Mathematics I- MATH F111

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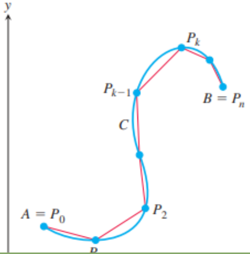
Arc length in Plane

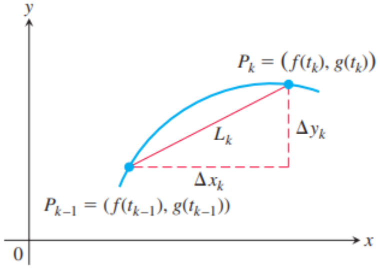
Let C be a curve given parametrically by equations

$$x = f(t), \; y = g(t), a \leq t \leq b.$$

Assume f and g are differentiable and have continuous first derivatives. We also assume that the derivatives $f'(t)$ and $g'(t)$ are not simultaneously zero, which prevents the curve C from having any corners or cusps. We subdivide the path (or arc) AB into n pieces at points $A = P_0, P_1, P_2, \cdots, P_n = B$.



These points correspond to a partition of the interval $[a, b]$ by $a = t_0, t_1, t_2 \cdots, t_n = b$, where $P_k = (f(t_k), g(t_k))$. Join successive points of this subdivision by straight-line segments



A representative line segment has length

$$L_k = \sqrt{(\Delta x_k)^2 + (\Delta y_k)^2} = \sqrt{(f(t_k) - f(t_{k-1}))^2 + (g(t_k) - g(t_{k-1}))^2}$$

If Δt_k is small, the length L_k is approximately the length of arc $P_{k-1}P_k$.

By mean value theorem, there are numbers t_k^* and t_k^{**} in $[t_{k-1}, t_k]$ such that

$$\Delta x_k = f(t_k) - f(t_{k-1}) = f'(t_k^*)\Delta t_k,$$
$$\Delta y_k = g(t_k) - g(t_{k-1}) = g'(t_k^{**})\Delta t_k.$$

Assuming the path from A to B is traversed exactly once as t increases from $t = a$ to $t = b$, with no retracing, an approximation to the (yet to be defined) "length" of the curve AB is the sum of all the lengths L_k:

$$\sum_{k=1}^{n} L_k = \sum_{k=1}^{n} \sqrt{(\Delta x_k)^2 + (\Delta y_k)^2} = \sum_{k=1}^{n} \sqrt{(f'(t_k^*))^2 + (g'(t_k^{**}))^2}\Delta t_k$$

$$\lim_{||P||\to 0} \sum_{k=1}^{n} \sqrt{(f'(t_k^*))^2 + (g'(t_k^{**}))^2}\Delta t_k = \int_a^b \sqrt{(f'(t))^2 + (g'(t))^2}dt$$

Definition

If a curve C is defined parametrically by $x = f(t)$ and $y = g(t)$, $a \leq t \leq b$, where f' and g' are continuous and not simultaneously zero on (a, b) and C is traversed exactly once as t increases from $t = a$ to $t = b$, then the length of C is the definite integral

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If $x = f(t)$ and $y = g(t)$, then using the Leibniz notation we have the following result for arc length

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$$

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This gives

$$L = \int_a^b \sqrt{(r)^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta.$$

Arc length in Space

To measure distance along a smooth curve in space, we add a z-term to the formula we use for curves in the plane.

Definition

The length of a smooth curve $\vec{r}(t) = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}, a \leq t \leq b$, that is traced exactly once as t increases from $t = a$ to $t = b$, is

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Arc Length Formula

$$L = \int_a^b |\vec{v}| dt.$$

Example

Find the arc length parameter of the following:

$$\vec{r}(t) = (4\cos t)\hat{i} + (4\sin t)\hat{j} + (3t)\hat{k}, 0 \leq t \leq \pi/2$$

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Solution: We have $\frac{dx}{dt} = -4\sin t$, $\frac{dy}{dt} = 4\cos t$, $\frac{dz}{dt} = 3$. Therefore $|\vec{v}(t)| = |\sqrt{16\sin^2 t + 16\cos^2 t + 9}| = \sqrt{25} = 5$.
Therefore $L = \int_0^{\frac{\pi}{2}} 5dt = 5[\pi/2 - 0] = 5\pi/2$.

Example (1)

A glider is soaring upward along the helix $\vec{r}(t) = (\cos t)\hat{i} + (\sin t)\hat{j} + t\hat{k}$*. How long is the glider's path from* $t = 0$ *to* $t = 2\pi$*?*

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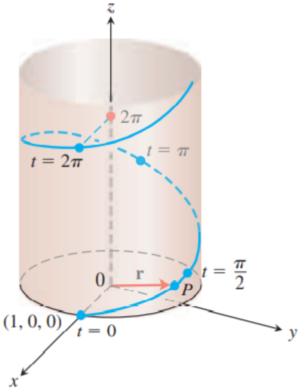

Ans: $2\pi\sqrt{2}$

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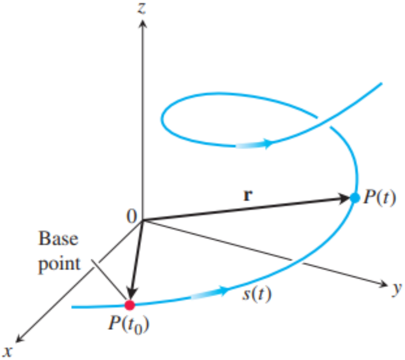
In general, it can be shown that when arc length formula is used to compute the length of any smooth curve, the arc length is independent of the parametrization that is used.

To illustrate that the length of a smooth space curve does not depend on the parametrization you use to compute it, calculate the length of one turn of the helix in earlier example, Example(1) with the following parametrizations.

1. $\vec{r}(t) = (\cos 4t)\hat{i} + (\sin 4t)\hat{j} + 4t\hat{k}, 0 \leq t \leq \frac{\pi}{2}$
2. $\vec{r}(t) = 3\cos(\frac{t}{2})\hat{i} + 3\sin(\frac{t}{2})\hat{j} + (\frac{t}{2})\hat{k}, 0 \leq t \leq 4\pi$
3. $\vec{r}(t) = (\cos t)\hat{i} - (\sin t)\hat{j} - t\hat{k}, -2\pi \leq t \leq 0.$

Arc Length Parameter



Arc length parameter

Definition

If we choose a base point $P(t_0)$ on a smooth curve C parametrized by t, each value of t determines a point $P(t) = (x(t), y(t), z(t))$ on C and a "directed distance"

$$s(t) = \int_{t_0}^{t} |\vec{v}(\tau)| d\tau.$$

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- Each value of s determines a point on C, and this parametrizes C with respect to s. We call s an **arc length parameter** for the curve.

Arc Length Parameter with Base Point $P(t_0)$

$$\boxed{s(t) = \int_{t_0}^{t} \sqrt{\left(\frac{dx}{d\tau}\right)^2 + \left(\frac{dy}{d\tau}\right)^2 + \left(\frac{dz}{d\tau}\right)^2}\, d\tau = \int_{t_0}^{t} |\vec{v}(\tau)|\, d\tau.}$$

- It is often useful to parametrize a curve with respect to arc length because arc length arises naturally from the shape of the curve and does not depend on a particular coordinate system
- If a curve $\vec{r}(t)$ is already given in terms of some parameter t and $s(t)$ is the arc length function given by the above equation, then we may be able to solve for t as a function of $s : t = t(s)$. Then the curve can be reparametrized in terms of s by substituting for $t : \vec{r} = \vec{r}(t(s))$.
- The new parametrization identifies a point on the curve with its directed distance along the curve from the base point.

Here is an example for which we can actually find the arc length parametrization of a curve.

Example

If $t_0 = 0$, find the arc length parameter along the helix

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Now solving the equation for t gives $t = \frac{s}{\sqrt{2}}$. Substituting into the position vector $\vec{r}$ gives

$$\vec{r}(t(s)) = \left(\cos\frac{s}{\sqrt{2}}\right)\hat{i} + \left(\sin\frac{s}{\sqrt{2}}\right)\hat{j} + \frac{s}{\sqrt{2}}\hat{k}.$$

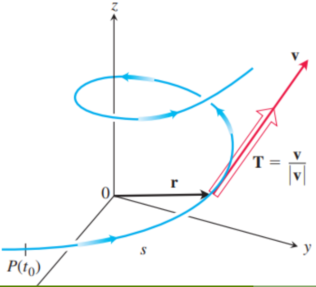
Unit Tangent Vector

Definition

The velocity vector $\vec{v} = d\vec{r}/dt$ is tangent to the curve $\vec{r}(t)$ and that the vector

$$\vec{T} = \frac{\vec{v}}{|\vec{v}|}$$

is called the unit tangent vector.



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Solution: We have $\vec{v} = \frac{d\vec{r}}{dt} = -3\sin t\hat{i} + (3\cos t)\hat{j} + 2t\hat{k}$. Then $\frac{\vec{v}}{|\vec{v}|} = -\frac{3\sin t}{\sqrt{9+4t^2}}\hat{i} + \frac{3\cos t}{\sqrt{9+4t^2}}\hat{j} + \frac{4t^2}{\sqrt{9+4t^2}}\hat{k}$.

Speed on a Smooth Curve

$$s(t) = \int_{t_0}^{t} \sqrt{\left(\frac{dx}{d\tau}\right)^2 + \left(\frac{dy}{d\tau}\right)^2 + \left(\frac{dz}{d\tau}\right)^2} d\tau = \int_{t_0}^{t} |\vec{v}(\tau)| d\tau.$$

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Remark

$ds/dt > 0$ since, by definition, $|\vec{v}|$ is never zero for a smooth curve.

We see once again that s is an increasing function of t.

Properties

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This makes $\vec{r}$ a differentiable function of s whose derivative can be calculated with the chain rule and we have

Remark

$$\frac{d\vec{r}}{ds} = \frac{d\vec{r}}{dt}\frac{dt}{ds} = \vec{v}\frac{1}{|\vec{v}|} = \vec{T}.$$

Exercises

1. Find the point on the curve $\vec{r}(t) = (5\sin t)\hat{i} + (5\cos t)\hat{j} + 12t\hat{k}$ at a distance 26π units along the curve from the point $(0,5,0)$ in the direction of increasing arc length.

2. Find the arc length parameter along the curve from the point where $t = 0$ by evaluating the integral

$$s(t) = \int_{t_0}^{t} |\vec{v}(\tau)| d\tau.$$

Then find the length of the indicated portion of the curve.

1. $\vec{r}(t) = (4\cos t)\hat{i} + (4\sin t)\hat{j} + 3t\hat{k}, 0 \leq t \leq \frac{\pi}{2}$
2. $r(t) = (\cos t + t\sin t)\hat{i} + (\sin t - t\cos t)\hat{j}, \frac{\pi}{2} \leq t \leq \pi.$