# Double and Triple Integrals

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# Double Integrals - Polar Form

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Solution:

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We sketch the region. Note that we have to write the region as disjoint union of the region $y \leq 2$ and the region $y \geq 2$. The final integral is

$$\int_0^2 \int_{\frac{y}{\pi}}^{y} \frac{1}{1+y^2} dx dy + \int_2^{2\pi} \int_{\frac{y}{\pi}}^{2} \frac{1}{1+y^2} dx dy,$$

which can be computed easily.

# Examples

1. Integrate the following by changing to the polar coordinates:
   - $\displaystyle\int_0^1 \int_x^1 \frac{y}{x^2+y^2}\,dy\,dx.$
   - $\displaystyle\int_0^1 \int_{-y/3}^{y/3} \frac{y}{\sqrt{x^2+y^2}}\,dx\,dy.$
2. Find the area of the region that lies inside the cardioid $r = 1 + \cos\theta$ and outside the circle $r = 1$. ($(8+\pi)/4$)
3. Find the volume of the solid cut from the first octant by the surface $z = 4 - x^2 - y$. (128/15)
4. Find the volume of the sphere of radius $a$ by double integration. ($\frac{4}{3}\pi a^3$ - compute the volume of the portion in the first octant and multiply the answer by $8$ )
5. Find the average height of the hemispherical surface $z = \sqrt{a^2 - x^2 - y^2}$ above the disk $x^2 + y^2 \leq a^2$ in the $xy$-plane. ($2a/3$)

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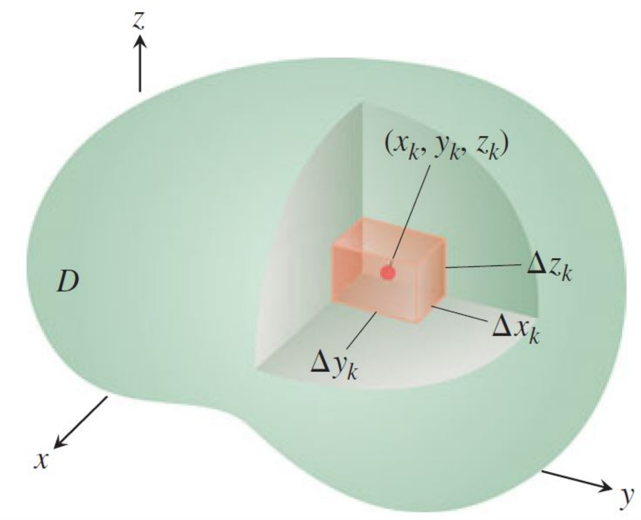
We partition a rectangular box like region containing $D$ into rectangular cells by planes parallel to the coordinate axes.

# Triple Integrals



## Triple Integrals

We number the cells that lie completely inside $D$ from $1$ to $n$ in some order, the $k$-th cell having dimensions $\Delta x_k \times \Delta y_k \times \Delta z_k$ and volume $\Delta V_k = \Delta x_k \Delta y_k \Delta z_k$. We choose a point $(x_k, y_k, z_k)$. Define the norm of the partition $P$ as $\|P\| = \max\{\Delta x_k, \Delta y_k, \Delta z_k\}$.

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Triple integral of $F$ over $D$ is defined by

$$\iiint\limits_D F(x,y,z)dV = \lim_{\|P\|\to 0} \sum_{k=1}^{n} F(x_k, y_k, z_k)\Delta V_k,$$

provided the limit exists.

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### Average

The average value of a function $F$ over a region $D$ in space is

$$\frac{1}{\text{Volume of } D} \iiint_D F\,dV.$$

# Triple Integral over Rectangular Region

## Fubini's Theorem

Suppose that $f(x, y, z)$ is continuous on the region $D = \{(x, y, z) : a \leq x \leq b,\ c \leq y \leq d,\ e \leq z \leq f\}$. Then, triple integral can be written as triple iterated integral:

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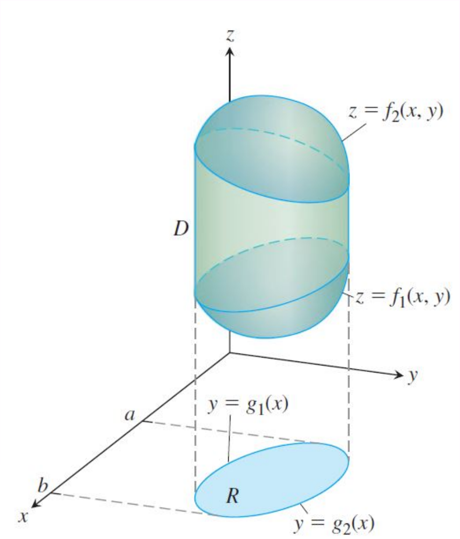
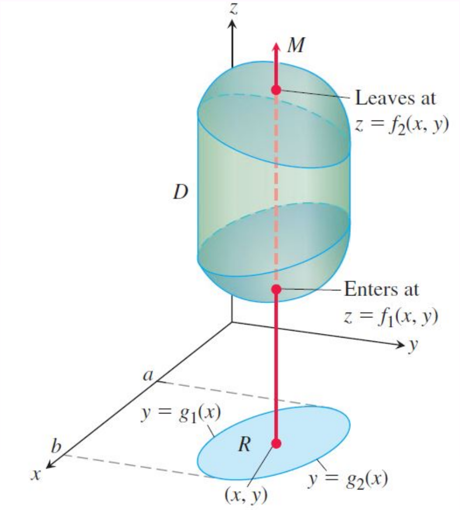
- Find the $x$ and $y$ limits for the projection in $xy$-plane as discussed in the double integrals in accordance of your order of integral.

# Triple Integrals - Step 1

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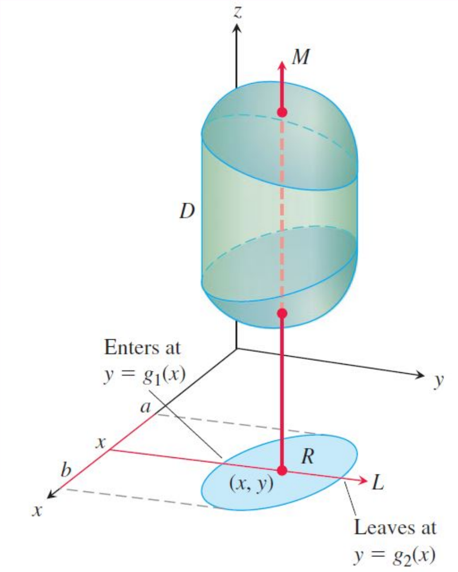

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# Examples

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Example 2: Evaluate the integral in Example 1 using $dydxdz$.

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$$V = \iiint_D dV = \int_{-3}^{3} \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} \int_{-\sqrt{9-x^2-y^2}}^{\sqrt{9-x^2-y^2}} dzdydx.$$

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After performing the integration with respect to $z$, transform the resulting integral into polar coordinates. The volume is $36\pi$ cubic units.

# Examples

Example 4: Let $D$ be the region bounded by the paraboloids $z = 8 - x^2 - y^2$ and $z = x^2 + y^2$. Write six different triple iterated integrals for the volume of $D$. Evaluate one of the integrals.

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## Examples

- Find the average value of $F(x, y, x) = x^2 + 9$ over the cube in the first octant bounded by the coordinate planes and the planes $x = 2$, $y = 2$ and $z = 2$. $(31/3)$
- Solve for $a$:

$$\int_0^1 \int_0^{4-a-x^2} \int_a^{4-x^2-y} dzdydx = \frac{4}{15}. \ (a = 13/3, 3)$$

Solved examples from Thomas' Calculus (please review the figures for guidance on how to sketch the region.)

- Find the volume of the region $D$ enclosed by the surface $z = x^2 + 3y^2$ and $z = 8 - x^2 - y^2$.
- Set up the limits of integration for evaluating the triple integral of a function $F(x, y, z)$ over the tetrahedron $D$ with vertices $(0, 0, 0), (1, 1, 0), (0, 1, 0)$, and $(0, 1, 1)$. Use the order of integration $dy\ dz\ dx$ and $dz\ dy\ dx$.