Functions of Several variables

Saranya G. Nair
Department of Mathematics

BITS Pilani

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Definition: Function of Several Variables

The cartesian product $\mathbb{R}^n$ is $\{(r_1, r_2, \ldots, r_n) : r_1, r_2, \ldots, r_n \in \mathbb{R}\}$.

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Remark

In this case, we often denote $w = f(x_1, x_2, \ldots, x_n)$.
Observe that w *is the* **dependent variable** *of* f *and* f *is called a function of the* n **independent variables** x_1 *to* x_n.
We often call $x_1, x_2, \ldots, x_n$ *as function's* **input variables** *and* w *as function's* **output variable**.

Example: Function of Several Variables

Let $f : \mathbb{R}^3 \to \mathbb{R}$ be defined by

$$f(x_1, x_2, x_3) = \sqrt{x_1^2 + x_2^2 + x_3^2}.$$

Then $f(0, 3, 4) = 5$ and $f(1, 1, 2) = \sqrt{6}$.

Finding Domain and Ranges Function of Several Variables

Example

1. $z = \sqrt{y - x^2}$
2. $z = \frac{1}{xy}$
3. $z = \sin xy$
4. $w = \sqrt{x^2 + y^2 + z^2}$
5. $w = \frac{1}{x^2+y^2+z^2}$
6. $w = xy \ln z$

Definitions

Definition (Open disk)

Let $r > 0$ be a real number and (x_0, y_0) be a point of $\mathbb{R}^2$. Then the set

$$\{(x, y) \in \mathbb{R}^2 : \sqrt{(x - x_0)^2 + (y - y_0)^2} < r\}$$

is called the **open disk** or **open ball** of radius r at (x_0, y_0) and is denoted by $B_r(x_0, y_0)$.

Open and Closed

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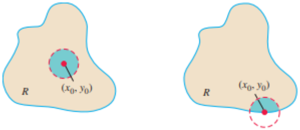

The set of all interior points of a region is called **interior** of the region.
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Definition

A region in the plane is bounded if it lies inside a disk of finite radius. A region is unbounded if it is not bounded.

Find and sketch the domain for each function.

1. $f(x,y) = \sqrt{y - x^2}$
2. $f(x,y) = \sqrt{y - x - 2}$
3. $f(x,y) = \frac{1}{\ln(4 - x^2 - y^2)}$
4. $f(x,y) = \sqrt{(x^2 - 4)(y^2 - 9)}$

For the following functions (a) find the function's domain, (b) find the function's range, (c) describe the function's level curves, (d) find the boundary of the function's domain, (e) determine if the domain is an open region, a closed region, or neither, and (f) decide if the domain is bounded or unbounded.

1. $f(x, y) = y - x$
2. $f(x, y) = \sqrt{y - x}$
3. $f(x, y) = \frac{1}{\sqrt{16 - x^2 - y^2}}$
4. $f(x, y) = e^{-(x^2 + y^2)}$

Level Curves

The set of all points $(x, y, f(x, y))$ in space, for (x, y) in the domain of f, is called the graph of f. The graph of f is also called the surface $z = f(x, y)$.

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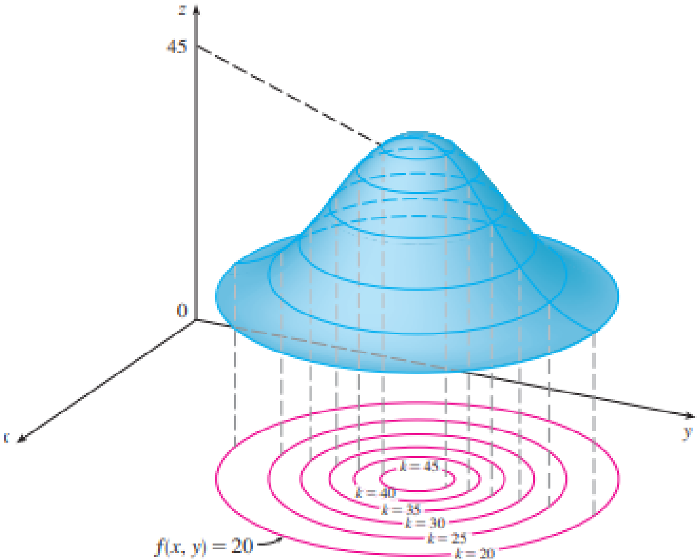
z
45
0
x
y
k = 45
k = 40
k = 35
k = 30
k = 25
k = 20
f(x, y) = 20

Consider $f(x, y) = x^2 + y^2$. The graph $(x, y, f(x, y))$ and level curves $f(x, y) = k$ for $k = 0, 1, 2, 3, 4, 5$ are

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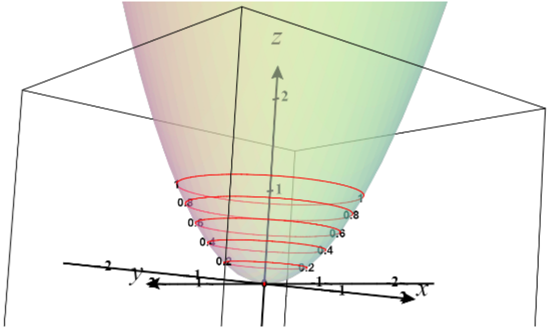

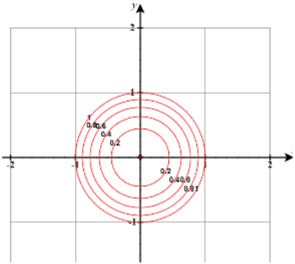

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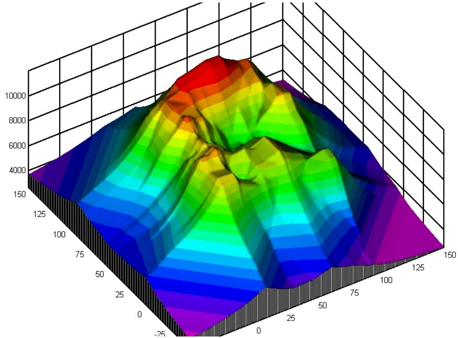

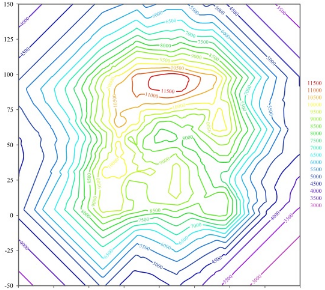

Sketch some level curves of the function $f(x, y) = 4x^2 + y^2$.

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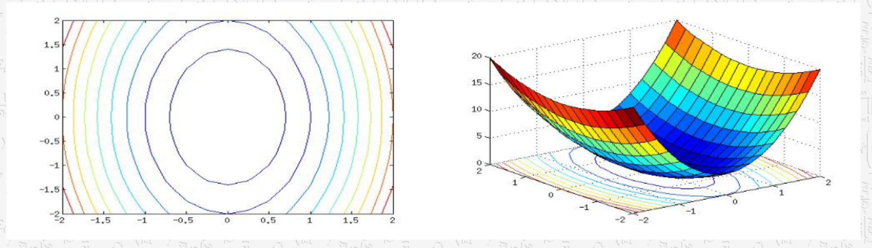

Example

Graph $f(x,y) = 1 - x^2 - y^2$ *and plot the level curves* $f(x,y) = 0$, $f(x,y) = 1$, $f(x,y) = 2$ *and* $f(x,y) = -2$ *in the domain of* f *in the plane.*

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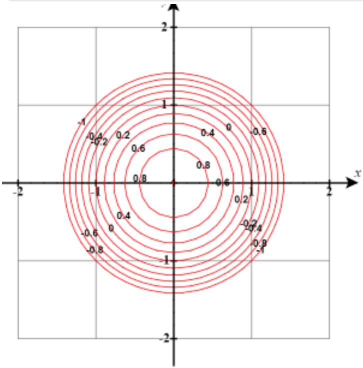
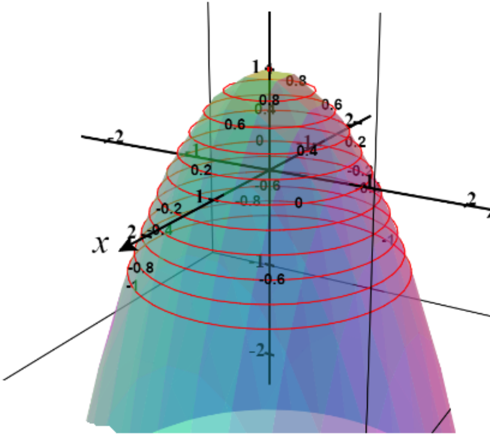
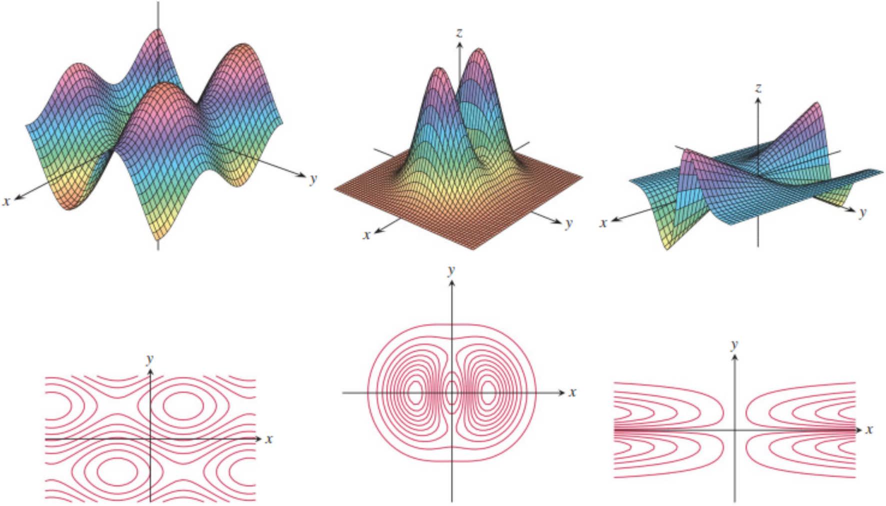




Some examples



(a) $z = \sin x + 2\sin y$ (b) $z = (4x^2 + y^2)e^{-x^2-y^2}$ (c) $z = xye^{-y^2}$

Definition (Level Surface)

Let f be a real valued function with domain $B \subseteq \mathbb{R}^3$; then for a real constant c, the set of all points (x, y, z) of B satisfying $f(x, y, z) = c$ is called the **level surface**.

Example

Describe the level surfaces of the function

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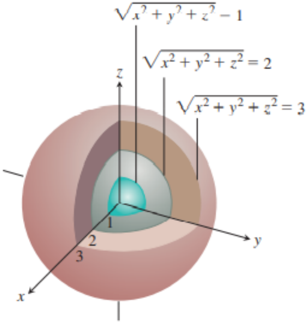

Find and sketch the level curves $f(x, y) = c$ on the same set of coordinate axes for the given values of c.

1. $f(x, y) = x + y - 1, \ c = -3, -2, -1, 0, 1, 2, 3$
2. $f(x, y) = x^2 + y^2, \ c = 0, 1, 4, 9, 16, 25$
3. $f(x, y) = xy, \ c = -9, -4, -1, 0, 1, 4, 9$

Open and Closed sets in $\mathbb{R}^3$

The definitions of interior, boundary, open, closed and unbounded for regions in space are similar to those of regions in plane. To accomodate the extra dimension, we use solid balls of positive radius instead of disks. An open ball B in $\mathbb{R}^3$ centered at (x_0, y_0, z_0) with radius r is the set of all points (x, y, z) such that

$$\sqrt{(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2} < r.$$