MATH F111- Mathematics I

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Properties of limits

Theorem

Uniqueness of Limits. *A sequence in* $\mathbb{R}$ *can have at most one limit.*

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Uniqueness of Limits. *A sequence in $\mathbb{R}$ can have at most one limit.*

Proof. Let (a_n) be a real sequence and suppose that ℓ_1 and ℓ_2 are both limits for (a_n) and let $\ell_1 \neq \ell_2$.

- Let $\epsilon := |\ell_1 - \ell_2|/2$. Since $\ell_1 \neq \ell_2, \epsilon > 0$.
- Since ℓ_1 is a limit of the sequence, for the chosen $\epsilon, \exists\ N_1 \in \mathbb{N}$ such that
$$|a_n - \ell_1| < \epsilon, \quad \text{for all} \quad n \geq N_1.$$
- Since ℓ_2 is a limit of the sequence, for the chosen $\epsilon, \exists\ N_2 \in \mathbb{N}$ such that
$$|a_n - \ell_2| < \epsilon, \quad \text{for all} \quad n \geq N_2.$$

Let $N = \max\{N_1, N_2\}$. Then:

$$|a_n - \ell_1| < \epsilon \text{ and } |a_n - \ell_2| < \epsilon \text{ for all } n \geq N,$$

and hence,

$$|\ell_1 - \ell_2| = |(a_N - \ell_2) - (a_N - \ell_1)| \leq |a_N - \ell_1| + |a_N - \ell_2| < \epsilon + \epsilon = |\ell_1 - \ell_2|$$

which is a contradiction. Hence, $\ell_1 = \ell_2$. ■

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$$|a_n - \ell| < 1 \text{ for all } n \geq N.$$

Hence

$$|a_n| \leq |a_n - \ell| + |\ell| < 1 + |\ell| \text{ for all } n \geq N.$$

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- Thus it remains to find a bound for $a_1, a_2 \cdots, a_{N-1}$. Choose $\beta = \max\{|a_1|, |a_2|, \cdots, |a_{N-1}|\}$. Then $|a_n| \leq \beta$, for all $1 \leq n \leq N-1$.
- Define $\alpha := \max\left\{|a_1|, \ldots, |a_{N-1}|, |\ell| + 1\right\}$. Then $|a_n| \leq \alpha$ for all $n \in \mathbb{N}$. Hence (a_n) is bounded.

If a_n is convergent, then a_n is bounded. Equivalently, if a_n is not bounded, then a_n is not convergent. This result can be used to show if a sequence is not bounded.

- The sequence $\{(-1)^n n : n \in \mathbb{N}\}$ divergent since it is not bounded.
- A bounded sequence need not be convergent. For example, the sequence $\{(-1)^n : n \in \mathbb{N}\}$ is bounded but not convergent.

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Limit theorems. *Let* $\{a_n\}$ *and* $\{b_n\}$ *be two convergent sequences that converge to* A *and* B *respectively. Then:*

- $\lim(a_n \pm b_n) = A \pm B.$
- $\lim(a_n b_n) = AB.$
  In particular, $\lim(ca_n) = cA$ *for* $c \in \mathbb{R}.$
- $\lim \frac{a_n}{b_n} = \frac{A}{B}$, *provided* (b_n) *is a sequence of non-zero real numbers and* $B \neq 0.$

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- $\lim_{n\to\infty} \frac{n+1}{n} = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right) = 1$.
- $\lim_{n\to\infty} \frac{4-7n^6}{n^6+3} = \lim_{n\to\infty} \frac{\left(\frac{4}{n^6} - 7\right)}{1+\frac{3}{n^6}} = \frac{0-7}{1+0} = -7$.

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Proof. Let $z_n := y_n - x_n$. Then (z_n) is convergent sequence of real numbers such that $z_n \geq 0$ for all $n \geq m$. It then follows from the preceding theorem that

$$\lim z_n = \lim(y_n - x_n) = \lim y_n - \lim x_n \geq 0.$$

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Sandwich Theorem. *Let $(a_n), (b_n), (c_n)$ be three sequences of real numbers and there is a natural number m such that*

$$a_n \leq b_n \leq c_n \text{ for all } n \geq m.$$

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as $0 \leq \frac{1}{2^n} \leq \frac{1}{n}$

(iii) $\lim\limits_{n\to\infty} (-1)^n \dfrac{1}{n} = 0$.

Examples (continued)

(iv) Let $a_n := \dfrac{n^3 + 3n^2 + 1}{n^4 + 8n^2 + 2}$ for $n \in \mathbb{N}$. Then $a_n \to 0$,

(iii) Let $a_n := \dfrac{1}{n} \sin\left(\dfrac{1}{n}\right)$ for $n \in \mathbb{N}$. Then $a_n \to 0$,

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