Tangent Planes

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BITS Pilani, K K Birla Goa Campus

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ANNOUNCEMENT:

An additional class will be held this **Saturday (26 October 2024)** from **12:00 PM to 1:00 PM** in **LT3**.

Recall - Directional derivative

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Example: Suppose that the temperature at a point (x, y, z) is given by $T(x, y, z) = 80/(1 + x^2 + 2y^2 + 3z^2)$, where T is measured in degree Celsius and x, y, z in meters. In which direction does the temperature increase fastest at the point $(1, 1, -2)$? What is the maximum rate of increase?

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Solution: $\nabla T_{|(1,1,-2)} = \frac{5}{8}(-\boldsymbol{i} - 2\boldsymbol{j} + 6\boldsymbol{k})$ and the maximum rate of increase of temperature is $\frac{5}{8}\sqrt{41}$.

Tangent Plane

Let $f(x, y, z)$ be a differentiable function. Suppose that $\boldsymbol{r}(t) = g(t)\boldsymbol{i} + h(t)\boldsymbol{j} + k(t)\boldsymbol{k}$ be a smooth curve on the level surface $f(x, y, z) = c$ passing through $P_0(x_0, y_0, z_0) = \boldsymbol{r}(t_0)$.

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At every point along the curve, ∇f is orthogonal to the curve's velocity vector/tangent vector.
This shows that the tangents to all the smooth curves on the level surface $f(x, y, z) = c$ through a fixed point P_0 lie in the plane through P_0 normal to $\nabla f|_{P_0}$.

Tangent Plane

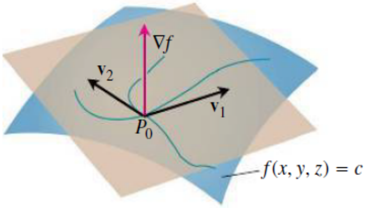

FIGURE 14.32 The gradient ∇f is orthogonal to the velocity vector of every smooth curve in the surface through P_0. The velocity vectors at P_0 therefore lie in a common plane, which we call the tangent plane at P_0.

Tangent Plane and Normal Line

Definition

The tangent plane at the point $P_0(x_0, y_0, z_0)$ on the level surface $f(x, y, z) = c$ of a differentiable function f is the plane through P_0 normal to $\nabla f|_{P_0}$.

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Equation of the tangent plane to $f(x, y, z) = c$ at $P_0(x_0, y_0, z_0)$ is

$$f_x(P_0)(x - x_0) + f_y(P_0)(y - y_0) + f_z(P_0)(z - z_0) = 0.$$

Equation of the normal line to $f(x, y, z) = c$ at $P_0(x_0, y_0, z_0)$ is

$$x = x_0 + f_x(P_0)t,\ y = y_0 + f_y(P_0)t,\ z = z_0 + f_z(P_0)t.$$

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Equivalently, the equation of the normal line is given by

$$\frac{x - x_0}{f_x(P_0)} = \frac{y - y_0}{f_y(P_0)} = \frac{z - z_0}{f_z(P_0)}.$$

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Tangent Plane to a Surface $z = f(x, y)$

Plane tangent to a surface $z = f(x, y)$ at a point $(x_0, y_0, f(x_0, y_0))$ of a differentiable function f is given by

$$f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) - (z - z_0) = 0.$$

Examples

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   $(\nabla f \times \nabla g = 2\boldsymbol{i} - 4\boldsymbol{j} + 2\boldsymbol{k};\ x = 1 + 2t, y = 1 - 4t, z = 1 + 2t.)$
2. Find parametric equations for the line tangent to the curve of intersection of $y = 1$ and $x + y^2 + z = 2$ at $(1/2, 1, 1/2)$.
3. Find the normal line at the point $(1, 0, 2)$ of the surface $x^2y + 2xz^2 = 8$.
4. Find the tangent plane to the surface $z = \sqrt{y - x}$ at $(1, 2, 1)$.

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For a given unit vector $\boldsymbol{u}$, we want to estimate how much the value of a function f changes if we move a small distance ds from a point P_0 to another point nearby.

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2. By about how much will $g(x, y, z) = x + x\cos z - y\sin z + y$ change if the point $P(x, y, z)$ moves from $P_0(2, -1, 0)$ a distance of $ds = 0.2$ unit toward the point $P_1(0, 1, 2)$?

Linearization for the functions of two variables

A differentiable function can be approximated by a linear function.

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Standard Linear Approximation

The linearization of a differentiable function $f(x, y)$ at a point (x_0, y_0) is the function

$$L(x, y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0).$$

The approximation $f(x, y) \approx L(x, y)$ is called the standard linear approximation of the function f at (x_0, y_0).

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Error in the Standard Linear Approximation

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Error in the standard linear approximation

If f has continuous second partial derivatives throughout an open set containing a rectangle R centered at (x_0, y_0) $(R := \{(x, y) : |x - x_0| \leq h,\ |y - y_0| \leq k\})$ and if M is an upper bound for the values of $|f_{xx}|, |f_{xy}|$ and $|f_{yy}|$ in the rectangle R (that is, $M = \max\{|f_{xx}|, |f_{xy}|, |f_{yy}|\}$ on R), then for any $(x, y) \in R$,

$$|E(x, y)| \leq \frac{1}{2} M \left(|x - x_0| + |y - y_0|\right)^2 .$$