Mathematics I- MATH F111

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Integrals of Vector Functions

Definition

A differentiable vector function $\overrightarrow{R}(t)$ is an antiderivative of a vector function $\overrightarrow{r}(t)$ on an interval I if $\frac{d\overrightarrow{R}}{dt} = \overrightarrow{r}$ at each point of I.

If $\overrightarrow{R}$ is an antiderivative of $\overrightarrow{r}$ on I, it can be shown, working one component at a time, that every antiderivative of $\overrightarrow{r}$ on I has the form $\overrightarrow{R} + \overrightarrow{C}$ for some constant vector $\overrightarrow{C}$. The set of all antiderivatives of $\overrightarrow{r}$ on I is the indefinite integral of $\overrightarrow{r}$ on I.

Basic Definition

Definition

The indefinite integral of $\overrightarrow{r}$ with respect to t is the set of all anti-derivatives of $\overrightarrow{r}$, denoted by $\int \overrightarrow{r}(t)dt$. If $\overrightarrow{R}$ is any antiderivative of $\overrightarrow{r}$, then

$$\int \overrightarrow{r}(t)dt = \overrightarrow{R}(t) + \overrightarrow{C}.$$

Example

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Integrate the vector function $\overrightarrow{r}(t) = \cos t\hat{i} + \hat{j} - 2t\hat{k}$.

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Solution: We have $\int \overrightarrow{r}(t)dt = \left(\int \cos t dt\right)\hat{i} + \left(\int dt\right)\hat{j} - \left(\int 2tdt\right)\hat{k} = \sin t\hat{i} + t\hat{j} - 2\frac{t^2}{2}\hat{k} + \overrightarrow{C} = \sin t\hat{i} + t\hat{j} - t^2\hat{k} + \overrightarrow{C}$, where $\overrightarrow{C} = c_1\hat{i} + c_2\hat{j} + c_3\hat{k}$.

Definite Integral

If the components of $\overrightarrow{r}(t) = f(t)\hat{i} + g(t)\hat{j} + h(t)\hat{k}$ are integrable over $[a, b]$, then so is $\overrightarrow{r}$ and the definite integral of $\overrightarrow{r}$ from a to b is

$$\int_a^b \overrightarrow{r}(t)dt = \left(\int_a^b f(t)dt\right)\hat{i} + \left(\int_a^b g(t)dt\right)\hat{j} + \left(\int_a^b h(t)dt\right)\hat{k}.$$

Example

Integrate the vector function $\overrightarrow{r}(t) = \cos t\hat{i} + \hat{j} - 2t\hat{k}$ *from* 0 *to* π.

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Solution: We have $\int_a^b \overrightarrow{r}(t)dt = \left(\int_a^b \cos t dt\right)\hat{i} + \left(\int_a^b dt\right)\hat{j} - \left(\int_a^b 2t dt\right)\hat{k}$
This implies $\int_0^\pi \overrightarrow{r}(t)dt = [\sin\pi - \sin 0]\hat{i} + \pi\hat{j} - \pi^2\hat{k} = \pi\hat{j} - \pi^2\hat{k}$.

Scalar valued function

Theorem

The Fundamental Theorem of Calculus: If f is continuous on $[a, b]$, then $F(t) = \int_a^t f(\tau)d\tau$ is continuous on $[a, b]$ and differentiable on (a, b) and its derivative is $f(t)$:

$$F^{'}(t) = \frac{d}{dt}\int_a^t f(\tau)d\tau = f(t).$$

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If f is continuous at every point in $[a, b]$ and F is any antiderivative of f on $[a, b]$, then

$$\int_a^b f(t)dt = F(b) - F(a).$$

Using Fundamental theorem of Calculus for scalar valued functions show that if a vector function $\vec{r}(t)$ is continuous for $a \leq t \leq b$, then

$$\frac{d}{dt}\int_a^t \vec{r}(\tau)d\tau = \vec{r}(t)$$

at every point t of (a, b).

Theorem

The Fundamental Theorem of Calculus for continuous vector functions: If $\overrightarrow{r}(t)$ is continuous for $a \leq t \leq b$ and $\overrightarrow{R}(t)$ is any antiderivative of $\overrightarrow{r}$ so that $\overrightarrow{R}'(t) = \overrightarrow{r}(t)$, then

$$\int_a^b \overrightarrow{r}(t)dt = \overrightarrow{R}(t)|_a^b = \overrightarrow{R}(b) - \overrightarrow{R}(a)$$

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The Fundamental Theorem of Calculus for continuous vector functions: If $\overrightarrow{r}(t)$ is continuous for $a \leq t \leq b$ and $\overrightarrow{R}(t)$ is any antiderivative of $\overrightarrow{r}$ so that $\overrightarrow{R}'(t) = \overrightarrow{r}(t)$, then

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Notice that an anti-derivative of a vector function is also a vector function, whereas a definite integral of a vector function is a single constant vector.

Properties

Example

Suppose we do not know the path of a hang glider, but only its acceleration vector $\overrightarrow{a}(t) = -(3\cos t)\hat{i} - (3\sin t)\hat{j} + 2\hat{k}$. *We also know that initially (at time* $t = 0$*) the glider departed from the point* $(4, 0, 0)$ *with velocity* $\overrightarrow{v}(0) = 3\hat{j}$. *Find the glider's position as a function of* t.

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Solution: We have $\overrightarrow{v}(t) = (-3\sin t)\hat{i} + (3\cos t)\hat{j} + 2t\hat{k} + c_1\hat{i} + c_2\hat{j} + c_3\hat{k}$.
Now $3\hat{j} = \overrightarrow{v}(0) = 3\hat{j} + c_1\hat{i} + c_2\hat{j} + c_3\hat{k}$.
Then $c_1 = c_2 = c_3 = 0$.
Therefore we have

$$\overrightarrow{v}(t) = (-3\sin t)\hat{i} + (3\cos t)\hat{j} + 2t\hat{k} \tag{1}$$

Properties

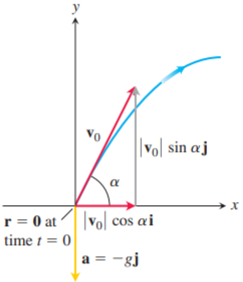
Now integrating once more, we have
$\overrightarrow{r}(t) = (3\cos t)\hat{i} + (3\sin t)\hat{j} + (2 \times \frac{t^2}{2})\hat{k} + d_1\hat{i} + d_2\hat{j} + d_3\hat{k}$. Now
$4\hat{i} = \overrightarrow{r}(0) = 3\hat{i} + d_1\hat{i} + d_2\hat{j} + d_3\hat{k}$.
This gives $d_1 = 1, d_2 = 0 = d_3$.
Thus $\overrightarrow{r}(t) = (3\cos t + 1)\hat{i} + (3\sin t)\hat{j} + (t^2)\hat{k}$

Understanding Projectile Motion

In physics, projectile motion describes how an object fired at some angle from an initial position, and acted upon by only the force of gravity, moves in a vertical coordinate plane. In the classic example, we ignore the effects of any frictional drag on the object, which may vary with its speed and altitude, and also the fact that the force of gravity changes slightly with the projectile's changing height.

To derive equations for projectile motion, we assume that the projectile behaves like a particle moving in a vertical coordinate plane and that the only force acting on the projectile during its flight is the constant force of gravity, which always points straight down.

We assume that the projectile is launched from the origin at time $t = 0$ into the first quadrant with an initial velocity v_0.



If v_0 makes an angle α with the horizontal, then

$$v_0 = (|v_0| \cos \alpha)\vec{i} + (|v_0| \sin \alpha)\vec{j}$$

The projectile's initial position is $\vec{r} = 0\vec{i} + 0\vec{j} = \vec{0}$.

Newton's second law of motion says that the force acting on the projectile is equal to the projectile's mass m times its acceleration. If the force is solely the gravitational force, $-mg\hat{j}$, then

$$m\overrightarrow{a}(t) = -mg\hat{j}, \text{ where } \overrightarrow{a}(t) = \frac{d^2\vec{r}}{dt^2}.$$

$$\implies \frac{d^2\vec{r}}{dt^2} = -g\hat{j} \text{ with initial conditions } \vec{r} = \vec{r}_0$$

and $\frac{d\vec{r}}{dt} = \vec{v}_0$ when $t = 0$.

$$\Rightarrow \overrightarrow{v}(t) = -gt\hat{j} + c_1\hat{i} + c_2\hat{j}$$

Now by putting $t = 0$, we have

$$(|v_0|\cos\alpha)\hat{i} + (|v_0|\sin\alpha)\hat{j} = \overrightarrow{v}(0) = c_1\hat{i} + c_2\hat{j}$$

$$c_1 = |v_0|\cos\alpha, \ c_2 = |v_0|\sin\alpha.$$

Understanding Projectile Motion

Hence $\overrightarrow{v}(t) = |v_0| \cos\alpha \hat{i} + (|v_0| \sin\alpha - gt)\hat{j}$.
Now by integrating once more, we have

$$\overrightarrow{r}(t) = (|v_0|\cos\alpha)t\hat{i} + \left(|v_0|\sin\alpha t - g\frac{t^2}{2}\right)\hat{j} + d_1\hat{i} + d_2\hat{j}$$

Now $0\hat{i} + 0\hat{j} = \overrightarrow{r}(0) = d_1\hat{i} + d_2\hat{j}$ implies $d_1 = d_2 = 0$. Thus we have
Ideal Projectile Motion Equation

$$\boxed{\overrightarrow{r}(t) = (|v_0|\cos\alpha)t\hat{i} + \left(|v_0|\sin\alpha t - \tfrac{1}{2}gt^2\right)\hat{j}.}$$

The components of $\vec{r}$ gives the parametric equations

$$x = (|v_0|\cos\alpha)t, \;\; y = (|v_0|\sin\alpha)t - \frac{1}{2}gt^2.$$

Understanding Projectile Motion

We now see the following example to Understand it better.

Example

A projectile is fired from the origin over horizontal ground at an initial speed of $500 m/sec$ *and a launch angle of* 60°. *Where will the projectile be* 10 *seconds later?*

Solution: We use $\alpha = 60^\circ, |v_0| = 500, g = 9.8$ and $t = 10$ in the above equation, we get

$$\overrightarrow{r}(t) = (500 \times \cos 60^\circ) \times 10\hat{i} + \left(500 \sin 60^\circ \times 10 - \frac{9.8 \times 100}{2}\right)\hat{j}$$

Remark

The ideal projectiles move along parabolas.

Substitute $t = x/(|v_0| \cos \alpha)$ to get

$$y = -\left(\frac{g}{2|v_0|^2 \cos^2 \alpha}\right)x^2 + (\tan \alpha)x.$$

This has the form $y = ax^2 + bx$, so the graph is a parabola.

Height, Flight time and Range of Projectile motion

$$\overrightarrow{r}(t) = (|v_0|\cos\alpha)t\hat{i} + \left(|v_0|\sin\alpha t - \frac{1}{2}gt^2\right)\hat{j}$$

A projectile reaches highest point when its vertical component of velocity vector is 0.

$$\frac{dr}{dt} = |v_0| \cos\alpha \hat{i} + (|v_0| \sin\alpha - gt)\hat{j}.$$

Thus at maximum height, $t = \frac{|v_0| \sin\alpha}{g}$, which implies maximum height, $y_{\max} = \frac{(|v_0| \sin\alpha)^2}{2g}$.
Once the object reaches ground, the y co-ordinate of position vector equals 0 which gives flight time, $t = \frac{2|v_0| \sin\alpha}{g}$. At this time, the x co-ordinate is given by

$$x = \frac{|v_0|^2 \sin 2\alpha}{g}$$

will give the range of projectile.

Properties

Height, Flight Time, and Range for Ideal Projectile Motion

For ideal projectile motion when an object is launched from the origin over a horizontal surface with initial speed v_0 and launch angle α:

$$\text{Maximum height : } \; y_{\max} = \frac{(v_0 \sin \alpha)^2}{2g}$$

$$\text{Flight time : } \; t = \frac{2v_0 \sin \alpha}{g}$$

$$\text{Range : } \; R = \frac{v_0^2}{g} \sin 2\alpha.$$

Let the projectile be fired from the point (x_0, y_0) instead of the origin. Then $m\overrightarrow{a}(t) = -mg\hat{j}$ with the initial condition $\overrightarrow{v}(0) = (v_0 \cos\alpha)\hat{i} + (v_0 \sin\alpha)\hat{j}$ and $\overrightarrow{r}(0) = x_0\hat{i} + y_0\hat{j}$.
Now

$$m\overrightarrow{a}(t) = -mg\hat{j}$$

$$\Rightarrow \overrightarrow{a}(t) = -g\hat{j}$$

$$\Rightarrow \overrightarrow{v}(t) = -gt\hat{j} + c_1\hat{i} + c_2\hat{j}$$

Now by putting $t = 0$, we have

$$(|v_0| \cos\alpha)\hat{i} + (|v_0| \sin\alpha)\hat{j} = \overrightarrow{v}(0) = c_1\hat{i} + c_2\hat{j}$$

$$c_1 = |v_0| \cos\alpha, c_2 = |v_0| \sin\alpha.$$

Understanding Projectile Motion

Hence $\overrightarrow{v}(t) = |v_0|\cos\alpha\hat{i} + (|v_0|\sin\alpha - gt)\hat{j}$.
Now by integrating once more, we have

$$\overrightarrow{r}(t) = (|v_0|\cos\alpha)t\hat{i} + \left(|v_0|\sin\alpha t - g\frac{t^2}{2}\right)\hat{j} + d_1\hat{i} + d_2\hat{j}$$

Now $x_0\hat{i} + y_0\hat{j} = \overrightarrow{r}(0) = d_1\hat{i} + d_2\hat{j}$ implies $d_1 = x_0$ and $d_2 = y_0$.
Thus we have

Ideal Projectile Motion Equation

$$\boxed{\overrightarrow{r}(t) = [(|v_0|\cos\alpha)t + x_0]\hat{i} + \left[\left(|v_0|\sin\alpha t - \frac{gt^2}{2}\right) + y_0\right]\hat{j}.}$$

Projectile motion with Wind gusts

A baseball is hit when it is 3 ft above the ground. It leaves the bat with initial speed of 152 ft/sec, making an angle of 20 degree with the horizontal. At the instant the ball is hit, an instantaneous gust of wind blows in the horizontal direction directly opposite the direction the ball is taking toward the outfield, adding a component of $-8.8\hat{i}$ (ft/sec) to the ball's initial velocity(8.8 ft/sec = 6 mph).

1. Find a vector equation (position vector) for the path of the baseball.
2. How high does the baseball go, and when does it reach maximum height?
3. Assuming that the ball is not caught, find its range and flight time.

Adding the effect of wind gust, the initial velocity of baseball

$$\begin{aligned} v_0 &= |v_0|\cos\alpha\hat{i} + |v_0|\sin\alpha\hat{j} - 8.8\vec{i} \\ &= (152\cos 20 - 8.8)\hat{i} + 152\sin 20\hat{j}. \end{aligned}$$

The initial position is $r_0 = 3\hat{j}$. Integration of $\frac{d^2r}{dt^2} = -g\hat{j}$ gives

$$\begin{aligned} \frac{dr}{dt} &= -(gt)\hat{j} + v_0 \text{and} \\ r &= -\frac{1}{2}gt^2\hat{j} + v_0 t + r_0 \end{aligned}$$

Substituting the values of v_0 and r_0,

$$r = (152\cos 20 - 8.8)\hat{i} + (3 + (152\sin 20)t - 16t^2)\hat{j}.$$

(b). The baseball reaches its highest point when the vertical component of velocity is zero, or $\frac{dy}{dt} = 0$ that $\implies t = \frac{152 sin20}{32} \equiv 1.62$sec. Substituting this time into the vertical component for $\vec{r}$ gives the maximum height

$$y_{max} = 3 + (152 \sin 20)(1.62) - 16(1.62)^2 \equiv 45.2 \text{ feet}$$

(c). To find when the baseball lands, we set the vertical component for r equal to 0 and solve for t:

$$3 + (152 \sin 20)t - 16t^2 = 0.$$

The values are at $t = 3.3$sec and $t = -0.06$ sec. Substituting the positive time into the horizontal component for $\vec{r}$, we find the range

$$R = (152 \cos 20 - 8.8)(3.3) \equiv 442 \text{ feet}.$$