Differentials & Extreme Values

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BITS Pilani, K K Birla Goa Campus

October 25, 2024

ANNOUNCEMENT:

An additional class will be held this **Saturday (26 October 2024)** from **12:00 PM to 1:00 PM** in **LT3**.

Recall

Let f be a function with derivatives of all orders throughout some interval containing x_0 as an interior point. The Taylors series expansion of $f(x)$ at $x = x_0$ is

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$$L(x, y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0).$$

The approximation $f(x, y) \approx L(x, y)$ is called the standard linear approximation of the function f at (x_0, y_0).

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Write down the equation of tangent plane of the surface $z = f(x, y)$ and compare it with $L(x, y)$ of $f(x, y)$.

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The error $E(x, y) = f(x, y) - L(x, y)$ in the standard linear approximation of a function f can be estimated in the following way.

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Error in the standard linear approximation

If f has continuous second partial derivatives throughout an open set containing a rectangle R centered at (x_0, y_0) $(R := \{(x, y) : |x - x_0| \leq h, |y - y_0| \leq k\})$ and if M is an upper bound for the values of $|f_{xx}|, |f_{xy}|$ and $|f_{yy}|$ in the rectangle R (that is, $M = \max\{|f_{xx}|, |f_{xy}|, |f_{yy}|\}$ on R), then for any $(x, y) \in R$,

$$|E(x, y)| \leq \frac{1}{2} M \left(|x - x_0| + |y - y_0|\right)^2 .$$

Example

Find the linearization of $f(x,y) = \ln x + \ln y$ at $(1,1)$ and the error of linearisation in the rectangle $R : |x-1| \leq 0.2, |y-1| \leq 0.2.$

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$\nabla f|_{(1,1)} = (1/x, 1/y)|_{(1,1)} = (1,1)$ and $L(x,y) = x + y - 2$.

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And $|f_{xx}| = |-1/x^2| \le \frac{1}{(0.8)^2} = 100/64$, since in the rectangle R, we have $0.8 \le x \le 1.2$ and $0.8 \le y \le 1.2$. Similarly, $|f_{yy}| \le 100/64$.

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$$|E(x,y)| \leq \frac{1}{2} M(|x - x_0| + |y - y_0|)^2.$$

Hence,
$|E(x,y)| \leq 1/2 \times 100/64 \times (0.2 + 0.2)^2 = 1/2 \times 100/64 \times 0.16 = 1/8.$

Total differential for the functions of two variables

Definition

If we move from the point (x_0, y_0) to a point $(x_0 + dx, y_0 + dy)$ nearby then the resulting change in the linearization of the function

$$df = f_x(x_0, y_0)dx + f_y(x_0, y_0)dy$$

is called the total differential of f and dx, dy are called differentials.

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Suppose that R is a closed rectangular solid centered at (x_0,y_0,z_0) and lying in an open region on which the second partial derivatives of f are continuous. Suppose also that$|f_{xx}|, |f_{xy}|, |f_{yy}|, |f_{xy}|, f_{yz}|$, and $|f_{xz}|$ are less than or equal to M throughout R (that is,
$M = \max\{|f_{xx}|, |f_{xy}|, |f_{yy}|, |f_{xy}|, f_{yz}|, |f_{xz}|\}$), then the error in the standard linear approximation is bounded by

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The total differential is given by

$$df = f_x(x_0, y_0, z_0)dx + f_y(x_0, y_0, z_0)dy + f_z(x_0, y_0, z_0)dz.$$

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2. Find an upper bound for the magnitude of the error in the approximation of $f(x, y, z) = xz - 3yz + 2$ at $P_0(1, 1, 2)$, where $R : |x - 1| \leq 0.01,\ |y - 1| \leq 0.01,\ |z - 2| \leq 0.02$.

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   $$|E(x,yz)| \leq \frac{1}{2}(3)(0.01 + 0.01 + 0.02)^2 = 0.0024.$$

Examples

3. Find an upper bound for the magnitude of the error of the linear approximation of the function $f(x, y) = 1 + y + x\cos y$ at the point (0,0) for the rectangle $|x| \leq 0.2$ and $|y| \leq 0.2$.

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   So we can choose our bound M to be 0.2.

   Hence, the error is given by

$$1/2 \times 0.2 \times (0.2 + 0.2)^2 = 1/2 \times 0.032 = 0.016.$$

Extreme Values

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Let $f(x, y)$ be defined on a region R containing the point (a, b).

1. $f(a, b)$ is a local maximum value of f if $f(a, b) \geq f(x, y)$ for all domain points (x, y) in an open disk centered at (a, b).
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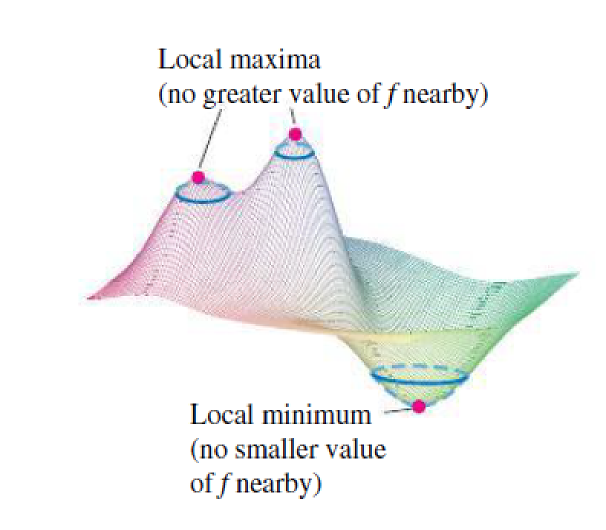
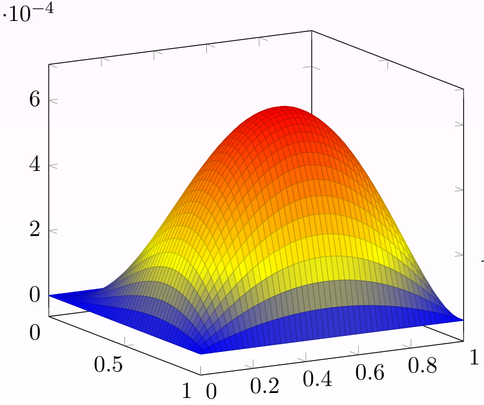
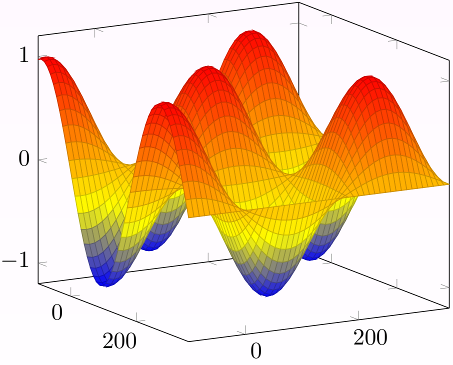
The local maxima and local minima are together some time called local extrema values or relative extrema.

Local maximum and minimum



Figures





Theorem (First Derivative Test for Local Extreme Values)

If $f(x, y)$ has a local maximum or minimum value at an interior point (a, b) of its domain and if the first partial derivatives exist there, then

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Hence by the single variable theory we know $g'(a) = 0 = f_x(a, b)$.
A similar arguments with $h(y) = f(a, y)$ shows that $f_y(a, b) = 0$.

Critical Points

Definition

An interior point of the domain of a function f is called a critical point of the function if

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Just as in the one variable situation we could have two dimensional analogue of "inflection points". These analogues are called "saddle points" of the function.

Saddle Points

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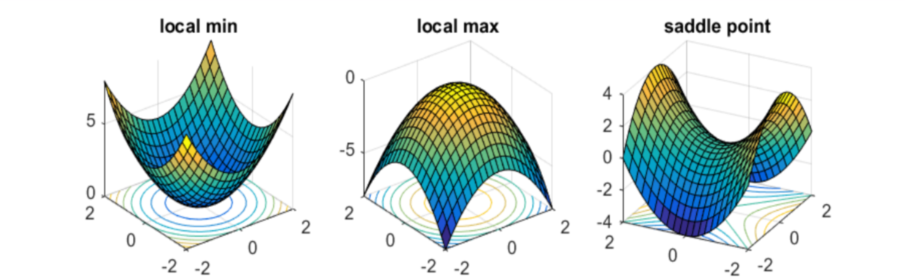

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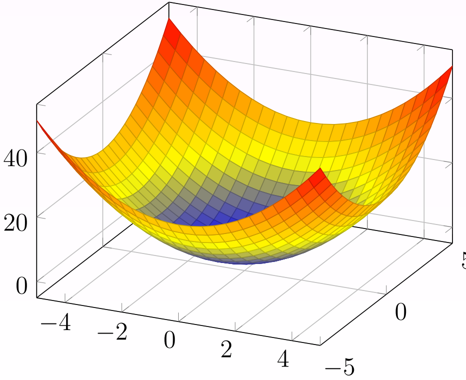

Examples

- Find the local extreme values of $f(x, y) = y^2 - x^2$.
- Find the critical points of $f(x, y) = x^2 + 2xy$.
- Find the critical points of $f(x, y) = y \sin x$.