Lecture 19

Simple Harmonic Motion

Forced Harmonic Motion

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  - Nuclear Magnetic Resonance (NMR)

Forced Oscillations:

Undamped oscillator: equation and solution

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Complexify: add $i(m\ddot{y} + ky = F_0 \sin \Omega t)$ to get $\quad m\ddot{z} + kz = F_0 e^{i\Omega t}$;
Trial exponential solution: $z = Re^{i(\omega t + \theta)}, \quad R, \theta$ real.

$$\implies (-mR\omega^2 + kR)e^{i(\omega t+\theta)} = F_0 e^{i\Omega t}$$

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(a) Imaginary part: $\sin((\Omega - \omega)t + \theta)) = 0.$

Since this must be true for all t, $\quad \Omega = \omega, \quad \theta = 0$

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- Solution to homogeneous eqn (undriven):

$$x_2(t) = B\cos(\omega_0 t + \phi):$$ "General solution"

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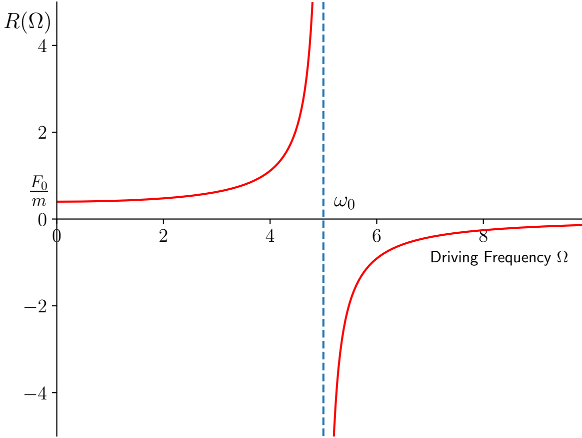
Full Solution: $x(t) = B\cos(\omega_0 t + \phi) + R\cos\Omega t.$

Analysis of solution:

- $$R(\Omega) = \frac{F_0/m}{\omega_0^2 - \Omega^2}, \quad \omega_0^2 = \frac{k}{m}.$$



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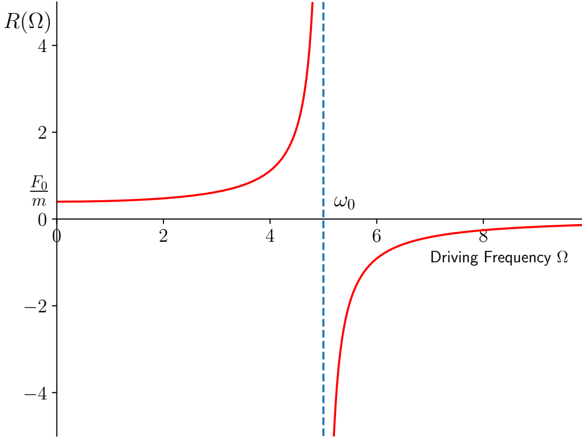

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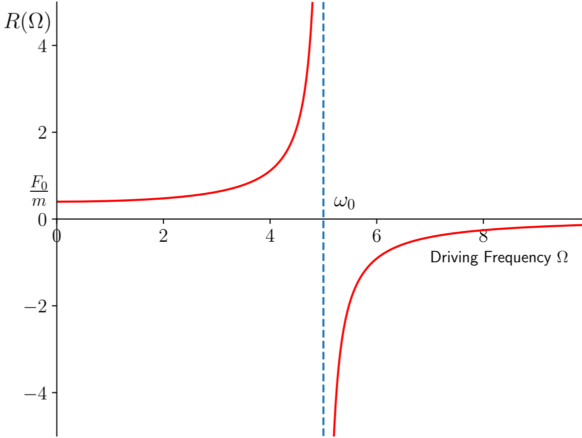

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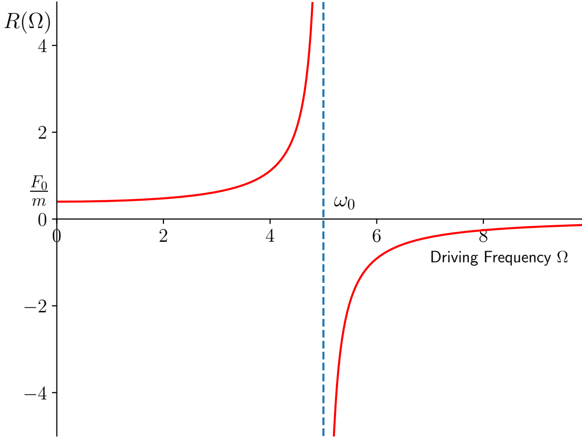
- R changes sign thru resonance: phase change of π.



Undamped system is unphysical

Amplitude has a problem at $\Omega = \omega_0$

Solution: consider damping.