Lecture 5

Length of polar curves and Sequences

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2. The area enclosed by one petal of $r = 6\cos(3\theta)$. **Ans:**
   $A = 1/2. \int_{-\pi/6}^{\pi/6} (6\cos(3\theta))^2 d\theta$

Examples

1. Find the area of the region in first quadrant bounded by $r = 4\sqrt{2}\cos(2\theta)$ and the line $\theta = \pi/8$.
Ans: $1/2 \int_{\pi/8}^{\pi/4} r^2 d\theta = \pi - 2$

2. Find the area of the region bounded by $r = 4\sin(2\theta)$ and $r = 4\cos(\theta)$
Ans: $\int_0^{\pi/6} (4\sin(2\theta))^2 d\theta + \int_{\pi/6}^{\pi/2} (4\cos(\theta))^2 d\theta = 4\pi - 3\sqrt{3}$
3. Find the area of the region bounded by $r = 4\sin(\theta)$ and $r = 4\cos(\theta)$
Ans: $1/2(\int_0^{\pi/4} (4\sin(\theta))^2 d\theta + \int_{\pi/4}^{\pi/2} (4\cos(\theta))^2 d\theta)$

4. Find the area of the region inside one loop of the lemniscate $r^2 = 4\sin(2\theta)$

Definition 1.

If $r = f(\theta)$ has a continuous first derivative for $\alpha \leq \theta \leq \beta$ and if the point $P(r, \theta)$ trace the curve exactly once as θ runs α to β then the length of the curve is

$$L = \int_{\alpha}^{\beta} \sqrt{r^2 + (\frac{dr}{d\theta})^2} d\theta$$

Example 8

Find the length of the curve $r = 4\sin\theta$

$$L = \int_0^{\pi} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

$$= 4\pi.$$

Sequences: Ch 10.1

- What is 'Sequence'?
  Ans: A list of objects (numbers) that are in order.

- Examples:
  1. $2, 4, 6, 8, 10, \ldots$ (Sequence of even positive integers)
  2. $1, 1/2, 1/3, \ldots$ (Sequence of reciprocals of $+$ve integers)

- Definition: A **sequence** of real numbers is a function $a : \mathbb{N} \to \mathbb{R}$ from the set $\mathbb{N}$ of natural numbers into the set $\mathbb{R}$ of real numbers.
- Sequences are useful in a number of mathematical disciplines for studying functions, spaces, and other mathematical structures. (In science and comuting e.g DNA sequencing and SQL database)

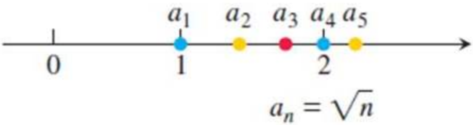
Representation of Sequences

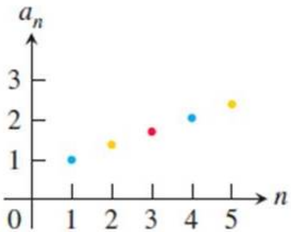
- How do we represent a sequence?
  $\{a_1, a_2, a_3, \ldots\}$, or $\left\{a_n\right\}_{n=1}^{\infty}$ or simply $\{a_n\}$.

- Examples:
  $\{a_n\} = \{\frac{n}{n+1}\}, \ \{b_n\} = \{\sqrt{n}\}, \ \{c_n\} = \{\frac{(-1)^n}{n}\}$ etc.

Graphical representation

- Representing a sequence graphically:





Important properties

- Consider the sequence

$$\{1, \frac{1}{4}, \frac{1}{9}, \frac{1}{16}, \ldots, \frac{1}{n^2}, \ldots\}.$$

Here the terms approach to 0 as n gets large.

- On the other hand, the sequence

$$\{\sqrt{1}, \sqrt{2}, \sqrt{3}, \sqrt{4}, \ldots, \sqrt{n}, \ldots\}$$

whose the terms get larger than any number as n increases.

- What about the sequence $\{1, -1, 1, \ldots, (-1)^{n+1}, \ldots\}$.??

Convergence and divergence

- The sequence $\{a_n\}$ **converges** to the number L if for every positive number ε there corresponds an integer N such that for all n,

$$n > N \qquad \Rightarrow \qquad |a_n - L| < \varepsilon.$$

- Here we call L as limit of the sequence. If no such number L exists, we say that $\{a_n\}$ **diverges**.
- If $\{a_n\}$ converges to L, we write $\lim_{n\to\infty} a_n = L$, or simply $a_n \to L$ as $n \to \infty$. We call L as the **limit** of the sequence $\{a_n\}$.

Convergence of Sequence

Examples

1. Show that (a) $\lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$; (b) $\lim_{n\to\infty} k = k$ (k is any real constant).

   Part (a): To show that $\lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$, for given $\varepsilon > 0$, we need to find an $N \in \mathbb{N}$ such that

   $$|1/\sqrt{n} - 0| < \varepsilon \text{ for every } n > N.$$

   It is clear that $|1/\sqrt{n} - 0| = 1/\sqrt{n} < \varepsilon$ if $n > 1/\varepsilon^2$. Hence we choose $N = [1/\varepsilon^2] + 1$.

Examples

1. Show that the sequence $\{(-1)^n\} = \{-1, 1, -1, \ldots (-1)^n, \ldots\}$ diverges.

   Proof: We will prove that the limit does not exist. Suppose the limit exists and let the limit be L.

   For $\varepsilon = 1/2 > 0$, by definition, there is a positive integer N such that for all $n > N$ we have that

$$\left|(-1)^n - L\right| < \frac{1}{2} \iff (-1)^n \in (L - \frac{1}{2}, L + \frac{1}{2})$$

   which is impossible as the interval $(L - \frac{1}{2}, L + \frac{1}{2})$ contains at most one of -1 and 1.

Diverging to $+\infty$

What can we say about the convergence of $\{n^2\}$?

- We say that the sequence $\{a_n\}$ **diverges to infinity** if for every number M there corresponds a positive integer N such that

$$a_n > M \quad \text{for all} \quad n > N.$$

- If this case we write

$$\lim_{n\to\infty} a_n = \infty \quad \text{or simply} \quad a_n \to \infty.$$

Diverging to $-\infty$

What can we say about the convergence of $\{-n\}$?

- We say that the sequence $\{a_n\}$ **diverges to negative infinity** if for every number m there is a positive integer N such that

$$a_n < m \quad \text{for all} \quad n > N$$

and in this case we write

$$\lim_{n \to -\infty} a_n = \infty \quad \text{or simply} \quad a_n \to -\infty.$$

Examples

1. Show that (a) $\lim_{n \to \infty} \sqrt{n} = \infty$; (b) $\lim_{n \to \infty} \frac{1 - n^3}{n^2} = -\infty$.