MATH F111- Mathematics I

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Definition

A power series about $x = 0$ is a series of the form

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n + \cdots$$

A power series about $x = a$ is a series of the form

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1 (x-a) + c_2 (x-a)^2 + \cdots + c_n (x-a)^n + \cdots$$

in which the center a and the coefficients $c_0, c_1, c_2, \cdots, c_n, \cdots$ are constants.

When we fix a value for x, say $x = 1$, the power series

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is an infinite series whose convergence or divergence can be investigated.

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For $x = 2$, the power series $\displaystyle\sum_{n=0}^{\infty} c_n x^n = c_0 + 2c_1 + 4c_2 + \cdots + 2^n c_n + \cdots$.

If for an x, the series $\displaystyle\sum_{n=0}^{\infty} c_n x^n$ converges we can use the limit of partial sequences to define a function f at x.

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- We will see that a power series defines a function $f(x)$ on a certain interval where it converges.
- Finding this interval of convergence is important. Moreover, this function will be shown to be continuous and differentiable inside the interval.

Let us consider some familiar power series.

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This is the geometric series with first term 1 and common ratio x. It converges to $\frac{1}{(1-x)}$ for $|x| < 1$. We express this fact by writing

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \cdots + x^n + \cdots , -1 < x < 1.$$

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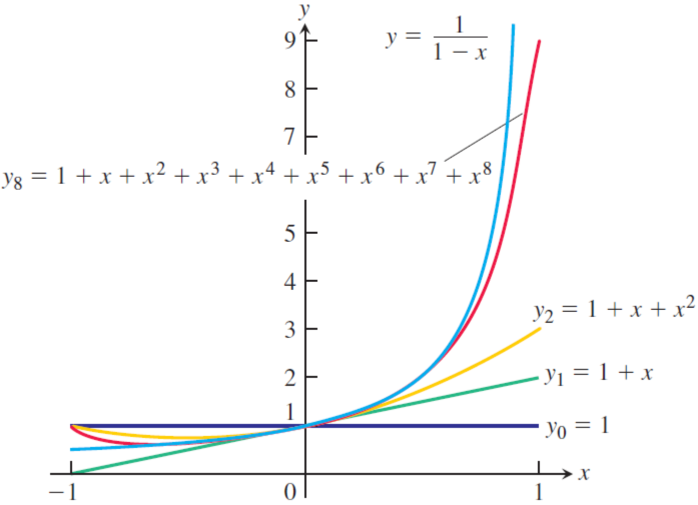
We think of the partial sums of the series on the right as polynomials $P_n(x)$ that approximate the function on the left.

$$\begin{aligned} P_1(x) &= 1 \\ P_2(x) &= 1 + x \\ P_3(x) &= 1 + x + x^2 \\ P_n(x) &= 1 + x + x^2 + \cdots + x^n \end{aligned}$$



For values of x near zero, we need take only a few terms of the series to get a good approximation. The approximations do not apply when $|x| > 1$.

Consider the power series

$$1 - \frac{1}{2}(x-2) + \frac{1}{4}(x-2)^2 + \cdots + (-\frac{1}{2})^n (x-2)^n + \cdots$$

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which simplifies to $0 < x < 4$. The sum is $\frac{1}{1-r} = \frac{2}{x}$. Thus

$$\frac{2}{x} = 1 - \frac{1}{2}(x-2) + \frac{1}{4}(x-2)^2 + \cdots + (-\frac{1}{2})^n(x-2)^n + \cdots, 0 < x < 4.$$

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- At $x = -1$, we get the harmonic series $1 + \frac{1}{2} + \frac{1}{3} + \cdots$ which diverges. Thus $\displaystyle\sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{n}$ converges for $-1 < x \leq 1$ and diverges elsewhere.

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- The series converges absolutely and hence converges for all $x \in \mathbb{R}$.

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The Convergence Theorem for Power Series

Theorem

If the power series

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converges at $x = c$, then it converges absolutely for all x with $|x| < |c|$.If the series diverges at $x = d$, then it diverges for all x with $|x| > |d|$.

- $\sum_{n=0}^{\infty}(-1)^{n-1}\frac{x^n}{n}$: Interval of convergence is $x \in (-1, 1]$.

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Definition

The maximum value of R such that the series $\sum_{n=0}^{\infty} a_n x^n$ converges for $x \in (-R, R)$ is called Radius of Converegence.

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- At $x = \pm R$, the series may or may not converge.
- When $x \in (-\infty, -R) \cup (R, \infty)$ the series diverges.

Radius of convergence

For series of the form $\sum_{n=0}^{\infty} a_n(x-a)^n$, we can replace $x-a$ by y and apply the results to the series $\sum_{n=0}^{\infty} a_n y^n$. The maximum range R for which the series $\sum_{n=0}^{\infty} a_n(x-a)^n$ converges for $a-R < x < a+R$ is called the radius of convergence of the power series, and the interval of radius R centered at $x=a$ is called the interval of convergence.

Possibilities of convergence

conv.png

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How to calculate R?

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How to calculate R?

- Use ratio test or root test to find R such that in $|x - a| < R$, the power series $\sum_{n=0}^{\infty} a_n (x-a)^n$, converges.
- Check for the convergence at $|x - a| = R$ to conclude if R also a part of interval of convergence.

  (This is already did for $\sum_{n=0}^{\infty}(-1)^{n-1}\frac{x^n}{n}$ and $\sum_{n=0}^{\infty}\frac{x^n}{n!}$)

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Theorem

If $A(x) = \sum_{n=0}^{\infty} a_n x^n$ *and* $B(x) = \sum_{n=0}^{\infty} b_n x^n$ *converge absolutely for* $|x| < R$,

and $c_n = a_0 b_n + a_1 b_{n-1} + a_2 b_{n-2} + \cdots + a_n b_0 = \sum_{k=0}^{n} a_k b_{n-k}$, *then*

$\sum_{n=0}^{\infty} c_n x^n$ *converges absolutely to* $A(x)B(x)$ *for* $|x| < R$.

$$\left(\sum_{n=0}^{\infty} a_n x^n\right)\left(\sum_{n=0}^{\infty} b_n x^n\right) = \sum_{n=0}^{\infty} c_n x^n$$

Theorem

If $\sum_{n=0}^{\infty} a_n x^n$ *converges absolutely for* $|x| < R$ *and* f *is a continuous function, then* $\sum_{n=0}^{\infty} a_n f(x)^n$ *converges absolutely on the set of points* x *where* $|f(x)| < R$.

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Remark

Note that $\sum_{n=0}^{\infty} a_n f(x)^n$ *may not be a power series, it will depend upon the choice of* f*. But using comparison test we can conclude about absolute convergence of* $\sum_{n=0}^{\infty} a_n f(x)^n$*, provided* $|f(x)| < R$*.*

1. Find the interval of convergence for $\sum_{n=0}^{\infty}(4x^2)^n$. To which function this series will converge in this interval?

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$\sum_{n=0}^{\infty}(4x^2)^n$ converges absolutely to $\frac{1}{1-4x^2}$ when x satisfies $|4x^2| < 1$ or equivalently when $|x| < \frac{1}{2}$.

Theorem

If $\sum_{n=0}^{\infty} a_n(x-a)^n$ *has radius of convergence* $R > 0$*, it defines a function* $f(x) = \sum_{n=0}^{\infty} a_n(x-a)^n$ *on the interval* $a - R < x < a + R$*. This function* f *has derivatives of all orders inside the interval, and the derivatives are obtained by differentiating the original series term by term*

$$f'(x) = \sum_{n=1}^{\infty} na_n(x-a)^{n-1}$$

$$f''(x) = \sum_{n=2}^{\infty} n(n-1)a_n(x-a)^{n-2}$$

and so on. Each of these derived series converges at every point of the interval $a - R < x < a + R$*.*

Find series for $f'(x)$ and $f''(x)$ if

$$f(x) = \frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots = \sum_{n=1}^{\infty} x^n.$$

We differentiate the power series on the right term by term:

$$f'(x) = \tfrac{1}{(1-x)^2} = 1 + 2x + 3x^2 + \cdots = \sum_{n=1}^{\infty} nx^{n-1}, -1 < x < 1$$

$$f''(x) = \tfrac{2}{(1-x)^3} = 2 + 6x + 12x^2 \cdots = \sum_{n=1}^{\infty} n(n-1)x^{n-2}, -1 < x < 1$$

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For example, the trigonometric series $\sum_{n=1}^{\infty} \frac{\sin(n!x)}{n^2}$ converges for all x. But if we differentiate term by term we get the series $\sum_{n=1}^{\infty} \frac{n!\cos(n!x)}{n^2}$ which diverges for all x. Note that this is not a power series since it is not a sum of positive integer powers of x.

Theorem

Suppose that $\sum_{n=0}^{\infty} a_n(x-a)^n$ *converges for* $a-R<x<a+R$. *Then*

$$\sum_{n=0}^{\infty} a_n \frac{(x-a)^{n+1}}{n+1} \quad \text{converges for} \quad a-R<x<a+R$$

and

$$\int f(x)dx = \sum_{n=0}^{\infty} a_n \frac{(x-a)^{n+1}}{n+1} + C \quad \text{for } a-R<x<a+R.$$

Identify the function

$$f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots, -1 \leq x \leq 1.$$

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Differentiating the original series term by term, we get

$$f'(x) = 1 - x^2 + x^4 - \cdots, -1 < x < 1.$$

This is a geometric series with first term 1 and ratio $-x^2$, so $f'(x) = \frac{1}{1+x^2}$. We can now integrate $f'(x) = \frac{1}{1+x^2}$ to get

$$\int f'(x)dx = \int \frac{dx}{1+x^2} = tan^{-1}x + C.$$

The series for $f(x)$ is 0 when $x = 0$, so $C = 0$. Thus

$$f(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots = tan^{-1}x, -1 < x < 1.$$

Note that the original series $x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots$, converges at both endpoints of the original interval of convergence, but our theorem can only guarantee the convergence of the differentiated series inside the interval.

The series

$$\frac{1}{1+t} = 1 - t + t^2 - t^3 + \cdots$$

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Therefore,

$$\ln(1+x) = \int_0^x \frac{1}{1+t} dt = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots, -1 < x < 1.$$

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Remark

- *If a function $f(x)$ has derivatives of all orders on an interval, can it be expressed as a power series on at least part of that interval?*

We have seen that within its interval of convergence I, the sum of a power series is a continuous function with derivatives of all orders. Now we ask the reverse question.

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If we can find power series representation of a function, they provide useful polynomial approximations of the original functions. Because approximation by polynomials is extremely useful to both mathematicians and scientists, we are interested to see when a function can have power series representation.