Directional Derivatives & Gradient Vectors

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ANNOUNCEMENT:

An additional class will be held this **Saturday (26 October 2024)** from **12:00 PM to 1:00 PM** in **LT3**.

Recall - Directional derivative

Definition

The derivative of $f(x, y)$ at $P_0(x_0, y_0)$ in the direction of the unit vector $\boldsymbol{u} = u_1\boldsymbol{i} + u_2\boldsymbol{j}$ is the number

$$(D_{\boldsymbol{u}}f)_{P_0} = \left(\frac{df}{ds}\right)_{\boldsymbol{u},P_0} := \lim_{s\to 0} \frac{f(x_0 + u_1 s, y_0 + u_2 s) - f(x_0, y_0)}{s}$$

provided the limit exists.

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- $f_x(x_0, y_0)$ - directional derivative of f at P_0 in the direction of $\boldsymbol{i}$
- $f_y(x_0, y_0)$ - directional derivative of f at P_0 in the direction of $\boldsymbol{j}$
- directional derivative $(D_{\boldsymbol{u}}f)_{P_0}$ - rate of change of f at P_0 in the direction of $\boldsymbol{u}$

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- $f_y(x_0,y_0)$ - directional derivative of f at P_0 in the direction of $\boldsymbol{j}$
- directional derivative $(D_{\boldsymbol{u}}f)_{P_0}$ - rate of change of f at P_0 in the direction of $\boldsymbol{u}$
- For an angle θ measured from the positive x-axis, $\boldsymbol{u} = \cos\theta\boldsymbol{i} + \sin\theta\boldsymbol{j}$.

Formula to Calculate the Directional Derivative

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Now, choose $s = 0$. Then $x = x_0$, $y = y_0$ and

$$g'(0) = f_x(x_0, y_0)u_1 + f_y(x_0, y_0)u_2 = (f_x|_{P_0}, f_y|_{P_0}) \cdot (u_1, u_2).$$

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Consequently, $(D_{\boldsymbol{u}} f)_{P_0} = (f_x|_{P_0}, f_y|_{P_0}) \cdot (u_1, u_2)$.
This says that the derivative of a differentiable function f in the direction of $\boldsymbol{u}$ at P_0 is the dot product of $\boldsymbol{u}$ with a special vector $(f_x|_{P_0}, f_y|_{P_0})$.

Definition

The gradient vector (or gradient) of $f(x, y)$ is the vector

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Theorem (The Directional Derivative is a Dot Product)

If $f(x, y)$ is differentiable, then

$$\left(\frac{df}{ds}\right)_{\boldsymbol{u}, P_0} = (\nabla f)_{P_0} \cdot \boldsymbol{u},$$

the dot product of ∇f at P_0 and $\boldsymbol{u}$. In brief, $D_{\boldsymbol{u}} f = \nabla f \cdot \boldsymbol{u}$.

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Consider

$$f(x,y) = \begin{cases} \frac{x^2 y}{x^4+y^2} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

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This function is not continuous at $(0,0)$ (choose $y = mx^2, m \neq 0$ and show that limit does not exist) and hence is not differentiable at $(0,0)$.
Let $\boldsymbol{u} = (u_1, u_2)$ be a unit vector. Then,

$$(D_{\boldsymbol{u}} f)_{|(0,0)} = \lim_{s \to 0} \frac{f(u_1 s, u_2 s) - f(0,0)}{s} = \lim_{s \to 0} \frac{s^3 u_1^2 u_2}{s(s^4 u_1^4 + s^2 u_2^2)}$$

$$== \lim_{s \to 0} \frac{u_1^2 u_2}{s^2 u_1^4 + u_2^2} = \begin{cases} \frac{u_1^2}{u_2} & \text{if } u_2 \neq 0, \\ 0 & \text{if } u_2 = 0. \end{cases}$$

This shows that the directional derivatives in all directions at $(0,0)$ exist.

Examples

1. Find the derivative of $f(x, y) = 2xy - 3y^2$ at $P_0(5, 5)$ in the direction of $\boldsymbol{u} = 4\boldsymbol{i} + 3\boldsymbol{j}$ using gradient.

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$$\begin{aligned}(D_{\boldsymbol{v}} f)_{P_0} &= \nabla f_{|P_0} \cdot \boldsymbol{v} = (f_x, f_y)_{|P_0} \cdot \boldsymbol{v} \\ &= (10, -20) \cdot \left(\frac{4}{5}, \frac{3}{5}\right) = 10\left(\frac{4}{5}\right) - 20\left(\frac{4}{5}\right) = -4.\end{aligned}$$

Examples (Using Gradient)

2. Find the derivative of $f(x, y, z) = xy + yz + zx$ at $P_0(1, -1, 2)$ in the direction of $\boldsymbol{u} = 3\boldsymbol{i} + 6\boldsymbol{j} - 2\boldsymbol{k}$. (Ans: 3)
3. Find the derivative of $g(x, y) = \frac{x-y}{xy+2}$ at $P_0(1, -1)$ in the direction of $\boldsymbol{u} = 12\boldsymbol{i} + 5\boldsymbol{j}$. (Ans: -4)
4. Find the derivative of $h(x, y) = \tan^{-1}(y/x) + \sqrt{3}\sin^{-1}(xy/2)$ at $P_0(1, 1)$ in the direction of $\boldsymbol{u} = 3\boldsymbol{i} - 2\boldsymbol{j}$. (Ans: $3/(2\sqrt{13})$)
5. Find the derivative of $g(x, y, z) = 3e^x \cos(yz)$ at $P_0(0, 0, 0)$ in the direction of $\boldsymbol{u} = 2\boldsymbol{i} + \boldsymbol{j} - 2\boldsymbol{k}$. (Ans: 2)

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For two vectors $\boldsymbol{v}$ and $\boldsymbol{w}$,

$$\boldsymbol{v} \cdot \boldsymbol{w} = |\boldsymbol{v}||\boldsymbol{w}| \cos\theta,$$

where $0 \leq \theta \leq \pi$ is the angle between $\boldsymbol{v}$ and $\boldsymbol{w}$.

Properties of $D_{\boldsymbol{u}} f = \nabla f \cdot \boldsymbol{u} = |\nabla f| \cos\theta$

1. The function f increases most rapidly when $\cos\theta = 1$ or when $\theta = 0$ and $\boldsymbol{u}$ is the direction of ∇f. That is, at each point P in its domain, f increases most rapidly in the direction of the gradient vector ∇f at P. The derivative in this direction is $D_{\boldsymbol{u}} f = |\nabla f| \cos 0 = |\nabla f|$.

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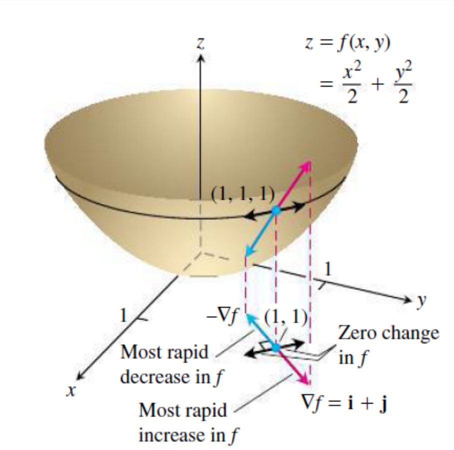
Hence,

$$D_{\boldsymbol{u}} f \in [-|\nabla f|, |\nabla f|] .$$

$D_{\boldsymbol{u}} f$ is maximum when $\theta = 0$ and minimum when $\theta = \pi$.

Figure for $f(x,y) = \frac{x^2}{2} + \frac{y^2}{2}$



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$$\nabla f_{|(1,0)} = (2xy + ye^{xy}\sin y)\boldsymbol{i} + (x^2 + xe^{xy}\sin y + e^{xy}\cos y)\boldsymbol{j}_{|(1,0)} = 2\boldsymbol{j}.$$

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f increases most rapidly in the direction $\boldsymbol{u} = \boldsymbol{j}$. The derivative of f in this direction is $D_{\boldsymbol{u}}(f) = |\nabla f| = 2$.

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   f increases most rapidly in the direction $\boldsymbol{u} = \boldsymbol{j}$. The derivative of f in this direction is $D_{\boldsymbol{u}}(f) = |\nabla f| = 2$.
   f decreases most rapidly in the direction $-\boldsymbol{u} = -\boldsymbol{j}$ and the derivative of f in this direction is $D_{-\boldsymbol{u}}(f) = -2$.

Examples

2. Find the directions in which $g(x, y, z) = xe^y + z^2$, increase and decrease most rapidly at $P_0(1, \ln 2, 1/2)$. Then find the derivatives of g in these directions. (Ans: 3, -3)

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3. In what direction is the derivative of $f(x, y) = xy + y^2$ at $P(3, 2)$ is zero? (Ans: $\frac{7}{\sqrt{53}}\boldsymbol{i} - \frac{2}{\sqrt{53}}\boldsymbol{j}$, $\frac{-7}{\sqrt{53}}\boldsymbol{i} + \frac{2}{\sqrt{53}}\boldsymbol{j}$)

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4. Is there a direction $\boldsymbol{u}$ in which the rate of change of the temperature function $T(x,y,z) = 2xy - yz$ (temperature in degrees Celsius, distance in feet) at $P(1,-1,1)$ is -$3°$C/ft? Give reasons for your answer.

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5. The derivative of $f(x, y)$ at $P_0(1, 2)$ in the direction of $\boldsymbol{i} + \boldsymbol{j}$ is $2\sqrt{2}$ and in the direction of $-2\boldsymbol{j}$ is -3. What is the derivative of f in the direction of $-\boldsymbol{i} - 2\boldsymbol{j}$?

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3. In what direction is the derivative of $f(x, y) = xy + y^2$ at $P(3, 2)$ is zero? (Ans: $\frac{7}{\sqrt{53}}\boldsymbol{i} - \frac{2}{\sqrt{53}}\boldsymbol{j}$, $\frac{-7}{\sqrt{53}}\boldsymbol{i} + \frac{2}{\sqrt{53}}\boldsymbol{j}$)

4. Is there a direction $\boldsymbol{u}$ in which the rate of change of the temperature function $T(x, y, z) = 2xy - yz$ (temperature in degrees Celsius, distance in feet) at $P(1, -1, 1)$ is -3°C/ft? Give reasons for your answer. (No - Compute max. and min. of $D_{\boldsymbol{u}}f$.)

5. The derivative of $f(x, y)$ at $P_0(1, 2)$ in the direction of $\boldsymbol{i} + \boldsymbol{j}$ is $2\sqrt{2}$ and in the direction of $-2\boldsymbol{j}$ is -3. What is the derivative of f in the direction of $-\boldsymbol{i} - 2\boldsymbol{j}$? (Ans: $\frac{-7}{\sqrt{5}}$)

Algebra Rules for Gradients

- Sum Rule: $\nabla(f+g) = \nabla f + \nabla g$.
- Difference Rule: $\nabla(f-g) = \nabla f - \nabla g$.
- Constant Multiple Rule: $\nabla(kf) = k\nabla(f)$ for any constant k
- Product Rule: $\nabla(fg) = f\nabla(g) + g\nabla(f)$.
- Quotient Rule: $\nabla\left(\frac{f}{g}\right) = \frac{g\nabla f - f\nabla g}{g^2}$

Funtions of Three Variables

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The derivative of $f(x, y, z)$ at $P_0(x_0, y_0, z_0)$ in the direction of the unit vector $\boldsymbol{u} = u_1\boldsymbol{i} + u_2\boldsymbol{j} + u_3\boldsymbol{k}$ is the number

$$(D_{\boldsymbol{u}}f)_{P_0} = \lim_{s\to 0} \frac{f(x_0 + u_1 s, y_0 + u_2 s, z_0 + u_3 s) - f(x_0, y_0, z_0)}{s}$$

provided the limit exists.

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$$(D_{\boldsymbol{u}}f)_{P_0} = \nabla f \cdot \boldsymbol{u}, \text{ where } \nabla f = (f_x, f_y, f_z).$$

At any given point, f increases most rapidly in the direction of ∇f and decreases most rapidly in the direction of $-\nabla f$. In any direction orthogonal to ∇f, the derivative is zero.

Gradients and Tangents to Level Curves

If a differentiable function $f(x, y)$ has a constant value c along a smooth curve $\boldsymbol{r} = g(t)\boldsymbol{i} + h(t)\boldsymbol{j}$, then $f(g(t), h(t)) = c$. Differentiating both sides of this equation with respect to t leads to

$$\frac{d}{dt} f(g(t), h(t)) = \frac{d}{dt}(c)$$

Using chain rule, $\dfrac{\partial f}{\partial x}\dfrac{dg}{dt} + \dfrac{\partial f}{\partial y}\dfrac{dh}{dt} = 0 \implies \nabla f \cdot \dfrac{d\boldsymbol{r}}{dt} = 0.$

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This says that ∇f is orthogonal to the tangent vector $d\boldsymbol{r}/dt$, so it is normal to the curve.

At every point (x_0, y_0) in the domain of a differentiable function $f(x, y)$, the gradient of f is normal to the level curve through (x_0, y_0).

Gradients to Level Curves

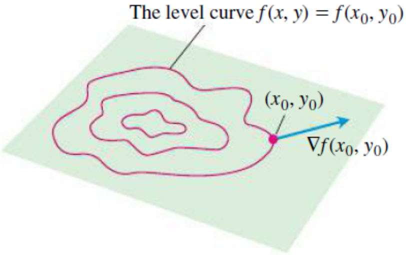

FIGURE 14.30 The gradient of a differentiable function of two variables at a point is always normal to the function's level curve through that point.

Tangent Lines

Tangent lines - lines that are tangent to the level curves

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If $\boldsymbol{N} = \nabla f$, then the equation of the tangent line to level curve $f(x, y) = c$ at $P_0(x_0, y_0)$ is given by

$$f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) = 0.$$

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- Write an equation for the tangent line $x^2 - xy + y^2 = 7$, $P_0(-1, 2)$.

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- Write an equation for the tangent line $x^2 - xy + y^2 = 7$, $P_0(-1, 2)$. (Ans: $y = x - 4$)
- Write an equation for the tangent line $xy = -4, P_0(2, -2)$. (Ans: $-4x + 5y - 14 = 0.$)