BIRLA INSTITUTE OF TECHNOLOGY AND SCIENCE PILANI - K. K. BIRLA GOA CAMPUS

Sem-1, 2024-2025

Course title: Mathematics-I — Tutorial Sheet 10

Course No. MATH F111 — Date: Oct 12, 2024

Textbook (14th Edition) Problems of 14.3

12, 20, 21, 22, 28, 33, 39, 46, 62, 67, 69, 73, 75, 81, 82, 89, 97, 98, 99, 101, 102, 104

Exercise 1: Partial derivatives and continuity

1. Let $f(x,y,z) = \ln\sqrt{x^2+y^2+z^2}$. Find the first-order partial derivatives $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$, and $\frac{\partial f}{\partial z}$.
2. Consider the function
$$f(x,y) = \begin{cases} \frac{xy}{x^2+y^2}, & \text{if } (x,y) \neq (0,0), \\ 0, & \text{if } (x,y) = (0,0). \end{cases}$$
   (a) Determine $f_x(0,0)$ and $f_y(0,0)$.
   (b) Analyze the continuity of f at $(0,0)$.
3. Consider the function
$$f(x,y) = \begin{cases} \frac{x^3y-xy^3}{x^2+y^2}, & \text{if } (x,y) \neq (0,0), \\ 0, & \text{if } (x,y) = (0,0). \end{cases}$$
   (a) Determine $f_x(0,0)$ and $f_y(0,0)$.
   (b) Analyze the continuity of f at $(0,0)$.

Exercise 2: Higher-Order Partial Derivatives

1. Compute the third-order partial derivative $\frac{\partial^3 f}{\partial x^2 \partial y}$ for $f(x,y) = x^3y^2$.
2. Consider the function
$$f(x,y) = \begin{cases} \frac{x^2y^2}{x^2+y^2}, & \text{if } (x,y) \neq (0,0), \\ 0, & \text{if } (x,y) = (0,0). \end{cases}$$
   Check whether $f_{xy}(0,0) = f_{yx}(0,0)$ or not. Justify.
3. Prove or disprove: If $z = f(x,y)$, then $f_{xy}(0,0) = f_{yx}(0,0)$ always holds.

Exercise 3: Applications of Partial Derivatives

1. The temperature at a point (x,y) on a plate is given by $T(x,y) = 100 - 4x^2 - 9y^2$.
   (a) Find the rate of change of temperature at the point $(2,1)$ in the x-direction.
   (b) Find the rate of change of temperature at the point $(2,1)$ in the y-direction.

Exercise 4: Differentiability and Continuity

1. Consider the function
$$f(x,y) = \begin{cases} \frac{xy}{\sqrt{x^2+y^2}}, & \text{if } (x,y) \neq (0,0), \\ 0, & \text{if } (x,y) = (0,0). \end{cases}$$
   (a) Check whether f is continuous at (0, 0) or not, justify.
   (b) Check whether f is differentiable at (0, 0) or not, justify.
2. Consider the function $f(x,y) = ||x| - |y|| - |x| - |y|$.
   (a) Check whether f is continuous at (0, 0) or not, justify.
   (b) Check whether f is differentiable at (0, 0) or not, justify.

Textbook (14th Edition) Problems of 14.4

5, 11, 22, 29, 39, 46, 50, 51, 52, 55, 58, 59

Exercise 5. Chain rule Problems

1. Let $z = \frac{x^2-y^2}{x^2+y^2}$, where $x = r\cos(\theta)$ and $y = r\sin(\theta)$. Find $\frac{\partial z}{\partial r}$ and $\frac{\partial z}{\partial \theta}$.
2. Let $w = \sqrt{x^2+y^2+z^2}$, where $x = u^2 - v^2$, $y = 2uv$, and $z = u + v$. Find $\frac{\partial w}{\partial u}$ and $\frac{\partial w}{\partial v}$.
3. Let $w = \tan^{-1}\left(\frac{y}{x}\right)$, where $x = r\cos(\theta)$ and $y = r\sin(\theta)$. Prove that $\frac{\partial w}{\partial r} = 0$ and find $\frac{\partial w}{\partial \theta}$.
4. Given $w = f(x,y,z)$ where $x = r\sin(\theta)\cos(\phi)$, $y = r\sin(\theta)\sin(\phi)$, and $z = r\cos(\theta)$, derive the formula for $\frac{\partial w}{\partial r}$ using the chain rule.
5. Let $w = \ln(xy)$, where $x = e^u\cos(v)$ and $y = e^u\sin(v)$. Find $\frac{\partial w}{\partial u}$ and $\frac{\partial w}{\partial v}$.
6. Suppose $z = f(x,y)$, where $x = g(s,t), y = h(s,t)$. then using the chain rule, show that
$$\frac{\partial^2 z}{\partial t^2} = \frac{\partial^2 f}{\partial x^2}\left(\frac{\partial g}{\partial t}\right)^2 + 2\frac{\partial^2 f}{\partial x \partial y}\frac{\partial g}{\partial t}\frac{\partial h}{\partial t} + \frac{\partial^2 f}{\partial y^2}\left(\frac{\partial h}{\partial t}\right)^2 + f_x\frac{\partial^2 g}{\partial t^2} + f_y\frac{\partial^2 h}{\partial t^2}$$
7. Express the Laplace equation in polar form using the chain rule.

Exercise 6: True/False Questions

1. If a function $f(x,y)$ has partial derivatives at $(0,0)$, then it must be continuous at $(0,0)$. (True/False)
2. A function that is continuous at $(0,0)$ and has partial derivatives at $(0,0)$ is necessarily differentiable at $(0,0)$. (True/False)
3. For the function $f(x,y) = \frac{x^2-y^2}{x^2+y^2}$ defined as 0 at $(0,0)$, f is discontinuous at $(0,0)$. (True/False)
4. If $f(x,y)$ has continuous second-order partial derivatives, then $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$. (True/False)
5. If $f_x(x,y)$ and $f_y(x,y)$ exist at a point (x_0,y_0), then f is differentiable at (x_0,y_0). (True/False)
6. Higher-order partial derivatives are always continuous. (True/False)
7. The mixed derivative theorem can be applied to $f(x,y) = \sqrt{x^2+y^2}$ at $(0,0)$. (True/False)

8. The function $f(x, y) = |x| + |y|$ is differentiable at $(0, 0)$. (True/False)

9. For $f(x, y) = \sqrt{x^2 + y^2}$, the partial derivatives exist at $(0, 0)$, so the function is differentiable there. (True/False)

10. For $w = f(x, y, z)$, where x, y, and z are functions of u and v, the partial derivative $\frac{\partial w}{\partial u}$ includes terms involving derivatives of x, y, and z with respect to u. (True/False)