MATHEMATICS-I

Anushaya Mohapatra

Department of Mathematics

BITS PILANI K K Birla Goa Campus, Goa

September 12, 2024

Lecture 14

Power Series

Definition: Let f be a function with derivatives of all orders throughout some interval containing a as an interior point.

Definition: Let f be a function with derivatives of all orders throughout some interval containing a as an interior point. Then the **Tayloar series generated by** f **at** $x = a$ is

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k = f(a)+f'(a)(x-a)+\frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \cdots .$$

Definition: the **Maclaurin series generated by** f is

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!}\, x^k = f(0) + f'(0)\, x + \frac{f''(0)}{2!}\, x^2 + \cdots + \frac{f^{(n)}(0)}{n!}\, x^n + \cdots ,$$

the Taylor series generated by f at $x = 0$.

Examples

Find the Taylor series generated by $f(x) = e^x$ at $x = 0$.

Find the Taylor series generated by $f(x) = e^x$ at $x = 0$.

Sol. Since $f^{(n)}(x) = e^x$ and $f^{(n)}(0) = 1$ for every $n = 0, 1, 2, \ldots$.

Find the Taylor series generated by $f(x) = e^x$ at $x = 0$.

Sol. Since $f^{(n)}(x) = e^x$ and $f^{(n)}(0) = 1$ for every $n = 0, 1, 2, \ldots$. The Taylor's generated by f at $x = 0$ (Maclaurin series) is given by

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} \frac{1}{n!} x^n$$

$$= 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots + \frac{1}{n!}x^n + \cdots .$$

Examples

The Taylor series representation of $\sin x$ is given by

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}.$$

Examples

The Taylor series representation of $\sin x$ is given by

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}.$$

The Taylor series representation of $\cos x$ is given by

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}.$$

Taylor Polynomial

Definition: Let $f(x)$ be a function with derivatives of order k for $k = 1, 2, \ldots, N$ in some interval containing $a \in \mathbb{R}$ as an interior point.

Definition: Let $f(x)$ be a function with derivatives of order k for $k = 1, 2, \ldots, N$ in some interval containing $a \in \mathbb{R}$ as an interior point.

Then for any integer n from 0 through N, the Taylor polynomial of order n generated by f at $x = a$ is the polynomial

$$\begin{aligned} P_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots \\ + \frac{f^{(k)}(a)}{k!}(x-a)^k + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n. \end{aligned}$$

Questions

Find the Taylor's series of $f(x)$ at $x = a$, when

1. $f(x) = \ln(x), \quad a = 1$

Questions

Find the Taylor's series of $f(x)$ at $x = a$, when

1. $f(x) = \ln(x), \quad a = 1$

   (Ans: $\sum_{n=1}^{n=\infty} (-1)^{n+1} \frac{(x-1)^n}{n}$)

Questions

Find the Taylor's series of $f(x)$ at $x = a$, when

1. $f(x) = \ln(x), \quad a = 1$

   (Ans: $\sum_{n=1}^{n=\infty}(-1)^{n+1}\frac{(x-1)^n}{n}$)
2. $f(x) = \sin\left((\pi x)\right) \quad a = 1/2.$

Questions

Find the Taylor's series of $f(x)$ at $x = a$, when

1. $f(x) = \ln(x), \quad a = 1$

   (Ans: $\sum_{n=1}^{n=\infty}(-1)^{n+1}\frac{(x-1)^n}{n}$)
2. $f(x) = \sin\left((\pi x)\right) \quad a = 1/2.$

   (Ans: $\sum_{n=0}^{\infty}(-1)^{n}\frac{(\pi)^{2n}(x-1/2)^{2n}}{(2n)!}$)
3. $f(x) = \sqrt{x}, \quad a = 1$

Questions

Find the Taylor's series of $f(x)$ at $x = a$, when

1. $f(x) = \ln(x), \quad a = 1$

   (Ans: $\sum_{n=1}^{n=\infty}(-1)^{n+1}\frac{(x-1)^n}{n}$)
2. $f(x) = \sin\left((\pi x)\right) \quad a = 1/2.$
   (Ans: $\sum_{n=0}^{\infty}(-1)^{n}\frac{(\pi)^{2n}(x-1/2)^{2n}}{(2n)!}$)
3. $f(x) = \sqrt{x}, \quad a = 1$

   (Ans: $1 + \frac{1}{2}(x-1) + \sum_{n=2}^{\infty}(-1)^{n+1}\frac{1.3\cdots(2n-3)}{2^n n!}$)

Theorem 0.1.

Let f be a function having derivatives of order n for $n = 1, 2, \dots N$ on an interval I containing 'a' as an interior point. Then for any $x \in I$ and $n = 1, 2, \dots, N-1$ we have that

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + R_n(x), \quad (0.1)$$

where $R_n(x) = \dfrac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$ for some c between a

Corollary 0.2.

Let f be a function having derivatives of all orders on an interval I containing 'a' as an interior point. Then for any point x in I, the Taylor series generated by f at $x = a$ converges to $f(x)$ if and only if $R_n(x) \to 0$ as $n \to \infty$.

Questions

1. Find the power series centered at $x = 0$ for $\arctan(x)$ and its radius of convergence.
2. Find the power series centered at $x = 0$ for $\ln(4 + 3x^2)$.
3. Compute the first four nonzero terms of the power series for $\frac{\ln(1+x)}{1+2x}$.
4. Find the Taylor polynomial of degree 8 (centered at $x = 0$) for the function $\ln(\cos(x))$.
5. Find the power series expansion and the radius of convergence of the function
$$f(x) = \frac{x^2}{1 - 2x + x^2}$$
centered at $x = -1$. .

Thank you