

Contact foliations and generalised Weinstein conjectures

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Abstract

We consider contact foliations: objects which generalise to higher dimensions the flow of the Reeb vector field on contact manifolds. We list several properties of such foliations and propose two conjectures about the topological types of their leaves, both of which coincide with the classical Weinstein conjecture in the case of contact flows. We give positive partial results for our conjectures in particular cases – when the holonomy of the contact foliation preserves a Riemannian metric, for instance – extending already established results in the field of Contact Dynamics.

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1 Introduction

A q -contact structure consists of a collection of q linearly independent 1-forms $\lambda_1, \dots, \lambda_q$ with the property that all the derivatives $d\lambda_i$ are non-degenerate in the intersection of their kernels. The concept first appeared in Almeida's doctoral dissertation [1]. The idea was to define the most general structure that could replicate the proprieties of Anosov contact flows in higher dimensions, allowing for the definition of a *contact action*. These structures were introduced in the setting of classification of Anosov actions of \mathbb{R}^q and the Verjovsky Conjecture for actions (c.f. [2]). As such, in Almeida's work, particular focus is given to the contact actions of \mathbb{R}^q induced by these structures on their ambient manifold and to matters revolving around the algebraicity of said action.

In this paper, we expand upon the basic theory laid down in [1] by focusing not on the contact action but on its underlying *contact foliation*. This foliation can be seen as a direct generalisation to higher dimensions of the Reeb flow (also called contact flow) defined by a co-orientable contact structure. We are particularly interested in the dynamics of contact foliations, especially the existence of closed orbits. In this sense, a famous question in the field of Contact Dynamics is the Weinstein Conjecture (WC) [3]. It states:

The Reeb vector field of every closed contact manifold admits a closed orbit.

Though not yet proven in its full generality, the conjecture is known to hold in several particular cases: in dimension three [4, 5]; for PS-overtwisted contact forms [6] and, more generally, for overtwisted contact manifolds [7]; and finally, when the Reeb vector field is Killing for some metric on M [8, 9]. Given the analogy between contact foliations and Reeb flows, it is reasonable that one asks oneself whether or not some Weinstein-like results exist for contact foliations as well. There are two straightforward generalisations to higher dimensions:

The weak generalised Weinstein conjecture (WGWC). *Every q -dimensional contact foliation on a closed manifold has a leaf homeomorphic to $\mathbb{T}^l \times \mathbb{R}^{q-l}$, for some $l \geq 1$.*

The strong generalised Weinstein conjecture (SGWC). *Every q -dimensional contact foliation on a closed manifold has a leaf homeomorphic to \mathbb{T}^q .*

In the one-dimensional case (Reeb flows), all three conjectures are the same. We will show that, under special conditions on the q -contact structure, the WGWC and the WC are equivalent (cf. Theorem 2.3). It is currently an open question whether other equivalences exist under other conditions. Moreover, if any of the conjectures hold, then there can be no *minimal* contact foliations, that is, contact foliations where all the leaves are dense subsets. This is clear for the WC and SGWC. For the WGWC, it will be a straightforward consequence (cf. Theorem 2.1) of a primarily algebraic construction, carried out in Proposition 2.6, which allows us to reduce a minimal contact action on a closed manifold M to a contact action *by planes* on a manifold M_0 , over which M is a principal torus bundle.

Since the closure of any leaf always contains a minimal set, one hopes that investigating leaf closures might lead to a positive answer to the strong and weak conjectures or eventually to a counter-example to them. For this to work, we need to ensure that the restriction of a contact foliation to an invariant set is also a *contact* foliation. One sufficient condition for that to happen is that the invariant subset is a *manifold of even codimension* in M (cf. Lemma 3.3).

This brings our interest to a class of foliations that is very well-behaved concerning leaf closures: Riemannian foliations. It turns out that for contact foliations, being Riemannian is equivalent to being isometric, i.e., the orbit foliation of an \mathbb{R}^q -action *via isometries* (cf. Proposition 3.3). It follows then, from classical results from Myers-Steenrod [10] and Kobayashi [11], that for Riemannian foliations the leaf closures have the necessary properties: they are submanifolds of even codimension. Following the steps laid down by Banyaga and Rukimbira in [9], we can show that Riemannian contact foliations satisfy the WGWC. Going even further, a reduction argument can be used to prove

Theorem A. *On a closed manifold, the SGWC is satisfied by every Riemannian contact foliation.*

The opposite case is when the contact foliation is also *Anosov*, so that instead of preserving distances, the holonomy transformations contract and expand simultaneously in different transverse directions. Since contact foliations first appeared in the setting of Anosov actions, it is only natural to ask if this class of foliations satisfy our conjectures. It turns out the problem is rather uninteresting, as the Anosov property alone almost implies the existence of closed orbits [12, Theorem 5]. Hence, every Anosov contact foliation on a closed manifold (in particular, the ones studied in [1]) satisfy the SGWC.

1.1 Structure of the paper

In Section 2, we give definitions and examples and expand the work started in [1, Section 3.2.1] by proving several properties of contact foliations. In particular, when studying their holonomies, a new object – which does not occur in the classical contact case – appears: the characteristic foliation associated with each form λ_i . It is the orbit foliation of an \mathbb{R}^{q-1} -action entirely determined by λ_i and the contact foliation, and its transversals become contact manifolds when equipped with the restriction of λ_i . In this section, we also introduce the two generalisations of the Weinstein conjecture and start pointing out scenarios where they hold or are equivalent to one another.

In the second section, we also investigate *uniform* q -contact structures, which are the ones such that all the derivatives $d\lambda_i$ are the same. Such objects were first studied by Bolle [13], who characterised them as compact submanifolds of a symplectic manifold satisfying what he calls the “ C_p condition”. In short, Bolle’s C_p condition generalises the concept of a contact hypersurface to submanifolds of higher codimension. Our approach has the advantage of being intrinsic, without the need for the ambient symplectic manifold. The existence of a uniform q -contact structure imposes severe restrictions on the topology of the manifold on which it is defined. For instance, using Tischler’s argument, we show that every such manifold fibres over the torus \mathbb{T}^{q-1} (cf. Theorem 2.2). More than that, we prove that for uniform q -contact structures, $q \geq 2$, every characteristic foliation is a suspension of an \mathbb{Z}^{q-1} -action on \mathbb{T}^{q-1} , which is interesting because no contact action can be a suspension, so in a sense, the characteristic foliations of a uniform q -contact structure are the orbit foliations of “maximal” suspended sub-actions. Moreover, as a corollary to the fact that characteristic foliations are suspensions, we obtain equivalence between the WC and the WGWC for uniform contact foliations (cf. Theorem 2.3).

We end with Subsection 2.5, which focuses on another property of contact foliations: their tautness. In our context, tautness (or harmonicity) is defined as the existence of a metric tensor for which all the leaves are minimal submanifolds, i.e., submanifolds whose normal curvature vanishes. Note that one-dimensional minimal submanifolds are simply geodesics, so in a sense this generalises to our higher dimensional objects the widely known fact that contact flows are geodesible.

The last section is Section 3, dedicated to partial results towards a positive answer to the SGWC. We briefly prove that it is satisfied by Anosov flows and then develop a theory for *isometric* contact foliations: ones that preserve a transverse Riemannian

metric. This is heavily inspired by the work of Rukimbira and Banyaga (cf. [8, 9]), and culminates in the proof of Theorem A. We show more, proving that an isometric contact foliation on a closed manifold must have at least *two* closed orbits and that the set of closed orbits decomposes into a disjoint union of invariant even-codimensional submanifolds (cf. Theorem 3.4).

1.2 Some remarks on nomenclature

To the best of the author’s knowledge, the objects of interest in this paper were first defined by Almeida in [1]. In his work, Almeida refers to q -contact structures as *generalised k -contact structures*. We changed the index letter from k to q to avoid confusion with the established notion of a K-contact structure. Moreover, Almeida’s notion of q -contact structure, taken from [13–15], is what we call here a *uniform q -contact structure* (Definition 2.5). Our decision to rename these concepts was based, first, on the belief that our nomenclature is less cumbersome and more informative.

Moreover, while preparing this paper, we stumbled upon the work of Gaset et al. [16], in which yet another object called a q -contact structure is defined. In the sense of Gaset et al., a q -contact structure is even more general than the ones present in this paper since they do not require the kernels of the derivatives of the defining 1-forms to be all the same. In light of that, we felt it made little sense to call the structures appearing in this work “generalised,” so we decided to abandon the adjective for good.

Besides the q -contact structures in the sense of Gaset et al., several other geometric structures are similar and/or related to the structures studied here. We want to give special mention to the notions of Contact Pair (cf. [17]); Multicontat Structures (cf. [18]); Pluricontact and Polycontact Structures (cf. [19, 20]). The interested reader is referred to [1, Section 3.2.2] for a more detailed comparison between all these different concepts. In this paper, we also briefly explore the concept of an f -structure, in the sense of Yano, which is somewhat related but not entirely similar to that of a q -contact structure, especially to the particular case of uniform q -contact structures. In particular, many of our results here have analogues in the case of f -structures, like the existence of Reeb vector fields, the characterisation of uniform q -contact manifolds (Theorem 2.2) and also our main result, Theorem A.

2 Contact Foliations

2.1 Definition and Examples

A contact foliation is a generalisation of the flow of the Reeb vector field determined by a co-orientable contact structure. The analogue structure determining the contact foliation is called a q -contact structure, defined as follows.

Definition 2.1 (*q -contact manifolds [1]*). *Let n, q be positive integers and consider a $2n + q$ dimensional differential manifold M . A **q -contact structure** on M is a collection $\vec{\lambda} = (\lambda_1, \dots, \lambda_q)$ of q (pointwise) linearly independent non-vanishing 1-forms λ_i , together with a splitting*

$$TM = \mathcal{R} \oplus \xi$$

of the tangent bundle, satisfying the following conditions:

- (i) $\xi := \cap_i \ker \lambda_i$;
- (ii) for every i , the restriction $d\lambda_i|_{\xi}$ is non-degenerate;
- (iii) for every i , one has $\ker d\lambda_i = \mathcal{R}$.

A manifold endowed with such structure is called a **q -contact manifold** and denoted by $(M, \vec{\lambda}, \mathcal{R} \oplus \xi)$, or simply by M when the context permits. We call the collection $\{\lambda_i\}$ an **adapted coframe** for the q -contact structure, and the q -form

$$\lambda := \lambda_1 \wedge \cdots \wedge \lambda_q \neq 0$$

is called the **characteristic form**. The bundles \mathcal{R} and ξ are called the **Reeb distribution** and **q -contact distribution**, respectively.

Remark 2.1. The q -contact and Reeb distributions, as well as the splitting $\mathcal{R} \oplus \xi$, have the same regularity class as the forms λ_i . From now on, we will always assume the splitting is at least of class C^1 .

The linear independence of the λ_i implies that ξ has constant rank $2n$; therefore, condition (ii) is equivalent to $d\lambda_i|_{\xi} \neq 0$, or, in other words, to $(\xi, d\lambda_i)$ being a symplectic bundle over M . Linear independence is also equivalent to the non-degeneracy of the characteristic form. From this, we can conclude that

$$dM_i := \lambda \wedge d\lambda_i^n$$

is a volume form on M for every i . In particular, M is orientable.

Remark 2.2. In [21], Yano began the study of manifolds M supporting a $(1,1)$ -tensor f satisfying the equation $f^3 + f = 0$. Such a structure induces a splitting $TM = \ker f \oplus \text{Im} f$, which is said to be an f -structure with complemented frames if $\ker f$ is parallelisable [22]. Given an f -manifold with complemented frames, additional hypotheses on the kernel $\ker f$ give rise to different geometric structures such as \mathcal{K} -manifolds and \mathcal{S} -manifolds. These last two are particular cases of q -contact structures as considered here (cf. [23]).

On the other hand, given a q -contact manifold $(M, \vec{\lambda}, \mathcal{R} \oplus \xi)$, for every defining 1-form λ_i , the symplectic bundle $(\xi, d\lambda_i)$ admits a compatible almost complex structure J_i . If we define f_i as equal to J_i on ξ and identically zero on \mathcal{R} , then it is clear that $f_i^3 + f_i = 0$. So M supports (at least) as many different f -structures with a complemented frame as there are different 2-forms $d\lambda_i$. The concept of an f -structure with complemented frames is more general than that of a q -contact structure.

As in the contact case, we can show that the q -contact distribution ξ is maximally non-integrable, and the Reeb distribution \mathcal{R} is a parallelisable bundle of constant rank q .

Proposition 2.1 ([1]). For any vector field $X \neq 0$ tangent to ξ , there is another vector field Y tangent to ξ such that $[X, Y]$ is not a section of ξ .

Proposition 2.2 ([1]). There is a unique collection of linearly independent vector fields R_1, \dots, R_q tangent to \mathcal{R} satisfying, for all $i, j = 1, \dots, q$, the relations

- (i) $\lambda_i(R_j) = \delta_{ij}$;
- (ii) $[R_i, R_j] = 0$;
- (ii) $\mathcal{R} = \text{Span}\{R_1, \dots, R_q\}$.

Remark 2.3. In [24, Corollary 2.4], Cabrerizo et al. showed the existence of these fields in the case of f -structures using techniques other than the ones in [1]. The vector fields R_i have the same regularity class as the splitting $TM = \mathcal{R} \oplus \xi$.

Definition 2.2 (Reeb vector fields). The q (unique) linearly independent vector fields from Proposition 2.2 are called the **Reeb vector fields** of the q -contact structure.

Each Reeb vector field preserves all the 1-forms λ_i and their derivatives:

$$\begin{aligned}\mathcal{L}_{R_j} \lambda_i &= d(\iota_{R_j} \lambda_i) + \iota_{R_j} d\lambda_i = 0; \\ \mathcal{L}_{R_j} d\lambda_i &= d(\iota_{R_j} d\lambda_i) + \iota_{R_j} d^2 \lambda_i = 0,\end{aligned}$$

and therefore, the Reeb flows preserve all of the volume forms dM_i , as well as the symplectic bundles $(\xi, d\lambda_i)$ over M .

Pairwise commutativity assures the following action of the Euclidean space $(\mathbb{R}^q, +)$ is well-defined.

Definition 2.3 (q -contact action). Let M be a q -contact manifold with Reeb vector fields R_1, \dots, R_q . We define a locally free action of $F : \mathbb{R}^q \rightarrow \text{Diff}(M)$ by

$$F : a \mapsto F^a := (\exp t_1 R_1) \circ \dots \circ (\exp t_q R_q),$$

where $a = (t_1, \dots, t_q) \in \mathbb{R}^q$ and $\exp t R_i$ the time- t flow map of R_i . The action F is called the **q -contact action on M** associated with the q -contact structure.

Remark 2.4. If the splitting $TM = \mathcal{R} \oplus \xi$ is C^r , then so are the Reeb fields, and therefore the action F is of class C^r as well.

We denote by \mathcal{F} the underlying foliation of F , that is, the q -dimensional foliation of M whose leaves are the orbits of F .

Definition 2.4 (Contact foliation). The foliation \mathcal{F} underlying a contact action of \mathbb{R}^q is called a **contact foliation**.

It is possible for distinct adapted coframes $\vec{\lambda}$ and $\vec{\eta}$ to give rise to the same Reeb and q -contact distributions (cf. Lemma 2.1) and, therefore, to define the same contact foliation. In this case, we say $\vec{\eta}$ is a **reparameterisation** of $\vec{\lambda}$.

Example 2.1 (Contact manifolds). Every contact manifold (M, ξ) with a transversely orientable non-integrable distribution ξ is a 1-contact manifold once a defining form λ is chosen. Here, \mathcal{R} is the span of the Reeb vector field, and the action F is the one induced by the Reeb field's flow. We will refer to this as the **contact case**.

Example 2.2 (Structures on \mathbb{R}^{2n+q}). There's a simple q -contact structure on \mathbb{R}^{2n+q} with coordinates $(x_1, y_1, \dots, x_n, y_n, z_1, \dots, z_q)$, given by the 1-forms

$$\lambda_i := dz_i + \sum_{j=1}^n x_j dy_j.$$

The Reeb vector fields are $R_i = \partial_{z_i}$, and therefore $\mathcal{R} = \text{Span}(\partial_{z_1}, \dots, \partial_{z_q})$. The contact action is translation on the last q coordinates, and the contact foliation \mathcal{F} consists of all planes parallel to $\{0\} \times \mathbb{R}^q$. The q -contact distribution is

$$\xi = \text{Span}(\partial_{x_1}, Y_1, \dots, \partial_{x_n}, Y_n),$$

where $Y_i := \partial_{y_i} - x_i \sum_j \partial_{z_j}$, and the volume form dM_i is the canonical volume form of \mathbb{R}^{2n+q} , for every i .

All of the above holds also for

$$\lambda_i := dz_i + \alpha_i,$$

assuming each $d\alpha_i$ is non-degenerate on ξ . In particular, the $d\lambda_i = d\alpha_i$ need not be equal.

Definition 2.5 (Uniform q -contact structures). An adapted coframe $\vec{\lambda}$ (and the q -contact structure it defines) is called **uniform** if it satisfies

$$d\lambda_i = d\lambda_1$$

for all $1 \leq i \leq q$. A contact foliation \mathcal{F} is said to be uniform if its adapted coframe is a reparameterisation of a uniform coframe.

Remark 2.5. Both \mathcal{K} -structures and \mathcal{S} -structures (cf. Remark 2.2) are particular cases of uniform q -contact structures. Uniform q -contact structures were also studied by Bolle in [13]. He defines them as codimension p submanifolds of a symplectic manifold (W, ω) satisfying what he calls “ C_p condition”. The concepts are the same because every compact uniform q -contact manifold is a codimension q submanifold of its symplectisation satisfying the C_q condition. The symplectisation of $(M, \vec{\lambda}, \mathcal{R} \oplus \xi)$ consists of the product $W := M \times \mathbb{R}^q$. Given coordinates (x, t_1, \dots, t_q) on W , if we equip it with the non-degenerate 2-form

$$\omega := \sum_{i=1}^q (dt_i \wedge \lambda_i + t_i d\lambda_1),$$

then the pair (W, ω) is symplectic, and $M \times 0$ satisfies the C_q condition. It is also worth noting that not every contact foliation is uniform. The algebraic contact foliations constructed by Almeida in [1] do not admit any uniform underlying coframe, for instance (see [1, Proposition 3.3.5] for further details). Another case of non-uniform structure is the construction from Example 2.3 bellow.

The following product-like constructions give us examples of closed q -contact manifolds other than closed contact manifolds.

Example 2.3 (Products of contact manifolds). Suppose (M_1, α_1) and (M_2, α_2) are contact manifolds, with X_i denoting the Reeb vector field of α_i , and $\xi_i = \ker \alpha_i$ the associated contact structure. Then the forms

$$\begin{aligned} \lambda_1 &:= \pi_1^* \alpha_1 - \pi_2^* \alpha_2 \\ \lambda_2 &:= \pi_1^* \alpha_1 + \pi_2^* \alpha_2 \end{aligned}$$

defined on the product $M := M_1 \times M_2$ by means of the canonical projections $\pi_i : M \rightarrow M_i$ are an adapted coframe for a 2-contact structure on M . The 2-contact distribution is

$$\xi := \ker \lambda_1 \cap \ker \lambda_2 = \pi_1^* \xi_1 \cap \pi_2^* \xi_2,$$

on which the $d\lambda_i$ are non-degenerate due to the non-degeneracy of the α_i . The Reeb fields are

$$\begin{aligned} R_1 &:= \frac{1}{2}(\pi_1^* X_1 - \pi_2^* X_2), \\ R_2 &:= \frac{1}{2}(\pi_1^* X_1 + \pi_2^* X_2). \end{aligned}$$

Example 2.4 (Flat \mathbb{T}^l -bundles over q -contact manifolds). *If $(M, \vec{\lambda}, \mathcal{R} \oplus \xi)$ is a q -contact manifold, the product $E := \mathbb{T}^l \times M$ supports a $(q + l)$ -contact structure. More generally, if $\pi : E \rightarrow M$ is a principal \mathbb{T}^l -bundle supporting a flat connection, then E admits a $(q + l)$ -contact structure. The construction, due to Almeida [1], is as follows. The connection induces a splitting $TE = H \oplus V$ into horizontal and vertical bundles. Flatness implies both bundles are involutive [25, Lemma 3.1]. Let $\{X_1, \dots, X_l\}$ be the commutative frame generating the \mathbb{T}^l -action on each fibre, so that $V = \text{Span}\{X_1, \dots, X_l\}$. Define forms α_i on M by setting*

$$\begin{cases} \alpha_i(X_j) = \delta_{ij}, \\ \alpha_i|_H = 0. \end{cases}$$

These are closed forms. Indeed, $[X_j, X_r] = 0$ for any choice of j, r . For any horizontal vector field W , the section $[X_j, W]$ is also horizontal [26, Lemma 3.12], hence

$$\begin{aligned} d\alpha_i(X_j, X_r) &= X_j\alpha_i(X_r) - X_r\alpha_i(X_j) - \alpha_i([X_j, X_r]) = 0, \\ d\alpha_i(X_j, W_1) &= X_j\alpha_i(W_1) - W_1\alpha_i(X_j) - \alpha_i([X_j, W_1]) = 0. \end{aligned}$$

Since H is involutive, given $W_1, W_2 \in H$,

$$d\alpha_i(W_1, W_2) = W_1\alpha_i(W_2) - W_2\alpha_i(W_1) - \alpha_i([W_1, W_2]) = 0,$$

hence $d\alpha_i|_H \equiv 0$, and therefore each α_i is closed.

Each fibre H_p of H is identifiable with the fibre $T_{\pi(p)}M = \mathcal{R}_{\pi(p)} \oplus \xi_{\pi(p)}$. Thus H has a splitting $H = \overline{\mathcal{R}} \oplus \tilde{\xi}$, where $\overline{\mathcal{R}} \approx \mathcal{R}$ and $\xi \approx \tilde{\xi}$. Write $\tilde{\mathcal{R}} := V \oplus \overline{\mathcal{R}}$ so that we have

$$TE = \tilde{\mathcal{R}} \oplus \tilde{\xi}.$$

For any choice of indices $\{j_1, \dots, j_l\} \subset \{1, \dots, q\}^l$, we set

$$\begin{aligned} \eta_i &:= \pi^* \lambda_i, & \text{for } 1 \leq i \leq q; \\ \eta_{q+i} &:= \alpha_i + \pi^* \lambda_{j_i}, & \text{for } 1 \leq i \leq l. \end{aligned}$$

These forms are such that, for $i = 1, \dots, q$

$$\ker \eta_i = \pi^* \xi = \tilde{\xi} \oplus \overline{\mathcal{R}};$$

and, for $i = 1, \dots, l$

$$\ker \tilde{\eta}_{q+i} = \text{Span}\{X_j\}_{j \neq i} \oplus \text{Span}\{X - \lambda_{j_i}(\pi_* X)X_i; X \in \Gamma(H)\} \supset \tilde{\xi}.$$

It then follows that

$$\begin{aligned} \bigcap_{i=1}^{q+l} \ker \eta_i &= (\tilde{\xi} \oplus \tilde{\mathcal{R}}) \cap \left(\bigcap_{i=1}^l \text{Span}\{X - \lambda_{j_i}(\pi_* X)X_i; X \in \Gamma(H)\} \right) \\ &= \tilde{\xi}, \end{aligned}$$

and $(E, \tilde{\eta}, \tilde{\mathcal{R}} \oplus \tilde{\xi})$ is therefore a $(l+q)$ -contact structure. Since the bundle $\overline{\mathcal{R}} \subset H$ is diffeomorphic to \mathcal{R} , we can think of the Reeb vector fields on M as vector fields on H . Hence the Reeb vector fields of the η_i are

$$\begin{aligned} \tilde{R}_i &:= R_i, & \text{for } 1 \leq i \leq q; \\ \tilde{R}_{q+i} &:= \frac{1}{2}(X_i + R_{j_i}), & \text{for } 1 \leq i \leq l. \end{aligned}$$

Note that the restriction of the flat connection on TE to $\tilde{\mathcal{R}}$ is again flat, providing us with two complementary integrable sub-bundles V and \mathcal{R} of $\tilde{\mathcal{R}}$. This means the leaves of $\tilde{\mathcal{F}}$ are the product of the leaves of \mathcal{F} with the torus \mathbb{T}^l .

A particular case of such bundles are simply the products $\mathbb{T}^l \times M$. We observe that by taking M to be a closed contact manifold and choosing the dimension of the fibre appropriately, we can construct closed manifolds supporting q -contact structures for any q larger than 1.

Finally, we have a classic example of algebraic nature from the theory of semi-simple Lie groups.

Example 2.5 (Weyl Chamber Actions). *Suppose G is a semi-simple Lie group and \mathfrak{a} is a Cartan subspace with centraliser $\mathfrak{a} \oplus \mathfrak{k}$. Let K be a compact subgroup of G associated with \mathfrak{k} and Γ a uniform lattice on G acting freely on G/K . The exponentials of elements in \mathfrak{a} act on $\Gamma \backslash (G/K)$ by right translation. This is called the **Weyl Chamber Action**. Almeida proved [1, Theorem 1.0.2] that this is a q -contact action.*

2.2 Local Charts and Reparameterisations

In a foliated chart $\psi : U \rightarrow \mathbb{R}^{2n} \times \mathbb{R}^q$, the tangent coordinates (z_1, \dots, z_q) can be chosen so that $\partial_{z_i} = R_i$ (cf [27, Theorem 8.3]). Evaluating the forms λ_i and their derivatives at these fields give rise to the following local characterisation for the adapted coframe $\vec{\lambda}$.

Proposition 2.3. *Around each point $p \in M$ there are coordinates $(x, y, z) \in \mathbb{R}^{2n} \times \mathbb{R}^q$, and, for each $1 \leq i \leq q$ and $1 \leq j \leq n$, functions $f_j^i, g_j^i : \mathbb{R}^{2n} \rightarrow \mathbb{R}$ such that:*

$$\lambda_i = dz_i + \sum_{j=1}^n (-f_j^i(x, y)dx_j + g_j^i(x, y)dy_j).$$

Motivated by the last proposition, one can ask whether q -contact structures are locally isomorphic or not, in the sense of a generalised Darboux Theorem. A partial result in this direction comes from the following proposition.

Proposition 2.4. *Let M be a manifold supporting a contact foliation of codimension 2, that is, $\dim M = 2 + q$. Then around each point of M there are coordinates $(U; x, y, z_1, \dots, z_q)$ and leaf-wise constant functions $a_i : U \rightarrow \mathbb{R}$ such that $\partial_x a_i$ is non-vanishing and*

$$\lambda_i = dz_i + a_i dy.$$

Moreover, we can take $a_1 = x$.

Proof. Around each point of M , one can find a neighbourhood U diffeomorphic to a product of discs $D^q \times D^2$. Let $\Phi : U \rightarrow D^2$ be the submersion defined by this diffeomorphism, the fibres of which are the plaques of \mathcal{F} on U . Since the symplectic form $d\lambda_i$ is holonomy invariant, there is a well-defined symplectic form ω_i on D^2 satisfying $\Phi^* \omega_i = d\lambda_i$. By Darboux's theorem, there is a coordinate chart (x', y') on D^2 such that $\omega_1 = dx' \wedge dy'$. Moreover, for every other i there is a non-vanishing function b_i on D^2 such that $\omega_i = b_i dx' \wedge dy'$.

Consider on U the unique fields X and Y on M that are Φ -related to the coordinate vector fields $\partial_{x'}$ and $\partial_{y'}$ and have no components in any leaf-wise direction. More precisely, they satisfy:

$$\begin{cases} X = \Phi^* \partial_{x'} \\ Y = \Phi^* \partial_{y'} \\ \mathcal{L}_{R_i} X = \mathcal{L}_{R_i} Y = 0 \text{ for all } i. \end{cases}$$

Thus, $\{R_1, \dots, R_q, X, Y\}$ is a local frame for the tangent bundle of M , satisfying $[R_i, R_j] = [X, R_j] = [Y, R_j] = 0$ for any i, j . As for $[X, Y]$, we have

$$\lambda_i([X, Y]) = X\lambda_i(Y) - Y\lambda_i(X) - d\lambda_i(X, Y) = b_i dx' \wedge dy'(\partial_{x'}, \partial_{y'}) = b_i \circ \Phi,$$

so that $[X, Y] = \sum_i (b_i \circ \Phi) R_i$. We define on D^2 functions $c_i(x', y') = \int_0^{x'} b_i(t, y') dt$ and note that, since X and $\partial_{x'}$ are Φ -related, these satisfy

$$X(c_i \circ \Phi) = \partial_{x'} c_i \circ \Phi = b_i \circ \Phi.$$

Thus, the vector field $\tilde{Y} := Y - \sum_i (c_i \circ \Phi) R_i$ is linearly independent from X and the R_i ; in addition, it is such that

$$[X, \tilde{Y}] = [X, Y] - \sum_i [X, (c_i \circ \Phi) R_i] = \sum_i (b_i \circ \Phi) R_i - \sum_i (b_i \circ \Phi) R_i = 0.$$

One can, therefore, find coordinates (x, y, z_1, \dots, z_q) on M whose coordinate vector fields are $\{X, \tilde{Y}, R_1, \dots, R_q\}$, obtaining

$$\begin{cases} X = \partial_x \\ Y = \partial_y + \sum_i (c_i \circ \Phi) \partial_{z_i} \\ R_i = \partial_{z_i}. \end{cases}$$

It follows from construction that $\Phi^* dx' = dx$ and $\Phi^* dy' = dy$. Hence

$$d\lambda_i = \Phi^* \omega_i = (b_i \circ \Phi) dx \wedge dy$$

and, since $d(c_i \circ \Phi) = (b_i \circ \Phi) dx + \partial_y(c_i \circ \Phi) dy$,

$$d((c_i \circ \Phi) dy) = (b_i \circ \Phi) dx \wedge dy.$$

Thus $d(\lambda_i - (c_i \circ \Phi) dy) = 0$, and there exists, on a possible smaller neighbourhood, a function f_i such that

$$\lambda_i = df_i - (c_i \circ \Phi) dy.$$

Finally, we see that

$$\begin{aligned} \delta_{ij} &= \lambda_i(R_j) = df_i(\partial_{z_j}) - (c_i \circ \Phi) dy(\partial_{z_j}) = \partial_{z_j} f_i; \\ 0 &= \lambda_i(X) = df_i(\partial_x) - (c_i \circ \Phi) dy(\partial_x) = \partial_x f_i; \\ 0 &= \lambda_i(Y) = df_i(\partial_y) + \sum_j (c_j \circ \Phi) df_i(\partial_{z_j}) - (c_i \circ \Phi) dy(\partial_y) - \sum_j (c_j \circ \Phi) dy(\partial_{z_j}) \\ &= \partial_y f_i + (c_i \circ \Phi) - (c_i \circ \Phi) = \partial_y f_i, \end{aligned}$$

and therefore $df_i = dz_i$, from where we conclude, setting $a_i := -(c_i \circ \Phi)$, that $\lambda_i = dz_i + a_i dy$, as we wanted. \square

A crucial fact in the construction above is the existence of the non-vanishing functions b_i . When the codimension is greater than 2, one can not generally find a single basis in which all the symplectic forms are written canonically, so this argument does not hold. When, however, all the derivatives $d\lambda_i$ coincide, then a similar construction in general codimension yields:

Proposition 2.5 ([28]). *If M is an uniform q -contact manifold, then around each point there are coordinates $(U; x, y, z_1, \dots, z_q)$ such that*

$$\lambda_i = dz_i + \sum_j x_j dy_j.$$

It is currently an open question whether or not any sort of local model exist for q -general contact structures. For instance, given q symplectic $n \times n$ matrices $S^l = \{s_{ij}^l\}_{ij}$, the 1-forms

$$dz_l + \sum_{ij} s_{ij}^l x_i dy_j$$

induce a q -contact structure on \mathbb{R}^{q+2n} . It is a natural question whether or not every q -contact structure is locally of this form.

Lemma 2.1 (Reparameterisation Lemma [1]). *Let $\vec{\lambda} = (\lambda_1, \dots, \lambda_q)$ be an adapted coframe for a q -contact structure on M , and $A = \{a_{ij}\} \in \text{Gl}_q(\mathbb{R})$ a matrix of real coefficients. If A is sufficient close to 0 then*

$$\vec{\eta} := (\text{id} - A)\vec{\lambda}$$

is also an adapted coframe for a q -contact structure on M , and the splittings associated with these two structures are the same.

In general, the Reeb vector fields of the η_i are not the same as the R_i (though they span the same invariant bundle), so the construction above might be thought of as a *reparameterisation of the contact action*.

When the action is transitive, one can show that a matrix only fails to take adapted coframes into adapted coframes if its entries are a zero of certain polynomials, that is, the set of such matrices form a Zariski closed subset of $\mathbb{R}^{q \times q}$. Thus, for transitive actions there is a dense subset of matrices in $\text{Gl}_q(\mathbb{R})$ corresponding to reparameterisations.

Lemma 2.2 ([1]). *Suppose the contact foliation \mathcal{F} is transitive, and let $\mathcal{B} \subset \text{Gl}_q(\mathbb{R})$ be the set of reparameterisations of the action. The complement of \mathcal{B} is closed and has empty interior. In particular, \mathcal{B} is dense in $\text{Gl}_q(\mathbb{R})$.*

2.3 Generalised Weinstein Conjectures

We wish to investigate whether or not some Weinstein-like conjecture holds for the more general objects of Definition 2.1. With this in mind, we propose two generalisations.

The weak generalised Weinstein conjecture (WGWC). *A contact foliation on a closed manifold M cannot be a foliation by planes.*

The strong generalised Weinstein conjecture (SGWC). *Every contact foliation has a closed leaf, that is, a leaf homeomorphic to a torus \mathbb{T}^q .*

Since F is a locally free action of \mathbb{R}^q , every leaf is of the form

$$\mathcal{F}(x) \approx \mathbb{R}^q / \mathbb{R}_x^q \approx \mathbb{R}^{q-l} \times \mathbb{T}^l$$

where $\mathbb{R}_x^q := \{a \in \mathbb{R}^q; F^a(x) = x\} \approx \mathbb{Z}^l$ is the isotropy subgroup of the leaf ($0 \leq l \leq q$). The WGWC states that $l \geq 1$ for some leaf, while the SGWC asks for the existence of a leaf for which $l = q$. In the contact case, that is, when $q = 1$, a leaf that is not homeomorphic to \mathbb{R} is automatically a closed curve, homeomorphic to S^1 . Therefore, both conjectures are equivalent to the Weinstein Conjecture on dimension 1 and comprise generalisations of this conjecture to higher dimensional contact foliations. They form a hierarchy

$$\text{SGWC} \implies \text{WGWC} \implies \text{WC},$$

with the converse implications holding when $q = 1$.

Recall that a subset of a foliated space is called *invariant* if it is closed and saturated. A foliation is *minimal* if its ambient manifold contains no invariant subset. Equivalently, a foliation is minimal when every leaf is a dense subset in the ambient manifold. Since closed leaves are invariant, it is clear that if the SGWC holds, then there can be no minimal contact foliations. A priori, there is nothing to prevent the existence of minimal contact foliations satisfying the WGWC. We wish to show that this scenario also can not happen. This is a corollary to the following “reduction” procedure, which can be seen as a partial converse to the “extension” construction of Example 2.4. First, let us recall that the kernel of F is a lattice in \mathbb{R}^q consisting the intersection of all the isotropy subgroups:

$$\ker F := \{a \in \mathbb{R}^q; F^a = \text{id}\} = \bigcap_{x \in M} \mathbb{R}_x^q \approx \mathbb{Z}^l, 0 \leq l \leq q.$$

Hence every leaf is a cylinder $\mathbb{T}^s \times \mathbb{R}^{q-s}$ for some $s \geq l$. In particular, if the action is non-fatihful, that is, its kernel is non-trivial, then no leaf is a plane.

Proposition 2.6 (Reduction of a contact action). *Let $(M, \vec{\lambda}, \mathcal{R} \oplus \xi)$ be a q -contact manifold. Suppose the contact action F non-faithful and that all the orbits have the same isotropy subgroup. Then M is a principal \mathbb{T}^l -bundle over a closed $(2n + q - l)$ -dimensional manifold M_0 , for a fixed $0 < l < q$. Additionally, F induces a faithful $(q - l)$ -contact action on M_0 .*

Proof. Let $\Gamma := \ker F \approx \mathbb{Z}^l$, where $0 < l < q$. By hypothesis, Γ is also the isotropy subgroup of each leaf of \mathcal{F} . We consider the vector spaces

$$\begin{aligned} G &:= \text{Span } \Gamma \approx \mathbb{R}^l, \\ H &:= \mathbb{R}^q / G \approx \mathbb{R}^{q-l}, \end{aligned}$$

and the natural isomorphism $\mathbb{R}^q \approx G \oplus H$.

First, let the torus $\mathbb{T}^l \approx \Gamma/G$ act on M by

$$(a + \Gamma) \cdot x := F^a(x).$$

This action is well defined because if $a + \Gamma = b + \Gamma$, their difference belongs to the kernel of F . It is also a free action, since $(a + \Gamma) \cdot x = x$ means F^a acts as the identity on the leaf $\mathcal{F}(x)$, hence $a \in \Gamma$, the isotropy group of $\mathcal{F}(x)$. Finally, the action is proper due to the compactness of \mathbb{T}^l . It follows that the leaf space

$$M_0 := M / \mathbb{T}^l$$

is a $(2n + q - l)$ -dimensional closed manifold M_0 , and $\mathbb{T}^l \hookrightarrow M \xrightarrow{\rho} M_0$ is a principal \mathbb{T}^l -bundle, where $\rho : M \rightarrow M_0$ is the canonical projection.

We define an action of \mathbb{R}^{q-l} on M_0 , via H , by

$$\begin{aligned} F_0 : H \times M_0 &\rightarrow M_0 \\ (\bar{a}, \rho(x)) &\mapsto \rho(F^a(x)). \end{aligned} \tag{1}$$

This does not depend on the first representative a , since $\rho(F^a(x)) = \rho(F^b(x))$ for every $a, b \in \mathbb{R}^q$. It also does not depend on the second representative x . Indeed, if $\rho(x) = \rho(y)$, then $y = F^b(x)$ for some $b \in G$, and consequently $F^a(y) = F^{a+b}(x)$ belong to the same leaf as $F^a(x)$.

It remains to show that F_0 is a contact action. We will achieve this by using integration along the fibres, possibly after choosing a suitable reparameterisation of the action F . Since the bundle \mathcal{R} is trivial, the isomorphism $\mathbb{R}^q \approx G \oplus H$ induces a splitting $\mathcal{R} = \mathcal{G} \oplus \mathcal{H}$. Note that \mathcal{G}_x is composed of the directions tangent to the fibre $\rho^{-1}(\rho(x))$, hence $\mathcal{G}_x = \ker d\rho_x$. Similarly, the tangent space at $\rho(x)$ of the orbit $F_0(H, \rho(x))$ is exactly $\rho_*(\mathcal{H}_x)$, so it is sufficient to show that $\mathcal{H}_0 := \rho_*\mathcal{H}$ can be realised as the Reeb bundle of a $(q-l)$ -contact structure on M_0 .

We begin by decomposing the Reeb fields of $\vec{\lambda}$ as

$$R_i = R_i^{\mathcal{G}} \oplus R_i^{\mathcal{H}}.$$

Using Lemma 2.1 to find a suitable reparameterisation if necessary (recall \mathcal{F} is a transitive foliation), we may assume without loss of generality that

$$\begin{aligned} \mathcal{H} &= \text{Span}\{R_1^{\mathcal{H}}, \dots, R_{k-l}^{\mathcal{H}}\}; \\ \lambda_i(R_i^{\mathcal{H}}) &\neq 0, \text{ for } i = 1, \dots, k-l. \end{aligned}$$

Let us further replace $R_i^{\mathcal{H}}$ by a suitable multiple X_i such that $\lambda_i(X_i) \equiv 1$. In addition, we consider a fibre-wise volume form ω , normalised as to satisfy

$$\int_{\rho^{-1}(y)} \omega = 1,$$

for every $y \in M_0$. We define $k-l$ differential 1-forms on M_0 by $\eta_i := \rho_*(\lambda_i \wedge \omega)$, using the linear morphism $\rho_* : \wedge^*(M) \rightarrow \wedge^{*-l}(M_0)$, determined by ρ via integration along the fibres. To be more precise,

$$\eta_i|_y(Z) = \int_{\rho^{-1}(y)} \iota_{\tilde{Z}}(\lambda_i \wedge \omega),$$

where the RHS is independent of the choice of lifting \tilde{Z} . The forms η_i are non-vanishing since $\eta_i(\rho_*X_i) \equiv 1$ for $i = 1, \dots, q-l$; it is clear from their construction that they are linearly independent forms whose restriction to $\xi_0 := \rho_*\xi$ is identically zero. Finally, the morphism ρ_* commutes with the exterior derivative, from where it follows that

each $d\eta_i$ is non-degenerate on ξ_0 , and has as its kernel the bundle $\rho_*\mathcal{R} = \rho_*\mathcal{H} =: \mathcal{H}_0$. Hence $(M_0, \vec{\eta}, \mathcal{H}_0 \oplus \xi_0)$ is a $(q-l)$ -contact manifold, as we wanted. \square

Theorem 2.1. *A minimal contact foliation (M, \mathcal{F}) is either a minimal foliation by planes, or it reduces to a minimal contact foliation by planes.*

Proof. Since (M, \mathcal{F}) is a minimal contact foliation, all leaves are dense in M , and their isotropy groups are all the same, namely the lattice $\Gamma := \ker F \approx \mathbb{Z}^l$. None of these leaves can be closed, hence $l < q$. If $l = 0$, then M is a foliation by planes. Otherwise, it follows from Proposition 2.6 that $\rho : M \rightarrow M_0$ is a principal bundle over a $(q-l)$ -contact manifold M_0 . We claim that the contact action defined by (1) is a minimal action whose orbits are all planes.

The action is minimal: given $\rho(x), \rho(y) \in M_0$, the leaf $\mathcal{F}(x)$ accumulates on y , hence there is a sequence $a_n \in \mathbb{R}^q$ such that

$$F^{a_n}(x) \rightarrow y.$$

Using the splitting $\mathbb{R} = G \oplus H$ from Proposition 1 above, we write $a_n = a_n^G + a_n^H$, and note that $F^{a_n^G}$ gets arbitrarily close to the identity. We choose a sub-sequence a_{n_j} such that $F^{a_{n_j}^G}(x) \rightarrow x$, from where we conclude

$$F^{a_{n_j}}(x) = F^{a_{n_j}^H} \circ F^{a_{n_j}^G}(x) \rightarrow y,$$

implying

$$F_0(\overline{a_{n_j}}, \rho(x)) = F_0\left(\overline{a_{n_j}}, \rho\left(F^{a_{n_j}^G}(x)\right)\right) = \rho\left(F^{a_{n_j}^H}\left(F^{a_{n_j}^G}(x)\right)\right) \rightarrow \rho(y).$$

Therefore, $\mathcal{F}_0(\rho(x))$ accumulates on $\rho(y)$, and every orbit of the action F_0 is dense.

The orbits are planes: this is simply a matter of noticing that F_0 is free. If $\rho(x) = F_0(\bar{a}, \rho(x)) := \rho(F^{\bar{a}}(x))$, then there is $b \in G$ such that

$$F^{\bar{a}}(x) = (b + \Gamma) \cdot x = F^b(x).$$

Thus \bar{a} acts like an element of G , representing the identity on H . In particular, minimality implies that the isotropy group of any leaf is $\ker F_0 = \{0\}$. Hence every leaf is a plane. \square

Corollary 2.1. *If the WGWC is true, then there can be no minimal contact foliations.*

2.4 Characteristic Foliations and transversals

In this subsection, we look briefly into the holonomy of a contact foliation. Moreover, we show that each 1-form λ_i can be restricted to a contact form on suitable submanifolds of M , which will be given as transversals to a characteristic foliation \mathcal{F}_i associated to λ_i . These first simple properties can be summed up in the following Proposition.

Proposition 2.7. *Let M be a q -contact manifold.*

- (i) The holonomy pseudogroup \mathcal{H} of the foliation \mathcal{F} tangent to the orbits of the action consists of symplectomorphisms.
- (ii) The holonomy pseudogroup \mathcal{H}_i of each characteristic foliation \mathcal{F}_i consists of contactomorphisms.

In particular, all the holonomy maps above are volume-preserving transformations.

We begin by constructing the foliations \mathcal{F}_i .

Definition 2.6 (*Characteristic distribution*). The **characteristic distribution** of λ_i is the distribution

$$\begin{aligned}\mathcal{C}_i(p) &:= \{X_p \in T_p M; \lambda_i(X_p) = 0 \text{ and } \iota_{X_p} d\lambda_i = 0\} \\ &= \ker \lambda_i|_p \cap \mathcal{R}_p \\ &= \text{Span}\{R_1|_p, \dots, \widehat{R_i|_p}, \dots, R_q|_p\}.\end{aligned}$$

Note that this distribution has a constant rank equal to $q - 1$ (in particular, they are trivial for contact manifolds). More than that, if the splitting $\mathcal{R} \oplus \xi$ is C^k , so are the characteristic distributions \mathcal{C}_i . Because the Reeb fields are pairwise commutative, the characteristic distributions are all integrable. In fact, the characteristic foliations are the orbit foliations of the \mathbb{R}^{q-1} “sub-actions” $F_i : \mathbb{R}^{q-1} \rightarrow \text{Diff}(M)$ defined by the flow of the Reeb fields $\{R_j\}_{j \neq i}$.

The underlying foliation \mathcal{F}_i associated with the characteristic distribution \mathcal{C}_i is called the **characteristic foliation of λ_i** .

Due to the nature of the q -contact action, the foliation \mathcal{F} admits no closed transversal.

Proposition 2.8. *There is no $2n$ -dimensional closed submanifold of M everywhere transverse to \mathcal{F} .*

Proof. Indeed, if N was one such manifold, then the exact $2n$ -form

$$\iota_{R_1}(\iota_{R_2} \cdots \iota_{R_q}(dM_i)) \cdots) = (d\lambda_i)^n = d(\lambda_i \wedge (d\lambda_i)^{n-1})$$

would be an exact volume form on N , contradicting Stokes’s theorem. \square

In particular, there are no complete transversals, so no contact foliation is equivalent to a suspension. On the other hand, in general, the characteristic foliations admit transverse submanifolds, and each of these transversals is a contact manifold.

Proposition 2.9. *If N is a transversal of \mathcal{F}_i , then $\lambda_i|_N$ is a contact form on N .*

Proof. The equality $\mathcal{C}_i = \text{Span}\{R_j\}_{j \neq i}$ implies that each leaf of \mathcal{F}_i is transverse to both R_i and ξ . Suppose that N is a transversal of \mathcal{F}_i . Then $\dim N = 2n + 1$ and, as before, the fact that N is transverse to R_j for every j other than i means

$$\iota_{R_1}(\iota_{R_2}(\cdots \widehat{\iota_{R_i}}(\cdots \iota_{R_q}(dM_i)) \cdots)) = \lambda_i \wedge d\lambda_i^n$$

is a volume form on N . Hence $\lambda_i|_N$ is a contact form. \square

Remark 2.6. *The contact structure on N is the intersection bundle*

$$\ker \lambda_i \cap TN = (T\mathcal{F}_i \oplus \xi) \cap TN = \xi \cap TN.$$

We emphasise that by $\lambda_i|_N$ we mean j^λ_i, where $j : N \rightarrow M$ is an embedding. In particular, R_i need not coincide with the Reeb vector field of $\lambda_i|_N$, though they share their closed orbits. Indeed, there is a bundle isomorphism $\Psi : \text{Span}\{R_i\} \oplus \xi \rightarrow TN$ determined by fibre-wise projection. In particular, the image ΨR_i is the unique vector field on N such that $X_i := \Psi R_i - R_i \in TN$. Therefore*

$$\lambda_i(\Psi R_i) = \lambda_i(R_i) + \lambda_i(X_i) = 1$$

and

$$d\lambda_i(\Psi R_i, \cdot) = d\lambda_i(R_i, \cdot) + d\lambda_i(X_i, \cdot) = 0,$$

so that ΨR_i is the Reeb vector field of $\lambda_i|_N$. Moreover, for any closed orbit $\gamma : \mathbb{R} \rightarrow N$ of ΨR_i of period T , we have

$$R_i(\gamma(s)) + X_i(\gamma(s)) = \Psi R_i(\gamma(s)) = \Psi R_i(\gamma(T+s)) = R_i(\gamma(T+s)) + X_i(\gamma(T+s))$$

and therefore

$$R_i(\gamma(s)) - R_i(\gamma(T+s)) = X_i(\gamma(T+s)) - X_i(\gamma(s)),$$

where the LHS belongs to $\text{Span}R_i \oplus \xi$ while the RHS belongs to $T\mathcal{F}_i$. Since these subspaces are complementary, it follows that γ is a closed orbit of period T for both R_i and X_i . In particular, R_i admits a closed orbit in M .

The holonomy maps of \mathcal{F}_i connect different transversal by following the leaves of \mathcal{F}_i , effectively “flowing” along the vector fields R_j , $j \neq i$. As all these fields preserve the q -contact distribution, so do the holonomy maps, implying:

Proposition 2.10. *Let $p \in M$ and $q \in \mathcal{F}_i(p)$, and N_p, N_q be transversals through p and q , respectively. Then there are small transverse neighbourhoods W_p and W_q around p and q such that the holonomy map $h : W_p \rightarrow W_q$ is a contactomorphism.*

Using these same arguments, we see that for any transversal N to the foliation \mathcal{F} , the restriction of $d\lambda_i$ to N is a volume form. Again, flowing along the leaves of the foliation \mathcal{F} also preserves the bundle ξ and the 2-forms $d\lambda_i$, so that we have

Proposition 2.11. *The holonomy maps of a contact foliation are symplectomorphisms.*

The last results are exactly the content of Proposition 2.7.

The fact that transversals of characteristic foliations are contact manifolds can be used to relate the Weinstein Conjecture with its generalised versions. For instance, in the uniform case, one can construct closed transversals using techniques first developed by Tischler in [29].

Proposition 2.12. *Let (M, \mathcal{F}) be a closed q -contact manifold with adapted coframe $\vec{\lambda}$. If there is i, j such that*

$$d\lambda_i = d\lambda_j,$$

then M is a fibre bundle over S^1 . Furthermore, if, for any choice of indices $i_1, \dots, i_l \subset \{1, \dots, q\}$, one has $d\lambda_{i_1} = \dots = d\lambda_{i_l}$, then $\dim H^1(M; \mathbb{R}) \geq l - 1$.

Proof. The 1-form $\alpha := \lambda_i - \lambda_j$ is closed and nowhere vanishing. Hence Tischler's theorem [29] proves the first assertion. For the second one, note that the forms $\alpha_j := \lambda_{i_1} - \lambda_{i_j}$ comprise a l.i. collection of closed forms representing non-trivial cohomology classes. Indeed, the forms α_j can not be exact, since any function $f : M \rightarrow \mathbb{R}$ has a critical point x , which would imply $df_x = \alpha_j|_x \equiv 0$, which can not happen as $\alpha_j(R_{i_1}) \equiv 1$. \square

Corollary 2.2. *A contact foliation defined by $\vec{\lambda} = (\lambda_1, \dots, \lambda_q)$ on a sphere S^{2n+q} can not have $d\lambda_{i_1} = d\lambda_{i_2}$ for more than one pair of indices i_1, i_2 . In particular, for $q > 2$, spheres admit no uniform q -contact structure.*

We can improve Proposition 2.12 in the case of uniform q -contact structures, obtaining M as a fibration over the torus \mathbb{T}^{q-1} .

Theorem 2.2. *Let $(M, \vec{\lambda}, \mathcal{R} \oplus \xi)$ be a closed uniform q -contact manifold, $q \geq 2$. Then M fibres over \mathbb{T}^{q-1} . More specifically, for each characteristic foliation \mathcal{F}_i there is a locally trivial fibration $\rho_i : M \rightarrow \mathbb{T}^{q-1}$ such that $(M, \mathcal{F}_i, \rho_i)$ is a foliated bundle. In other words, the foliation \mathcal{F}_i is the suspension of an action of \mathbb{Z}^{q-1} on a closed contact manifold N_0 whose dimension is $2n + 1$.*

Remark 2.7. *We agreed that our manifolds be at least of class C^1 , and so are the fibrations ρ_i . If M is C^2 or higher, ρ_i can be taken to be at least C^2 .*

Proof. We follow the steps of [29, Corollary II]. For simplicity, let us work with the characteristic foliation \mathcal{F}_q , associated with λ_q , noting that this choice implies no loss of generality. We define

$$\alpha_i := \lambda_i - \lambda_q, \quad i = 1, \dots, q - 1.$$

As in Proposition 2.12, these are linearly independent, non-vanishing closed forms. Moreover, α_i is a leaf-wise volume form for the foliation \mathcal{L}_i defined by the flow of R_i , since $\alpha_i(R_i) \equiv 1$. In other words, the closed form α_i is *transverse* to \mathcal{L}_i (cf. [30]).

First, let us show that one can perturb the 1-forms α_i without losing linear independence. Given a finite atlas for M , we can define suitable C^r norms on $\wedge^1(M)$ as the maximum over all charts of the sum of C^r norms of coordinate functions. We consider on $\wedge^1(M)$ the topology obtained as direct limit of the C^r norms, and denote by \mathcal{D}_1 the corresponding topological \mathbb{R} -vector space (cf. [31]). Due to the linear independence of $\{\alpha_i\}$, we can use the Hahn-Banach theorem to find for each α_i a corresponding continuous dual (a *current*) $f_i : \mathcal{D}_1 \rightarrow \mathbb{R}$ in the strong dual space \mathcal{D}_1^* , satisfying

$$f_i(\alpha_j) = \delta_{ij}.$$

We define a mapping from the $(q - 1)$ -fold product of \mathcal{D}_1 with itself to \mathbb{R} by

$$\begin{aligned} \Phi : \mathcal{D}_1 \times \dots \times \mathcal{D}_1 &\rightarrow \mathbb{R} \\ (\omega_1, \dots, \omega_{q-1}) &\mapsto \det\{f_i(\omega_j)\}. \end{aligned}$$

This mapping is continuous, and since $\Phi(\alpha_1, \dots, \alpha_{q-1}) = 1$, we can find an open neighbourhood U of $(\alpha_1, \dots, \alpha_{q-1})$ on which Φ is always positive. This means the matrix $\{f_i(\omega_j)\}$ is invertible for any choice of $(\omega_1, \dots, \omega_{q-1}) \in U$, and consequently the set $\{\omega_1, \dots, \omega_{q-1}\}$ is linearly independent for every element of U .

Now, each α_i can be arbitrarily well approximated by a C^2 closed 1-form ω_i with the properties:

- (i) the foliation \mathcal{N}_i integral to the bundle $\ker \omega_i$ comprises the fibres of a C^2 fibration $\rho_i : M \rightarrow S^1$, that is, $\mathcal{N}_i(p) = \rho_i^{-1}(\rho(p))$;
- (ii) the leaves of \mathcal{N}_i are transverse to \mathcal{L}_i .

See, for instance, [29, Theorem 1] and [31, Theorem 9.4.2 and Proposition 10.3.14]) for better exposition of these results. Here we limit ourselves to emphasise that our contact foliations are always of class at least C^1 so that all such theorems can be applied to our setting.

Since the linear independence of a set of $(q-1)$ 1-forms is an open property, we can choose the ω_i to be linearly independent. We set

$$\begin{aligned} \rho : M &\rightarrow \mathbb{T}^{q-1} \approx S^1 \times \dots \times S^1 \\ x &\mapsto (\rho_1(x), \dots, \rho_{q-1}(x)). \end{aligned}$$

The mapping ρ is a C^2 submersion since the ω_i are linearly independent; moreover, it is proper, being a map between compact manifolds. Ehresmann Fibration Theorem shows that ρ is a (locally trivial) fibration. Each fibre $\mathcal{N}(p)$ of this fibration is an intersection

$$\mathcal{N}(p) := \bigcap_{i=1}^{q-1} \mathcal{N}_i(p).$$

By construction, the fibres are transverse to each of the Reeb fields R_1, \dots, R_{q-1} , and consequently to the characteristic foliation \mathcal{F}_q . Since the fibres are compact, it follows that the restriction of ρ to each leaf of \mathcal{F}_q is a covering map of the torus, and that \mathcal{F}_i is the suspension of an action of $\pi_1(\mathbb{T}^{q-1}) \approx \mathbb{Z}^{q-1}$ on the typical fibre N_0 (cf. [32, Chapter V]). Moreover, since M is boundaryless, so is the fibre N_0 . As in Proposition 2.9, the restriction of λ_q to N_0 induces a contact structure on the closed $(2n+1)$ -manifold N_0 . \square

Corollary 2.3. *If $(M, \vec{\lambda}, \mathcal{R} \oplus \xi)$ is a closed C^1 uniform q -contact manifold and $q \geq 2$, then every characteristic foliation \mathcal{F}_i admits a complete closed transversal.*

Remark 2.8. *Theorem 2.2 is somewhat similar in taste to the characterisation of metric f -K-structures provided by Goertsches and Loiudice in [33, Theorem 4.4]. Note that a metric f -K-structures induces a uniform q -contact structure for which all the Reeb fields are Killing for some metric. We will deal with such structures in more detail in Section 3.3.*

In light of Proposition 2.9, we have the following equivalence of Weinstein Conjectures:

Theorem 2.3. *For closed uniform q -contact manifolds, the Weak Generalised Weinstein Conjecture is equivalent to the Weinstein Conjecture.*

Proof. The implication $\text{WGWC} \implies \text{WC}$ is evident since a flow line that is not homeomorphic to \mathbb{R} has to be closed. Conversely, if the Weinstein Conjecture is true and T_i is a closed transversal to \mathcal{F}_i , then $(T_i, \lambda_i|_{T_i})$ has a closed Reeb orbit, being a closed contact manifold (cf. Proposition 2.9). This, in turn, implies the existence of a closed orbit γ for \mathcal{L}_i , as pointed out in Remark 2.6. The leaf of \mathcal{F} containing the closed orbit γ is not a plane. Hence M satisfy the WGWC. \square

Since the WC holds in dimension 3 [4], we have

Corollary 2.4. *Every Reeb field of a codimension 2 uniform contact foliation on a closed manifold has a closed orbit. In particular, every such contact foliation satisfies the WGWC.*

Another consequence of Corollary 2.3 is the following, a direct application of a famous result of Plante [34, Theorem 1.1].

Corollary 2.5. *If a closed manifold M admits a uniform q -contact structure, then $H_{2n}(M; \mathbb{R})$ is non-zero.*

Recall that an overtwisted contact 3-manifold is one which contains an embedded overtwisted disk. In higher dimensions, the role of the overtwisted disk can be played by a *Plastikstufe* [35], or, more generally, by overtwisted $2n$ -closed balls [7]. In any case, it is known that such contact manifolds satisfy the WC [6, 7]. If we define a q -contact structure to be overtwisted when it “contains” an overtwisted contact structure in the classical sense, we can conclude that all such structures satisfy the WGWC. More precisely:

Definition 2.7 (*Overtwisted q -contact structures*). *A q -contact structure is **overtwisted** when, for any of the defining 1-forms λ_i , the characteristic foliation \mathcal{F}_i admits a transversal T such that (T, λ_i) is an overtwisted contact manifold.*

From Corollary 2.3 and Theorem 2.3 it follows:

Theorem 2.4. *Every closed uniform overtwisted q -contact manifold satisfies the WGWC.*

2.5 Tautness

This section shows that contact foliations satisfy a particular rigidity property. Namely, they are taut in the sense of Sullivan. We recall the concept of tautness for foliations of arbitrary codimension.

Definition 2.8 (*Harmonic foliations*). *A foliation is **geometrically taut** (also called **harmonic**) if there is a Riemannian metric on the ambient manifold for which every leaf is a minimal submanifold.*

It is a simple calculation to check that the characteristic form λ satisfies

$$d\lambda(X_1, \dots, X_{q+1}) \equiv 0$$

whenever at least q of the fields X_1, \dots, X_{q+1} are tangent to \mathcal{R} . In other words, λ is \mathcal{F} -closed. Hence, by Rummier’s criterion [36], \mathcal{F} is taut, and the 2-tensor on \mathcal{R}

$$g^\tau = \sum_{i=1}^q \lambda_i \otimes \lambda_i$$

is a leaf-wise Riemannian metric with respect to which every leaf of \mathcal{F} is a minimal submanifold (has 0 mean normal curvature). Adapting the calculations from [31, Lemma 10.5.6] we obtain:

Proposition 2.13. *There is a Riemannian metric on M such that any submanifold of dimension $2 \leq d \leq q$ realisable as an orbit of the action of \mathbb{R}^d induced by Reeb vector fields R_{i_1}, \dots, R_{i_d} is a minimal submanifold.*

The metric, in this case, consists of the sum of g^τ with any metric g^\perp on ξ . Since minimal curves are geodesics, the case $d = 1$ says that the integral curves of R_i , when parameterised by arclength, are geodesics of g .

Corollary 2.6. *There is a metric on M such that each Reeb vector field is of unit length, their flow lines are geodesics, and the foliations \mathcal{F} and \mathcal{F}_i are harmonic foliations.*

There are several links between harmonic foliations, harmonic functions, and harmonic differential forms (cf. [37]). We are particularly interested in the following improved version of Proposition 2.3.

Proposition 2.14. *Around each point $p \in M$ one can find coordinates $(x, y, z) \in \mathbb{R}^{2n} \times \mathbb{R}^q$ and, for each $1 \leq i \leq q$ and $1 \leq j \leq n$, functions $f_j^i, g_j^i : \mathbb{R}^{2n} \rightarrow \mathbb{R}$ such that:*

- (i) $\lambda_i = dz_i + \sum_j (-f_j^i(x, y)dx_j + g_j^i(x, y)dy_j)$ for every i ;
- (ii) each coordinate function z_i and differential form dz_i is harmonic.

Proof. Item (i) is just Proposition 2.3. Given $p \in (M, g)$, let U be an open geodesic ball of radius ϵ around p , and choose a foliated chart $\psi : U \rightarrow \mathbb{R}^{2n} \times \mathbb{R}^q$ such that $R_i = \partial_{z_i}$ and $\psi(p) = 0$. Then the inclusion

$$t \mapsto (0, \dots, 0, z_j(t), 0, \dots, 0)$$

is an arclength parameterisation of the integral of R_j through the point p , that is, a geodesic of g . In particular, the function z_j is a Riemannian immersion. Its image is a minimal submanifold, hence z_j is harmonic (cf. [38, Paragraph §2]), and therefore so is its derivative dz_j (see also [38, Paragraph §3]). This proves (ii). \square

3 Partial results on the existence of compact leaves

3.1 Anosov Contact Foliation

We briefly recall what it means for an action of \mathbb{R}^q to be Anosov.

Definition 3.1 (*Contact Elements and Anosov Actions*). *Consider an C^2 action $\phi : \mathbb{R}^q \times M \rightarrow M$ of \mathbb{R}^q on a closed Riemannian manifold (M, g) . A point $a \in \mathbb{R}^q$ is said to be an **Anosov element of the action** if $f = \phi^a := \phi(a, \cdot)$ acts **normally hyperbolically** on M , meaning that there are constants $A, C > 0$ and a df -invariant splitting*

$$TM = E_a^{ss} \oplus T\phi \oplus E_a^{uu}$$

of the tangent bundle of M such that:

- (i) $\| (df|_{E_a^{ss}})^n \| \leq Ce^{-An}$, for all $n > 0$;
- (ii) $\| (df|_{E_a^{uu}})^n \| \leq Ce^{An}$, for all $n < 0$.

The action ϕ is called an **Anosov action of \mathbb{R}^q** if it has an Anosov element $a \in \mathbb{R}^q$.

Anosov elements do not form a discrete set. Each connected component of the set $\mathcal{A}(\phi)$ of all Anosov elements of the action ϕ is an open cone in \mathbb{R}^q , called a **chamber** [12]. An open convex subcone in a chamber is called a **regular subcone of the action ϕ** .

Given subsets $K \subset M$, $E \subset \mathbb{R}^q$, we say K is E -transitive if N is ϕ -invariant and there is a point x in K such that

$$\overline{\{\phi(v, x); v \in E\}} = K,$$

that is, the invariant set K contains a dense E -orbit.

Proposition 3.1 (Spectral Decomposition [12]). *Let M be a closed smooth manifold and ϕ be an Anosov action on M . Then there is a finite collection $\{\Lambda_i\}_{i=1}^l$ of pairwise disjoint, locally connected, closed ϕ -invariant subsets of M such that*

- (i) *For every $1 \leq i \leq l$, the union of closed orbits of ϕ inside Λ_i is dense in Λ_i ;*
- (ii) *For any regular subcone \mathcal{C} the set Λ_i is \mathcal{C} -transitive;*
- (iii) *the non-wandering set of M decomposes*

$$\text{nw}(\phi) = \bigcup_{i=1}^l \Lambda_i.$$

Theorem 3.1. *If a contact action $F : \mathbb{R}^q \rightarrow \text{Diff}(M)$ is Anosov, then the underlying foliation satisfies the SGWC.*

Proof. In light of items (i) and (iii) of the Spectral Decomposition 3.1, it is sufficient to show that $\text{nw}(F) \neq \emptyset$. To that end, given a point $x \in M$ and an open set U containing x , we consider an element $a \in \mathbb{R}^q$ with $|a| > 1$. The transformation $f = F^a$ is volume preserving (it preserves the volume forms dM_i). Hence, by Poincaré's recurrence theorem, there exists a positive natural number j such that $f^j(U) \cap U \neq \emptyset$. But $f^j = F^{ja}$, where, by construction, $|ja| > 1$, which means exactly that x is a non-wandering point. Thus $\text{nw}(F) = M$, and the theorem follows. \square

3.2 Compatible Toric Actions

In this section, we define a particular class of toric actions on a q -contact manifold. Adapting arguments from Banyaga and Rukimbira [9], we show that contact manifolds supporting such actions satisfy the WGWC. In the next section, we will show that every Riemannian contact foliation on a compact manifold supports an compatible toric action of the same dimension.

Definition 3.2 (*Compatible toric action*). *Let $(M, \vec{\lambda}, \mathcal{R} \oplus \xi)$ be a contact foliation, and $\mathbb{T}^l \rightarrow \text{Diff}(M)$ be a locally free action of the torus \mathbb{T}^l on M . We will say this action is **compatible** with the contact foliation \mathcal{F} if it preserves the non-degenerate forms $d\lambda_i$. More precisely, let X_1, \dots, X_l be infinitesimal generators of the toric action. The action is compatible if*

$$\mathcal{L}_{X_i} d\lambda_j = 0 \quad \text{for any } i, j.$$

Compatible toric actions are helpful when searching for closed orbits because whenever a generator X_i is tangent to a leaf $\mathcal{F}(x)$, the leaf in question is not a plane. **Proposition 3.2.** *Let $(M, \vec{\lambda}, \mathcal{R} \oplus \xi)$ be a closed q -contact manifold, and $\mathbb{T}^s \rightarrow \text{Diff}(M)$ be a compatible action with generators X_1, \dots, X_s . Then each generator is tangent to \mathcal{R} in at least two points.*

Proof. We consider on M a q -form $\bar{\lambda}$, obtained from the leaf-wise volume element $\lambda := \lambda_1 \wedge \dots \wedge \lambda_q$ by averaging it with respect to a Haar measure μ on \mathbb{T}^s , in the following sense:

$$\bar{\lambda}_p(Y_1, \dots, Y_q) := \int_{g \in \mathbb{T}^s} g^* \lambda_p(Y_1, \dots, Y_q) d\mu(g).$$

Note that g is being used to write both the element of \mathbb{T}^l and the diffeomorphism of M it represents. Due to the invariance of μ under multiplication on \mathbb{T}^s , the form $\bar{\lambda}$ is invariant under the action of \mathbb{T}^s . In particular, for any $i \leq q$, any $l \leq s$ and any set of vector fields Y_1, \dots, Y_l we have

$$\mathcal{L}_{X_i}(\iota_{Y_1} \dots \iota_{Y_l} \bar{\lambda}) = 0.$$

We also define $\bar{\lambda}_i$ by

$$\bar{\lambda}_i|_p(Y) := \int_{g \in \mathbb{T}^q} g^* \lambda_i|_p(Y) d\mu(g).$$

We consider, for each $i \leq s$ and $j \leq q$, the functions

$$\phi_{ij}(p) := (-1) \iota_{X_i} \bar{\lambda}_j|_p,$$

Note that since $g^* X_i = X_i$ for any $g \in \mathbb{T}^q$ and generator X_i , these functions are constant along the orbits of the toric action. We claim that

$$d\phi_{ij} = \iota_{X_i} d\lambda_j, \tag{2}$$

To see this, begin defining auxiliary forms $\alpha_j = \lambda_j - \bar{\lambda}_j$, and notice that since the action preserves $d\lambda_j$, the forms α_j is closed. Together with $\mathcal{L}_{X_i} \bar{\lambda}$, the closeness of α_j yields, for any generator X_i :

$$\iota_{X_i} d\lambda_j + d\iota_{X_i} \lambda_j = \mathcal{L}_{X_i} \lambda_j = \mathcal{L}_{X_i} \alpha_j + \mathcal{L}_{X_i} \bar{\lambda}_j = d\iota_{X_i} \alpha_j = d\iota_{X_i} \lambda_j - d\iota_{X_i} \bar{\lambda}_j,$$

that is

$$d\phi_{ij} = -d\iota_{X_i} \bar{\lambda}_j = \iota_{X_i} d\lambda_j, \quad i, j = 1, \dots, q,$$

which proves 2. Moreover, for any i, j, l , the relation $\iota_{[R_i, X_j]} = [\mathcal{L}_{R_i}, \iota_{X_j}]$, together with Equation 2, yields

$$\iota_{[R_i, X_j]} d\lambda_l = \mathcal{L}_{R_i} \iota_{X_j} d\lambda_l$$

$$\begin{aligned}
&= \iota_{R_i} d(\iota_{X_j} d\lambda_l) + d(\iota_{R_i} \iota_{X_j} d\lambda_l) \\
&= \iota_{R_i} dd\phi_{ij} \\
&= 0.
\end{aligned}$$

Hence $[R_i, X_j] \in \ker d\lambda_l = \mathcal{R}$, that is, the generators X_i are *foliate vector fields* of \mathcal{F} . In particular, their flows preserve the leaves of \mathcal{F} . Finally, consider the function ϕ_{ij} . Since M is closed, ϕ_{ij} has at least two critical points, where ϕ_{ij} attains its maximum and minimum, respectively. At a critical point p , one has

$$0 = d\phi_{ij}|_p = d\lambda_j|_p(X_i|_p, \cdot) \iff X_i|_p \in \mathcal{R}_p,$$

that is, the critical points of ϕ_{ij} are exactly the points where X_i is tangent to \mathcal{R} . \square

We can improve this and show that for such a critical point p , the vector field X_i is everywhere tangent to $\mathcal{F}(p)$ and, moreover, that $\mathcal{F}(p)$ is not a plane. In order to do that, we need the following fact.

Lemma 3.1. *There exists no locally free actions of S^1 on the Euclidean space \mathbb{R}^q .*

Proof. It is known that any homeomorphism $\phi : \mathbb{R}^q \rightarrow \mathbb{R}^q$ satisfying the relation $\phi^n = \text{id}$ for some n has at least one fixed point (cf. Theorem 13.1, Appendix B, in [39]). Consequently, there are no free actions of finite groups on \mathbb{R}^q . Now, if S^1 were to act locally freely on \mathbb{R}^q , then for some large enough prime number p the element $e^{2\pi i/p} \in S^1$ would act on \mathbb{R}^q without fixed points. But then the finite subgroup $\langle e^{2\pi i/p} \rangle \approx \mathbb{Z}_p$ would act freely on \mathbb{R}^q , which can not happen. \square

Theorem 3.2. *If a closed q -contact manifold admits a compatible toric action, then the contact foliation can not be a foliation by planes. In other words, closed q -contact manifolds supporting compatible toric actions satisfy the WGWC.*

Proof. As we saw in the last proposition, the generating fields X_i are foliate. Recall that the set of foliate vector fields is the normaliser

$$\mathfrak{L}(\mathcal{F}) := \{X \in \Gamma(M); [X, Y] \in \Gamma(\mathcal{F}) \text{ for all } Y \in \Gamma(\mathcal{F})\}$$

of the Lie algebra $\Gamma(\mathcal{F})$ of vector fields tangent to \mathcal{F} , that is, sections of \mathcal{R} . The Lie algebra of sections of \mathcal{R} is clearly a sub-algebra of $\mathfrak{L}(\mathcal{F})$, and the quotient

$$\mathfrak{t}(\mathcal{F}) := \mathfrak{L}(\mathcal{F}) / \Gamma(\mathcal{F})$$

is the Lie algebra of *transverse vector fields*. We denote by \overline{X} the image of a field X under the projection $\mathfrak{L}(\mathcal{F}) \rightarrow \mathfrak{t}(\mathcal{F})$. Each transverse field acts on the leaf space M/\mathcal{F} via its flow. According to Proposition 3.2 above, if p is a critical point of the function ϕ_{ij} , then the generating field X_i is tangent to \mathcal{F} at p . Consequently, $\overline{X_i}$ is zero at the point (that is, the *leaf*) $\mathcal{F}(p)$.

On the other hand, the flow of a foliate field preserves the foliation, that is, it sends leaf into leaf. Hence, the fact that $\overline{X_i}$ is zero at $\mathcal{F}(p)$ means that X_i is tangent to $\mathcal{F}(p)$

at every point. In particular, the flow of X_i preserves the entire leaf $\mathcal{F}(p)$, and can be restricted to a flow on the leaf.

Finally, since we can choose the generators of the toric action to be all periodic fields, the restriction of X_i to $\mathcal{F}(p)$ induces a locally free action of S^1 . Hence, according to Lemma 3.1 above, $\mathcal{F}(p)$ must be a cylinder $\mathbb{R}^{q-l} \times \mathbb{T}^l$ for some $l > 0$. \square

In particular, for $q = 1$ Theorem 3.2 is a result on circle invariant pre-symplectic forms by Banyaga and Rukimbira [9].

3.3 Riemannian Contact Foliations

In this last section, we consider the case of contact foliations preserving a Riemannian metric on the ambient manifold. We show that all such contact foliations support a special compatible toric action, and therefore satisfy the WGWC. More over, the existence of an invariant metric will allow us, by studying the closures of leaves, to conclude that in the class of isometric contact foliations the WGWC implies the SGWC.

Following the work of Rukimbira, (cf. [8, 40]), we define the following notion.

Definition 3.3 (*Isometric contact foliation*). *A foliation \mathcal{F} induced by a q -contact structure is called a **isometric contact foliation** if there is a metric g for which the Reeb vector fields are Killing, that is*

$$\mathcal{L}_{R_i}g = 0.$$

In other words, \mathcal{F} is isometric if it is induced by an action of \mathbb{R}^q through isometries.

There is always a metric on the bundle \mathcal{R} for which the Reeb vector fields are Killing, namely

$$g^\tau = \sum_i \lambda_i \otimes \lambda_i,$$

that is, the metric defined by setting $g^\tau(R_i, R_j) = \delta_{ij}$, and extending it linearly. It satisfies $\mathcal{L}_{R_j}g^\tau = \sum_i (\mathcal{L}_{R_j}\lambda_i \otimes \lambda_i + \lambda_i \otimes \mathcal{L}_{R_j}\lambda_i) = 0$. Therefore, asking for a foliation to be isometric amounts to assert the existence of a transverse invariant metric, that is, a metric g^\perp on the q -contact distribution such that $\mathcal{L}_{R_j}g^\perp = 0$ for every Reeb vector field R_j . Since every transverse invariant metric is associated with a bundle-like metric [41, Proposition 3.3], it follows that every Riemannian contact foliation is, in fact, isometric:

Proposition 3.3. *For a q -contact manifold $(M, \vec{\lambda}, \mathcal{R} \oplus \xi)$ the following are equivalent*

- (i) *The contact foliation \mathcal{F} is isometric;*
- (ii) *There exists a metric g on M with respect to which the Reeb vector fields R_i are Killing, and*

$$\lambda_i(X) = g(R_i, X)$$

for any vector field on M .

Following the terminology of [40], we define

Definition 3.4 (*R-metric*). *On a q -contact manifold $(M, \vec{\lambda}, \mathcal{R} \oplus \xi)$ a metric g satisfying*

- (i) $\mathcal{L}_{R_i}g = 0$;

(ii) $g(R_i, \cdot) = \lambda_i$,

is called an R -metric of the foliation \mathcal{F} .

Remark 3.1. Note that Proposition 3.3 can be restated as "a contact foliation is isometric if and only if it admits an R -metric". From now on, whenever we say the triple (M, \mathcal{F}, g) is an isometric contact foliation, it is implicitly assumed that g is an R -metric.

Example 3.1 (The canonical contact structure of \mathbb{R}^{2n+q}). Consider on \mathbb{R}^{2n+q} the structure of Example 2.2. Let g^\perp be the metric on ξ obtained as the restriction of the canonical metric of \mathbb{R}^{2n+q} . On \mathcal{R} , let $g^\tau = \sum_i \lambda_i \otimes \lambda_i$, as usual. Then the metric g^\perp is invariant under the Reeb vector fields, as the ∂_{z_i} are Killing vector fields of the canonical metric on \mathbb{R}^{2n+q} , and therefore $g := g^\tau \oplus g^\perp$ satisfies

$$\mathcal{L}_{\partial_{z_i}} g = 0,$$

that is, g is an R -metric.

Example 3.2 (Toric extensions). Let $(M, \vec{\lambda}, \mathcal{R} \oplus \xi)$ be a q -contact manifold admitting an R -metric $g = g^\tau \oplus g^\perp$. Suppose $\pi : E \rightarrow M$ is a principal \mathbb{T}^l -bundle equipped with a flat connection $TE = H \oplus V$. As it is shown in detail in Example 2.4, E admits a $(q+l)$ -contact structure determined by 1-forms

$$\begin{aligned} \eta_i &= \pi^* \lambda_i, \quad \text{for } 1 \leq i \leq q; \\ \eta_{q+i} &= \alpha_i + \pi^* \lambda_{j_i}, \quad \text{for } 1 \leq i \leq l, \end{aligned}$$

where the α_i are 1-forms that are identically zero on the horizontal bundle and on the vertical bundle are given by

$$\alpha_i(X_j) = \delta_{ij},$$

with the X_i being the fields generating the toric action on the fibres. The identification of H with TM via π induces a splitting $H = \tilde{\mathcal{R}} \oplus \tilde{\xi}$, where $\mathcal{R} \approx \tilde{\mathcal{R}}$ and $\xi \approx \tilde{\xi}$, both diffeomorphisms being a restriction of $d\pi$. Thus $TE = \tilde{\mathcal{R}} \oplus \tilde{\xi}$, where $\tilde{\mathcal{R}} = V \oplus \mathcal{R}$ is the bundle tangent to the $(q+l)$ -dimensional contact foliation $\tilde{\mathcal{F}}$ on E . The Reeb vector fields are

$$\begin{aligned} \tilde{R}_i &= R_i, \quad \text{for } 1 \leq i \leq q; \\ \tilde{R}_{q+i} &= \frac{1}{2}(X_i + R_{j_i}), \quad \text{for } 1 \leq i \leq l. \end{aligned} \tag{3}$$

We want to show that $(E, \vec{\eta}, TE = \tilde{\mathcal{R}} \oplus \tilde{\xi})$ defines an isometric contact foliation for a suitable choice of metric \tilde{g} on E . For this, let g_0 be a metric on the tangent bundle $\tilde{\mathcal{R}}$ defined by

$$g_0 := \sum_{i=1}^{q+l} \eta_i \otimes \eta_i.$$

This makes $\{\tilde{R}_i\}$ into an orthonormal basis for $\tilde{\mathcal{R}}$. Finally, we let

$$\tilde{g} := g_0 \oplus g^\perp$$

be a Riemannian metric on $TE = \tilde{\mathcal{R}} \oplus \tilde{\xi}$.

We claim that \tilde{g} is an R-metric for E . First, it is clear from the definition of \tilde{g} that

$$\tilde{g}(X, \tilde{R}_i) = g_0(X, \tilde{R}_i) = \eta_i(X). \quad (4)$$

It is straightforward that

$$\left(\mathcal{L}_{\tilde{R}_i} g_0 \right) (\tilde{R}_j, \tilde{R}_l) = \mathcal{L}_{\tilde{R}_i} (\iota_{\tilde{R}_l} \iota_{\tilde{R}_j} g_0) - g_0([\tilde{R}_i, \tilde{R}_j], \tilde{R}_l) - g_0(\tilde{R}_j, [\tilde{R}_i, \tilde{R}_l]) = \mathcal{L}_{\tilde{R}_i} \delta_{ij} = 0,$$

hence $\mathcal{L}_{\tilde{R}_i} g_0 \equiv 0$. Now, from the characterisation of the Reeb fields in 3, we have

$$\begin{aligned} \mathcal{L}_{\tilde{R}_i} \tilde{g} &= \mathcal{L}_{R_i} \tilde{g} = \mathcal{L}_{R_i} g_0 + \mathcal{L}_{R_i} g^\perp = 0, \text{ for } 1 \leq i \leq q; \\ \mathcal{L}_{\tilde{R}_{q+i}} \tilde{g} &= \mathcal{L}_{\tilde{R}_{q+i}} g^\perp = \frac{1}{2} \mathcal{L}_{X_i} g^\perp + \frac{1}{2} \mathcal{L}_{R_i} g^\perp = \frac{1}{2} \mathcal{L}_{X_i} g^\perp, \text{ for } 1 \leq i \leq l. \end{aligned} \quad (5)$$

Recall, from Example 2.4, that for every generator field X_1 it holds that

$$[X_i, Z] \text{ is horizontal whenever } Z \text{ is horizontal.}$$

Moreover, every vector field Z tangent to $\tilde{\xi}$ is foliate. Together, these two conditions imply

$$[X_i, Z] \in \overline{\mathcal{R}} \text{ whenever } Z \in \Gamma(\tilde{\xi}).$$

In particular, for Y, Z tangent to $\tilde{\xi}$ we have

$$\begin{aligned} (\mathcal{L}_{X_i} g^\perp)(Y, Z) &= \mathcal{L}_{X_i} (\iota_Z \iota_Y g^\perp) - g^\perp([X_i, Y], Z) - g^\perp(Y, [X_i, Z]) \\ &= \mathcal{L}_{X_i} (\iota_Z \iota_Y g^\perp) = 0, \end{aligned}$$

as the function $\iota_Z \iota_Y g^\perp$ can be thought as a lift to E of a leaf-wise function of (M, \mathcal{F}) to a leaf-wise function of $(E, \tilde{\mathcal{F}})$. The leaves of $\tilde{\mathcal{F}}$ are all of the form

$$\tilde{\mathcal{F}}(x) \approx \mathbb{T}^l \times \mathcal{F}(\pi(x)),$$

with the X_i being vectors on the \mathbb{T}^l directions, hence $\mathcal{L}_{X_i} (\iota_Z \iota_Y g^\perp) = 0$. Thus, Equation 5 reduces to

$$\mathcal{L}_{\tilde{R}_i} \tilde{g} \equiv 0,$$

which together with 4 means that \tilde{g} is an R-metric for $(E, \vec{\eta}, TE = \tilde{\mathcal{R}} \oplus \tilde{\xi})$, and $\tilde{\mathcal{F}}$ is an isometric contact foliation.

In particular, by taking products of tori with manifolds supporting R-flows, we can produce isometric contact foliations of any dimension $q \geq 2$. We remark that the resulting manifold is closed if the torus is multiplied by a closed manifold. Examples of R-flows include regular and almost regular contact manifolds (the canonical contact structures on odd spheres S^{2n+1} are specific examples of regular contact manifolds, cf. [42, 43] for more details and examples), and also every K-contact manifold. This last

class includes, in particular, compact contact hypersurfaces [44], Kähler manifolds of constant positive holomorphic sectional curvature, and Brieskorn manifolds [45].

The set $\text{Diff}(M)$ is a Lie Group when equipped with the C^1 compact-open topology τ_1 , its Lie Algebra being that of vector fields on M . Our main tool in studying closed orbits for isometric contact foliations will be the following object.

Definition 3.5 (C^1 -enveloping group). *Let $F : \mathbb{R}^q \rightarrow \text{Diff}(M)$ be an action. The C^1 -enveloping group of F is the closure*

$$E_F^1 := \overline{F(\mathbb{R}^q)} = \overline{\text{Span} \left\{ \bigcup_{i=1}^q \bigcup_{t \in \mathbb{R}} \exp tR_i \right\}}.$$

in the Lie group $(\text{Diff}(M), \tau_1)$.

The C^1 -enveloping group acts on the manifold M in a natural way, and its orbits are exactly the closures of the leaves of \mathcal{F} .

Lemma 3.2. *Given a C^1 -equicontinuous action $F : \mathbb{R}^q \rightarrow \text{Diff}(M)$ and $x \in M$, the orbit of x under the action of E_F^1 is exactly $\overline{\mathcal{F}(x)}$.*

For isometric actions, E_F^1 is actually a subset of $\text{Iso}(M, g)$. The latter is a compact Lie group when M is compact, due to the famous Myers-Steenrod theorem [10]. Therefore, for isometric contact foliations E_F^1 is the smallest compact Abelian subgroup containing the image of \mathbb{R}^q , hence $E_F^1 \approx \mathbb{T}^l$ for some $l \geq q$.

We note that the action of E_F^1 preserves the forms $d\lambda_i$: since all the elements in E_F^1 commute, for any element $h \in \mathbb{T}^q$ we have $h \circ (\exp tR_i) = (\exp tR_i) \circ h$. This implies the action preserves the Reeb vector fields, that is, $dh_p(R_i|_p) = R_i|_{h(p)}$. Therefore, the action preserves the adapted coframe:

$$\begin{aligned} h^* \lambda_i|_p(X) &= g_{h(p)}(R_i|_{h(p)}, dh_p(X)) \\ &= h^* g_p(R_i, X) \\ &= \lambda_i|_p(X), \end{aligned}$$

and from the naturality of the pullback operation:

$$h^* d\lambda_i = dh^* \lambda_i = d\lambda_i.$$

We can use the action of E_F^1 to induce compatible toric actions on $(M, \vec{\lambda}, g)$.

Proposition 3.4. *If \mathcal{F} is an isometric contact foliation on a compact manifold M , then \mathcal{F} admits a free compatible action by isometries $\mathbb{T}^q \rightarrow \text{Iso}(M, g)$. Moreover, the action of \mathbb{T}^q and the contact action of \mathbb{R}^q commute with one another.*

Proof. We consider the Lie algebra \mathfrak{e} of the enveloping group $E := E_F^1$. Then \mathfrak{e} is a subalgebra of the family of Killing fields $\mathfrak{iso}(M)$ containing $\Gamma(\mathcal{R})$, the space of fields tangent to the Reeb distribution \mathcal{R} .

For each point $p \in M$, the isotropy subalgebra \mathfrak{e}_p is defined as the set of elements of \mathfrak{e} whose value at p is zero, i.e., the fields in \mathfrak{e} whose flow fixes p . This is exactly the Lie subalgebra associated to the isotropy subgroup $E_p \subset E$ of p . Recall the type of the orbit $E(p) = \overline{\mathcal{F}(p)}$ is the conjugacy class of its isotropy group E_p . Now, E is an

Abelian group and therefore each such class has exactly one element, that is, there is exactly one isotropy subgroup in E for each orbit type. On the other hand, since M is compact, it follows from the Slice Theorem that there are only finitely many orbit types (see Theorem I.2.1 and Proposition I.2.4 in [46], for instance). Consequently, there are only finitely many distinct subalgebras \mathfrak{e}_p .

Note that \mathfrak{e}_p has dimension at most $\dim E - q$ since the Reeb fields never belong to it. This means there are only finitely many isotropy subgroups in E , the codimension of each of them being at least q . Thus, one can find an embedding $f : \mathbb{T}^q \rightarrow E$ avoiding all such subgroups. Such inclusion $\mathbb{T}^q \hookrightarrow E$ induces the desired free compatible action $\mathbb{T}^q \rightarrow \text{Iso}(M, g)$. \square

Remark 3.2. *In particular, since the codimension of each isotropy subgroup is at least q , in the case $E_F^1 \approx \mathbb{T}^q$ all of its orbits have trivial isotropy subgroups, and the action of E_F^1 is already free. In such a scenario, all the orbits of F are closed submanifolds of M .*

As a consequence to Proposition 3.4 and Theorem 3.2 we obtain the following.

Theorem 3.3. *Every isometric contact foliation on a closed manifold satisfies the WGWC.*

Now we generalise a result from [9], proving that an isometric contact action of \mathbb{R}^q must have at least two closed orbits. To that end, we first need the following Proposition, due to Caramello Jr., which we state here for completeness.

Proposition 3.5 (Proposition 3.1 in [47]). *Let (M, \mathcal{F}) be a Riemannian foliation and \bar{X} be a transverse Killing vector field. Then each connect component N of the set of zeros of \bar{X} is an even codimensional closed submanifold of M saturated by the leaves of \mathcal{F} . Moreover, N is horizontally totally geodesic, and if \mathcal{F} is transversely orientable, so is $(N, \mathcal{F}|_N)$.*

This is a generalisation to Riemannian foliations of a result for Riemannian manifolds regarding zero sets of Killing fields, that is, sets of points where the vector field vanishes (cf. [48, Theorem 5.3, Chapter II], and also [11]). We will need the following simple lemma as well.

Lemma 3.3. *Let $(M, \vec{\lambda}, \mathcal{R} \oplus \xi)$ be a contact foliation. If $N \subset M$ is an \mathcal{F} -saturated submanifold of even codimension, then $(N, \vec{\lambda}|_N, \mathcal{R}|_N \oplus \xi \cap TN)$ is a q -contact structure on N .*

Proof. First note that, since N is saturated (that is, $\mathcal{F}(p) \subset N$ for every $p \in N$), we have $\dim N \geq q$, and therefore $\text{codim} N = 2m$ for some $m \leq n$. Moreover, all Reeb vector fields are everywhere tangent to N , hence $\mathcal{R}|_N \subset TN$. One can also check that

$$\ker(\lambda_i|_N) = \ker \lambda_i \cap TN,$$

for any i and, consequently,

$$\bigcap_i \ker(\lambda_i|_N) = \xi \cap TN.$$

Moreover, the derivatives satisfy

$$\ker(d\lambda_i|_N) = \mathcal{R} \cap TN = \mathcal{R}|_N,$$

for whichever i we choose. Hence

$$\begin{aligned} TN &= (TN \cap \mathcal{R}) \oplus (TN \cap \xi) \\ &= \mathcal{R}|_N \oplus (TN \cap \xi) \\ &= \ker(d\lambda_i|_N) \oplus \left(\bigcap_j \ker(\lambda_j|_N) \right), \end{aligned}$$

and therefore $(N, \lambda_1|_N, \dots, \lambda_q|_N, \mathcal{R}|_N \oplus \xi \cap TN)$ is a q -contact structure on N , as claimed. \square

Theorem 3.4. *Every isometric contact foliation \mathcal{F} on a closed manifold M satisfies the SGWC. Moreover, \mathcal{F} has at least 2 closed orbits; and the set $\mathcal{C}_{\mathcal{F}}$ of closed orbits of \mathcal{F} consists of a union*

$$\mathcal{C}_{\mathcal{F}} = \bigcup N_i,$$

where each N_i is an even-codimension totally geodesic closed submanifold of (M, g) carrying a closed contact foliation, that is, one where every leaf is closed.

Proof. Let $\mathbb{T}^q \rightarrow \text{Iso}(M, g)$ be a compatible action by isometries as in Proposition 3.4. As remarked before in Proposition 3.2, since the generators X_1, \dots, X_q of the compatible toric action are foliated vector fields, to find a close orbit, it is sufficient to find a point where all the generators are tangent to \mathcal{R} . The main idea here is that the set of points where these generators are tangent to the foliation can be realised as a zero set of a Killing vector field. Hence it has even codimension and therefore carries a q -contact structure, according to Lemma 3.3.

Given a generator X_i , we denote by \overline{X}_i its image under the projection onto the algebra of transverse fields. By construction, each \overline{X}_i is a *transverse Killing vector field*, that is, one that preserves the metric g^\perp on ξ . Consider the set of zeros of \overline{X}_1 , i.e.,

$$\{p \in M; \overline{X}_1|_p = 0\} \subset M.$$

This is precisely the set of points $p \in M$ where X_1 is tangent to the foliation, and, due to Proposition 3.2, it has at least two elements. Let M_1 be a connected component of this set. It follows from Proposition 3.5 that M_1 is a closed submanifold of even codimension saturated by \mathcal{F} . According to Lemma 3.3, M_1 admits a q -contact structure inducing on M_1 the same foliation as $\mathcal{R}|_{M_1}$, and whose Reeb vector fields are $R_i|_{M_1}$. By construction, X_1 is everywhere tangent to the leaves on M_1 .

Now, since M_1 is closed, the function $\phi_{2j} = -\lambda_j(X_2)$ has a critical point on M_1 , and hence the set

$$\{p \in M_1; \overline{X}_2|_p = 0\}$$

is non-empty. We choose a connected component M_2 on this set. It follows again from proposition 3.5 that M_2 is a closed submanifold of even codimension (both in M and

M_1), which is saturated by \mathcal{F} and on which X_2 restricts to a vector field everywhere tangent to the leaves.

Iterating this procedure, we eventually arrive at an \mathcal{F} -saturated submanifold M_{q-1} , of even codimension, on which X_1, \dots, X_{q-1} are everywhere tangent to the leaves, by construction. On this closed submanifold, the functions $\phi_{q,j}$ have at least two distinct critical points p and q , corresponding to the extrema of the function, so that both $X_q|_p$ and $X_q|_q$ are tangent to \mathcal{R} . By construction, all the X_i are tangent to \mathcal{R} at p and q , hence, by Proposition 3.2, the leaves $\mathcal{F}(p)$ and $\mathcal{F}(q)$ are closed. This proves the subset $\mathcal{C}_{\mathcal{F}}$ of points in M belonging to a closed leaf is non-empty.

To obtain the decomposition of $\mathcal{C}_{\mathcal{F}}$, we simply note that if p belongs to a closed leaf $L \approx \mathbb{T}^q$, then L is fixed by the entire isotropy subgroup E_p . Now, there are only finite many such isotropy groups, say E_1, \dots, E_l . The leaf L is a set of collective zeros of the Killing fields in the correspondent isotropy algebra \mathfrak{e}_j . Hence, it follows from [11, Corollary 1] that L is contained in an even-codimensional totally geodesic submanifold N of M . The finite collection of all these submanifolds gives us the desired partition. Moreover, as pointed out before, $p \in N$ implies $\mathcal{F}(p) \subset N$. Hence each N is a saturated submanifold of even codimension, so that Lemma 3.3 applies, allowing us to conclude that the restriction of $\tilde{\lambda}$ to N gives rise to a contact foliation, the leaves of which are all closed by construction. \square

4 Conclusion

The theory of q -contact structures, as dealt with here, started with Almeida's work [1]. He focused mainly on the contact action, and at that, in the particular case when such action is partially hyperbolic. In this work, we focus instead on the perspective of the *contact foliation*. We dived into the fundamental theory of contact foliations, greatly expanding the foundations laid out by Almeida. In particular, we provided new examples, local descriptions for these foliations (Propositions 2.3 and 2.14), canonical forms for the adapted coframe in particular cases (Propositions 2.4 and 2.5), and pointed out some basic properties of the holonomy groups of various distinct foliations related to a q -contact structure (Proposition 2.7).

Two novel results of ours are the reduction procedure of Theorem 2.6 and the characterisation of uniform contact manifolds in Theorem 2.2. The former is a partial converse to the construction of contact structures in flat toric bundles introduced by Almeida. It allowed us to prove that non-trivial minimal contact foliations can not take place if any of the Weinstein conjectures are valid (cf. Theorem 2.1), a result not all obvious at first sight. The latter imposes substantial restrictions on the existence of uniform q -contact structures for q greater than 2, hinting that such foliations might not occur so frequently. It is also our first result specifically concerning contact *foliations*, in that it does not apply for contact flows, i.e., for the classic case. It is a result nicely related to a Theorem of Goertsches and Loiudice [33] stating that every metric f -K-contact manifold can be constructed from a K-contact manifold utilising mapping tori and the so-called “type II” deformations (cf. [33, Theorem 4.4]).

We also introduced two possible generalisations for the Weinstein conjecture in the context of contact foliation. We were able to show that the strongest of such

generalisations holds true for Anosov and isometric contact actions. In particular, the results in the isometric case expand results previously known in Contact Dynamics literature, due to Rukimbira and Banyaga [8, 9].

In the follow-up paper [49] we investigate the geometrical properties of quasiconformal contact foliations, as well as the existence of closed leaves, effectively weakening the hypothesis of Theorem A. This theory could also be applied to prove classification theorems for quasiconformal Anosov bundles, generalising established results from Fang [50]. Moreover, there are questions regarding the existence of local models for q -contact structures as discussed in Section 2.2, as well as the topology of the ambient manifold. For instance, it is known that K-contact and R-contact manifolds, as well as for Sasakian structures, any harmonic form is orthogonal to the Reeb distribution, which has a number of different consequences in the cohomology and curvature (cf. [42, 44, 51]). It is currently an open question if the same is true of harmonic forms in an isometric contact manifold.

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