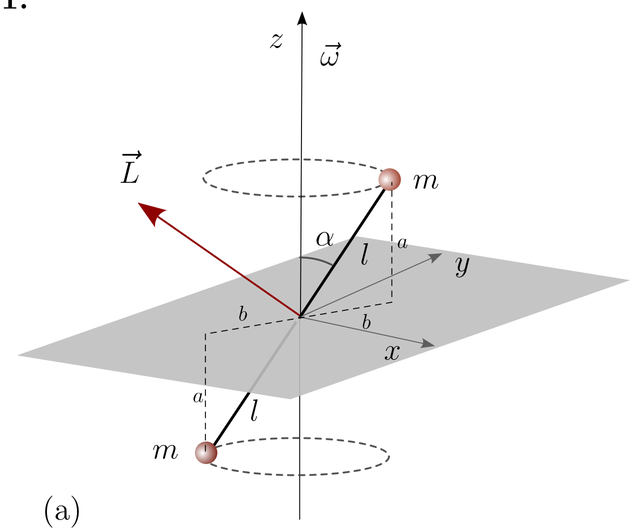
Tutorial 5 **Moment of Inertia** 22 August 2024

P1.



(a)

Let's label the upper mass A and the lower mass B. It will reduce clutter to call the projections $l\cos\alpha = a, l\sin\alpha = b$.

$$\begin{aligned}
\vec{\boldsymbol{L}}_O &= \sum m_i \vec{\boldsymbol{r}}_i \times \vec{\boldsymbol{v}}_i; \\
\vec{\boldsymbol{r}}_A &= m_A\left(a\hat{\boldsymbol{k}} + b\hat{\boldsymbol{\rho}}_A\right); \vec{\boldsymbol{r}}_B = m_B\left(-a\hat{\boldsymbol{k}} + b\hat{\boldsymbol{\rho}}_B\right) \\
\vec{\boldsymbol{v}}_A &= b\omega\hat{\boldsymbol{\theta}}_A; \quad \vec{\boldsymbol{v}}_B = b\omega\hat{\boldsymbol{\theta}}_B \\
|\vec{\boldsymbol{L}}_O^A| &= |\vec{\boldsymbol{L}}_O^B| = m\omega l^2\sin\alpha \equiv L; \\
\vec{\boldsymbol{L}}_O^A &= L\left(-\cos\alpha\hat{\boldsymbol{\rho}}_A + \sin\alpha\hat{\boldsymbol{k}}\right) \\
&\text{where } \hat{\boldsymbol{\rho}}_A = \cos\omega t\hat{\boldsymbol{\imath}} + \sin\omega t\hat{\boldsymbol{\jmath}}. \\
\vec{\boldsymbol{L}}_O^B &= L_B\left(\cos\alpha\hat{\boldsymbol{\rho}}_B + \sin\alpha\hat{\boldsymbol{k}}\right) \\
&\text{where } \hat{\boldsymbol{\rho}}_B = -\cos\omega t\hat{\boldsymbol{\imath}} - \sin\omega t\hat{\boldsymbol{\jmath}}.
\end{aligned}$$

$$\vec{\boldsymbol{L}}_O = -2L\cos\alpha\left(\cos\omega t\hat{\boldsymbol{\imath}} + \sin\omega t\hat{\boldsymbol{\jmath}}\right) + 2L\sin\alpha\hat{\boldsymbol{k}}.$$

(b) $|\vec{\boldsymbol{L}}_O| = 2L = 2m\omega l^2\sin\alpha, \qquad \dfrac{d\vec{\boldsymbol{L}}}{dt} = 2L\omega\cos\alpha(-\sin\omega t\hat{\boldsymbol{\imath}} + \cos\omega t\hat{\boldsymbol{\jmath}}).$

(c) $\vec{\boldsymbol{\tau}} = \dfrac{d\vec{\boldsymbol{L}}}{dt}$, origin is the contact forces at point of pivot.

(d) Moment of inertia tensor of this object: let's use some simplifying notation.

$$\begin{aligned}
I_{xx} &= 2m(a^2 + b^2\sin^2\omega t); \quad & I_{yy} &= 2m(a^2 + b^2\cos^2\omega t); \quad & I_{zz} &= 2mb^2, \\
I_{xy} &= -2mb^2\sin\omega t\cos\omega t, \quad & I_{yz} &= -2mab\cos\omega t \quad & I_{zx} &= -2mab\sin\omega t.
\end{aligned}$$

You can check that $\vec{\boldsymbol{L}} = I\vec{\boldsymbol{\omega}}$.

P2. $x = 2\cos\alpha, y = 2\sin\alpha, z = 3.$

$$I(\alpha) = \begin{pmatrix} 9 & -4\alpha & -6 \\ -4\alpha & 13 & -6\alpha \\ -6 & -6\alpha & 4 \end{pmatrix}$$

P3.

$$I = \begin{pmatrix} 12 & 6 & -6 \\ 2 & 16 & 2 \\ -6 & 2 & 16 \end{pmatrix} kgm^2, \vec{\boldsymbol{L}} = -18\hat{\boldsymbol{\jmath}} + 42\hat{\boldsymbol{k}}\ kgm^2s^{-1}$$