$

$n = \frac{1}{\varepsilon a}$

Given $\varepsilon > 0$, choose $\boxed{N = \left[\frac{1}{\varepsilon a}\right] + 1}$

s.t. $|a_n - 0| = |r^n - 0|$

$= |r^n| < \varepsilon \Rightarrow |r|^n < \varepsilon$

$|r| < 1$

$\Rightarrow \frac{1}{|r|} > 1 \Rightarrow \frac{1}{|r|} = 1 + a$ for $a \in \mathbb{R}^+$

$\Rightarrow |r| = \frac{1}{1+a} \Rightarrow |r|^n = \frac{1}{(1+a)^n} < \frac{1}{na}$

$< \varepsilon$

Example

$\forall n \geq N$

Prove that $\lim_{n\to\infty} r^n = 0$ for $|r| < 1$.

Case 1. $r = 0$
In this case the sequence is $\{0, 0, 0, \ldots\}$ which converges to 0.

Case 2. $r \neq 0$ and $|r| < 1$.
Since $|r| < 1$, $\frac{1}{|r|} > 1$. Let $\frac{1}{|r|} = 1 + a$ where $a > 0$. Then

$$|r^n - 0| = |r|^n = \frac{1}{(1+a)^n}.$$

We have $(1+a)^n > na$ for all $n \in \mathbb{N}$ and hence,

$$|r^n - 0| < \frac{1}{na} \text{ for all } n \in \mathbb{N}.$$

Let $\epsilon > 0$ be given. Then

$$|r^n - 0| < \epsilon \text{ holds if } n > \frac{1}{a\epsilon}.$$

Choose any $N \in \mathbb{N}$ such that $N > \frac{1}{a\epsilon}$. Then

$$\forall n \geq N, \ |r^n - 0| < \epsilon.$$

Since $\epsilon > 0$ is arbitrary,

$$\lim_{n \to \infty} r^n = 0.$$

Example

Prove that the sequence $((-1)^n : n \in \mathbb{N}) = (-1, 1, -1, 1, \ldots)$ is not convergent.

Prove by contradiction.

Assume limit of the seq $\{(-1)^n\}$ exist

say $\{(-1)^n\} \longrightarrow x \in \mathbb{R}$

$\Rightarrow$ For $\varepsilon = \frac{1}{2}$, $\exists$ $k \in \mathbb{N}$ s.t

$$|(-1)^n - x| < \frac{1}{2} \quad \forall \; n \geq k$$

i.e For $n = 2k$ $\Rightarrow$ $|(-1)^{2k} - x| < \frac{1}{2}$

$n = 2k+1$ $\quad |(-1)^{2k+1} - x| < \frac{1}{2}$ — (*)

clearly, $(-1)^{2k} = 1$, $(-1)^{2k+1} = -1$

$$2 = |1 + 1| = |(-1)^{2k} - (-1)^{2k+1}|$$

Example

$= |((-1)^{2k} - x) - ((-1)^{2k+1} - x)|$
$< \frac{1}{2} + \frac{1}{2} < 1$

$\to \epsilon$

Prove that the sequence $((-1)^n : n \in \mathbb{N}) = (-1, 1, -1, 1, \ldots)$ is not convergent.

Let $x_n := (-1)^n$ be convergent and converges to the real number x. Then definition of convergence must hold for every ϵ.

- In particular choose $\epsilon = \frac{1}{2}$. Then there exists a natural number k such that

$$|x_n - x| = |(-1)^n - x| < \frac{1}{2} \text{ for all } n \geq k.$$

Since $(-1)^{2k} = 1$ and $(-1)^{2k+1} = -1$,

$$\begin{aligned} 2 &= |(-1)^{2k} - (-1)^{2k+1}| \\ &= |((-1)^{2k} - x) - ((-1)^{2k+1} - x)| \\ &\leq |((-1)^{2k} - x)| + |((-1)^{2k+1} - x)| < \frac{1}{2} + \frac{1}{2} = 1, \end{aligned}$$

which is a contradiction. Therefore, the sequence $(-1)^n$ is not convergent.

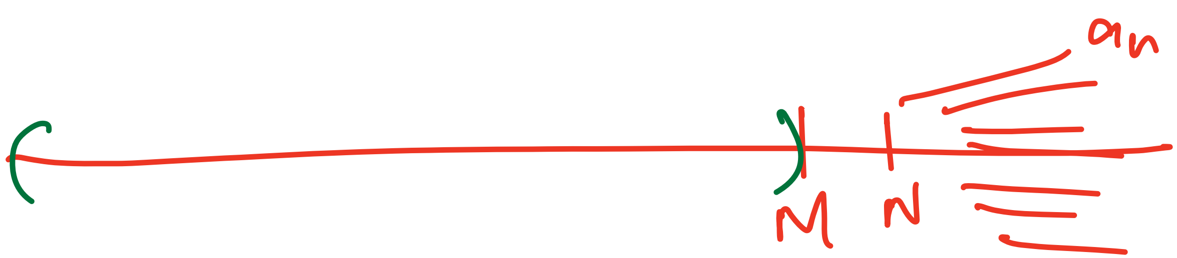
Definition

The sequence (a_n) diverges to infinity if for every number $M \in \mathbb{R}$ there is an integer N such that $\forall n > N$, we have $a_n > M$. If this holds, we write

$$\lim_{n\to\infty} a_n = \infty \quad \text{or } a_n \to \infty.$$

Similarly if for every number $m \in \mathbb{R}$, there is an integer N such that $\forall n > N$, we have $a_n < m$, then we say (a_n) diverges to negative infinity. We write

$$\lim_{n\to\infty} a_n = -\infty \text{ or } a_n \to -\infty.$$



$\{\sqrt{n}\}$

Properties of limits

Uniqueness of Limits. A sequence in $\mathbb{R}$ can have at most one limit.

Let $\{a_n\}$ be a sequence which converges to l_1 and l_2

s.t $l_1 \neq l_2$

Given $\varepsilon = \frac{|l_1 - l_2|}{2}$, $\exists$ $N_1(\varepsilon)$ and $N_2(\varepsilon)$ s.t

$|a_n - l_1| < \varepsilon \quad \forall \quad n \geq N_1$

and $|a_n - l_2| < \varepsilon \quad \forall \quad n \geq N_2$

$\Rightarrow \quad |a_n - l_1| < \varepsilon$ and $|a_n - l_2| < \varepsilon \quad \forall \quad n \geq \underbrace{\max\{N_1, N_2\}}_{= N}$

Properties of limits

Uniqueness of Limits. A sequence in $\mathbb{R}$ can have at most one limit.

Proof. Let (a_n) be a real sequence and suppose that ℓ_1 and ℓ_2 are both limits for (a_n) and let $\ell_1 \neq \ell_2$.

- Let $\epsilon := |\ell_1 - \ell_2|/2$. Since $\ell_1 \neq \ell_2, \epsilon > 0$.
- Since ℓ_1 is a limit of the sequence, for the chosen $\epsilon, \exists\ N_1 \in \mathbb{N}$ such that
$$|a_n - \ell_1| < \epsilon, \quad \text{for all} \quad n \geq N_1.$$
- Since ℓ_2 is a limit of the sequence, for the chosen $\epsilon, \exists\ N_2 \in \mathbb{N}$ such that
$$|a_n - \ell_2| < \epsilon, \quad \text{for all} \quad n \geq N_2.$$

Let $N = \max\{N_1, N_2\}$. Then:

$$|a_n - \ell_1| < \epsilon \text{ and } |a_n - \ell_2| < \epsilon \text{ for all } n \geq N,$$

and hence,

$$|\ell_1 - \ell_2| = |(a_N - \ell_2) - (a_N - \ell_1)| \leq |a_N - \ell_1| + |a_N - \ell_2| < \epsilon + \epsilon = |\ell_1 - \ell_2|$$

which is a contradiction. Hence, $\ell_1 = \ell_2$. ■

Theorem

A convergent sequence is bounded.

Suppose $a_n \to \ell$. Let $\epsilon := 1$. There is $N \in \mathbb{N}$ such that

$$|a_n - \ell| < 1 \text{ for all } n \geq N.$$

Hence

$$|a_n| \leq |a_n - \ell| + |\ell| < 1 + |\ell| \text{ for all } n \geq N.$$

- Thus it remains to find a bound for $a_1, a_2 \cdots, a_{N-1}$. Choose $\beta = \max\{|a_1|, |a_2|, \cdots, |a_{N-1}|\}$. Then $|a_n| \leq \beta$, for all $1 \leq n \leq N-1$.
- Define $\alpha := \max\{|a_1|, \dots, |a_{N-1}|, |\ell| + 1\}$. Then $|a_n| \leq \alpha$ for all $n \in \mathbb{N}$. Hence (a_n) is bounded.

$\Rightarrow -\alpha \leq a_n \leq \alpha$