MATHEMATICS-I

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Lecture 7

Infinite Sequences

Important Limits:

1. $\lim_{n\to\infty} \frac{\ln n}{n} = 0$
2. $\lim_{n\to\infty} n^{1/n} = 1$
3. $\lim_{n\to\infty} x^{1/n} = 1 \quad (x > 0)$
4. $\lim_{n\to\infty} x^n = 0 \quad (|x| < 1)$
5. $\lim_{n\to\infty} (1 + \frac{x}{n})^n = e^x \quad (\text{any } x \in \mathbb{R})$
6. $\lim_{n\to\infty} \frac{x^n}{n!} = 0 \quad (\text{any } x \in \mathbb{R})$

Sequences defined recursively: Sequences are aften defined recursively by giving

1. The value(s) of the initial term(s) and
2. A rule, called a **recursion formula**, for calculating any later term from terms that precede it.

Examples

- Let $a_1 = 1$ and $a_n = a_{n-1} + 1$ for $n > 1$. This recursion formula with $a_1 = 1$ defines the sequence $\{1, 2, 3, \ldots\}$.
- Let $a_1 = 1$, $a_2 = 1$ and $a_n = a_{n-1} + a_{n-2}$ for $n > 2$ define the sequence $1, 1, 2, 3, 5, \ldots$ of **Fibonacci numbers**.
- $a_1 = 0$, $a_2 = 1$ and $a_n = (a_{n-1} + a_{n-2})/2$ for $n > 2$.

Bounded Sequences:

- A sequence $\{a_n\}$ is said to be **bounded from above** if there exists a number M such that $a_n \leq M$ for all n. The number M is an **upper bound** for $\{a_n\}$.
- If u is an upper bound for $\{a_n\}$ but any number less than u is not an upper bound for $\{a_n\}$, then u is called the **least upper bound** (lup) for $\{a_n\}$.
- **Examples:** $\{n/(n+2)\}$ is bounded from above. 2 is an upper bound. What is the least upper bound for this sequence?

Bounded Sequences

- A sequence $\{a_n\}$ is said to be **bounded from below** if there exists a number m such that $a_n \geq m$ for all n. The number m is a **lower bound** for $\{a_n\}$.
- If ℓ is a lower bound for $\{a_n\}$ but no number greater than ℓ is a lower bound for $\{a_n\}$, then ℓ is called the **greatest lower bound** (glb) for $\{a_n\}$.
- **Example:** $\{n/(n-2)\}$ is bounded from below. 0 is a lower bound. What is the greatest lower bound for this sequence?

Boundedness

- If $\{a_n\}$ is bounded from above and below, then $\{a_n\}$ is called **bounded**.
- If $\{a_n\}$ is not bounded, then we say that $\{a_n\}$ is an **unbounded sequence**.
- **Example:** $n + 1$ is bounded from below by 2 but it is not bounded above, so it is unbounded.
- $\{n/(n - \frac{1}{2})\}$ is bounded from below by 1 and above by 3. Hence it is bounded.

Boundedness

Theorem 0.1.

Every convergent sequence $\{a_n\}$ *is bounded (bounded from above and bounded from below) equivalently there exist* $M, m \in \mathbb{R}$ *such that* $m \leq a_n \leq M$ *for all* n.

Monotonic Sequences

Monotonic Sequences: A sequence $\{a_n\}$ is said to be

- **increasing** if $a_n \leq a_{n+1}$ for all n, i.e., $a_1 \leq a_2 \leq a_3 \leq \cdots$.
- **decreasing** if $a_n \geq a_{n+1}$ for all n, i.e., $a_1 \geq a_2 \geq a_3 \geq \cdots$.
- **monotonic** if the sequence $\{a_n\}$ is either decreasing or increasing.
- **Examples:** The sequence $\{1, 2, 3, \ldots\}$ is increasing.
- And the seqeunce $\{1/n\}$ is decreasing.

- The constant sequence $\{2, 2, 2, \ldots\}$ is both decreasing and increasing.
- The sequence $\{1, -1, 1, -1, \ldots\}$ is not monotonic but bounded.

Theorem 0.2 (The Monotonic Sequence Theorem).

If a sequence $\{a_n\}$ is both **bounded** *and* **monotonic***, then the sequence $\{a_n\}$ is convergent.*

- If the sequence $\{a_n\}$ is increasing and bounded then it converges to its *lub*.
- If the sequence $\{a_n\}$ is decreasing and bounded then it converges to its *glb*.

Subsequences:

- Consider the sequences $\{1, 1/2, 1/3, \dots\}$.
- $\{1, 1/3, 1/5, \dots\}$; $\{1, 1/2, 1/4, \dots\}$ or $\{1/3, 1/6, 1/9, \dots\}$.
- Let $n_1 < n_2 < n_3 < \cdots$ be strictly increasing sequence of positive integers and $\{a_n\}$ be a sequence of real numbers then the sequence $\{a_{n_k}\}_{k=1}^{\infty} = \{a_{n_1}, a_{n_2}, a_{n_3}, \dots\}$ is called a subsequence of $\{a_n\}$.
- Examples: $\{1, 1, 1, \dots\}$ and $\{-1, -1, -1, \dots\}$ are subsequences of $\{(-1)^n\}$.
- $\{1/2^{n^2}\}$ is a subsequence of $\{1/2^n\}$.

Theorem 0.3.

If the sequence $\{a_n\}$ converges to L then all the subsequences $\{a_{n_k}\}$ converges to L.

Corollary 0.4.

If one of the subsequences of $\{a_n\}$ diverges then the sequence $\{a_n\}$ also diverges.

Theorem 0.5.

If the subsequences $\{a_{2n}\}$ and $\{a_{2n+1}\}$ of $\{a_n\}$ converge to same limit L, then the sequence $\{a_n\}$ also converges to L.

Questions

1. Find the limit of the followings as $n \to \infty$.
   - $\sqrt{n+1} - \sqrt{n}$
   - $\ln(\frac{n+2}{1+4n})$
   - $\frac{\sin n}{n}$
   - $\frac{\cos n^2}{n^2}$
2. Suppose a_n is sequence of real number converging to a. Show that the sequence $\{\frac{a_1+a_2+\cdots+a_n}{n}\}$ is also converging to the same limit a.
3. If x_n is a sequence of real numbers such that $\{x_{n+1} - x_n\}$ convergens to some $x \in R$. Is the sequence x_n/n convergent ? If so find the limit.
4. Let $x_1 = 1$ and $x_{n+1} = (\frac{n}{n+1})x_n^2$ for all n. Examine whether the sequence x_n is convergent. Also, find the limit if it is convergent.

Thank you