Triple Integrals in Rectangular and Cylindrical Coordinates

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Examples - Triple Integrals

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Example 2: Rewrite the following integral as an equivalent iterated integral in the order $dy\ dz\ dx$ and $dx\ dy\ dz$.

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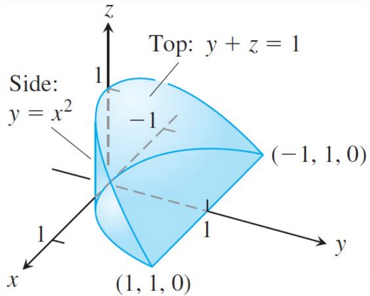

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Cylindrical coordinates represent a point P in space by ordered triples (r, θ, z) in which

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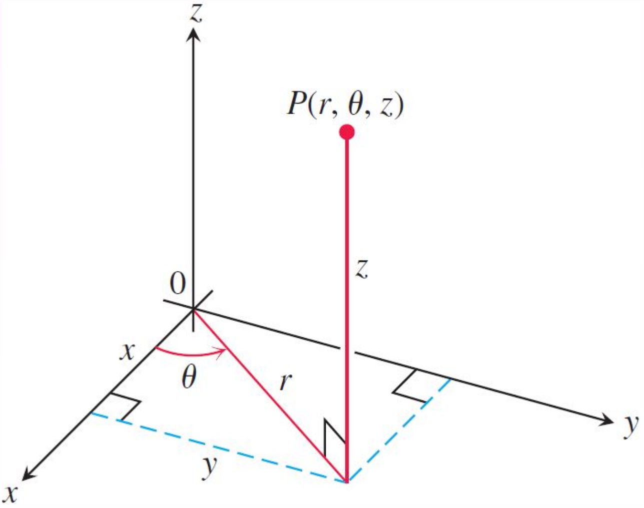
Equations relating rectangular (x, y, z) and cylindrical (r, θ, z) coordinates

$$x = r\cos\theta, \;\; y = r\sin\theta, \;\; z = z,$$
$$r^2 = x^2 + y^2, \;\; \tan\theta = y/x.$$

Cylindrical Coordinates



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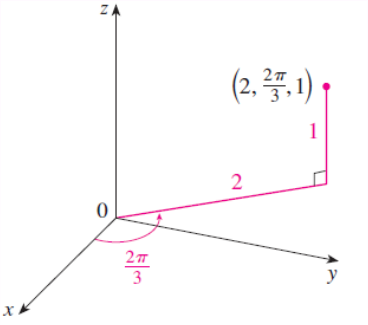

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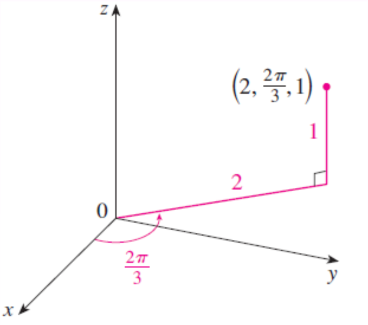

$$x = 2\cos(2\pi/3) = 2(-1/2) = -1$$
$$y = 2\sin(2\pi/3) = 2(\sqrt{3}/2) = \sqrt{3}$$
$$z = 1$$

Rectangular coordinates: $(-1, \sqrt{3}, 1)$

Examples

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Solution:

$$r = \sqrt{3^2 + (-3)^2} = 3\sqrt{2}$$

$$\tan\theta = -1 \implies \theta = \frac{7\pi}{4} + 2n\pi$$

$$z = -7$$

Therefore one set of cylindrical coordinates is $(3\sqrt{2}, \frac{7\pi}{4}, -7)$. Another is $(3\sqrt{2}, \frac{-\pi}{4}, -7)$. As with polar coordinates, there are infinitely many choices.

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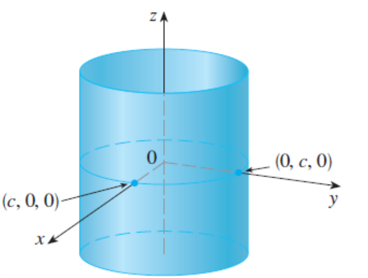

Figure: $r = c$, a cylinder

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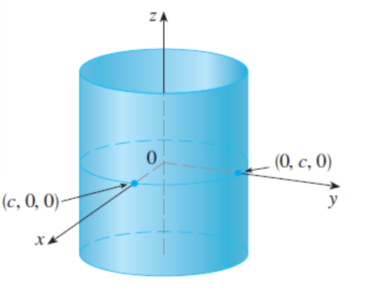


Figure: $r = c$, a cylinder

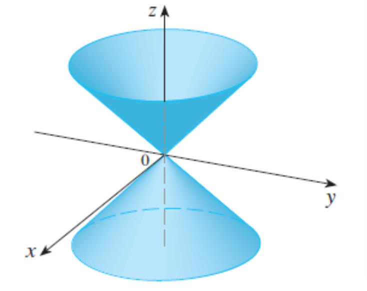
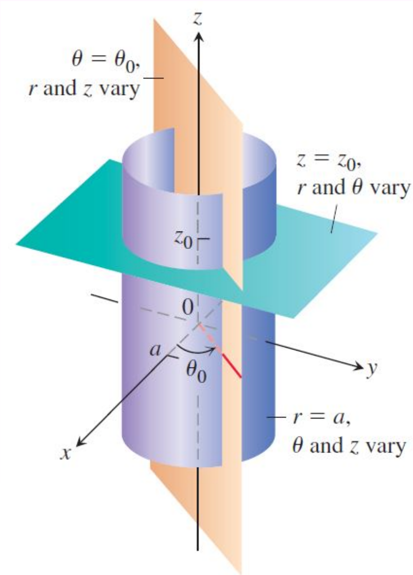


Figure: $z = r$, a cone

Constant-coordinate equations in cylindrical coordinates



How to find the limits in cylindrical coordinates

When computing triple integrals over a region D in cylindrical coordinates, we partition the region into n small cylindrical wedges, rather than into rectangular boxes. In cylindrical coordinates the volume of the wedge is approximated by the product

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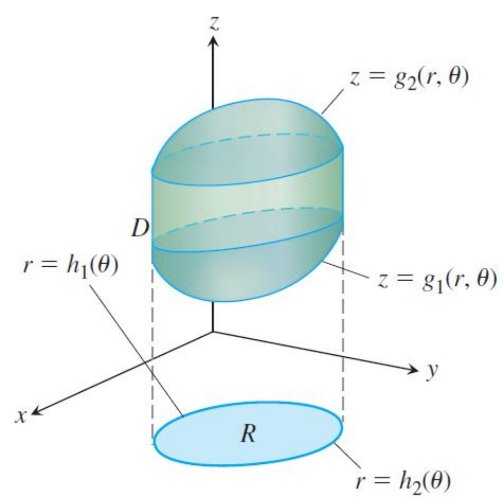
To find the limits for the triple integral $\iiint\limits_D f dV$:

- First sketch the solid D in three space.
- Find the z limits of the region.
- Find the projection of the solid R on the XY plane, from the projected region find the coordinates r, θ as one would do for polar coordinates.

Once the limits are found then substitute dV or $dz\ dy\ dx$ with $rdz\ dr\ d\theta$ to convert the integral.

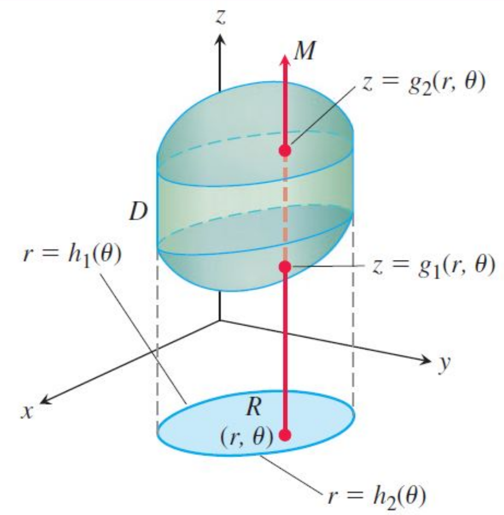
Step 1

Sketch the region D along with its projection R on the xy-plane. Label the surrounding faces and curves that bound D and R.



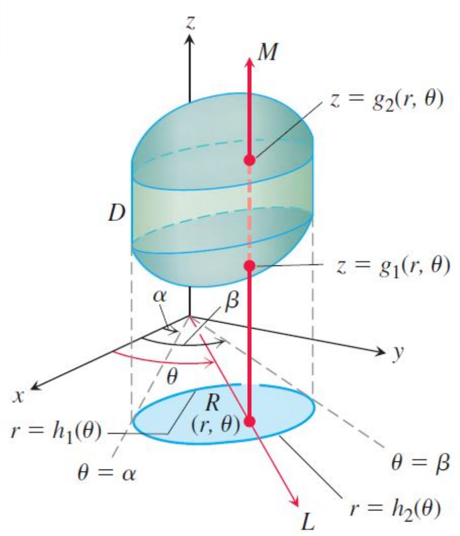
Step 2

Draw a line M through a typical point (r, θ) of R parallel to the z axis.



Step 3

Draw a ray L through (r, θ) from the origin.



Examples

The integral is

$$\iiint\limits_D F(x,y,z)dV = \int_{\alpha}^{\beta}\int_{h_1(\theta)}^{h_2(\theta)}\int_{g_1(r,\theta)}^{g_2(r,\theta)} F(r\cos\theta, r\sin\theta, z) r dz dr d\theta.$$

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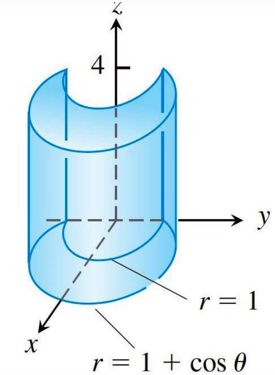

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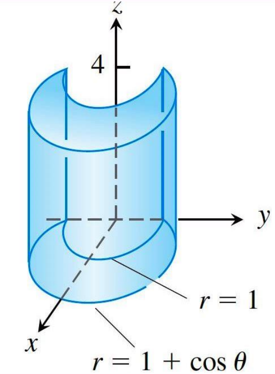

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$$0 \leq z \leq 4$$

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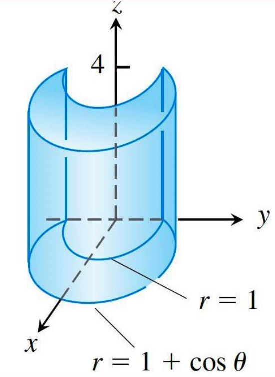

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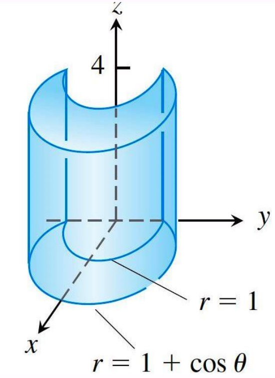

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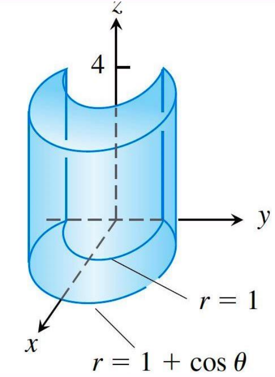

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So the integral turns out to be:

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{1}^{1+\cos\theta} \int_{0}^{4} r dz dr d\theta = 8 + \pi.$$