Differentiability and Chain Rule

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Differentiability of a Function of Two Variables

Definition

A function $z = f(x, y)$ is differentiable at (x_0, y_0) if $f_x(x_0, y_0)$ and $f_y(x_0, y_0)$ exist and $\triangle z = f(x_0 + \triangle x, y_0 + \triangle y) - f(x_0, y_0)$ satisfies

$$\triangle z = f_x(x_0, y_0)\triangle x + f_y(x_0, y_0)\triangle y + \epsilon_1 \triangle x + \epsilon_2 \triangle y$$

in which each of $\epsilon_1, \epsilon_2 \to 0$ as both $\triangle x, \triangle y \to 0$.

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If f is differentiable at (x_0, y_0), then

$$\begin{aligned} f(x, y) = {} & f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) \\ & + \epsilon_1 (x - x_0) + \epsilon_2 (y - y_0) \end{aligned}$$

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and the approximation

$$f(x, y) \approx f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

is called the linear approximation of f at (x_0, y_0).

Differentiability of a Function of Two Variables

$f(x, y)$ is differentiable at (x_0, y_0) if

- $f_x(x_0, y_0)$ exists
- $f_y(x_0, y_0)$ exists
- the change in f satisfies the linearization property:

$$f(x_0+h, y_0+k)-f(x_0, y_0) = f_x(x_0, y_0)h+f_y(x_0, y_0)k+\epsilon_1 h+\epsilon_2 k,$$

where

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Dividing the equation in f by $\sqrt{h^2+k^2}$ and letting $(h, k) \to (0, 0)$,

$$\lim_{(h,k)\to(0,0)} \frac{f(x_0+h, y_0+k) - f(x_0, y_0) - f_x(x_0, y_0)h - f_y(x_0, y_0)k}{\sqrt{h^2+k^2}} = 0.$$

Alternate Definition of Differentiability

Definition

Suppose $D \subset \mathbb{R}^2$ and let (x_0, y_0) be an interior point of D. A function $f : D \to \mathbb{R}$ is said to be differentiable at (x_0, y_0) if there is $(\alpha_1, \alpha_2) \in \mathbb{R}^2$ such that

$$\lim_{(h,k)\to(0,0)} \frac{f(x_0 + h, y_0 + k) - f(x_0, y_0) - \alpha_1 h - \alpha_2 k}{\sqrt{h^2 + k^2}} = 0.$$

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The pair (α_1, α_2) is called the total derivative of f at (x_0, y_0).

If f is differentiable at (x_0, y_0) and (α_1, α_2) be the total derivative of f at (x_0, y_0), then $\alpha_1 = f_x$ (choose $k = 0$ and use the limit definition for f_x) and $\alpha_2 = f_y$ (choose $h = 0$ and use the limit definition for f_y).

Theorem - The Increment Theorem for Functions of Two Variables

Suppose that the first partial derivatives of $f(x, y)$ are defined throughout an open region R containing the point (x_0, y_0) and that f_x and f_y are continuous at (x_0, y_0). Then the change

$$\triangle z = f(x_0 + \triangle x, y_0 + \triangle y) - f(x_0, y_0)$$

in the value of f that results from moving from (x_0, y_0) to another point $(x_0 + \triangle x, y_0 + \triangle y)$ in R satisfies an equation of the form

$$\triangle z = f_x(x_0, y_0)\triangle x + f_y(x_0, y_0)\triangle y + \epsilon_1 \triangle x + \epsilon_2 \triangle y$$

in which each of $\epsilon_1, \epsilon_2 \to 0$ as both $\triangle x, \triangle y \to 0$.

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If the partial derivatives f_x and f_y of a function $f(x, y)$ are continuous throughout an open region R, then f is differentiable at every point of R.

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- f is differentiable $\Longrightarrow$ f is continuous, converse is not true
- f_x and f_y are continuous $\Longrightarrow$ f is differentiable.

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- f_x, f_y exist $\not\Longrightarrow$ $f(x, y)$ is continuous
- $f(x, y)$ is continuous $\not\Longrightarrow$ f_x and f_y exist
- f is differentiable $\implies$ partial derivatives exist, converse is not true
- f is differentiable $\implies$ f is continuous, converse is not true
- f_x and f_y are continuous $\implies$ f is differentiable. What about the converse?

Examples

1. Discuss the existence of partial derivatives and differentiability of f at a point (x_0, y_0):
   - $f(x, y) = x^2 + 2xy$ at (x_0, y_0) (differentiable)
   - $f(x, y) = x^2 + y^2$ at (x_0, y_0) (differentiable)
   - $f(x, y) = \sqrt{x^2 + y^2}$ at $(0, 0)$ ($f_x(0, 0)$ does not exist $\implies$ f is not differentiable)
   - $f(x, y) = |xy|$ at $(0, 0)$ ($f_x(0, 0) = f_y(0, 0) = 0$ and f is differentiable)

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2. Discuss the differentiability of f at $(0,0)$:

$$f(x,y) = \begin{cases} \frac{xy}{\sqrt{x^2+y^2}} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

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Solution: $f_x(0,0) = f_y(0,0) = 0$, but f is not differentiable at $(0,0)$.

Examples

3. Show that the below function is differentiable at $(0,0)$ but the partial derivatives f_x and f_y of f are not continuous at $(0,0)$.

$$f(x,y) = \begin{cases} (x^2+y^2)\sin\left(\frac{1}{\sqrt{x^2+y^2}}\right) & \text{when } (x,y) \neq (0,0), \\ 0 & \text{when } (x,y) = (0,0). \end{cases}$$

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$$= \lim_{(h,k)\to(0,0)} \sqrt{h^2+k^2}\sin\left(\frac{1}{\sqrt{h^2+k^2}}\right) = 0.$$

(Use $\epsilon - \delta$ definition to show the above limit.)

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$$\begin{aligned} f_x(x,y) &= \lim_{h\to 0} \frac{f(x+h,y)-f(x,y)}{h} \\ &= 2x\sin\left(\frac{1}{\sqrt{x^2+y^2}}\right) - \frac{x}{\sqrt{x^2+y^2}}\cos\left(\frac{1}{\sqrt{x^2+y^2}}\right) \end{aligned}$$

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$$= 2x \sin\left(\frac{1}{\sqrt{x^2+y^2}}\right) - \frac{x}{\sqrt{x^2+y^2}} \cos\left(\frac{1}{\sqrt{x^2+y^2}}\right)$$

$\implies \lim\limits_{(x,y)\to(0,0)} f_x(x,y)$ does not exist (to see this, choose $y = mx$ and simplify for the second term)

$\implies f_x$ is not continuous at $(0,0)$.

Similarly, show that f_y is not continuous at $(0,0)$.

This example shows that differentiability does not guarantee continuous partial derivatives.

4. Let

$$f(x,y) = \begin{cases} 0 & \text{if } x^2 < y < 2x^2, \\ 1 & \text{else.} \end{cases}$$

Show that $f_x(0,0)$ and $f_y(0,0)$ exist, but f is not differentiable at $(0,0)$.

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Note that if f is not continuous at $(0,0)$, then f is not differentiable at $(0,0)$.

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Along $y = x^2$, $\lim\limits_{(x,y)\to(0,0)} f(x,y) = 1$.

Along $y = 1.5x^2$, $\lim\limits_{(x,y)\to(0,0)} f(x,y) = 0$.

$\implies$ f is not continuous at $(0,0)$ $\implies$ f is not differentiable at $(0,0)$.

This example shows that existence of f_x and f_y does not imply the differentiability of f.

Chain Rule for Functions of Two Variables

Theorem

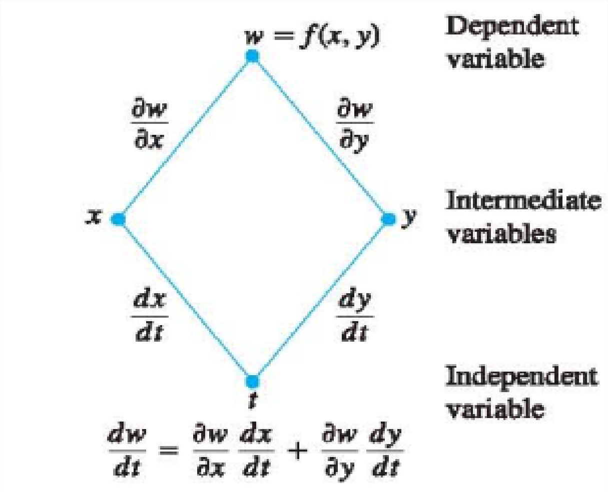
If $w = f(x, y)$ *is differentiable and if* $x = x(t), \; y = y(t)$ *are differentiable functions of* t, *then the composition* $w = f(x(t), y(t))$ *is a differentiable function of* t *and*

$$\frac{dw}{dt} = f_x(x(t), y(t))x'(t) + f_y(x(t), y(t))y'(t),$$

or

$$\frac{dw}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}.$$

Branch diagram for Chain Rule



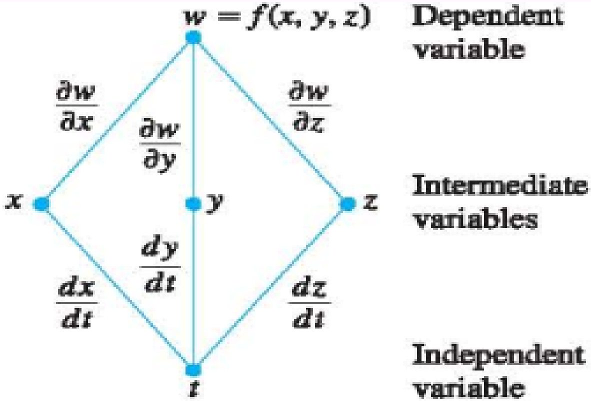
Chain Rule for Functions of Three Variables

Theorem

Let $w = f(x, y, z)$ *be a differentiable function, and let* $x = x(t)$, $y = y(t)$, $z = z(t)$ *be three differentiable functions of* t. *Then the function* $w(t) = f(x(t), y(t), z(t))$ *is a differentiable function of* t *and the derivative is given by:*

$$\frac{dw}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}$$

Branch Diagram



$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt}$$

Chain Rule for Two Independent Variables and Three Intermediate Variables

Theorem

Suppose that $w = f(x, y, z)$, $x = g(r, s)$, $y = h(r, s)$ *and* $z = k(r, s)$. *If all four functions are differentiable, then* w *has partial derivatives with respect to* r *and* s, *given by the formulas*

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial r},$$

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If $w = f(x)$ and $x = g(r, s)$, then

$$\frac{\partial w}{\partial r} = \frac{dw}{dx}\frac{\partial x}{\partial r} \text{ and } \frac{\partial w}{\partial s} = \frac{dw}{dx}\frac{\partial x}{\partial s}.$$

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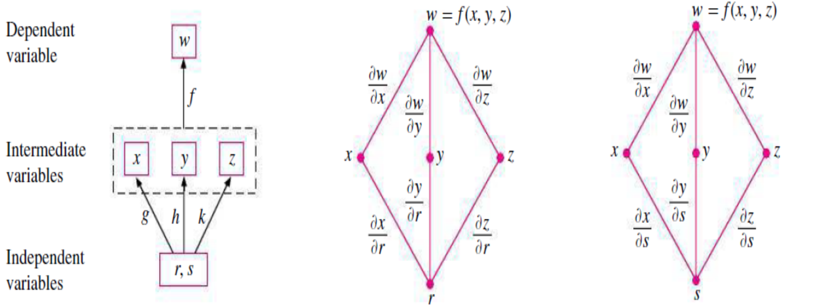

$$w = f(g(r,s), h(r,s), k(r,s))$$

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial r}$$

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(a) (b) (c)

Examples (Using Chain Rule)

1. Evaluate dw/dt where $w = x^2 + y^2$ with $x = \cos t$, $y = \sin t$ at $t = \pi$.

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3. Find $\partial w/\partial u$ and $\partial w/\partial v$ where $w = xy + yz + xz$ with $x = u + v$, $y = u - v$ and $z = uv$ at $(u, v) = (1/2, 1)$.
4. Find $\partial w/\partial v$ when $u = v = 0$ if $w = x^2 + (y/x)$, $x = u - 2v + 1$, $y = 2u + v - 2$.

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5. Let $w = f(x, y)$ and let (r, θ) denotes standard polar coordinates. Then,
   a) Show that
$$\frac{\partial w}{\partial r} = f_x \cos\theta + f_y \sin\theta \text{ and } \frac{1}{r}\frac{\partial w}{\partial \theta} = -f_x \sin\theta + f_y \cos\theta.$$
   b) Express f_x and f_y in terms of $\partial w/\partial r$ and $\partial w/\partial \theta$.
   c) Show that
$$(f_x)^2 + (f_y)^2 = \left(\frac{\partial w}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial w}{\partial \theta}\right)^2.$$