MATHEMATICS-I

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Lecture 13

Power series

Corollary 0.1.

The convergence of the series $\sum a_n(x-a)^n$ *is described by one of the following three possibilities:*

- *There is a positive number* R *such that the series diverges for* x *with* $|x-a| > R$ *but converges absolutely for* x *with* $|x-a| < R$. *The series may or may not converge at either of the endpoints* $x = a - R$ *and* $x = a + R$.

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- *The series converges absolutely for every* x $(R = \infty)$.
- *The series converges at* $x = a$ *and diverges elsewhere* $(R = 0)$.

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2. The interval of radius R centered at $x = a$ is called the **interval of convergence**. The interval of convergence may be open, closed, or half-open, depending on the particular series.

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- Therefore, the radius of convergence is $R = 1$ and $(-1, 1]$ is the interval of convergence

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- Therefore, the radius of convergence is $R = 1/24$ and $[5/24, 7/24)$ is the interval of convergence

Theorem 0.3 (Multiplication Theorem for Power Series).

If $A(x) = \sum a_n x^n$ and $B(x) = \sum b_n x^n$ coverge absolutely for all $|x| < r$ and

$$c_n = a_n b_0 + a_{n-1} b_1 + a_{n-2} b_2 + \cdots + a_0 b_n = \sum_{k=0}^{n} a_{n-k} b_k,$$

then $\sum c_n x^n$ converges to $A(x)B(x)$ absolutely for $|x| < r$:

$$\left(\sum^{\infty} a_n x^n\right) \cdot \left(\sum^{\infty} b_n x^n\right) = \sum^{\infty} c_n x^n.$$

The Term-by-Term Differentiation Theorem

Theorem 0.4.

If $\sum a_n(x-a)^n$ has radius of convergence $R > 0$, it defines a function

$$f(x) = \sum_{n=0}^{\infty} a_n(x-a)^n \quad \textit{on the interval} \quad a - R < x < a + R.$$

This function f has derivatives of all orders inside the interval, and we obtain the derivatives by differentiating the original series term by term:

$$f'(x) = \sum_{n=1}^{\infty} na_n(x-a)^{n-1}; \; f''(x) = \sum_{n=2}^{\infty} n(n-1)a_n(x-a)^{n-2},$$

and so on. Each of these derived series converges at evert point of the interval $a - R < x < a + R$.

Theorem 0.5.

Suppose that $f(x) = \sum_{n=0}^{\infty} a_n (x-a)^n$ *converges for all* $|x-a| < R$*. Then* $\sum_{n=0}^{\infty} a_n \frac{(x-a)^{n+1}}{n+1}$ *converges for all* $|x-a| < R$ *and*

$$\int f(x)\,dx = \sum_{n=0}^{\infty} a_n \frac{(x-a)^{n+1}}{n+1} + C$$

for $|x-a| < R$.

Example

By knowing that $\dfrac{1}{1-x} = 1 + x + x^2 + \cdots$ for $|x| < 1$, show that the following:

1. $\dfrac{1}{(1-x)^2} = 1 + 2x + 3x^2 + \cdots = \displaystyle\sum_{n=1}^{\infty} nx^{n-1}$ for $|x| < 1$.

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2. $$\frac{1}{(1-x)^3} = 1 + 3x + 6x^2 + \cdots = \sum_{n=2}^{\infty} \frac{n(n-1)}{2} x^{n-2} \text{ for } |x| < 1.$$

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2. $\frac{1}{(1-x)^3} = 1 + 3x + 6x^2 + \cdots = \sum_{n=2}^{\infty} \frac{n(n-1)}{2} x^{n-2}$ for $|x| < 1$.
3. $\ln(1-x) = x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots = \sum_{n=1}^{\infty} \frac{1}{n} x^n$ for $|x| < 1$.

Questions

1. Find a power series representation for the following function and determine its interval of convergence.

$$f(x) = \frac{1}{1+x^3}$$

2. Find the interval of convergence and radius of convergence:
   - (a) $\displaystyle\sum_{n=1}^{\infty} (-3)^{n-1} \frac{(x-1)^n}{n}$

10.8 Taylor and Maclaurin Series

Introduction

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$$g(x) = \sum_{n=1}^{\infty} a_n(x-a)^n$$

is differentiable for infinitely many times on $(a-R, a+R)$.

But what about the other way around?

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If a function $f(x)$ has derivatives of all orders on an interval, can it be expressed as a power series on at least some part of that interval?

And if it can, what are its coefficients?

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$$\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k = f(a)+f'(a)(x-a)+\frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \cdots .$$

Definition: the **Maclaurin series generated by** f is

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!}\, x^k = f(0) + f'(0)\, x + \frac{f''(0)}{2!}\, x^2 + \cdots + \frac{f^{(n)}(0)}{n!}\, x^n + \cdots \,,$$

the Taylor series generated by f at $x = 0$.