# Limits and Continuity in Higher Dimensions

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October 2, 2024

# Lecture-22

# Limits of functions of two variables

**Definition 0.1 (Limit of functions of two variables).**

We say that a function $f(x, y)$ approaches the **limit** $L$ as $(x, y)$ approaches $(x_0, y_0)$ and we write

$$\lim_{(x,y)\to(x_0,y_0)} f(x, y) = L,$$

if, for every number $\varepsilon > 0$, there exists a corresponding number $\delta > 0$ such that for all $(x, y)$ in the domain of $f$,

$$|f(x, y) - L| < \varepsilon \quad \text{whenever} \quad 0 < \sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta.$$

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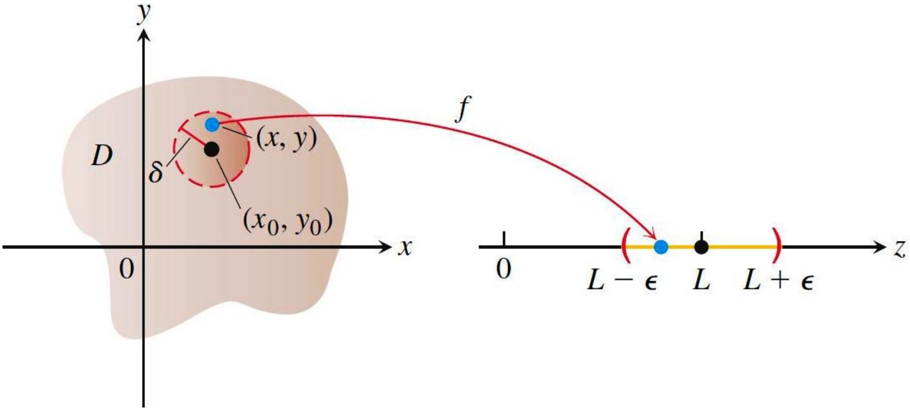

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3. $$\lim_{(x,y)\to(x_0,y_0)} k = k \quad (k \text{ is any constant})$$

**THEOREM 1—Properties of Limits of Functions of Two Variables** The following rules hold if $L$, $M$, and $k$ are real numbers and

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L \qquad \text{and} \qquad \lim_{(x,y)\to(x_0,y_0)} g(x,y) = M.$$

1. *Sum Rule:* $\displaystyle\lim_{(x,y)\to(x_0,y_0)} (f(x,y) + g(x,y)) = L + M$
2. *Difference Rule:* $\displaystyle\lim_{(x,y)\to(x_0,y_0)} (f(x,y) - g(x,y)) = L - M$
3. *Constant Multiple Rule:* $\displaystyle\lim_{(x,y)\to(x_0,y_0)} kf(x,y) = kL$ (any number $k$)
4. *Product Rule:* $\displaystyle\lim_{(x,y)\to(x_0,y_0)} (f(x,y)\cdot g(x,y)) = L\cdot M$
5. *Quotient Rule:* $\displaystyle\lim_{(x,y)\to(x_0,y_0)} \frac{f(x,y)}{g(x,y)} = \frac{L}{M}, \qquad M \neq 0$
6. *Power Rule:* $\displaystyle\lim_{(x,y)\to(x_0,y_0)} [f(x,y)]^n = L^n$, $n$ a positive integer
7. *Root Rule:* $\displaystyle\lim_{(x,y)\to(x_0,y_0)} \sqrt[n]{f(x,y)} = \sqrt[n]{L} = L^{1/n}$,
   $n$ a positive integer, and if $n$ is even, we assume that $L > 0$.

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3. Show that $\lim_{(x,y)\to(0,0)} \frac{4xy^2}{x^2 + y^2} = 0$ by using $\varepsilon - \delta$ definition.

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4. If $f(x,y) = \frac{y}{x}$, does $\lim_{(x,y)\to(0,0)} f(x,y)$ exist ?

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A function $f(x, y)$ is said to be **continuous**, if it is continuous at every point of its domain.

Show that the following functions are continuous at given point.

1. $\dfrac{2x - xy + 5}{x^2y + xy - y^2}, \quad (0, 1).$

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Find the all the points in $xy$-plane where the following functions are continuous:

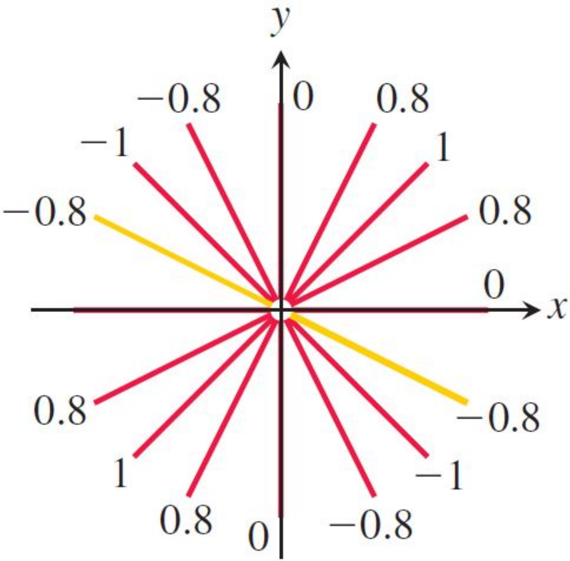
1. $f(x, y) = \dfrac{x}{y^2 + 1}.$
2. $f(x, y) = \dfrac{1}{x^2 - 1}.$
3. $f(x, y) = \sin(x + y).$

# Examples

Show that

$$f(x,y) = \begin{cases} \frac{2xy}{x^2+y^2} & \text{for } (x,y) \neq (0,0); \\ \\ 0 & \text{for } (x,y) = (0,0), \end{cases}$$

is continuous at every point except the origin.


y
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−0.8
−1
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## Two path test for nonexistence of a limit

**Theorem 0.3.**

*Let* $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}$ *is any continuous curve passing through the point* $(x_0, y_0)$, $\mathbf{r}(t_0) = (x_0, y_0)$ *and* $\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L$, *then*

$$\lim_{t\to t_0} f(\mathbf{r}(t)) = \lim_{t\to t_0} f(x(t), y(t)) = L.$$

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$$\lim_{t\to t_0} f(\mathbf{r}(t)) = \lim_{t\to t_0} f(x(t), y(t)) = L.$$

*Remark 0.4.*

If a function $f(x, y)$ has two different limits along two different paths in the domain of $f$ as $(x, y)$ approaches $(x_0, y_0)$, then $\lim_{(x,y)\to(x_0,y_0)} f(x, y)$ does not exist.

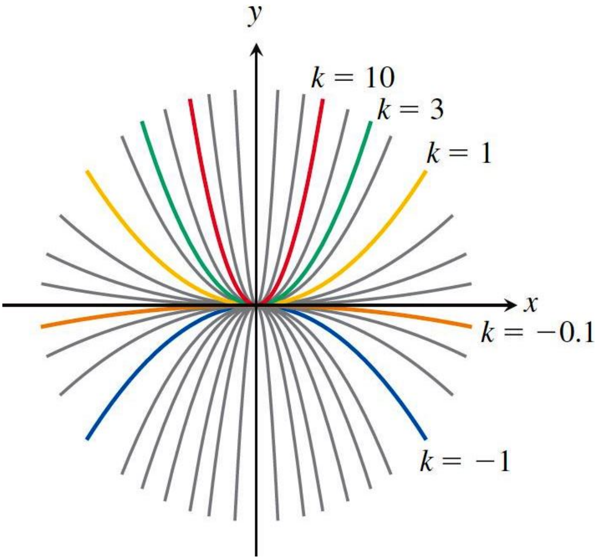
# Two path test for nonexistence of a limit

Show that if

$$f(x,y) = \begin{cases} \frac{10x^2y}{x^4+y^2} & \text{for } (x,y) \neq (0,0); \\ \\ 0 & \text{for } (x,y) = (0,0), \end{cases}$$

then $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist.

# Two path test for nonexistence of a limit



**Theorem 0.5 (Continuity of Composites).**

*If $f$ is continuous at $(x_0, y_0)$ and $g$ is a single-variable function continuous at $f(x_0, y_0)$, then the composite function $h = g \circ f$ defined by $h(x, y) = g(f(x, y))$ is continuous at $(x_0, y_0)$.*

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**Examples:** The functions

$$e^{x-y}, \quad \cos\frac{xy}{x^2+1}, \quad \ln(1+x^2y^2)$$

are continuous at every point $(x, y)$

**Theorem 0.6 (The Sandwhich Theorem).**

*Let $f$, $g$ and $h$ be functions of two variables such that*

$$g(x, y) \leq f(x, y) \leq h(x, y)$$

*for all $(x, y) \neq (x_0, y_0)$ in a disk centered at $(x_0, y_0)$ and if*

$$\lim_{(x,y)\to(x_0,y_0)} g(x, y) = L = \lim_{(x,y)\to(x_0,y_0)} h(x, y)$$

*for a finite limit $L \in \mathbb{R}$, then*

$$\lim_{(x,y)\to(x_0,y_0)} f(x, y) = L.$$

# Examples

Find the limits (if they exist):

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2. $$\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2 + y^2}.$$

   (Here $-|x| \leq f(x, y) \leq |x|$)

*Remark 0.7 (Changing Variables to Polar Coordinates).*

If $f(x, y)$ is a function of two variables, $L \in \mathbb{R}$ and for given any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|f(r\cos\theta, r\sin\theta) - L| < \varepsilon \quad \text{whenever} \quad 0 < |r| < \delta$$

for all $\theta$ with $(r\cos\theta, r\sin\theta)$ in the domain of $f$, then

$$\lim_{(x,y)\to(0,0)} f(x, y) = L.$$

In otherwords, if $\lim_{r\to 0} f(r\cos\theta, r\sin\theta) = L$ where is $L$ is a constant independent of $\theta$, then

$$\lim_{(x,y)\to(0,0)} f(x, y) = L.$$

# Changing Variables to Polar Coordinates

Find the limits, (if they exist):

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## Functions of more than two variables

The definitions of limit and continuity for functions of two variables and the conlusions about limits and continuity for sums, product, quotients, powers and composites all extend to functions of three or more variables.

**Definition 0.8 (Limit of functions of two variables).**

We say that a function $f(x,y,z)$ approaches the **limit** $L$ as $(x,y,z)$ approaches $(x_0,y_0,z_0)$ and we write

$$\lim_{(x,y,z)\to(x_0,y_0,z_0)} f(x,y,z) = L,$$

if, for every number $\varepsilon > 0$, there exists a corresponding number $\delta > 0$ such that for all $(x,y,z)$ in the domain of $f$,

$$|f(x,y,z) - L| < \varepsilon$$

$$\text{whenever} \quad 0 < \sqrt{(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2} < \delta.$$

## Continuity of functions of three variables

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A function $f(x, y.z)$ is said to be **continuous**, if it is continuous at every point of its domain.

## Problems:

1. Using $\epsilon - \delta$ definition show that
$$\lim_{(x,y,z)\to(0,0,0)} \frac{x+y+z}{x^2+y^2+z^2+1} = 0.$$

2. At what points $(x, y, z)$ in space are the following functions are continuous?
    1. $f(x, y, z) = e^{x+y+z} \sin z$

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   2. $g(x, y, z) = \frac{1}{|xy|+|z|}$
   3. $h(x, y, z) = \frac{1}{1-\sqrt{x^2+y^2+z^2-4}}$.