MATH F111- Mathematics I

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Notations

$\mathbb{N}$- Set of Natural numbers
$\mathbb{Q}$- Set of rational numbers
$\mathbb{R}$- Set of real numbers
$\forall$ - For all
$\exists$- There exists

Intervals

Definition

A subset I of $\mathbb{R}$ is said to be an interval if $a, b \in I$ and $a < x < b \implies x \in I$.

Let $a, b \in \mathbb{R}$ and $a < b$.

- $(a, b) := \{x \in \mathbb{R} : a < x < b\}$ (open interval)
- $[a, b] := \{x \in \mathbb{R} : a \leq x \leq b\}$ (closed interval)
- $[a, b) := \{x \in \mathbb{R} : a \leq x < b\}$ and $(a, b] := \{x \in \mathbb{R} : a < x \leq b\}$ are half-open (or half-closed) intervals.
- $(a, \infty) := \{x \in \mathbb{R} : x > a\}$ and $(-\infty, a) := \{x \in \mathbb{R} : x < a\}$ are infinite open intervals.
- $[a, \infty) := \{x \in \mathbb{R} : x \geq a\}$ and $(-\infty, a] := \{x \in \mathbb{R} : x \leq a\}$ are infinite closed intervals.

Let $a \in \mathbb{R}$ and $\epsilon > 0$. Then $(a - \epsilon, a + \epsilon)$ is called the **ϵ-neighborhood of a.**

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- The values x_n are also called the terms or the elements of the sequence and x_n (that is, the value of x at n) is called the n-th term of the sequence.

We will denote this sequence by the notations

$$(x_n), \quad \text{or} \quad (x_n : n \in \mathbb{N}).$$

In this course, we will consider only Real sequences.

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- $\big((-1)^n : n \in \mathbb{N}\big) = (-1, 1, -1, 1, \ldots)$
- $x_1 := 1, x_2 := 1$ *and* $x_n := x_{n-1} + x_{n-2}$ *for* $n \geq 3$*:* $(1, 1, 2, 3, 5, 8, 13, 21, 34, 55, \ldots)$ *This sequence is known as the Fibonacci sequence.*

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If a sequence is not bounded, it is said to be **unbounded**. eg. $(a_n) = (-1)^n n$

- $(a_n) = \{1, \frac{1}{2}, \frac{1}{3}, \cdots, \frac{1}{n} \cdots\}$

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Remark

Question: What do we mean by a sequence converges?

It says that if we go far enough out in the sequence, the difference between a_n and the limit of the sequence becomes less than any preselected number $\epsilon > 0$.

Let us see this with $(a_n) = \frac{1}{n}$.

- Can you find an integer N such that $|a_n - 0| < \frac{1}{2}, \forall n \geq N$?

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- For any preselected positive number, say $\epsilon > 0$, can you find an integer N such that $|a_n - 0| < \epsilon, \forall n \geq N$?

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- Can you find an integer N such that $|a_n - 0| < \frac{1}{10000}, \forall n \geq N$? Yes, choose $N = 10001$.
- For any preselected positive number, say $\epsilon > 0$, can you find an integer N such that $|a_n - 0| < \epsilon, \forall n \geq N$? Yes, choose $N > \frac{1}{\epsilon}$. In particular choose $N = \lfloor \frac{1}{\epsilon} \rfloor + 1$.

The Limit of a Sequence

Definition

A sequence (a_n) in $\mathbb{R}$ is said to converge to $\ell \in \mathbb{R}$, or ℓ is said to be a limit of (a_n),
if for every $\epsilon > 0$, there exists an integer $N \in \mathbb{N}$ such that

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Remark

The choice of N depends on the value of ϵ.

When a sequence (a_n) has limit ℓ, we will use the notation

$$\lim a_n = \ell.$$

We will sometimes use the symbolism $a_n \to \ell$, which indicates the intuitive idea that the values a_n "approach" the number ℓ as $n \to \infty$.

- If a sequence has a limit, we say that the sequence is **convergent**
- If a sequence has no limit, we say that the sequence is **divergent**.

There is also a notion of divergence if the sequence is not bounded.

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Similarly we say (a_n) diverges to negative infinity or

$$\lim_{n\to\infty} a_n = -\infty$$

if for every number m, there is an integer N such that $\forall n > N$, we have $a_n < m$.

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Examples:

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(ii) $a_n := 1/n$ for all $n \in \mathbb{N}$. Then $a_n \to 0$.

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Choose $N \in \mathbb{N}$ such that $N > 5/3\epsilon$. Then $|a_n - 0| < \frac{5}{3n} < \epsilon$ for all $n \geq N$. Then $a_n \to 0$.

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$$|r^n - 0| = |r|^n = \frac{1}{(1+a)^n}.$$

We have $(1+a)^n > na$ for all $n \in \mathbb{N}$ and hence,

$$|r^n - 0| < \frac{1}{na} \text{ for all } n \in \mathbb{N}.$$

Let $\epsilon > 0$ be given. Then

$$|r^n - 0| < \epsilon \text{ holds } \text{ if } n > \frac{1}{a\epsilon}.$$

Choose any $N \in \mathbb{N}$ such that $N > \frac{1}{a\epsilon}$. Then

$$\forall n \geq N, \ |r^n - 0| < \epsilon.$$

Since $\epsilon > 0$ is arbitrary,

$$\lim_{n \to \infty} r^n = 0.$$