Differentiability

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Recall

The partial derivative of $f(x, y)$ with respect to x at (x_0, y_0) is

$$\left.\frac{\partial f}{\partial x}\right|_{(x_0,y_0)} := \lim_{h\to 0} \frac{f(x_0 + h, y_0) - f(x_0, y_0)}{h},$$

provided the limit exists, otherwise we say that the partial derivative does not exist at the point.

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Other notations: $\dfrac{\partial f}{\partial x}(x_0, y_0),\ f_x(x_0, y_0).$

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$f_x(x_0, y_0)$ is the slope of the tangent line to the curve of intersection of surface $z = f(x, y)$ and plane $y = y_0$ at $P(x_0, y_0, f(x_0, y_0))$.

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$f_x(x_0, y_0)$ gives the rate of change of $f(x, y)$ with respect to x at (x_0, y_0) when $y = y_0$ is fixed.

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Second Order Partial Derivatives:

- $\left.\dfrac{\partial^2 f}{\partial x^2}\right|_{(x_0,y_0)} = \left.\dfrac{\partial}{\partial x}\left(\dfrac{\partial f}{\partial x}\right)\right|_{(x_0,y_0)} := \lim\limits_{h\to 0} \dfrac{f_x(x_0+h, y_0) - f_x(x_0, y_0)}{h}$,
- $\left.\dfrac{\partial^2 f}{\partial y^2}\right|_{(x_0,y_0)} = \left.\dfrac{\partial}{\partial y}\left(\dfrac{\partial f}{\partial y}\right)\right|_{(x_0,y_0)} := \lim\limits_{h\to 0} \dfrac{f_y(x_0, y_0+h) - f_y(x_0, y_0)}{h}$,
- $\left.\dfrac{\partial^2 f}{\partial x \partial y}\right|_{(x_0,y_0)} = \left.\dfrac{\partial}{\partial x}\left(\dfrac{\partial f}{\partial y}\right)\right|_{(x_0,y_0)} := \lim\limits_{h\to 0} \dfrac{f_y(x_0+h, y_0) - f_y(x_0, y_0)}{h}$,
- $\left.\dfrac{\partial^2 f}{\partial y \partial x}\right|_{(x_0,y_0)} = \left.\dfrac{\partial}{\partial y}\left(\dfrac{\partial f}{\partial x}\right)\right|_{(x_0,y_0)} := \lim\limits_{h\to 0} \dfrac{f_x(x_0, y_0+h) - f_x(x_0, y_0)}{h}$,

provided the limit exists.

A Counterexample

Example: Let

$$f(x,y) = \begin{cases} \frac{xy(x^2-y^2)}{x^2+y^2} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{else.} \end{cases}$$

1. Show that $\frac{\partial f}{\partial y}(x,0) = x$ for all x, and $\frac{\partial f}{\partial x}(0,y) = -y$ for all y.
2. Show that $\frac{\partial^2 f}{\partial y \partial x}(0,0) \neq \frac{\partial^2 f}{\partial x \partial y}(0,0)$.

Theorem - The Mixed Derivative Theorem

If $f(x, y)$ and its partial derivatives $f_x, f_y, f_{xy},$ and f_{yx} are defined throughout an open region containing a point (a, b) and are all continuous at (a, b), then

$$f_{xy}(a, b) = f_{yx}(a, b).$$

Differentiability of a Function of Two Variables

Definition

A function $z = f(x, y)$ is differentiable at (x_0, y_0) if $f_x(x_0, y_0)$ and $f_y(x_0, y_0)$ exist and $\triangle z = f(x_0 + \triangle x, y_0 + \triangle y) - f(x_0, y_0)$ satisfies an equation of the form

$$\triangle z = f_x(x_0, y_0)\triangle x + f_y(x_0, y_0)\triangle y + \epsilon_1 \triangle x + \epsilon_2 \triangle y$$

in which each of $\epsilon_1, \epsilon_2 \to 0$ as both $\triangle x, \triangle y \to 0$.

We call f differentiable if it is differentiable at every point in its domain, and say that its graph is a smooth surface.

Differentiability of a Function of Two Variables

$f(x, y)$ is differentiable at (x_0, y_0) if

- $f_x(x_0, y_0)$ exists
- $f_y(x_0, y_0)$ exists
- the change in f satisfies the linearization property:

$$f(x_0+h, y_0+k)-f(x_0, y_0) = f_x(x_0, y_0)h+f_y(x_0, y_0)k+\epsilon_1 h+\epsilon_2 k,$$

where

$$\lim_{(h,k)\to(0,0)} \epsilon_1 = \lim_{(h,k)\to(0,0)} \epsilon_2 = 0.$$

Dividing the equation in f by $\sqrt{h^2+k^2}$ and letting $(h, k) \to (0, 0)$,

$$\lim_{(h,k)\to(0,0)} \frac{f(x_0+h, y_0+k)-f(x_0, y_0)-f_x(x_0, y_0)h-f_y(x_0, y_0)k}{\sqrt{h^2+k^2}} = 0.$$

Examples

1. Find if the function f is differentiable at a point (x_0, y_0):
   - $f(x,y) = x^2 + 2xy$ at (x_0, y_0)
   - $f(x,y) = x^2 + y^2$ at (x_0, y_0)
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   - $f(x,y) = |xy|$ at $(0,0)$.

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    - $f(x,y) = |xy|$ at $(0,0)$.
2. Find if the following function is differentiable at $(0,0)$:

$$f(x,y) = \begin{cases} \frac{xy}{\sqrt{x^2+y^2}} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$