MATH F111- Mathematics I

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- Course Name: Mathematics I- MATH F111
- Introducing Handout
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Notations

We recall/denote the following notations:

- $\mathbb{C} =$ the set of all complex numbers.
- $\mathbb{R} =$ the set of all real numbers.
- $\mathbb{Z} = \{0, \pm 1, \pm 2, \ldots\}$ the set of all integers.
- $\mathbb{N} = \{1, 2, \ldots\}$ the set of all natural numbers.
- $\mathbb{Q} = \{\frac{p}{q} : p, q \in \mathbb{Z}, q \neq 0\}$ the set of all rational numbers.
- $\mathbb{R} \setminus \mathbb{Q}$ the set of all irrational numbers.

Polar Coordinates

In this section we study Polar Coordinates and their relation to Cartesian coordinates. We first formally introduce polar coordinates.

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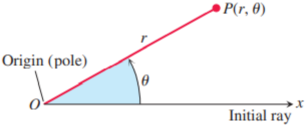
Definition (Polar Coordinates)

Let us fix an origin O and an **initial ray** from O. Now every point P in the plane can be determined by a pair (r, θ) (say) where r is the directed distance from O to P and θ is the directed angle between the initial ray and the line segment OP. This coordinate system is known as **polar coordinate system**.



Note

When we say $P = (r, \theta)$ is a point in a plane, then

- *r is the distance.*
- *θ is the angle.*

As in trigonometry, we calculate θ in the anti-clockwise direction.
Therefore θ is negative implies that we count in the clockwise direction.
In the xy-plane, we often consider the intial ray as the positive x-axis.

Example

Example

- *The point* $(1, 0)$ *in the xy-plane is described as* $(1, 0)$ *in polar coordinates. The same point can also be described as* $(1, 2\pi)$ *or* $(1, -2\pi)$. *In fact the point can generally be described as* $(1, 2n\pi)$, *where n is an integer.*

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- *The point* $(0, 1)$ *in the xy-plane is described as* $(1, \frac{\pi}{2})$ *in polar coordinates. In general, the point is described in the polar coordinate as* $(1, 2n\pi + \frac{\pi}{2})$, *where* n *is an integer.*

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- *Finally the point* $(1, 1)$ *in the* xy*-plane is* $(\sqrt{2}, \frac{\pi}{4} + 2n\pi)$ *in the polar coordinates.*

Properties

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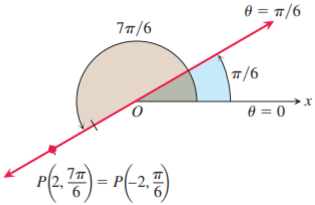

Example

Find all the polar coordinates of the point $P(2, \frac{\pi}{6})$.

Solution: We have already seen that we can represent the point P as $(2, \frac{\pi}{6})$ and $(-2, \frac{7\pi}{6})$.
The other representations are $(2, 2n\pi + \frac{\pi}{6})$ and $(-2, 2n\pi + \frac{7\pi}{6})$, where n is any integer.

Example

Fix $r = a \neq 0$ and vary θ over $[0, 2\pi]$. Then $P(r, \theta)$ traces a circle of radius $|a|$.

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Example (Example 2(a))

Both $r = 1$ and $r = -1$ are equations for the circle of radius 1 centered at O.

Example

Remark

If we fix $\theta = \theta_0$ and vary r between $-\infty$ and ∞, then we get a line passing through origin that makes an angle of θ with the initial ray.

Example (Example 2(b))

A line can have more than one polar equation.

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Example (Example 2(b))

A line can have more than one polar equation.

$\theta = \frac{\pi}{6}, \theta = \frac{7\pi}{6}$ and $\theta = -\frac{5\pi}{6}$ are equations of the same line.

Example

Graph the sets of points whose polar coordinates satisfy the following conditions.

(i). $1 \leq r \leq 2, 0 \leq \theta \leq \frac{\pi}{2}$.

(ii). $-3 \leq r \leq 2, \theta = \frac{\pi}{4}$.

(iii). $\frac{2\pi}{3} \leq \theta \leq \frac{5\pi}{6}$.

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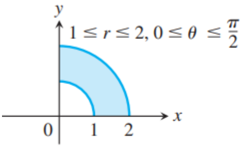
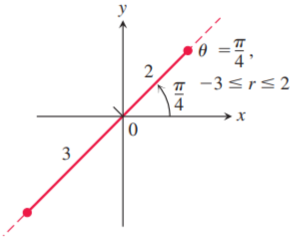
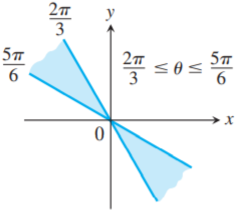
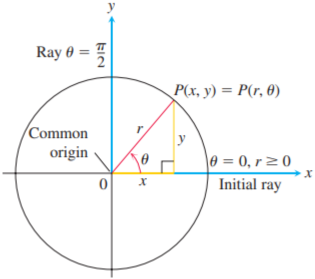






Relating Polar and Cartesian Coordinates

When we use both polar and Cartesian coordinates in a plane, we place the two origins together and let the initial polar ray be the positive x-axis.



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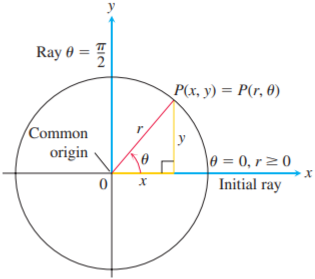

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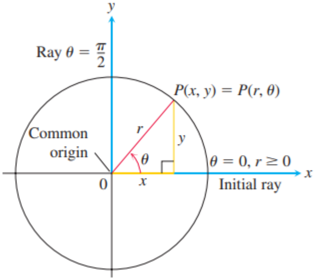

The ray $\theta = \frac{\pi}{2}$, becomes the positive y-axis.

- $x = r\cos\theta$,$y = r\sin\theta$.
- $r^2 = x^2 + y^2$, $\theta = \tan^{-1}\left(\frac{y}{x}\right)$.

Example

Find a polar equation for the circle $x^2 + (y-3)^2 = 9$.

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Solution: Putting $x = r\cos\theta$ and $y = r\sin\theta$, we have

$$r^2\cos^2\theta + (r\sin\theta - 3)^2 = 9$$

$$\Leftrightarrow r^2\cos^2\theta + r\sin^2\theta + 9 - 6r\sin\theta = 9$$

$$\Leftrightarrow r^2 - 6r\sin\theta = 0$$

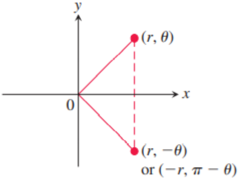
$$\Rightarrow r(r - 6\sin\theta) = 0$$

Thus $r = 6\sin\theta$.

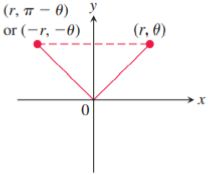
Graphing in Polar Coordinates

We will see how symmetries and tangents help in graphing the equation in polar coordinates. **Symmetry Tests for Polar Graphs in the Cartesian xy-Plane** To draw a graph, we first see the followings:

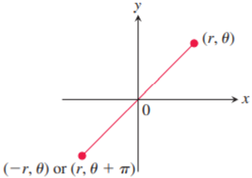
- *Symmetry about the x-axis*: If the point (r, θ) lies on a graph, then we check whether the point $(r, -\theta)$ or $(-r, \pi - \theta)$ lies on the graph or not.



- *Symmetry about the y-axis*: If the point (r, θ) lies on the graph, then check for the point $(r, \pi - \theta)$ or $(-r, -\theta)$ that lies on the graph or not.
- *Symmetry about the origin*: If the point (r, θ) lies on the graph, then we check for the point $(-r, \theta)$ or $(r, \pi + \theta)$ lies on the graph or not.



(b) About the y-axis



Slope of a Polar Curve

Let $r = f(\theta)$ be a polar curve. Then $x = r\cos\theta = f(\theta)\cos\theta$ and $y = r\sin\theta = f(\theta)\sin\theta$. If f is a differentiable function of θ, then so are x and y and when $\frac{dx}{d\theta} \neq 0$, we have

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{\frac{d}{d\theta}(f(\theta)\sin\theta)}{\frac{d}{d\theta}(f(\theta)\cos\theta)} = \frac{f'(\theta)\sin\theta + f(\theta)\cos\theta}{f'(\theta)\cos\theta - f(\theta)\sin\theta} \tag{1}$$

where $f'(\theta) = \frac{df}{d\theta}$.

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Remark

$\frac{dy}{dx} \neq \frac{dr}{d\theta}$.

Properties

If the curve $r = f(\theta)$ passes through the origin at $\theta = \theta_0$, then $f(\theta_0) = 0$, and the slope equation gives

Remark (Slope of the Curve $r = f(\theta)$ in the Cartesian xy-Plane)

$$\frac{dy}{dx}_{(0,\theta_0)} = \frac{f'(\theta_0)\sin\theta_0}{f'(\theta_0)\cos\theta_0} = \frac{\sin\theta_0}{\cos\theta_0} = \tan\theta_0.$$

That is, the slope at $(0, \theta_0)$ is $\tan\theta_0$. The reason we say "slope at $(0, \theta_0)$" and not just "slope at the origin" is that a polar curve may pass through the origin (or any point) more than once, with different slopes at different θ values.