

CH3: Flight Mechanics**International standard Atmosphere (ISA)**

ISA assumes the relation of temperature (T) in K, and the height (h) in km as follows:

$$T = \begin{cases} T_0 - hL_T, & 0 < h < 11 \\ T_0 - 11L_T, & 11 < h < 20 \end{cases}$$

Where:

- $L_T = 6.5 \text{ K/km}$
- $T_0 = 288 \text{ K}$ (temperature at sea level)

With this, one can find the pressure (p) and density (ρ) variation along the height $h \leq 11\text{km}$ by:

$$p = p_0 \left(\frac{T}{T_0} \right)^{\frac{g}{RL_T}}$$
$$\rho = \rho_0 \left(\frac{T}{T_0} \right)^{\frac{g}{RL_T} - 1}$$

Where:

- $R = 0.287 \text{ kJ/(kg K)}$
- $p_0 = 101 \text{ kPa}$
- $\rho_0 = 1.225 \text{ kg/m}^3$

Most flights operate at a height $h \leq 11\text{km}$, if $h > 11\text{km}$:

The pressure becomes:

$$p = p_{11} e^{\frac{g}{RT_{11}}(h-h_{11})}$$

The air density becomes:

$$\rho = \frac{p}{RT_{11}}$$

Where:

- p_{11} = pressure at 11 km
- T_{11} = temperature at 11 km
- $h_{11} = 11 \text{ km}$ or $3280 \times 11 \text{ ft}$ ($1 \text{ km} = 3280 \text{ ft}$)**

Dimensional analysis of the Lift force (L)

The lift force (L) depends on 7 parameters:

$$L = f(s, c, \alpha, \rho, \mu, V, V_{\text{sound}})$$

Where:

- s = span
- c = chord
- α = incidence angle
- ρ = air density
- μ = air viscosity
- V = velocity
- V_{sound} = speed of sound

Then we know that the lift coefficient (C_L):

$$C_L = \frac{L}{\frac{1}{2} \rho V_{\text{flight}}^2 A_w}$$

Where

- $A_w = cs$

Since lift force is a force, and force have the SI unit of $[\text{kgm/s}^2]$, and the shape (s, c) is fixed in analysis problems. Also, the effect of Reynolds number (ρ, μ, V, c) is large only when the Reynolds number is small, the influence is very small for the speed range during flight. With these, we can conclude that the lift coefficient C_L only depends on two parameters:

$$\begin{aligned} C_L &= f\{(s, c) \rightarrow \text{fixed shape}, \alpha, (\rho, \mu) \rightarrow \text{part of Re}, (V, V_{\text{sound}}) \rightarrow \mathbf{M}\} \\ &= f(\alpha, M) \end{aligned}$$

Mach number:

$$M = \frac{V_{\text{flight}}}{V_{\text{sound}}}$$

The speed of sound:

$$V_{\text{sound}} = \sqrt{kRT}$$

Wing aspect ratio:

$$\text{Aspect ratio} = \frac{s}{c}$$

Viscosity:

$$\mu = \frac{1}{3} \rho \lambda u_{\text{rms}}$$

Where:

$\lambda = 0.1 \times 10^{-6} \text{ m}$ (mean free path)

$u_{\text{rms}} = 450 \text{ m/s}$ (molecular speed of air on ground)

Reynolds number:

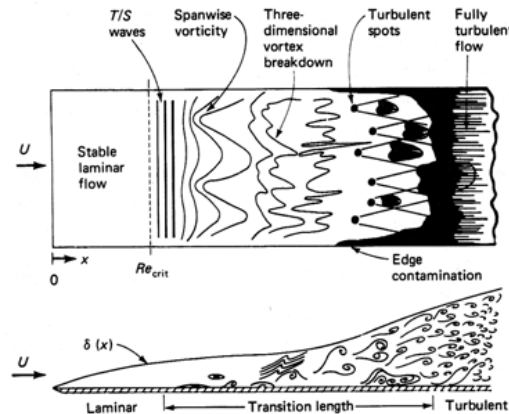
$$\text{Re} = \frac{\rho V c}{\mu}$$

Here, the hydraulic diameter $D_h = c$, as refer to fluid mechanics.

Sources of drag

There are 3 types of sources for drag:

- Skin friction: arise from the shear stress in the boundary layer of the flow, exist on all parts of exposed aircraft surfaces.



- Induced / Vortex drag: induced by tip vortex on a 3D body like a wing.

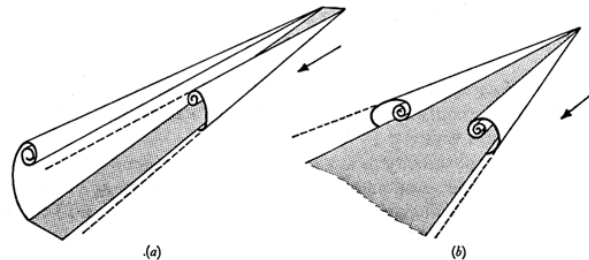


Figure 7.8.7. Sketches showing the rolling-up of the sheet vortex shed from the sides of a flat elongated lifting surface.

- Form drag: the drag derived from normal pressure acting on a lifting surface with vortex drag deduced.
 - Normal airfoil has form drag equals skin friction when max. thickness of the airfoil = $\frac{1}{3}$ chord
 - Flow separation and shock wave cause high form drag (because result in high pressure and low flow speed).
 - Flow turbulence can energize flow and prevent or delay flow separation.

For the whole airplane, there is also drag on other components:

Wing	Basic aircraft
Body	Undercarriage
Empennage	Flaps
Engine instn	Lift-dependent drag
Interference leaks etc.	
Lift-dependent drag	
Cruise $M_\infty = 0.8$	Takeoff

The drag polar

It is not difficult to design a wing that has a great lift. The question is how much drag it comes with. Therefore, the crucial parameter is the lift to drag ratio.

Since it is impractical to separate drag into components due to pressure and shear stress for flight analysis, we would rather separate into two parts:

- Drag that dependent on lift
- Drag that independent on lift

We know that the drag coefficient (C_D):

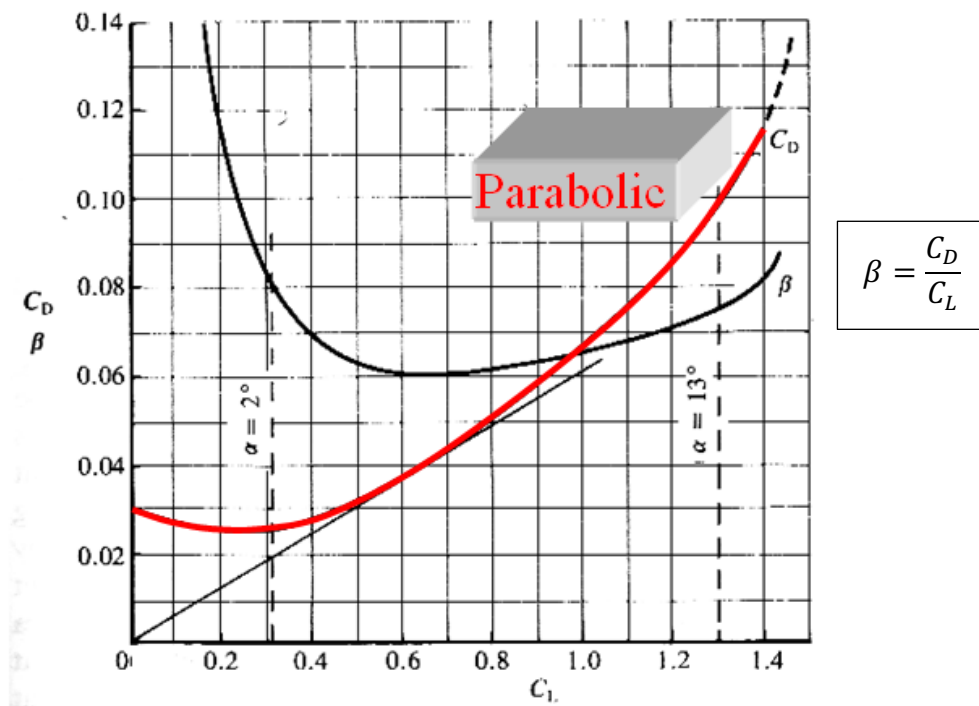
$$C_D = \frac{D}{\frac{1}{2} \rho V_{\text{flight}}^2 A_w}$$

Together with C_L , is related to the incidence angle α . Therefore, we can conclude that C_D is also a function of C_L and Mach number (M), and is found best described using a parabolic curve:

$$C_D = C_{D0} + K(C_L - C_{L, \min(D)})^2$$

where:

C_{D0} is the minimum drag coefficient which collects all drag components that are not sensitive to the incidence angle and the lift change.

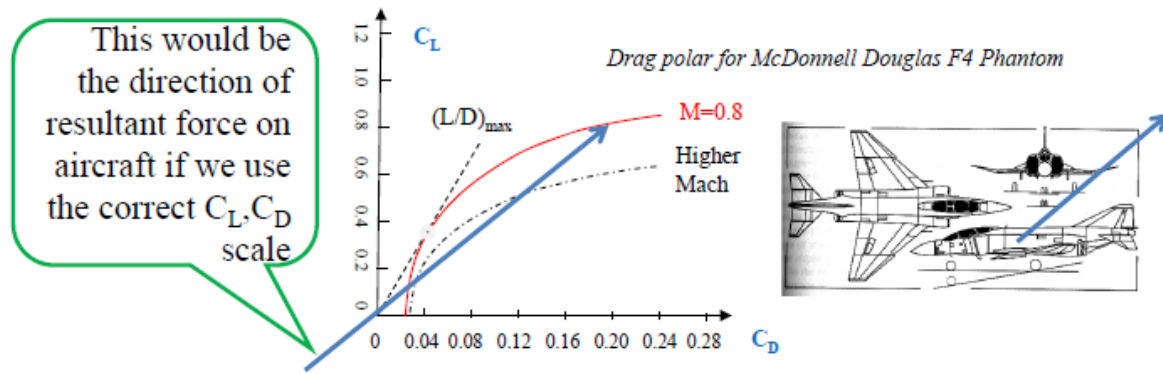


Which gives:

$$C_{D0} = 0.025$$

$$C_{L, \min(D)} = 0.27$$

If we flip the above figure about the origin:



It becomes the polar coordinate plot in which the line linking the origin and any point on the curve is exactly the force direction. It is thus called a drag polar.

The tangent of this line will give the design point where the slope $\frac{L}{D}$ is maximum.

If we assume $C_{L, \min(D)} = 0$:

$$C_D = C_{D0} + KC_L^2$$

This gives the simplified form of drag polar equation which used in early design stage of aircrafts. In this form:

- C_{D0} is no longer the true zero-lift drag but a mere fitting constant.
- When engine thrust changes, the flow pressure around nacelle changes, causing changes in total aircraft drag. Hence C_{D0} is now not purely a wing aerodynamics characterization.

Recall that C_D and C_L are both functions of Mach number (M) and incidence angle (α), as proposed by Prandtl Glauert the lift coefficient C_L :

$$C_L = \frac{C_{L0}}{\sqrt{1 - M^2}}$$

It is shown that C_L significantly percentage change with M :

$$\frac{1}{C_L} \frac{\partial C_L}{\partial M} = \frac{M}{1 - M^2}$$

When $M = 0.85$, Rate = 3.06

When $M = 0.62$, Rate = 1.00

This implies that C_D and C_{D0} also significantly change with M , because $C_D = C_{D0} + KC_L^2$. Therefore, the lift variation is not only due to the dynamic pressure $\frac{1}{2}\rho V^2$.

The above suggest that there is a peak exist for the lift drag ratio $\frac{C_D}{C_L}$:

At max $\frac{L}{D}$ ratio:

$$\frac{d}{dC_L} \left(\frac{C_D}{C_L} \right) = \frac{d}{dC_L} \left(\frac{C_{D0}}{C_L} + KC_L \right) = -\frac{C_{D0}}{C_L^2} + KC_L^* = 0$$

$$C_{D0} = KC_L^{*2}$$

Where C_L^* is the lift coefficient at the attack angle with max $\frac{L}{D}$ ratio.

This shows that the max $\frac{C_D}{C_L}$ occurs when $C_{D0} = KC_L^{*2}$.

With C_L^* , the drag polar equation can be expressed as:

$$C_D = C_{D0} \left[1 + \left(\frac{C_L}{C_L^*} \right)^2 \right]$$

In real case, we often need to optimize (maximize) C_D and C_L in the form of $\frac{C_D^m}{C_L^n}$:

$$\begin{aligned} \frac{d}{dC_L} \left(\frac{C_D^m}{C_L^n} \right) &= 0 \\ m \frac{C_D^{m-1}}{C_L^n} \frac{\partial C_D}{\partial C_L} - n \frac{C_D^m}{C_L^{n+1}} &= 0 \\ \frac{\partial C_D}{\partial C_L} &= \frac{n}{m} \frac{C_D}{C_L} \\ 2C_{D0} \frac{C_L}{C_L^{*2}} &= \frac{n}{m} \frac{C_D}{C_L} \\ 2 \left(\frac{C_L}{C_L^*} \right)^2 &= \frac{n}{m} \frac{C_D}{C_{D0}} \\ 2 \left(\frac{C_L}{C_L^*} \right)^2 &= \frac{n}{m} \left[1 + \left(\frac{C_L}{C_L^*} \right)^2 \right] \\ \frac{C_L}{C_L^*} &= \sqrt{\frac{n}{2m-n}} \end{aligned}$$

$$C_D = C_{D0} \left[1 + \left(\frac{C_L}{C_L^*} \right)^2 \right]$$

$$\frac{\partial C_D}{\partial C_L} = 2C_{D0} \frac{C_L}{C_L^{*2}}$$

This means that the maximum $\frac{C_D^m}{C_L^n}$ occurs when $\frac{C_L}{C_L^*} = \sqrt{\frac{n}{2m-n}}$

Then:

$$\begin{aligned} \frac{C_D}{C_{D0}} &= 1 + \left(\frac{C_L}{C_L^*} \right)^2 \\ \frac{C_D}{C_{D0}} &= \frac{2m}{2m-n} \end{aligned}$$

Example:

Fuel saving:

$$m = 1, n = \frac{1}{2}$$

$$\frac{C_L}{C_L^*} = \frac{1}{\sqrt{3}}, \quad \frac{C_D}{C_{D0}} = \frac{4}{3}$$

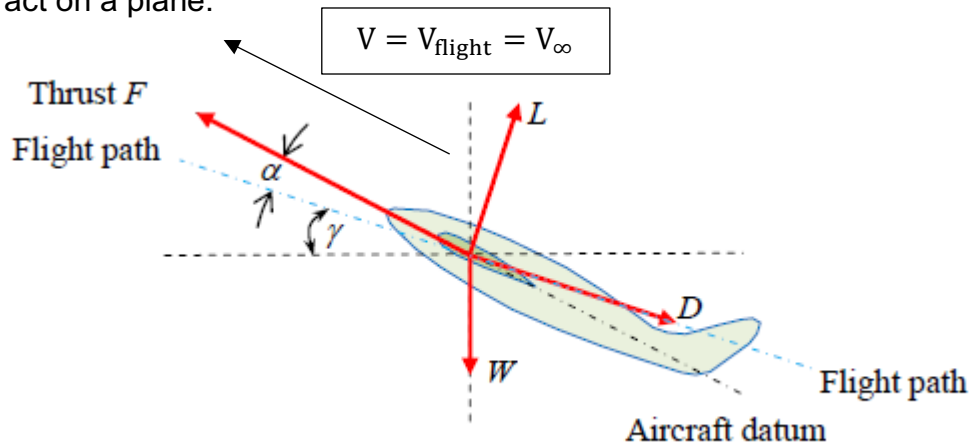
Obtain good lift during gliding:

$$m = 1, n = \frac{3}{2}$$

$$\frac{C_L}{C_L^*} = \sqrt{3}, \quad \frac{C_D}{C_{D0}} = 4$$

General flight dynamics

There are 4 forces act on a plane:



For analysis purpose, we assume very small the incidence angle (α) so that it is insignificant, then for steady flight ($\frac{dV}{dt} = 0$):

$$\lim_{\alpha \rightarrow 0} \cos(\alpha) = 1 \rightarrow \begin{aligned} L &= W \cos(\gamma) \\ F \cos(\alpha) &= D + W \sin(\gamma) \\ \frac{F}{W} &= \frac{D}{W} + \sin(\gamma) \\ &= \frac{D}{L} \cos(\gamma) + \sin(\gamma) \end{aligned}$$

Gliding

For gliding / unpowered flight, we have the thrust $\mathbf{F} = \mathbf{0}$, then:

$$\begin{aligned} \frac{D}{L} \cos(\gamma_{\text{glide}}) + \sin(\gamma_{\text{glide}}) &= 0 \\ \gamma_{\text{glide}} &= -\tan^{-1}\left(\frac{D}{L}\right) \end{aligned}$$

which suggest that gliding has to follow a certain angle (γ_{glide}).

For gliding, the plane goes down and the minimum height sinking rate ($-\frac{dh}{dt}$) is:

$$\begin{aligned} \begin{aligned} W &\approx L \\ \frac{C_D}{C_L} &= \frac{D}{L} = \frac{D}{W} \end{aligned} &\rightarrow \begin{aligned} -\frac{dh}{dt} &= -V \sin(\gamma) \\ -\frac{dh}{dt} &= \frac{D}{W} V \end{aligned} &\leftarrow \begin{aligned} F &= 0 \\ \frac{D}{W} + \sin(\gamma) &= 0 \end{aligned} \\ &\rightarrow \begin{aligned} -\frac{dh}{dt} &= \frac{C_D}{C_L} \sqrt{\frac{2W}{\rho A_w C_L}} \\ -\frac{dh}{dt} &= \frac{C_D}{C_L^{\frac{3}{2}}} \sqrt{\frac{2W}{\rho A_w}} \end{aligned} &\leftarrow \begin{aligned} W &\approx L \\ W &= \frac{1}{2} \rho V^2 A_w C_L \\ V &= \sqrt{\frac{2W}{\rho A_w C_L}} \end{aligned} \end{aligned}$$

So, by optimizing $\frac{C_D^m}{C_L^n}$ with $m = 1$, $n = \frac{3}{2}$, one can obtain the minimum sinking rate:

$$\frac{C_L}{C_L^*} = \sqrt{3}, \quad \frac{C_D}{C_{D0}} = 4$$

Climbing:

Climbing occurs when $F > D$. Therefore, we have the climbing rate ($\frac{dh}{dt}$):

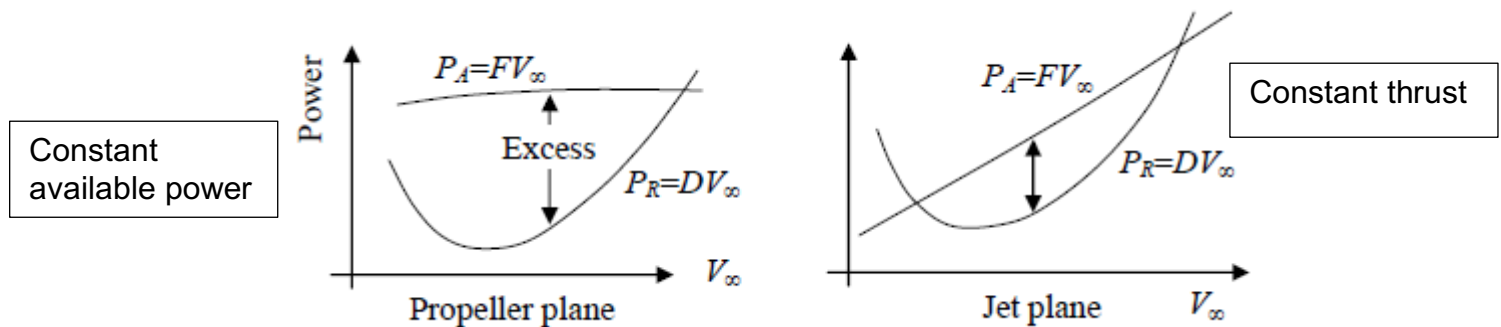
$$\begin{aligned}\frac{dh}{dt} &= V \sin(\gamma) \\ W \frac{dh}{dt} &= (F - D)V \\ W \frac{dh}{dt} &= P_A - P_R\end{aligned}$$

$$\begin{aligned}\frac{F}{W} &= \frac{D}{W} + \sin(\gamma) \\ \sin(\gamma) &= \frac{1}{W} (F - D)\end{aligned}$$

Where:

- $P_A = FV$ is the power available
- $P_R = DV$ is the power required

The above means that the potential energy ($W \frac{dh}{dt} = mg \Delta h$) gained is supplied by the engine power.



So the maximum climbing rate occurs where $(P_A - P_R)$ is maximum.

As airplane climbs, W drops with fuel consumption (\dot{m}), F drops due to air density, eventually $\frac{dh}{dt} = 0$.

In practice, the ceiling is defined at a height h such that the climbing rate $\frac{dh}{dt} = 1.5 \text{ m/s}$ is maintained.

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The equation:

$$W \frac{dh}{dt} = (F - D)V$$

also suggest that there is a flight velocity that can gives a peak climbing rate, for jet plane:

$$\begin{aligned}\frac{d}{dV} [(F - D)V] &= 0 \\ F - D - V \frac{dD}{dV} &= 0 \\ \frac{dD}{dV} &= \frac{1}{V} (F - D)\end{aligned}$$

This in turns means that the peak climbing rates occurs when the plane fly a at velocity which the drag becomes minimum. The above will be different for propulsion because F is not constant, but the available power is constant.

Jet engine characteristics

Engine performance depends on flight speed (V_{flight}) and height (h) and the rotational speed (N). For now, we assume N is independent.

When **the plane flies at a very low Mach number** ($M = \frac{V_{\text{flight}}}{V_{\text{sound}}}$) [low V_{flight}], the contribution of V_{flight}^2 is small to total energy $h_2 = h_1 + \frac{V_{\text{flight}}^2}{2}$.

For such low Mach number is considered, we also assume the stagnation properties, the entry properties and the combustion properties of this flying engine behave the same as that on ground test. Since such low V_{flight} will not affect the core engine, the engine will still produce a jet velocity V_{jet} .

Here, we model the effect of slows down of air (stagnation) in the diffuser as a choked nozzle with a throat area A_{thr} , the mass flow rate of air through the core engine (or anywhere we have a nozzle) to be:

$$\begin{aligned}\dot{m}_{\text{air, core}} &= \rho V A_{\text{thr}} \\ &= \left(\frac{p_1}{RT_1} \right) (\sqrt{kRT_1}) (A_{\text{thr}}) \\ &= 1.281 \frac{p_1 A_{\text{thr}}}{\sqrt{c_p T_1}}\end{aligned}$$

Where p_1 and T_1 are the stagnation pressure and temperature which are determined by the entry properties plus what the combustion has added. (because it can apply to the core engine and the nozzle at the end of the bypass)

And the thrust developed is:

$$F = \dot{m}_{\text{air, core}} (V_{\text{jet}} - V_{\text{flight}})$$

With the above setting, as we increase the flight velocity (V_{flight}), the thrust will decrease due to the increase of ram drag $= \dot{m}_{\text{air, core}} V_{\text{flight}}$.

When **the plane flies at a high Mach number**, the contribution of V_{flight}^2 is significant to total energy

$$h_2 = h_1 + \frac{V_{\text{flight}}^2}{2}.$$

As $h_2 = c_p T_2$, one can conclude that the temperature level of the whole core engine is raised. This lead to a higher efficiency, so a higher V_{jet} is expected. At the same time, the total pressure possessed by the turbine exhaust air is higher and it will expand to greater nozzle velocity. Therefore, the mass flow passing through the core engine:

$$\dot{m}_{\text{air, core}} = 1.281 \frac{p_1 A_{\text{thr}}}{\sqrt{c_p T_1}}$$

is also increased due to the increased term $\frac{p_1}{\sqrt{T_1}}$. (power 1 over power $\frac{1}{2}$)

Therefore, as:

$$F = \dot{m}_{\text{air, core}} (V_{\text{jet}} - V_{\text{flight}})$$

We can conclude that the thrust F also increased.

For an engine from Rolls Royce (RR), an empirical formula for thrust (F) was established for sea level flight test:

$$\frac{F}{F_{\text{static}}} = 1 - \frac{V_{\text{flight}}}{397[\text{m/s}]} + \left(\frac{V_{\text{flight}}}{480[\text{m/s}]} \right)^2$$

where:

- F_{static} = the sea-level thrust with zero flight speed (ground test)
- $\frac{V_{\text{flight}}}{397[\text{m/s}]}$ = ram drag
- $\left(\frac{V_{\text{flight}}}{480[\text{m/s}]} \right)^2$ = ram energy or pressure

The equation is illustrated below with Mach number:

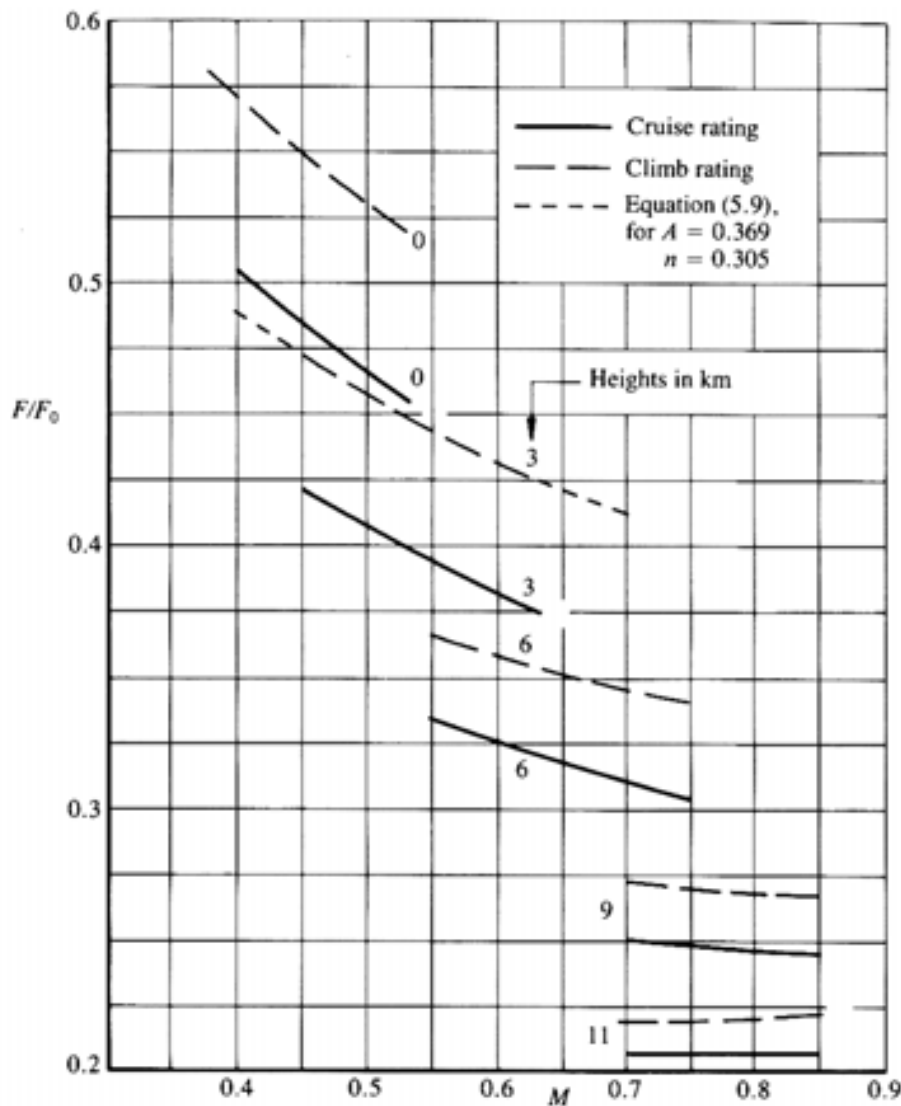


Figure 5.8. Variation of maximum thrust with height and Mach number. Rolls-Royce RB211-535E4 turbofan.

When we move to higher altitude, the speed of sound (V_{sound}) drops, so the plane flies at a higher Mach number with the same flight speed (V_{flight}) at higher altitude.

From the above figure, we can see that the thrust become constant at a height = 11 km.

When the bypass ratio further increases, or for the turbo-prop engines, the trend of thrust reduction with flight speed is sharper because propeller engine tends to have constant available power:

$$P_A = FV_{\text{flight}}$$
$$F \propto \frac{1}{V_{\text{flight}}}$$

On the other hand, pure turbo-jet engine has less reduction of thrust with flight speed.

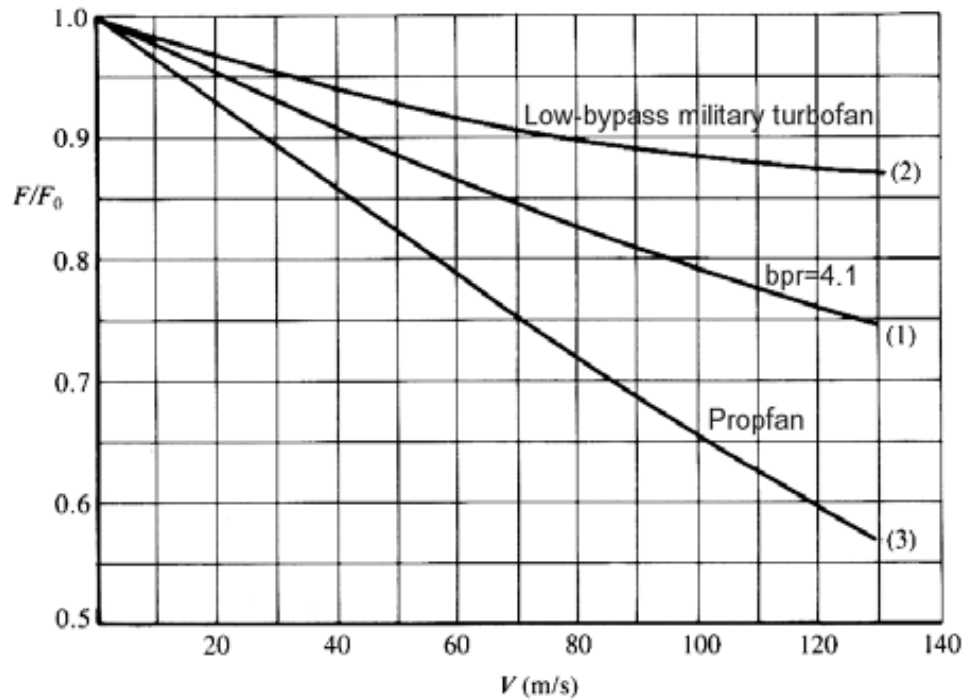
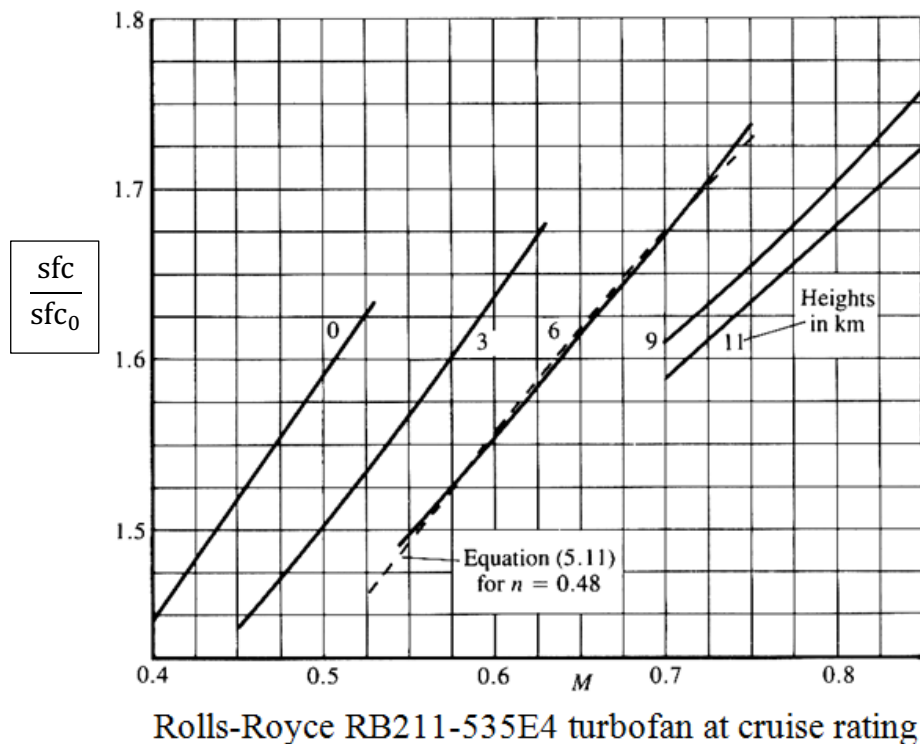


Figure 5.7. Variation of maximum take-off thrust with forward speed at sea level.

The thrust specific fuel consumption for the RR engine, with $bpr = 4.3$, is illustrated below with Mach number (M):



Here, sfc_0 is the value of sfc obtained during static test at the sea-level.

It is found that the increase of sfc with respect to M is significant at 11 km, so the percentage change of $\frac{sfc}{M}$:
By 2 point form:

$$\begin{aligned}\frac{sfc}{sfc_0} &= 0.9833 + 0.8667M \\ \frac{1}{sfc_0} \frac{\partial(sfc)}{\partial M} &= 0.8667 \\ \frac{M}{sfc_0} \frac{\partial(sfc)}{\partial M} &= 0.8667M \\ \frac{0.9833 + 0.8667M}{sfc} \times M \frac{\partial(sfc)}{\partial M} &= 0.8667M \\ \frac{\frac{\partial(sfc)}{\partial M}}{\frac{sfc}{M}} &= \frac{0.8667M}{0.9833 + 0.8667M}\end{aligned}$$

When $M = 0.85$:

$$\frac{M}{sfc} \frac{\partial(sfc)}{\partial M} = 0.4283$$

This increase is due to the decrease of thrust, and the actual increase in fuel flow with the air mass flow when ram pressure (KE) takes effect at high Mach number. (Higher Mach number, higher velocity, air mass flow increase through whole engine, burnt more fuel)

This means **as flight velocity (V_{flight}) increase**, the **specific fuel consumption (sfc) will increase significantly**.

Determination of steady level flight speed

From last section, we know that the (available) thrust is a function of flight velocity in the form of:

$$F = F_{\text{static}} - aV_{\text{flight}} + bV_{\text{flight}}^2$$

As well as the drag and lift:

$$D = \frac{1}{2} \rho V_{\text{flight}}^2 A_w C_D, \quad L = \frac{1}{2} \rho V_{\text{flight}}^2 A_w C_L$$

To find the flight speed at steady flow, we must choose an incidence angle (α) such that the lift supports the weight:

$$L = W = \frac{1}{2} \rho V_{\text{flight}}^2 A_w C_L$$

$$C_L = \frac{2W}{\rho V_{\text{flight}}^2 A_w}$$

As C_L is also a function of incidence angle (α), we can use the above equation to determine α . Then for the drag:

$$D = \frac{D}{L} L$$

$$D = \frac{D}{L} W$$

$$D = W \beta(\alpha | V_{\text{flight}})$$

$$\frac{C_D}{C_L} = \frac{D}{L} = \beta(\alpha | V_{\text{flight}})$$

$\alpha | V_{\text{flight}}$ denotes β can be a function of incidence angle and flight velocity in an equivalent manner

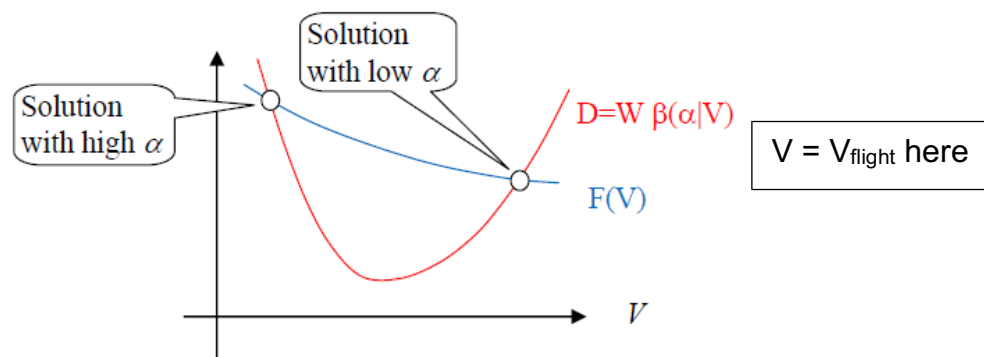
Note that the drag (D) here is not to be confused with the drag of an airfoil at constant incidence while flow velocity increases. This is a drag with various incidence angle, required by $L = W$.

With the above we solve:

$$F = D$$

$$F_{\text{static}} - aV_{\text{flight}} + bV_{\text{flight}}^2 = W\beta(\alpha | V_{\text{flight}})$$

This gives 2 solutions for steady flight:



Determination of steady-level flight velocity with the condition of $L=W$ embedded

The one with low α and high flight velocity (V_{flight}) is better because:

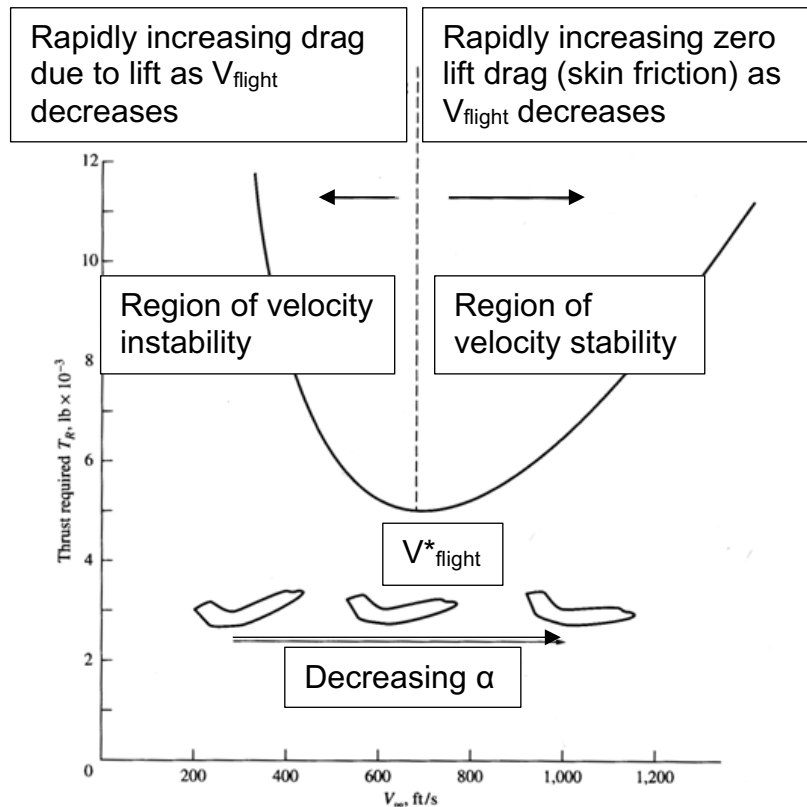
- Aircraft flies faster
- The flow over the wing is far away from stall region, so more stable

The one with high α have high risk to stall.

The equation:

$$F = F_0 - aV_{\text{flight}} + bV_{\text{flight}}^2$$

also indicates that when the velocity is very low or very high, the thrust (F) is high. The low-end runs the risk of compressor stall, and the high-end runs the risk of developing high drag due to transonic flow if the engine and aircraft wing are not properly designed.



The minimal drag occurs when the flight speed (V_{flight}) coincides with α^* for the highest lift-drag ratio.

Explanation of the region of velocity instability: when a gust (陣風) cause decrease in V_{flight} , L and C_L increase to support W , then D increase. Since $F = D$, so F also increase a bit. However, $F < D$ leads to further decrease in V_{flight}

In reality, throttle (油門) may be increased to keep steady V_{flight} , or allow drop in height to get more lift.

In any case, higher thrust requires more fuel input and the temperature limitation for turbine blade must impose a sustainable peak thrust (F_{peak}). If a peak thrust is set, the peak velocity is determined easily by:

$$F_{\text{peak}} = D$$

$$F_{\text{peak}} = W\beta(\alpha|V_{\text{flight}})$$

Cruise performance – Breguet range

Fuel-saving is important for economy, for pollution control, and for maximizing payload. Although fuel cost only about 20% of the airline operation, the weight of the fuel account for 45% of the whole plane, which also needs a large space for storage.

Therefore, if less fuel can be used for the same travel distance, the number of passenger or cargo load can be increased. This gain in payload would yield considerable profit.

A relation between the specific fuel consumption (sfc) and the travel distance is then developed.

For propeller (constant available power):

We use Power (P) specific fuel consumption (sfc_P):

$$P \times \text{sfc}_P = \dot{m}_{\text{fuel}} g$$

$$F V_{\text{flight}} \times \text{sfc}_P = \dot{m}_{\text{fuel}} g$$

The reduction of aircraft weight is:

$$\frac{ds}{dt} = V_{\text{flight}}$$

$$\frac{1}{dt} = \frac{V_{\text{flight}}}{ds}$$

\rightarrow

$$F = D, \quad L = W$$

$$F = D = \frac{D}{L} L = \frac{D}{L} W$$

$$\frac{dW}{dt} = -\dot{m}_{\text{fuel}} g$$

$$\frac{dW}{dt} = -\text{sfc}_P \times FV$$

$$V_{\text{flight}} \frac{dW}{ds} = -\text{sfc}_P \times \frac{D}{L} W \times V_{\text{flight}}$$

$$-\frac{L}{D \times \text{sfc}_P} \int_{W_0}^{W_{\text{end}}} \frac{1}{W} dW = \int_0^{S_{\text{prop}}} ds$$

$$S_{\text{prop}} = \frac{L}{D \times \text{sfc}_P} \int_{W_{\text{end}}}^{W_0} \frac{1}{W} dW$$

This gives the travel range (S_{prop}) as a function of Power specific fuel consumption (sfc_P) to be:

$$S_{\text{prop}} = \left(\frac{L}{D \times \text{sfc}_P} \right)_{\text{avg}} \times \ln \left(\frac{W_0}{W_0 - W_{\text{fuel}}} \right)$$

With:

$$W_{\text{end}} = W_0 - W_{\text{fuel}}$$

Where:

- W_0 = initial weight
- W_{end} = end weight

For jet-powered airplane (constant thrust):

We use Thrust (F) specific fuel consumption (sfc):

$$F \times \text{sfc} = \dot{m}_{\text{fuel}} g$$

The reduction of aircraft weight is:

$$\frac{ds}{dt} = V_{\text{flight}}$$

$$\frac{1}{dt} = \frac{V_{\text{flight}}}{ds}$$

\rightarrow

$$F = D, \quad L = W$$

$$F = D = \frac{D}{L} L = \frac{D}{L} W$$

$$\frac{dW}{dt} = -\dot{m}_{\text{fuel}} g$$

$$\frac{dW}{dt} = -\text{sfc} \times F$$

$$V_{\text{flight}} \frac{dW}{ds} = -\frac{D}{L} W \times \text{sfc}$$

$$-\frac{LV_{\text{flight}}}{D \times \text{sfc}} \int_{W_0}^{W_{\text{end}}} \frac{1}{W} dW = \int_0^{S_{\text{jet}}} ds$$

$$S_{\text{jet}} = \frac{LV_{\text{flight}}}{D \times \text{sfc}} \int_{W_{\text{end}}}^{W_0} \frac{1}{W} dW$$

This gives the travel range (S_{jet}) as a function of Thrust specific fuel consumption (sfc) to be:

$$S_{\text{jet}} = \left(\frac{LV_{\text{flight}}}{D \times \text{sfc}} \right)_{\text{avg}} \times \ln \left(\frac{W_0}{W_0 - W_{\text{fuel}}} \right)$$

With:

$$W_{\text{end}} = W_0 - W_{\text{fuel}}$$

Where:

- W_0 = initial weight
- W_{end} = end weight

General cruise studies

There are in total 4 variables:

- Cruise height, h , which fixes the inlet air density and pressure and affects a lot of performance parameters of engine and wing.
- Flight speed (V_{flight}) or Mach number (M) which affects engine pressure at stage 2 (p_2), thrust, lift, drag, etc.
- Incidence angle (α) of the wing and other controlling surfaces of the aircraft.
- Engine throttle / engine rotational speed (N).

With the assumption of $L = W$, these variables are included in the **general range equation using thrust specific fuel consumption**:

$$S = - \int_{W_0}^{W_{\text{end}}} \frac{V_{\text{flight}} C_L}{\text{sfc} \times C_D} \times \frac{1}{W} dW$$

Note that:

- sfc is a function of height (h), **flight speed (V_{flight})** and throttle (N)
- $\frac{C_D}{C_L}$ is a function of incidence angle (α) and **Mach number (M)**, and $M = \frac{V_{\text{flight}}}{V_{\text{sound}}}$
- **Flight velocity (V_{flight}), lift coefficient (C_L) and weight (W) are related through $L = W$**

This means the term $\frac{VC_L}{\text{sfc} \times C_D}$ can be vary with weight (W), so we cannot solve the integrand without specifying the course of flight. 3 cases are studied.

Constant height, constant flight velocity

This case assumes sfc to be constant and satisfies $L = W$:

$$L = W = \frac{1}{2} \rho V_{\text{flight}}^2 A_w C_L$$
$$C_L = \frac{2W}{\rho V_{\text{flight}}^2 A_w}$$

Now we define W_V^* :

$$W_V^* = \frac{1}{2} \rho V_{\text{flight}}^2 A_w C_L^*$$

Where W_V^* is the weight supported by the reference lift coefficient C_L^* at the current flight dynamic pressure. We use C_L^* because it is the lift coefficient at the attack angle with max $\frac{L}{D}$ ratio.

The above gives:

$$\frac{C_L}{C_L^*} = \frac{W}{W_V^*}$$

Then the drag polar equation:

$$C_D = C_{D0} \left[1 + \left(\frac{C_L}{C_L^*} \right)^2 \right]$$
$$C_D = C_{D0} \left[1 + \left(\frac{W}{W_V^*} \right)^2 \right]$$

The range equation then becomes:

$$\begin{aligned}
 S &= - \int_{W_0}^{W_{\text{end}}} \frac{V_{\text{flight}} C_L}{\text{sfc} \times C_D} \times \frac{1}{W} dW \\
 &= - \int_{W_0}^{W_{\text{end}}} \frac{V_{\text{flight}} \left[C_L^* \frac{W}{W_V^*} \right]}{\text{sfc} \times C_{D0} \left[1 + \left(\frac{W}{W_V^*} \right)^2 \right]} \times \frac{1}{W} dW \\
 &= \int_{W_{\text{end}}}^{W_0} \frac{V_{\text{flight}} C_L^*}{\text{sfc} \times C_{D0} W_V^* \left[1 + \left(\frac{W}{W_V^*} \right)^2 \right]} dW \\
 &= \frac{V_{\text{flight}} C_L^*}{\text{sfc} \times C_{D0} W_V^*} \int_{W_{\text{end}}}^{W_0} \frac{1}{\left[1 + \left(\frac{W}{W_V^*} \right)^2 \right]} dW \\
 S &= \frac{V_{\text{flight}} C_L^*}{\text{sfc} \times C_{D0} W_V^*} \left\{ W_V^* \left[\tan^{-1} \left(\frac{W_0}{W_V^*} \right) - \tan^{-1} \left(\frac{W_{\text{end}}}{W_V^*} \right) \right] \right\}
 \end{aligned}$$

Finally gives the range equation for the case of **constant height, constant flight velocity** as:

$$S = \frac{V_{\text{flight}} C_L^*}{\text{sfc} \times C_{D0}} \left[\tan^{-1} \left(\frac{W_0}{W_V^*} \right) - \tan^{-1} \left(\frac{W_{\text{end}}}{W_V^*} \right) \right]$$

With:

$$W_{\text{end}} = W_0 - W_{\text{fuel}}$$

In this flight, with sfc assumed as constant, the lift coefficient (C_L) and incidence angle (α) will decrease with decreasing weight according to:

$$C_L = f(\alpha) = \frac{2W}{\rho V_{\text{flight}}^2 A_w}$$

Therefore, the lift will also decrease accordingly because:

Using drag polar equation:

$$\begin{aligned}
 D &= \frac{1}{2} \rho V_{\text{flight}}^2 A_w C_D \\
 D &= \frac{1}{2} \rho V_{\text{flight}}^2 A_w C_{D0} \left[1 + \left(\frac{C_L}{C_L^*} \right)^2 \right]
 \end{aligned}$$

Constant height and constant incidence angle

This case assumes sfc and the incidence angle (α) to be constants. Therefore, C_L and C_D also constant, and the flight velocity (V_{flight}) will decrease with weight (W) and drag (D) because:

$$\boxed{L = W} \longrightarrow C_L = f(\alpha) = \frac{W}{\frac{1}{2} \rho V_{\text{flight}}^2 A_w}, \quad C_D = f(\alpha) = \frac{D}{\frac{1}{2} \rho V_{\text{flight}}^2 A_w}$$

The coefficient of lift (C_L) gives:

$$V_{\text{flight}} = \sqrt{\frac{2W}{\rho C_L A_w}}$$

The range equation then becomes:

$$\begin{aligned} S &= - \int_{W_0}^{W_{\text{end}}} \frac{V_{\text{flight}} C_L}{\text{sfc} \times C_D} \times \frac{1}{W} dW \\ &= \int_{W_{\text{end}}}^{W_0} \frac{C_L}{\text{sfc} \times C_D} \sqrt{\frac{2W}{\rho C_L A_w}} \times \frac{1}{W} dW \\ S &= \int_{W_{\text{end}}}^{W_0} \frac{\sqrt{C_L}}{\text{sfc} \times C_D} \sqrt{\frac{2}{\rho A_w}} \times \frac{1}{\sqrt{W}} dW \end{aligned}$$

Since the maximum $\frac{C_D^m}{C_L^n}$ occurs when:

$$\frac{C_L}{C_L^*} = \sqrt{\frac{n}{2m-n}}, \quad \frac{C_D}{C_{D0}} = \frac{2m}{2m-n}$$

Now, we maximize $\frac{\sqrt{C_L}}{C_D}$ by using $m = -1$ and $n = -\frac{1}{2}$ gives:

$$\sqrt{\frac{C_L}{C_L^*}} = \frac{1}{\sqrt{\sqrt{3}}}, \quad \frac{C_D}{C_{D0}} = \frac{4}{3}$$

Combining gives:

$$\begin{aligned} \frac{\sqrt{C_L}}{\sqrt{C_L^*}} \times \frac{C_{D0}}{C_D} &= \frac{1}{\frac{4}{3} \sqrt{\sqrt{3}}} \\ \frac{\sqrt{C_L}}{C_D} &= \frac{3}{4\sqrt{\sqrt{3}}} \frac{\sqrt{C_L^*}}{C_{D0}} \end{aligned}$$

The range equation then further becomes:

$$\begin{aligned}
 S &= \int_{W_{\text{end}}}^{W_0} \frac{\sqrt{C_L}}{\text{sfc} \times C_D} \sqrt{\frac{2}{\rho A_w}} \times \frac{1}{\sqrt{W}} dW \\
 &= \frac{3\sqrt{C_L^*}}{4\sqrt{\sqrt{3}} C_{D0} \times \text{sfc}} \sqrt{\frac{2}{\rho A_w}} \int_{W_{\text{end}}}^{W_0} \frac{1}{\sqrt{W}} dW \\
 &= \frac{3\sqrt{2C_L^*}}{4\sqrt{\sqrt{3}} \sqrt{\rho A_w} \times C_{D0} \times \text{sfc}} \int_{W_{\text{end}}}^{W_0} \frac{1}{\sqrt{W}} dW \\
 &= \frac{3\sqrt{2C_L^*}}{4\sqrt{\sqrt{3}} \sqrt{\rho A_w} \times C_{D0} \times \text{sfc}} \times 2[\sqrt{W_0} - \sqrt{W_{\text{end}}}] \\
 &= \frac{3\sqrt{2C_L^*}}{2\sqrt{\sqrt{3}} \sqrt{\rho A_w} \times C_{D0} \times \text{sfc}} \times \sqrt{W_0} \left[1 - \sqrt{\frac{W_{\text{end}}}{W_0}} \right] \\
 &= \frac{3\sqrt{2C_L^*}}{2\sqrt{\sqrt{3}} \sqrt{\rho A_w} \times C_{D0} \times \text{sfc}} \times V_0 \sqrt{\frac{\rho C_L^* A_w}{2\sqrt{3}}} \left[1 - \sqrt{\frac{W_{\text{end}}}{W_0}} \right] \\
 S &= \frac{3\sqrt{2}}{2\sqrt{\sqrt{3}} \times \sqrt{2\sqrt{3}}} \frac{V_0 C_L^*}{C_{D0} \times \text{sfc}} \left[1 - \sqrt{\frac{W_{\text{end}}}{W_0}} \right]
 \end{aligned}$$

$$\begin{aligned}
 V_0 &= \sqrt{\frac{2W_0}{\rho C_L A_w}} \\
 V_0 &= \sqrt{\frac{2\sqrt{3}W_0}{\rho C_L^* A_w}} \\
 \sqrt{W_0} &= V_0 \sqrt{\frac{\rho C_L^* A_w}{2\sqrt{3}}}
 \end{aligned}$$

Finally gives the range equation for the case of **constant height and constant incidence angle** as:

$$S = \frac{\sqrt{3}}{2} \frac{V_0 C_L^*}{C_{D0} \times \text{sfc}} \left[1 - \sqrt{\frac{W_{\text{end}}}{W_0}} \right]$$

With:

$$W_{\text{end}} = W_0 - W_{\text{fuel}}$$

Where:

- V_0 = flight velocity at initial weight

Cruise climb

In this case, cruise height is allowed to change continuously, so sfc, C_L and C_D are no longer constant. Recall that:

- sfc is a function of height (h), **flight speed (V_{flight})** and throttle (N)
- $\frac{C_D}{C_L}$ is a function of incidence angle (α) and **Mach number (M)**, and $M = \frac{V_{\text{flight}}}{V_{\text{sound}}}$
- **Flight velocity (V_{flight}), lift coefficient (C_L) and weight (W) are related through $L = W$**

With $L = W$, we can only express the flight velocity (V_{flight}) as:

$$C_L = \frac{W}{\frac{1}{2} \rho V_{\text{flight}}^2 A_w}$$
$$V_{\text{flight}} = \sqrt{\frac{2W}{\rho C_L A_w}}$$

Therefore, the range equation can only become:

$$S = - \int_{W_0}^{W_{\text{end}}} \frac{V_{\text{flight}} C_L}{\text{sfc} \times C_D} \times \frac{1}{W} dW$$
$$= \int_{W_{\text{end}}}^{W_0} \frac{C_L}{\text{sfc} \times C_D} \sqrt{\frac{2W}{\rho C_L A_w}} \times \frac{1}{W} dW$$
$$S = \int_{W_{\text{end}}}^{W_0} \frac{\sqrt{C_L(\alpha)}}{\text{sfc}(h, V_{\text{flight}}) \times C_D(\alpha)} \times \sqrt{\frac{2}{\rho A_w}} \times \frac{1}{\sqrt{W}} dW$$

and become very difficult to further solve.

When height (h) is high, air density (ρ) is low, then sfc will also be low. The travel distance can therefore be longer.

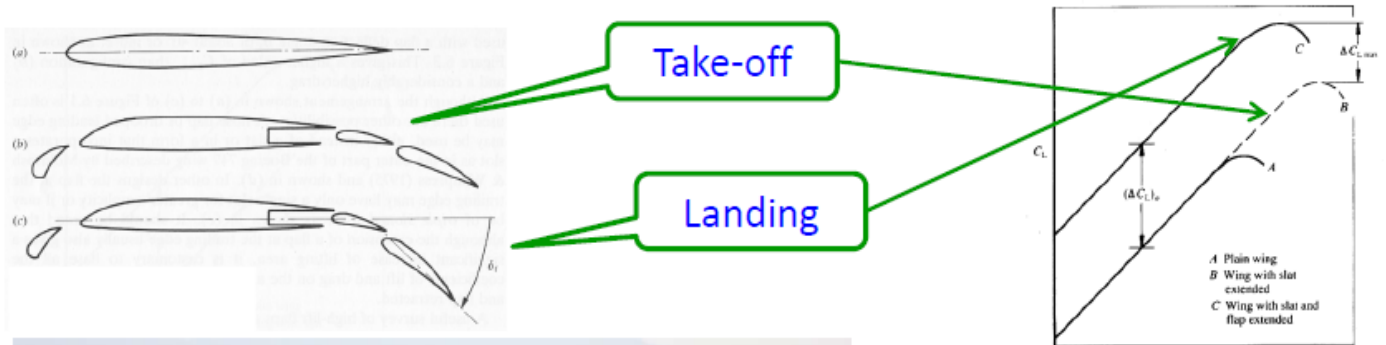
However, the benefit will not always grow with height. When the plane is too high, the required flight velocity (V_{flight}) is high and the Mach number (M) increases with height (h) rapidly due to the decreasing speed of sound, which will increase sfc in the end.

The drag coefficient may also diverge when the Mach number (M) approaches the transonic range. In the subsonic range like $M = 0.85$, it seems one does benefit from height increase.

Introduction to take-off and landing

Since the speed during take-off and landing is slow, the following two high-lift devices is required:

1. Slats at the leading edge
2. Flaps at the trailing edge



The x-axis in the graph on the right-hand side is stall angle.

- Line **A** is plain wing with no extension,
- Line **B** refers to wing with slat extent, we can see that the maximum stall angle is larger (the maximum point translates to the right side), while the lift coefficient remains unchanged (the line does not move up)
- Line **C** show that extending the wing with both slat and flap can provide significant lift to the air plane.

During take-off and landing:

- To maintain a sufficient thrust during take-off, the increment of drag needed to be limited. So the flap will only turn for around 15-20° during take-off.
- Deceleration is needed during landing, so in this stage a larger drag is required, so the flap angle can be as large as 40°.

Where the extra drag, expressed as coefficient of drag (C_D) caused by under carriage (landing gear) is estimated by:

$$\Delta C_D = \frac{wK_{uc}}{m^{0.215}}$$

Where:

- w is the wing loading in N/m^2
- m is the maximum take-off mass in kg
- K_{uc} is a coefficient:
 - $K_{uc} = 5.81 \times 10^{-5}$ for zero flap
 - $K_{uc} = 3.16 \times 10^{-5}$ for full flap

Ground effect:

Ground effect is the increased lift (force) and decreased aerodynamic drag that an aircraft's wings generate when they are close to a fixed surface. When taking off, ground effect may temporarily reduce the stall speed.

The ground does not allow vertical flow velocity. This slows down the flow and hence reduce both lift and skin friction (zero drag). This also reduce vortex drag by interrupting part of the downwash. The ground effect coefficient describes the above scenario and is expressed as:

$$\text{Ground effect} = \frac{1}{\left(\frac{s}{16h}\right)^2 + 1}$$

Where:

- h = the distance from wing to ground
- s = the wing span

Stages of take-off

Normal take-off (no engine failure):

- When the plane on the ground accelerate to a certain speed V_R , take-off begins.
- Nose-up configuration due to the elevator rotate upward.
- Nose wheel begins to leave the ground.
- Incidence angle increase to a point that there is enough ground clearance for the aircraft tail (so that the tail would not hit the ground).
- Main wheels leave the ground.
- Keep accelerating to a certain speed that the plane can still keep climbing even with one engine failed after climb over a hypothetical screen height (35 ft for passenger aircraft, 50 ft for military aircraft).

If one engine failed after passing the recognition speed V_1 , the take-off will keep continue and the rudder at the rear will keep operating to provide a side-force to against the unbalance due to one engine failed and prevent yawing.

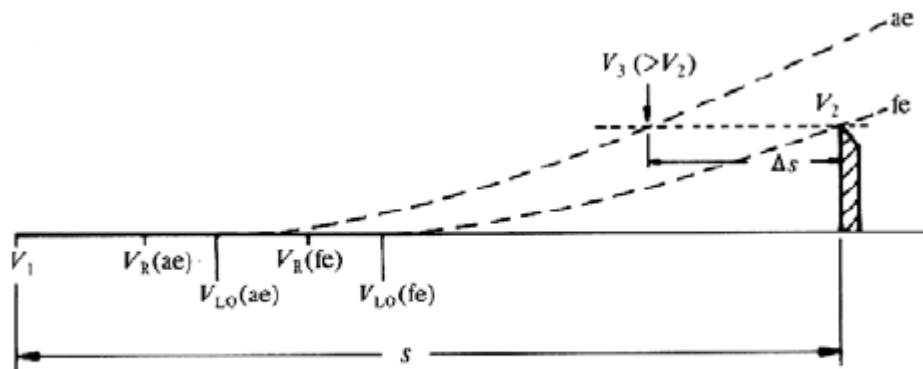


Figure 6.8. Distinction between normal take-off performance with all engines operating (ae) and minimum performance after recognition of the failure of one engine (fe) at speed V_1 (not to scale).

Mile stone velocities

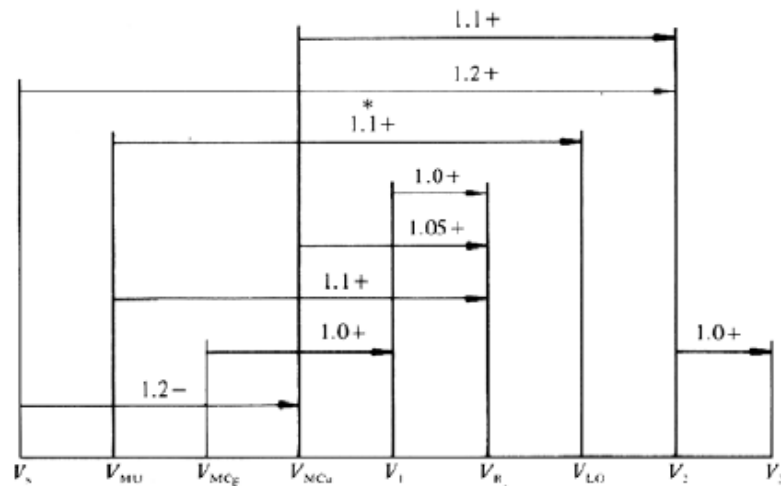


Figure 6.7. Multiplying factors relating take-off reference speeds. (No sense of scale is implied.)

Figure: Scaling between each milestone speed

Velocity	Description
V_s	Stall velocity, minimum speed to maintain a steady flight with maximum lift condition.
V_{MU}	Minimum unstick velocity, the speed that the plane can be rotate and leave the ground.
V_{MCG}	Minimum control speed on ground when one engine fails, since yawing and steering is not allowed on the ground (the plane is almost lifted up), attaining this speed means that the plane can operate rudder and counter yaw when engine fails.
V_{MCa}	Minimum control speed when engine fails during airborne.
V_1	Decision speed: If engine fail before this speed: take-off aborted. If engine fail after this speed: take-off continued.
V_R	Rotation speed.
V_{LO}	Lift-off speed.
V_2	Take-off safety speed, the speed needed to be reached after the hypothetical screen (35 ft / 50 ft) is cleared.
V_3	Take-off safety speed, can keep accelerate and climb-out if no engine fails after reaching this speed.

Ground roll distance (starting point to lift off or landing) is calculated by Newton 2nd law:

$$m \frac{dV}{dt} = F - [D + \mu(W - L)]$$

$$mV \frac{dV}{ds} = F - [D + \mu(W - L)]$$

μ is about 0.05 during take-off and 0.4 during landing brake.

The term $D + \mu(W - L)$ is negligible, because it is small when take-off.

For turbojet, the thrust (F):

$$F = F_{\text{static}} - aV + bV^2$$

By using linear velocity correction, we have:

$$F = F_{\text{static}} \left(1 - \frac{V}{V_L}\right)$$

where V_L is the linear velocity in m/s (constant).

We set the lift off speed $V_{LO} = 1.1V_{MU}$. At V_{MU} , the plane can leave the ground, it starts to roll and have an initial lift $L = W_0$, and negligible drag (so no consideration of C_D here):

$$\begin{aligned} C_L &= \frac{W_0}{\frac{1}{2} \rho V_{MU}^2 A_w} \\ V_{MU}^2 &= \frac{2W_0}{\rho (C_L A_w)_{\text{roll}}} \\ V_{LO}^2 &= \frac{2 \times 1.1^2 W_0}{\rho (C_L A_w)_{\text{roll}}} \\ V_{LO} &= \sqrt{\frac{2.42 W_0}{\rho (C_L A_w)_{\text{roll}}}} \end{aligned}$$

With the above, the rolling distance is:

$$\frac{1}{1 - \frac{V}{V_L}} \approx 1 + \frac{V}{V_L}$$

$$\begin{aligned} mV \frac{dV}{ds} &= F \\ mV \frac{dV}{ds} &= F_{\text{static}} \left(1 - \frac{V}{V_L}\right) \\ \int_0^{V_{LO}} \frac{mV}{F_{\text{static}} \left(1 - \frac{V}{V_L}\right)} dV &= \int_0^{S_{LO}} ds \\ \frac{m}{F_{\text{static}}} \int_0^{V_{LO}} V \left(1 + \frac{V}{V_L}\right) dV &= \int_0^{S_{LO}} ds \\ S_{LO} &= \frac{mV_{LO}^2}{2F_{\text{static}}} \left(1 + \frac{2V_{LO}}{3V_L}\right) \\ S_{LO} &= \frac{mV_{LO}^2}{2F_{\text{static}}} \left(1 + \frac{2V_{LO}}{3V_L}\right) \quad \leftarrow \boxed{W_0 = mg} \\ S_{LO} &= \frac{1.21 W_0^2}{F_{\text{static}} \rho g (C_L A_w)_{\text{roll}}} \left(1 + \frac{2V_{LO}}{3V_L}\right) \end{aligned}$$

Although S_{LO} is proportion to W_0^2 , the wing area A_w is also proportion to W_0^2 . That's why it is more difficult for an airport to serve the heavy aircraft unless special high-lift device is part of design.

Since $F_{\text{static}} \propto \rho$ ($F = \dot{m}_{air} \dots = \rho V A \dots$) and $S_{LO} \propto \frac{1}{\rho^2}$, we can conclude that the airplane need a longer distance to take-off at airport with high attitude. This also explain why take-off against the wind is easier than with the wind.

Landing

Landing procedure:

- Landing gear (flaps and slats) and under-carriages are lowered at least 300 m above ground.
- Constant 3° angle with horizon approach.
- The speed should remain at about 1.3 times of the stalling speed after passing hypothetical screen height.
- Flare-up to reduce touchdown impact to a velocity about 0.5 m/s (undercarriage designed to withstand 3 m/s).
- Main wheels touchdown before the nose-wheel.
- The brake is then applied, aerodynamics device (e.g. wing spoilers) are deployed.

Ground effect on landing:

- When landing, ground effect can give the pilot the feeling that the aircraft is "floating".
- Ground effect could increase lift (by ram effect) and trigger auto-flare, or pilot overshoot flare, leading to significant "floating" distance.

Landing distance theoretically shorter than take-off due to:

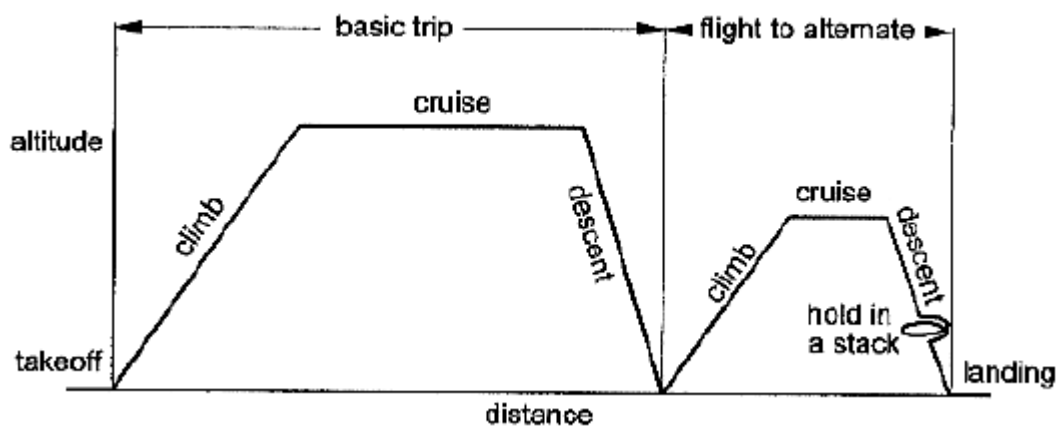
- Reduced weight due to fuel consumption.
- Brake power and aerodynamic devices (e.g. spoiler) for deployment.

Conventional auto-landing system is about the same as manual.

Nowadays, a module system called Direct Lift Control (DLC) allows a much more accurate and effective landing by changing lift rapidly through control surfaces without changing aircraft posture.

Sometimes, the destination airport is not suitable for landing.

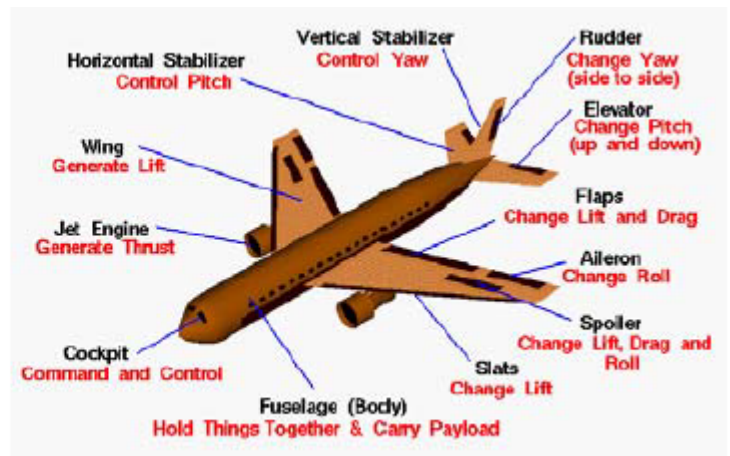
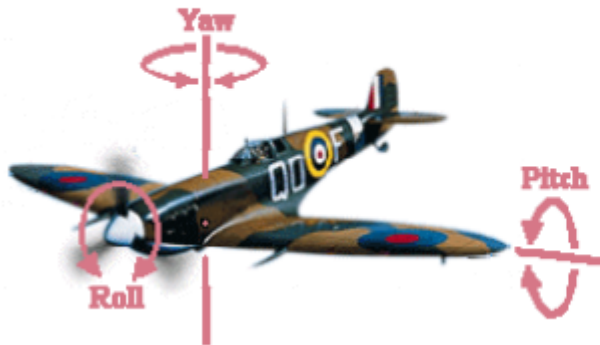
So usually more fuel is needed for landing at an alternative airport (within 350 km) and can hold for a period of 30 minutes before a permit to land is given (hold in a stack).



Introduction to flight manoeuvre

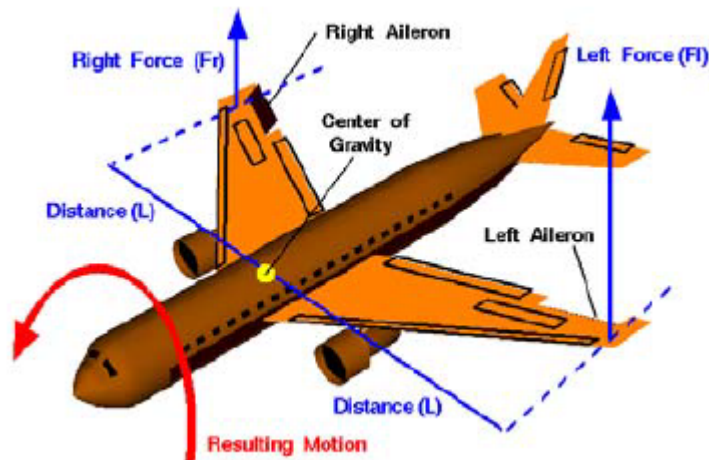
There are 3 types of air rotation:

- Roll (required during flight)
- Yaw (required when one engine failed)
- Pitch (required in normal take-off)



Rolling:

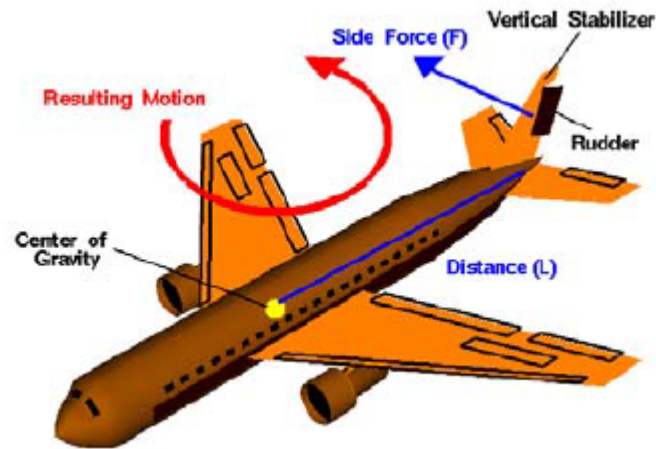
- Rolling is controlled by aileron, ailerons are installed at the trailing edge with the flaps but more closer to the wing tip.
- The side that aileron turns up would have a smaller effective incidence angle and causing less lift force on the plane and vice versa. The unbalance between two side of wings cause the rolling.



For example, right aileron turns up and left aileron turn down will result in rolling clockwise.

Yaw:

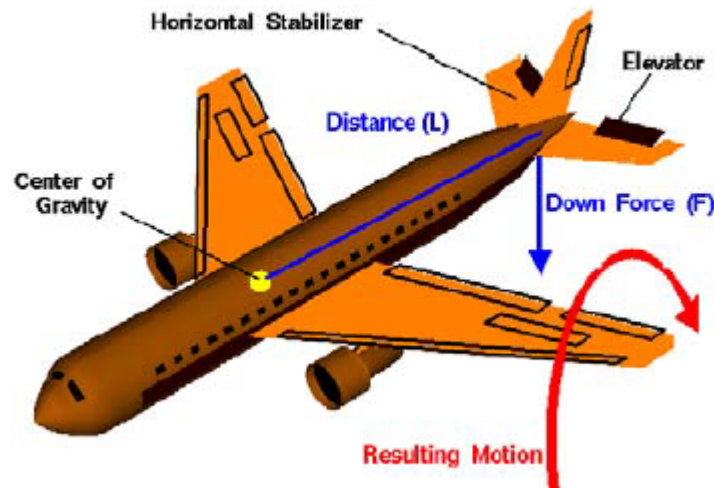
- Yaw is controlled by the rudder
- The yaw control is the main steering action when engine(s) fail.
- This steering required enough speed so that the side force generated is sufficient.



For example, the rudder turn left and the plane will take side force from the left. Resulting in yawing anti clockwise.

Pitching:

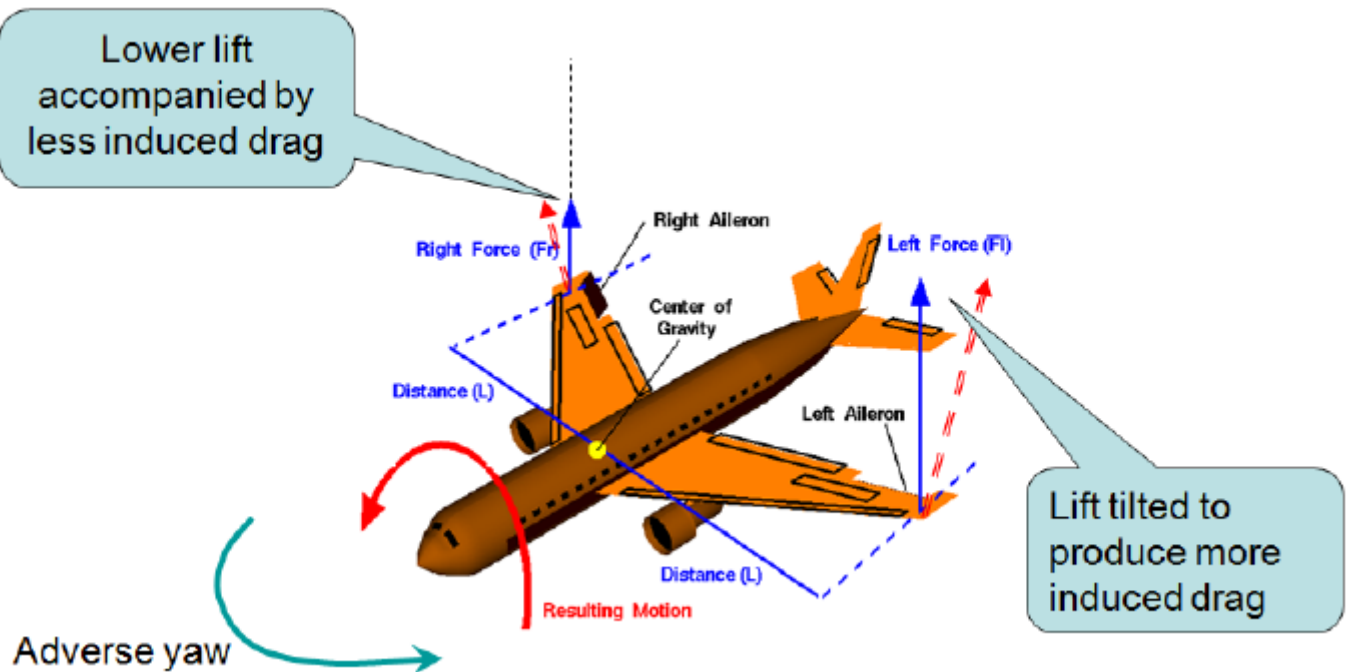
- Pitching is controlled by the elevator.
- The elevator is at the trailing edge of the horizontal stabilizer.
- During take-off and landing, the elevator should rotate up.



When the elevator is up, the trailing edge is taking a down force and thus pitching towards back.

Adverse yaw:

- Occurs during a roll
- The difference of the lift over the two wings also causes difference in the vortex drag. Therefore, the higher lift side is tilted backwards, while the lower lift side is tilted forward, causing yaw.



Stability

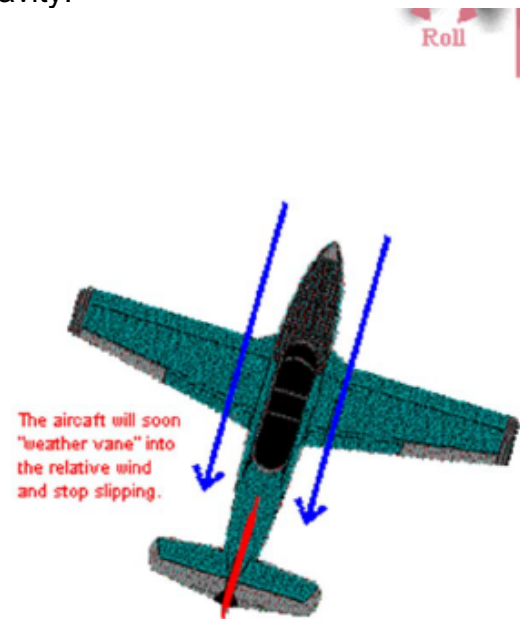
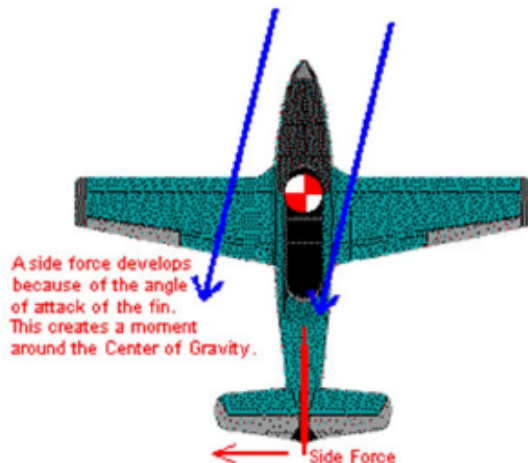
Ability of aircraft to return to a state of equilibrium ("trim") after a disturbance without pilot intervention. Stability and maneuverability is two sides of a coin.

There are 3 types of stability:

- Directional stability (yaw)
- Longitudinal stability (pitch)
- Lateral stability (roll)

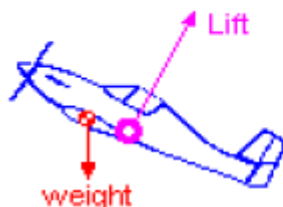
Directional stability:

- When the plane is yawing due to tilted flow, a side force will be created on the vertical stabilizer to automatically correct this.
- The stabilizer should be installed behind the centre of gravity.

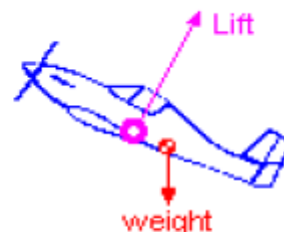


Longitudinal stability:

- Depends on the position of centre of gravity (CG).
- If CG is in front of the lift centre, when a disturbance causes a nose-up posture, the incidence angle is increased leading to higher lift. Then the weight tends to push the aircraft nose-down. So the plane become stable again.
- All the things happen opposite if CG is behind the lift centre.



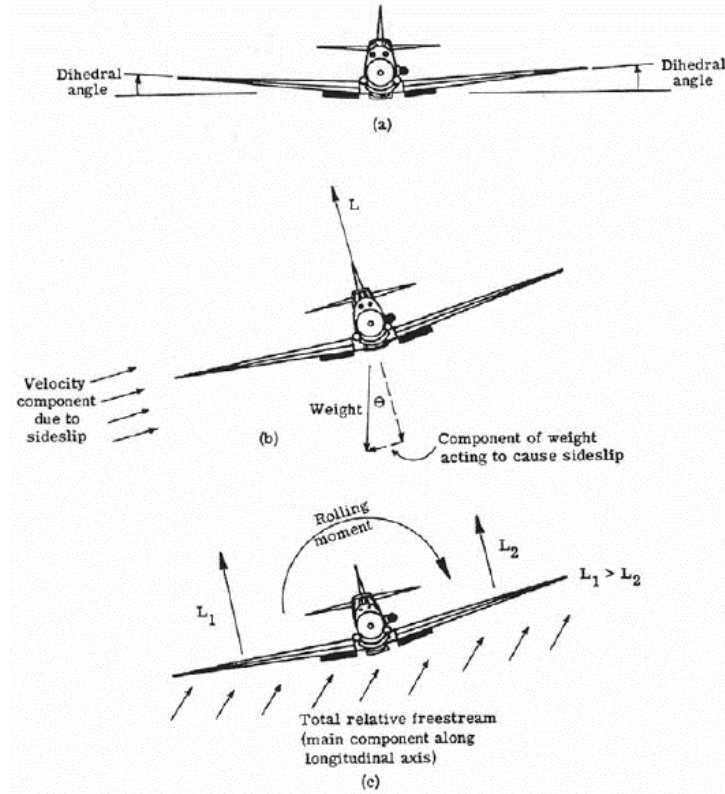
When the CG is ahead of NP the weight tends to correct the upset = Stable



When the CG is behind NP the weight worsens the upset = Unstable

Lateral stability:

- Dihedral design (the wing bent upward): When rolling, a side-slip velocity is caused by the gravity and an incoming velocity is created approaching the wing tip. It balances the lift at the wing and tending to eliminate the original roll.
- Anhedral design (the wing bent downward): Unstable, high maneuverability (機動性), military use.



Keel design:

- Used in high wing design (wing above the fuselage) for the military plane.
- The weight is hung on the wing and the fuselage behaves like a stable pendulum
- So stable that diminish roll maneuverability.
- Anhedral design is used for compensation

Keel and sweep

