14. Partial Derivatives

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Recall

$z = f(x,y)$, $w = f(x,y,z)$

Given a $\underbrace{\text{domain}}_{D}$ (or Region) of such functions :

Interior pt of D : Any pt (x_0, y_0) (or (x_0, y_0, z_0)) is an It pt. of D if $\exists$ an open disk (or open ball) around (x_0, y_0) (or (x_0, y_0, z_0)) which is entirely contain inside D.

$\text{Int } D = \{\text{Interior pts of Domain}\}$

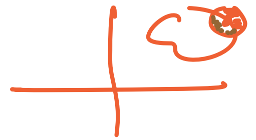
open set : D is open if $\text{Int } D = D$

Boundary pt : Any pt (x_0, y_0) (or (x_0, y_0, z_0)) is a boundary pt. of D, if $\forall$ open disk (or open ball) B around the point, we have $B \cap D \neq \phi$ and $B \cap D^c \neq \phi$

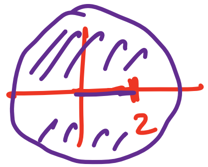
$\text{Bd } D = \{\text{Boundary pts. of domain}\}$

closed set : D is closed if $\text{Bd } D \subseteq D$

Bounded and Unbounded Regions



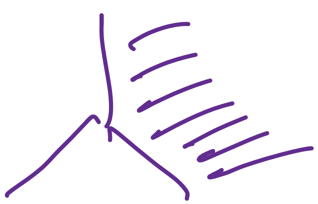
A region in the *xy*-plane is **bounded** if it lies inside a disk of fixed radius.



Examples: Line segments, triangles, interior of triangles, rectangles, circles and disks are bounded regions (sets) in *xy*-plane.

A region in the space is **bounded** if it lies inside an open ball of fixed radius.

Examples: Line segments, triangles, rectangles, open balls are bounded regions (sets) in space.



A region is unbounded if it is not bounded.

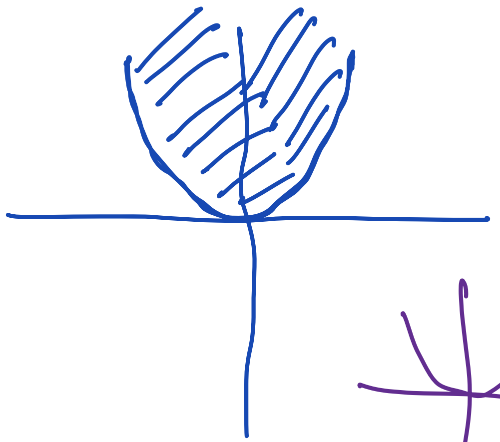
The lines, coordinate axes, octants, half-spaces, and the full space itself are unbounded regions.

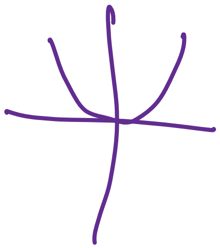
The lines, coordinate axes, quadrants, half-planes, and the full plane itself are unbounded region

Examples

1. Describe the domain of the function $f(x, y) = \sqrt{y - x^2}$.

Domain $D = \{(x,y) \in \mathbb{R}^2 \mid y - x^2 \geq 0\} = \{(x,y) \in \mathbb{R}^3 \mid y \geq x^2\}$





Clearly D is unbounded

$\text{Int } D = \{(x,y) \in \mathbb{R}^2 \mid y - x^2 > 0\}$

$\Rightarrow \text{Int } D \neq D \Rightarrow D$ is not open

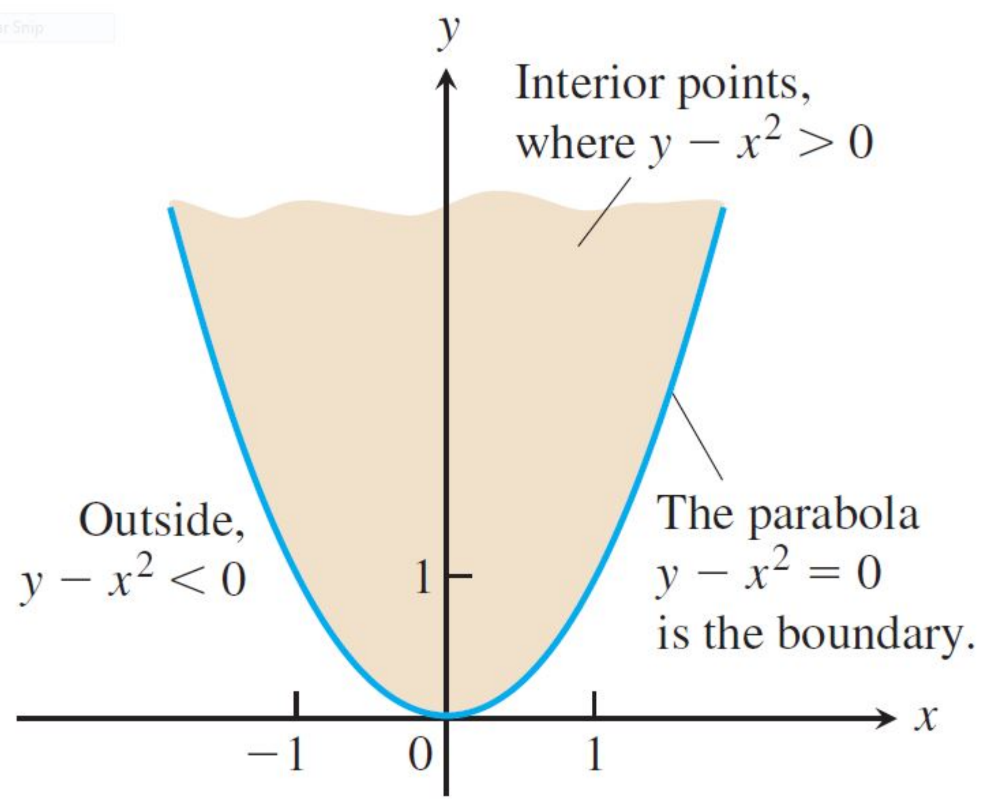
$\text{Bd } D = \{(x,y) \in \mathbb{R}^2 \mid y = x^2\}$

clearly $\text{Bd } D \subset D$

$\Rightarrow D$ is closed.

Examples

1. Describe the domain of the function $f(x, y) = \sqrt{y - x^2}$.



The domain is given by $D = \{(x, y) \ : \ y - x^2 \geq 0\}$. It is closed, not open and it is unbounded.

2. Describe the domain of the function

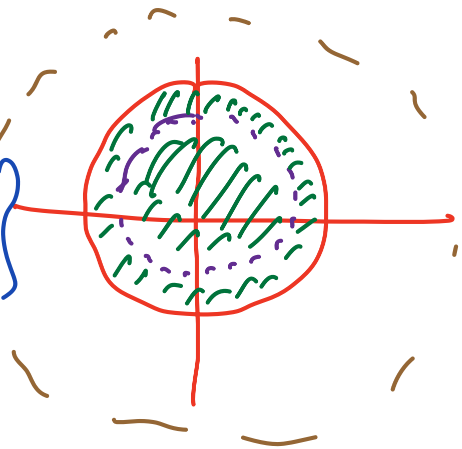
$$f(x,y)=\frac{1}{\ln\left(25-x^2-y^2\right)}.$$

Domain $D = \{(x,y)\in\mathbb{R}^2 \mid 25-x^2-y^2>0,\ 25-x^2-y^2\neq 1\}$

$= \{(x,y)\in\mathbb{R}^2 \mid x^2+y^2<25,\ x^2+y^2\neq 24\}$

$\text{Int } D = D \Rightarrow D$ is open

$\text{Bd } D = \{(x,y)\in\mathbb{R}^2 \mid x^2+y^2=25,\ x^2+y^2=24\}$



$\text{Bd } D \not\subseteq D \Rightarrow D$ is not closed

D is clearly bdd.

2. Describe the domain of the function

$$f(x, y) = \frac{1}{\ln(25 - x^2 - y^2)}.$$

Ans. The domain is given by $D = \{(x, y) \ : \ x^2 + y^2 < 25, \ x^2 + y^2 \neq 24\}$. It is open but not closed and it is bounded.

Graphs, Level Curves and Contours of Functions of Two Variables

How to draw the graph of two variable function $z = f(x, y)$?

There are two standard ways:

- One is to sketch the surface $z = f(x, y)$ in space.

  The set of all points $(x, y, f(x, y))$ in space, for (x, y) in the domain of f, is called <u>the graph of f</u>.

  The graph of f is also called surface $z = f(x, y)$.

Definition

- the other is to draw and label the curves in the domain on which f has a constant value.

  The set of points in the plane (xy-plane) where a function $f(x, y)$ has a constant value c i.e., $f(x, y) = c$ where $c \in Range(f)$ is called a level curve of f.

Level Curves and Graph of Two Variable Function

Example. $z = 100 - x^2 - y^2$

$100 - x^2 - y^2 = c$ Range of z

$x^2 + y^2 = 100 - c$

$c = 0, \quad x^2 + y^2 = 100$

$c = 75, \quad x^2 + y^2 = 75$

$\vdots \qquad \vdots$

$c = 90, \quad x^2 + y^2 = 10$

$z = c = -75$

$100 - x^2 - y^2 = -75$

$175 = x^2 + y^2$

$f(x, y) = z$

fixing z value, gives different level curves

$f(x, y) = \text{constant}$

$x^2 + y^2 \geq 0$

$-x^2 - y^2 \leq 0$

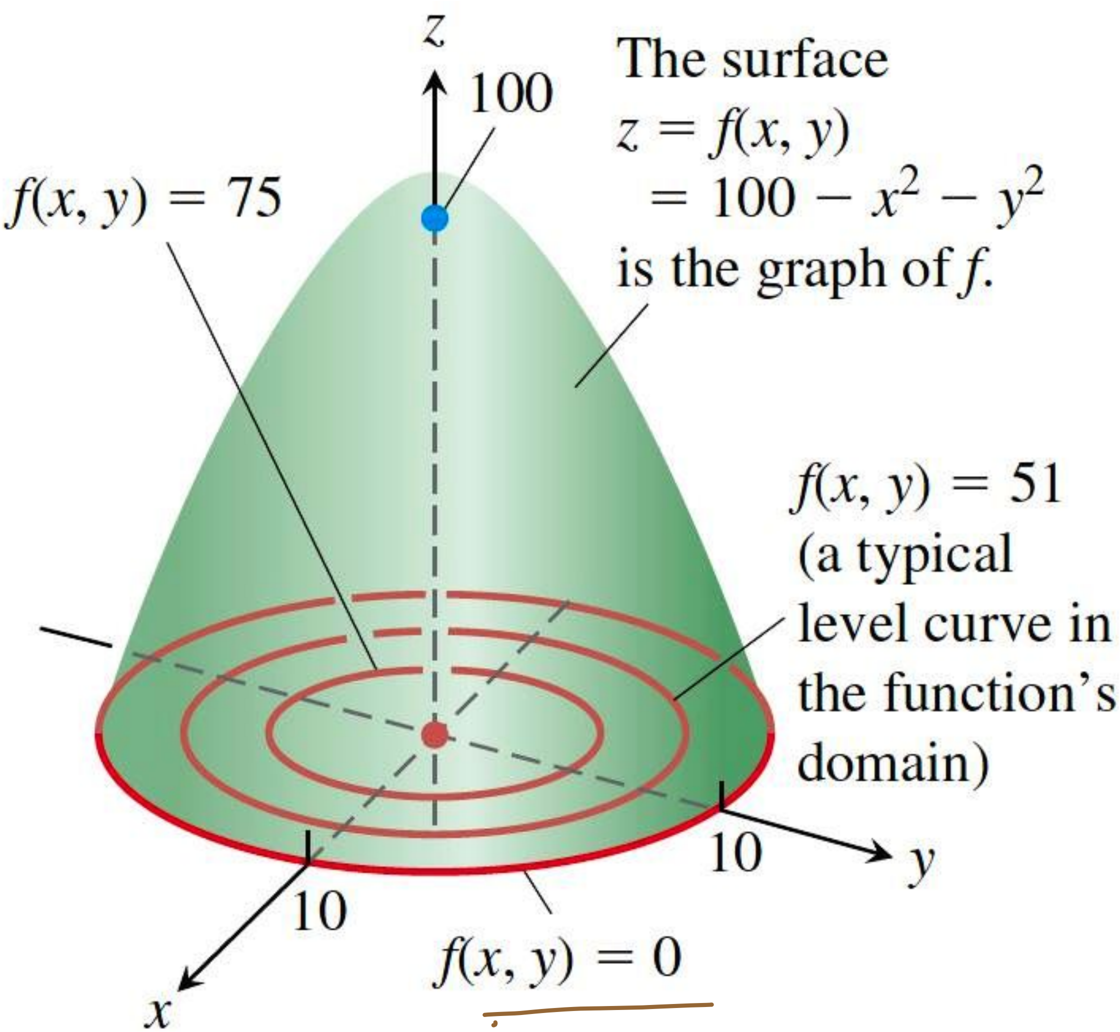
$100 - x^2 - y^2 \leq 100$

$\therefore \boxed{c \in (-\infty, 100]}$

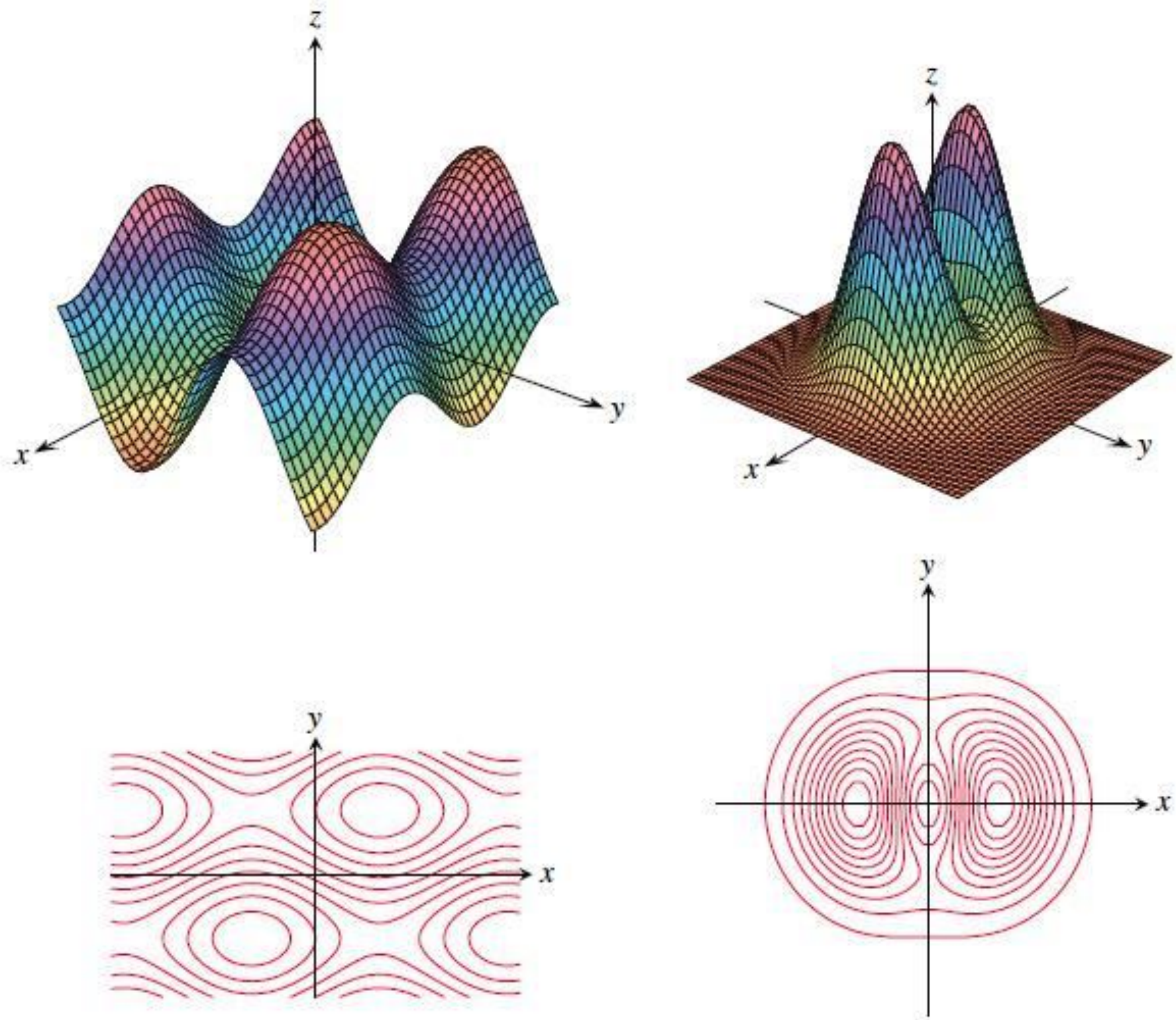
$\mathbb{R}^2 \to \mathbb{R}$

Level Curves and Graph of Two Variable Function

Example. $z = 100 - x^2 - y^2$



Level Curves and Graph of Two Variable Function



(a) $z = \sin x + 2\sin y$

(b) $z = (4x^2 + y^2)e^{-x^2-y^2}$

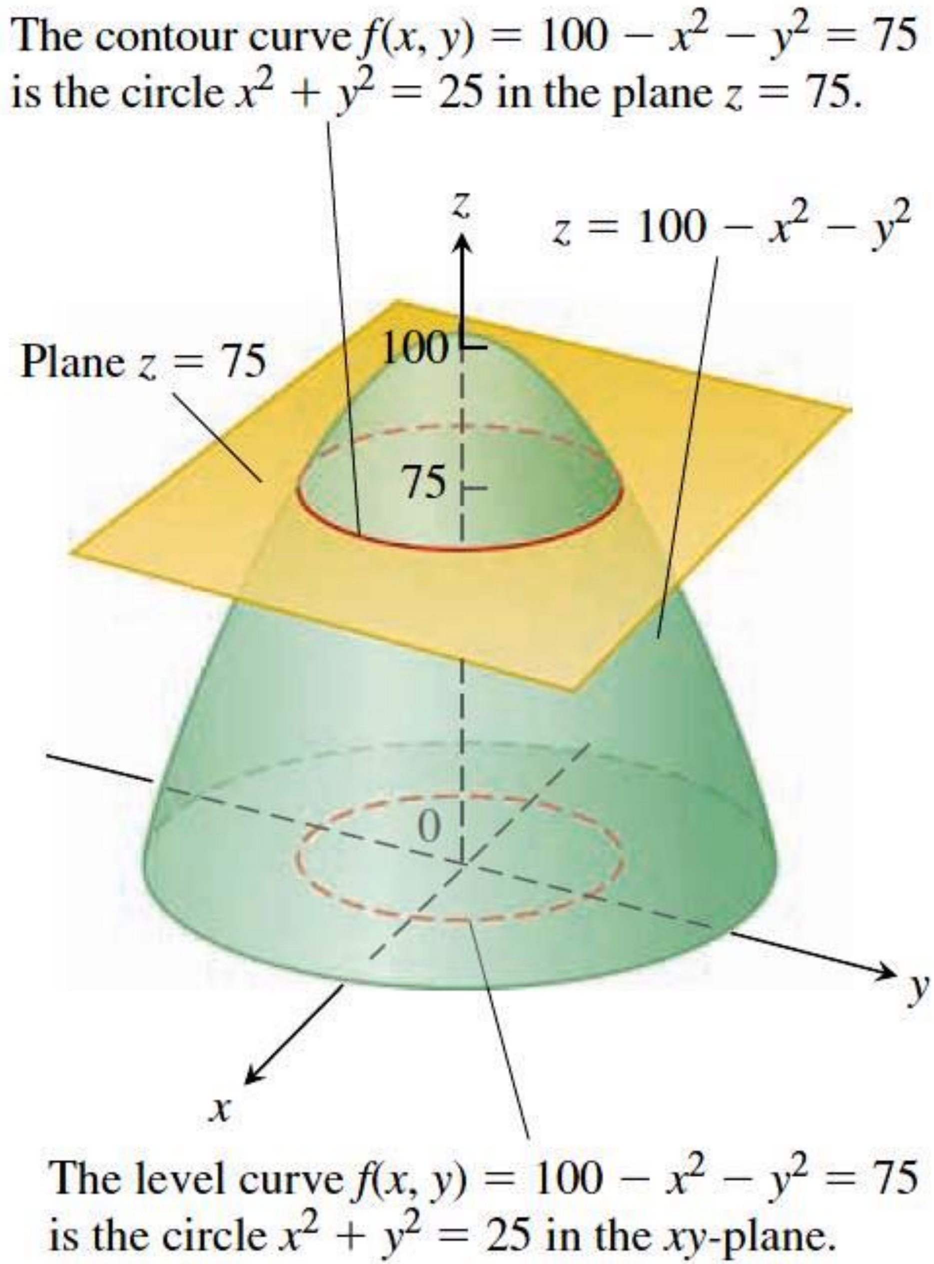
Contours of Functions of Two Variables

Definition

- The curve in the space in which the plane $z = c$ cuts a surface $z = f(x, y)$ is made up of the points that represent the function value $f(x, y) = c$.

- It is called the contour curve $f(x, y) = c$ to distinguish it from the level curve $f(x, y) = c$ in the domain of f.

Contour of Two Variable Function



Find the domain, range and the level curve for the following functions passing through the given point.

1. $f(x, y) = 16 - x^2 - y^2, \quad (2\sqrt{2}, \sqrt{2})$.

Domain $= \mathbb{R}^2$, Range $= (-\infty, 16]$

Eq. of level curve is

$16 - x^2 - y^2 = c$, $c \in (-\infty, 16]$

at $(2\sqrt{2}, \sqrt{2})$

$c = 16 - 8 - 2 = 6$

$\Rightarrow 16 - x^2 - y^2 = 6 \Rightarrow x^2 + y^2 = 10$

Examples

Find the domain, range and the level curve for the following functions passing through the given point.

1. $f(x, y) = 16 - x^2 - y^2, \quad (2\sqrt{2}, \sqrt{2})$.

   Ans. Domain is $\mathbb{R}^2$, Range is $(-\infty, 16]$ and a typical level curve is $x^2 + y^2 = 16 - c$ where $c \in (-\infty, 16]$ and the level curve that is passing through the point $(2\sqrt{2}, \sqrt{2})$ is $x^2 + y^2 = 10$.

2. $f(x, y) = \sqrt{x^2 - 1}, \quad (1, 0)$.

Domain $= \{ (x,y) \in \mathbb{R}^2 \mid x^2 \geq 1 \}$, Range $= [0, \infty)$

eq. of level curve

$\sqrt{x^2 - 1} = c, \quad c \in [0, \infty)$

$x^2 - 1 = c^2$ passes through $(1,0) \Rightarrow c = 0$

$\Rightarrow x^2 = 1 \Rightarrow x = 1, x = -1$

Level surfaces of functions of three variables

Definition

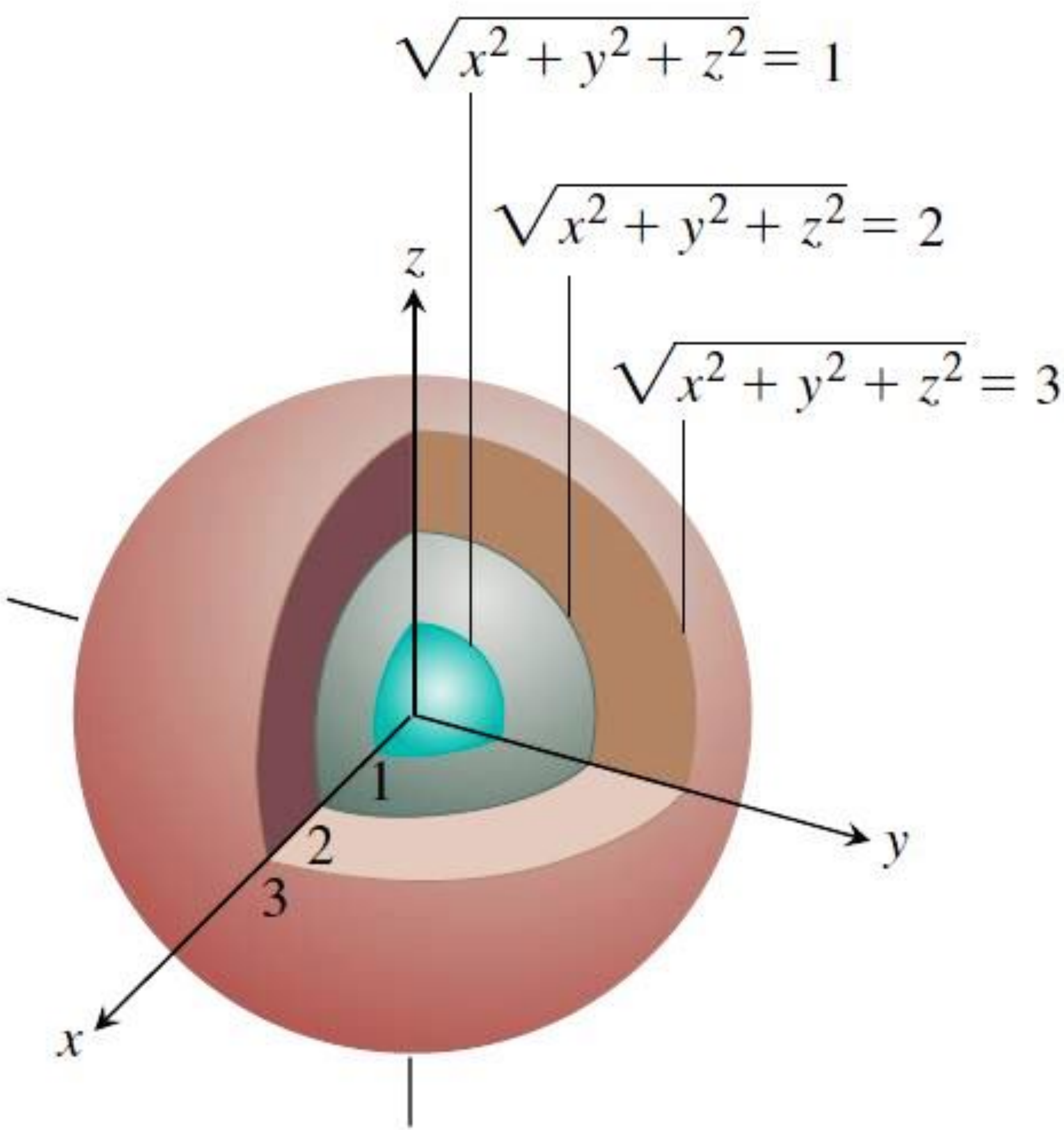
The set of points (x, y, z) in space where a function $f(x, y, z)$ of three independent variables has a constant value i.e., $f(x, y, z) = c$ where c is from the range of f is called a level surface of f.

Example. Describe the level surfaces of the function

$$f(x, y, z) = \sqrt{x^2 + y^2 + z^2}.$$

$x^2+y^2+z^2=c^2$, $c \in [0,\infty)$

Level surfaces of functions of three variables



Level surfaces of functions of three variables

Find the domain, range and sketch a typical level surface for the following functions

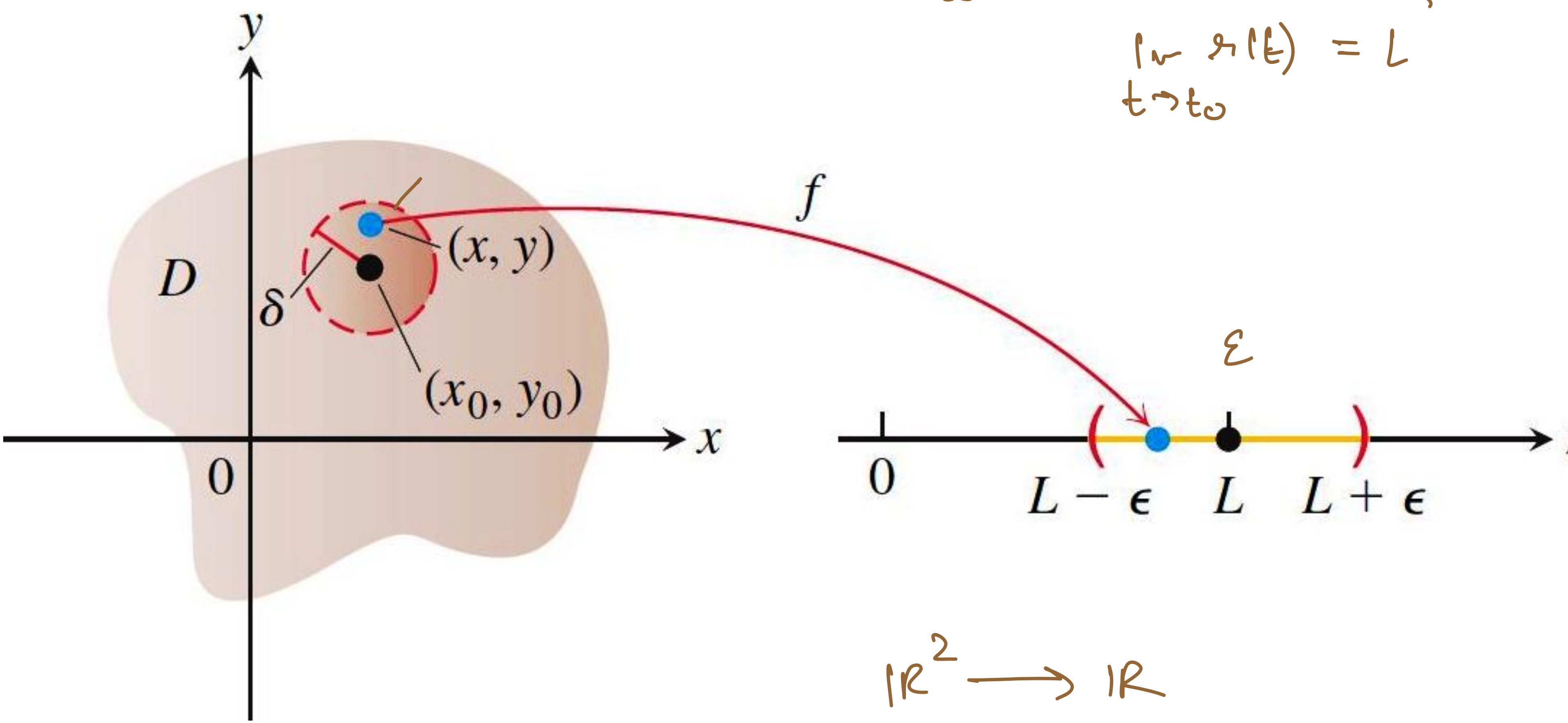
1. $f(x, y, z) = \ln(x^2 + y^2 + z^2)$.
2. $f(x, y, z) = x + z$.
3. $f(x, y, z) = z - x^2 - y^2$

Limits of functions of two variables

$\mathbb{R} \to \mathbb{R}^3$

t_0

$\lim_{t \to t_0} r(t) = L$



$\mathbb{R}^2 \to \mathbb{R}$

$\mathbb{R}^n \to \mathbb{R}^m$

Limits of functions of two variables

Definition (Limit of functions of two variables)

We say that a function $f(x,y)$ approaches the **limit** L as (x,y) approaches (x_0, y_0) and we write

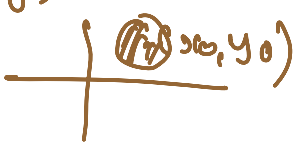
$$\boxed{\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L,}$$

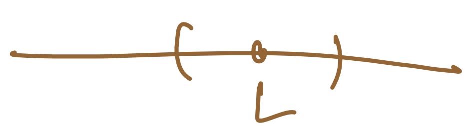
if, for every number $\varepsilon > 0$, there exists a corresponding number $\delta > 0$ such that for all (x,y) in the domain of f,

$$|f(x,y) - L| < \varepsilon \quad \text{whenever} \quad 0 < \sqrt{(x-x_0)^2 + (y-y_0)^2} < \delta. \checkmark$$

$\Updownarrow$

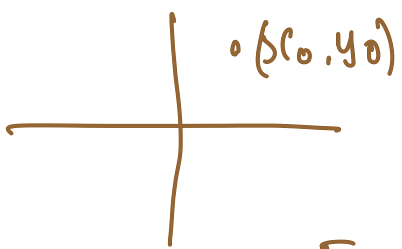
$f(x,y) \in (-\varepsilon + L, \varepsilon + L)$

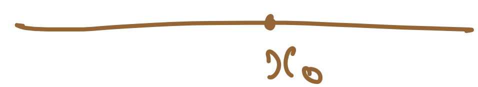




Examples

1. $\lim\limits_{(x,y)\to(x_0,y_0)} x = \underline{x_0} = L$





Given $\varepsilon>0$, Find $\delta>0$ s.t. Choose $\varepsilon=\delta$

$$0<\sqrt{(x-x_0)^2+(y-y_0)^2}<\delta \overset{?}{\Longrightarrow} |x-x_0|<\varepsilon$$

$$\text{L.H.S} = |x-x_0| = \sqrt{(x-x_0)^2}$$
$$< \sqrt{(x-x_0)^2+(y-y_0)^2}$$
$$< \varepsilon$$

Examples

1. $\lim_{(x,y)\to(x_0,y_0)} x = x_0$

2. $\lim_{(x,y)\to(x_0,y_0)} y = y_0$ ✓

3. $\lim_{(x,y)\to(x_0,y_0)} k = k$ (k is any constant) ✓

Given $\varepsilon > 0$, Find δ, choose $\delta = \varepsilon$

s.t

$\sqrt{(x-x_0)^2 + (y-y_0)^2} < \delta \overset{?}{\Longrightarrow} |k-k| < \varepsilon$

$|k-k| = 0 < \varepsilon$

Example

4. Show that $\lim\limits_{(x,y)\to(0,0)} \overbrace{\frac{4xy^2}{x^2+y^2}}^{f(x,y)} = 0$ $=L$ by using $\varepsilon - \delta$ definition.

Given $\varepsilon > 0$, Find $\delta > 0$ s.t

$\sqrt{(x-0)^2+(y-0)^2} < \delta \overset{?}{\Longrightarrow} |f(x,y) - L| < \varepsilon$

$\Rightarrow \sqrt{x^2+y^2} < \delta$ — ①

i.e $\left|\frac{4xy^2}{x^2+y^2}\right| < \varepsilon$

$y^2 < x^2+y^2$

$\frac{y^2}{x^2+y^2} < 1$

$\underline{L.H.S} = \left|\frac{4xy^2}{x^2+y^2}\right| = \frac{4|x|y^2}{x^2+y^2}$

$< 4|x| = 4\sqrt{x^2}$

$< 4\sqrt{x^2+y^2} < 4\delta$ Using ①

$= \varepsilon$

choose $\delta = \varepsilon/4$

$\Rightarrow \varepsilon = 4\delta$

Definition (Limit of functions of three variables)

We say that a function $f(x, y, z)$ approaches the **limit** L as (x, y, z) approaches (x_0, y_0, z_0) and we write

$$\lim_{(x,y,z)\to(x_0,y_0,z_0)} f(x, y, z) = L,$$

if, for every number $\varepsilon > 0$, there exists a corresponding number $\delta > 0$ such that for all (x, y, z) in the domain of f,

$$|f(x, y, z) - L| < \varepsilon$$

$$\text{whenever} \quad 0 < \sqrt{(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2} < \delta.$$

Using $\epsilon - \delta$ definition show that

$$\lim_{(x,y,z)\to(0,0,0)} \frac{x+y+z}{x^2+y^2+z^2+1} = 0.$$

Given $\varepsilon > 0$, Find $\delta > 0$ s.t

$$\sqrt{x^2+y^2+z^2} < \delta \overset{?}{\Longrightarrow} \left| \frac{x+y+z}{x^2+y^2+z^2+1} \right| < \varepsilon$$

$$\text{L.H.S} = \frac{|x+y+z|}{x^2+y^2+z^2+1} < |x+y+z|$$

$x^2+y^2+z^2+1 > 1$

$\frac{1}{x^2+y^2+z^2+1} < 1$

$$< |x| + |y| + |z|$$

$$= \sqrt{x^2} + \sqrt{y^2} + \sqrt{z^2}$$

$$< \sqrt{x^2+y^2+z^2} + \sqrt{x^2+y^2+z^2} + \sqrt{x^2+y^2+z^2}$$

$$< 3\sqrt{x^2+y^2+z^2} \leq 3\delta = \varepsilon$$

choose $\delta = \varepsilon/3$

$\Rightarrow \varepsilon = 3\delta$

THEOREM 1—Properties of Limits of Functions of Two Variables The following rules hold if L, M, and k are real numbers and

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L \qquad \text{and} \qquad \lim_{(x,y)\to(x_0,y_0)} g(x,y) = M.$$

1. *Sum Rule:* $\lim_{(x,y)\to(x_0,y_0)} (f(x,y) + g(x,y)) = L + M$
2. *Difference Rule:* $\lim_{(x,y)\to(x_0,y_0)} (f(x,y) - g(x,y)) = L - M$
3. *Constant Multiple Rule:* $\lim_{(x,y)\to(x_0,y_0)} kf(x,y) = kL$ (any number k)
4. *Product Rule:* $\lim_{(x,y)\to(x_0,y_0)} (f(x,y) \cdot g(x,y)) = L \cdot M$ ✓
5. *Quotient Rule:* $\lim_{(x,y)\to(x_0,y_0)} \dfrac{f(x,y)}{g(x,y)} = \dfrac{L}{M}, \quad M \neq 0$
6. *Power Rule:* $\lim_{(x,y)\to(x_0,y_0)} [f(x,y)]^n = L^n$, n a positive integer
7. *Root Rule:* $\lim_{(x,y)\to(x_0,y_0)} \sqrt[n]{f(x,y)} = \sqrt[n]{L} = L^{1/n}$, n a positive integer, and if n is even, we assume that $L > 0$.

Examples

$$\lim_{(x,y)\to(1,1)} x^2+y^2$$
$$= \lim x^2 + \lim y^2$$
$$= (\lim x)^2 + (\lim y)^2$$

Find the limits for the following:

1. $$\lim_{(x,y)\to(0,1)} \frac{x - xy + 3}{x^2y + 5xy - y^3}$$ $= -3$

2. $$\lim_{(x,y)\to(0,0)} \frac{x^2 - xy}{\sqrt{x} - \sqrt{y}}.$$

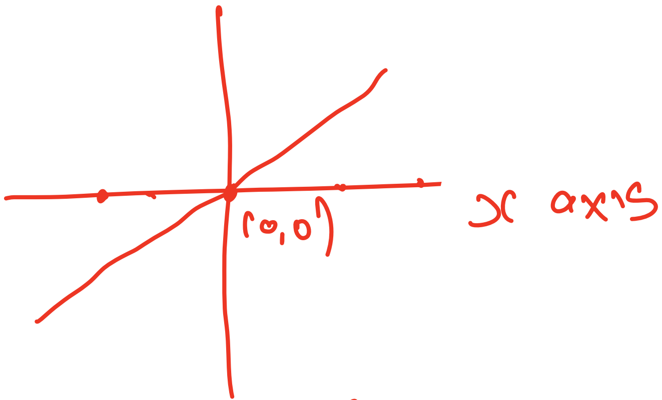
$$= \lim \frac{x(x-y)(\sqrt{x}+\sqrt{y})}{(\sqrt{x}-\sqrt{y}) \times (\sqrt{x}+\sqrt{y})}$$
$$= \lim \frac{x\cancel{(x-y)}}{\cancel{x-y}} (\sqrt{x}+\sqrt{y})$$
$$= 0$$

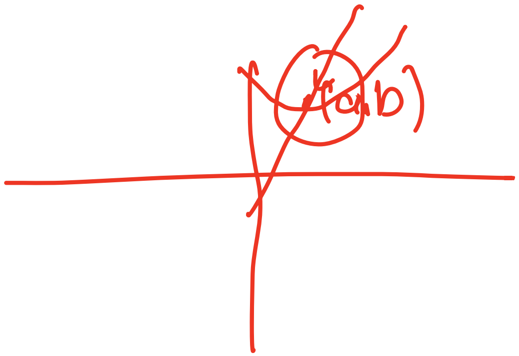
Examples

If $f(x,y) = \dfrac{y}{x}$, does $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ exist ?

$f : \mathbb{R}^2 \to \mathbb{R}$

$(x,y) \to (a,b) \Rightarrow f(x,y) \to L$





$\therefore$ lim doesn't exist

$\lim\limits_{x\to 0} f(x,0)$ { along x axis i.e $y=0$

$= \lim\limits_{x\to 0} \dfrac{0}{x} = 0$

$\neq$

along $\underline{y = x}$,

$f(x,y) = f(t)$

$\lim\limits_{x\to 0} f(x,x) = \lim\limits_{x\to 0} \dfrac{x}{x} = \lim\limits_{x\to 0} 1 = 1$

Two path test for nonexistence of a limit

Theorem

Let $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}$ *is any continuous curve passing through the point* (x_0, y_0), $\mathbf{r}(t_0) = (x_0, y_0)$ *and* $\lim_{(x,y)\to(x_0,y_0)} f(x, y) = L$, *then*

$$\lim_{t\to t_0} f(\mathbf{r}(t)) = \lim_{t\to t_0} f(x(t), y(t)) = L.$$

p

q

Remark

If a function $f(x, y)$ has two different limits along two different paths in the domain of f as (x, y) approaches (x_0, y_0), then $\lim_{(x,y)\to(x_0,y_0)} f(x, y)$ does not exist.

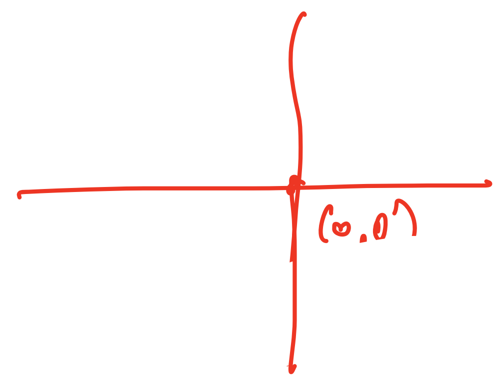
Example

Show that if

$$f(x,y) = \begin{cases} \frac{10x^2y}{x^4+y^2} & \text{for } (x,y) \neq (0,0); \\ 0 & \text{for } (x,y) = (0,0), \end{cases}$$

then $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist.

$y=0$, $\lim_{x\to 0} f(x,0) = \lim_{x\to 0} \frac{0}{x^4} = 0$



$y=x^2$, $\lim_{x\to 0} f(x,x^2) = \lim_{x\to 0} \frac{10x^2\cdot x^2}{2x^4}$

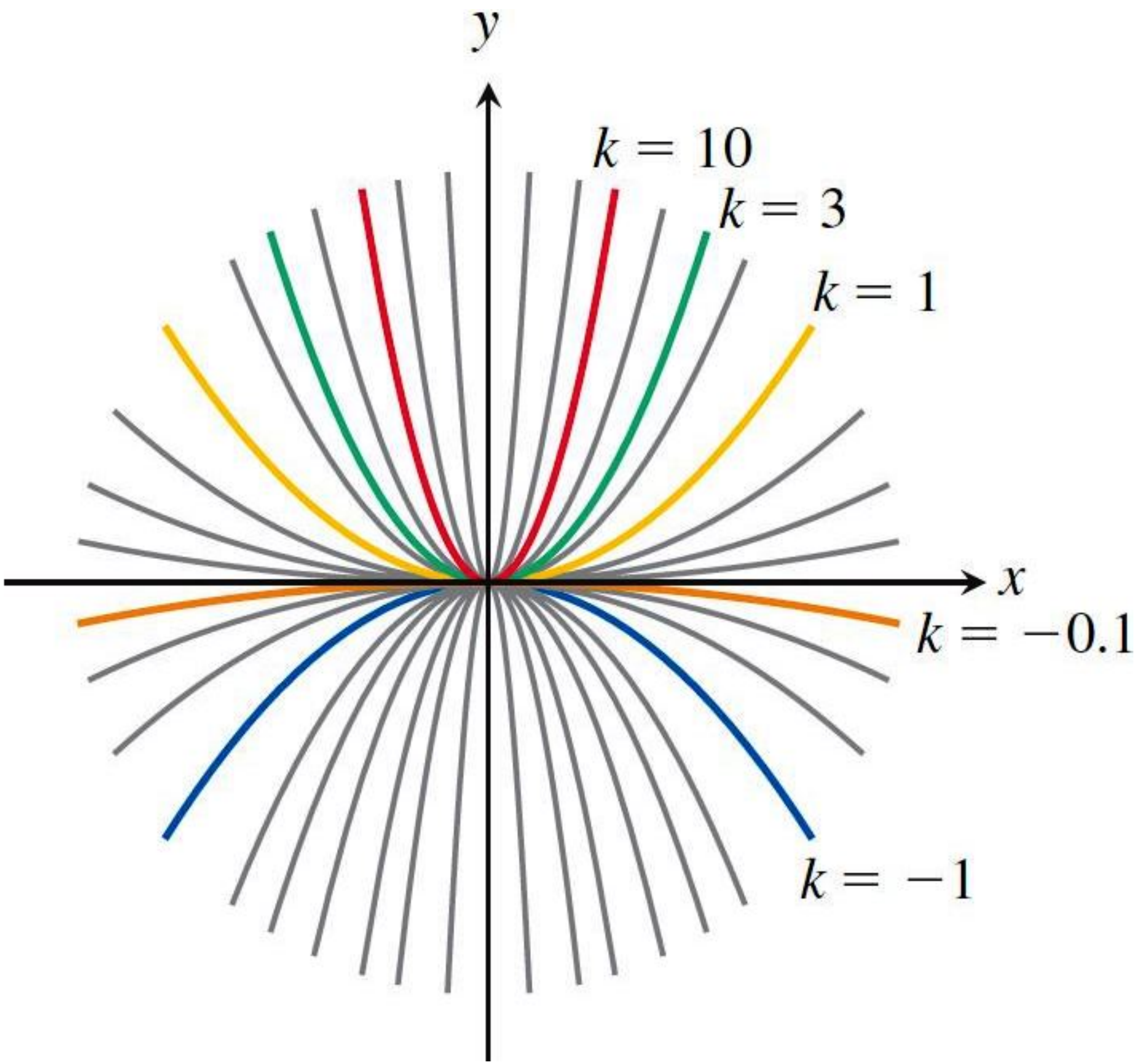
$y\to 0$

$x\to 0$

$= \lim_{x\to 0} 5 = 5$

$y=2x^2$, $\lim_{x\to 0} f(x,2x^2) = 10$

Two path test for nonexistence of a limit



Theorem (The Sandwich Theorem)

Let f, g and h be functions of two variables such that

$$g(x,y) \leq f(x,y) \leq h(x,y)$$

for all $(x,y) \neq (x_0,y_0)$ in a disk centered at (x_0,y_0) and if

$$\lim_{(x,y)\to(x_0,y_0)} g(x,y) = L = \lim_{(x,y)\to(x_0,y_0)} h(x,y)$$

for a finite limit $L \in \mathbb{R}$, then

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L.$$

Find the limits (if they exist):

1. $\lim\limits_{(x,y)\to(0,0)} y \sin \dfrac{1}{x}.$ $= 0$

$$-1 \leq \sin\frac{1}{x} \leq 1$$

$$-y \leq y\sin\frac{1}{x} \leq y$$

$$\underbrace{\lim_{(x,y)\to(0,0)} (-y)}_{=0} \leq \underbrace{\lim_{(x,y)\to(0,0)} y\sin\frac{1}{x}}_{=0} \leq \underbrace{\lim_{(x,y)\to(0,0)} y}_{=0}$$

Examples

Find the limits (if they exist):

1. $\lim_{(x,y)\to(0,0)} y \sin \frac{1}{x}$.
2. $\lim_{(x,y)\to(0,0)} \frac{x^2 \sin^2 y}{x^2 + 2y^2}$.
3. $\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2 + y^2}$.
4. $\lim_{(x,y)\to(0,0)} xy \frac{x^2 - y^2}{x^2 + y^2}$.

$0 < \frac{x^2}{x^2+2y^2} < 1$

$0 \le \sin^2 y \le y^2$

$0 < \frac{x^2}{x^2+2y^2} \sin^2 y < y^2$

$0 \qquad 0 \qquad 0$

Remark (Changing Variables to Polar Coordinates)

If $f(x,y)$ is a function of two variables, $L \in \mathbb{R}$ and for given any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|f(r\cos\theta, r\sin\theta) - L| < \varepsilon \quad \text{whenever} \quad 0 < |r| < \delta$$

for all θ with $(r\cos\theta, r\sin\theta)$ in the domain of f, then

$$\lim_{(x,y)\to(0,0)} f(x,y) = L.$$

In other words, if $\lim_{r\to 0} f(r\cos\theta, r\sin\theta) = L$ uniformly in θ, where is L is a constant independent of θ, then

$$\lim_{(x,y)\to(0,0)} f(x,y) = L.$$

Find the limits, (if they exist):

1. $$\lim_{(x,y)\to(0,0)} \frac{x^3 - xy^2}{x^2+y^2}.$$

$x = r\cos\theta$

$y = r\sin\theta$

$$\lim_{r\to 0} \frac{r^3\cos^3\theta - r^3\cos\theta\,\sin^2\theta}{r^2} = \lim_{r\to 0} r(\cos^3\theta - \cos\theta\sin^2\theta)$$

$= 0 \quad \forall$ values of θ

Changing Variables to Polar Coordinates

Find the limits, (if they exist):

1. $$\lim_{(x,y)\to(0,0)} \frac{x^3 - xy^2}{x^2 + y^2}.$$

2. $$\lim_{(x,y)\to(0,0)} \tan^{-1}\left(\frac{|x| + |y|}{x^2 + y^2}\right).$$

3. $$\lim_{(x,y)\to(0,0)} \frac{x^2 - y^2}{x^2 + y^2}.$$

Continuity of functions of two variables

Definition

A function $f(x, y)$ is said to be **continuous at the point (x_0, y_0)**, if

1. f is defined at (x_0, y_0),
2. $\lim\limits_{(x,y)\to(x_0,y_0)} f(x, y)$ exists,
3. $\lim\limits_{(x,y)\to(x_0,y_0)} f(x, y) = f(x_0, y_0)$.

A function $f(x, y)$ is said to be **continuous**, if it is continuous at every point of its domain.

Show that the following functions are continuous at given point.

1. $\dfrac{x - xy + 3}{x^2y + 5xy - y^3} = f(x,y)$, $\quad (0, 1)$.

1) Is function defined at (0,0) , $f(0,1) = -3$

2) Does $\lim_{(x,y)\to(0,1)} f(x,y)$ exist ?

Using arithmetic operations

$\lim_{(x,y)\to(0,1)} f(x,y) = -3 = L$

3) $L = f(0,1) = -3$

Show that the following functions are continuous at given point.

1. $\dfrac{x - xy + 3}{x^2y + 5xy - y^3}$, $(0, 1)$.

2. $\sqrt{x^2 + y^2}$, $(3, -4)$. ✓

1) $f(x,y) = \sqrt{x^2+y^2}$, $f(3,-4) = 5$ defined ✓

2) Prove that $\lim\limits_{(x,y)\to(3,-4)} \sqrt{x^2+y^2} = 5$ Using ε-δ

Given $\varepsilon > 0$, Find δ s.t

$\sqrt{(x-3)^2+(y+4)^2} < \delta \overset{?}{\Longrightarrow} \left|\sqrt{x^2+y^2} - 5\right| < \varepsilon$

i.e $x-3 < \delta$ ✓
and $y+4 < \delta$ ✓

$$L.H.S = \frac{\left|\left(\sqrt{x^2+y^2}-5\right)\left(\sqrt{x^2+y^2}+5\right)\right|}{\sqrt{x^2+y^2}+5}$$

$$= \left| \frac{x^2+y^2-25}{\sqrt{x^2+y^2}+5} \right| < \left| x^2+y^2-25 \right|$$

$$= \left| (x-3)^2 + (y+4)^2 + 6x - 8y - 50 \right|$$

$$\leq \left| (x-3)^2 + (y+4)^2 \right| + \left| 6(x-3) \right| + 8|y+4|$$

$$< \delta^2 + \delta^2 + 6\delta + 8\delta$$

$$= 2\delta^2 + 14\delta = \varepsilon$$

choose δ

s.t.

$$\boxed{2\delta^2 + 14\delta = \varepsilon}$$ (solve for δ)

Show that the following functions are continuous at given point.

1. $\dfrac{x - xy + 3}{x^2y + 5xy - y^3}, \quad (0, 1).$

2. $\sqrt{x^2 + y^2}, \quad (3, -4).$

Find the all the points in xy-plane where the following functions are continuous:

1. $f(x, y) = \dfrac{x}{y^2 + 1}.$

Domain = $\mathbb{R}^2$

Show that the following functions are continuous at given point.

1. $\dfrac{x - xy + 3}{x^2y + 5xy - y^3}, \quad (0, 1).$

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Find the all the points in xy-plane where the following functions are continuous:

1. $f(x, y) = \dfrac{x}{y^2 + 1}.$

2. $f(x, y) = \dfrac{1}{x^2 - 1}.$

3. $f(x, y) = \sin(x + y).$

Examples

Show that

$$f(x,y) = \begin{cases} \frac{2xy}{x^2+y^2} & \text{for } (x,y) \neq (0,0); \\ 0 & \text{for } (x,y) = (0,0), \end{cases}$$

is continuous at every point except the origin.

$y = mx$ $\quad f(x, mx) = \dfrac{2m}{1+m^2}$

Since it is independent of (x, y).

For different values of m, we get different limits

Continuity of Composites

Theorem (Continuity of Composites)

If f is continuous at (x_0, y_0) and g is a single-variable function continuous at $f(x_0, y_0)$, then the composite function $h = g \circ f$ defined by $h(x, y) = g(f(x, y))$ is continuous at (x_0, y_0).

$$\mathbb{R}^2 \xrightarrow{f} \mathbb{R} \xrightarrow{g} \mathbb{R}$$

$(x_0, y_0) \qquad f(x_0, y_0)$

Theorem (Continuity of Composites)

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Examples: The functions

$$e^{x-y}, \quad \cos\frac{xy}{x^2+1}, \quad \ln(1 + x^2y^2)$$

are continuous at every point (x, y)

$$\mathbb{R}^2 \longrightarrow \mathbb{R} \xrightarrow{g} \mathbb{R}$$

$f(x,y) = x - y \qquad g(t) = e^t$

$g \circ f(x,y)$

$= g(x-y)$

$= e^{x-y}$

Continuity of functions of three variables

Definition

A function $f(x, y, z)$ is said to be **continuous at the point (x_0, y_0, z_0)**, if

1. f is defined at (x_0, y_0, z_0),
2. $\lim\limits_{(x,y,z)\to(x_0,y_0,z_0)} f(x, y, z)$ exists,
3. $\lim\limits_{(x,y,z)\to(x_0,y_0,z_0)} f(x, y, z) = f(x_0, y_0, z_0)$.

A function $f(x, y, z)$ is said to be **continuous**, if it is continuous at every point of its domain.

Problems:

1. At what points (x, y, z) in space are the following functions continuous?
   1.1 $f(x, y, z) = e^{x+y} \cos z$
   1.2 $g(x, y, z) = \frac{1}{|xy|+|z|}$
   1.3 $h(x, y, z) = \frac{1}{4-\sqrt{x^2+y^2+z^2-1}}$.

$$\lim_{(x,y,z)\to(0,0,0)} e^{x+y} \cos z$$

$$= \lim_{(x,y,z)\to(0,0,0)} e^{x+y} \lim_{x,y,z\to(0,0,0)} \cos z$$

$$= e^0 \cdot \cos 0 = 1$$

$$|xy| + |z| \neq 0$$