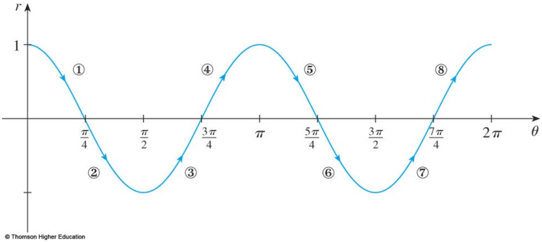
Lecture 3

Polar Coordinates

Text book chapter: 11.4

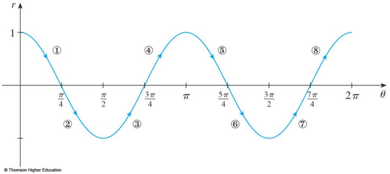
Example 3

Sketch the curve $r = \cos(2\theta)$.
We first sketch in cartesian coordinates.



In the figure part (1) shows that as θ increases from 0 to $\pi/4$, r decreases from 1 to 0. Similarly other parts corresponds to respective r values. It shows how r varies for $0 \leq \theta \leq 2\pi$.

Example-3



r
1
①
②
③
④
⑤
⑥
⑦
⑧
$\frac{\pi}{4}$
$\frac{\pi}{2}$
$\frac{3\pi}{4}$
π
$\frac{5\pi}{4}$
$\frac{3\pi}{2}$
$\frac{7\pi}{4}$
2π
θ
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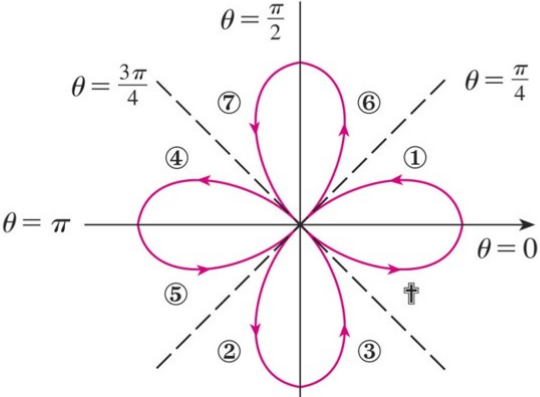


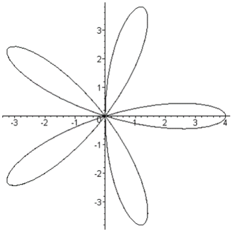
$\theta = \frac{\pi}{2}$
$\theta = \frac{3\pi}{4}$
$\theta = \frac{\pi}{4}$
$\theta = \pi$
$\theta = 0$
①
②
③
④
⑤
⑥
⑦

Roses

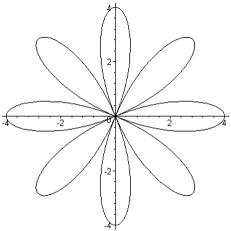
Roses are graphs of polar equations of the forms

$$r = a\sin(m\theta), \quad r = a\cos(m\theta)$$

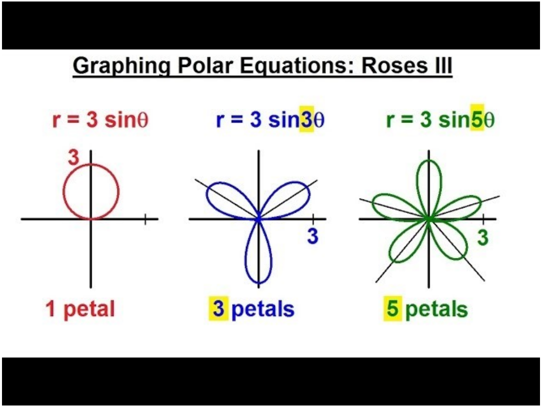
There are m petals if n is odd and 2m petals if n is even.



$r = 4\cos(5\theta)$

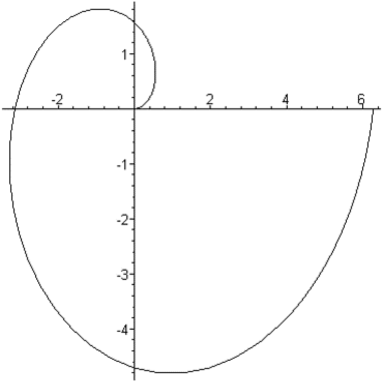


$r = 4\cos(4\theta)$


Graphing Polar Equations: Roses III
r = 3 sinθ
r = 3 sin3θ
r = 3 sin5θ
3
3
3
1 petal
3 petals
5 petals

Spirals

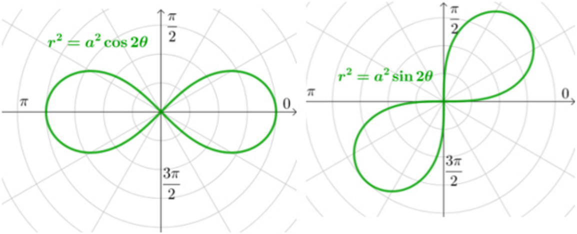
$$r = \pm\theta, r = e^{\theta}$$



Lemniscates

Lemniscates are graphs of polar equations of the form

$$r^2 = a^2 \sin(\theta), \quad r^2 = a\cos(\theta)$$



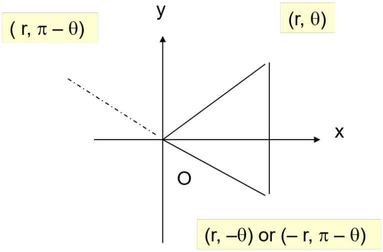
Symmetry

If the symmetries of the curve are known then the computation time can be reduced.

Some of the symmetries are:

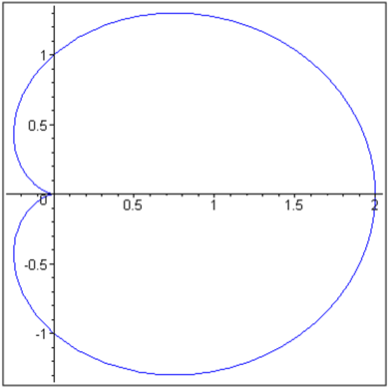
- Symmetry about the x-axis.
- Symmetry about the y-axis.
- Symmetry about the origin.

Symmetry about x-axis: If the point (r, θ) lies on the graph, then the point $(r, -\theta)$ or the point $(-r, \pi - \theta)$ also lies on the graph.

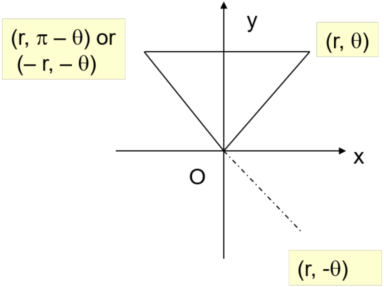


Example

The polar curve $r = 1 + \cos(\theta)$ is symmetric about $x-$axis.

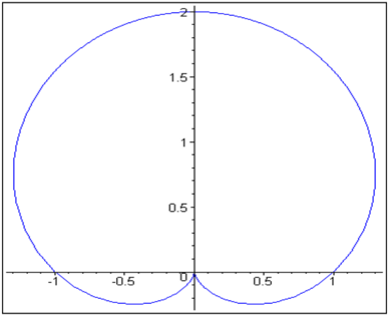


If the point (r, θ) lies on the graph, then the point $(-r, -\theta)$ or the point $(r, \pi - \theta)$ also lies on the graph.



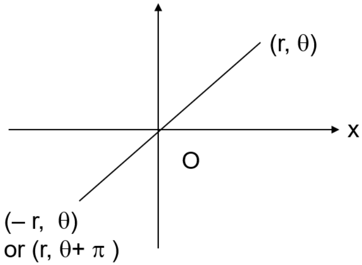
Example

The polar curve $r = 1 + \sin(\theta)$ is symmetric about $y-$axis.



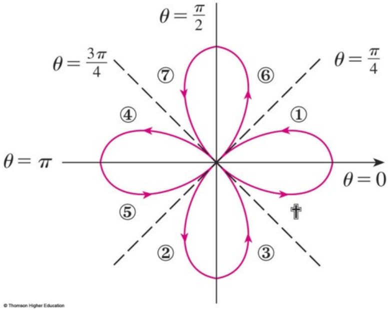
Symmetry about origin

If the point (r, θ) lies on the graph, then the point $(-r, \theta)$ or the point $(r, \pi + \theta)$ also lies on the graph.



Example

The polar curve of $r = \cos(2\theta)$ is symmetric about origin.



Symmetry Test

1. A curve in polar coordinate is symmetric about x- axis if we replace θ by $-\theta$ in the equation results an equivalent equation.
2. A curve in polar coordinate is symmetric about y- axis if we replace θ by $\pi - \theta$ in the equation results an equivalent equation.
3. A curve in polar coordinate is symmetric about origin if we replace θ by $\pi + \theta$ in the equation results an equivalent equation.

Tangent to polar curves

To find a tangent line to a polar curve $r = f(\theta)$, we regard θ as a parameter and write its parametric equations as:

$$x = r\cos\theta = f(\theta)\cos\theta$$

$$y = r\sin\theta = f(\theta)\sin\theta$$

Tangent to polar curve

To find the slope of the tangent to parametric curves, we have product rule:

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{dr/d\theta \sin\theta + r\cos\theta}{dr/d\theta \cos\theta - r\sin\theta}$$

Tangent to the polar curves

- Horizontal tangents can be found by setting $dy/d\theta = 0$ provided $dx/d\theta \neq 0$
- Vertical tangents can be found by setting $dx/d\theta = 0$ provided $dy/d\theta \neq 0$

Remark:

When we are finding the tangent line at the pole, then $r = 0$ and $\frac{dy}{dx} = \tan(\theta)$ provided $\frac{dr}{d\theta} \neq 0$. There could be many tangents with different slope at origin.

Example 4

1. Find the slope of the tangent line for the cardioid $r = 1 + \sin(\theta)$ at $\theta = \pi/3$
2. Find the points on the cardioid where the tangent line is horizontal or vertical.

Example 4

We have

$$\frac{dy}{dx} = \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta}$$

and $r = 1 + \sin\theta$, simplifying, we got

$$\frac{dy}{dx} = \frac{\cos\theta(1 + 2\sin\theta)}{(1 + \sin\theta)(1 - 2\sin\theta)}$$

The slope of the tangent at $\pi/3$ is $\frac{dy}{dx}$ evaluated at $\pi/3$ which is -1.

Example 4

Hence there are horizontal tangents at the points

$$(2, \pi/2), (1/2, 7\pi/6), (1/2, 11\pi/6)$$

and vertical tangents at

$$(3/2, \pi/6), (3/2, 5\pi/6)$$

At $\theta = 3\pi/2$, both $dy/d\theta$ and $dx/d\theta$ are zero, so we find the limit next.

Example 4

We have

$$\lim_{\theta\to(3\pi/2)^-} \frac{dy}{dx}$$
$$= \left(\lim_{\theta\to(3\pi/2)^-} \frac{1+2\sin\theta}{1-2\sin\theta}\right)\left(\lim_{\theta\to(3\pi/2)^-} \frac{\cos\theta}{1+\sin\theta}\right)$$
$$= -\frac{1}{3}\lim_{\theta\to(3\pi/2)^-} \frac{\cos\theta}{1+\sin\theta}$$
$$= -\frac{1}{3}\lim_{\theta\to(3\pi/2)^-} \frac{-\sin\theta}{\cos\theta} = \infty$$

Thus, there is a vertical tangent line at the pole.

Example 4

We have