MATHEMATICS-I

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Lecture 8

Subsequences and Infinite series

Subsequences:

- Consider the sequences $\{1, 1/2, 1/3, \ldots\}$.
- $\{1, 1/3, 1/5, \ldots\}$; $\{1, 1/2, 1/4, \ldots\}$ or $\{1/3, 1/6, 1/9, \ldots\}$.
- Let $n_1 < n_2 < n_3 < \cdots$ be strictly increasing sequence of positive integers and $\{a_n\}$ be a sequence of real numbers then the sequence $\{a_{n_k}\}_{k=1}^{\infty} = \{a_{n_1}, a_{n_2}, a_{n_3}, \ldots\}$ is called a subsequence of $\{a_n\}$.
- Examples: $\{1, 1, 1, \ldots\}$ and $\{-1, -1, -1, \ldots\}$ are subsequences of $\{(-1)^n\}$.
- $\{1/2^{n^2}\}$ is a subsequence of $\{1/2^n\}$.

Properties of Subsequences

Theorem 0.1.

If the sequence $\{a_n\}$ converges to L then all the subsequences $\{a_{n_k}\}$ converges to L.

Corollary 0.2.

If one of the subsequences of $\{a_n\}$ diverges then the sequence $\{a_n\}$ also diverges.

Theorem 0.3.

If the subsequences $\{a_{2n}\}$ and $\{a_{2n+1}\}$ of $\{a_n\}$ converge to same limit L, then the sequence $\{a_n\}$ also converges to L.

Questions

1. Find the limit of the followings as $n \to \infty$.
   - $\sqrt{n+1} - \sqrt{n}$
   - $\ln(\frac{n+2}{1+4n})$
   - $\frac{\sin n}{n}$
   - $\frac{\cos n^2}{n^2}$
2. Suppose a_n is sequence of real number converging to a. Show that the sequence $\{\frac{a_1+a_2+\cdots+a_n}{n}\}$ is also converging to the same limit a.
3. If x_n is a sequence of real numbers such that $\{x_{n+1} - x_n\}$ convergens to some $x \in R$. Is the sequence x_n/n convergent ? If so find the limit.
4. Let $x_1 = 1$ and $x_{n+1} = (\frac{n}{n+1})x_n^2$ for all n. Examine whether the sequence x_n is convergent. Also, find the limit if it is convergent.

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- In this section we want to understand the meaning of such an infinite sum and to develop methods to calculate it.
- In order to give meaning for the infinite sum, we just consider the sum of the first n terms

$$s_n = a_1 + a_2 + \cdots + a_n = \sum_{k=1}^{n} a_k.$$

Infinite Series

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$$\sum_{k=1}^{\infty} a_k = a_1 + a_2 + a_3 + \cdots = \lim_{n\to\infty} s_n$$

whenever the later limit exists.

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- For example consider the series $\sum_{k=1}^{\infty} 1/2^{k-1}$.

- $s_n = \sum_{k=1}^{n} 1/2^{k-1} = 2 - \frac{1}{2^{n-1}}$, therefore we can say that

$$\sum_{k=1}^{\infty} 1/2^{k-1} = \lim_{n\to\infty} \left(2 - \frac{1}{2^{n-1}}\right) = 2$$

Infinite Series Conti.

- Given a sequence of numbers $\{a_n\}$, an expression of the form

$$a_1 + a_2 + a_3 + \cdots + a_n + \cdots .$$

is called an **infinite series.** The number a_n is called nth term of the series.

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- The sequence $\{s_n\}$ defined by

$$s_n = \sum_{k=1}^{n} a_k = a_1 + a_2 + \cdots a_n$$

is called the sequence of partial sums of the series and s_n is called ***n*th partial sum**.

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- If the sequence $\{s_n\}$ of partial sums converges to a limit L, we say the series converges and its **sum** is L. In this case we also write

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- If the sequence $\{s_n\}$ of partial sums does not converge, we say the the series **diverges.**

Geometric Series: Geometric series are series of the form (for $a, r \in \mathbb{R}$)

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Theorem 0.4.

If $|r| < 1$ then the above geometric series converges and

$$a + ar + ar^2 + \cdots + ar^{n-1} + \cdots = \sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}.$$

if $|r| \geq 1$, the series diverges.

Thank you