Lecture 11

ANGULAR MOMENTUM II

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1. Angular Momentum in Lab frame vs CM frame
2. Some theorems on Torque

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3. **Translation of axis of rotation**

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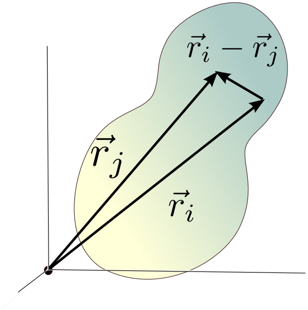
1. **Angular Momentum in Lab frame vs CM frame**
2. **Some theorems on Torque**
3. **Translation of axis of rotation**
4. **Angular Momentum & KE in Rolling**
5. **Examples**
   - Downhill Race..
   - Yo-Yo
   - Rolling Wheel rotating about axle

Radhika Vathsan, BITS-Goa, 2010-11

System of Particles: Rigid Bodies

Definition

Rigid Body: mutual separation of points within body remain unchanged as the body moves.

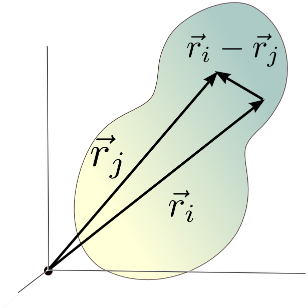


System of Particles: Rigid Bodies

Definition

Rigid Body: mutual separation of points within body remain unchanged as the body moves.

Chasle's theorem:
The most general motion of a rigid body is
Translation of CM + Pure rotation about CM



Angular Momentum of a system of particles

$$\overrightarrow{L} = \sum_i (\vec{r}_i \times \vec{p}_i)$$

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$$\overrightarrow{\boldsymbol{L}} = \sum_i \left(\vec{\boldsymbol{r}}_i \times \vec{\boldsymbol{p}}_i\right) = \sum_i m_i \left(\vec{\boldsymbol{r}}_i \times \vec{\boldsymbol{v}}_i\right)$$

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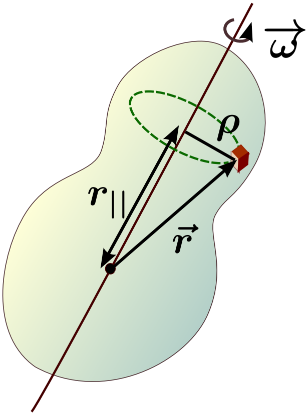
- Rigid body rotating about axis thru CM



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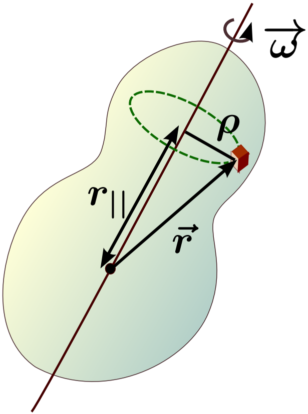

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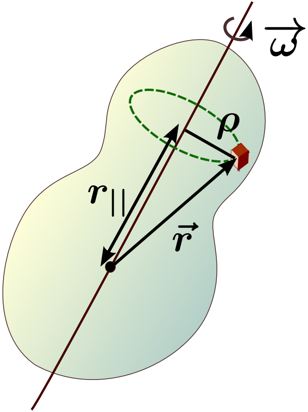

$$\begin{aligned} \vec{L} &= \int dm \;\; \vec{r} \times \vec{v} \\ &= \int dm \;\; \vec{r} \times (\vec{\omega} \times \vec{r}) \end{aligned}$$

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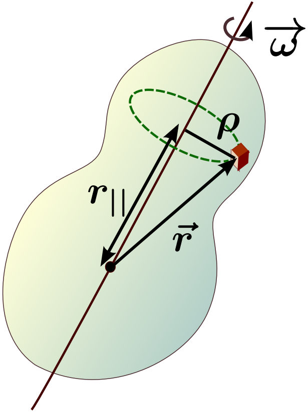
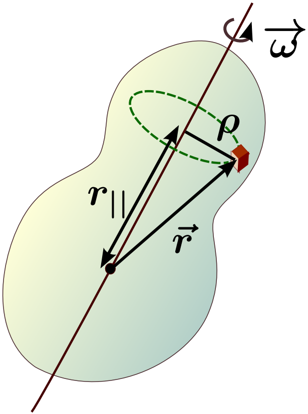


$$\begin{aligned} \vec{L} &= \int dm \;\; \vec{r} \times \vec{v} \\ &= \int dm \;\; \vec{r} \times (\vec{\omega} \times \vec{r}) \\ &= \int dm \;\; [\vec{\omega} r^2 - \vec{r}(\vec{\omega} \cdot \vec{r})] \end{aligned}$$

Angular Momentum of a system of particles

Component of $\overrightarrow{L}$ along $\vec{\omega}$:

$$L_{||} = \overrightarrow{L} \cdot \hat{\omega}$$



Angular Momentum of a system of particles

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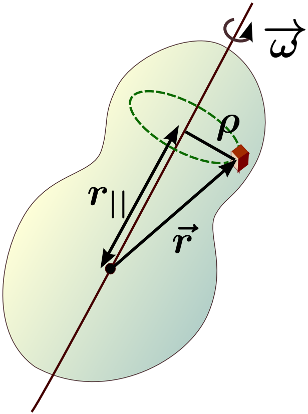
$$\begin{aligned} L_{||} &= \overrightarrow{\boldsymbol{L}} \cdot \hat{\boldsymbol{\omega}} \\ &= \omega \int dm \; [r^2 - r_{||}^2] \end{aligned}$$



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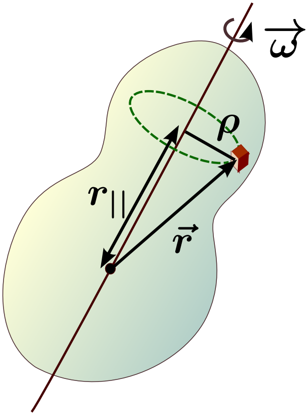

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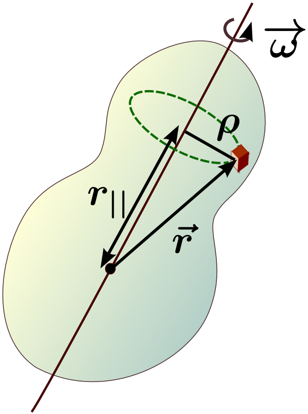

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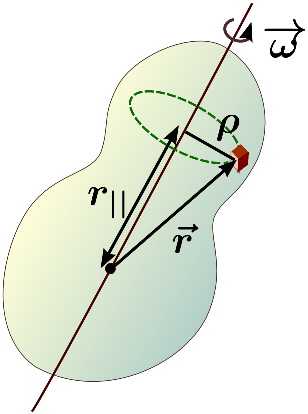
$$\overrightarrow{\boldsymbol{L}}_{||} = I\vec{\boldsymbol{\omega}}$$



$\vec{L}$ in Lab frame vs CM frame

An important (and useful) result

$\overrightarrow{L}$ in Lab frame vs CM frame

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$$\vec{\boldsymbol{r}}_i = \overrightarrow{\boldsymbol{R}}_{\text{cm}} + \vec{\boldsymbol{r}}'_i \qquad \vec{\boldsymbol{v}}_i = \overrightarrow{\boldsymbol{V}}_{\text{cm}} + \vec{\boldsymbol{v}}'_i$$

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$$\overrightarrow{\boldsymbol{L}}_{\text{lab}} = \sum_i m_i \left(\overrightarrow{\boldsymbol{R}}_{\text{cm}} + \vec{\boldsymbol{r}}'_i\right) \times \left(\overrightarrow{\boldsymbol{V}}_{\text{cm}} + \vec{\boldsymbol{v}}'_i\right)$$

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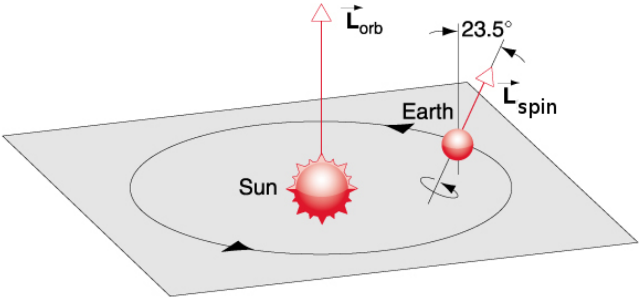
\end{aligned}
$$

$\vec{L}$ of system in Lab frame = $\vec{L}$ of CM in Lab frame + $\vec{L}$ of system about CM

Example:

What is the net angular momentum of Earth about sun?



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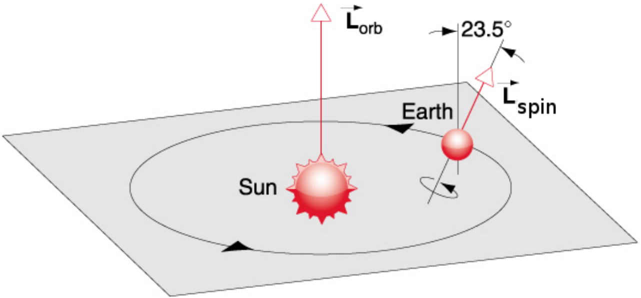

$$\vec{L} = \vec{L}_{\text{orbit}} + \vec{L}_{\text{spin}}$$

Some Theorems on Torque on a System

- $\vec{\tau}_{\text{net}} =$

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- $\vec{\boldsymbol{\tau}}_{\text{net}} = \sum_i \left(\vec{r}_i \times \vec{\boldsymbol{f}}_{\text{net},i} \right)$

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Total Torque on a system = Total **external** Torque

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if internal mutual forces are along line joining particles

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- $\vec{\tau}_{\text{net}} = \vec{R}_{\text{CM}} \times \vec{F}_{\text{net,ext}} + \sum_i \left(\vec{r}'_i \times \vec{f}_{\text{ext},i} \right)$

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Torque in Lab Frame $=$

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Torque in Lab Frame $=$ Torque on CM

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Proof Page 263, Eq 6.15 (Kleppner)

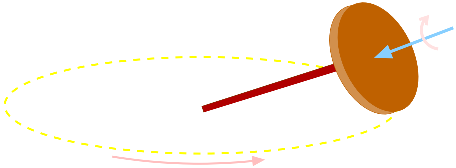
Rolling: translation of axis of rotation

Rotation about CM + translation of CM along some curve

Rolling: translation of axis of rotation

Rotation about CM + translation of CM along some curve

- Eg: rotating wheel about a point on axle



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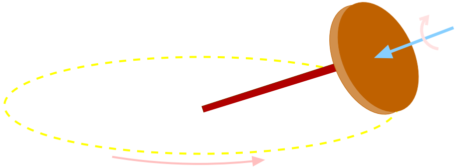
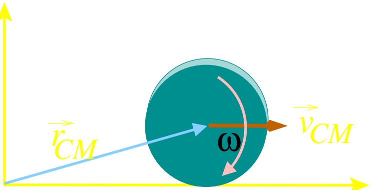


- Simpler case: rolling wheel: translation of CM



Rotation axis moves parallel to itself ($\vec{v}_{CM} \perp \vec{\omega}$)

$\overrightarrow{L}$ & KE in Rolling (Pure Translation!)

- Angular Momentum

$$\overrightarrow{\boldsymbol{L}}_O = \overrightarrow{\boldsymbol{L}}_{CM} + \overrightarrow{\boldsymbol{L}}_{about\ CM}$$

$\overrightarrow{L}$ & KE in Rolling (Pure Translation!)

- Angular Momentum

$$\begin{aligned}\overrightarrow{\boldsymbol{L}}_O &= \overrightarrow{\boldsymbol{L}}_{CM} + \overrightarrow{\boldsymbol{L}}_{about\ CM} \\ &= M\vec{\boldsymbol{r}}_{\text{CM}} \times \overrightarrow{\boldsymbol{V}}_{\text{CM}} + I_0\vec{\boldsymbol{\omega}}\end{aligned}$$

- Angular Momentum

$$
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\overrightarrow{\boldsymbol{L}}_O &= \overrightarrow{\boldsymbol{L}}_{CM} + \overrightarrow{\boldsymbol{L}}_{about\ CM} \\
&= M\vec{\boldsymbol{r}}_{\text{CM}} \times \overrightarrow{\boldsymbol{V}}_{\text{CM}} + I_0\vec{\boldsymbol{\omega}} \\
&= MRv_{CM} + I_0\omega \ \text{ into page}
\end{aligned}
$$

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\end{aligned}
$$

$\vec{L}$ & KE in Rolling (Pure Translation!)

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\end{aligned}
$$

- Kinetic Energy of Rolling:

$\vec{L}$ & KE in Rolling (Pure Translation!)

- Angular Momentum

$$
\begin{aligned}
\vec{\boldsymbol{L}}_O &= \vec{\boldsymbol{L}}_{CM} + \vec{\boldsymbol{L}}_{about\ \ CM} \\
&= M\vec{\boldsymbol{r}}_{\text{CM}} \times \vec{\boldsymbol{V}}_{\text{CM}} + I_0\vec{\boldsymbol{\omega}} \\
&= MRv_{CM} + I_0\omega \ \text{ into page} \\
&= MR^2\omega + I_0\omega \ \text{ : rolling w/o slipping}
\end{aligned}
$$

- Kinetic Energy of Rolling:

$$KE = KE_{\text{trans}} + KE_{\text{rot}}$$

$\vec{L}$ & KE in Rolling (Pure Translation!)

$$KE = \sum_i \frac{1}{2} m_i \vec{v}_i^2$$

$\overrightarrow{L}$ & KE in Rolling (Pure Translation!)

$$KE = \sum_i \frac{1}{2} m_i \vec{\boldsymbol{v}}_i^2 = \sum_i \frac{1}{2} m_i \left(\vec{\boldsymbol{v}}_{CM} \right.$$

$\overrightarrow{L}$ & KE in Rolling (Pure Translation!)

$$KE = \sum_i \frac{1}{2} m_i \vec{\boldsymbol{v}}_i^2 = \sum_i \frac{1}{2} m_i \left(\vec{\boldsymbol{v}}_{CM} + (\vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{r}}_i') \right)^2$$

$\overrightarrow{L}$ & KE in Rolling (Pure Translation!)

$$
\begin{aligned}
KE &= \sum_i \frac{1}{2} m_i \vec{\boldsymbol{v}}_i^2 = \sum_i \frac{1}{2} m_i \left(\vec{\boldsymbol{v}}_{CM} + (\vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{r}}_i')\right)^2 \\
&= \frac{1}{2} \sum_i m_i v_{CM}^2
\end{aligned}
$$

$\vec{L}$ & KE in Rolling (Pure Translation!)

$$
\begin{aligned}
KE &= \sum_i \frac{1}{2} m_i \vec{\boldsymbol{v}}_i^2 = \sum_i \frac{1}{2} m_i \left(\vec{\boldsymbol{v}}_{CM} + (\vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{r}}'_i) \right)^2 \\
&= \frac{1}{2} \sum_i m_i v_{CM}^2 + \frac{1}{2} \sum_i m_i \left(\omega^2 {r'}_i^2 - (\vec{\boldsymbol{\omega}} \cdot \vec{\boldsymbol{r}}'_i)^2 \right)
\end{aligned}
$$

$\vec{L}$ & KE in Rolling (Pure Translation!)

$$
\begin{aligned}
KE &= \sum_i \frac{1}{2} m_i \vec{\boldsymbol{v}}_i^2 = \sum_i \frac{1}{2} m_i \left(\vec{\boldsymbol{v}}_{CM} + (\vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{r}}'_i)\right)^2 \\
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\end{aligned}
$$

(cross term=0)

$\overrightarrow{L}$ & KE in Rolling (Pure Translation!)

$$
\begin{aligned}
KE &= \sum_i \frac{1}{2} m_i \vec{\boldsymbol{v}}_i^2 = \sum_i \frac{1}{2} m_i \left(\vec{\boldsymbol{v}}_{CM} + (\vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{r}}'_i) \right)^2 \\
&= \frac{1}{2} \sum_i m_i v_{CM}^2 + \frac{1}{2} \sum_i m_i \left(\omega^2 r'^2_i - (\vec{\boldsymbol{\omega}} \cdot \vec{\boldsymbol{r}}'_i)^2 \right)
\end{aligned}
$$

(cross term=0)

$$
= \frac{1}{2} M v_{CM}^2
$$

$\vec{L}$ & KE in Rolling (Pure Translation!)

$$
\begin{aligned}
KE &= \sum_i \frac{1}{2} m_i \vec{\boldsymbol{v}}_i^2 = \sum_i \frac{1}{2} m_i \left(\vec{\boldsymbol{v}}_{CM} + (\vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{r}}'_i) \right)^2 \\
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\end{aligned}
$$

(cross term=0)

$$
= \frac{1}{2} M v_{CM}^2 + \frac{1}{2} \sum_i m_i \left(\omega^2 \rho_i^2 \right)
$$

$$
\begin{aligned}
KE &= \sum_i \frac{1}{2} m_i \vec{\boldsymbol{v}}_i^2 = \sum_i \frac{1}{2} m_i \left(\vec{\boldsymbol{v}}_{CM} + (\vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{r}}'_i)\right)^2 \\
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$$

(cross term=0)

$$
\begin{aligned}
&= \frac{1}{2} M v_{CM}^2 + \frac{1}{2} \sum_i m_i \left(\omega^2 \rho_i^2\right) \\
&= \frac{1}{2} M v_{CM}^2 + \frac{1}{2} \left(\int dm \rho^2\right) \omega^2 i
\end{aligned}
$$

$\vec{L}$ & KE in Rolling (Pure Translation!)

$$
\begin{aligned}
KE &= \sum_i \frac{1}{2} m_i \vec{\boldsymbol{v}}_i^2 = \sum_i \frac{1}{2} m_i \left(\vec{\boldsymbol{v}}_{CM} + (\vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{r}}'_i) \right)^2 \\
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\end{aligned}
$$

(cross term=0)

$$
\begin{aligned}
&= \frac{1}{2} M v_{CM}^2 + \frac{1}{2} \sum_i m_i \left(\omega^2 \rho_i^2 \right) \\
&= \frac{1}{2} M v_{CM}^2 + \frac{1}{2} \left(\int dm \rho^2 \right) \omega^2 i \\
&= \frac{1}{2} M v_{CM}^2 + \frac{1}{2} I_0 \omega^2
\end{aligned}
$$

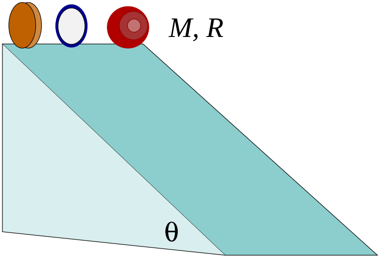
Race down an inclined plane

Which object reaches first?

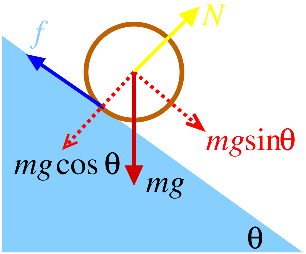
A) The sphere, since it has the least I_0

B) The hoop since it has the largest I_0

C) The disc since its I_0 is intermediate

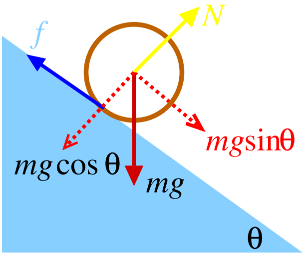


Down the incline plane....



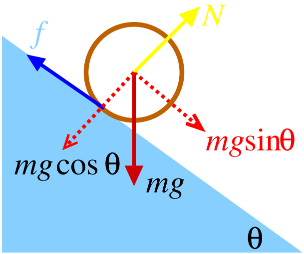
$$N - mg\cos\theta \quad = \quad 0$$

Down the incline plane....



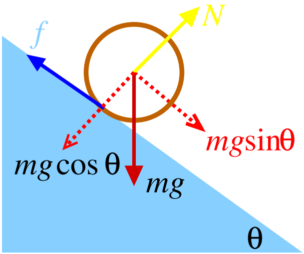
$$
\begin{aligned}
N - mg\cos\theta &= 0 \\
mg\sin\theta - f &= ma
\end{aligned}
$$

Down the incline plane....



$$
\begin{aligned}
N - mg\cos\theta &= 0 \\
mg\sin\theta - f &= ma \\
fR = I\alpha &= Ia/R \quad \text{(No Slipping)}
\end{aligned}
$$

Down the incline plane....



$$
\begin{aligned}
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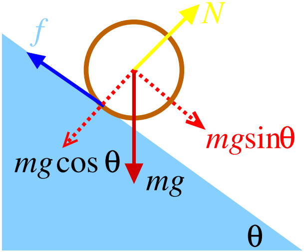
\end{aligned}
$$

$$\Longrightarrow \; f = I_0 a/R^2$$

Down the incline plane....



$$
\begin{aligned}
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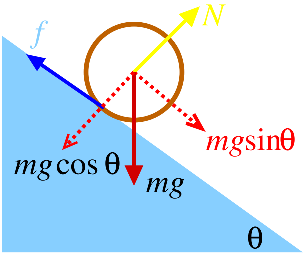
\end{aligned}
$$

$$\Longrightarrow f = I_0 a/R^2$$

$$\Longrightarrow a = \frac{g\sin\theta}{1 + I_0/mR^2}$$

Down the incline plane....



$$
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$$

$$\Longrightarrow \; f = I_0 a/R^2$$

$$\Longrightarrow \; a = \frac{g\sin\theta}{1 + I_0/mR^2}$$

Hoop	Disc	Sphere
$I_0 = mR^2$	$I_0 = mR^2/2$	$I_0 = 2mR^2/5$
$a = \frac{1}{2} g\sin\theta$	$a = \frac{2}{3} g\sin\theta$	$a = \frac{5}{7} g\sin\theta$

On friction in rolling w/o slipping

- Friction fixed by rolling w/o slipping condition

$$f_{\text{fr}} = I_0 a/R^2 \quad (\text{ for rolling down incline })$$

On friction in rolling w/o slipping

- Friction fixed by rolling w/o slipping condition

$$f_{\text{fr}} = I_0 a / R^2 \quad (\text{ for rolling down incline })$$

- this value has to be within the static friction limit

$$f_{\text{fr}} \leq \mu_s N$$

otherwise slipping

On friction in rolling w/o slipping

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- this value has to be within the static friction limit

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otherwise slipping

- $f_{\text{fr}} = 0$ when no acceleration

On friction in rolling w/o slipping

- Friction fixed by rolling w/o slipping condition

$$f_{\text{fr}} = I_0 a / R^2 \quad (\text{ for rolling down incline })$$

- this value has to be within the static friction limit

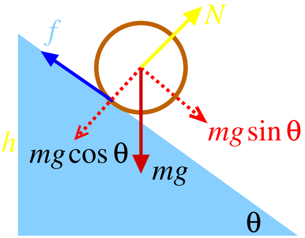
$$f_{\text{fr}} \leq \mu_s N$$

otherwise slipping

- $f_{\text{fr}} = 0$ when no acceleration
- f_{fr} does no work (point of action stationary)

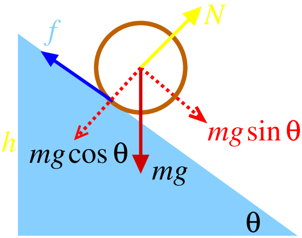
Down the incline plane: Energy method

Energy is conserved! (Friction does no work when no slipping)



Down the incline plane: Energy method

Energy is conserved! (Friction does no work when no slipping)

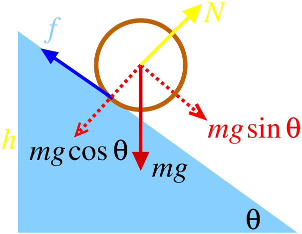


Work energy theorem:

$$mgh \quad = \quad \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2$$

Down the incline plane: Energy method

Energy is conserved! (Friction does no work when no slipping)

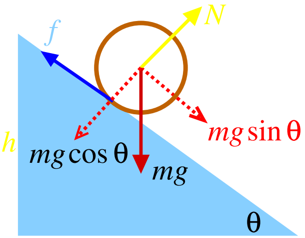


Work energy theorem:

$$
\begin{aligned}
mgh &= \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2 \\
gh &= \frac{1}{2}\left(1 + \frac{I}{mR^2}\right)v^2
\end{aligned}
$$

Down the incline plane: Energy method

Energy is conserved! (Friction does no work when no slipping)

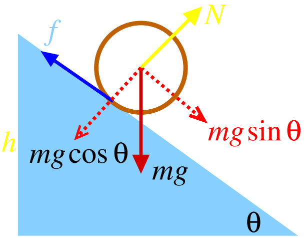


Work energy theorem:

$$\begin{aligned} mgh &= \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2 \\ gh &= \frac{1}{2}\left(1 + \frac{I}{mR^2}\right)v^2 \\ \Longrightarrow \qquad & v^2 = \frac{2gh}{1 + I/(mR^2)} \end{aligned}$$

Down the incline plane: Energy method

Energy is conserved! (Friction does no work when no slipping)



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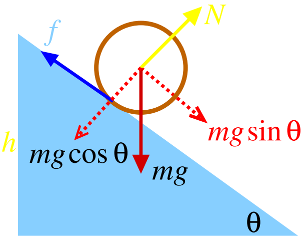
\Longrightarrow &\qquad v^2 = \frac{2gh}{1 + I/(mR^2)} \\
\Longrightarrow &\qquad a = \frac{v^2}{s} = \frac{g\sin\theta}{1 + I/(mR^2)}
\end{aligned}
$$

Down the incline plane: Energy method

Energy is conserved! (Friction does no work when no slipping)



Work energy theorem:

$$mgh = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2$$

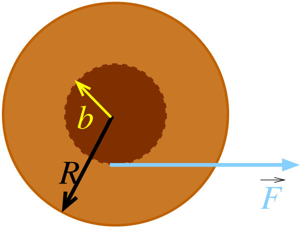
$$gh = \frac{1}{2}\left(1 + \frac{I}{mR^2}\right)v^2$$

$$\implies \quad v^2 = \frac{2gh}{1 + I/(mR^2)}$$

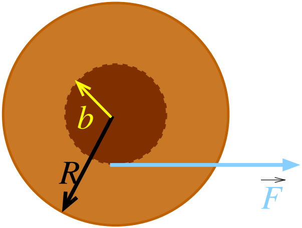
$$\implies \quad a = \frac{v^2}{s} = \frac{g\sin\theta}{1 + I/(mR^2)}$$

Hoop	Disc	Sphere
$v^2 = gh$	$v^2 = \frac{2gh}{3}$	$v^2 = \frac{10gh}{7}$
$a = \frac{1}{2}g\sin\theta$	$a = \frac{2}{3}g\sin\theta$	$a = \frac{5}{7}g\sin\theta$

Yo-Yo



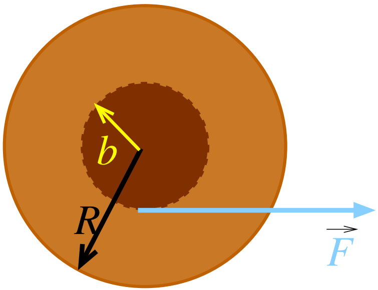
Yo-Yo



Which way will the Yo-Yo roll?

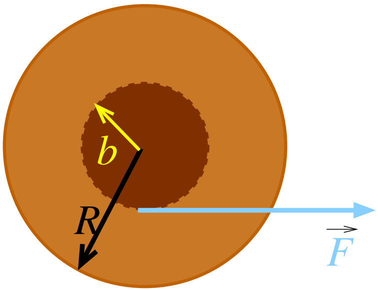
- **A)** Forward (String Winds)
- **B)** Backward (String Unwinds)

Yo-Yo



Yo-Yo

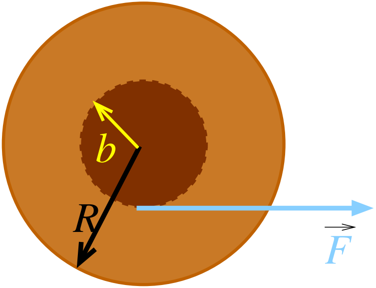
Which way is the friction?



Yo-Yo

Which way is the friction?

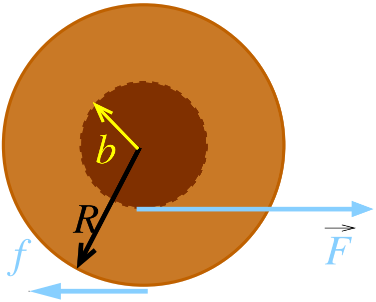
Choose any direction



Yo-Yo

Which way is the friction?

Choose any direction

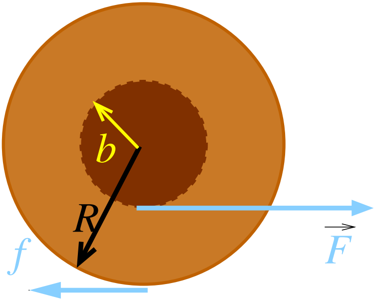


Yo-Yo

Which way is the friction?

Choose any direction

Which way is the acceleration?



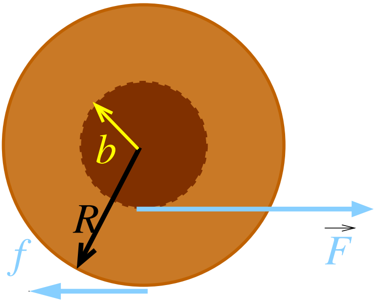
Yo-Yo

Which way is the friction?

Choose any direction

Which way is the acceleration?

Choose any direction



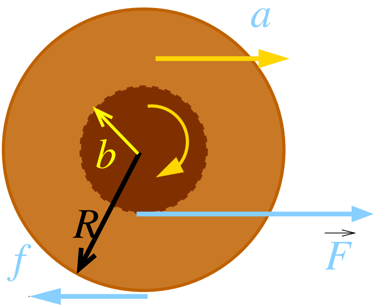
Yo-Yo

Which way is the friction?

Choose any direction

Which way is the acceleration?

Choose any direction



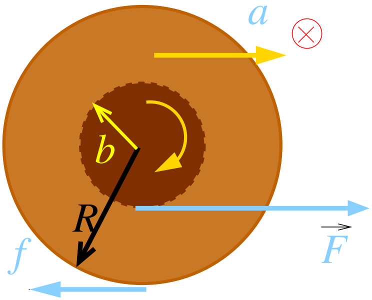
Yo-Yo

Which way is the friction?

Choose any direction

Which way is the acceleration?

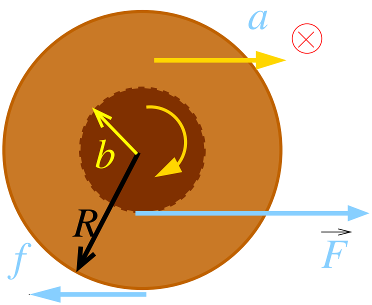
Choose any direction



Yo-Yo

Which way is the friction?
Choose any direction
Which way is the acceleration?
Choose any direction

$$F - f = ma$$



Yo-Yo

Which way is the friction?

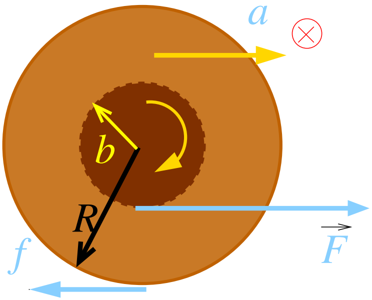
Choose any direction

Which way is the acceleration?

Choose any direction

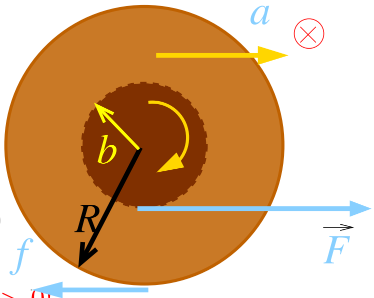
$$F - f = ma$$

$$fR - Fb = I\alpha = \left(\frac{MR^2}{2}\right)\left(\frac{a}{R}\right)$$



Which way is the friction?
Choose any direction
Which way is the acceleration?
Choose any direction

$$
\begin{aligned}
F - f &= ma \\
fR - Fb = I\alpha &= \left(\frac{MR^2}{2}\right)\left(\frac{a}{R}\right) \\
\implies a &= \frac{2F}{3}\left(1 - \frac{b}{R}\right) > 0!
\end{aligned}
$$



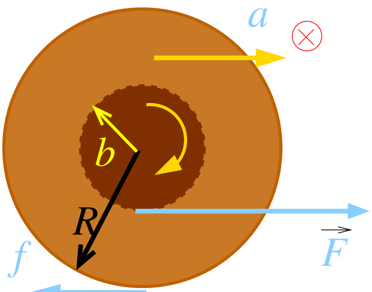
Yo-Yo

Which way is the friction?

Choose any direction

Which way is the acceleration?

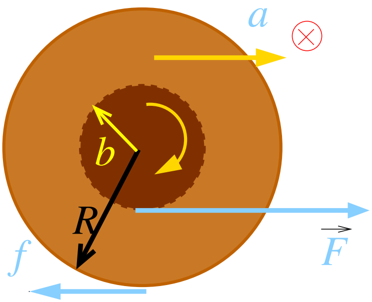
Choose any direction



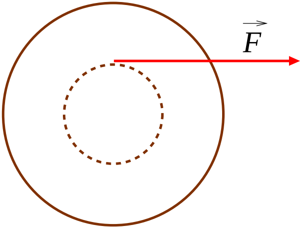
$$
\begin{aligned}
F - f &= ma \\
fR - Fb = I\alpha &= \left(\frac{MR^2}{2}\right)\left(\frac{a}{R}\right) \\
\Longrightarrow a &= \frac{2F}{3}\left(1 - \frac{b}{R}\right) > 0! \\
f &= \frac{F}{3}\left(1 + \frac{2b}{R}\right) > 0!
\end{aligned}
$$

$$\begin{aligned} f &= \frac{F}{3}\left(1+\frac{2b}{R}\right) \\ &\leq \mu m g \\ \Longrightarrow F &\leq \frac{3\mu m g}{(1+2b/R)} \end{aligned}$$

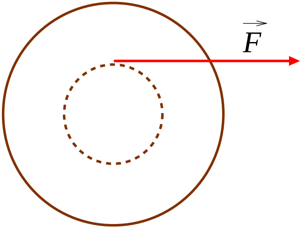
otherwise slipping



Yo-Yo



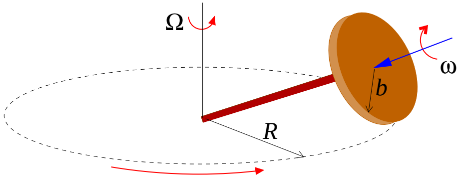
Yo-Yo



Which way will the Yo-Yo roll?

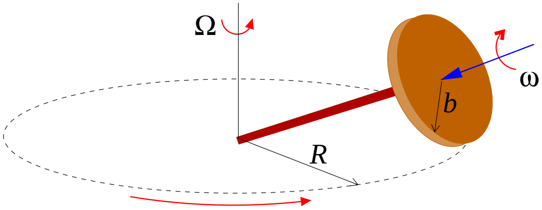
A)Forward (String Winds)
B)Backward (String Unwinds)

Rolling Wheel rotating about axle

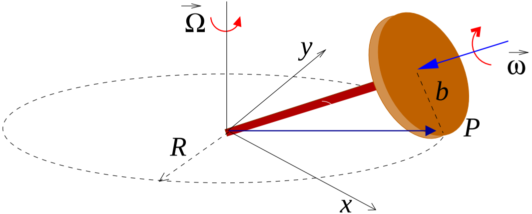


Q: What is the condition for rolling w/o slipping?

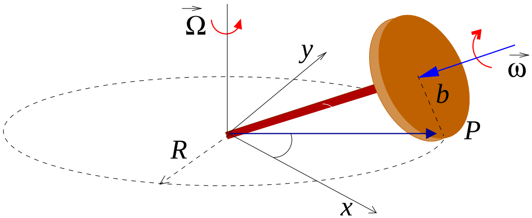
Rolling Wheel rotating about axle



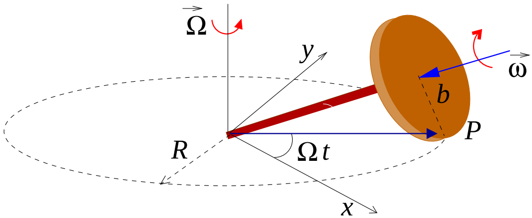
Rolling Wheel rotating about axle



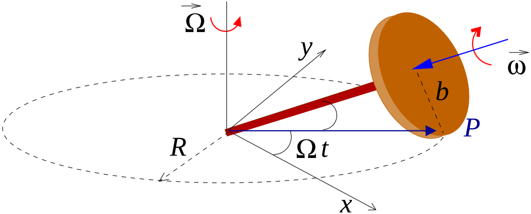
Rolling Wheel rotating about axle



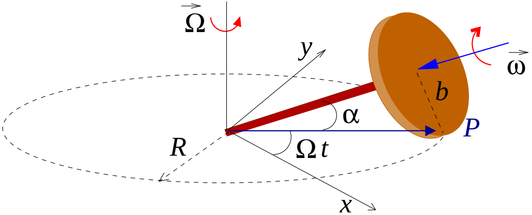
Rolling Wheel rotating about axle



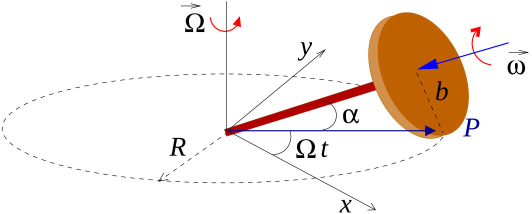
Rolling Wheel rotating about axle



Rolling Wheel rotating about axle



Rolling Wheel rotating about axle



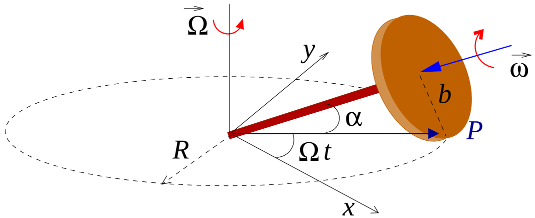
$$\vec{\boldsymbol{r}}_P = R\cos(\Omega t)\hat{i} + R\sin(\Omega t)\hat{j}$$

$$\vec{\boldsymbol{\omega}}_{tot}(t) = (\Omega - \omega\sin\alpha)\,\hat{k} - \omega\cos\alpha\left(\cos(\Omega t)\hat{i} + \sin(\Omega t)\hat{j}\right)$$

$$\vec{\boldsymbol{v}}_P(t) = \vec{\boldsymbol{\omega}}_{tot} \times \vec{\boldsymbol{r}}_P \implies$$

$$\vec{\boldsymbol{v}}_P(t) = (\Omega - \omega\sin\alpha)\left(R\cos(\Omega t)\hat{j} - R\sin(\Omega t)\hat{i}\right)$$

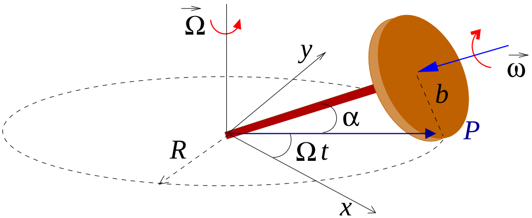
Rolling Wheel rotating about axle



$$
\begin{aligned}
\vec{r}_P &= R\cos(\Omega t)\hat{i} + R\sin(\Omega t)\hat{j} \\
\vec{\omega}_{tot}(t) &= (\Omega - \omega\sin\alpha)\,\hat{k} - \omega\cos\alpha\left(\cos(\Omega t)\hat{i} + \sin(\Omega t)\hat{j}\right) \\
\vec{v}_P(t) &= \vec{\omega}_{tot} \times \vec{r}_P \implies \\
\vec{v}_P(t) &= (\Omega - \omega\sin\alpha)\left(R\cos(\Omega t)\hat{j} - R\sin(\Omega t)\hat{i}\right)
\end{aligned}
$$

$\vec{v}_P(t) = 0$ (no slipping) $\implies \Omega = \omega\sin\alpha = \omega\dfrac{b}{R}$

Rolling Wheel rotating about axle



$$
\begin{aligned}
\vec{\boldsymbol{r}}_P &= R\cos(\Omega t)\hat{i} + R\sin(\Omega t)\hat{j} \\
\vec{\boldsymbol{\omega}}_{tot}(t) &= (\Omega - \omega\sin\alpha)\,\hat{k} - \omega\cos\alpha\left(\cos(\Omega t)\hat{i} + \sin(\Omega t)\hat{j}\right) \\
\vec{\boldsymbol{v}}_P(t) &= \vec{\boldsymbol{\omega}}_{tot} \times \vec{\boldsymbol{r}}_P \implies \\
\vec{\boldsymbol{v}}_P(t) &= (\Omega - \omega\sin\alpha)\left(R\cos(\Omega t)\hat{j} - R\sin(\Omega t)\hat{i}\right)
\end{aligned}
$$

$\vec{\boldsymbol{v}}_P(t) = 0$ (no slipping) $\implies \Omega = \omega\sin\alpha = \omega\dfrac{b}{R}$

Qn: What if the Axle is not perpendicular to Wheel?