Curvature and Normal Vector of a Curve

Anushaya Mohapatra

Department of Mathematics

BITS PILANI K K Birla Goa Campus, Goa

September 24, 2024

Chapter 13.4

Curvature

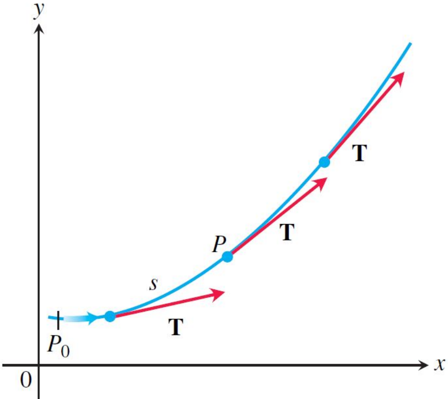
- In this section we study how a curve turns or bends.

Curvature

- In this section we study how a curve turns or bends.
- To gain perspective, we look first at curves in the coordinate plane. Then we consider curves in space.

Curvature

- In this section we study how a curve turns or bends.
- To gain perspective, we look first at curves in the coordinate plane. Then we consider curves in space.



Curvature of A Plane Curve

- When a particle moves along a smooth curve in the plane, the unit tangent $\mathbf{T} = d\mathbf{r}/ds$ changes its direction (turns) wherever the curve bends.

Curvature of A Plane Curve

- When a particle moves along a smooth curve in the plane, the unit tangent $\mathbf{T} = d\mathbf{r}/ds$ changes its direction (turns) wherever the curve bends.
- The rate at which $\mathbf{T}$ turns per unit of length along the curve is called the curvature.

- When a particle moves along a smooth curve in the plane, the unit tangent $\mathbf{T} = d\mathbf{r}/ds$ changes its direction (turns) wherever the curve bends.

- The rate at which $\mathbf{T}$ turns per unit of length along the curve is called the curvature.

Definition 0.1.

If $\mathbf{T}$ is the unit tangent of a smooth curve, the curvature of the curve is define by

$$\kappa = \left|\frac{d\mathbf{T}}{ds}\right|.$$

Curvature of A Plane Curve

- Since $|\mathbf{T}| = 1$, its length remains constant and only its direction changes as the particle moves along the curve.

Curvature of A Plane Curve

- Since $|\mathbf{T}| = 1$, its length remains constant and only its direction changes as the particle moves along the curve.
- The curvature for straight lines and circles should be constant by the intution of curvature's defination. Let us test these.

- Since $|\mathbf{T}| = 1$, its length remains constant and only its direction changes as the particle moves along the curve.

- The curvature for straight lines and circles should be constant by the intution of curvature's defination. Let us test these.
- Any straight line can be parametrized by

$$\mathbf{r}(t) = t\mathbf{v}_0 + \mathbf{a}$$

Curvature of A Plane Curve

- Since $|\mathbf{T}| = 1$, its length remains constant and only its direction changes as the particle moves along the curve.
- The curvature for straight lines and circles should be constant by the intution of curvature's defination. Let us test these.
- Any straight line can be parametrized by

$$\mathbf{r}(t) = t\mathbf{v}_0 + \mathbf{a}$$

where $\mathbf{v}_0$ and $\mathbf{a}$ constant vectors.

Curvature of A Plane Curve

- Since $|\mathbf{T}| = 1$, its length remains constant and only its direction changes as the particle moves along the curve.

- The curvature for straight lines and circles should be constant by the intution of curvature's defination. Let us test these.
- Any straight line can be parametrized by

$$\mathbf{r}(t) = t\mathbf{v}_0 + \mathbf{a}$$

where $\mathbf{v}_0$ and $\mathbf{a}$ constant vectors. Then $\mathbf{T} = \frac{\mathbf{v}_0}{|\mathbf{v}_0|}$, therefore $\kappa = 0$.

Curvature formula

Formula for calculating curvature: If $\mathbf{r}(t)$ is smooth curve, then the curvature is the scalar function

$$\kappa = \frac{1}{|\mathbf{v}|}\left|\frac{d\mathbf{T}}{dt}\right|,$$

where $\mathbf{T} = \mathbf{v}/|\mathbf{v}|$ is the unite tangent vector.

Curvature formula

Formula for calculating curvature: If $\mathbf{r}(t)$ is smooth curve, then the curvature is the scalar function

$$\kappa = \frac{1}{|\mathbf{v}|}\left|\frac{d\mathbf{T}}{dt}\right|,$$

where $\mathbf{T} = \mathbf{v}/|\mathbf{v}|$ is the unite tangent vector.

Now we will find the curvature of a circle.

Curvature formula

Formula for calculating curvature: If $\mathbf{r}(t)$ is smooth curve, then the curvature is the scalar function

$$\kappa = \frac{1}{|\mathbf{v}|}\left|\frac{d\mathbf{T}}{dt}\right|,$$

where $\mathbf{T} = \mathbf{v}/|\mathbf{v}|$ is the unite tangent vector.

Now we will find the curvature of a circle.

A parametrization of the circle of radius 'a' with center at the origin O is given by

$$\mathbf{r}(t) = (a\cos t)\mathbf{i} + (a\sin t)\mathbf{j}.$$

Example

- We first find the velocity vector

$$\mathbf{v} = \mathbf{r}'(t) = -(a\sin t)\mathbf{i} + (a\cos t)\mathbf{j}.$$

Example

- We first find the velocity vector
$$\mathbf{v} = \mathbf{r}'(t) = -(a \sin t)\mathbf{i} + (a \cos t)\mathbf{j}.$$

- The speed is given by
$$|\mathbf{v}| = \sqrt{(-a \sin t)^2 + (a \cos t)^2} = |a| = a.$$

Example

- We first find the velocity vector

$$\mathbf{v} = \mathbf{r}'(t) = -(a \sin t)\mathbf{i} + (a \cos t)\mathbf{j}.$$

- The speed is given by

$$|\mathbf{v}| = \sqrt{(-a \sin t)^2 + (a \cos t)^2} = |a| = a.$$

- Therefore, the unit tangent

$$\mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|} = -(\sin t)\mathbf{i} + (\cos t)\mathbf{j}.$$

Example

- We first find the velocity vector
$$\mathbf{v} = \mathbf{r}'(t) = -(a \sin t)\mathbf{i} + (a \cos t)\mathbf{j}.$$

- The speed is given by
$$|\mathbf{v}| = \sqrt{(-a \sin t)^2 + (a \cos t)^2} = |a| = a.$$

- Therefore, the unit tangent
$$\mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|} = -(\sin t)\mathbf{i} + (\cos t)\mathbf{j}.$$

- $\dfrac{d\mathbf{T}}{dt} = -(\cos t)\mathbf{i} + (\sin t)\mathbf{j} \Rightarrow \left|\dfrac{d\mathbf{T}}{dt}\right| = 1$

Example

- Hence, for any value of the parameter t, the curvature of the circle is

$$\kappa = \frac{1}{|\mathbf{v}|}\left|\frac{d\mathbf{T}}{dt}\right| = \frac{1}{a}(1) = \frac{1}{a} = \frac{1}{\text{radius}}$$

Example

- Hence, for any value of the parameter t, the curvature of the circle is

$$\kappa = \frac{1}{|\mathbf{v}|}\left|\frac{d\mathbf{T}}{dt}\right| = \frac{1}{a}(1) = \frac{1}{a} = \frac{1}{\text{radius}}$$

- Find the curvature of the parabola $y = x^2$ at the points $(0,0),\ (1,1),\ (2,4)$.

Example

- Hence, for any value of the parameter t, the curvature of the circle is

$$\kappa = \frac{1}{|\mathbf{v}|}\left|\frac{d\mathbf{T}}{dt}\right| = \frac{1}{a}(1) = \frac{1}{a} = \frac{1}{\text{radius}}$$

- Find the curvature of the parabola $y = x^2$ at the points $(0,0),\ (1,1),\ (2,4)$.
  Ans.

$$\kappa(t) = \frac{2}{(1+4t^2)^{3/2}}.$$

Principal Unit Normal

- Since $\mathbf{T}$ has constant length (as $|\mathbf{T}| = 1$), the derivative $d\mathbf{T}/ds$ is orthogonal to $\mathbf{T}$.

Principal Unit Normal

- Since $\mathbf{T}$ has constant length (as $|\mathbf{T}| = 1$), the derivative $d\mathbf{T}/ds$ is orthogonal to $\mathbf{T}$.
- Therefore, $\dfrac{d\mathbf{T}/ds}{|d\mathbf{T}/ds|}$ is unit vector orthogonal to $\mathbf{T}$.

Principal Unit Normal

- Since $\mathbf{T}$ has constant length (as $|\mathbf{T}| = 1$), the derivative $d\mathbf{T}/ds$ is orthogonal to $\mathbf{T}$.
- Therefore, $\dfrac{d\mathbf{T}/ds}{|d\mathbf{T}/ds|}$ is unit vector orthogonal to $\mathbf{T}$.

  Also note that $|d\mathbf{T}/ds| = \kappa$.

- Since $\mathbf{T}$ has constant length (as $|\mathbf{T}| = 1$), the derivative $d\mathbf{T}/ds$ is orthogonal to $\mathbf{T}$.
- Therefore, $\dfrac{d\mathbf{T}/ds}{|d\mathbf{T}/ds|}$ is unit vector orthogonal to $\mathbf{T}$.

  Also note that $|d\mathbf{T}/ds| = \kappa$.

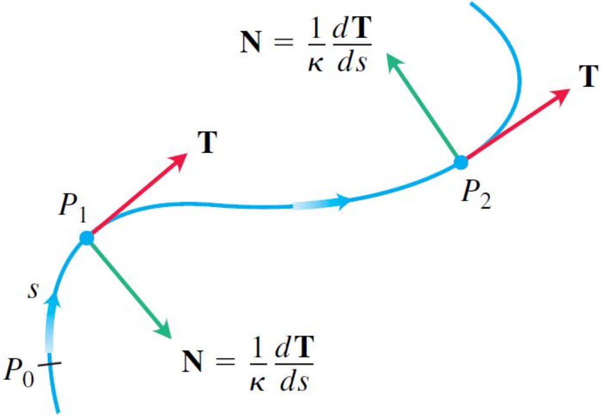
Definition 0.2 (Principal unit normal).

At a point where $\kappa \neq 0$, the principal unit normal vector for a smooth curve in the plane is

$$\mathbf{N} = \frac{d\mathbf{T}/ds}{|d\mathbf{T}/ds|} = \frac{1}{\kappa}\frac{d\mathbf{T}}{ds}.$$

Principal unit normal

Note that the principal unit normal **N** points the direction in which the unit tangent is turning.



]

Principal unit normal

Formula for calculating $\mathbf{N}$: If $\mathbf{r}(t)$ is smooth curve, then the principal unit normal is

$$\mathbf{N} = \frac{d\mathbf{T}/dt}{|d\mathbf{T}/dt|}$$

where $\mathbf{T} = \mathbf{v}/|\mathbf{v}|$ is the unit tangent vector.

Example

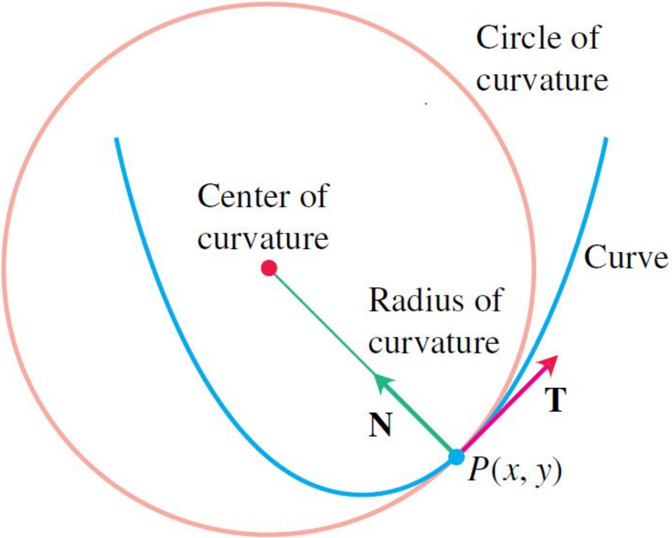
- Find the principal unit normal to the curve $\mathbf{N}$ for the circle
$$\mathbf{r}(t) = (a\cos t)\mathbf{i} + (a\sin t)\mathbf{j}$$
at $t = \pi/4$.

Example

- Find the principal unit normal to the curve $\mathbf{N}$ for the circle
$$\mathbf{r}(t) = (a\cos t)\mathbf{i} + (a\sin t)\mathbf{j}$$
at $t = \pi/4$.

- Find the $\mathbf{T}$, $\mathbf{N}$ and κ for the plane curve
$$\mathbf{r}(t) = (\cos t + t\sin t)\mathbf{i} + (\sin t - t\cos t)\mathbf{j}, \ \ t > 0.$$

Circle of curvature for plane curves



Circle of curvature for plane curves

The **circle of curvature** or **osculating circle** at a point P on a plane curve where $\kappa \neq 0$ is the circle in the plane of the curve that

Circle of curvature for plane curves

The **circle of curvature** or **osculating circle** at a point P on a plane curve where $\kappa \neq 0$ is the circle in the plane of the curve that

1. is tangent to the curve at P (has the same tangent line the curve has)

Circle of curvature for plane curves

The **circle of curvature** or **osculating circle** at a point P on a plane curve where $\kappa \neq 0$ is the circle in the plane of the curve that

1. is tangent to the curve at P (has the same tangent line the curve has)
2. has the same curvature the curve has at P

Circle of curvature for plane curves

The **circle of curvature** or **osculating circle** at a point P on a plane curve where $\kappa \neq 0$ is the circle in the plane of the curve that

1. is tangent to the curve at P (has the same tangent line the curve has)
2. has the same curvature the curve has at P
3. has center that lies toward the concave or inner side of the curve

Definition 0.3.
The **radius of curvature** of the curve at P is the radius of the circle of curvature, which is

$$\text{Radius of curvature} = \rho = \frac{1}{\kappa}.$$

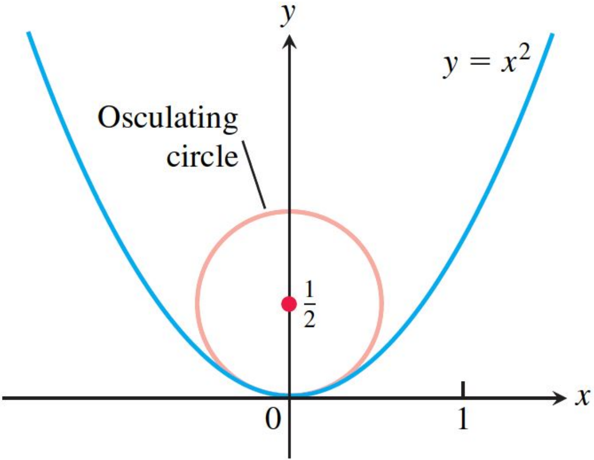
The **center of curvature** of the curve at P is the center of the circle of curvature.

Example

Find and graph the osculating circle of the parabola $y = x^2$ at the origin.

Example

Find and graph the osculating circle of the parabola $y = x^2$ at the origin.



Let $\mathbf{r}(t)$ be a smooth curve in space, and if s is the arc length parameter of the curve, then:

Curvature and Normal Vectors for Space Curves

Let $\mathbf{r}(t)$ be a smooth curve in space, and if s is the arc length parameter of the curve, then:

1. The unit tangent vector $\mathbf{T}$ is $d\mathbf{r}/ds = \mathbf{v}/|\mathbf{v}|$.

Curvature and Normal Vectors for Space Curves

Let $\mathbf{r}(t)$ be a smooth curve in space, and if s is the arc length parameter of the curve, then:

1. The unit tangent vector $\mathbf{T}$ is $d\mathbf{r}/ds = \mathbf{v}/|\mathbf{v}|$.

2. The curvature in space is then defined to be

$$\kappa = \left|\frac{d\mathbf{T}}{ds}\right| = \frac{1}{|\mathbf{v}|}\left|\frac{d\mathbf{T}}{dt}\right|$$

Curvature and Normal Vectors for Space Curves

Let $\mathbf{r}(t)$ be a smooth curve in space, and if s is the arc length parameter of the curve, then:

1. The unit tangent vector $\mathbf{T}$ is $d\mathbf{r}/ds = \mathbf{v}/|\mathbf{v}|$.
2. The curvature in space is then defined to be

$$\kappa = \left|\frac{d\mathbf{T}}{ds}\right| = \frac{1}{|\mathbf{v}|}\left|\frac{d\mathbf{T}}{dt}\right|$$

3. The principal unit normal to be

$$\mathbf{N} = \frac{1}{\kappa}\frac{d\mathbf{T}}{ds} = \frac{d\mathbf{T}/dt}{|d\mathbf{T}/dt|}.$$

Example

Find the curvature κ and $\mathbf{N}$ for the helix

$$\mathbf{r}(t) = (a\cos t)\mathbf{i} + (a\sin t)\mathbf{j} + (bt)\mathbf{k}, \ \ a, b > 0.$$

Example

Find the curvature κ and $\mathbf{N}$ for the helix

$$\mathbf{r}(t) = (a\cos t)\mathbf{i} + (a\sin t)\mathbf{j} + (bt)\mathbf{k}, \ \ a, b > 0.$$

Example

Solution.

The velocity vector: $\mathbf{v}(t) = (-a\sin t)\mathbf{i} + (a\cos t)\mathbf{j} + b\mathbf{k}$.

Example

Solution.

The velocity vector: $\mathbf{v}(t) = (-a\sin t)\mathbf{i} + (a\cos t)\mathbf{j} + b\mathbf{k}$.

$|\mathbf{v}| = \sqrt{a^2 + b^2}$ and the unit tangent is given by

$$\mathbf{T} = \frac{1}{\sqrt{a^2 + b^2}}[(-a\sin t)\mathbf{i} + (a\cos t)\mathbf{j} + b\mathbf{k}]$$

Example

Solution.

The velocity vector: $\mathbf{v}(t) = (-a\sin t)\mathbf{i} + (a\cos t)\mathbf{j} + b\mathbf{k}$.

$|\mathbf{v}| = \sqrt{a^2 + b^2}$ and the unit tangent is given by

$$\mathbf{T} = \frac{1}{\sqrt{a^2+b^2}}[(-a\sin t)\mathbf{i} + (a\cos t)\mathbf{j} + b\mathbf{k}]$$

which implies

$$\frac{d\mathbf{T}}{dt} = \frac{a}{\sqrt{a^2+b^2}}[(-\cos t)\mathbf{i} - (\sin t)\mathbf{j}].$$

Example

Therefore, we have (as $a > 0$)

$$\left|\frac{d\mathbf{T}}{dt}\right| = \frac{a}{\sqrt{a^2 + b^2}}.$$

Example

Therefore, we have (as $a > 0$)

$$\left|\frac{d\mathbf{T}}{dt}\right| = \frac{a}{\sqrt{a^2 + b^2}}.$$

The curvature and the principal unit normal are given by

$$\kappa = \frac{1}{|\mathbf{v}|}\left|\frac{d\mathbf{T}}{dt}\right| = \frac{a}{a^2 + b^2},$$

and

$$\mathbf{N} = (-\cos t)\mathbf{i} - (\sin t)\mathbf{j}.$$

Questions

1. Compute the curvature $\kappa(t)$ of the ellipse $x^2/a^2 + y^2/b^2 = 1$. Suppose $a > b$, when is the curvature maximal? Try to think geometrically why this must be the case.

2. Show that the curvature of the curve $y = f(x)$ in xy-plane is given by

$$\kappa(x) = \frac{|f''(x)|}{[1 + (f'(x))^2]^{3/2}}$$

where $f(x)$ is twice-differentiable function of x.

3. Show that the curvature of the smooth curve $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}$ in xy-plane plane is given by the formula

$$\kappa = \frac{|(x'y'' - x'y'')|}{[(x')^2 + (y')^2]^{3/2}}.$$