MATH F111- Mathematics I

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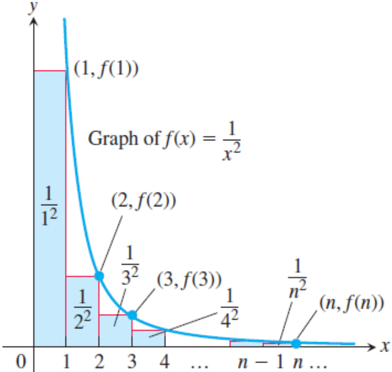
Integral test for Series

Does the series $\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} + \cdots$ converge?

Let us compare the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ with $\int_1^{\infty} \frac{1}{x^2} dx$.



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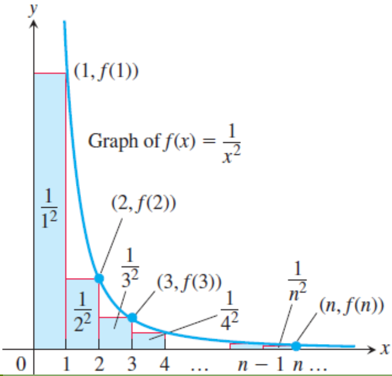


$$
\begin{aligned}
s_n &= 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} \\
&= f(1) + f(2) + \cdots + f(n) \\
&< f(1) + \int_1^n \frac{1}{x^2}\,dx \\
&< f(1) + \int_1^\infty \frac{1}{x^2}\,dx = 1 + 1 = 2
\end{aligned}
$$

Thus s_n is monotonically increasing and bounded above. Thus the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges.

Remark

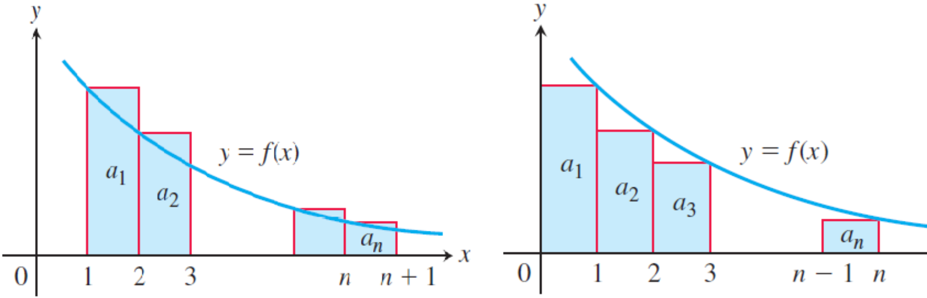
The series and integral need not have the same value in the convergent case as $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2} \neq \int_1^\infty \frac{1}{x^2}\,dx = 1.$

Theorem

The Integral Test: *Let a_n be a sequence of positive terms ($a_n > 0$). Suppose that $a_n = f(n)$, where f is a continuous, positive, decreasing function of x for all $x \geq N$ (N a positive integer). Then the series $\sum_{n=N}^{\infty} a_n$ and the integral $\int_N^{\infty} f(x)dx$ both converge or both diverge.*

Let us prove this result when $N = 1$. The proof for general N is similar. Thus we know that

- f is decreasing and $f(n) = a(n), \forall n$.



$$\int_1^{n+1} f(x)dx \leq a_1 + a_2 + a_3 + \cdots + a_n \leq a_1 + \int_1^n f(x)dx$$

- If $\int_1^\infty f(x)dx = \lim_{n\to\infty} \int_1^n f(x)dx$ is finite, the RHS inequality shows that $\sum_{n=1}^\infty a_n$ is finite.
- If $\lim_{n\to\infty} \int_1^n f(x)dx$ is infinite, the LHS inequality shows that $\sum_{n=1}^\infty a_n$ is infinite.
- Hence the series and the integral are either both finite or both infinite.

Using integral test discuss the convergence or divergence of the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p} = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \cdots + \frac{1}{n^p} + \cdots +$ for any fixed real number p.

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\int_1^{\infty} \frac{1}{x^p} dx = \lim_{b\to\infty} \int_1^b x^{-p} dx &= \lim_{b\to\infty} \left. \frac{x^{-p+1}}{-p+1} \right|_1^b \\
&= \frac{1}{p-1} \lim_{b\to\infty} \left(\frac{1}{b^{p-1}} - 1 \right) \\
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- Thus the series converges by the Integral test when $p > 1$.

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Then $1 - p > 0$ and

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Case IV: If $p = 1$, we have the (divergent) harmonic series $1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} + \cdots +$

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- For instance, it takes more than 178 million terms of the harmonic series to move the partial sums beyond 20.

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Comparison tests

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Theorem

Suppose that $a_n > 0$ *and* $b_n > 0$ *for all* $n \geq N$ *for some* $N \in \mathbb{N}$.

(i) If $\lim_{n\to\infty} \frac{a_n}{b_n} = c > 0$, *then* $\sum a_n$ *and* $\sum b_n$ *both converge or diverge.*

(ii) If $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$ *and* $\sum b_n$ *converges, then* $\sum a_n$ *converges.*

(iii) If $\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$ *and* $\sum b_n$ *diverges, then* $\sum a_n$ *diverges.*

Limit comparison test

Proof of (i) : Since $\lim_{n\to\infty} \frac{a_n}{b_n} = c$, we have $\exists\ N \in \mathbb{N}$ such that

$$\left|\frac{a_n}{b_n} - c\right| < \frac{c}{2}.$$

i.e taking $\epsilon = \frac{c}{2}$. Thus for $n \geq N$,

$$\frac{c}{2} b_n < a_n < \frac{3c}{2} b_n$$

Use Comparison tests to conclude.

Which of the following series converge and, which diverge?

$$(i) \sum_{n=1}^{\infty} \frac{2n+1}{(n+1)^2}$$

$$(ii) \sum_{n=1}^{\infty} \frac{1}{2^n - 1}$$

$$(iii) \sum_{n=1}^{\infty} \frac{1 + n \ln n}{n^2 + 5}$$

$$(iv) \sum_{n=1}^{\infty} \frac{\ln n}{n^{\frac{3}{2}}}$$