Q1 A planet of mass m is in a circular orbit about a star of mass M. The radius of the orbit is R

a) Determine the i) time period T of the planet

ii) The total energy $E = KE + PE$ of the planet

iii) the angular momentum L of the planet

Solution (i) $\frac{G\cancel{m}M}{R^2} = \cancel{m}\omega^2 R \Rightarrow \omega^2 = \frac{GM}{R^3}, \; \omega = \sqrt{\frac{GM}{R^3}}$

$\Rightarrow T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{R^3}{GM}} = \frac{2\pi}{\sqrt{GM}} R^{3/2}$

ii) $v = \frac{2\pi R}{T} = \frac{\cancel{2\pi} R}{\cancel{2\pi} R^{3/2}} \sqrt{GM} = \sqrt{\frac{GM}{R}}$

Kinetic energy $KE = \frac{1}{2}mv^2 = \frac{1}{2}m\frac{GM}{R} = \frac{GmM}{2R}$

Potential energy ~~is~~ & $PE = -\frac{GmM}{R}$

total energy $= KE + PE = \frac{GmM}{2R} - \frac{GmM}{R} = -\frac{GmM}{2R}$

iii) $L = mvR = m\sqrt{\frac{GM}{R}} \times R = m\sqrt{GMR}$ $(3+4+3) = 10$

b) An asteroid hits the planet & provides it an inward impulse J in the radial ~~direction~~ ~~outward~~ direction such that the new trajectory is a parabola. Determine

i) The value of J

ii) The distance of closest approach between the planet and the star $(4+6=10)$

Solution – (i) $E' = E + \frac{J^2}{2m} = 0 \Rightarrow \frac{J^2}{2m} = -E = \frac{GmM}{2R}$

$\Rightarrow J^2 = \frac{GM}{R} m^2 \Rightarrow J = m\sqrt{\frac{GM}{R}}$

ii) $L' = L = m\sqrt{GMR} = mv_c r_c$ where

v_c = velocity at the point of closest approach &

r_c = distance of closest approach

$$E = 0 = \frac{1}{2} m v_c^2 - \frac{GmM}{r_c} = \frac{1}{2} m \left(\frac{L}{m r_c}\right)^2 = \frac{GmM}{r_c}$$

$$\Rightarrow \frac{L^2}{2 m r_c^{\cancel{2}}} = \frac{GmM}{\cancel{r_c}} \Rightarrow r_c = \frac{L^2}{2 G M m^2}$$

Put $L = m\sqrt{GMR} \Rightarrow r_c = \frac{\cancel{m^2} \cancel{GM} R}{2 \cancel{GM} \cancel{m^2}} = R/2$

The distance of closest approach $= r_c = R/2$