Vector Valued Functions and Motion in Space

Gunja Sachdeva

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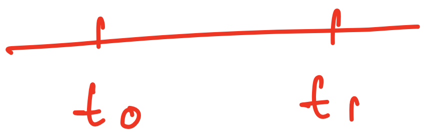
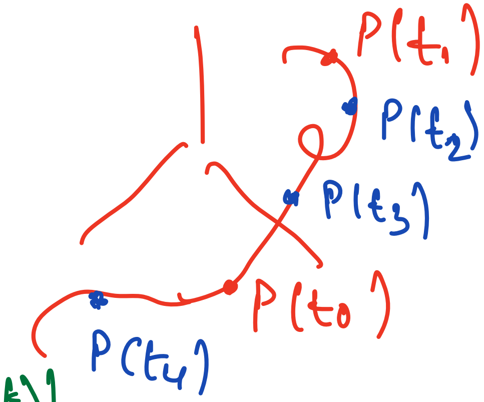
Recall

If both the end pts are fixed say $r(t_0)$, $r(t_1)$





$$S = \int_{t_0}^{t_1} |v(z)|\, dz$$

$$s(t) = \int_{t_0}^{t} |v(z)|\, dz \implies \frac{ds}{dt} = |v(t)|$$

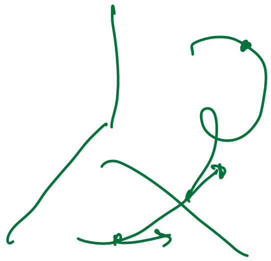
$$s : \mathbb{R} \longrightarrow \mathbb{R}$$
$$t \longmapsto s(t)$$

$$r(t) \xrightarrow[\text{value of } t \text{ in terms of } s]{\text{Put the}} r(s)$$

$$v(t) = \frac{dr}{dt}$$

$$T\text{- unit tangent vector} = \frac{v(t)}{|v(t)|} = \frac{dr}{dt} \Big/ \frac{ds}{dt} = \frac{dr}{ds}$$

$$\kappa = \left|\frac{dT}{ds}\right| \text{ - curvature.}$$



Exs

$$\kappa(\text{straight line}) = 0$$

$$\kappa(\text{circle in a plane of radius } a) = \frac{1}{a}$$

Recall

$$|r(t)| = c$$

$$\Rightarrow \quad r'(t) \cdot r(t) = 0$$

$$r(t) = r'(t_0)t + r(t_0)$$

$$s = \int_0^t |r'(t_0)|\,dz$$

$$v(t) = r'(t_0)$$

$$|v(t)| = |r'(t_0)| \qquad = t|r'(t_0)|$$

$$r\left(s/|r'(t_0)|\right) = r'(t_0)\frac{s}{|r'(t_0)|} + r(t_0) \qquad t = s\Big/|r'(t_0)|$$

$$T = \frac{dr}{ds} = \frac{r'(t_0)}{|r'(t_0)|},$$

non unit normal vector $= 0$

Principal Unit Normal

clearly $|T(s)| = 1$

$$\Rightarrow \frac{dT}{ds} \cdot T = 0$$

Principal Unit Normal

- Since $\mathbf{T}$ has constant length (as $|\mathbf{T}| = 1$), the derivative $d\mathbf{T}/ds$ is orthogonal to $\mathbf{T}$.
- Therefore, $\dfrac{d\mathbf{T}/ds}{|d\mathbf{T}/ds|}$ is unit vector orthogonal to $\mathbf{T}$.

  Also note that $|d\mathbf{T}/ds| = \kappa$.

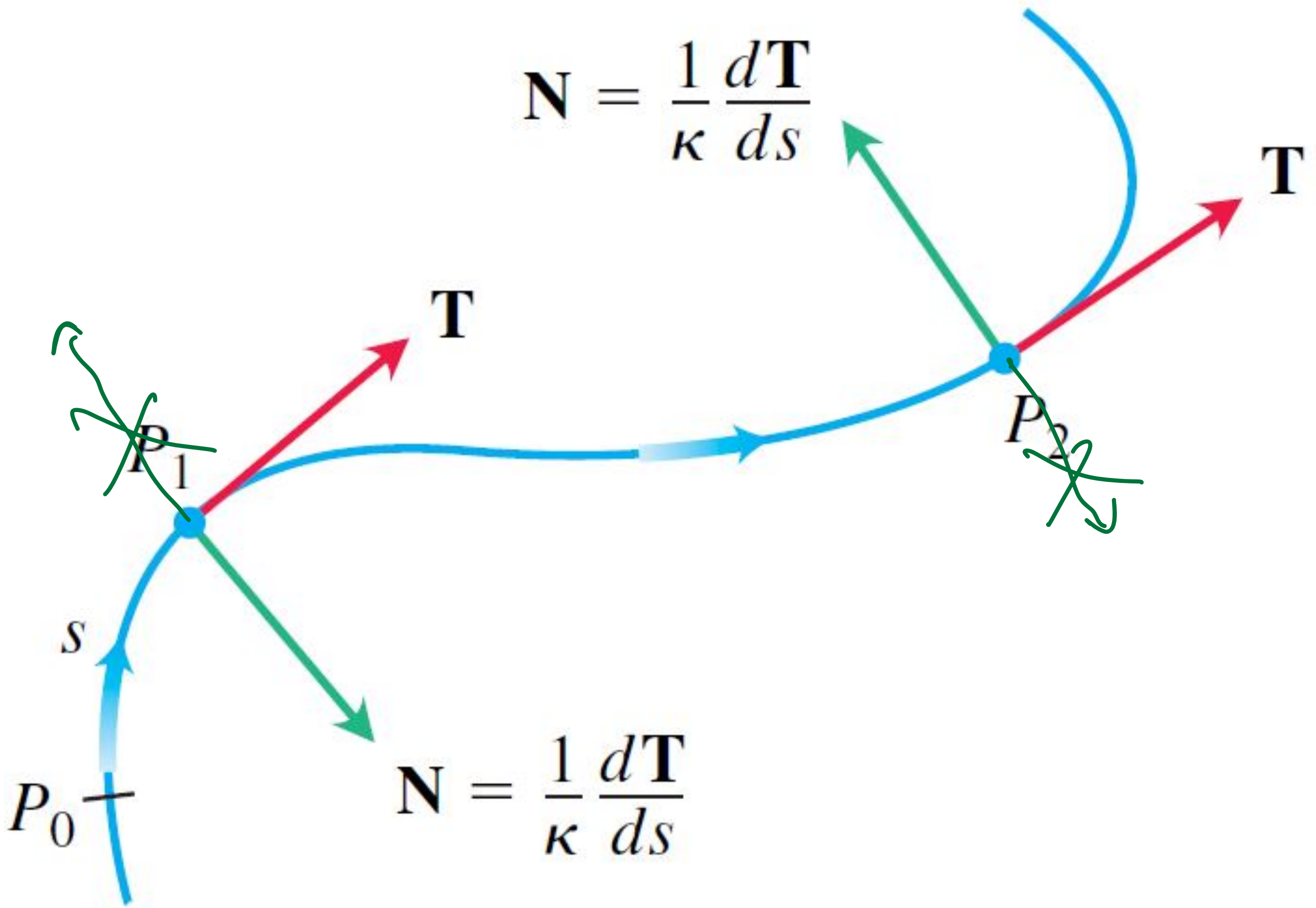
Definition 0.1 (Principal unit normal).

At a point where $\kappa \neq 0$, the principal unit normal vector for a smooth curve in the plane is

$$\mathbf{N} = \frac{d\mathbf{T}/ds}{|d\mathbf{T}/ds|} = \frac{1}{\kappa}\frac{d\mathbf{T}}{ds}.$$

Principal unit normal

Note that the principal unit normal **N** points the direction in which the unit tangent is turning. In other words it will point towards the concave side of the curve.



Principal unit normal

Remark 0.2 (Formula for calculating **N***).*

If $\mathbf{r}(t)$ is smooth curve, then the principal unit normal is

$$\mathbf{N} = \frac{d\mathbf{T}/dt}{|d\mathbf{T}/dt|}$$

where $\mathbf{T} = \mathbf{v}/|\mathbf{v}|$ is the unit tangent vector.

Summary

Let $\mathbf{r}(t)$ be a smooth curve in space, and if s is the arc length parameter of the curve, then:

1. The unit tangent vector $\mathbf{T}$ is $d\mathbf{r}/ds = \mathbf{v}/|\mathbf{v}|$.
2. The curvature in space is then defined to be

$$\kappa = \left|\frac{d\mathbf{T}}{ds}\right| = \frac{1}{|\mathbf{v}|}\left|\frac{d\mathbf{T}}{dt}\right|$$

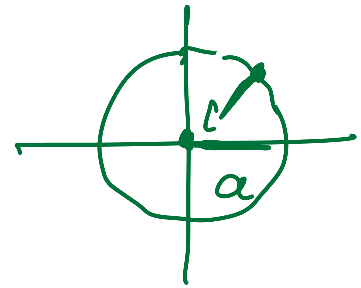
3. The principal unit normal to be

$$\mathbf{N} = \underline{\frac{1}{\kappa}\frac{d\mathbf{T}}{ds}} = \underline{\frac{d\mathbf{T}/dt}{|d\mathbf{T}/dt|}}.$$

Example

- Find the principal unit normal to the curve **N** for the circle

$$\mathbf{r}(t) = (a\cos t)\mathbf{i} + (a\sin t)\mathbf{j}$$



at $t = \pi/4$.

$t = \pi/4$, $r(\pi/4) = \frac{a}{\sqrt{2}}\hat{i} + \frac{a}{\sqrt{2}}\hat{j}$

$T = \frac{v(t)}{|v(t)|}$ $\qquad v(t) = r'(t) = -a\sin t\,\hat{i} + a\cos t\,\hat{j}$

$|v(t)| = a$

$= -\sin t\,\hat{i} + \cos t\,\hat{j}$

$\frac{dT}{dt} = -\cos t\,\hat{i} - \sin t\,\hat{j}$ $\qquad \left|\frac{dT}{dt}\right| = 1$

$N\Big|_{\pi/4} = \left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$

Example

- Find the principal unit normal to the curve $\mathbf{N}$ for the circle

$$\mathbf{r}(t) = (a\cos t)\mathbf{i} + (a\sin t)\mathbf{j}$$

at $t = \pi/4$.

- Find the $\mathbf{T}$, $\mathbf{N}$ and κ for the plane curve

$$\mathbf{r}(t) = (\cos t + t\sin t)\mathbf{i} + (\sin t - t\cos t)\mathbf{j}, \;\; t > 0.$$

$$T = \frac{t\cos t\,\hat{i} + t\sin t\,\hat{j}}{t} = (\cos t, \sin t)$$

$$|v| = t$$

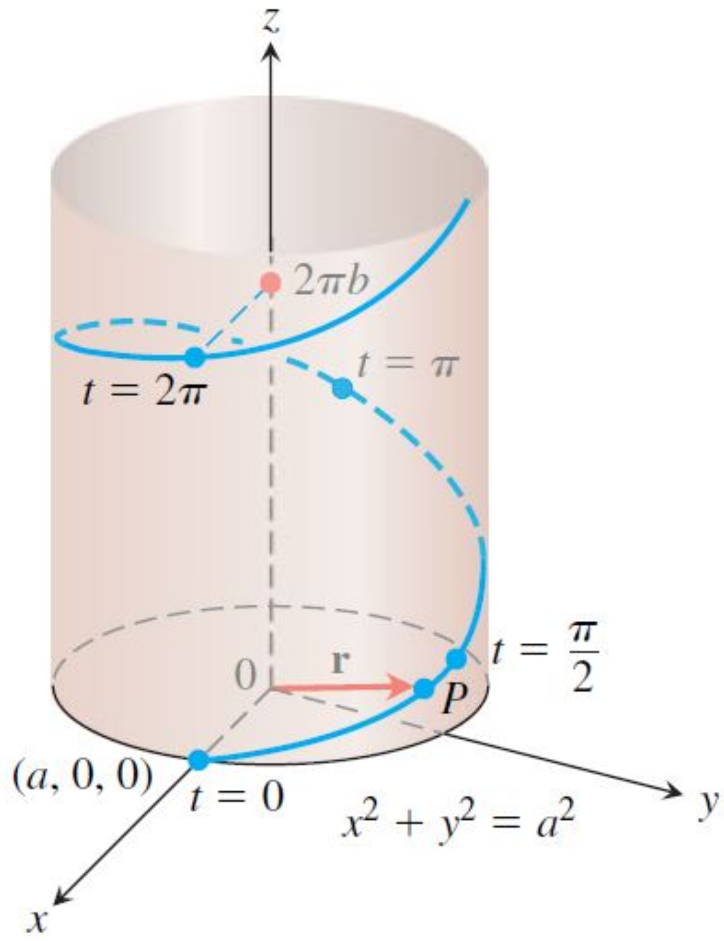
$$\boxed{\kappa = \frac{1}{t}}$$

$$N = (-\sin t, \cos t)$$

Example

Find the curvature κ and **N** for the helix

$$\mathbf{r}(t) = (a\cos t)\mathbf{i} + (a\sin t)\mathbf{j} + (bt)\mathbf{k}, \;\; a, b > 0.$$



Example

Solution.

$$\kappa = \frac{1}{|v|}\left|\frac{dT}{dt}\right| \qquad |v| = \sqrt{a^2+b^2}$$

$$T = \frac{(-a\sin t,\ a\cos t,\ b)}{\sqrt{a^2+b^2}}, \quad \frac{dT}{dt} = \left(\frac{-a\cos t}{\sqrt{a^2+b^2}}, \frac{-a\sin t}{\sqrt{a^2+b^2}}, 0\right)$$

$$\left|\frac{dT}{dt}\right| = \frac{a}{\sqrt{a^2+b^2}}$$

$$\kappa = \frac{1}{\sqrt{a^2+b^2}} \times \frac{a}{\sqrt{a^2+b^2}} = \frac{a}{a^2+b^2}$$

$$N = (-\cos t, -\sin t, 0)$$

Example

Solution. The velocity vector: $\mathbf{v}(t) = (-a\sin t)\mathbf{i} + (a\cos t)\mathbf{j} + b\mathbf{k}$.

$|\mathbf{v}| = \sqrt{a^2 + b^2}$ and the unit tangent is given by

$$\mathbf{T} = \frac{1}{\sqrt{a^2 + b^2}}[(-a\sin t)\mathbf{i} + (a\cos t)\mathbf{j} + b\mathbf{k}]$$

which implies

$$\frac{d\mathbf{T}}{dt} = \frac{a}{\sqrt{a^2 + b^2}}[(-\cos t)\mathbf{i} - (\sin t)\mathbf{j}].$$

Example

Therefore, we have (as $a > 0$)

$$\left|\frac{d\mathbf{T}}{dt}\right| = \frac{a}{\sqrt{a^2+b^2}}.$$

The curvature and the principal unit normal are given by

$$\kappa = \frac{1}{|\mathbf{v}|}\left|\frac{d\mathbf{T}}{dt}\right| = \frac{a}{a^2+b^2},$$

and

$$\mathbf{N} = (-\cos t)\mathbf{i} - (\sin t)\mathbf{j}. \quad +0\hat{k}$$

Example

1. Show that the curvature of the curve $y = f(x)$ in xy-plane is given by

$$\kappa(x) = \frac{|f''(x)|}{[1 + (f'(x))^2]^{3/2}}$$

where $f(x)$ is twice-differentiable function of x.

$$\Rightarrow \boxed{x = t} \quad y = f(t)$$

$$r(t) = (t, f(t)) = t\hat{i} + f(t)\hat{j}$$

$$r'(t) = \hat{i} + f'(t)\hat{j}$$

$$|r'(t)| = \sqrt{1 + f'(t)^2}$$

$$\kappa = \frac{1}{|v|}\left|\frac{dT}{dt}\right|, \qquad T = \frac{\hat{i} + f'(t)\hat{j}}{\sqrt{1 + f'(t)^2}}$$

$$= (\hat{i} + f'(t)\hat{j})\left((1 + f'(t)^2)\right)^{-1/2}$$

$$\frac{dT}{dt} = -\frac{1}{\cancel{2}}(1 + f'(t)^2)^{-3/2}\,\cancel{2}f'(t)f''(t)$$

$$f''(t)(1 + f'(t)^2)^{-1/2} + f'(t)\left(-\frac{1}{\cancel{2}}(1 + f'(t)^2)^{-3/2}\right), \quad f'(t)(1 + f'(t)^2)^{-1/2}$$

Example

1. Show that the curvature of the curve $y = f(x)$ in xy-plane is given by

$$\kappa(x) = \frac{|f''(x)|}{[1 + (f'(x))^2]^{3/2}}$$

where $f(x)$ is twice-differentiable function of x.

Solution: $T = \frac{\mathbf{i}+f'(x)\mathbf{j}}{(1+f'(x)^2)^{1/2}}$

$$\frac{dT}{dt} = f''(x)[1 + f'(x)^2]^{-3/2}[-f'(x)\mathbf{i} + \mathbf{j}]$$

Example

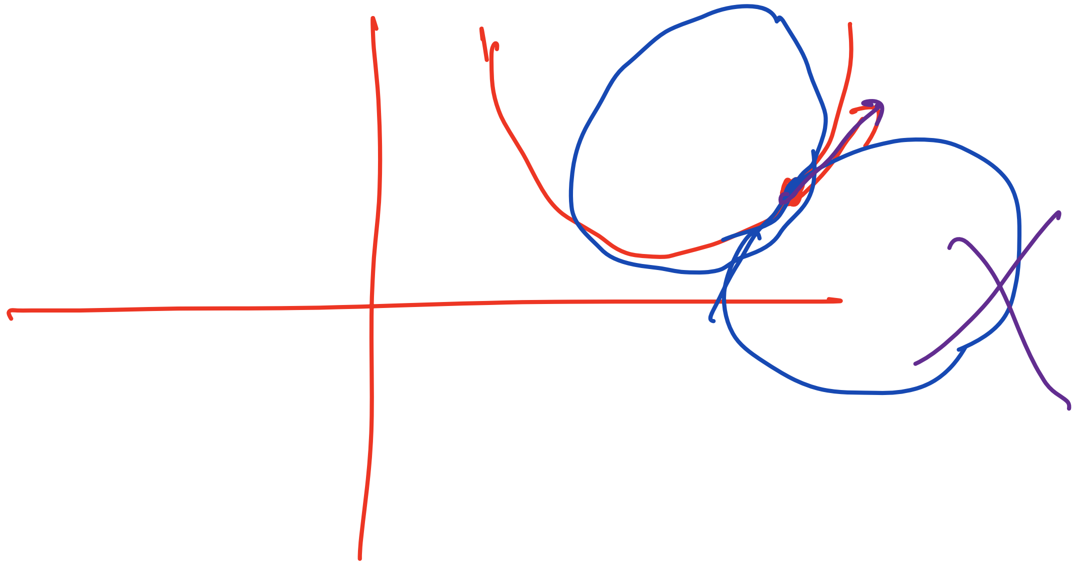
2. Show that the curvature of the smooth curve $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}$ in xy-plane is given by the formula

$$\kappa = \frac{|(x'y'' - x''y')|}{[(x')^2 + (y')^2]^{3/2}}.$$

Solution: $T = \dfrac{x'(t)\mathbf{i} + y'(t)\mathbf{j}}{(x'^2 + y'^2)^{1/2}}$

$$\frac{dT}{dt} = [x'^2 + y'^2]^{-3/2}[x''y' - x'y''][y'\mathbf{i} - x'\mathbf{j}]$$

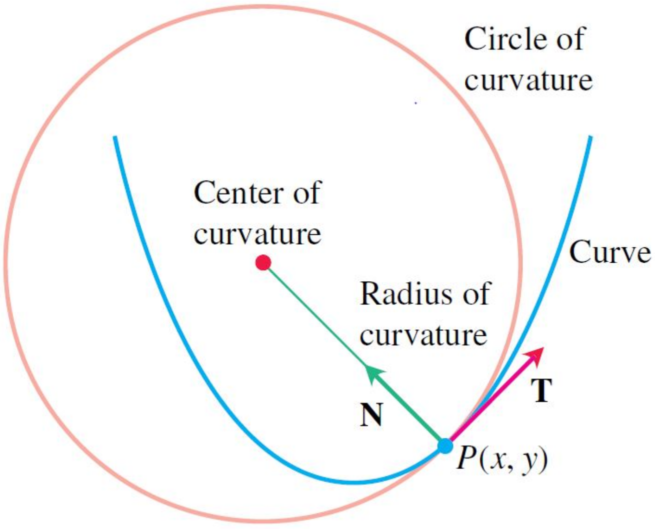
Circle of curvature for plane curves



Circle of curvature for plane curves

The **circle of curvature** or **osculating circle** at a point P on a plane curve where $\kappa \neq 0$ is the circle in the plane of the curve that

1. is tangent to the curve at P (has the same tangent line the curve has)
2. has the same curvature the curve has at P
3. has center that lies toward the concave or inner side of the curve



Radius and center of curvature

Definition 0.3.

The **radius of curvature** of the curve at P is the radius of the circle of curvature, which is

$$\text{Radius of curvature} = \rho = \frac{1}{\kappa}.$$

The **center of curvature** of the curve at P is the center of the circle of curvature.