Lecture 14

Moment of Inertia Tensor II

MOMENT OF INERTIA TENSOR II

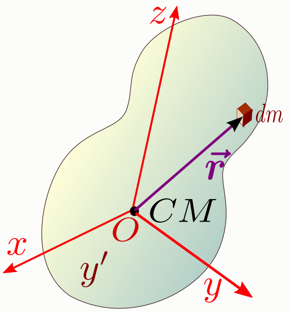
Moment of Inertia Tensor II

1. Parallel Axis Theorem for MI Tensor
2. Rotational Kinetic Energy
3. Principal Axes

Parallel Axis Theorem for MI Tensor

Parallel Axis Theorem for MI Tensor

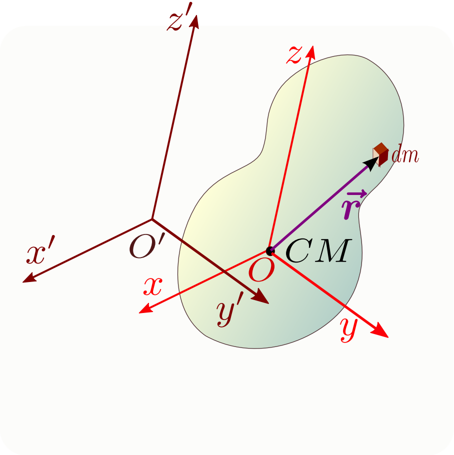
Know $\bar{I}$ about CM ((x, y, z) axes



Parallel Axis Theorem for MI Tensor

Know $\bar{I}$ about CM ((x, y, z) axes

What is $\bar{I}'$ about origin O' and parallely shifted axes?

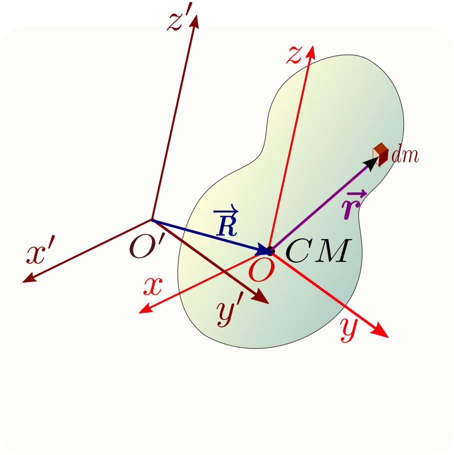


Parallel Axis Theorem for MI Tensor

Know $\bar{I}$ about CM ((x, y, z) axes

What is $\bar{I}'$ about origin O' and parallely shifted axes?

$$\overrightarrow{\boldsymbol{O'O}} = \overrightarrow{\boldsymbol{R}} = (X, Y, Z)$$

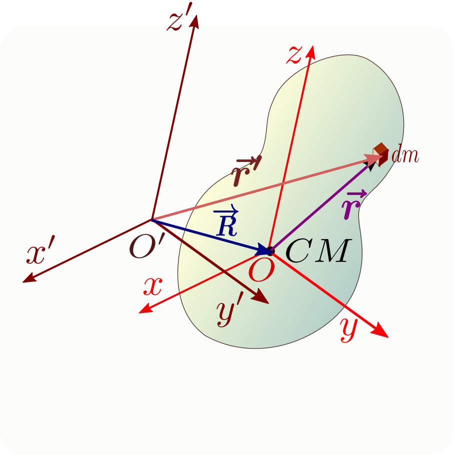


Parallel Axis Theorem for MI Tensor

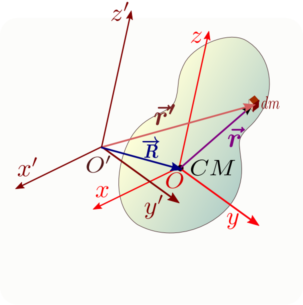
Know $\bar{I}$ about CM ((x, y, z) axes
What is $\bar{I}'$ about origin O' and parallely shifted axes?

$$\overrightarrow{\boldsymbol{O'O}} = \overrightarrow{\boldsymbol{R}} = (X, Y, Z)$$

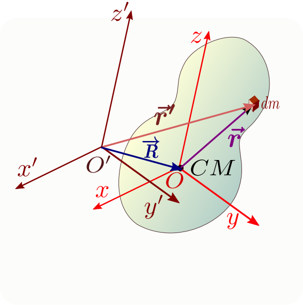
$$\vec{\boldsymbol{r}}' = \vec{\boldsymbol{r}} + \overrightarrow{\boldsymbol{R}}$$



"Parallel Axis Theorem" for MI Tensor

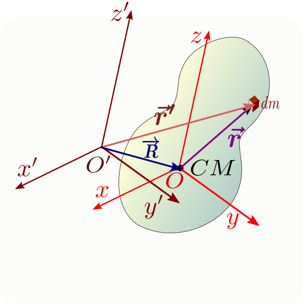


”Parallel Axis Theorem” for MI Tensor



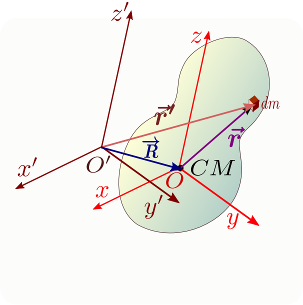
$$I'_{xx} = \iiint dx'dy'dz'\rho(y'^2 + z'^2)$$

"Parallel Axis Theorem" for MI Tensor



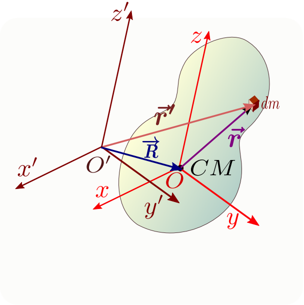
$$
\begin{aligned}
I'_{xx} &= \iiint dx'dy'dz'\rho(y'^2 + z'^2) \\
&= \iiint dxdydz\rho\left((y+Y)^2 + (z+Z)\right.
\end{aligned}
$$

"Parallel Axis Theorem" for MI Tensor



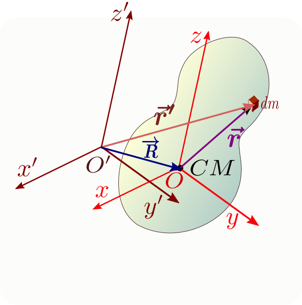
$$
\begin{aligned}
I'_{xx} &= \iiint dx'dy'dz'\rho(y'^2 + z'^2) \\
&= \iiint dxdydz\rho\left((y+Y)^2 + (z+Z)\right. \\
&= I_{xx} + M(Y^2 + Z^2)
\end{aligned}
$$

”Parallel Axis Theorem” for MI Tensor



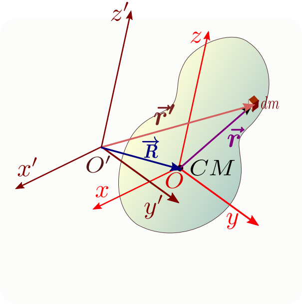
$$
\begin{aligned}
I'_{xx} &= \iiint dx'dy'dz'\rho(y'^2 + z'^2) \\
&= \iiint dxdydz\rho\left((y+Y)^2 + (z+Z)\right. \\
&= I_{xx} + M(Y^2 + Z^2) \\
I'_{xy} &= -\iiint dx'dy'dz'\rho(x'y')
\end{aligned}
$$

"Parallel Axis Theorem" for MI Tensor



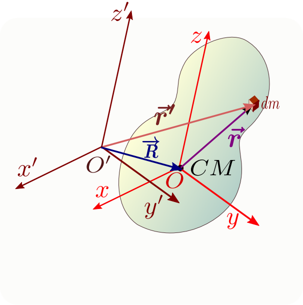
$$
\begin{aligned}
I'_{xx} &= \iiint dx'dy'dz'\rho(y'^2 + z'^2) \\
&= \iiint dxdydz\rho\left((y+Y)^2 + (z+Z)\right. \\
&= I_{xx} + M(Y^2 + Z^2) \\
I'_{xy} &= -\iiint dx'dy'dz'\rho(x'y') \\
&= -\iiint dxdydz\rho(x+X)(y+Y)
\end{aligned}
$$

"Parallel Axis Theorem" for MI Tensor



$$
\begin{aligned}
I'_{xx} &= \iiint dx'dy'dz'\rho(y'^2 + z'^2) \\
&= \iiint dxdydz\rho\left((y+Y)^2 + (z+Z)\right. \\
&= I_{xx} + M(Y^2 + Z^2) \\
I'_{xy} &= -\iiint dx'dy'dz'\rho(x'y') \\
&= -\iiint dxdydz\rho(x+X)(y+Y) \\
&= I_{xy} - MXY
\end{aligned}
$$

"Parallel Axis Theorem" for MI Tensor

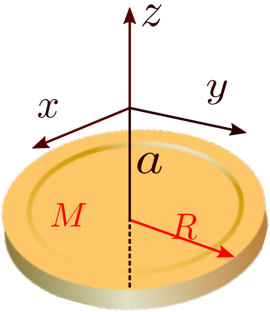


$$
\begin{aligned}
I'_{xx} &= \iiint dx'dy'dz'\rho(y'^2 + z'^2) \\
&= \iiint dxdydz\rho\left((y+Y)^2 + (z+Z)\right. \\
&= I_{xx} + M(Y^2 + Z^2) \\
I'_{xy} &= -\iiint dx'dy'dz'\rho(x'y') \\
&= -\iiint dxdydz\rho(x+X)(y+Y) \\
&= I_{xy} - MXY \\
& etc
\end{aligned}
$$

Parallel axis theorem for MI Tensor

$$\begin{bmatrix} I'_{xx} & I'_{xy} & I'_{xz} \\ I'_{yx} & I'_{yy} & I'_{yz} \\ I'_{zx} & I'_{zy} & I'_{zz} \end{bmatrix} = \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{bmatrix} + \begin{bmatrix} M(Y^2+Z^2) & -MXY & -MXZ \\ -MXY & M(X^2+Z^2) & -MYZ \\ -MXZ & -MYZ & M(X^2+Y^2) \end{bmatrix}$$

Example

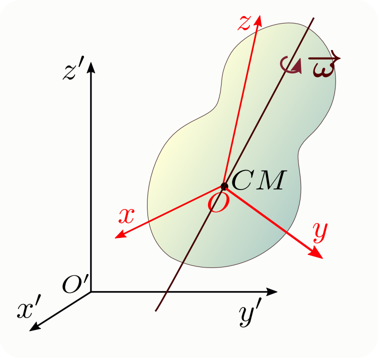


$$\bar{I}' = \begin{bmatrix} MR^2/4 & 0 & 0 \\ 0 & MR^2/4 & 0 \\ 0 & 0 & MR^2/2 \end{bmatrix}$$

$$+ \begin{bmatrix} Ma^2 & 0 & 0 \\ 0 & Ma^2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

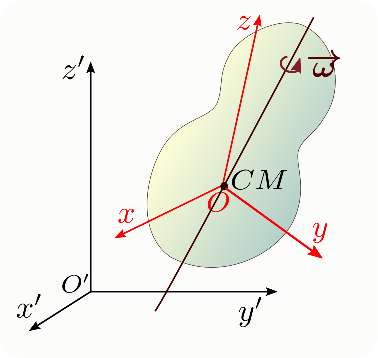
Eq of Motion for Rigid Body

Eq of Motion for Rigid Body



Eq of Motion for Rigid Body

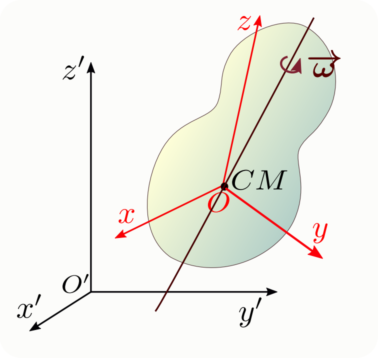
- Space fixed Axes $((x', y', z')$ (Inertial Frame):



Eq of Motion for Rigid Body

- Space fixed Axes $((x', y', z')$ (Inertial Frame):

$$\left.\frac{d\vec{\boldsymbol{L}}}{dt}\right|_{Space-fixed} = \vec{\boldsymbol{\tau}}$$

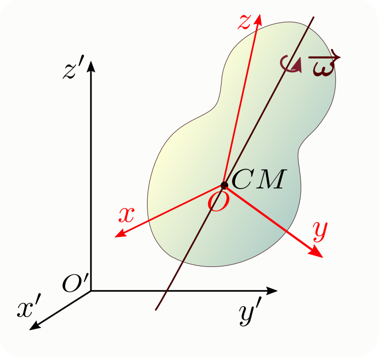


Eq of Motion for Rigid Body

- Space fixed Axes $((x', y', z')$ (Inertial Frame):

$$\left.\frac{d\overrightarrow{\boldsymbol{L}}}{dt}\right|_{Space-fixed} = \vec{\boldsymbol{\tau}}$$

$\overrightarrow{\boldsymbol{L}}$, $\vec{\boldsymbol{\tau}}$ assumed about CM



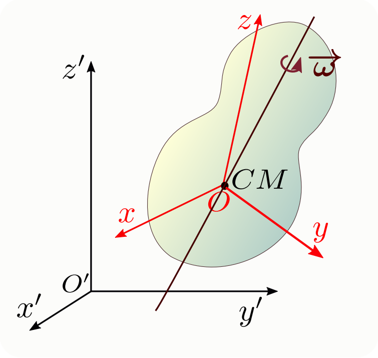
Eq of Motion for Rigid Body

- Space fixed Axes $((x', y', z')$ (Inertial Frame):

$$\left.\frac{d\vec{L}}{dt}\right|_{Space-fixed} = \vec{\tau}$$

$\vec{L}$, $\vec{\tau}$ assumed about CM

- Body Fixed Axes (x, y, z) (Non-Inertial Frame)



Eq of Motion for Rigid Body

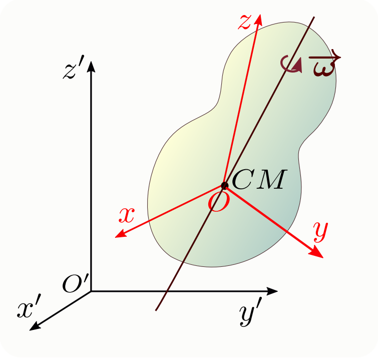
- Space fixed Axes ((x', y', z') (Inertial Frame):

$$\left.\frac{d\vec{\boldsymbol{L}}}{dt}\right|_{Space-fixed} = \vec{\boldsymbol{\tau}}$$

$\vec{\boldsymbol{L}}$, $\vec{\boldsymbol{\tau}}$ assumed about CM

- Body Fixed Axes (x, y, z) (Non-Inertial Frame)

$$\left.\frac{d\vec{\boldsymbol{L}}}{dt}\right|_{I} = \left.\frac{d\vec{\boldsymbol{L}}}{dt}\right|_{NI} + \vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{L}}$$



Eq of Motion for Rigid Body

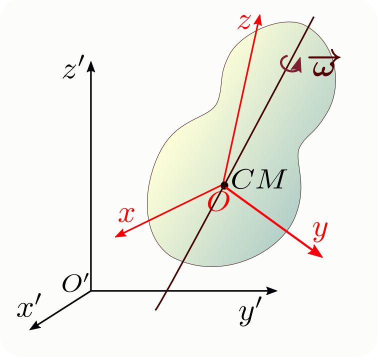
- Space fixed Axes $((x', y', z')$ (Inertial Frame):

$$\left.\frac{d\vec{\boldsymbol{L}}}{dt}\right|_{Space-fixed} = \vec{\boldsymbol{\tau}}$$

$\vec{\boldsymbol{L}}$, $\vec{\boldsymbol{\tau}}$ assumed about CM

- Body Fixed Axes (x, y, z) (Non-Inertial Frame)

$$\left.\frac{d\vec{\boldsymbol{L}}}{dt}\right|_{I} = \left.\frac{d\vec{\boldsymbol{L}}}{dt}\right|_{NI} + \vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{L}}$$



Rigid body Eq. in Body Axes

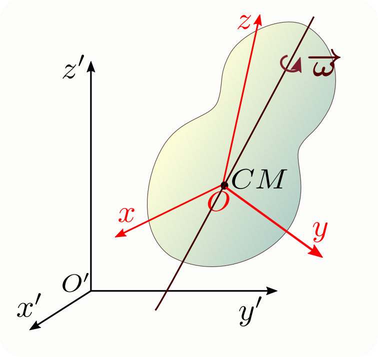
- Space fixed Axes $((x', y', z')$ (Inertial Frame):

$$\left.\frac{d\vec{\boldsymbol{L}}}{dt}\right|_{Space-fixed} = \vec{\boldsymbol{\tau}}$$

$\vec{\boldsymbol{L}}$, $\vec{\boldsymbol{\tau}}$ assumed about CM

- Body Fixed Axes (x, y, z) (Non-Inertial Frame)

$$\left.\frac{d\vec{\boldsymbol{L}}}{dt}\right|_{I} = \left.\frac{d\vec{\boldsymbol{L}}}{dt}\right|_{NI} + \vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{L}}$$

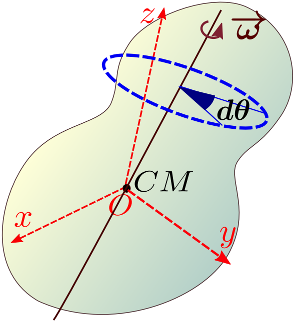


Rigid body Eq. in Body Axes

$$\frac{d\vec{\boldsymbol{L}}}{dt} + \vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{L}} = \vec{\boldsymbol{\tau}}$$

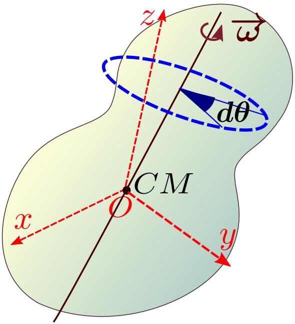
Rotational KE

Work-Energy Theorem



Rotational KE

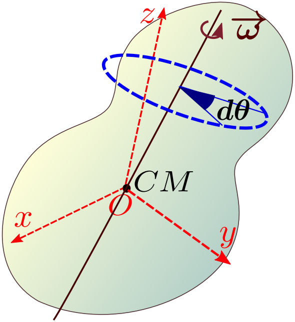
Work-Energy Theorem



$$\frac{d\vec{L}}{dt} + \vec{\omega} \times \vec{L} = \vec{\tau}$$

Rotational KE

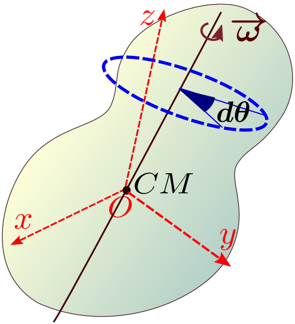
Work-Energy Theorem



$$\frac{d\vec{L}}{dt} + \vec{\omega} \times \vec{L} = \vec{\tau} \implies$$

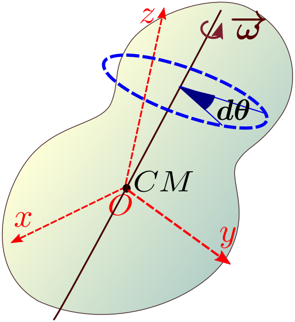
Rotational KE

Work-Energy Theorem



$$\frac{d\vec{\boldsymbol{L}}}{dt} + \vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{L}} = \vec{\boldsymbol{\tau}} \implies$$

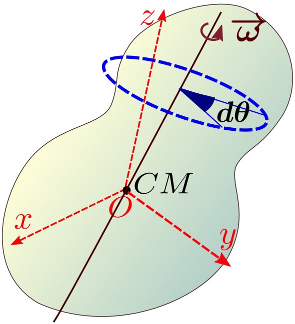
$$\overrightarrow{\boldsymbol{d\theta}} \cdot \frac{d\vec{\boldsymbol{L}}}{dt} + \overrightarrow{\boldsymbol{d\theta}} \cdot \vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{L}} = \overrightarrow{\boldsymbol{d\theta}} \cdot \vec{\boldsymbol{\tau}}$$



$$\frac{d\vec{\boldsymbol{L}}}{dt} + \vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{L}} = \vec{\boldsymbol{\tau}} \implies$$

$$\vec{\boldsymbol{d\theta}} \cdot \frac{d\vec{\boldsymbol{L}}}{dt} + \vec{\boldsymbol{d\theta}} \cdot \vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{L}} = \vec{\boldsymbol{d\theta}} \cdot \vec{\boldsymbol{\tau}}$$

$$\vec{\boldsymbol{\omega}} \cdot d\vec{\boldsymbol{L}}$$

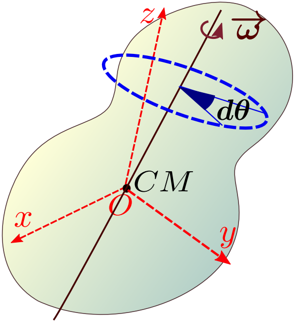


$$\frac{d\vec{\boldsymbol{L}}}{dt} + \vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{L}} = \vec{\boldsymbol{\tau}} \implies$$

$$\overrightarrow{d\boldsymbol{\theta}} \cdot \frac{d\vec{\boldsymbol{L}}}{dt} + \overrightarrow{d\boldsymbol{\theta}} \cdot \vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{L}} = \overrightarrow{d\boldsymbol{\theta}} \cdot \vec{\boldsymbol{\tau}}$$

$$\vec{\boldsymbol{\omega}} \cdot d\vec{\boldsymbol{L}} = d(\frac{1}{2}\vec{\boldsymbol{\omega}} \cdot \vec{\boldsymbol{L}})$$

Rotational KE
Work-Energy Theorem

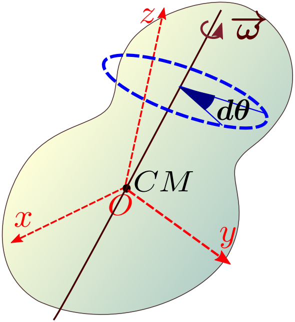


$$\frac{d\vec{\boldsymbol{L}}}{dt} + \vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{L}} = \vec{\boldsymbol{\tau}} \implies$$

$$\overrightarrow{\boldsymbol{d\theta}} \cdot \frac{d\vec{\boldsymbol{L}}}{dt} + \overrightarrow{\boldsymbol{d\theta}} \cdot \vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{L}} = \overrightarrow{\boldsymbol{d\theta}} \cdot \vec{\boldsymbol{\tau}}$$

$$\vec{\boldsymbol{\omega}} \cdot d\vec{\boldsymbol{L}} = d\left(\frac{1}{2}\vec{\boldsymbol{\omega}} \cdot \vec{\boldsymbol{L}}\right) = \vec{\boldsymbol{\tau}} \cdot \overrightarrow{\boldsymbol{d\theta}}$$

Rotational KE
Work-Energy Theorem



$$\frac{d\vec{L}}{dt} + \vec{\omega} \times \vec{L} = \vec{\tau} \implies$$

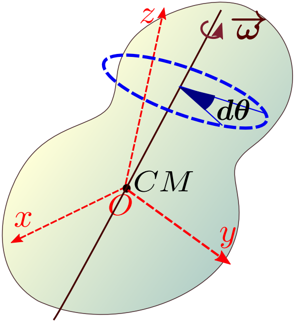
$$\vec{d\theta} \cdot \frac{d\vec{L}}{dt} + \vec{d\theta} \cdot \vec{\omega} \times \vec{L} = \vec{d\theta} \cdot \vec{\tau}$$

$$\vec{\omega} \cdot d\vec{L} = d\left(\frac{1}{2}\vec{\omega} \cdot \vec{L}\right) = \vec{\tau} \cdot \vec{d\theta}$$

$$W_\tau =$$

Rotational KE

Work-Energy Theorem

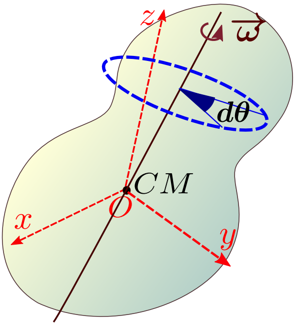


$$\frac{d\vec{\boldsymbol{L}}}{dt} + \vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{L}} = \vec{\boldsymbol{\tau}} \implies$$

$$\vec{\boldsymbol{d\theta}} \cdot \frac{d\vec{\boldsymbol{L}}}{dt} + \vec{\boldsymbol{d\theta}} \cdot \vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{L}} = \vec{\boldsymbol{d\theta}} \cdot \vec{\boldsymbol{\tau}}$$

$$\vec{\boldsymbol{\omega}} \cdot d\vec{\boldsymbol{L}} = d(\frac{1}{2}\vec{\boldsymbol{\omega}} \cdot \vec{\boldsymbol{L}}) = \vec{\boldsymbol{\tau}} \cdot \vec{\boldsymbol{d\theta}}$$

$$W_\tau = \int \vec{\boldsymbol{\tau}} \cdot \vec{\boldsymbol{d\theta}} =$$



$$\frac{d\vec{L}}{dt} + \vec{\omega} \times \vec{L} = \vec{\tau} \implies$$

$$\vec{d\theta} \cdot \frac{d\vec{L}}{dt} + \vec{d\theta} \cdot \vec{\omega} \times \vec{L} = \vec{d\theta} \cdot \vec{\tau}$$

$$\vec{\omega} \cdot d\vec{L} = d(\frac{1}{2}\vec{\omega} \cdot \vec{L}) = \vec{\tau} \cdot \vec{d\theta}$$

$$W_\tau = \int \vec{\tau} \cdot \vec{d\theta} = \Delta(\frac{1}{2}\vec{\omega} \cdot \vec{L})$$

Rotational KE

Rotational KE = $\frac{1}{2}\vec{\omega} \cdot \vec{L}$

Rotational KE

Rotational KE

$$\overrightarrow{\boldsymbol{L}} = \bar{I}\vec{\boldsymbol{\omega}}$$

Rotational KE

$$\overrightarrow{\boldsymbol{L}} = \bar{I}\vec{\boldsymbol{\omega}} \implies$$

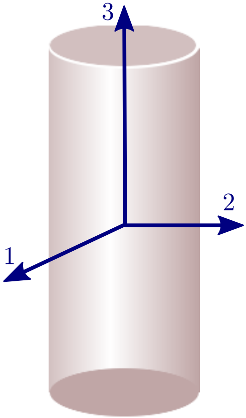
$$\begin{aligned} KE_{Rot} &= \frac{1}{2}\vec{\boldsymbol{\omega}} \cdot \overrightarrow{\boldsymbol{L}} \\ &= \frac{1}{2}\left[\omega\right]_{1\times 3}\left[I\right]_{3\times 3}\left[\omega\right]_{3\times 1} \\ &= \frac{1}{2}\left[\begin{array}{ccc} \omega_x & \omega_y & \omega_z \end{array}\right]\left[\begin{array}{ccc} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{array}\right]\left[\begin{array}{c} \omega_x \\ \omega_y \\ \omega_z \end{array}\right] \end{aligned}$$

Principal Axes

Principal Axes $(1, 2, 3)$: Symmetry axes of rigid body

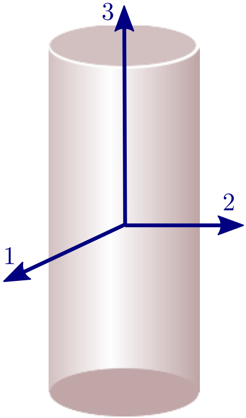
Principal Axes

Principal Axes $(1, 2, 3)$: Symmetry axes of rigid body



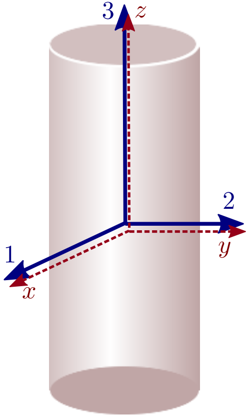
Principal Axes

Principal Axes $(1, 2, 3)$: Symmetry axes of rigid body
When Body Axes chosen to coincide with the Principal axes:



Principal Axes

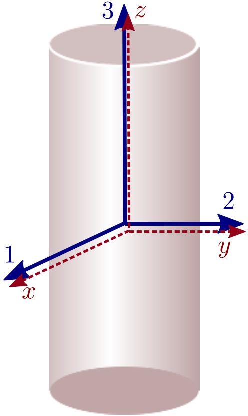
Principal Axes $(1,2,3)$: Symmetry axes of rigid body
When Body Axes chosen to coincide with the Principal axes:



Principal Axes

Principal Axes $(1, 2, 3)$: Symmetry axes of rigid body
When Body Axes chosen to coincide with the Principal axes:

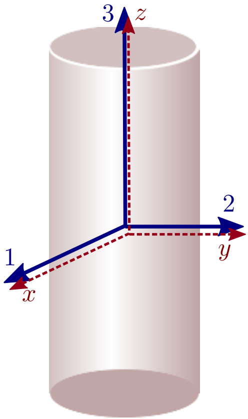
$$\mathcal{I} = \begin{bmatrix} I_{xx} & 0 & 0 \\ 0 & I_{yy} & 0 \\ 0 & 0 & I_{zz} \end{bmatrix}$$



Principal Axes

Principal Axes $(1,2,3)$: Symmetry axes of rigid body
When Body Axes chosen to coincide with the Principal axes:

$$\mathcal{I} = \begin{bmatrix} I_{xx} & 0 & 0 \\ 0 & I_{yy} & 0 \\ 0 & 0 & I_{zz} \end{bmatrix} = \begin{bmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{bmatrix}$$

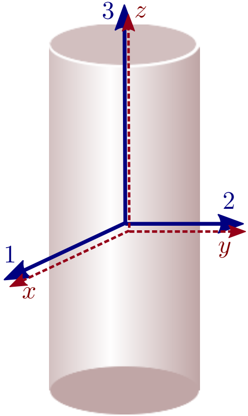


Principal Axes

Principal Axes $(1, 2, 3)$: Symmetry axes of rigid body
When Body Axes chosen to coincide with the Principal axes:

$$\mathcal{I} = \begin{bmatrix} I_{xx} & 0 & 0 \\ 0 & I_{yy} & 0 \\ 0 & 0 & I_{zz} \end{bmatrix} = \begin{bmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{bmatrix}$$

In Principal Axes:



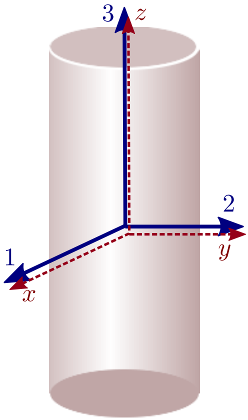
Principal Axes

Principal Axes $(1, 2, 3)$: Symmetry axes of rigid body
When Body Axes chosen to coincide with the Principal axes:

$$\mathcal{I} = \begin{bmatrix} I_{xx} & 0 & 0 \\ 0 & I_{yy} & 0 \\ 0 & 0 & I_{zz} \end{bmatrix} = \begin{bmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{bmatrix}$$

In Principal Axes:

$$\begin{aligned} L_1 &= I_1\omega_1 \\ L_2 &= I_2\omega_2 \\ L_3 &= I_3\omega_3 \end{aligned}$$



Principal Axes

Theorem

Principal Axes

Theorem

Principal Axes can be calculated for any arbitrary unsymmetric body

Principal Axes

Theorem

Principal Axes can be calculated for any arbitrary unsymmetric body
It is possible to find coordinate axes in which MI tensor can be diagonalized for any arbitrarily shaped body

Principal Axes

Theorem

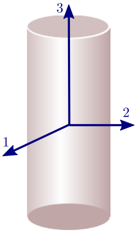
Principal Axes can be calculated for any arbitrary unsymmetric body
It is possible to find coordinate axes in which MI tensor can be diagonalized for any arbitrarily shaped body

Any symmetric tensor can be diagonalized

Examples

Examples

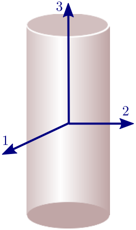
- Cylinder (M, R, L)



Examples

- Cylinder (M, R, L)

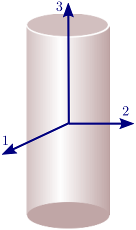
$$I = \begin{bmatrix} \frac{MR^2}{4} + \frac{ML^2}{12} & 0 & 0 \\ 0 & \frac{MR^2}{4} + \frac{ML^2}{12} & 0 \\ 0 & 0 & \frac{MR^2}{2} \end{bmatrix}$$



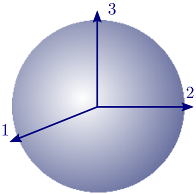
Examples

- Cylinder (M, R, L)

$$I = \begin{bmatrix} \frac{MR^2}{4} + \frac{ML^2}{12} & 0 & 0 \\ 0 & \frac{MR^2}{4} + \frac{ML^2}{12} & 0 \\ 0 & 0 & \frac{MR^2}{2} \end{bmatrix}$$



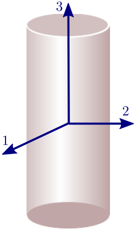
- Sphere (M, R)



Examples

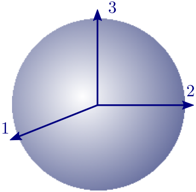
- Cylinder (M, R, L)

$$I = \begin{bmatrix} \frac{MR^2}{4} + \frac{ML^2}{12} & 0 & 0 \\ 0 & \frac{MR^2}{4} + \frac{ML^2}{12} & 0 \\ 0 & 0 & \frac{MR^2}{2} \end{bmatrix}$$



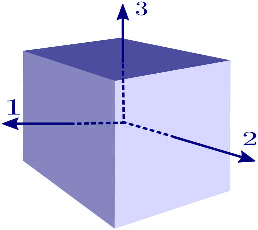
- Sphere (M, R)

$$I = \begin{bmatrix} \frac{2MR^2}{5} & 0 & 0 \\ 0 & \frac{2MR^2}{5} & 0 \\ 0 & 0 & \frac{2MR^2}{5} \end{bmatrix}$$



Examples

- Cube (a)



Examples

- Cube (a)

$$I = \begin{bmatrix} \frac{Ma^2}{6} & 0 & 0 \\ 0 & \frac{Ma^2}{6} & 0 \\ 0 & 0 & \frac{Ma^2}{6} \end{bmatrix}$$