Sequence and Series

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Recall

$\sum_{n=1}^{\infty} a_n$ if $a_n > 0$ then we have the following test

1. Integral test: Find f s.t f is cont., decreasing, +ve s.t $f(n) = a_n$ if $n \geq N$ then

$\int_N^{\infty} f(x)\,dx$ converge (or diverge) $\Rightarrow$ $\sum_{n=N}^{\infty} a_n$ converge (or diverge)

2. Direct comparision test

if $\sum a_n \leq \sum b_n$ and $\sum b_n$ converge then $\sum a_n$ converge

if $\sum a_n \leq \sum b_n$ and $\sum a_n$ diverge then $\sum b_n$ diverge

3. Limit comparision test

4. Alternating series test: $\sum_{n=1}^{\infty} (-1)^{n+1} a_n$ if $a_n > 0$, $a_{n+1} < a_n$, $a_n \to 0$ as $n \to \infty$

then this series converges

Absolute convergence

Examples $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ converges

$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^p}$ $0<p<1$

– converges

Definition

A series $\sum a_n$ converges absolutely (is absolutely convergent) if the corresponding series of absolute values, $\sum |a_n|$ converges.

$\sum \left(\frac{-1}{4}\right)^n$ converges absolutely as $\sum \left(\frac{1}{4}\right)^n$ converges.

$\sum \left|\left(\frac{-1}{4}\right)^n\right| = \sum \left(\frac{1}{4}\right)^n$ converges

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Theorem

Absolute convergent test: *If $\sum |a_n|$ converges, then $\sum a_n$ converges.*

Absolute convergence $\Rightarrow$ Convergence.

$\sum \left(-\frac{1}{4}\right)^n$ converges.

Ex.

Proof

$$\boxed{-|a_n| \leq a_n \leq |a_n|}$$

$\Rightarrow \quad 0 \leq a_n + |a_n| \leq 2|a_n|$

<u>Given</u> $\quad \sum |a_n|$ converges $\Rightarrow \quad \sum |2a_n|$ converges

$\Rightarrow \quad \sum (a_n + |a_n|)$ converges   Using direct comparison test.

$a_n = a_n + |a_n| - |a_n|$

$\sum a_n = \sum (a_n + |a_n|) - \sum |a_n|$ — converges

Proof

$$-|a_n| \leq a_n \leq |a_n| \implies 0 \leq a_n + |a_n| \leq 2|a_n|.$$

$\displaystyle\sum_{n=1}^{\infty} |a_n|$ converges $\implies \displaystyle\sum_{n=1}^{\infty} 2|a_n|$ converges. Thus By direct comparison test, $\displaystyle\sum_{n=1}^{\infty} a_n + |a_n|$ converges.

Now $a_n = (a_n + |a_n| - |a_n|)$ and $\displaystyle\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} (a_n + |a_n|) - \sum_{n=1}^{\infty} |a_n|$.

Thus $\sum a_n$ converges.

Examples

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$$

$\sum \frac{1}{n^2}$ converges

$\Rightarrow \sum \frac{(-1)^{n+1}}{n^2}$ is absolutely convergent

$\Rightarrow \sum \frac{(-1)^{n+1}}{n^2}$ convergent.

Examples

$\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$ converges absolutely and hence $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$ converges by the absolute convergent theorem.

$$\sum_{n=1}^{\infty} \frac{\sin n}{n^2}$$

$$0 \le \frac{|\sin n|}{n^2} \le \frac{1}{n^2}$$

$\because \sum \frac{1}{n^2}$ converges $\Rightarrow \sum \frac{|\sin n|}{n^2}$ converges

$\Rightarrow \sum \frac{\sin n}{n^2}$ converges absolutely

$\Rightarrow \sum \frac{\sin n}{n^2}$ converges

Examples

$\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$ converges absolutely and hence $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$ converges by the absolute convergent theorem.

$\displaystyle\sum_{n=1}^{\infty} \frac{\sin n}{n^2}$ converges absolutely and hence $\displaystyle\sum_{n=1}^{\infty} \frac{\sin n}{n^2}$ converges.

What about converse?

Does convergence $\Longrightarrow$ absolute convergence

No!

Ex $\sum \frac{(-1)^{n+1}}{n}$ converges

whereas $\sum \frac{(-1)^{n+1}}{n}$ is not absolute convergent..

Examples

$\sum_{n=1}^{\infty} \left| \frac{(-1)^{n+1}}{n^2} \right|$ converges absolutely and hence $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$ converges by the absolute convergent theorem.

$\sum_{n=1}^{\infty} \left| \frac{\sin n}{n^2} \right|$ converges absolutely and hence $\sum_{n=1}^{\infty} \frac{\sin n}{n^2}$ converges.

<u>What about converse?</u>

Remark

Converse not true: $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ *converges, but* $\sum_{n=1}^{\infty} \frac{1}{n}$ *doesn't converge, hence* $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ *doesn't converge absolutely.*

Definition

A series that is convergent but not absolutely convergent is called conditionally convergent.

$\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{n}$ is conditionally convergent.(converges, but not absolutely convergent.)

$\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{n^p}, 0 < p < 1$ is conditionally convergent.

Rearranging terms in a series

We know $\sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{n}$ converges and say it converges to L. (hence conditional convergent, but not absolute convergent).

$$L = \sum \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \frac{1}{9} - \frac{1}{10} + \frac{1}{11} + \cdots$$

$$2L = 2 - 1 + \frac{2}{3} - \frac{1}{2} + \frac{2}{5} - \frac{1}{3} + \frac{2}{7} - \frac{1}{4} + \frac{2}{9} - \frac{1}{5} + \frac{2}{11} - \cdots$$

$$= (2-1) - \frac{1}{2} + \left(\frac{2}{3} - \frac{1}{3}\right) - \frac{1}{4} + \left(\frac{2}{5} - \frac{1}{5}\right) - \frac{1}{6} + - \cdot$$

$$= 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - $$

$$= L$$

Rearranging terms in a series

We know $\sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{n}$ converges and say it converges to L. (hence conditional convergent, but not absolute convergent).

$$\begin{aligned}
2L =& 2\sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{n} = 2\left(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \frac{1}{9} - \frac{1}{10} \cdots\right) \\
=& 2 - 1 + \frac{2}{3} - \frac{1}{2} + \frac{2}{5} - \frac{1}{3} + \frac{2}{7} - \frac{1}{4} + \frac{2}{9} - \frac{1}{5} + \frac{2}{11} \cdots \\
=& (2-1) - \frac{1}{2} + \left(\frac{2}{3} - \frac{1}{3}\right) - \frac{1}{4} + \left(\frac{2}{5} - \frac{1}{5}\right) - \frac{1}{6} + \left(\frac{2}{7} - \frac{1}{7}\right) - \frac{1}{8} + \cdots \\
=& 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots \\
=& L.
\end{aligned}$$

This shows that we cannot rearrange the terms of a conditionally convergent series and expect the new series to be the same as the original one. Can we rearrange the terms of an absolute convergent series?

Theorem

The Rearrangement Theorem for Absolutely Convergent Series: *If* $\sum_{n=1}^{\infty} a_n$ *converges absolutely, and* $b_1, b_2, \cdots, b_n, \cdots$ *is any arrangement of the sequence* (a_n)*, then* $\sum_{n=1}^{\infty} b_n$ *converges absolutely and* $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} b_n$.

Example

$$\sum_{n=0}^{\infty}(-1)^n\left(\frac{1}{3}\right)^n \quad - \text{ absolutely convergent.}$$

$$= 1 - \frac{1}{3} + \frac{1}{3^2} - \frac{1}{3^3} + \frac{1}{3^4} - \cdots$$

$$= 1 + \frac{1}{3^2} + \frac{1}{3^4} + \frac{1}{3^6} + \cdots - \left(\frac{1}{3} + \frac{1}{3^3} + \frac{1}{3^5} + \cdots\right)$$

$$= \frac{1}{1-\frac{1}{9}} - \left(\frac{1/3}{1-\frac{1}{9}}\right)$$

$$= \frac{9}{8} - \frac{1}{\cancel{3}} \times \frac{\cancel{9}^{\,3}}{8} = \frac{6}{8} = \frac{3}{4}$$

Ratio test

Theorem

Let $\sum_{n=0}^{\infty} a_n$ *be any series and suppose that* $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = r$. *Then*

- (a) *the series converges absolutely if* $r < 1$*,*
- (b) *the series diverges if* $r > 1$ *or* r *is infinite,*
- (c) *the test is inconclusive if* $r = 1$.

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If we apply ratio test for $\sum_{n=0}^{\infty} \frac{1}{n}$ (diverges) and $\sum_{n=0}^{\infty} \frac{1}{n^2}$ (converges), both cases $r = 1$.

$a_n = \frac{1}{n}$

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \frac{n}{n+1} = \lim_{n\to\infty} \frac{1}{1+1/n} = 1$$

Examples.

Investigate the convergence of the following series:

- $\displaystyle\sum_{n=0}^{\infty} \frac{2^n+5}{3^n}$,

$$a_n = \frac{2^n+5}{3^n}$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{2^{n+1}+5}{3^{n+1}} \times \frac{3^n}{2^n+5}$$

$$= \lim_{n\to\infty} \frac{1}{3} \times \frac{2+5/2^n}{1+5/2^n} = \frac{2}{3} < 1$$

Examples.

Investigate the convergence of the following series:

- $\displaystyle\sum_{n=0}^{\infty} \frac{2^n+5}{3^n}$, $|\frac{a_{n+1}}{a_n}| \to \frac{2}{3}$, so converges.
- $\displaystyle\sum_{n=0}^{\infty} \frac{(2n)!}{(n!)^2}$,

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \frac{(2(n+1))!}{(n+1)! \times (n+1)!} \times \frac{n! \times n!}{2n!}$$

$$= \lim_{n\to\infty} \frac{2(n+1)(2n+1)}{(n+1)(n+1)}$$

$$= 4 > 1$$

$\Rightarrow$ diverges

Examples.

Investigate the convergence of the following series:

- $\displaystyle\sum_{n=0}^{\infty} \frac{2^n + 5}{3^n}$, $\left|\frac{a_{n+1}}{a_n}\right| \to \frac{2}{3}$, so converges.
- $\displaystyle\sum_{n=0}^{\infty} \frac{(2n)!}{(n!)^2}$, $\left|\frac{a_{n+1}}{a_n}\right| \to 4$, so diverges.
- $\displaystyle\sum_{n=0}^{\infty} \frac{4^n (n!)^2}{(2n!)}$,

✓

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{4^{n+1}}{4^n} \left\{ \frac{(n+1)!\,(n+1)!}{(2n+2)!} \times \frac{2n!}{n!\times n!} \right.$$

$$= 1$$

Examples.

Investigate the convergence of the following series:

- $\displaystyle\sum_{n=0}^{\infty} \frac{2^n + 5}{3^n}$, $|\frac{a_{n+1}}{a_n}| \to \frac{2}{3}$, so converges.
- $\displaystyle\sum_{n=0}^{\infty} \frac{(2n)!}{(n!)^2}$, $|\frac{a_{n+1}}{a_n}| \to 4$, so diverges.
- $\displaystyle\sum_{n=0}^{\infty} \frac{4^n (n!)^2}{(2n!)}$, $|\frac{a_{n+1}}{a_n}| \to 1$, so ratio test is inconclusive. Can you apply any other test to conclude?

$$a_n = \frac{4^n (n!)^2}{(2n)!}$$

$\Rightarrow \sum \frac{4^n (n!)^2}{2n!}$ diverges

check a_n increasing when n increasing

$\Rightarrow a_n \not\to 0$ as $n \to \infty$

Root test

Theorem

Let $\sum_{n=0}^{\infty} a_n$ *be any series and suppose that* $\lim_{n\to\infty} \sqrt[n]{|a_n|} = r$. *Then*

- *(a) the series converges absolutely if* $r < 1$,
- *(b) the series diverges if* $r > 1$ *or* r *is infinite,*
- *(c) the test is inconclusive if* $r = 1$.

Examples. Investigate the convergence of the following series:

- $\sum_{n=0}^{\infty} \frac{n^2}{2^n}$,

$$\lim_{n\to\infty} |a_n|^{1/n} = \lim_{n\to\infty} \left(\frac{n^2}{2^n}\right)^{1/n}$$

$$= \lim_{n\to\infty} \frac{(n^{1/n})^2}{2} = \frac{1}{2} < 1$$

$$\boxed{\lim_{n\to\infty} n^{1/n} = 1}$$

Root test

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Examples. Investigate the convergence of the following series:

- $\sum_{n=0}^{\infty} \frac{n^2}{2^n}$, $\lim_{n\to\infty} \sqrt[n]{|a_n|} = \frac{1}{2}$, so converges.
- $\sum_{n=0}^{\infty} \frac{2^n}{n^3}$,

$$\lim_{n\to\infty} |a_n|^{1/n} = \lim_{n\to\infty} \left(\frac{2^n}{n^3}\right)^{1/n}$$

$$= \lim_{n\to\infty} \frac{2}{(n^{1/n})^3} = 2 > 1$$

Root test

Theorem

Let $\sum_{n=0}^{\infty} a_n$ *be any series and suppose that* $\lim_{n\to\infty} \sqrt[n]{|a_n|} = r$. *Then*

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- $\sum_{n=0}^{\infty} \frac{n^2}{2^n}$, $\lim_{n\to\infty} \sqrt[n]{|a_n|} = \frac{1}{2}$, so converges.
- $\sum_{n=0}^{\infty} \frac{2^n}{n^3}$, $\lim_{n\to\infty} \sqrt[n]{|a_n|} = 2$, so diverges.