Double Integrals

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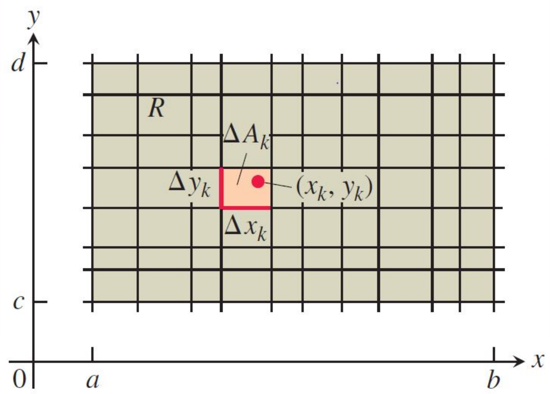
Let us divide R into rectangles as shown in the figure below.



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$$\lim_{\|P\|\to 0} \sum_{k=1}^{n} f(x_k, y_k)\Delta A_k = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k, y_k)\Delta A_k$$

exists, then f is said to be integrable over R and this limit is called the double integral of f over R, written as

$$\iint\limits_R f(x, y)dA \quad \text{or} \quad \iint\limits_R f(x, y)\, dxdy.$$

Double Integral as Volume

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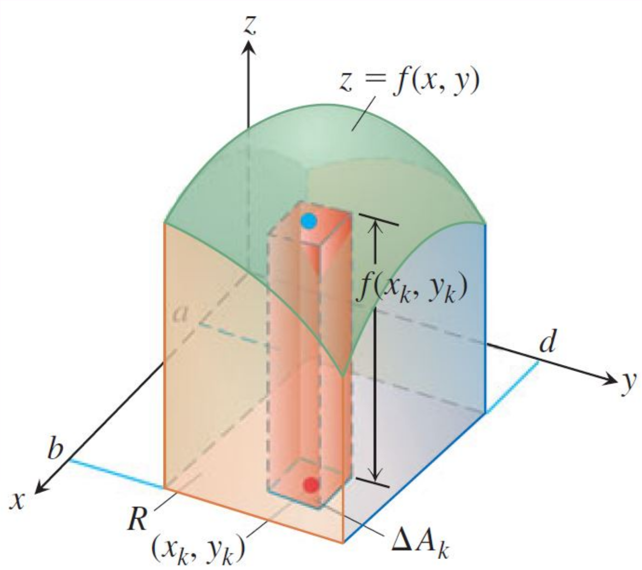
If $f(x, y)$ is a positive continuous function on a rectangular region R, then the volume of the 3-dimensional solid region over the xy-plane bounded below by R and above by the surface $z = f(x, y)$ is given by the double integral

$$\text{Volume} = \iint\limits_R f(x, y)\,dA.$$

Double Integral as Volume



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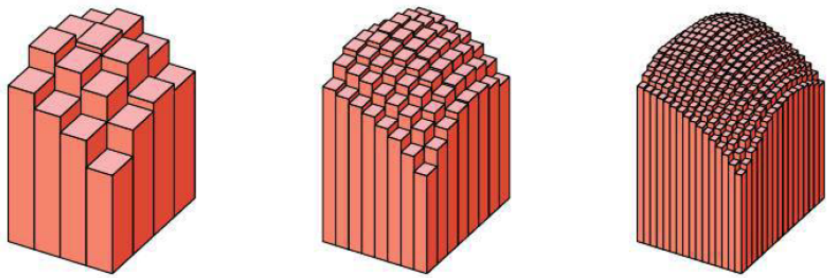

(a) $n = 16$ (b) $n = 64$ (c) $n = 256$

Fubini's Theorem

Fubini's Theorem (First Version)

If $f(x, y)$ is continuous throughout the rectangular region $R: \; a \leq x \leq b, c \leq y \leq d$, then

$$\iint\limits_R f(x,y)dA = \int_a^b \int_c^d f(x,y)dydx = \int_c^d \int_a^b f(x,y)dxdy.$$

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- Fubini's theorem says that the double integral of any continuous function over a rectangle can be calculated as an iterated or repeated integral in either order of integration.

Examples

Example 1. $\displaystyle\iint_R (4-y^2)\,dA$, where $R: 0 \le x \le 3,\ 0 \le y \le 2$.

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Solution: Use Fubini's Theorem.

$$\begin{aligned}
\iint_R (4-y^2)\,dA &= \int_0^3 \int_0^2 (4-y^2)\,dy\,dx \\
&= \int_0^3 \left(4y - \frac{1}{3}y^3\right)\Big|_0^2 dx \\
&= \int_0^3 \frac{16}{3}\,dx = \frac{16}{3}(x)\Big|_0^3 = 16.
\end{aligned}$$

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\end{aligned}$$

If we reverse the order of integration, then

$$\int_0^2 (4-y^2)(x)\Big|_0^3 dy = 3\int_0^2 (4-y^2)\,dy = 3 \times 16/3 = 16.$$

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$$\iint_R xy\cos y\,dA = \int_{-1}^{1}\int_{0}^{\pi} xy\cos y\,dy\,dx$$

$$= \int_{-1}^{1} (xy\sin y + x\cos y)\Big|_0^{\pi} dx = \int_{-1}^{1} -2x\,dx = 0.$$

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Example 3. $\displaystyle\iint_R \frac{y}{x^2y^2+1}\,dA$, where $R: 0 \le x \le 1,\ 0 \le y \le 1.$

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$$\iint_R \frac{y}{x^2y^2+1}dA = \int_0^1\int_0^1 \frac{y}{(xy)^2+1}dx\,dy$$

$$= \int_0^1 (\tan^{-1}(xy))\Big|_0^1 dy = y\tan^{-1} y - \frac{1}{2}\ln(1+y^2)\Big|_0^1 = \frac{\pi}{4} - \frac{1}{2}\ln 2.$$

Example 4. Find the volume of the region bounded above by the plane $z = 2 - x - y$ and below by the square R: $0 \leq x \leq 1$, $0 \leq y \leq 1$.

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Solution: Volume V is given by

$$\begin{aligned} V &= \iint\limits_R (2 - x - y) dA \\ &= \int_0^1 \int_0^1 (2 - x - y)\, dy dx = \int_0^1 (2y - xy - \frac{y^2}{2})|_0^1 dx \\ &= \int_0^1 (\frac{3}{2} - x)\, dx = (\frac{3}{2}x - \frac{x^2}{2})\Big|_0^1 = 1. \end{aligned}$$

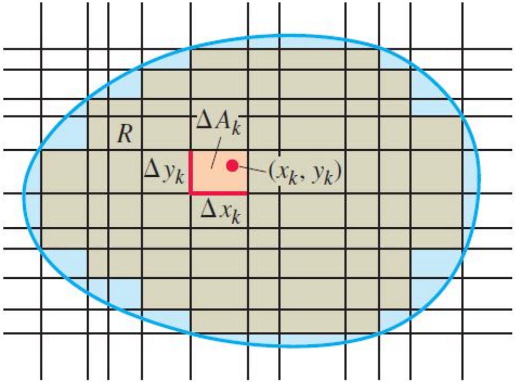

FIGURE 15.8 A rectangular grid partitioning a bounded, nonrectangular region into rectangular cells.

Double Integrals over Bounded, Nonrectangular Regions

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A partition of R is formed by taking the rectangles that lie completely inside it, not using any that are either partly or completely outside. For commonly arising regions, more and more of R is included as the norm of a partition approaches zero.

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Once we have a partition of R, we number the rectangles in some order from 1 to n and let ΔA_k be the area of the kth rectangle. We then choose a point (x_k, y_k) in the kth rectangle and form the Riemann sum:

$$S_n = \sum_{k=1}^{n} f(x_k, y_k)\Delta A_k.$$

As the norm of the partition forming S_n goes to zero, $\|P\| \to 0$, the width and height of each enclosed rectangle goes to zero and their number goes to infinity. If $f(x, y)$ is a continuous function, then these Riemann sums converge to a limiting value, not dependent on any of the choices we made. This limit is called the double integral of $f(x, y)$ over R:

$$\iint\limits_R f(x,y)dA := \lim_{\|P\|\to 0} \sum_{k=1}^{n} f(x_k, y_k)\Delta A_k.$$

Fubini's Theorem General Version

Theorem

Let $f(x, y)$ be a continuous function on a region R.

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In a double integral, the outer limits must be constant, but the inner limits can depend on the outer variable.

Examples

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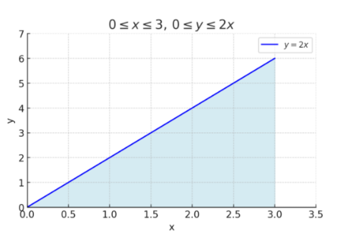

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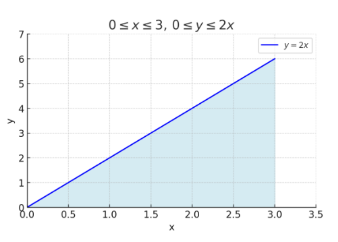

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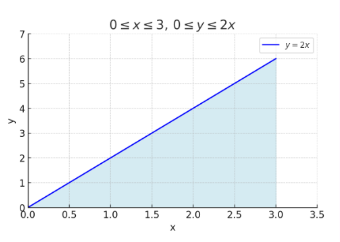

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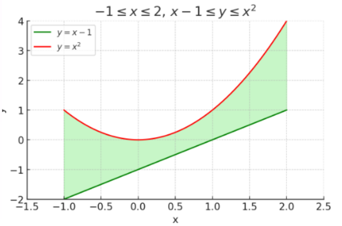

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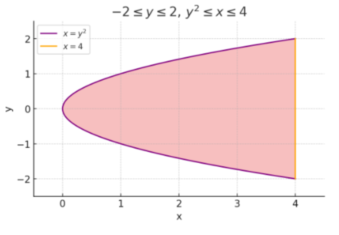

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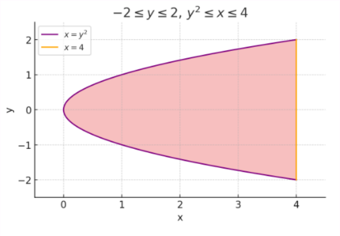

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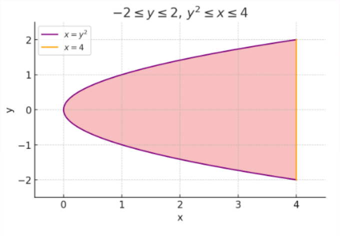

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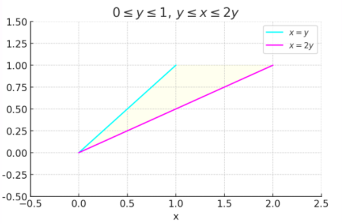

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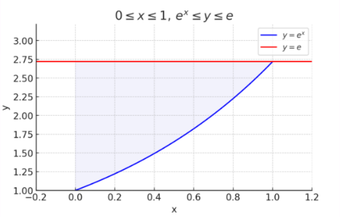

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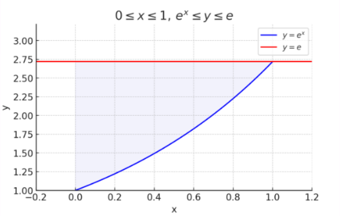

Example 2. Sketch the region of integration and evaluate the integral

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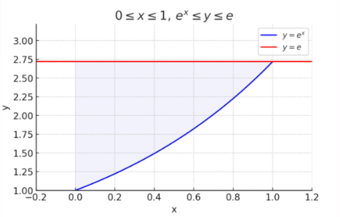

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Solution:

$$\int_0^1 \int_0^{y^2} 3y^3 e^{xy}\,dx\,dy = \int_0^1 \left[3y^3 \frac{e^{xy}}{y}\right]_0^{y^2} dy$$

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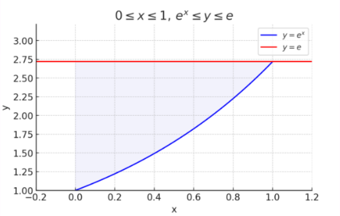

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$$= \int_0^1 (3y^2 e^{y^3} - 3y^2) dy = e^{y^3} - y^3 \Big|_0^1 = e - 2.$$

How to find the limits of integration

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Using horizontal cross-sections: If we want to integrate with respect to x first, we simply choose horizontal lines to find the x limits as functions of y.

Properties of Double Integrals

If $f(x,y)$ and $g(x,y)$ are continuous on the bounded region R, then the following properties hold:

1. Constant Multiple: $\displaystyle\iint\limits_R cf(x,y)dA = c\iint\limits_R f(x,y)dA$ ($c \in \mathbb{R}$)
2. Sum and Difference:
$$\iint\limits_R (f(x,y) \pm g(x,y))dA = \iint\limits_R f(x,y)dA \pm \iint_R g(x,y)dA$$
3. Monotonicity: $\displaystyle\iint\limits_R f(x,y)dA \geq \iint\limits_R g(x,y)dA$, if $f(x,y) \geq g(x,y)$
4. Additivity: $\displaystyle\iint\limits_R f(x,y)dA = \iint\limits_{R_1} f(x,y)dA + \iint\limits_{R_2} f(x,y)dA,$
if R is the union of two nonoverlapping regions R_1 and R_2.

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$$(a) \int_0^9 \int_0^{\sqrt{x}} dydx,$$

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we run into a problem because $\int \frac{\sin x}{x} dx$ cannot be expressed in terms of elementary functions (there is no simple antiderivative)!

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$$\int_0^1 \int_0^{1-v} (v - \sqrt{u})\,du\,dv = \int_0^1 \left[vu - \frac{2}{3}u^{\frac{3}{2}}\right]_0^{1-v} dv$$
$$= \int_0^1 \left(v - v^2 - \frac{2}{3}(1-v)^{\frac{3}{2}}\right) dv$$
$$= \frac{1}{2}v^2 - \frac{1}{3}v^3 + \frac{4}{15}(1-v)^{\frac{5}{2}}\Big|_0^1 = -\frac{1}{10}.$$

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Solution. The volume is by definition given by the integral $\displaystyle\iint_R (x^2 + y^2)dA$ where the region R is stated as above.

Examples

Let us say that we want to first integrate with respect to y first then let us find the limits of y as functions of x. The limits are $y = x$ and $y = 2 - x$, and x varies from 0 to 1.

$$\begin{aligned}\int_0^1 \int_x^{2-x} (x^2 + y^2)dydx &= \int_0^1 \left[\frac{1}{3}y^3 + x^2 y\right]_x^{2-x} dx \\ &= \int_0^1 \left(2x^2 - \frac{7}{3}x^3 + \frac{1}{3}(2-x)^3\right)dx \\ &= \left[\frac{2}{3}x^3 - \frac{7}{3}x^4 - \frac{1}{12}(2-x)^4\right]_0^1 = \frac{4}{3}.\end{aligned}$$

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Note: If you want to integrate w.r.t x first, consider two regions $R = R_1 \cup R_2$, first region where $R_1 : 0 \leq x \leq y$, $0 \leq y \leq 1$ and the second region where $R_2 : 0 \leq x \leq 2 - y$, $0 \leq y \leq 1$.