Comments/Corrections for "Semigroup congruences: computation techniques and theoretical applications"

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October 24, 2018

Page 19, Definition 1.11: The set X needs to be non-empty in this definition. Otherwise, the least subsemigroup containing X might not exist.

Page 21, Paragraph after 1.19: A monoid homomorphism (that is, a homomorphism in the variety of monoids) must be required to map the identity of the domain to the identity of the codomain. The proof given here that a semigroup homomorphism will also do this is incorrect (and indeed the claimed result is false).

Page 21, Definition 1.20: X should be S in the first line.

Page 23, Line 5: $\subseteq$ is a relation, not an operation.

Page 23, Definition 1.28: This is not the definition of a lattice: It is the definition of a *complete* lattice. The condition for a lattice is that every pair of elements of X has a greatest lower bound and a least upper bound.

Page 27, Second displayed equation: This doesn't really use Lemma 1.37. We discussed this in the viva: It probably follows by explicit use of the original definition (Definition 1.34).

Page 29, Definition 1.47: Woud the following be a more accurate definition?: "A semigroup if **finitely presented** if it *can be* defined by a finite presentation." (i.e., I think of finite presentability as a property of the semigroup, rather than a property of its presentation.)

Page 30, Definition 1.50: You need to assume I is non-empty in this definition.

Page 31, Definition 1.52: S should be S^1 in the first two definition (at the end of both lines). Similarly in the proof of Proposition 1.54, a and b should be from S^1, not necessarily S.

Page 31, Green's relations: Establishing that $\mathscr{D}$ is an equivalence relation requires a bit of work and is probably not obvious. It relies upon the fact that $\mathscr{L} \circ \mathscr{R} = \mathscr{R} \circ \mathscr{L}$, which is 10-line proof in James's lecture notes for MT5823. The fact that $\mathscr{D} \subseteq \mathscr{J}$ is quite easy. I agree that the equality of these two relations in a finite semigroup is not obvious.

Page 33, Figure 1.56: Is there anything to be gained by saying how many points the given transformation semigroup acts upon? Saying that it has cardinality 63 is one thing, but I'd also like to know how many points we are acting upon.

Page 35, Proof of Theorem 1.63, 6th line: This should say: "...it is isomorphic to *a subsemigroup of* $\mathcal{T}_n$."

Page 35, Line 14: I don't think you have defined the term "transformation semigroup." You mean any subsemigroup of the full transformation monoid $\mathcal{T}_n$.

Page 37, Subsection 1.11.3: "Order-preserving" only makes sense if the set X upon which we are acting is ordered. Implicitly here, I think you are assuming that $X = \{1, 2, \ldots, n\} \subseteq \mathbb{N}$ in this subsection (or some assumption along those lines).

Page 38, Line 2: You don't define the term "regular $*$-semigroup". Perhaps you should if it is significant enough to mention here? (The term also appears in Chapter 5, I think.)

Moreoever, the assertion appearing here is repeated on page 40. Why include it twice?

Page 38, Line 15: I don't think this definition for Π correct? I think it should be $(\alpha^{\wedge} \cup \beta^{\vee})^{e}$. (Compare with Example 1.75 and its diagrams. Though in the viva you suggested a different fix. Please fix in some appropriate way.)

In Line 1 of Example 1.75, the first block of α should be $\{1, 1', 2'\}$.

Page 39, Definition 1.77 and Example 1.78: As discussed in the viva, the definition of cokernel in 1.77 does not agree with what is given in Example 1.78. Clarify whether $\operatorname{coker} \alpha$ is a relation on $\mathbf{n}$ or on $\mathbf{n}'$.

Page 40, Line 4 of Example 1.81: What do you mean by "minimal" for this counterexample? Are you claiming that there is no smaller possible counterexample? If so, how do you know this? (Though perhaps that would become clearer, if I knew what "smaller" meant here.)

Page 40, Line 9 of Example 1.81: How do you know these restrictions of f are monomorphisms? Perhaps provide a reference?

Page 40, Line -5: In what sense do (v) follow "naturally" from (iv)? Would not "immediately" be a more precise formulation? ["Naturally" usually has a specific meaning in mathematics, e.g., when we speak of the *natural homomorphism* from a semigroup S to the Rees quotient S/I.]

Page 44ff, Section 1.13: Please explain this section more clearly. I would like to see clearly explained statements of (i) how the data structures are set up (you say τ is a "table" but don't say what a table is); (ii) which steps you precisely take when processing the elements in X and the relations in $\mathbf{R}$.

Can you make direct reference to the algorithms (as given in the pseudocode) when explaining the steps. Describe exactly what the output is or the state of the data structures after running each of these algorithms. What do they do and how do you know they do what you want them to do? How do you know they stop?

As a specific example, describe more clearly what Algorithm 1.92 is doing. It would help the reader to note that having set n equal to τ_n, that τ_n may have a different value (since n has changed) and so we would have to repeat application of line 4 of the algorithm. (Why does this loop terminate? I believe because n descreases throughout the loop.)

Page 51, Definition 2.1: The definition of "concrete" is problematic. It sort of depends on what "known in advance" means.

In reality, "concrete" needs to refer to the way that a semigroup is *represented*, so you probably want to define the term "concrete representation" of a semigroup. Whatever this is, it needs to provide a precise description of what we can assume once we have a concrete representation of a semigroup S. You seem to want to assume that with a concrete representation, you know that S is finite, you can solve the word problem in S in an effective manner, and can perhaps run an algorithm that loops over the elements of S.

This needs to be clarified and made precise. Doing so will have knock-on effects on the mathematics that follow. You will have to chase down everything that depends upon this definition and fix the subsequent algorithms, theorems, etc. (As one example, the second paragraph of Section 2.3, on page 52, does not make sense without the term "concrete" clarified. This is just the first of many paragraphs that will need adjustment.)

Page 51, Line −7ff: To see that not all finitely presented semigroups are concrete, is it not enough to just find an example of a finitely presented semigroup that is infinite; e.g., the monogenic semigroup $\langle x \mid \varnothing \rangle$. One does not need to rely on the undecidable word problem to produce such an example.

Page 53ff: In Section 2.4, you describe what you want to achieve with the algorithms that you present. However, you don't really come back to this after you describe them. As we discussed in the viva, there are consequently problems: the Todd–Coxeter Algorithm, for example, appears to calculate the congruence classes, in the sense that we can determine when two elements belong to the same class. It does not determine the congruence in the sense of computing all the pairs in the relation. As we discussed during the viva, if the semigroup is sufficiently small (possibly "sufficiently concrete"?) then we can, with repeated running of the algorithms, answer some of the questions on page 53. This should be explained more clearly. One must clarify exactly what the Todd–Coxeter Algorithm actually does compute and not claim that it computes something that it does not.

Page 53, Line −3: What do you mean by "class number"?

Page 55, Line −3: I would greatly prefer writing "The Froidure–Pin Algorithm…" rather than "Froidure–Pin…". Yes, we often use the latter informally when giving talks, but in formal written mathematics I think one tends to be a bit more careful. I recommend this be implemented throughout the thesis. (e.g., Page 74, Line 3, "Knuth–Bendix" → "the Knuth–Bendix algorithm"; also Todd–Coxeter in line −2 on Page 76 and in Example 2.49 on Page 77; etc. At this point, I've stopped checking for these.)

All Algorithms: (Those on pages 56 and 57 are examples, but this comments applies to all of them.)

Explain in more detail what the algorithm is doing. Make direct reference to the pseudocode so that one can see that they work. Verify that the algorithms do what you say they should do. Address the issue of termination and correctness for each algorithm.

Running a loop over a set that one changes during the course of the loop is bad form, in my opinion. Can you rephrase the appropriate algorithms (e.g., Algorithms 2.7 and 2.8) so that you don't do this. (You explained in the viva that your actual implementation does not do this, so we know it isn't necessary.)

Page 56, Subsection 2.6.1, Line 1: What is a "pair orbit"? What exactly are you enumerating?

Page 57, Theorem 2.9: In this theorem, make more precise reference to the steps in the Algorithms (e.g., 2.8) when proving the result. (Similarly for all theorems that make claims about what the Algorithms do.)

Page 58, Line −15ff: How do you justify the claimed run-time?

Page 59: Can you clarify what is your work in this chapter? What is new? If I recall, you said that much of what you say about the Todd–Coxeter algorithm is known, but that the "pre-filling the table" is new.

Page 65, Second Paragraph: Is it also worth noting that this is an application of Cayley's Theorem? I presume that is how you conclude S is isomorphic to the semigroup generated by these transformations.

Page 65, Example 2.25: This is wrong. The relation $a^2 = a$ does not hold in this semigroup.

Page 67, Line 17: Very strange hyphenation of "ToddCoxeterTwoSided"!

Page 68, Displayed Equation: Missing brace at the end of the definition of the set.

Page 68, End of page and following subsection: Can you explain what pre-filling the table is doing and what it achieves? In the viva we noted that the second paragraph (lines 11ff) on page 69 is the key. State clearly that you are computing the lines corresponding to the Cayley graph of S first and using that as a start point for then calculating the equivalence classes of our relation.

Page 73, Definition 2.40: The definition needs to be fixed, as we discussed in the viva. We thought that actually conditions (i) and (ii) define what is meant by a critical pair, rather than the second sentence of the definition. If, however, you use that second sentence, then you need to quote a result to justify that all critical pairs arise in the described manner. Please fix appropriately, with perhaps a relevant reference if necessary.

Page 74, Line 3: What does the Corollary 12.21 in [HEO05] say?

Page 74, Line −1: Note that there is a difference between a presentation being finite and the semigroup given by the presentation being finite. To say that the presentation $\langle X \mid R \rangle$ is finite is to require that the set X of generators is finite and the set R of relators is finite. However, I think what you actually want is the semigroup with the given presentation to be finite. Perhaps the sentence should be reworded to read: "...unless the semigroup given by the presentation $\langle X \mid R \rangle$ is known to be finite." (However, perhaps the concept of "concrete" raises its head again at this point.)

Page 92, Definitions 3.3–3.5: These definitions need to be fixed to take into account the trivial semigroup as we discussed in the viva. In particular, should the definition for 0-simple be that the semigroup has precisely two ideals? However, to fully address the issue of whether the trivial semigroup satisfies these definitions might require more attention than just that fix.

Page 93, Paragraph after 3.7: Do we need to worry or care about detection or recognition of a completely 0-simple semigroup? Or indeed about algorithmically converting into Rees 0-matrix semigroup format once we have established that such exists? Should you say something about algorithmic implementations of such things (e.g., that what you need already exists in GAP's relevant package)?

Page 94, Definition 3.10: You should not assume as a hypothesis anything about ρ. Otherwise Γ is not actually a function. Delete the clause "and let ρ be a non-universal congruence on S." Change the next sentence to: The **linked function** of S is defined, for ρ a non-universal congruence on S, by ....

Page 94, Formula for $\mathbf{\Gamma^{-1}}$: Do you need to explicitly verify all the necessary conditions here?

Page 97, Line 10–12: This sentence is too complicated and convoluted. It should be broken up into at least two, possible more sentences, since it is trying to say too many things at once.

Page 102, Line 11: $\mathbf{R}_I$ and $\mathbf{R}_\Lambda$ were actually denoted $\mathbf{R}|_I$ and $\mathbf{R}|_\Lambda$ in Definition 3.19.

Page 102, Line −17: Although it is apparently permitted to sometimes start a sentence with "And", I don't see any reason for doing such an unusual thing here. The sentence would have the same meaning with "and" deleted and would look less strange to the reader. [Similarly on page 133, line 15.]

Page 102, Lines −7ff: Explain how you are viewing the equivalence relation $\mathcal{S}$ as a (complete) graph and hence what you mean by a "minimal spanning tree" here. Clarify what you are achieving here.

Page 105, Subsection 3.2.5, First paragraph: Why is it faster if we know the kernel-trace pair? Can you demonstrate that this is the case?

Page 109, Line −14: You have not defined what it means for an ideal to be normal. We decided in the viva that you probably mean that the congruence τ is normal in the sense of Definition 3.15, rather that I_τ being normal.

Page 109, Line −5: Typo: "…implies ρ_I" should be "…implies $(x, y) \in \rho_I$", I think.

Page 116, Line −13: "center" should be "centre".

Page 116, Line −13: What do you mean by "splitting of a normal subgroup into simple groups"? Fix this to what you actually mean. (Some reference to finding composition factors, maybe?)

Page 117, Algorithm 4.2: Are lines 8–9 necessary? Could you now just replace lines 8–11, by something that says "If $(a, b) \notin (x, y)^\sharp$ then set P equal to $P \cup \{((x, y)^\sharp, (a, b)^\sharp)\}$"?

Page 130, Proof of Lemma 5.9: Correct the second sentence to: "The upper blocks of a product are equal to those of its first component." (Or maybe "…first term" or "…first factor.")

Page 130, Section 5.1, Last Sentence: What is the significance of $\alpha\beta\gamma = \alpha\gamma$? Why do you point it out here?

Page 130, Section 5.2: Are the terms "retractable ideals" and "liftable congruences" new? Or have they already appeared in the literature? Where do these concepts first arise? (Provide a reference?)

Page 131, Definition 5.11: You claim that the three conditions are equivalent, but provide no verification. Show that they are equivalent.

Page 131, Lemma 5.12: The proof of this lemma is incorrect. Specifically, $s\phi$ is not defined for $s \in S \setminus I$, but is explicitly used in the second displayed equation.

Page 133, Proof of Proposition 5.17, Line −3: We don't know that $x \in I_0$. So how is application of Lemma 5.9 valid? There seemed to be two possible fixes that we discussed in the viva. Either observe that Lemma 5.9 only needs $\alpha \in I_0$, not that $\beta \in I_0$, or alternatively apply Lemma 5.9 to the elements a and ax and note that $a = a(xa)a = axa$ (using $a^2 = a$).

Page 133, Corollary 5.13: The proof doesn't seem to work. You have assumed that e_l and e_r are left and right identies for $x\phi$, respectively, but your calculations at the top of page 132 use $x\psi = x\psi \cdot e_r$ and $x\psi = e_l \cdot x\psi$. Potentially you need these elements to be one-sided identities for $x\psi$, but you need to check exactly what you use and how your argument should go in this proof.

Page 134, Proof of Proposition 5.20: In the first paragraph, are the particular properties linking the B_i and C_j? Do you need to know this in order to perform this calculation of $\alpha\beta$? Perhaps clarify that the formula for $\alpha\beta$ that you give, in the first case of β, depends upon whether or not $B_0 \cap C_0$ is empty.

In the second paragraph, why do you refer to the minimal ideal $\mathcal{I}_0$ as being regular? That is not a necessary hypothesis for the application of Corollary 5.13.

Alsi $\mathcal{I}_0$ should be I_0.

Page 135, Formulae at top of page: How did you establish these formulae? It seems to involve interpreting what the equation $\hat{\alpha} = \hat{\beta}$ means. Perhaps provide a proof of Proposition 5.22.

Page 135, Lemma 5.26: Should Δ actually be $\Delta_{\mathcal{M}_n}$? (Similarly on page 137.)

Page 136, Line −10: "Proposition" is mis-typed.

Page 136, Line −2: I find that writing $\gamma = \tau\alpha\gamma \ \sigma \ \tau\beta\gamma$ looks a bit clunky (primarily because σ is a relation whereas the other greek letters represent elements of $\mathcal{M}_n$. I'd be tempted to clean these two sentences into a single one as: "Since σ is left- and right-compatible, we deduce that $\gamma = \tau\alpha\gamma$ is σ-related to $\tau\beta\gamma$.

Page 137, items (b) and (c): What do you mean by "replacing" here? I didn't see what you were saying.

Page 137, Line −12: What are a and b when you define τ_{ab}?

In Line −10, is this the definition of $\tau_{\varnothing}$ (as the bipartition consisting entirely of singletons)?

Page 138, Second line after (e): Strictly speaking this calculation of $\beta\gamma$ only makes sense in Case (a). There is no C_0 in Cases (b)–(e).

Page 142, Lemmas 5.32 and 5.33: I think both these lemmas should be moved earlier in the chapter. This is so that you have already established them at the point where you want to use them. For example, you make use of these lemmas during the proof of Lemma 5.27. One shouldn't really rely on later results until they have been verified (particularly to avoid having circular arguments).

Page 153, Lines −8ff: Insert the missing commas into the sets $\{1, \dots, n\}$ and $\{1, \dots, k\}$. (At least five occurrences need to be fixed in this paragraph.)

Page 154, Lines 2–3: Why only for *some* $k > 1$? Will this not hold for all choices of k? Also can you note that this defines the group G, the index sets I and Λ and the matrix P? Something along the lines "... let $\overline{D_k^n} = \mathcal{M}^0[G; I, \Lambda, P]$ for some some group G, some index sets I and Λ and some matrix P."

Page 154, Line −16: Have you already made the observation that $p_{\lambda i} \neq 0$ if and only if the corresponding $\mathscr{H}$-class is a group? Should you provide a reference to this fact? Perhaps it belongs in the preliminaries chapter?