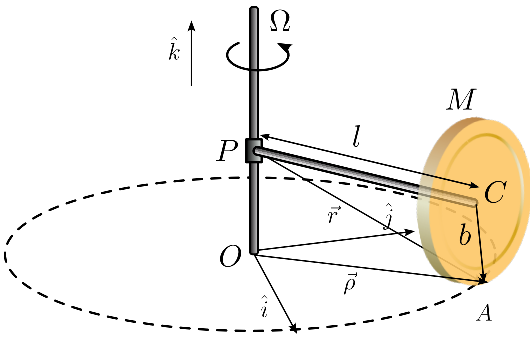
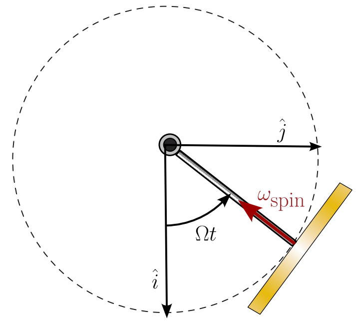
Tutorial 6 **Rotation** 23 August 2024

P1. Set up coordinates system with origin at P, planar x-y coordinates in the plane of rolling.





(a) $\vec{\boldsymbol{\omega}}_{\text{total}} = \overrightarrow{\boldsymbol{\Omega}}_{\text{roll}} + \vec{\boldsymbol{\omega}}_{\text{spin}}$.

Let $\vec{\boldsymbol{\omega}}_s = -\omega_s \hat{\boldsymbol{\rho}}$ be the spin angular velocity of the wheel, where $\hat{\boldsymbol{\rho}} = \cos\Omega t\hat{\boldsymbol{\imath}} + \sin\Omega t\hat{\boldsymbol{\jmath}}$.

The angular velocity of the wheel is

$$\vec{\boldsymbol{\omega}}_{\text{total}} = -\omega_s\left(\cos\Omega t\hat{\boldsymbol{\imath}} + \sin\Omega t\hat{\boldsymbol{\jmath}}\right) + \Omega\hat{\boldsymbol{k}}.$$

(b) For rolling without slipping, the point of contact A is at rest, i.e $\vec{\boldsymbol{v}}_A = 0$.

If the origin is at P,

$$\begin{aligned}
\vec{\boldsymbol{v}}_A &= \vec{\boldsymbol{\omega}}_{\text{total}} \times \vec{\boldsymbol{r}}_A, \quad \vec{\boldsymbol{r}}_A = l\cos\Omega t\hat{\boldsymbol{\imath}} + l\sin\Omega t\hat{\boldsymbol{\jmath}} + b\hat{\boldsymbol{k}}.\\
\vec{\boldsymbol{v}}_A &= -\omega_s\ l\ \cos\Omega t\ \sin\Omega t\ \hat{\boldsymbol{k}} - \omega_s\ b\ \cos\Omega t\ (-\hat{\boldsymbol{\jmath}})\\
&\quad -\omega_s\ l\ \sin\Omega t\ \cos\Omega t\ (-\hat{\boldsymbol{k}}) - \omega_s\ b\ \sin\Omega t\ \hat{\boldsymbol{\imath}}\\
&\quad +\Omega\ l\ (\cos\Omega t\ \hat{\boldsymbol{\jmath}} - \sin\Omega t\ \hat{\boldsymbol{\imath}})\\
&= (\Omega\ l - \omega_s\ b)(\sin\Omega t\ \hat{\boldsymbol{\imath}} + \cos\Omega t\ \hat{\boldsymbol{\jmath}}).
\end{aligned}$$

For this to be zero, we need $\boxed{\omega_s = \frac{l}{b}\Omega.}$ $\implies$ $\boxed{\vec{\boldsymbol{\omega}}_{\text{total}} = \frac{l}{b}\Omega\left(\cos\Omega t\hat{\boldsymbol{\imath}} + \sin\Omega t\hat{\boldsymbol{\jmath}}\right) + \Omega\hat{\boldsymbol{k}}.}$

(c) Motion of this sort can always be resolved into (translation of center of mass) + (rotation of body about center of mass).

$$\begin{aligned}
\overrightarrow{\boldsymbol{L}}_P &= \overrightarrow{\boldsymbol{L}}_{\text{of CM}} + \overrightarrow{\boldsymbol{L}}_{\text{about CM}}\\
&= Ml^2\Omega\hat{\boldsymbol{k}} + \overrightarrow{\boldsymbol{L}}_C.
\end{aligned}$$

(d) The instantaneous angular momentum of the wheel with respect to its center of mass C is a bit tricky. Imagine a point in space, coinciding with the instantaneous CM, attached to an *inertial frame*. It is clear that the wheel is spinning about this point with angular momentum

$$\overrightarrow{\boldsymbol{L}}_{\text{spin}} = \frac{1}{2}Mb^2\vec{\boldsymbol{\omega}}_{\text{spin}}.$$

In addition, you will realize that the orientation of the wheel is changing as it goes around the circle: this is equivalent to the wheel rotating about a vertical diameter with angular velocity Ω!

$$\text{So, } \overrightarrow{\boldsymbol{L}}_C = \frac{1}{2}Mbl\Omega(\cos\Omega t\hat{\boldsymbol{\imath}} + \sin\Omega t\hat{\boldsymbol{\jmath}}) + \frac{1}{4}Mb^2\Omega\hat{\boldsymbol{k}}$$

$$\boxed{\vec{\boldsymbol{L}}_P = \frac{1}{2}Mbl\Omega(\cos\Omega t\hat{\boldsymbol{i}} + \sin\Omega t\hat{\boldsymbol{j}}) + M\left(l^2 + \frac{b^2}{4}\right)\Omega\hat{\boldsymbol{k}}.}$$

(e) The physical forces are contact forces: at the point of contact A and at the pivot P, and the weight acting through C. The contact forces are expected to have components along $\hat{\boldsymbol{k}}$ as well as in the perpendicular plane. At the ground, $\vec{\boldsymbol{F}} = f\hat{\boldsymbol{\theta}} + N\hat{\boldsymbol{k}}$.

$$\vec{\boldsymbol{\tau}}_P = (lN - Mgl)\hat{\boldsymbol{\theta}} + lf\hat{\boldsymbol{k}}.$$

(f) τ_P is causing the angular momentum to change.

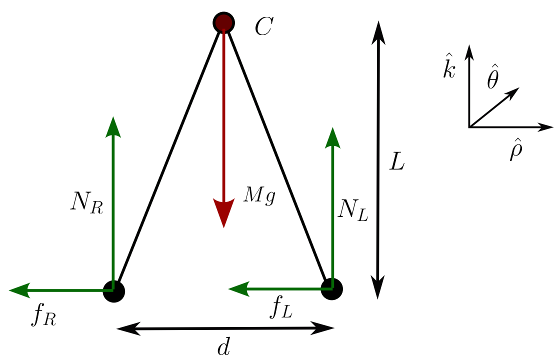
$$\frac{d\vec{\boldsymbol{L}}}{dt} = \frac{1}{2}Mbl\Omega^2\hat{\theta}.$$

Thus we have

$$lN - Mgl = \frac{1}{2}Mbl\Omega^2$$

$$\implies \boxed{N = Mg + \frac{1}{2}Mb\Omega^2.}$$

P2.



Irrespective of the coordinate system or origin, the forces on the man are (1) weight Mg downward at C (2) contact forces on each foot: normal reaction upward on each foot N_R, N_L and friction towards the center of curvature of the track on each foot f_R, f_L.
We will use cylindrical coordinates.

$$a_\rho = -\left(R + \frac{d}{2}\right)\dot{\theta}^2, a_\theta = 0, a_k = 0. \quad \text{Also, } \dot{\theta} = \frac{v}{R + \frac{d}{2}}.$$

$$\text{No vertical acceleration} \implies N_R + N_L = Mg.$$

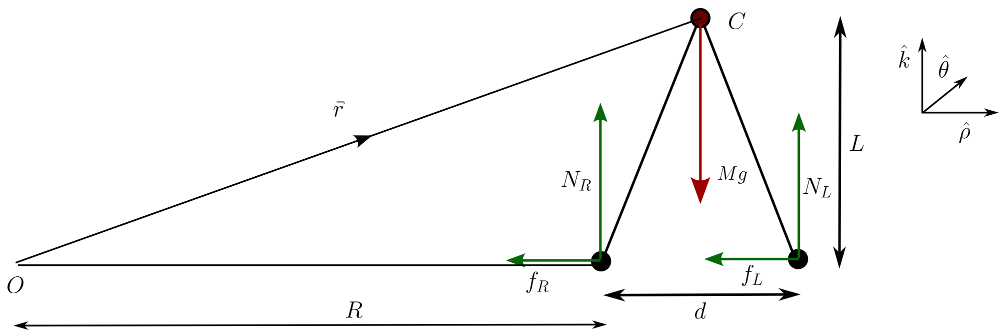
$$\text{Radial acceleration} \implies f_R + f_L = M\left(\frac{v^2}{R + \frac{d}{2}}\right).$$

(a) With respect to the center of mass, the instantaneous angular momentum is along $\hat{\boldsymbol{k}}$ (the man rotates about the vertical axis) which is constant. So the net torque must be zero. Torque on the CM is

$$\begin{aligned} \vec{\boldsymbol{\tau}}_{CM} &= (N_R - N_L)\frac{d}{2}\hat{\boldsymbol{\theta}} + (f_R + f_L)L\hat{\boldsymbol{\theta}} = 0 \\ \implies N_R &= \frac{Mg}{2} - \frac{Mv^2L}{d\left(R + \frac{d}{2}\right)}. \end{aligned}$$

(b) With respect to O, Torque is not zero:

$$\vec{\boldsymbol{\tau}}_O = MgR'(\hat{\boldsymbol{\theta}}) - N_R R\hat{\boldsymbol{\theta}} - N_L(R + d)\hat{\boldsymbol{\theta}} \quad = -Mg\frac{d}{2} + N_R d.$$



Angular momentum of CM

$$\vec{\boldsymbol{J}}_O^{CM} = M\left((R+\frac{d}{2})\hat{\boldsymbol{\rho}} + L\hat{\boldsymbol{k}}\right)\times v(-\hat{\boldsymbol{\theta}}) = -Mv\left(R+\frac{d}{2}\right)\hat{\boldsymbol{k}} + MvL\hat{\boldsymbol{\rho}}.$$

Angular momentum about CM

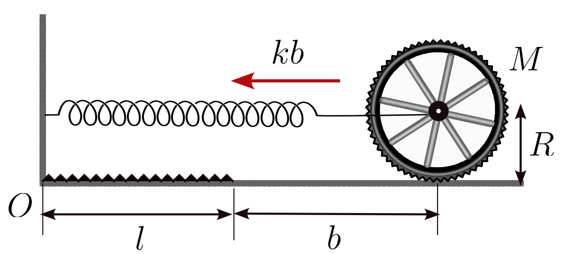
$$\vec{\boldsymbol{J}}_{CM} = I\frac{v}{R+d/2}(-\hat{\boldsymbol{k}}).$$

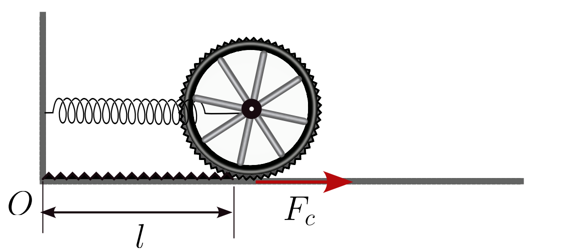
Torque causes $$\frac{d\vec{\boldsymbol{J}}}{dt} = MvL\frac{d\hat{\boldsymbol{\rho}}}{dt} = -MvL.\frac{v}{R+d/2}\hat{\boldsymbol{\theta}}$$

$$-\frac{Mv^2L}{R+\frac{d}{2}} = -Mg\frac{d}{2} + N_Rd \implies N_R = M\left(\frac{g}{2} - \frac{Lv^2}{R'd}\right).$$

The man topples when $N_R = 0$.

P3.





The wheel first slides under the restoring force of the spring, and when it hits the toothed portion of the track, its velocity can be calculated from energy conservation:

$$\frac{1}{2}kb^2 = \frac{1}{2}mv^2 \implies v = \sqrt{\frac{k}{m}}b.$$

Once it hits the wheel torque due to the teeth cause it to roll.
At collision, force at the teeth implies no conservation of linear momentum. But torque about the origin due to this force $\vec{\boldsymbol{F}}_c$ being zero, $\vec{\boldsymbol{L}}_O$ is conserved.

Initial angular momentum $L_i = MvR$.

Immediately after collision, $L_f = MR^2\omega + Mv_{cm}R = 2MR^2\omega$.

$L_i = L_f \implies v_{cm} = \frac{v}{2}$.

(a) While rolling over the gears, F_c does no work. So energy is conserved. The closest distance to the wall is x at which point $v = 0$.

Initial KE $= \frac{1}{2}Mv_{cm}^2 + \frac{1}{2}MR^2\omega^2 = MR^2\omega^2 = \frac{1}{4}Mv^2$.

Final KE $= \dfrac{1}{2}kx^2$.

$\implies x^2 = \dfrac{Mv^2}{2k} = \dfrac{b^2}{2} \implies x = \dfrac{b}{\sqrt{2}}$.

(b) On the first outward trip, part from x to l is rolling on the teeth, while part from l to say x_1 is on the smooth part while rolling and slipping.

KE conservation from x to $l \implies \dfrac{1}{2}kx^2 = \dfrac{1}{2}Mv_{cm}^2 + \dfrac{1}{2}MR^2\omega^2 = Mv_{cm}^2$. Thus at the end of the toothed track,

$v_{cm}^2 = \dfrac{kx^2}{2M} = \dfrac{kb^2}{4M}$.

At this point, the rotation of the wheel is $\omega = \dfrac{v_{cm}}{R} = \dfrac{b}{2R}\sqrt{\dfrac{k}{M}}$, which stays throughout the smooth journey out.

So while rotational KE remains the same, translational KE goes to zero by the time the wheel reached x_1.

$\dfrac{1}{2}Mv_{cm}^2 = \dfrac{1}{2}kx_1^2 \implies x_1 = \dfrac{b}{2}$.

(c) On the next trip, when the wheel hits the geared track, $v = \sqrt{\dfrac{k}{m}}\dfrac{b}{2}$.

Angular momentum wrt O is intially $L_i = MvR$ (due to motion of CM) $-MR^2\omega$ (due to rolling in clockwise direction) $= 0$! Thus the final angular momentum of the wheel must also be zero $\implies v_{\text{final}} = 0$: wheel does not roll onto the gears at all. So motion stops.