Implicit Differentiation & Directional Derivative

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Recall

If $w = f(x, y)$ is differentiable and if $x = x(t),\ y = y(t)$ are differentiable functions of t, then the composition $w = f(x(t), y(t))$ is a differentiable function of t and

$$\frac{dw}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

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Similar arguments hold for other forms. Check!

Examples (Using Chain Rule)

1. Evaluate dw/dt where $w = x^2 + y^2$ with $x = \cos t$, $y = \sin t$ at $t = \pi$.

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3. Find $\partial w/\partial u$ and $\partial w/\partial v$ where $w = xy + yz + xz$ with $x = u + v$, $y = u - v$ and $z = uv$ at $(u, v) = (1/2, 1)$. (Ans: -3/2)
4. Find $\partial w/\partial v$ when $u = v = 0$ if $w = x^2 + (y/x)$, $x = u - 2v + 1$, $y = 2u + v - 2$. (Ans: -7)

5. Let $w = f(x, y)$ and let (r, θ) denotes standard polar coordinates. Then,
   - Show that
$$\frac{\partial w}{\partial r} = f_x \cos\theta + f_y \sin\theta \text{ and } \frac{1}{r}\frac{\partial w}{\partial \theta} = -f_x \sin\theta + f_y \cos\theta.$$

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   - Express f_x and f_y in terms of $\partial w/\partial r$ and $\partial w/\partial \theta$.
   Solution:
   $$f_x = \cos\theta w_r - \frac{\sin\theta}{r} w_\theta,\ f_y = \sin\theta w_r + \frac{\cos\theta}{r} w_\theta.$$
   - Show that
   $$(f_x)^2 + (f_y)^2 = \left(\frac{\partial w}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial w}{\partial \theta}\right)^2.$$

Examples

5. Show that if $w = f(u, v)$ satisfies the Laplace equation $w_{uu} + w_{vv} = 0$ and if $u = (x^2 - y^2)/2$ and $v = xy$, then w satisfies the Laplace equation $w_{xx} + w_{yy} = 0$.

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$$w_x = \frac{\partial w}{\partial x} \quad = \frac{\partial w}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial w}{\partial v}\frac{\partial v}{\partial x} = x\frac{\partial w}{\partial u} + y\frac{\partial w}{\partial v}$$

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&= \frac{\partial w}{\partial u} + x\frac{\partial}{\partial x}\left(\frac{\partial w}{\partial u}\right) + y\frac{\partial}{\partial x}\left(\frac{\partial w}{\partial v}\right) = \frac{\partial w}{\partial u} + x\frac{\partial w_u}{\partial x} + y\frac{\partial w_v}{\partial x}\\
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&= w_u + x^2 w_{uu} + xyw_{uv} + xyw_{vu} + y^2 w_{vv}.
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$$= w_u + x^2 w_{uu} + xyw_{uv} + xyw_{vu} + y^2 w_{vv}.$$

Similarly, compute w_{yy} and hence,

$$w_{xx} + w_{yy} = (x^2 + y^2)(w_{uu} + w_{vv}) = 0.$$

A Formula for Implicit Differentiation

Theorem

Suppose that $F(x, y)$ is differentiable and that the equation $F(x, y) = 0$ defines y as a differentiable function of x. Then at any point where $F_y \neq 0$,

$$\frac{dy}{dx} = -\frac{F_x}{F_y}.$$

Here, x acts as both an intermediate variable and an independent variable. Since $F(x, y(x)) = 0$, the derivative dF/dx must be zero. Computing the derivative from chain rule, we have

$$0 = \frac{dF}{dx} = \frac{\partial F}{\partial x}\frac{dx}{dx} + \frac{\partial F}{\partial y}\frac{dy}{dx}$$

Simplifying this would give us desired expression.

Extension to Three Variables

Suppose that the equation $F(x, y, z) = 0$ defines the variable z implicitly as a function $z = f(x, y)$. Then for all (x, y) in the domain of f, we have $F(x, y, f(x, y)) = 0$. Assume that F and f are differentiable functions.

Extension to Three Variables

Suppose that the equation $F(x,y,z)=0$ defines the variable z implicitly as a function $z=f(x,y)$. Then for all (x,y) in the domain of f, we have $F(x,y,f(x,y))=0$. Assume that F and f are differentiable functions. Since $F(x,y,f(x,y))=0$, $\partial F/\partial x=0$. Using chain rule for $F(x,y,z)=0$ with respect to the independent variable x, we get

$$0=\frac{\partial F}{\partial x}\frac{\partial x}{\partial x}+\frac{\partial F}{\partial y}\frac{\partial y}{\partial x}+\frac{\partial F}{\partial z}\frac{\partial z}{\partial x}=\frac{\partial F}{\partial x}\cdot 1+\frac{\partial F}{\partial y}\cdot 0+\frac{\partial F}{\partial z}\frac{\partial z}{\partial x}.$$

Hence, we obtain

$$F_x+F_z\frac{\partial z}{\partial x}=0.$$

Similarly, $F_y+F_z\dfrac{\partial z}{\partial y}=0$. Using these, we conclude that whenever $F_z\neq 0$, we have

$$\frac{\partial z}{\partial x}=-\frac{F_x}{F_z} \text{ and } \frac{\partial z}{\partial y}=-\frac{F_y}{F_z}.$$

Implicit Function Theorem

Theorem

If the partial derivatives F_x, F_y, and F_z are continuous throughout an open region R in space containing the point (x_0, y_0, z_0), and if for some constant c, $F(x_0, y_0, z_0) = c$ and $F_z(x_0, y_0, z_0) \neq 0$, then the equation $F(x, y, z) = c$ defines z implicitly as a differentiable function of x and y near (x_0, y_0, z_0), and the partial derivatives of z are given by

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} \text{ and } \frac{\partial z}{\partial y} = -\frac{F_y}{F_z}.$$

Examples (Using Implicit Differentiation)

1. Assuming y as a differentiable function of x, find the value of dy/dx at
   - $(1, 1)$ where $x^3 - 2y^2 + xy = 0$.

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   - $(1, 1)$ where $x^3 - 2y^2 + xy = 0$. ($4/3$)
   - $(0, \ln 2)$ where $xe^y + \sin(xy) + y - \ln 2 = 0$. (-$(2 + \ln 2)$)
2. Find $\partial z/\partial x$ and $\partial z/\partial y$ at (π, π, π) where $\sin(x + y) + \sin(y + z) + \sin(x + z) = 0$. ($-1,\ -1$)
3. Find $\partial z/\partial x$ and $\partial z/\partial y$ at $(2, 3, 6)$ where $\dfrac{1}{x} + \dfrac{1}{y} + \dfrac{1}{z} - 1 = 0$. ($-9,\ -4$)

Directional derivative

Definition

The derivative of $f(x, y)$ at $P_0(x_0, y_0)$ in the direction of the unit vector $\boldsymbol{u} = u_1\boldsymbol{i} + u_2\boldsymbol{j}$ is the number

$$\left(\frac{df}{ds}\right)_{\boldsymbol{u},P_0} := \lim_{s\to 0} \frac{f(x_0 + u_1 s, y_0 + u_2 s) - f(x_0, y_0)}{s}$$

provided the limit exists.

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- $f_x(x_0, y_0)$ - directional derivative of f at P_0 in the direction of $\boldsymbol{i}$
- $f_y(x_0, y_0)$ - directional derivative of f at P_0 in the direction of $\boldsymbol{j}$

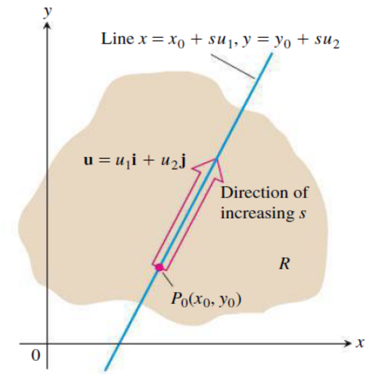

FIGURE 14.26 The rate of change of f in the direction of $\mathbf{u}$ at a point P_0 is the rate at which f changes along this line at P_0.

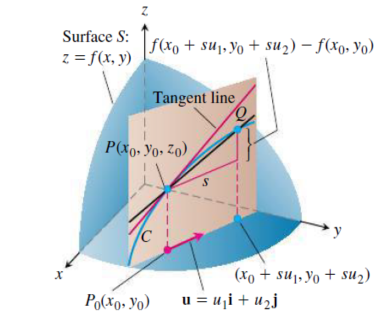

FIGURE 14.27 The slope of curve C at P_0 is $\lim_{Q \to P}$ slope (PQ); this is the directional derivative

$$\left(\frac{df}{ds}\right)_{\mathbf{u}, P_0} = (D_{\mathbf{u}} f)_{P_0}.$$

Examples

Find the derivative of the function at P_0 in the direction of $\boldsymbol{u}$.

1. $f(x, y) = 2xy - 3y^2,\ P_0(5, 5),\ \boldsymbol{u} = 4\boldsymbol{i} + 3\boldsymbol{j}.$

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$$\boldsymbol{v} = \frac{\boldsymbol{u}}{|\boldsymbol{u}|} = \frac{4\boldsymbol{i} + 3\boldsymbol{j}}{\sqrt{4^2 + 3^2}} = \frac{4}{5}\boldsymbol{i} + \frac{3}{5}\boldsymbol{j}.$$

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   The directional derivative of f at $(5, 5)$ in the direction of $\boldsymbol{v}$ is

$$\left(\frac{df}{ds}\right)_{\boldsymbol{v}, P_0} := \lim_{s \to 0} \frac{f(5 + (4/5)s, 5 + (3/5)s) - f(5, 5)}{s} = -4.$$

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2. $f(x,y,z) = xy + yz + zx,\ P_0(1,-1,2),\ \boldsymbol{u} = 3\boldsymbol{i} + 6\boldsymbol{j} - 2\boldsymbol{k}.$
   (Ans: 3)