Polar Coordinates

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August 5, 2024

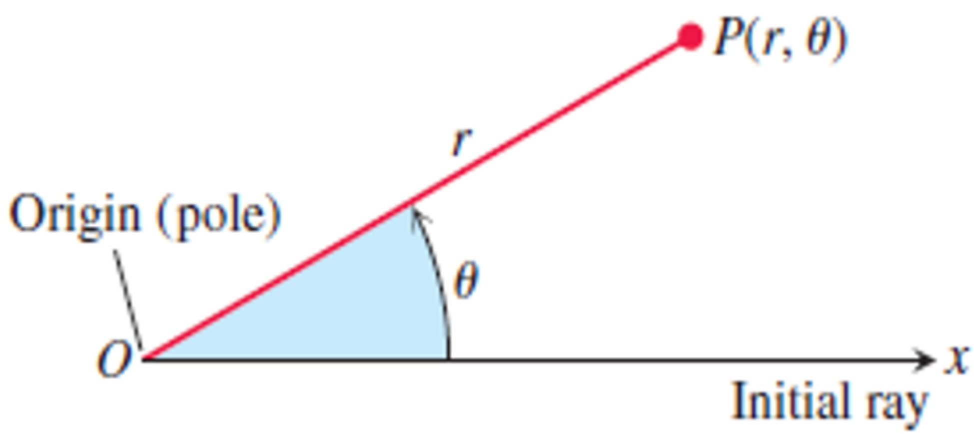
- We know how to specify the location of a point in the plane by means of coordinates relative to two perpendicular coordinates axes.
  Such a system is called as Cartesian (or rectangular) coordinate systems.
- Some time a moving point has special affinity for some fixed point, such as a planet moving in an orbit under the central attraction of Sun.
- In such cases the path of particle is best described by its angular direction and its distance from the fixed point.

  This representation of a point is called Polar coordinates.

Mathematical definition

Polar coordinates for a point P.

1. Fix an origin O, called the pole, and an initial ray from O (initial ray is called polar axis).
2. Let r be the 'directed' distance from O to P and θ be the 'directed' angle (counterclockwise, usually measured in radians) from the polar axis to the ray OP.
3. P is represented by the ordered pair (r, θ). Here r, θ are called polar coordinates of the point P.



Types of coordinate systems

cartesian (x, y)

Two dimensional

Polar (r, θ)

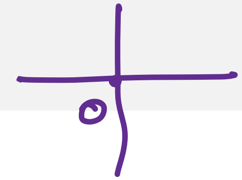
cartesian — (x, y, z)

Three dimensional

spherical Polar (r, θ, ϕ)

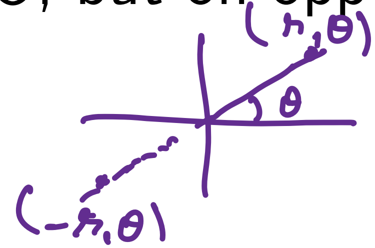
cylindrical Polar (r, θ, z)

Remarks

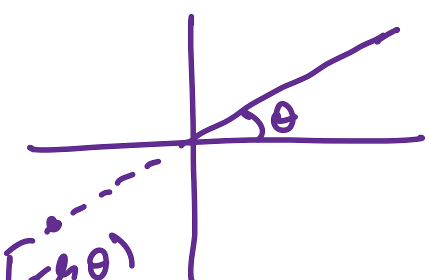


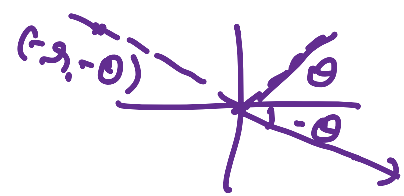
1. If $P = O$, then $r = 0$ and $(0, \theta)$ represents the origin (pole) for any value of θ.

2. The point (r, θ) and $(-r, \theta)$ lie on the same line through the origin O and the same distance $|r|$ from O, but on opposite sides of O.



3. Note that $(-r, \theta)$ represents the same point as $(r, \theta + \pi)$.

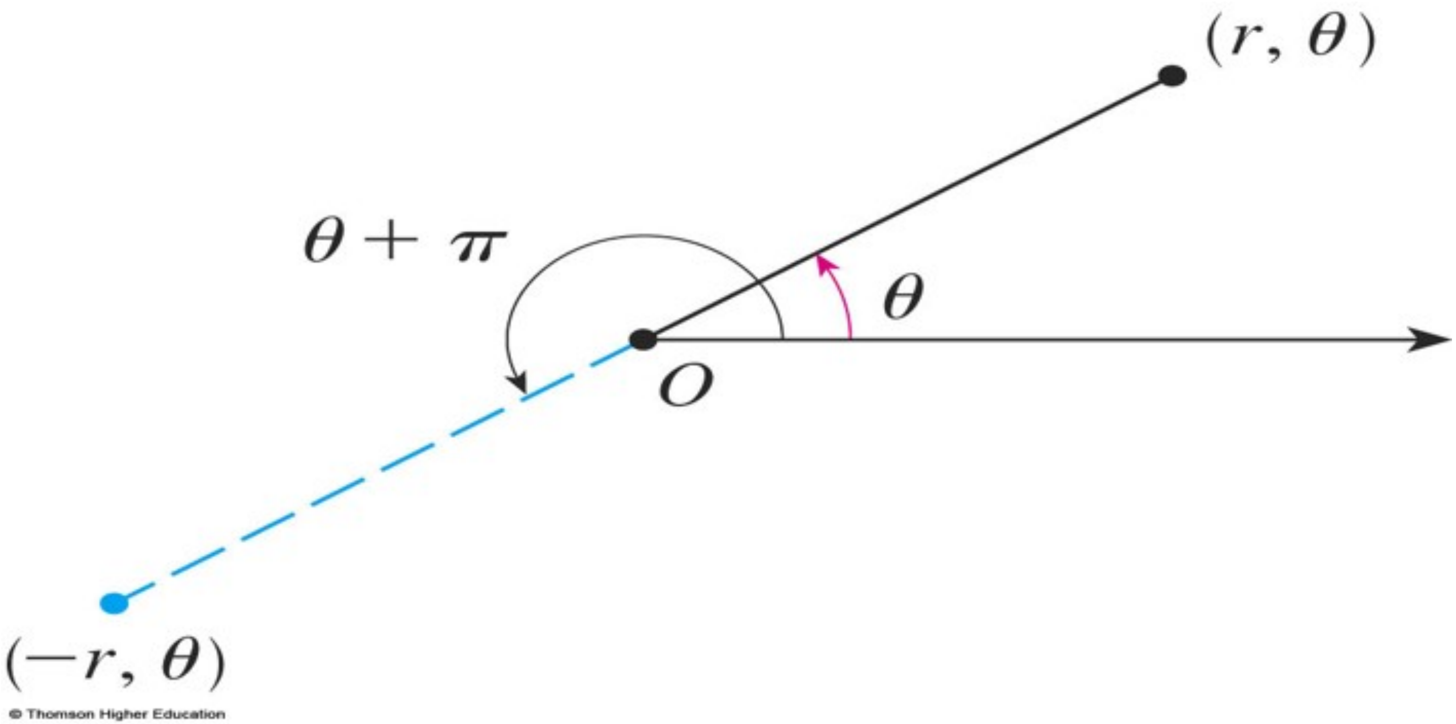




$(-r, \theta + 2\pi n) = (-r, \theta)$
$= (r, \theta + \pi) = (r, \theta + \pi + 2\pi n)$

4. If $r > 0$, then the point (r, θ) lies in the same quadrant as θ.

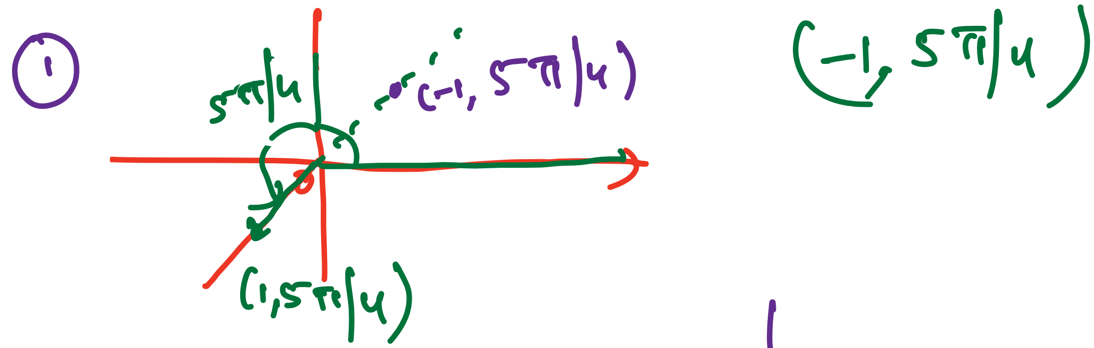
5. If $r < 0$, then the point (r, θ) lies in the quadrant on the opposite side of the pole.

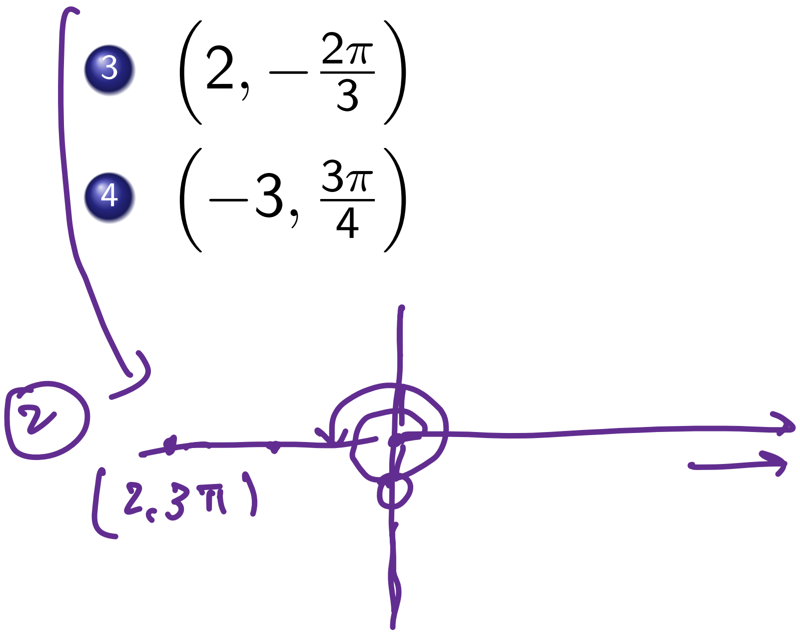


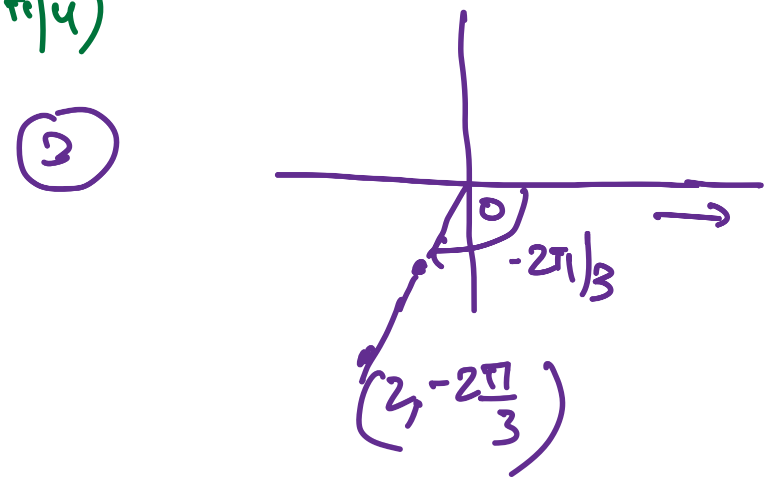
Examples

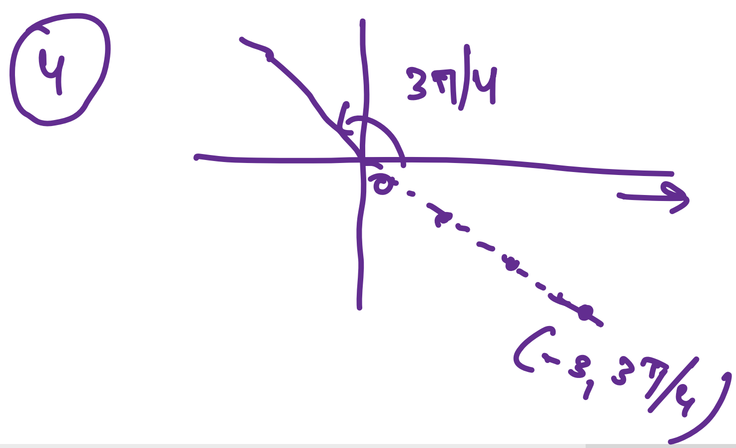
Plot the points whose polar coordinates are given as follows:

1. $\left(1, \frac{5\pi}{4}\right)$
2. $(2, 3\pi)$
3. $\left(2, -\frac{2\pi}{3}\right)$
4. $\left(-3, \frac{3\pi}{4}\right)$

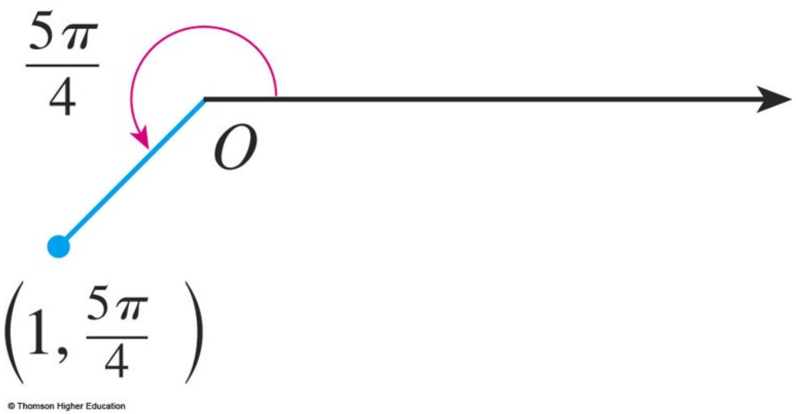








Solutions



$\frac{5\pi}{4}$
O
$\left(1, \frac{5\pi}{4}\right)$
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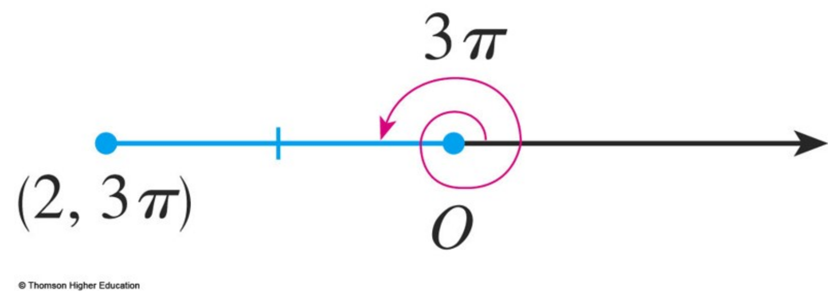

3π
$(2, 3\pi)$
O
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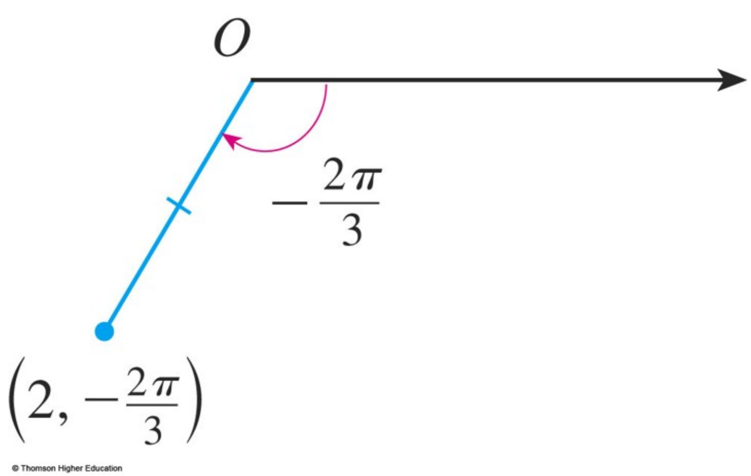

O
$-\frac{2\pi}{3}$
$\left(2, -\frac{2\pi}{3}\right)$
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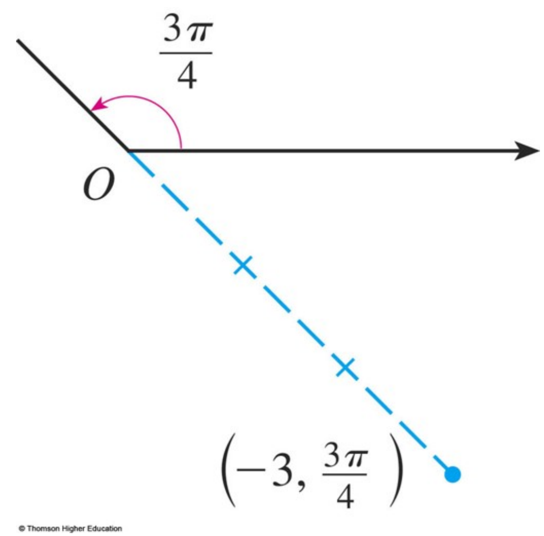

$\frac{3\pi}{4}$
O
$\left(-3, \frac{3\pi}{4}\right)$
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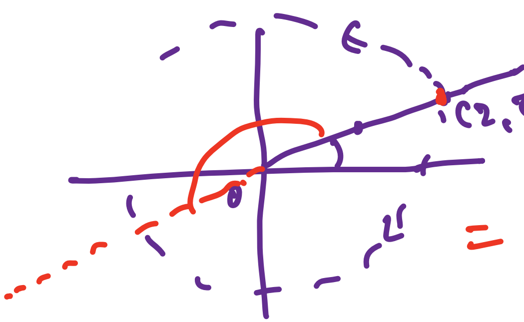
Cartesian Vs Polar coordinates

In the Cartesian coordinate system, every point has a unique representation.

Whereas, in the polar coordinate system, each point has many representations. For instance, the point $\left(2, \frac{\pi}{6}\right)$.



$(2, \pi/6) = \left(2, \frac{\pi}{6} + 2n\pi\right)$

$= \left(-2, \pi + \frac{\pi}{6}\right) = \left(-2, \frac{\pi}{6} + (2n+1)\pi\right)$

$= \left(-2, -\frac{5\pi}{6}\right) = \left(-2, \frac{\pi}{6} - \pi\right)$

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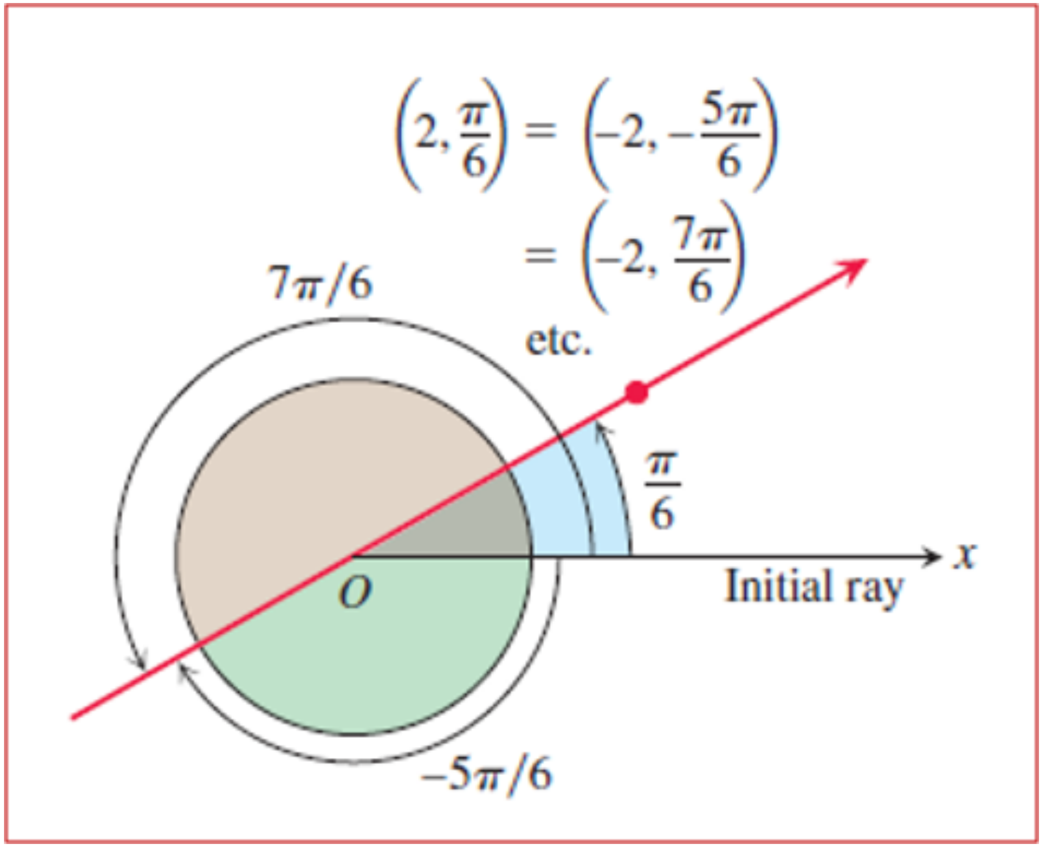


Different Polar representations of a point

Since a complete counterclockwise rotation is given by an angle 2π, the point represented by polar coordinates (r, θ) is also represented by

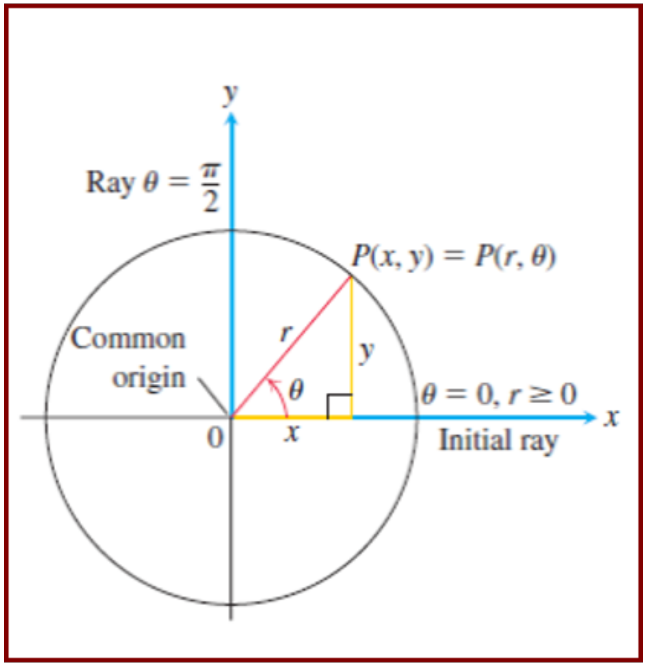
$$(r, \theta + 2n\pi) \text{ and } (-r, \theta + (2n+1)\pi)$$

where n is any integer.

Therefore every point has infinite polar representations.

Relation between Cartesian and Polar coordinates

From Polar to Cartesian



- The pole corresponds to the origin.
- The polar axis coincides with the positive x-axis.
- If the point P has Cartesian coordinates (x, y) and polar coordinates (r, θ), then from the figure, we have

$$\cos\theta = \frac{x}{r}, \quad \sin\theta = \frac{y}{r} \quad \Rightarrow \boxed{x = r\cos\theta, \text{ and } y = r\sin\theta.}$$

Note. Although the above equations, deduced from the figure, illustrates the case where $r > 0$ and $0 < \theta < \frac{\pi}{2}$, but these equations are valid for all values of r and θ.

Example. Convert the point $\left(2, \frac{\pi}{3}\right)$ from Polar to Cartesian coordinates.

$r = 2, \theta = \frac{\pi}{3}$

$x = r\cos\theta$

$= 2\cos\frac{\pi}{3} = 2 \times \frac{1}{\cancel{2}} = 1$

$y = r\sin\theta = 2\sin\frac{\pi}{3} = \sqrt{3}$

$= (1, \sqrt{3})$

$// (-1, -\sqrt{3})$

Exercise. Find the Cartesian coordinates of $(-2, \frac{\pi}{3})$ and $(1, \frac{\pi}{4})$. $= (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$

From Cartesian to Polar

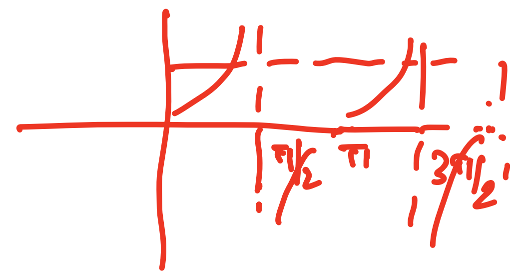
$$\boxed{r^2 = x^2 + y^2, \quad \tan\theta = \frac{y}{x}}$$

Example. Represent the point with Cartesian coordinates $(-1, -1)$ in terms of polar coordinates.

Guess $(-\sqrt{2}, \frac{\pi}{4})$

Given x, y want to obtain r, θ

$\boxed{r = +\sqrt{x^2+y^2}}$,



$x = -1, \; y = -1$

$r = \sqrt{2}$

$\tan\theta = \frac{1}{1} = 1$

$\theta = \cancel{\frac{\pi}{4}}, \frac{5\pi}{4}$

Since $(-1,-1)$ is in IIIrd quadrant, $\therefore$ choose $\theta = \frac{5\pi}{4}$

$(\sqrt{2}, 5\pi/4)$

While finding the θ

- The above equations do not uniquely determine θ for a positive r, when x and y are given.
- This is because, as θ increases through the interval $0 \leq \theta \leq 2\pi$, each value of $\tan\theta$ occurs twice.
- So, in converting from Cartesian to polar coordinates, it is not good enough just to find r and θ that satisfy the equations $r^2 = x^2 + y^2, \tan\theta = \frac{y}{x}$.
- Rather we must choose θ so that the point r, θ lies in the correct quadrant.

General formula

If the Cartesian coordinates (x, y) are given we can use the following formula to find the polar coordinates (r, θ):

$$r = +\sqrt{x^2 + y^2}$$

$$\theta = \begin{cases} \tan^{-1}(\frac{y}{x}) & \text{if } x > 0; \\ \tan^{-1}(\frac{y}{x}) + \pi \text{ or } \tan^{-1}(\frac{y}{x}) - \pi & \text{if } x < 0; \\ \frac{\pi}{2} & \text{if } x = 0, y > 0; \\ -\frac{\pi}{2} & \text{if } x = 0, y < 0. \end{cases}$$

Note that for the origin, θ can take any value.

Exercise

Convert the Cartesian coordinates $(\sqrt{3},1),(1,-\sqrt{3})$ and $(-\sqrt{3},-1)$ into polar coordinates.

$(\sqrt{3},1)$ — $x=\sqrt{3}$, $y=1$

$r=+\sqrt{3+1}=2$

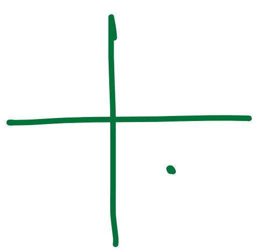
$\tan\theta=\frac{1}{\sqrt{3}} \Rightarrow \theta=\frac{\pi}{6}, \cancel{\frac{7\pi}{6}}$

$\therefore$ polar coordinates will be $\left(2,\frac{\pi}{6}\right)$

$(1,-\sqrt{3})$ — $x=1$, $y=-\sqrt{3}$

$r=+\sqrt{1+3}=2$

$\tan\theta=-\sqrt{3} \Rightarrow \theta=\cancel{120}, 300$

$\therefore$  $\Rightarrow (r,\theta)=\left(2,\frac{5\pi}{3}\right)$