On derived Galois deformation rings in characteristic 0 and a conjecture of Venkatesh

Douglas Molin

October 7, 2025

Abstract

Let Π be a regular algebraic cuspidal automorphic representation of GL_n over a CM field F with associated characteristic 0 Galois representation ρ. We use the Taylor–Wiles method to define a free action of the derived deformation ring of ρ on the Π-part of the p-adic cohomology of GL_n / F, as predicted by a conjecture of Venkatesh. As part of the proof, we establish an equivalence between the completion of the characteristic p derived deformation ring at a characteristic 0 point and the characteristic 0 derived deformation ring of the corresponding representation. This allows us to avoid restrictive assumptions on the reduction $\bar{\rho}$ and the $\bmod p$ cohomology of GL_n / F.

1 Introduction

Langlands philosophy posits a correspondence between suitable subclasses of automorphic representations and Galois representations. This correspondence is expected to be compatible with p-adic deformations in the sense that there should exist an isomorphism of rings $R \to T$ where R is a deformation ring parametrising p-adic deformations of a Galois representation and T is a p-adic Hecke algebra which acts on p-adic automorphic forms. Restriction of scalars along $R \to T$ defines an action of R on p-adic automorphic forms. This article is concerned with defining a derived enhancement of this action.

To explain the motivation for this question, let us describe what we will mean by p-adic automorphic forms. Fix a prime p, a CM field F and an isomorphism $\iota\colon \mathbb{C} \cong \overline{\mathbb{Q}}_p$. Let $\mathbf{G} = \mathrm{GL}_n / F$ and suppose Π is a regular algebraic cuspidal automorphic representation of $\mathbf{G}(\mathbb{A}_F)$ of weight λ. Fix an open compact subgroup $U \subset \mathbf{G}(\mathbb{A}_F^\infty)$ (assumed small enough) and consider the locally symmetric space Y_U of level U, which is a smooth manifold of dimension $\dim Y_U = \frac{1}{2}[F\colon \mathbb{Q}](n^2-1)$ equipped with a locally constant sheaf of $\mathbb{Q}_p$-vector spaces $\mathcal{V}_\lambda$ determined by the weight λ. The cohomology group $H^*(Y_U, \mathcal{V}_\lambda)$ is a finite-dimensional $\mathbb{Q}_p$-vector space equipped with an action of Hecke operators and serves as our space of p-adic automorphic forms. The system of eigenvalues associated to Π defines a generalised eigenspace

$$H^*(Y_U, \mathcal{V}_\lambda)_\Pi \subset H^*(Y_U, \mathcal{V}_\lambda),$$

which is non-zero in the degrees $[\frac{1}{2}(d-l_0), \frac{1}{2}(d+l_0)]$, where $d = \dim Y_U$ and $l_0 = \frac{1}{2}[F\colon \mathbb{Q}](n-1) \in \mathbb{N}$ is the defect of $\mathbf{G}$. Disregarding a multiplicity factor depending on the prime-to-p level U^p, the dimension in degree $q_0 + i$ is $\binom{l_0}{i}$. This suggestive numerology indicates that the Π-contribution is a free graded module over an exterior algebra $\wedge^* V_\Pi$, where V_Π is a vector space of dimension l_0. A conjecture of Venkatesh (or rather, the p-adic realisation thereof) predicts that this does indeed hold in a natural way, where V_Π is a certain Selmer group defined in terms of the Galois representation attached to Π via the Langlands correspondence.

By work of Harris–Lan–Taylor–Thorne [HLTT16] and Scholze [Sch15], on can attach to Π a continuous Galois representation

$$\rho = \rho_{\Pi,\iota} \colon \Gamma_S \to \mathrm{GL}_n(E).$$

Here, S is a finite set of finite places of F containing the ramified places of Π, $\Gamma_S = \mathrm{Gal}(\overline{F}_S/F)$ is the Galois group of the maximal extension of F which is unramified outside of S and $E/\mathbb{Q}_p$ is a finite extension. For the purpose of this introduction, we will ignore the places in S not above p.

The representation ρ and its Hodge–Tate weights $\mathrm{HT}(\rho)$ define a deformation functor $D_\rho^{\mathrm{ss}} \colon \mathrm{Art}_E \to \mathrm{Set}$ given by

$$B \mapsto \{r_B \colon \Gamma_S \to \mathrm{GL}_n(B) \mid r_B \text{ unramified outside } S,\ \text{potentially semistable},\ \mathrm{HT}(r_B) = \mathrm{HT}(\rho)\}/\sim$$

where Art_E is the category of artinian E-algebras and $\sim$ denotes strict equivalence. When ρ is irreducible, D_ρ^{ss} is pro-represented by a complete Noetherian local E-algebra R_ρ^{ss}.

If we let R_ρ^{ur} denote the unrestricted deformation ring of ρ and $R_{\mathrm{loc}}^{\mathrm{ss}}$ deformation problem without the conditions at $v \mid p$, we can view the spectrum of $R_{\mathbf{S}}$ as an intersection

$$\mathrm{Spec}(R_\rho^{\mathrm{ss}}) \cong \mathrm{Spec}(R_\rho \otimes_{R_\rho^{\mathrm{loc}}} R_\rho^{\mathrm{loc,ss}}),$$

namely the intersection inside local deformations between the loci of global deformations and potentially semistable local deformations with preserved Hodge–Tate weights. The existence of ρ attached to Π determines a ring homomorphism

$$R_\rho^{\mathrm{ss}} \to T_\lambda$$

where T_λ is a p-adic Hecke algebra which acts on the Π-part $H^*(Y_U, \mathcal{V}_\lambda)_\Pi$.

The tangent space of R_ρ^{ss} is given by the geometric Bloch–Kato Selmer group $H^1_g(\Gamma_S, \mathrm{ad}\,\rho)$, which in our case is known to vanish (as the Bloch–Kato conjecture predicts) by the main result of [A'C24]. In other words, $\mathrm{Spec}\, R_\rho^{\mathrm{ss}}$ is a point. However, tangent-obstruction theory predicts the dimension of R_ρ^{ss} to be given by the Euler characteristic, which comes out to $-l_0$. This suggests that the intersection above is non-transverse and that one ought to look for a derived version of R_ρ^{ss} which accounts for 'redundant' obstructions.

Such a derived enhancement of classical deformation rings was introduced for $\mathrm{mod}\, p$ representations by Galatius–Venkatesh [GV18], and for characteristic 0 representations a construction has been given by Zhu [Zhu21]. In both setups, derived deformation rings are simplicial (or animated) rings whose underlying static ring is canonically isomorphic to a classical deformation ring. For example, we can define a derived version of the functor D_ρ^{ss} above and obtain a *derived* deformation ring $\mathcal{R}_\rho^{\mathrm{ss}}$ such that

$$\pi_0(\mathcal{R}_\rho^{\mathrm{ss}}) \cong R_\rho^{\mathrm{ss}}.$$

The positive degree part of the graded ring $\pi_*(\mathcal{R}_\rho^{\mathrm{ss}})$ encodes derived structure which is hidden from view in the classical setup.

Thus, to explain the dimension formula for the Π-contribution, we have two goals: define a free graded action

$$\pi_*(\mathcal{R}_\rho^{\mathrm{ss}}) \circlearrowright H^*(X_U, \mathcal{L}_\lambda)_\Pi, \tag{$\star$}$$

and prove that $\pi_*(\mathcal{R}_\rho^{\mathrm{ss}})$ is an exterior algebra on a vector space V_Π of dimension l_0, a natural candidate being the dual adjoint Selmer group $H^1_g(\Gamma_S, \operatorname{ad}\rho(1))$. Without further ado, let us state a compact version of our main result.

Theorem 1.1 (Theorem 4.1)**.** *With notation as above, and under various assumptions on F and ρ, there exists a free graded action of $\pi_*(\mathcal{R}_{\mathbf{S}})$ on $H^*(Y_U, \mathcal{V}_\lambda)_\Pi$.*

In [GV18], such an action is defined *integrally* in the Fontaine–Laffaille setting and assuming the Calegari–Geraghty vanishing conjecture for $\bmod p$ cohomology. Inverting p then yields the desired action. However, beyond the Fontaine-Laffaille setting, the naive generalisation of the integral result is not expected to hold in general.

Our proof relies on a Taylor–Wiles argument, which means we will be interested in The proof of the main theorem relies on the following theorem which relates the de To analyse this morphism, we prove analogues of results by Kisin ([Kis09, Lem. 2.3.3, Prop. 2.3.5])which relate classical deformation rings of $\bmod p$ representations to those of characteristic 0 representations. The following corollary is a consequence of Theorems 3.39 and 3.40.

Corollary 1.2. *Let $\bar\rho\colon \Gamma \to \mathrm{GL}_n(k)$ be an absolutely irreducible representation of a profinite group Γ, and let $\mathcal{R}_{\bar\rho} \in \mathcal{CNL}_k$ denote its derived deformation ring in the sense of [GV18] (or [Zhu21]).*

Suppose $\rho\colon \Gamma \to \mathrm{GL}_n(E)$ is a characteristic 0 lift of $\bar\rho$ and $\xi \subset \rho$ an $\mathcal{O}$-lattice. Then the derived completion

$$(\mathcal{R}_{\bar\rho} \otimes_{\mathcal{O}} E)^\wedge_\xi \coloneqq \lim_r (\mathcal{R}_{\bar\rho} \otimes_{\mathcal{O}} E)/^{\mathbf{L}} I_\xi^r$$

of $\mathcal{R}_{\bar\rho} \otimes_{\mathcal{O}} E$ at the ideal $I_\xi = \ker(\pi_0(\mathcal{R}_{\bar\rho}) \otimes_{\mathcal{O}} E \to E)$ defined by ξ represents the derived deformation ring of ρ in the sense of [Zhu21].

Our strategy builds on that of [GV18], and uses derived deformation rings for both ρ and its $\bmod p$ reduction $\bar\rho\colon \Gamma_S \to \mathrm{GL}_n(k)$. In particular, we prove the following generalisation
The main idea is to approximate $\mathcal{R} \coloneqq \mathcal{R}_{\mathbf{S}}$ by a derived tensor product of static rings using the Taylor–Wiles method, in the following sense. For every $N \geq 1$ one has an augmented deformation problem $\mathbf{S}_N$ of 'Taylor–Wiles level N' and a corresponding derived deformation ring $\mathcal{R}_N \coloneqq \mathcal{R}_{\mathbf{S}_N}$ with underlying static ring $R_N \coloneqq \pi_0(\mathcal{R}_N)$. Descending from Taylor–Wiles level yields a weak equivalence

$$\mathcal{R} \simeq \mathcal{R}_N \otimes^{\mathbf{L}}_{S_N} \mathcal{O},$$

where S_N is the group ring (over $\mathcal{O}$) of a finite p-group. Via this equivalence, we construct an almost Artinian approximation of $\mathcal{R}$ by

$$\mathcal{R} \simeq \mathcal{R}_N \otimes^{\mathbf{L}}_{S_N} \mathcal{O} \to R_N \otimes^{\mathbf{L}}_{S_N} \mathcal{O} \to \overline{R}_N \otimes^{\mathbf{L}}_{\overline{S}_N} \mathcal{O}_N \eqqcolon \mathcal{C}_N$$

where $R_N \twoheadrightarrow \overline{R}_N$ and $S_N \twoheadrightarrow \overline{S}_N$ are certain Artinian quotients and $\mathcal{O}_N \coloneqq \mathcal{O}/\varpi^N$. In this way, we obtain 'small' approximations $\mathcal{C}_N$ of $\mathcal{R}$ for every $N \geq 1$. Via a compactness argument one passes to a non-canonical limit

$$\mathcal{R}_{\mathbf{S}} \to \mathcal{C}_\infty \coloneqq \lim_N \mathcal{C}_N.$$

Under their assumptions, Galatius–Venkatesh prove a weak equivalence

$$\mathcal{C}_\infty \simeq R_\infty \otimes^{\mathbf{L}}_{S_\infty} \mathcal{O},$$

where R_∞ is a formally smooth algebra over and S_∞ are power series rings such that $\dim_{\mathcal{O}} S_\infty - \dim_{\mathcal{O}} R_\infty = \ell_0$. Thus, the derived deformation ring $\mathcal{R}_{\mathbf{S}}$ is obtained from R_∞ by imposing $\dim_{\mathcal{O}}(R_\infty) + \ell_0$ relations (in the derived sense). The ring R_∞ acts freely on a patched cohomology module. Descending from Taylor–Wiles level , a graded action of $\pi_*(R_\infty \otimes^{\mathbf{L}}_{S^\circ_\infty} \mathcal{O})$ on the integral cohomology is obtained; inverting p then implies the sought result. A priori, this action might depend on the choices made in the Taylor–Wiles process; to prove this is not the case, the authors relate the action of $\mathcal{R}_{\bar\rho}$ to the derived Hecke algebra introduced in [Ven19].

Let us give an outline of our proof. The idea is to approximate $\mathcal{R}$ by a derived tensor product of classical deformation rings using a Taylor–Wiles argument. Roughly speaking, we construct a morphism from $\mathcal{R}$ to $R_\infty \otimes^{\mathbf{L}}_{S^\circ_\infty} \mathcal{O}$ and prove that it is a weak equivalence after localisation and completion at the characteristic 0 point corresponding to ρ. Then, applying an '$R = T$' theorem in characteristic 0 of A'Campo [A'C24] suffices to define an action

$$\pi_*((\mathcal{R}_{\mathbf{S}})^\wedge_{\mathfrak{p}}) \circlearrowleft H^*(X_U, \mathcal{V}_\lambda)_\Pi.$$

For every $N \geq 0$, there is a set of Taylor–Wiles places Q_N and an augmented deformation problem of Taylor–Wiles level N and a corresponding deformation ring $\mathcal{R}_N$. Our choice of Taylor–Wiles places is delicate since we require vanishing of $\bmod p^r$ Selmer groups; our method is inspired by that of [NT23]. Now, for every $N \geq 1$, one has a weak equivalence

$$\mathcal{R} \simeq \mathcal{R}_N \otimes^{\mathbf{L}}_{\mathcal{S}_N} \mathcal{S}^{\mathrm{ur}}_N$$

where $\mathcal{S}_N, \mathcal{S}^{\mathrm{ur}}_N$ are deformation rings for torus-valued representations of Taylor–Wiles level N. This equivalence corresponds to the tautological fact that a representation which is unramified outside $S \cup Q_N$ and unramified at every $v \in Q_N$ is unramified outside S.

Letting $R_N = \pi_0(\mathcal{R}_N)$ denote the underlying classical deformation ring and similarly for $\mathcal{S}_N$ and $\mathcal{S}^{\mathrm{ur}}_N$, we use the weak equivalence above to define a map

$$\mathcal{R} \xrightarrow{\sim} \mathcal{R}_N \otimes^{\mathbf{L}}_{\mathcal{S}_N} \mathcal{S}^{\mathrm{ur}}_N \to R_N \otimes^{\mathbf{L}}_{S_N} S^{\mathrm{ur}}_N \to \overline{R}_N \otimes^{\mathbf{L}}_{\overline{S}_N} \overline{S}^{\mathrm{ur}}_N =: \mathcal{C}_N$$

where the rings $\overline{R}_N, \overline{S}_N, \overline{S}^{\mathrm{ur}}_N$ are Artinian quotients of the classical deformation rings $R_N, S_N, S^{\mathrm{ur}}_N$. A compactness argument and A'Campo's patching result [A'C24, Lem. 2.6.4] allows us to define a morphism to an inverse limit

$$g \colon \mathcal{R} \to \lim_N \mathcal{C}_N$$

where N runs over an infinite sequence of natural numbers in such a way that we retain some amount of control over the morphism of tangent complexes. So far, our approach closely follows that of Galatius–Venkatesh but at this point the proofs diverge. We pass to characteristic 0 by localising and completing at the point corresponding to ρ, thus obtaining a morphism of E-algebras

$$(\mathcal{R} \otimes_{\mathcal{O}} E)^\wedge_\xi \to (\lim_N \mathcal{C}_N)^\wedge_\xi.$$

To analyse this morphism, we prove analogues of results by Kisin ([Kis09, Lem. 2.3.3, Prop. 2.3.5])which relate classical deformation rings of $\bmod p$ representations to those of characteristic 0 representations. The following corollary is a consequence of Theorems 3.39 and 3.40.

Corollary 1.3. *Let $\bar{\rho}\colon \Gamma \to \mathrm{GL}_n(k)$ be an absolutely irreducible representation of a profinite group Γ, and let $\mathcal{R}_{\bar{\rho}} \in \mathcal{CNL}_k$ denote its derived deformation ring in the sense of [GV18] (or [Zhu21]).*

Suppose $\rho\colon \Gamma \to \mathrm{GL}_n(E)$ is a characteristic 0 lift of $\bar{\rho}$ and $\xi \subset \rho$ an $\mathcal{O}$-lattice. Then the derived completion

$$(\mathcal{R}_{\bar{\rho}} \otimes_{\mathcal{O}} E)^{\wedge}_{\xi} \coloneqq \lim_r (\mathcal{R}_{\bar{\rho}} \otimes_{\mathcal{O}} E)/^{\mathbf{L}} I_{\xi}^r$$

of $\mathcal{R}_{\bar{\rho}} \otimes_{\mathcal{O}} E$ at the ideal $I_{\xi} = \ker(\pi_0(\mathcal{R}_{\bar{\rho}}) \otimes_{\mathcal{O}} E \to E)$ defined by ξ represents the derived deformation ring of ρ in the sense of [Zhu21].

This result allows us to identify the ring $(\mathcal{R} \otimes_{\mathcal{O}} E)^{\wedge}_{\xi}$ above with a characteristic 0 deformation ring, whose tangent complex is computable in terms of Galois cohomology, whereas the tangent complex of the other ring is readily computed. To complete the proof, we leverage our control of the induced map of g on tangent complexes and our choice of Taylor–Wiles primes. From the weak equivalence of E-algebras, the action on cohomology $(\star)$ is straightforwardly obtained using the main result in [A'C24].

Let us describe the content of the sections below. In Section 4.3, we introduce some notation and recall some basic facts concerning ∞-categories, the cohomology of locally symmetric spaces and animated rings. Section 3 contains the needed facts about derived deformation theory in general as developed by Lurie, and Zhu's derived deformation rings of Galois representations. Subsection 3.8 contains results connecting derived deformations of a representation ρ and its reduction $\bar{\rho}$. In Section 4, we prove the main result (Theorem 4.1).

2 Preliminaries

F is a CM field. E is a finite extension of $\mathbb{Q}_p$ with ring of integers $\mathcal{O}$, uniformiser ϖ and residue field $\mathcal{O}/\varpi = k$. For any $r \geq 1$, we let $\mathcal{O}_r = \mathcal{O}/\varpi^r$. Throughout the paper, we use homological indexing.

Over-∞-categories. If Y is an object of an ∞-category $\mathcal{C}$, there is an over-∞-category $\mathcal{C}_{/Y}$ whose objects are given by morphisms $(X \to Y)$. The object $(1_Y : Y \to Y)$ is final in $\mathcal{C}_{/Y}$, i.e. for any $(X \to Y) \in \mathcal{C}_{/Y}$, the mapping space $\mathrm{Map}_{\mathcal{C}_{/Y}}((X \to Y), (Y \to Y))$ is contractible.

Sub-∞-categories. Given an ∞-category $\mathcal{C}$ and a full subcategory $h'\mathcal{C} \hookrightarrow h\mathcal{C}$ of its homotopy category, we define an ∞-category $\mathcal{C}'$ by forming the pullback of

$$N(h'\mathcal{C}) \to N(h\mathcal{C}) \leftarrow \mathcal{C}.$$

Equivalently, $\mathcal{C}'$ is the simplicial subset of $\mathcal{C}$ spanned by the vertices lying in $h'\mathcal{C}$ [Cis19, Ex. 3.9.5].

2.1 Locally symmetric spaces

In this section, we recall some basic facts about the locally symmetric spaces associated to GL_n/F. See [KT17, §6.2] for a reference.

Notation 2.1. We fix the following notation:

- $\mathbf{G} = \mathrm{GL}_n/F$ with centre and torus $\mathbf{Z} \subset \mathbf{T} \subset \mathbf{G}$

- $G_\infty = \mathbf{G}(F \otimes_{\mathbb{Q}} \mathbb{R})$ with maximal compact subgroup $K_\infty \subset G_\infty$
- $Z_\infty = \mathbf{Z}(F \otimes_{\mathbb{Q}} \mathbb{R})$

Given an open compact subgroup $U \subset \mathbf{G}(\mathbb{A}_F^\infty)$, we define the locally symmetric space of level U as the double quotient

$$Y_U = \mathbf{G}(F)\backslash\mathbf{G}(\mathbb{A}_F)/UZ_\infty K_\infty.$$

We say that U is a *good* subgroup if $U = \prod_v U_v$ and U is neat. If U is good, then Y_U is a smooth manifold of dimension $\frac{1}{2}[F\colon \mathbb{Q}](n^2-1)$. The defect of $\mathbf{G}$ is the positive integer

$$l_0 = \operatorname{rank} G_\infty - \operatorname{rank} Z_\infty K_\infty = \frac{1}{2}[F\colon \mathbb{Q}](n-1).$$

Given a dominant weight $\lambda \in X^*(\mathbf{T})$, there exists a sheaf $\mathcal{V}_\lambda$ such that the cohomology $H^*(Y_U, \mathcal{V}_\lambda) \coloneqq H^*(R\Gamma(Y_U, \mathcal{V}_\lambda))$ carries an action of the Hecke algebra $\mathcal{H}(\mathbf{G}(\mathbf{A}_F^{\infty,S}), U^S)$ where S is a finite set of places such that $U_v = \mathrm{GL}_n(\mathcal{O}_{F_v})$ for $v \in S$ and $U^S = \prod_{v\notin S} U_v$. If Π is a regular algebraic cuspidal representation of $\mathbf{G}$, we let $\mathfrak{p} = \ker(\mathcal{H}(\mathbf{G}(\mathbf{A}_F^{\infty,S}), U^S) \to \mathrm{End}_{\mathbb{C}}(\otimes'_{v\notin S}\Pi_v^{K_v}))$ and define the Π-part as

$$H^*(Y_U, \mathcal{V}_\lambda)_\Pi \coloneqq H^*(Y_U, \mathcal{V}_\lambda)_{\mathfrak{p}},$$

i.e. the generalised eigenspace corresponding to the system of Hecke eigenvalues of Π. Thi

Proposition 2.2. *[HT17, Prop. 4.2] Let $2q_0 = \dim Y_U - l_0$, and fix a regular algebraic cuspidal representation Π of weight λ. Then $H^i(Y_U, \mathcal{V}_\lambda)_\Pi = 0$ for $i \notin [q_0, q_0 + l_0]$ and if $m_\Pi = \dim_{\mathbb{C}}(\Pi^\infty)^{U^p}$,*

$$\dim H^{q_0+i}(Y_U, \mathcal{V}_\lambda)_\Pi = m_\Pi \binom{l_0}{i}, \text{ for } i = 0, \dots, l_0.$$

2.2 Animated algebras

We begin by recalling the definition of animated algebras and its relation to simplicial rings, following the presentation in [ČS24, §5.1]. Let $\mathrm{Alg}_{\mathcal{O}}$ denote the ordinary category of commutative $\mathcal{O}$-algebras and let

$$\mathcal{A}lg_{\mathcal{O}} = \mathcal{A}ni(\mathrm{Alg}_{\mathcal{O}})$$

denote the ∞-category of animated $\mathcal{O}$-algebras, i.e. the ∞-category generated under sifted colimits by the subcategory $\mathrm{Alg}_{\mathcal{O}}^{\mathrm{ffg}}$ of finite free $\mathcal{O}$-algebras. It is the ∞-category underlying the simplicial model category

$$\mathrm{Alg}_{\mathcal{O}}^{\Delta} = \mathcal{F}un(\Delta^{\mathrm{op}}, \mathrm{Alg}_{\mathcal{O}})$$

of simplicial $\mathcal{O}$-algebras. That is, we have an equivalence of ∞-categories

$$\mathcal{A}lg_{\mathcal{O}} \simeq N\big((\mathrm{Alg}_{\mathcal{O}}^{\Delta})^\circ\big)[W^{-1}]$$

where N is the nerve functor, $(\mathrm{Alg}_{\mathcal{O}}^{\Delta})^\circ$ the full subcategory of bifibrant objects and W the class of weak equivalences [Lur09, Cor. 5.5.9.3]. We will freely alternate between the simplicial and animated viewpoints.

If $A \in \mathcal{A}lg_{\mathcal{O}}$ is an animated $\mathcal{O}$-algebra, an animated A-module is a connective module over the underlying $\mathbb{E}_1$-ring of A (see [Lur04, p. 19]). When $A = \pi_0(A)$ is static, the category

$$\mathcal{M}od_A \coloneqq \mathcal{A}ni(\mathrm{Mod}_A)$$

of animated A-modules is equivalent to the derived category $D^{\geq 0}(A)$ of connective complexes of A-modules, which is equivalent to the localisation of the simplicial model category Mod_A^{Δ} of simplicial A-modules at weak equivalences.

Definition 2.3. [Lur04, Def. 2.5.9] An animated $\mathcal{O}$-algebra $A \in \mathcal{Alg}_{\mathcal{O}}$ is:

(1) Noetherian if $\pi_0(A)$ is a Noetherian ring and $\pi_i(A)$ is a finitely generated $\pi_0(A)$-module for every $i \geq 0$.

(2) Local if $\pi_0(A)$ is local.

A local Noetherian algebra A is complete if A is derived $\mathfrak{m}_{\pi_0(A)}$-adically complete. We let $\mathcal{CNL}_k \subset (\mathcal{Alg}_{\mathcal{O}})_{/k}$ and $\mathcal{CNL}_E \subset (\mathcal{Alg}_{\mathcal{O}})_{/E}$ denote the full subcategories of complete Noetherian local algebras over k and E, respectively.

Proposition 2.4. *[Lur04, p. 32] Let $A \in \mathcal{Alg}_{\mathcal{O}}$ and suppose B lies in the smallest subcategory of $\mathcal{Alg}_A$ containing $A[T]$ which is closed under finite colimits. Then $\mathrm{Map}_A(B, -)$ commutes with filtered colimits.*

If the assumption of Proposition 2.4 holds, we say that B is a finitely presented A-algebra.

Proposition 2.5. *[Lur04, Prop. 3.1.5] Let $A \in \mathcal{Alg}_{\mathcal{O}}$ be Noetherian, $j \geq 0$ and suppose B is an A-algebra. The following are equivalent:*

(1) The functor $\mathrm{Map}_A(B, -)$ commutes with filtered colimits when restricted to the subcategory $\tau_{\leq j}(\mathcal{Alg}_A)$ of j-truncated A-algebras.

(2) The A-algebra $\tau_{\leq j}B$ is Noetherian and $\pi_0(B)$ is a finitely presented $\pi_0(A)$-algebra.

If the equivalent conditions in Proposition 2.5 hold, we say that B is of finite presentation to order j as an A-algebra. If the conditions hold for every $j \geq 0$, we say that B is almost of finite presentation over A. From (2), we deduce that any Noetherian static ring $B \in \mathrm{Alg}_{\mathcal{O}}$ is almost of finite presentation.

Derived quotients. Fix an animated $\mathcal{O}$-algebra $A \in \mathcal{Alg}_{\mathcal{O}}$ and an element $a \in \pi_0(A)$. We define a morphism of $\mathcal{O}$-algebras $\mathcal{O}[t] \to A$ by choosing a lift $\tilde{a} \in A_0$, and let

$$A/^{\mathbf{L}}a := A \otimes^{\mathbf{L}}_{\mathcal{O}[t]} \mathcal{O},$$

where t acts as 0 on $\mathcal{O}$. The algebra $A/^{\mathbf{L}}a$ is well-defined up to weak equivalence, and for every $r \geq 1$ we have a natural map

$$A/^{\mathbf{L}}a^{r+1} \to A/^{\mathbf{L}}a^r.$$

More generally, given $a_1, \ldots, a_m \in \pi_0(A)$ we define

$$A/^{\mathbf{L}}(a_1, \ldots, a_m) = A \otimes^{\mathbf{L}}_{\mathcal{O}[t_1,\ldots,t_m]} \mathcal{O} \simeq A/^{\mathbf{L}}a_1/^{\mathbf{L}} \ldots /^{\mathbf{L}}a_m.$$

There are natural maps

$$A/^{\mathbf{L}}(a_1^{r+1}, \ldots, a_m^{r+1}) \to A/^{\mathbf{L}}(a_1^r, \ldots, a_m^r)$$

forming an inverse system of animated $\mathcal{O}$-algebras and the limit depends only on the closed subspace of $\mathrm{Spec}\, \pi_0(A)$ defined by $(a_1, \ldots, a_m)$, as the following result shows.

Proposition 2.6. *[Lur04, Prop. 6.1.1] Let $A \in \mathcal{A}lg_{\mathcal{O}}$ and let $J \subseteq \pi_0(A)$ be a finitely generated ideal. Fix a set of generators $\{a_1, \ldots, a_m\}$ for J. For any $B \in \mathcal{A}lg_{\mathcal{O}}$, the subspace*

$$\operatorname*{colim}_r \mathrm{Map}_{\mathcal{A}lg_{\mathcal{O}}}(A/^{\mathbf{L}}(a_1^r, \ldots a_m^r), B) \to \mathrm{Map}_{\mathcal{A}lg_{\mathcal{O}}}(A, B)$$

is the space spanned by morphisms $f\colon A \to B$ such that the induced morphism $\pi_0(A) \to \pi_0(B)$ factors through $\pi_0(A)/J^N$ for some $N \geq 1$.

Definition 2.7. [ČS24, 5.6.1] Let $A \in \mathcal{A}lg_{\mathcal{O}}$ be Noetherian and let $I = (a_1, \ldots, a_m) \subset \pi_0(A)$. Then A is derived I-adically complete if the natural map

$$A \to \lim_r A/^{\mathbf{L}}(a_1^r, \ldots, a_m^r)$$

is a weak equivalence.

Lemma 2.8. *[BS15, Lem. 3.4.13] Let $A \in \mathrm{Mod}_{\mathcal{O}}$ and suppose $\pi_i(A)$ is ϖ-adically complete in the usual sense for every $i \geq 0$. Then A is derived ϖ-adically complete.*

3 Derived deformation theory

3.1 Artinian algebras

In this section, we recall the notion of Artinian animated algebras following [Lur04].

Definition 3.1. The category $\mathcal{A}rt_k$ (resp. $\mathcal{A}rt_E$) of Artinian animated $\mathcal{O}$-algebras over k (resp. over E) is defined as the sub-∞-category of $(\mathcal{A}lg_{\mathcal{O}})_{/k}$ (resp. $(\mathcal{A}lg_{\mathcal{O}})_{/E}$) spanned by $(B \to k)$ (resp. $(B \to E)$) such that

(1) $\pi_0(B)$ is an Artinian ring.

(2) $\pi_*(B)$ is a finitely generated $\pi_0(B)$-module.

(3) The induced map $\pi_0(B) \to k$ (resp. $\pi_0(B) \to E$) is surjective.

Any Artinian animated $\mathcal{O}$-algebra (over k or E) is truncated. Note that if $B \in \mathcal{A}rt_E$ then $\pi_0(B)$ is an E-algebra, and B is truncated since $\pi_*(B)$ is a finite-dimensional E-vector space. A useful description of Artinian algebras allowing for proofs by induction is given in terms of small extensions.

Definition 3.2. [Lur04, Def. 3.3.1] Let $B \in \mathcal{A}lg_{\mathcal{O}}$ and $M \in \mathrm{Mod}_B^{\mathrm{cn}}$. A small extension of B by M is a pullback

$$\begin{tikzcd} B' \arrow[r] \arrow[d] & B \arrow[d, "s"] \\ B \arrow[r, "(\mathrm{id},0)"'] & B \oplus M[1] \end{tikzcd}$$

where $B \oplus M[1]$ is the trivial square-zero extension [Lur04, p.32] and $s\colon B \to B \oplus M[1]$ is a section of the projection.

Lemma 3.3. *[Lur04, Lem. 6.2.6] Let $(B \to E) \in \mathcal{A}rt_E$. Then there exists a factorisation*

$$B \simeq B_m \to B_{m-1} \to \cdots \to B_0 \simeq E$$

such that B_j is a small extension of B_{j-1} by $E[n_j]$ for an increasing sequence $n_j \geq 0$ and $\dim_E(\pi_*(B_j)) = j+1$. The analogous statement is true for $(B \to k) \in \mathcal{A}rt_k$.

3.2 Integral models of Artinian E-algebras

In this section, we introduce a notion of integral models of Artinian E-algebras. The definition is made in analogy with [Kis09, 2.3.5] and will be used in Definition 3.38.

Note that we have a canonical equivalence of ∞-categories

$$\big((\mathcal{A}lg_{\mathcal{O}})_{/E}\big)_{/(B\to E)} \simeq (\mathcal{A}lg_{\mathcal{O}})_{/B}.$$

Definition 3.4. Let $(B \to E) \in \mathcal{A}rt_E$. The ∞-category $\operatorname{Int} B$ is defined as the sub-∞-category of $(\mathcal{A}lg_{\mathcal{O}})_{/B}$ spanned by all $(A \to B)$ such that:

(1) $\pi_*(A)$ is a finitely generated $\mathcal{O}$-module.

(2) The induced map $\pi_*(A) \to \pi_*(B)$ is injective.

(3) The induced map $A \otimes^{\mathbf{L}}_{\mathcal{O}} E \to B \otimes^{\mathbf{L}}_{\mathcal{O}} E$ is an weak equivalence.

The central result about the category $\operatorname{Int} B$ that we will need is the following.

Proposition 3.5. *For any $B \in \mathcal{A}rt_E$, the ∞-category $\operatorname{Int} B$ is filtered.*

Note that conditon (1) in the definition implies that any $A \in \operatorname{Int} B$ is finitely presented. When B is clear from context, we sometimes denote objects of $\operatorname{Int} B$ simply by A. Note that, if $A, A' \in \operatorname{Int} B$ then by definition

$$\operatorname{Map}_{\operatorname{Int} B}(A, A') = \operatorname{Map}_{(\mathcal{A}lg_{\mathcal{O}})_{/B}}(A, A').$$

Lemma 3.6. *Let $(B \to E) \in \mathcal{A}rt_E$ and $(A \to B) \in \operatorname{Int} B$. The natural map*

$$\operatorname{Map}_{\mathcal{A}lg_{\mathcal{O}}}(A, B \times_E \mathcal{O}) \to \operatorname{Map}_{\mathcal{A}lg_{\mathcal{O}}}(A, B)$$

is a weak equivalence.

Proof. Since $\mathcal{O}$ and E are static and $\pi_0(A)$ is a finitely generated $\mathcal{O}$-module, we have equivalences of mapping spaces

$$\operatorname{Map}_{\mathcal{A}lg_{\mathcal{O}}}(A, E) \simeq \operatorname{Map}_{\mathcal{A}lg_{\mathcal{O}}}(\pi_0(A), E) \simeq \operatorname{Map}_{\mathcal{A}lg_{\mathcal{O}}}(\pi_0(A), \mathcal{O}) \simeq \operatorname{Map}_{\mathcal{A}lg_{\mathcal{O}}}(A, \mathcal{O}).$$

Consequently,

$$\operatorname{Map}_{\mathcal{A}lg_{\mathcal{O}}}(A, B \times_E \mathcal{O}) \simeq \operatorname{Map}_{\mathcal{A}lg_{\mathcal{O}}}(A, B) \times_{\operatorname{Map}_{\mathcal{A}lg_{\mathcal{O}}}(A,E)} \operatorname{Map}_{\mathcal{A}lg_{\mathcal{O}}}(A, \mathcal{O}) \simeq \operatorname{Map}_{\mathcal{A}lg_{\mathcal{O}}}(A, B).$$

$\square$

To prove Proposition 3.5, we will prove an analogue of Lemma 3.3 for the category $\operatorname{Int} B$.

Lemma 3.7. *Let $B \in \mathcal{A}rt_E$ be given by a sequence of small extensions*

$$B \simeq B_m \to B_{m-1} \to \cdots \to B_0 \simeq E$$

as in Lemma 3.3. Then for any $A \in \operatorname{Int} B$, the morphism $A \to \mathcal{O}$ factors as

$$A \simeq A_m \to A_{m-1} \to \cdots \to A_0 \simeq \mathcal{O},$$

where, for every $j \geq 1$, $A_j \in \operatorname{Int} B_j$ is a small extension of A_{j-1} by $\varpi^{-r_j}\mathcal{O}[n_j]$ for a non-decreasing sequence $0 \leq n_1 \leq \cdots \leq n_m$ and some $r_j \geq 0$. Moreover, $B \times_E \mathcal{O}$ is equivalent in $(\mathcal{A}lg_{\mathcal{O}})_{/\mathcal{O}}$ to a filtered colimit of algebras of the form A_m.

Before embarking on the proof of the Lemma, let us describe the homotopy groups of a small extension. Let $A \in \mathcal{A}lg_{\mathcal{O}}$ be n-truncated and suppose $M \in \mathcal{A}ni(\operatorname{Mod} A)$ has homotopy concentrated in degree $n-1$. Then $M[1]$ has homotopy concentrated in degree n, and any small extension $A' = A \times_{A \oplus M[1]} A$ gives rise to a long exact sequence

$$\cdots \to \pi_{i+1}(A) \to \pi_{i-1}(M) \to \pi_i(A') \to \pi_i(A) \to \pi_{i-2}(M) \to \dots$$

which implies that $\pi_i(A') \cong \pi_i(A)$ for every $i \neq n$ and that

$$\pi_{n-1}(M) \cong \ker(\pi_n(A') \to \pi_n(A)).$$

Thus, a single small extension of the form considered in Lemma 3.7 only changes the homotopy groups in the top degree by adding a module which is free of rank 1 over $\mathcal{O}$.

Proof of Lemma 3.7. We prove the lemma by induction on $m = \dim_E(\pi_*(B)) - 1$. Let $n = n_m$ denote the largest integer such that $\pi_n(B) \neq 0$. The case $m = 0$ is trivial. For the induction step, suppose the statement holds for any length $m-1$ sequence $B'_{m-1} \to \cdots \to B'_0$.

The kernel

$$I = \ker(\pi_*(A) \hookrightarrow \pi_*(B_m) \to \pi_*(B_{m-1})) \subset \pi_n(A)$$

is a $\pi_0(A)$-submodule of $\pi_*(A)$, which is free of rank 1 as an $\mathcal{O}$-module. Using [Lur04, Prop. 3.3.3, Prop. 3.3.5], we construct a morphism in $\mathcal{A}lg_{\mathcal{O}}$

$$A \to A_{m-1}$$

where A_{m-1} ('A/I' in the notation of loc. cit.) satisfies the following properties:

(1) For any $Y \in \mathcal{A}lg_{\mathcal{O}}$,

$$\operatorname{Map}_{\mathcal{A}lg_{\mathcal{O}}}(A_{m-1}, Y) \hookrightarrow \operatorname{Map}_{\mathcal{A}lg_{\mathcal{O}}}(A, Y)$$

is the subspace spanned by maps for which the induced map on homotopy groups maps I to $0 \in \pi_*(Y)$.

(2) The induced map $\pi_*(A) \to \pi_*(A_{m-1})$ is an isomorphism in degree $i \neq n$ and for $i = n$ is the map $\pi_n(A) \to \pi_n(A)/I$.

(3) A is a small extension of A_{m-1} by $I[n]$, i.e. there is an equivalence

$$A \simeq A_m = A_{m-1} \times_{A_{m-1} \oplus I[n+1]} A_{m-1}.$$

From (1)-(3), we deduce that the map $A_{m-1} \to B_{m-1}$ defines an object of $\operatorname{Int} B_{m-1}$ and that the square

$$\begin{tikzcd} A \arrow[r] \arrow[d] & A_{m-1} \arrow[d] \\ B \arrow[r] & B_{m-1} \end{tikzcd}$$

commutes up to homotopy. By the induction hypothesis, $A_{m-1} \to \mathcal{O}$ factors as a series of small extensions of the required form, and we are done. $\square$

Lemma 3.8. *Let $B \in \mathcal{A}rt_E$. Then $B \times_E \mathcal{O}$ is equivalent in $(\mathcal{A}lg_{\mathcal{O}})_{/\mathcal{O}}$ to a filtered colimit of algebras of the form of A_m in Lemma 3.7.*

Proof. To prove the second claim, we use induction on m again. The case $m = 0$ is trivial. Suppose the statement is true for any B' given by a length $m-1$ sequence of small extensions. Note that for any j,

$$E[j] = \operatorname*{colim}_{r\geq 0} \varpi^{-r}\mathcal{O}[j].$$

Since filtered colimits commute with finite limits, under the induction hypothesis we have

$$\begin{aligned} B_m \times_E \mathcal{O} &\simeq (B_{m-1} \times_{B_{m-1}\oplus E[n_m+1]} B_{m-1}) \times_E \mathcal{O} \\ &\simeq (B_{m-1} \times_E \mathcal{O}) \times_{(B_{m-1}\times_E \mathcal{O})\oplus E[n_m+1]} (B_{m-1} \times_E \mathcal{O}) \\ &\simeq \operatorname*{colim}_{A_{m-1}} (A_{m-1} \times_{A_{m-1}\oplus E[n_m+1]} A_{m-1}) \end{aligned}$$

where the colimit runs over a filtered diagram of $A_{m-1} \in \operatorname{Int} B_{m-1}$ of the sought form. Now,

$$\begin{aligned} A_{m-1} \oplus E[n_m+1] &\simeq A_{m-1} \oplus \operatorname*{colim}_{r\geq 0} \varpi^{-r}\mathcal{O}[n_m+1] \\ &\simeq \operatorname*{colim}_{r\geq 0} A_{m-1} \oplus \varpi^{-r}\mathcal{O}[n_m+1] \end{aligned}$$

since $\pi_*(A_{m-1}) \to \pi_*(B_{m-1})$ is injective and $\pi_*(B_{m-1})$ is an E-algebra. It follows that

$$\operatorname*{colim}_{A_{m-1}} (A_{m-1} \times_{A_{m-1}\oplus E[n_m+1]} A_{m-1}) \cong \operatorname*{colim}_{A_{m-1}} \operatorname*{colim}_{r\geq 0} (A_{m-1} \times_{A_{m-1}\oplus \varpi^{-r}\mathcal{O}[n_m+1]} A_{m-1}).$$

Thus, B_m is a filtered colimit of the sought form. □

We are now in a position to prove the main result of this section, Proposition 3.5.

Proof of Proposition 3.5. By the local characterisation of filtered ∞-categories [Lur18, Tag 02PS], it suffices to prove the following statement: Given $A, A' \in \operatorname{Int} B$ and a morphism of simplicial sets

$$\gamma\colon \partial\Delta^n \to \operatorname{Map}_{/B}(A, A')$$

for some $n \geq 0$, there exists an $A'' \in \operatorname{Int} B$ and a morphism $g\colon A' \to A''$ such that the composition

$$g \circ \gamma\colon \partial\Delta^n \to \operatorname{Map}_{/B}(A, A') \to \operatorname{Map}_{/B}(A, A'')$$

is null-homotopic. In other words, we would like to define the dotted arrow in the diagram

$$\begin{tikzcd} \partial\Delta^n \arrow[r, "\gamma"] \arrow[d] & \operatorname{Map}_{/B}(A, A') \arrow[d, "g_*"] \\ \Delta^n \arrow[r, dotted] & \operatorname{Map}_{/B}(A, A'') \end{tikzcd}$$

Since B is final in $(\mathcal{A}lg_{\mathcal{O}})_{/B}$ and by Lemma 3.8, we have

$$\{*\} \simeq \operatorname{Map}_{/B}(A, B) \simeq \operatorname{Map}_{/B}(A, B \times_E \mathcal{O}) \simeq \operatorname{Map}_{/B}(A, \operatorname*{colim}_{A''} A''),$$

where A'' runs over a filtered subcategory of $\operatorname{Int} B$. Thus, the composite

$$\partial\Delta^n \to \operatorname{Map}_{/B}(A, A') \to \operatorname{Map}_{/B}(A, \operatorname*{colim}_{A''} A'')$$

is null-homotopic. Now, since $A' \in \operatorname{Int} B$ is a finitely presented $\mathcal{O}$-algebra by Proposition 2.5, the map $A' \to \operatorname{colim} A''$ factors through some A''. It follows that there exists a $g\colon A' \to A''$ such that the composition $g \circ \gamma$ is null-homotopic. □

Lemma 3.9. *Let $B \in \mathcal{A}rt_E$. We have the following:*

(1) There is a natural weak equivalence $\operatorname{colim}_{A \in \operatorname{Int} B} A \to B \times_E \mathcal{O}$.

(2) There is a canonical isomorphism of graded rings $\pi_*(B \times_E \mathcal{O}) \cong \pi_*(B) \times_E \mathcal{O}$.

(3) Any $A \in \operatorname{Int} B$ is derived ϖ-adically complete.

(4) For any $A \in \operatorname{Int} B$ and $r \geq 1$, $(A/^{\mathbf{L}}\varpi^r \to k) \in \mathcal{A}rt_k$.

Proof. (1) By Lemma 3.6, any $(A \to B) \in \operatorname{Int} B$ defines a map $A \to B \times_E \mathcal{O}$. Taking the colimit over $\operatorname{Int} B$, we produce

$$\operatorname{colim}_{A \in \operatorname{Int} B} A \to B \times_E \mathcal{O},$$

and we claim that this map is a weak equivalence. Since $\operatorname{Int} B$ is filtered and filtered colimits commute with π_*, it is enough to prove that the map

$$\operatorname{colim}_{A \in \operatorname{Int} B} \pi_*(A) \to \pi_*(B \times_E \mathcal{O})$$

is an isomorphism. Injectivity follows from condition (3) in the definition of $\operatorname{Int} B$. To see that the map is surjective, we use the isomorphism provided by Lemma 3.8, i.e.

$$\pi_*(B \times_E \mathcal{O}) \cong \operatorname{colim}_{A''} \pi_*(A''),$$

a filtered colimit over objects $A'' \in \operatorname{Int} B$. Surjectivity follows.
(2) There is a Mayer-Vietoris sequence

$$\cdots \to \pi_n(B \times_E \mathcal{O}) \to \pi_n(B) \times \pi_n(\mathcal{O}) \to \pi_n(E) \to \ldots,$$

and hence for every $n \geq 1$, we have

$$\pi_n(B \times_E \mathcal{O}) \cong \pi_n(B) \times \pi_n(\mathcal{O}) \cong \pi_n(B)$$

since $\pi_n(B) \to E$ is the zero map. For $n = 0$, the sequence implies

$$\pi_0(B \times_E \mathcal{O}) \cong \ker(\pi_0(B) \times \mathcal{O} \to E) \cong \pi_0(B) \times_E \mathcal{O}.$$

(3) By definition, $\pi_*(A)$ is a finitely generated $\mathcal{O}$-module and hence A is ϖ-adically complete by Lemma 2.8.
(4) Let $A \in \operatorname{Int} B$ and $r \geq 1$. Since $\pi_*(A)$ is a finitely generated and free $\mathcal{O}$-module, Quillen's spectral sequence ([Qui67, Thm. 2.6])

$$E^2_{i,j} \colon \operatorname{Tor}_i^{\mathcal{O}[t]}\left(\pi_j(A), \mathcal{O}\right) \implies \pi_{i+j}(A/^{\mathbf{L}}\varpi^r)$$

amounts to an isomorphism

$$\pi_*(A/^{\mathbf{L}}\varpi^r) \cong \pi_*(A) \otimes_{\mathcal{O}} \mathcal{O}/\varpi^r.$$

It follows that $\pi_0(A/^{\mathbf{L}}\varpi^r)$ is Artinian and that $\pi_*(A/^{\mathbf{L}}\varpi^r)$ is finitely generated over $\pi_0(A/^{\mathbf{L}}\varpi^r)$. $\square$

Lemma 3.10. *Let $(A \to B) \in \operatorname{Int} B$. Then the map $A \to E$ factors through $\mathcal{O} \to E$, and we have a lifting*

$$\begin{tikzcd}
\operatorname{Int} B \arrow[r, dashed] \arrow[d] & (\mathcal{A}lg_{\mathcal{O}})_{/\mathcal{O}} \arrow[d] \\
(\mathcal{A}lg_{\mathcal{O}})_{/B} \arrow[r] & (\mathcal{A}lg_{\mathcal{O}})_{/E}
\end{tikzcd}$$

which reflects colimits.

Proof. The functor $\operatorname{Int} B \to (\mathcal{Alg}_{\mathcal{O}})_{/B}$ factors through

$$(\mathcal{Alg}_{\mathcal{O}})_{/\operatorname{colim}_{A\in \operatorname{Int} B} A} \simeq (\mathcal{Alg}_{\mathcal{O}})_{/B\times_E \mathcal{O}} \simeq (\mathcal{Alg}_{\mathcal{O}})_{/B} \times_{(\mathcal{Alg}_{\mathcal{O}})_{/E}} (\mathcal{Alg}_{\mathcal{O}})_{/\mathcal{O}},$$

where we have used Lemma 3.9(1) in the first equivalence. Thus, we obtain the sought lifting. The second part follows from that colimits in a slice ∞-category $\mathcal{C}/X$ are computed in $\mathcal{C}$ [Lur09, Prop. 1.2.13.8]. $\square$

3.3 Topologies on animated algebras

In this section, we discuss some notions of continuity (or topology) in the derived setting which we, following [Zhu21], will use later on to define derived moduli stacks of continuous representations.

Given a static $\mathcal{O}$-module M, we can equip it with an ind-ϖ-adic topology via the isomorphism

$$M \cong \varinjlim_{M'\subset M} \varprojlim_{r\geq 1} M'/\varpi^r M',$$

where the filtered colimit runs over the finitely generated submodules M' of M. In general, the ind-ϖ-adic topology is finer than the ϖ-adic, and if M is finitely generated, they coincide. As a topological space, M equipped with the ind-ϖ-adic topology is the image of M under the functor $\operatorname{Mod}_{\mathcal{O}} \to \operatorname{Ind}(\operatorname{Pro}(\operatorname{Set}^{\mathrm{cp}})) \hookrightarrow \operatorname{Top}$ where $\operatorname{Set}^{\mathrm{cp}}$ is the category of finite sets.

Lemma 3.11. *[Zhu21, 2.4.18] Let Γ be a profinite set and define*

$$\begin{aligned} C_{\mathrm{sc}}(\Gamma, -)\colon\ \operatorname{Mod}_{\mathcal{O}} &\to \operatorname{Mod}_{\mathcal{O}} \\ M &\mapsto \operatorname{Map}_{\operatorname{Ind}(\operatorname{Pro}(\operatorname{Set}^{\mathrm{cp}}))}(\Gamma, M) \end{aligned}$$

Then $C_{\mathrm{sc}}(\Gamma, -)$ is exact and lax symmetric monoidal and therefore extends to a t-exact functor $\mathcal{Ani}(\operatorname{Mod}_{\mathcal{O}}) \to \mathcal{Ani}(\operatorname{Mod}_{\mathcal{O}})$ which preserves colimits. Moreover, $C_{\mathrm{sc}}(\Gamma, -)$ lifts to a nilcomplete functor $\mathcal{Alg}_{\mathcal{O}} \to \mathcal{Alg}_{\mathcal{O}}$ which preserves finite limits and sifted colimits.

Lemma 3.12. *Let $A \in \operatorname{Mod}_{\mathcal{O}}$ be an animated $\mathcal{O}$-module and suppose $\pi_*(A)$ is a finitely generated $\mathcal{O}$-module. Then there is a natural weak equivalence*

$$C_{\mathrm{sc}}(\Gamma, A) \to \lim_{r\geq 1} C_{\mathrm{sc}}(\Gamma, A/^{\mathbf{L}}\varpi^r)$$

which respects the monoidal structure and hence if $A \in \mathcal{Alg}_{\mathcal{O}}$ then the equivalence holds in $\mathcal{Alg}_{\mathcal{O}}$.

Proof. For any $A \in \mathcal{Ani}(\operatorname{Mod}_{\mathcal{O}}) \simeq \mathcal{D}^{\geq 0}(\mathcal{O})$, we have natural morphisms (of $\mathcal{O}$-algebras, if $A \in \mathcal{Alg}_{\mathcal{O}}$)

$$C_{\mathrm{sc}}(\Gamma, A) \to C_{\mathrm{sc}}(\Gamma, \lim A/^{\mathbf{L}}\varpi^r) \to \lim C_{\mathrm{sc}}(\Gamma, A/^{\mathbf{L}}\varpi^r)$$

If A is derived ϖ-adically complete – in particular, if $\pi_*(A)$ is finitely generated over $\mathcal{O}$ (Lemma 2.8) – the first map is an equivalence. Let $P(A)$ be the property that the second map is a weak equivalence. Then $P(\mathcal{O}[0])$ holds, $P(A)$ implies $P(A[1])$ and for any fiber sequence $A \to B \to C \to A[1]$, if P holds for two of A, B, C then it holds for the third by the five lemma. Moreover, for any A, B, $P(A\oplus B)$ is implied by $P(A)$ and $P(B)$ since $C_{\mathrm{sc}}(\Gamma, -)$ commutes with colimits. Finally, any retract $A \to B \to A$ induces a commutative diagram

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & A \arrow[d] \\
\lim_r C_{\mathrm{sc}}(\Gamma, A/^{\mathbf{L}}\varpi^r) \arrow[r] & \lim_r C_{\mathrm{sc}}(\Gamma, B/^{\mathbf{L}}\varpi^r) \arrow[r] & \lim_r C_{\mathrm{sc}}(\Gamma, A/^{\mathbf{L}}\varpi^r)
\end{tikzcd}$$

where the composition of horizontal arrows is the identity, so that if $P(B)$ holds then the induced map $\pi_*(A) \to \pi_*(\lim_r C_{\mathrm{sc}}(\Gamma, B/^{\mathbf{L}}\varpi^r))$ is both injective and surjective, i.e. $P(A)$ holds. It follows that $P(A)$ holds for any perfect animated $\mathcal{O}$-module, since the category $\mathcal{D}^{\geq 0}_{\mathrm{perf}}(\mathcal{O})$ of perfect complexes is the smallest strictly full, saturated, triangulated subcategory of $\mathcal{D}^{\geq 0}(\mathcal{O})$ containing $\mathcal{O}[0]$ ([Sta23, Tag 0ATI]). $\square$

3.4 Moduli stacks of Galois representations

In this section, we discuss moduli functors of representations of a profinite group Γ valued in $\mathbf{G} = \mathrm{GL}_n / \mathcal{O}$, following [Zhu21]. In applications, we will have $\Gamma = \mathrm{Gal}(\overline{F}|F)$ or $\mathrm{Gal}(\overline{F_v}|F)$.

For any $n \in \mathbb{N}$, we let $\mathcal{O}[\mathbf{G}^n]$ denote the coordinate ring of $\mathbf{G}^n$. The functor of points $\mathbf{G}\colon \mathrm{Alg}_{\mathcal{O}} \to \mathrm{Grp}$ extends via animation to a functor

$$\mathbf{G}\colon \mathcal{A}lg_{\mathcal{O}} \to \mathcal{A}ni(\mathrm{Grp}).$$

As a simplicial space, $\mathbf{G}(A)$ is the functor $[n] \mapsto \mathrm{Map}_{\mathcal{A}lg_{\mathcal{O}}}(\mathcal{O}[\mathbf{G}^n], A)$ with boundary maps defined using the group law on $\mathbf{G}$ and degeneracies defined by inclusions $\mathbf{G}^{n_1} \times \{\mathrm{id}\} \times \mathbf{G}^{n_2} \to \mathbf{G}^{n_1+n_2+1}$. These maps define a cosimplicial animated $\mathcal{O}$-algebra $\mathcal{O}[\mathbf{G}^\bullet] \in \mathcal{F}un(\Delta, \mathcal{A}lg_{\mathcal{O}}) = \mathcal{F}un(\Delta, \mathcal{A}lg_{\mathcal{O}})$ which is finitely presented in the following sense.

Lemma 3.13. *The functor* $\mathrm{Map}_{\mathcal{F}un(\Delta,\mathcal{A}lg_{\mathcal{O}})}(\mathcal{O}[\mathbf{G}^\bullet], -)$ *commutes with filtered colimits.*

Proof. Since $\mathcal{O}[\mathbf{G}^n] \cong \mathcal{O}[\mathbf{G}] \otimes_{\mathcal{O}} \cdots \otimes_{\mathcal{O}} \mathcal{O}[\mathbf{G}]$ and

$$\mathcal{O}[\mathbf{G}] = \mathcal{O}[X_{i,j}, T]/(T\det(X_{i,j}) - 1) = \mathcal{O}[X_{i,j}, T]/^{\mathbf{L}}(T\det(X_{i,j}) - 1)$$

is a finitely presented $\mathcal{O}$-algebra, the statement follows from Proposition 2.4 and [Lur04, Prop. 5.3.4.13]. $\square$

Given $A \in \mathcal{A}lg_{\mathcal{O}}$, Zhu [Zhu21, Remark 2.2.2] proves the equivalence

$$\mathrm{Map}_{\mathcal{A}ni(\mathrm{Grp})}(\Gamma, \mathbf{G}(A)) \simeq \mathrm{Map}_{\mathcal{F}un(\Delta,\mathcal{A}lg_{\mathcal{O}})}(\mathcal{O}[\mathbf{G}^\bullet], C(\Gamma^\bullet, A))$$

where for any $[n] \in \Delta$, $C(\Gamma^n, -)$ is the functor which maps a static $\mathcal{O}$-module M to the set of all functions $\Gamma^t \to M$ (cf. the definition of $C_{\mathrm{sc}}(\Gamma^\bullet, -)$ in Lemma 3.11). Replacing $C(\Gamma^\bullet, -)$ by $C_{\mathrm{sc}}(\Gamma^\bullet, -)$, we obtain the following definition.

Definition 3.14. [Zhu21, Def. 2.4.21] The derived moduli stack of framed $\mathbf{G}$-valued Γ-representations is the functor

$$\begin{aligned} \mathcal{X}^{\square}_{\Gamma,\mathbf{G}}\colon \mathcal{A}lg_{\mathcal{O}} &\to \mathcal{A}ni \\ A &\mapsto \mathrm{Map}_{\mathcal{F}un(\Delta,\mathcal{A}lg_{\mathcal{O}})}\big(\mathcal{O}[\mathbf{G}^\bullet], C_{\mathrm{sc}}(\Gamma^\bullet, A)\big). \end{aligned}$$

Example 3.15. [Zhu21, Lem. 2.4.22] If $A \in \mathcal{A}lg_{\mathcal{O}}$ is static, then $\mathcal{X}^{\square}_{\Gamma,\mathbf{G}}(A)$ is the set of homomorphisms $\rho_A\colon \Gamma \to \mathrm{GL}_n(A)$ such that A^n is a filtered colimit of Γ-stable finite $\mathcal{O}$-modules with continuous Γ-action.

Definition 3.16. Let $\mathcal{F}\colon \mathcal{A}lg_{\mathcal{O}} \to \mathcal{A}ni$ be a functor and fix $\bar\rho \in \mathcal{F}(k)$. We define

$$\begin{aligned} \mathcal{F}_{\bar\rho}\colon \mathcal{A}rt_k &\to \mathcal{A}ni \\ (B \to k) &\mapsto \mathcal{F}(B) \times_{\mathcal{F}(k)} \{\bar\rho\}. \end{aligned}$$

Similarly, if $\rho \in \mathcal{F}(E)$, we obtain a functor $\mathcal{F}_\rho\colon \mathcal{A}rt_E \to \mathcal{A}ni$ given by $\mathcal{F}(B \to E) = \mathcal{F}(B) \times_{\mathcal{F}(E)} \{\rho\}$.

The functor in Definition 3.14 is the functor of framed deformations. To construct the 'unframed' versions, let $\mathbf{Z} \subset \mathbf{G}$ be the center of $\mathbf{G}$ and consider the conjugation action of $\mathbf{PG} = \mathbf{G}/\mathbf{Z}$ on $\mathcal{X}$, which amounts to a simplicial presheaf

$$(\mathbf{PG}^\bullet \times \mathcal{X}^\square)\colon (\ldots \substack{\rightarrow\\\rightarrow\\\rightarrow} \mathbf{PG} \times \mathcal{X}^\square \rightrightarrows \mathcal{X}^\square).$$

Definition 3.17. [Zhu21, Def 2.2.14] The derived moduli stack of unframed $\mathbf{G}$-valued Γ-representations is the functor

$$\mathcal{X}_{\Gamma,\mathbf{G}} = |\mathbf{PG}^\bullet \times \mathcal{X}^\square|$$

where $|\cdot|$ denotes geometric realisation in the category of presheaves $\mathrm{PSh}(\mathcal{A}lg_{\mathcal{O}})$.

Remark 3.18. Zhu [Zhu21] defines $\mathcal{X}_{\Gamma,\mathbf{G}}$ as the geometric realisation in the category of étale sheaves. We simplify the definition in order to prove the final statement in the following lemma. The difference between presheaves and sheaves is immaterial for our purposes since we are only interested in the local geometry of $\mathcal{X}_{\Gamma,\mathbf{G}}$.

Lemma 3.19. *The functors $\mathcal{X}^\square_{\Gamma,\mathbf{G}}$ and $\mathcal{X}_{\Gamma,\mathbf{G}}$ of Definitions 3.14 and 3.17 commute with filtered colimits and finite limits. If $A \in \mathcal{A}lg_{\mathcal{O}}$ is an animated $\mathcal{O}$-algebra such that $\pi_i(A)$ is a finitely generated $\mathcal{O}$-module for every $i \geq 0$, then*

$$\mathcal{X}^\square_{\Gamma,\mathbf{G}}(\lim A/^{\mathbf{L}}\varpi^r) = \lim \mathcal{X}^\square_{\Gamma,\mathbf{G}}(A/^{\mathbf{L}}\varpi^r), \quad \mathcal{X}_{\Gamma,\mathbf{G}}(\lim A/^{\mathbf{L}}\varpi^r) = \lim \mathcal{X}_{\Gamma,\mathbf{G}}(A/^{\mathbf{L}}\varpi^r).$$

Proof. By definition,

$$\mathcal{X}^\square_{\Gamma,\mathbf{G}} = \mathrm{Hom}_{\mathcal{F}un(\Delta,\mathcal{A}lg_{\mathcal{O}})}(\mathcal{O}[\mathbf{G}^\bullet], -) \circ C_{\mathrm{sc}}(\Gamma, -).$$

The functor $C_{\mathrm{sc}}(\Gamma, -)$ preserves filtered colimits and finite limits by Lemma 3.11, and limits of the form $\lim A/^{\mathbf{L}}\varpi^r$ for A as in the statement by Lemma 3.11. The functor $\mathrm{Hom}_{\mathcal{F}un(\Delta,\mathcal{A}lg_{\mathcal{O}})}(\mathcal{O}[\mathbf{G}^\bullet], -)$ commutes with all limits and filtered colimits since the cosimplicial algebra $\mathcal{O}[\mathbf{G}^\bullet]$ is a compact object in $\mathcal{F}un(\Delta, \mathcal{A}lg_{\mathcal{O}})$ by Lemma 3.13. This completes the proof for the $\mathcal{X}^\square_{\Gamma,\mathbf{G}}$. Since geometric realisation of simplicial sets is a Quillen equivalence, it commutes with all colimits, finite limits and inverse limits. Thus, the same holds for presheaves and the statement for $\mathcal{X}_{\Gamma,\mathbf{G}}$ follows. ☐

Remark 3.20. Given a residual representation $\bar\rho \in \mathcal{X}^\square_{\Gamma,\mathbf{G}}(k)$, the functor $(\mathcal{X}^\square_{\Gamma,\mathbf{G}})_{\bar\rho}$ is the same as the one defined by Galatius–Venkatesh in [GV18, Def. 5.4] (see [Zhu21, p.20]).

3.5 Deformation problems

In this section, we discuss the notion of deformation problems and T-framed deformations in the derived setting. We let $\Gamma = \mathrm{Gal}(\overline{F}/F)$. Note that if $\Gamma_v \subset \Gamma$ is a decomposition group at a place v of F, there is a natural morphism $\mathcal{X}^\square_{\Gamma,\mathbf{G}} \to \mathcal{X}^\square_{\Gamma_v,\mathbf{G}}$, and similarly for the other deformation functors.

We fix an absolutely irreducible $\bar\rho \in \mathcal{X}^\square_{\Gamma,\mathbf{G}}(k)$ with restrictions $\bar\rho_v \coloneqq \bar\rho|_{\Gamma_v} \in \mathcal{X}^\square_{\Gamma_v,\mathbf{G}}(k)$ and simplify the notation by defining:

$$\mathcal{X}^\square_{\bar\rho} = (\mathcal{X}^\square_{\Gamma,\mathbf{G}})_{\bar\rho}$$
$$\mathcal{X}^\square_{\bar\rho_v} = (\mathcal{X}^\square_{\Gamma_v,\mathbf{G}})_{\bar\rho_v}$$

Definition 3.21. A deformation problem is a tuple

$$\mathbf{S} = (\bar\rho, S, \{\mathcal{D}_v\}_{v\in S})$$

where $\bar\rho\colon \Gamma \to \mathrm{GL}_n(k)$ is an absolutely irreducible representation, S is a finite set of finite places of F and $\{\mathcal{D}_v\}_{v\in S}$ is a set of functors $\mathcal{D}_v\colon \mathcal{A}rt_k \to \mathcal{A}ni$ equipped with actions of $\mathbf{PG}$ and equivariant maps $\mathcal{D}_v \to \mathcal{X}^\square_{\bar\rho_v}$.

Definition 3.22. Let $\mathbf{S} = (\bar{\rho}, S, \{\mathcal{D}_v\}_{v\in S})$ be a deformation problem. We define

$$\mathcal{X}_{\mathbf{S}}^{\square} = \mathcal{X}_{\bar{\rho}}^{\square} \times_{\prod_{v\in S} \mathcal{X}_{\bar{\rho}_v}^{\square}} \prod_{v\in S} \mathcal{D}_v$$

which we call the stack of framed deformations of type $\mathbf{S}$. Since the morphisms $\mathcal{D}_v \to \mathcal{X}_v^{\square}$ are equivariant under the conjugation action, we may pass to the quotient as in Definition 3.17 to obtain the stack of deformations of type $\mathbf{S}$, denoted

$$\mathcal{X}_{\mathbf{S}} = |\mathcal{X}_{\mathbf{S}}^{\square} \times \mathbf{PG}^{\bullet}|.$$

We let $\hat{\mathbf{G}} \colon \mathcal{A}rt_k \to \mathcal{A}ni$ be the functor $\mathbf{G}(B \to k) = \mathbf{G}(B) \times_{\mathbf{G}(k)} \{1\}$.

Definition 3.23. Let $\mathbf{S} = (\bar{\rho}, S, \{\mathcal{D}_v\}_{v\in S})$ be a deformation problem and suppose $T \subseteq S$ is a subset. We define the functor of T-framed liftings of type $\mathbf{S}$ as

$$\mathcal{X}_{\mathbf{S}}^{\square,\square_T} = \mathcal{X}_{\mathbf{S}}^{\square} \times \prod_{v\in T} \hat{\mathbf{G}}$$

There exists an action of $\hat{\mathbf{G}}$ on $\mathcal{X}_{\bar{\rho}}^{\square,\square_T}$ given by

$$\gamma \colon (\rho_A, \{\alpha_v\}_{v\in T}) \mapsto (\gamma\rho_A\gamma^{-1}, \{\gamma\alpha_v\}_{v\in T}).$$

We define the stack of T-framed deformations of type $\mathbf{S}$ as the geometric realisation (in $\mathrm{PSh}(\mathcal{A}rt_k)$)

$$\mathcal{X}_{\bar{\rho}}^{\square_T} = |\mathcal{X}_{\bar{\rho}}^{\square,\square_T} \times \hat{\mathbf{G}}^{\bullet}|.$$

Proposition 3.24. *Let $\mathbf{S} = (\bar{\rho}, S, \{\mathcal{D}_v\}_{v\in S})$ be a deformation problem and $T \subset S$. Then the natural morphism $\mathcal{X}_{\mathbf{S}}^{T} \to \mathcal{X}_{\mathbf{S}}$ is formally smooth of relative dimension $n^2|T| - 1$.*

Example 3.25. Let $v \in S_p$ and denote by $R_v^{\square}$ the usual framed deformation ring of $\bar{\rho}_v$. Then $\mathcal{X}_{\bar{\rho}_v}^{\square}$ is represented by $R_v^{\square}$ by [BIPs23, Thm. 1.1] and [GV18, Lem. 7.5], and any conjugation-invariant quotient $R_v^{\square} \twoheadrightarrow R_v^{\square,D}$ defines a local condition $\mathcal{D}_v$ using the morphisms

$$\mathcal{D}_v = \mathrm{Map}(R_v^{\square,D}, -) \to \mathrm{Map}(R_v^{\square}, -) \simeq \mathcal{X}_{\bar{\rho}_v}^{\square}.$$

Although Definition 3.22 is quite general, we will only consider local conditions arising in this way.

At places not dividing p, we will have trivial local conditions, simply denoted $\mathcal{D}_v^{\square}$. At the places dividing p, our local conditions are provided by the following theorem.

Theorem 3.26. *Let $v \mid p$ and $r \geq 0$. Suppose $\bar{\rho}_v$ is torsion semistable with Hodge–Tate weights contained in $[-r, r]$. Then we have the following.*

(1) There exists a well-defined local condition $\mathcal{D}_v^{ss,r} \to \mathcal{D}_v^{\square}$ represented by a quotient $R_v^{\square} \twoheadrightarrow R_v^{ss,r}$.

(2) If ρ_v is a characteristic 0 lift of $\bar{\rho}_v$ such that the attached Weil–Deligne representation $\mathrm{WD}(\rho_v)$ is generic, then the corresponding point of $\operatorname{Spec} R_v^{ss,r}[1/\varpi]$ is smooth with tangent space given by the geometric Selmer group $H_g^1(\Gamma_v, \operatorname{ad}\rho)$.

Proof. See the main theorem of [Liu07]. For the tangent space, see [All16, Prop. 1.3.12]. $\square$

3.6 Tangent complexes and representability

Definition 3.27. Let $\mathcal{X}\colon \mathcal{A}lg_{\mathcal{O}} \to \mathcal{A}ni$ be a functor and fix $x \in \mathcal{X}(A)$ for some static $A \in \mathcal{A}lg_{\mathcal{O}}$. The tangent complex of $\mathcal{X}$ at x is the unique A-module $\mathrm{T}^\bullet_A(\mathcal{X})$ such that for every connective A-module $M \in \mathcal{M}od_A$,

$$\mathrm{Map}_{/A}(\mathcal{R}, A \oplus M) \simeq \mathrm{Map}_{\mathcal{M}od(A)}\big(\mathrm{T}^\bullet_A(\mathcal{X}), M\big)$$

If $\mathcal{X}$ is represented by an algebra $\mathcal{R}$ and $x\colon \mathcal{R} \to A$, we write $\mathrm{T}^\bullet_A(\mathcal{R}) \coloneqq \mathrm{T}^\bullet_A(\mathcal{X})$.

Definition 3.28. Let $\mathcal{R} \in \mathcal{A}lg_{\mathcal{O}}$ and let $x\colon \mathcal{R} \to A$ be a morphism. The tangent complex of $\mathcal{R}$ at x is defined as

$$\mathrm{T}^\bullet_A(\mathcal{R}) = \mathrm{Map}_{\mathcal{M}od_{\mathcal{R}}}(\mathrm{L}_{\mathcal{R}/\mathcal{O}}, A) \cong \mathrm{Map}_{\mathcal{M}od A}(\mathrm{L}_{\mathcal{R}/\mathcal{O}} \otimes^{\mathbf{L}}_{\mathcal{R}} A, A)$$

where $\mathrm{L}_{\mathcal{R}/\mathcal{O}}$ is the algebraic cotangent complex of $\mathcal{R}$ over $\mathcal{O}$.

Proposition 3.29. *[Lur04, Def. 6.2.1] A functor $\mathcal{F}\colon \mathcal{A}rt_k \to \mathcal{A}ni$ (resp. $\mathcal{F}\colon \mathcal{A}rt_E \to \mathcal{A}ni$) is formally cohesive if:*

(1) $\mathcal{F}(k)$ is contractible (resp. $\mathcal{F}(E)$ is contractible).

(2) $\mathcal{F}$ preserves pullbacks.

Proposition 3.30. *Let $(\mathcal{C}_N)$ be an inverse system in $\mathcal{A}lg_{\mathcal{O}}$ and $\mathcal{C}_\infty = \lim_N \mathcal{C}_N$. Then*

$$\mathrm{T}^\bullet_{\mathcal{O}_r}(\mathcal{C}_\infty) = \operatorname*{colim}_N \mathrm{T}^\bullet_{\mathcal{O}_r}(\mathcal{C}_N).$$

Proof. We have

$$\begin{aligned}\mathrm{T}^i_{\mathcal{O}_r}(\mathcal{C}_\infty) &= \pi_0\big(\mathrm{Map}_{/\mathcal{O}_r}(\mathcal{C}_\infty, \mathcal{O}_r \oplus \mathcal{O}_r[i])\big)\\ &\simeq \operatorname*{colim}_N \pi_0\big(\mathrm{Map}_{/\mathcal{O}_r}(\mathcal{C}_N, \mathcal{O}_r \oplus \mathcal{O}_r[i])\big)\\ &\simeq \operatorname*{colim}_N \mathrm{T}^i_{\mathcal{O}_r}(\mathcal{C}_N).\end{aligned}$$

□

Theorem 3.31. *[Lur04, 6.2.14] Let $\mathcal{F}\colon \mathcal{A}rt_k \to \mathcal{A}ni$ be a formally cohesive functor. The following are equivalent:*

(1) There exists an $\mathcal{R} \in \mathcal{CNL}_k$ and an equivalence of functors

$$\mathcal{F} \simeq \mathrm{Hom}_{(\mathcal{A}lg_{\mathcal{O}})_{/k}}(\mathcal{R}, -)$$

(2) The k-vector spaces $\mathrm{T}^i(\mathcal{F}, k)$ vanish for $i < 0$ and are finite dimensional for $i \geq 0$.

The analogous statement holds for a formally cohesive functor $\mathcal{F}\colon \mathcal{A}rt_E \to \mathcal{A}ni$.

Given $\mathcal{R} \in \mathcal{CNL}_k$ with $\mathfrak{m} \subset \pi_0(\mathcal{R})$ its maximal ideal and an arbitrary $B \in \mathcal{A}rt_k$, we have natural equivalences

$$\mathrm{Map}_{/k}(\mathcal{R}, B) \simeq \mathrm{Map}_{/k}(\lim_r \mathcal{R}/^{\mathbf{L}}\mathfrak{m}^r, B) \simeq \operatorname*{colim}_r \mathrm{Map}_{/k}(\mathcal{R}/^{\mathbf{L}}\mathfrak{m}^r, B).$$

The first isomorphism holds because $\mathcal{R}$ is derived $\mathfrak{m}$-adically complete and the second holds by Prop. 2.6 and the assumption on B. Since $\mathcal{R}$ is Noetherian, by [Lur04, Prop. 6.1.8] we have an isomorphism

$$\pi_0(\lim \mathcal{R}/^{\mathbf{L}}\mathfrak{m}^r) \cong \varprojlim \pi_0(\mathcal{R}).$$

The algebras $\mathcal{R}/^{\mathbf{L}}\mathfrak{m}^r$ might not lie in $\mathcal{A}rt_k$ as they need not be truncated, but we can associate a pro-object $\{\tau_{\leq i}(\mathcal{R}/^{\mathbf{L}}/\mathfrak{m}^r)\} \in \mathrm{Pro}(\mathcal{A}rt_k)$. In this way, we can rephrase statement (1) of Theorem 3.31 in terms of pro-representability. This is the viewpoint chosen throughout [GV18], but we will use complete Noetherian local rings.

Proposition 3.32. *Let $\mathcal{R} \in \mathcal{CNL}_k$ and let $\mathcal{R} \to \mathcal{O}_r$ be a morphism over k. Then for every $i \geq 0$, $\mathrm{T}^i_{\mathcal{O}_r}(\mathcal{R})$ is a finitely generated $\mathcal{O}_r$-module.*

Proof. The case $r = 1$ is Theorem 3.31. By considering the short exact sequence

$$0 \to \mathcal{O}_r \to \mathcal{O}_{r+1} \to k \to 0$$

we deduce the analogous statement for $\mathcal{O}_r$-coefficients using induction. $\square$

Corollary 3.33. *Let $\mathcal{R} \in \mathcal{CNL}_k$ and let $\mathcal{R} \to \mathcal{O}$ be a morphism over k. Then*

$$\mathrm{T}^i_{\mathcal{O}}(\mathcal{R}) \cong \varprojlim_r \mathrm{T}^i_{\mathcal{O}_r}(\mathcal{R})$$

Proof. By definition,

$$\mathrm{T}^\bullet_{\mathcal{O}}(\mathcal{R}) = \mathrm{Map}_{\mathrm{Mod}\,\mathcal{R}}(\mathrm{L}_{\mathcal{R}/\mathcal{O}}, \mathcal{O}) \simeq \lim_r \mathrm{Map}_{\mathrm{Mod}\,\mathcal{R}}(\mathrm{L}_{\mathcal{R}/\mathcal{O}}, \mathcal{O}_r).$$

By Prop. 3.32, the derived functors of $\varprojlim_r$ vanish and hence $\mathrm{T}^*_{\mathcal{O}}(\mathcal{R}) = \varprojlim_r \mathrm{T}^*_r(\mathcal{R})$. $\square$

The tangent complexes of the moduli functors introduced above are given by Galois cohomology groups.

Proposition 3.34. *[Zhu21, Prop. 2.4.24] Let $\bar{\rho} \in \mathcal{X}^\square_{\Gamma,\mathbf{G}}(k)$. The tangent complex of $\mathcal{X}^\square_{\Gamma,\mathbf{G}}$ at $\bar{\rho}$ is given by*

$$\mathrm{T}^\bullet_k\left((\mathcal{X}^\square_{\Gamma,\mathbf{G}})_{\bar{\rho}}\right) \simeq \overline{C}^\bullet_{\mathrm{sc}}(\Gamma, \mathrm{ad}\,\bar{\rho})[1]$$

where $\overline{C}^\bullet(\Gamma, \mathrm{ad}\,\bar{\rho})[1]$ is the cofibre of the natural map $C^\bullet(\Gamma, \mathrm{ad}\,\bar{\rho}) \to \mathrm{ad}\,\bar{\rho}$. In particular,

$$\mathrm{T}^i_k(\mathcal{X}^\square_{\Gamma,\mathbf{G}}) \cong \pi_{-i}(\overline{C}^\bullet(\Gamma, \mathrm{ad}\,\bar{\rho})[1]) \cong \begin{cases} Z^1(\Gamma, \mathrm{ad}\,\bar{\rho}) & \text{if } i = 0, \\ H^2(\Gamma, \mathrm{ad}\,\bar{\rho}) & \text{if } i = 1, \\ 0 & \text{otherwise.} \end{cases}$$

The tangent complex of $\mathcal{X}_{\Gamma,\mathbf{G}}$ at the point corresponding to $\bar{\rho}$ is given by

$$\mathrm{T}^i_k(\mathcal{X}^\square_{\Gamma,\mathbf{G}}) \cong \pi_{-i}(\overline{C}^\bullet(\Gamma, \mathrm{ad}\,\bar{\rho})[1]) \cong \begin{cases} H^0(\Gamma, \mathrm{ad}^0\,\bar{\rho}) & \text{if } i = -1 \\ H^1(\Gamma, \mathrm{ad}\,\bar{\rho}) & \text{if } i = 0, \\ H^2(\Gamma, \mathrm{ad}\,\bar{\rho}) & \text{if } i = 1, \\ 0 & \text{otherwise.} \end{cases}$$

and in particular, if $\bar{\rho}$ is absolutely irreducible then

$$\mathrm{T}^i_k(\mathcal{X}^\square_{\Gamma,\mathbf{G}}) \cong H^{i+1}(\Gamma, \mathrm{ad}\,\bar{\rho}).$$

The analogous statements are true for $\overline{\xi}_r \in \mathcal{X}^\square_{\Gamma,\mathbf{G}}(\mathcal{O}_r)$ and $\rho \in \mathcal{X}^\square_{\Gamma,\mathbf{G}}(E)$.

Corollary 3.35. *Let $\Gamma = \mathrm{Gal}(F_S/F)$ or $\mathrm{Gal}(\overline{F}_v/F_v)$ and fix $\bar\rho \in \mathcal{X}^{\square}_{\Gamma,\mathbf{G}}(k)$. The functor $(\mathcal{X}^{\square}_{\Gamma,\mathbf{G}})_{\bar\rho}$ is representable and if $\bar\rho$ is Schur, $(\mathcal{X}_{\Gamma,\mathbf{G}})_{\bar\rho}$ is representable. The same statement is true for $\rho \in \mathcal{X}^{\square}_{\Gamma,\mathbf{G}}(E)$.*

Proof. This follows from Proposition 3.34 and Theorem 3.31. □

Lemma 3.36. *Suppose we have morphisms $A \to B \to \mathcal{O}$ in $\mathcal{A}lg_{\mathcal{O}}$ and that B is finitely presented over A. Then, if $\mathrm{T}^i_{\mathcal{O}}(A)$ is a finitely generated $\mathcal{O}$-module for every $i \geq 0$, the same holds for $\mathrm{T}^i_{\mathcal{O}}(B)$.*

Proof. By [Lur04, Prop. 3.2.14], the relative cotangent complex $\mathrm{L}_{B/A}$ is an almost perfect animated B-module, i.e. $\pi_i(\mathrm{L}_{B/A})$ is a finitely generated $\pi_0(B)$-module for every $i \geq 0$. It follows that $\mathrm{L}_{B/A} \otimes^{\mathbf{L}}_B \mathcal{O}$ is a perfect animated $\mathcal{O}$-module. Dualising the fibre sequence

$$\mathrm{L}_{A/\mathcal{O}} \otimes^{\mathbf{L}}_A \mathcal{O} \to \mathrm{L}_{B/\mathcal{O}} \otimes^{\mathbf{L}}_B \mathcal{O} \to \mathrm{L}_{B/A} \otimes^{\mathbf{L}}_B \mathcal{O} \xrightarrow{+1}$$

we obtain a fibre sequence of tangent complexes with $\mathcal{O}$-coefficients

$$\mathrm{T}^{\bullet}_{\mathcal{O}}(B/A) \to \mathrm{T}^{\bullet}_{\mathcal{O}}(B) \to \mathrm{T}^{\bullet}_{\mathcal{O}}(A) \xrightarrow{+1} .$$

By assumption, $\mathrm{T}^i_{\mathcal{O}}(A)$ is finitely generated over $\mathcal{O}$ for every i. The same now follows for $\mathrm{T}^{\bullet}_{\mathcal{O}}(B)$. □

3.7 A computation in Galois cohomology

Proposition 3.37. *Let $\rho\colon \Gamma_{F,S} \to \mathrm{GL}_n(E)$ be a continuous representation, such that:*

(1) ρ is irreducible.

(2) For every $v \in S \smallsetminus S_p$, the Weil-Deligne representation $\mathrm{WD}(\rho_v)$ is generic.

(3) For every $v \in S_p$, ρ_v is de Rham with distinct Hodge-Tate weights.

Then

$$h^1_g(\Gamma_{F,S}, \mathrm{ad}\,\rho) = h^1_g(\Gamma_{F,S}, \mathrm{ad}\,\rho(1)) - l_0.$$

Proof. By the Greenberg–Wiles formula,

$$\begin{aligned} &h^1_g(\Gamma_{F,S}, \mathrm{ad}\,\rho) - h^1_g(\Gamma_{F,S}, \mathrm{ad}\,\rho(1)) = \\ &\qquad \sum_{v\in S} \big(h^1_g(\Gamma_v, \mathrm{ad}\,\rho) - h^0(\Gamma_v, \mathrm{ad}\,\rho)\big) + h^0(\Gamma, \mathrm{ad}\,\rho) - h^0(\Gamma, \mathrm{ad}\,\rho(1)) - \sum_{v\mid\infty} h^0(\Gamma_v, \mathrm{ad}\,\rho). \end{aligned}$$

Assumptions (1) and (2) imply the vanishing of the 'global terms' and terms in the first sum corresponding to $v \in S \smallsetminus S_p$, respectively. Hence, we have

$$h^1_g(\Gamma_{F,S}, \mathrm{ad}\,\rho) - h^1_g(\Gamma_{F,S}, \mathrm{ad}\,\rho(1)) = \sum_{v\in S_p} \big(h^1_g(\Gamma_v, \mathrm{ad}\,\rho) - h^0(\Gamma_v, \mathrm{ad}\,\rho)\big) - \sum_{v\mid\infty} h^0(\Gamma_v, \mathrm{ad}\,\rho).$$

Now, the difference $h^1_g(\Gamma_v, \mathrm{ad}\,\rho) - h^0(\Gamma_v, \mathrm{ad}\,\rho)$ equals the number of negative Hodge-Tate weights of $\mathrm{ad}\,\rho|_{\Gamma_v}$, and by assumption (3) this number is $n(n+1)/2$ for every $v \in S_p$. The result now follows from the formula for l_0 (see §2.2). □

3.8 Localisation at a characteristic 0 point

In this section, we prove two results which generalise results of Kisin [Kis09, Lem. 2.3.3, Prop. 2.3.5] relating the deformation ring of a characteristic 0 representation to that of its reduction mod ϖ.

In the following definition, we use the functor $\operatorname{Int} B \to (\mathcal{A}lg_{\mathcal{O}})_{/\mathcal{O}}$ of Lemma 3.10 to view $A \in \operatorname{Int} B$ as an object of $(\mathcal{A}lg_{\mathcal{O}})_{/\mathcal{O}}$.

Definition 3.38. Let $\mathcal{F}_{\bar{\rho}} \colon \mathcal{A}rt_k \to \mathcal{A}ni$ be a functor and $\xi \in \mathcal{F}_{\bar{\rho}}(\mathcal{O}) = \lim \mathcal{F}_{\bar{\rho}}(\mathcal{O}_r)$. We define

$$\begin{aligned} \mathcal{F}_{\bar{\rho},(\xi)} \colon \mathcal{A}rt_E &\to \mathcal{A}ni \\ (B \to E) &\mapsto \operatorname*{colim}_{A \in \operatorname{Int} B} \mathcal{F}(A) \times_{\mathcal{F}(\mathcal{O})} \{\xi\} \end{aligned}$$

Theorem 3.39. *Let $\xi \in \mathcal{X}^{\square}_{\bar{\rho}}(\mathcal{O})$ and $\rho = \xi[1/\varpi] \in \mathcal{X}^{\square}(E)$. Then there are natural weak equivalences of functors*

$$\mathcal{X}^{\square}_{\bar{\rho},(\xi)} \simeq \mathcal{X}^{\square}_{\rho} \text{ and } \mathcal{X}_{\bar{\rho},(\xi)} \simeq \mathcal{X}_{\rho}.$$

Proof. Let $(B \to E) \in \mathcal{A}rt_E$. By definition, Lemma 3.9, Lemma 3.19, and the fact that filtered colimits commute with finite limits,

$$\begin{aligned} \mathcal{X}_{\bar{\rho},(\xi)}(B) &= \operatorname*{colim}_{A \in \operatorname{Int} B} (\mathcal{X}_{\bar{\rho}}(A) \times_{\mathcal{X}_{\bar{\rho}}(\mathcal{O})} \{\xi\}) \\ &\simeq \operatorname*{colim}_{A \in \operatorname{Int} B} (\mathcal{X}(A) \times_{\mathcal{X}(\mathcal{O})} \{\xi\}) \\ &\simeq (\mathcal{X}(\operatorname*{colim}_{A \in \operatorname{Int} B} A) \times_{\mathcal{X}(\mathcal{O})} \{\xi\}) \\ &\simeq \mathcal{X}(B \times_E \mathcal{O}) \times_{\mathcal{X}(\mathcal{O})} \{\xi\} \\ &\simeq \mathcal{X}(B) \times_{\mathcal{X}(E)} \{\rho\} \end{aligned}$$

The proof for the framed functor is identical. $\square$

Theorem 3.40. *Suppose $\mathcal{X}_{\bar{\rho}} \colon \mathcal{A}rt_k \to \mathcal{A}ni$ is a formally cohesive functor represented by $\mathcal{R} \in \mathcal{CNL}_k$, and let $\xi \in \mathcal{X}_{\bar{\rho}}(\mathcal{O})$. Then if $\mathcal{X}_{\bar{\rho}}$ commutes with filtered colimits, $\mathcal{X}_{\bar{\rho},(\xi)}$ is represented by*

$$(\mathcal{R} \otimes_{\mathcal{O}} E)^{\wedge}_{\xi},$$

the completion of $\mathcal{R} \otimes_{\mathcal{O}} E$ at the maximal ideal of $\pi_0(\mathcal{R} \otimes_{\mathcal{O}} E)$ defined by ξ.

Proof. Let $(B \to E) \in \mathcal{A}rt_E$ and let $\xi_r \in \mathcal{X}_{\bar{\rho}}(\mathcal{O}_r)$ denote the mod ϖ^r reduction of ξ. By definition,

$$\mathcal{X}_{\bar{\rho},(\xi)}(B) = \operatorname*{colim}_{A \in \operatorname{Int} B} (\mathcal{X}_{\bar{\rho}}(A) \times_{\mathcal{X}_{\bar{\rho}}(\mathcal{O})} \{\xi\}).$$

Now, by Lemmas 3.9 and 3.19,

$$\begin{aligned} \mathcal{X}_{\bar{\rho}}(A) \times_{\mathcal{X}_{\bar{\rho}}(\mathcal{O})} \{\xi\} &\simeq \lim_r \left(\mathcal{X}_{\bar{\rho}}(A/^{\mathbf{L}}\varpi^r) \times_{\mathcal{X}_{\bar{\rho}}(\mathcal{O}/^{\mathbf{L}}\varpi^r)} \{\xi_r\} \right) \\ &\simeq \lim_r \left(\operatorname{Map}_{/k}(\mathcal{R}_{\bar{\rho}}, A/^{\mathbf{L}}\varpi^r) \times_{\operatorname{Map}_{/k}(\mathcal{R}_{\bar{\rho}}, \mathcal{O}/^{\mathbf{L}}\varpi^r)} \{\xi_r\} \right) \\ &\simeq \operatorname{Map}_{/k}(\mathcal{R}_{\bar{\rho}}, A) \times_{\operatorname{Map}_{/k}(\mathcal{R}_{\bar{\rho}}, \mathcal{O})} \{\xi\} \\ &\simeq \operatorname{Map}_{/\mathcal{O}}(\mathcal{R}_{\bar{\rho}}, A) \end{aligned}$$

Thus, we have

$$\mathcal{X}_{\bar{\rho},(\xi)}(B) = \operatorname*{colim}_{A \in \operatorname{Int} B} \operatorname{Map}_{/\mathcal{O}}(\mathcal{R}_{\bar{\rho}}, A).$$

Now, Proposition 2.6 and the injectivity of $\pi_0(A) \to \pi_0(B)$ for any $A \in \operatorname{Int} B$ together imply that

$$\operatorname{Map}_{/\mathcal{O}}(\mathcal{R}_{\bar\rho}, A) \simeq \operatorname*{colim}_r \operatorname{Map}_{/\mathcal{O}}(\mathcal{R}_{\bar\rho}/^{\mathbf{L}}(I_\xi^\circ)^r, A)$$

where $I_\xi^\circ \subset \pi_0(\mathcal{R}_{\bar\rho})$ is the inverse image of $I_\xi = \ker(\pi_0(\mathcal{R}_{\bar\rho}) \otimes_{\mathcal{O}} E \to E)$. Since every $A \in \operatorname{Int} B$ is n-truncated for n sufficiently big and depending only on B, and $\mathcal{R}_{\bar\rho}/^{\mathbf{L}}(I_\xi^\circ)^r$ is almost of finite presentation in $\mathcal{A}lg_{\mathcal{O}}$ we therefore have (using Lemma 3.9)

$$\begin{aligned}\operatorname*{colim}_{A\in \operatorname{Int} B} \operatorname{Map}_{/\mathcal{O}}(\mathcal{R}_{\bar\rho}, A) &\simeq \operatorname*{colim}_r \operatorname{Map}_{/\mathcal{O}}(\mathcal{R}_{\bar\rho}/^{\mathbf{L}}(I_\xi^\circ)^r, \operatorname*{colim}_{A\in \operatorname{Int} B} A) \\ &\simeq \operatorname*{colim}_r \operatorname{Map}_{/\mathcal{O}}(\mathcal{R}_{\bar\rho}/^{\mathbf{L}}(I_\xi^\circ)^r, B \times_E \mathcal{O}) \\ &\simeq \operatorname*{colim}_r \operatorname{Map}_{/E}((\mathcal{R}_{\bar\rho} \otimes_{\mathcal{O}} E)/^{\mathbf{L}} I_\xi^r, B).\end{aligned}$$

Thus,

$$\mathcal{X}_{\bar\rho,(\xi)}(B \to E) \simeq \operatorname{Map}_{/E}\left((\mathcal{R} \otimes_{\mathcal{O}} E)^\wedge_\xi, B\right)$$

as required. □

Proposition 3.41. *Let $F\colon \mathcal{A}rt_k \to \mathcal{A}ni$ be a formally cohesive functor and let $\xi \in F(\mathcal{O})$. Then $F_{(\xi)}\colon \mathcal{A}rt_E \to \mathcal{A}ni$ has tangent complex*

$$\left(\lim_r \mathrm{T}^\bullet_{\mathcal{O}_r}(F_{(\xi)})\right) \otimes_{\mathcal{O}} E$$

In particular, if $F = \operatorname{Map}_{/k}(\mathcal{R}, -)$ for $\mathcal{R} \in \mathcal{CNL}_k$ then, with notation as in Proposition 3.40,

$$\mathrm{T}^\bullet_E\left((\mathcal{R} \otimes_{\mathcal{O}} E)^\wedge_\xi\right) = \left(\lim_r \mathrm{T}^\bullet_{\mathcal{O}_r}(\mathcal{R})\right) \otimes_{\mathcal{O}} E$$

Proof. Let $L = L_{F_{(\xi)}}$ denote the cotangent complex of $F_{(\xi)}$ over $\mathcal{O}$. Then

$$\begin{aligned}T^\bullet_E(F_{(\xi)}) &= \operatorname{Map}_E(L_{F_{(\xi)}} \otimes^{\mathbf{L}} E, E) \\ &= \operatorname{Map}_{\mathcal{O}}(L_{F_{(\xi)}} \otimes^{\mathbf{L}} E, \mathcal{O}) \otimes_{\mathcal{O}} E \\ &= \operatorname{Map}(L_{F_{(\xi)}}, \mathcal{O}) \otimes_{\mathcal{O}} E \\ &= \left(\lim_r \operatorname{Map}(L_{F_{(\xi)}}, \mathcal{O}_r)\right) \otimes_{\mathcal{O}} E \\ &= (\lim_r \mathrm{T}^\bullet_{\mathcal{O}_r}(F_{(\xi)})) \otimes_{\mathcal{O}} E.\end{aligned}$$

□

4 Patching and Venkatesh's conjecture

In this section, we prove the following result.

Theorem 4.1. *Let Π be a regular algebraic cuspidal automorphic representation of $\mathrm{GL}_n(\mathbb{A}_F)$, cohomological with respect to an algebraic representation $\mathcal{V}_\lambda$ where $\lambda \in (\mathbb{Z}^n)^{\operatorname{Hom}(F,E)}$, and let $\rho = \rho_{\Pi,\iota}\colon \Gamma_F \to \mathrm{GL}_n(E)$ denote the Galois representation attached to (Π, ι) where $\iota\colon \overline{\mathbb{Q}}_p \xrightarrow{\sim} \mathbb{C}$ and $E/\mathbb{Q}_p$ is a finite extension containing all embeddings $F \to \overline{\mathbb{Q}}_p$. Suppose the following:*

(1) $p \nmid 2n$.

(2) The reduction $\bar{\rho}$ is absolutely irreducible and decomposed generic [ACC$^+$23, Def 4.3.1].

(3) $\zeta_p \notin F$ and $\bar{\rho}|_{\Gamma_{F(\zeta_p)}}$ has enormous image [ACC$^+$23, Def 6.2.28].

(4) For every $v \in S \setminus S_p$, the Weil-Deligne representation $\mathrm{WD}(\rho|_{\Gamma_{F_v}})$ is generic.

(5) For every $v \in S_p$, $\rho|_{\Gamma_{F_v}}$ is de Rham and the Weil-Deligne representation $\mathrm{WD}(\rho|_{\Gamma_{F_v}})$ is generic.

(6) For each embedding $\tau : F \hookrightarrow E$, we have $-\lambda_{\tau c,1} - \lambda_{\tau,1} \geq 0$.

(7) For every $v \in S_p$, the group K_v contains the Iwahori subgroup Iw_v.

(8) For every pair $\bar{v} \neq \bar{v}'$ of places of F^+ above p, we have

$$\sum_{\bar{w} \leq \bar{v} \in \bar{S}_p} [F^+_{\bar{w}} : \mathbb{Q}_p] > \frac{1}{2}[F^+ : \mathbb{Q}].$$

(9) Π is tempered at all finite places.

Let $\mathcal{R}_{\mathbf{S}}$ denote the derived deformation ring representing the deformation problem

$$\mathbf{S} = (\bar{\rho}, S, \{\mathcal{D}_v\}_{v \in S}), \quad \mathcal{D}_v = \begin{cases} \mathcal{D}_v^{ss,r} & \text{if } v \in S_p \\ \mathcal{D}_v^{\square} & \text{if } v \in S \smallsetminus S_p. \end{cases}$$

Then, for $\mathfrak{p} \subset \pi_0(\mathcal{R})$ the prime ideal corresponding to ρ and $\mathcal{R}_\rho = (\mathcal{R} \otimes_{\mathcal{O}} E)^\wedge_\xi$, there is an action of $\mathcal{R}_\rho$ on $R\Gamma(K, \mathcal{V}_\lambda)_\mathfrak{p}$ such that the induced action

$$\pi_*(\mathcal{R}_\rho) \circlearrowright H^*(X_K, \mathcal{V}_\lambda)_\mathfrak{p}$$

is free.

Remark 4.2. The action of $\mathcal{R}_\rho$ is not proven to be natural and a priori depends on the non-canonical choice inherent to the Taylor–Wiles method. With stronger assumptions on $\bar{\rho}$, the naturality is proven in [GV18, §15] by identifying $\mathcal{R}_{\bar{\rho}}$ with the derived Hecke algebra.

Remark 4.3. Conditions (1)-(8) of the theorem are required to apply the main result of [A'C24], parts of which we state in Theorem 4.9. The conditions (6)-(8) are required for the local-global compatibility result [A'C24, Thm. 4.3.1]; we expect that this result can be replaced by [Hev24, Thm. 1.4] and thereby relaxing conditions (6)-(8). We require the purity assumption (9) to prove Proposition 4.5; the Ramanujan conjecture predicts that (9) always holds and the case $n = 2$ was recently proved in [BCG$^+$25].

For the remainder of the paper, fix a continuous representation $\xi \colon \Gamma_F \to \mathrm{GL}_n(\mathcal{O})$ such that

$$\xi \otimes E \colon \Gamma_F \to \mathrm{GL}_n(E)$$

satisfies the conditions in Theorem 4.1.

4.1 Taylor–Wiles primes

The basis of our patching argument later on is a delicate choice of Taylor–Wiles places, detailed in this subsection.

Definition 4.4. Let $\mathbf{S} = (\bar{\rho}, S, \{\mathcal{D}_v\}_{v\in S})$ be a deformation problem. A Taylor–Wiles datum of level N for $\mathbf{S}$ is a tuple $(Q_N, \{(\alpha_{v,1}, \dots, \alpha_{v,n})\}_{v\in Q_N})$ where:

(1) Q_N is a finite set of places of F, disjoint from S and such that for every $v \in Q_N$, $v \equiv 1 \bmod p^N$.

(2) For every $v \in Q_N$, $(\alpha_{v,1}, \dots, \alpha_{v,n}) \in k^n$ are the eigenvalues of $\bar{\rho}(\mathrm{Frob}_v)$, assumed to be pairwise distinct and k-rational.

Our main result Theorem 4.1 contains the assumption (3) that $\bar{\rho}\mid_{\Gamma_{F(\zeta_{p^\infty})}}$ has enormous image. In particular, In [NT23, Def. 2.23], an analogous property is defined for characteristic 0 representations.

math

Proposition 4.5. *Let $\mathbf{S}' = (\bar{\rho}, S, \{\mathcal{D}_v^{\square}\}_{v\in S})$ be the deformation problem with no conditions at $v \in S$, and suppose that*

- $p \nmid 2n$
- $\zeta_p \notin F$ *and* $\rho|_{\Gamma_{F(\zeta_p)}}$ *is enormous [NT23, Def. 2.23].*
- Π *is tempered at all finite places.*

Then there exists $q \geq 0$ and $l \geq 0$ such that for any $N \geq 1$, there exists a Taylor–Wiles datum $(Q_N, \{(\alpha_{v,1}, \dots, \alpha_{v,n})\}_{v\in Q_N})$ for $\mathbf{S}'$ such that

(1) $|Q_N| = q$

(2) For every $v \in Q_N$, $v \equiv 1 \bmod p^N$ and the rational prime below v splits in F_0.

(3) There is a local $\mathcal{O}$-algebra surjection $R_{\mathbf{S}}^{S,\mathrm{loc}}[[X_1, \dots, X_g]] \twoheadrightarrow R^S_{\mathbf{S}_{Q_N}}$.

(4) For every $r \leq N$,
$$\mathrm{length}_{\mathcal{O}} H^1(\Gamma_{SQ_N}, \mathrm{ad}\, \xi_r(1)) \leq l,$$
hence if $\mathcal{L}_S = \{\mathcal{L}_{v,r}\}_{v\in S}$ is a Selmer system for $\mathrm{ad}\,\xi_r$ and $\mathcal{L}_{SQ_N} = \mathcal{L}_S \cup \{H^1(\Gamma_v, \mathrm{ad}\,\xi_r)\}_{v\in Q_N}$,
$$\mathrm{length}_{\mathcal{O}} H^1_{\mathcal{L}^{\perp}_{SQ_N}}(\Gamma_{SQ_N}, \mathrm{ad}\,\xi_r(1)) \leq l.$$

Proof. By [A'C24, Prop. 6.2.32], for q large enough we can find, for every $N \geq 1$, a Taylor–Wiles datum $(Q_N, \{(\alpha_{v,1}, \dots, \alpha_{v,n})\}_{v\in Q_N})$ such that (1)-(3) hold. To achieve (4), we will enlarge the sets Q_N by adding suitable Taylor–Wiles primes, following the proof of [NT23, Lem. 2.30]. Thus, in (1) we replace if necessary q by a larger number. The splitting condition in (2) defines a set of Dirichlet density 1, so we may restrict our attention to places satisfying the condition. Enlarging Q_N does not affect (3), as the statement is equivalent to the vanishing of

$$H^1(\Gamma_{SQ_N}, \mathrm{ad}\,\bar{\rho}(1)) = \ker\left(H^1(\Gamma_{SQ_N}, \mathrm{ad}\,\bar{\rho}(1)) \to \prod_{v\in Q_N} H^1(\Gamma_v, \mathrm{ad}\,\bar{\rho}(1))\right)$$

and adding more factors to the product over v preserves the vanishing of this kernel.

The second statement in (4) follows from the first, since we have an inclusion

$$H^1_{\mathcal{L}^\perp_{SQ_N}}(\Gamma_{SQ_N}, \operatorname{ad}\xi_r(1)) \hookrightarrow H^1(\Gamma_{SQ_N}, \operatorname{ad}\xi_r(1)).$$

Let

$$\operatorname{ad}_{E/\mathcal{O}} = \operatorname{ad}\xi \otimes_{\mathcal{O}} E/\mathcal{O}$$

and similarly for $\operatorname{ad}_{\mathcal{O}}, \operatorname{ad}_E, \operatorname{ad}_{\mathcal{O}_r}$ so that $\operatorname{ad}_{E/\mathcal{O}} = \operatorname{colim}\operatorname{ad}_{\mathcal{O}_r}$. Fix an integer

$$s \geq \operatorname{corank}_{\mathcal{O}} H^1(\Gamma_S, \operatorname{ad}_{E/\mathcal{O}}(1)) \coloneqq \operatorname{rank}_{\mathcal{O}} \operatorname{Hom}_{\mathcal{O}}\left(H^1(\Gamma_S, \operatorname{ad}_{E/\mathcal{O}}(1)), E/\mathcal{O}\right).$$

Our goal is to find Frobenius elements $\sigma_1, \ldots, \sigma_s \in G_{F(\zeta_{p^\infty})}$ such that:

(i) $\rho(\sigma_i)$ has n distinct eigenvalues in E.

(ii) The kernel of the map

$$H^1(\Gamma_S, \operatorname{ad}_{E/\mathcal{O}}(1)) \to \bigoplus_{i=1}^s H^1(\langle\sigma_i\rangle, \operatorname{ad}_{E/\mathcal{O}}(1)) \cong \bigoplus_{i=1}^s \operatorname{ad}_{E/\mathcal{O}}/(\sigma_i - 1)\operatorname{ad}_{E/\mathcal{O}}$$

is of finite length as an $\mathcal{O}$-module.

Then, for every $r \leq N$, we have a diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H^0(\Gamma_S, \operatorname{ad}_{E/\mathcal{O}}(1))/\varpi^r & \longrightarrow & H^1(\Gamma_S, \operatorname{ad}_{\mathcal{O}_r}(1)) & \longrightarrow & H^1(\Gamma_S, \operatorname{ad}_{E/\mathcal{O}}(1))[\varpi^r] & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \oplus_{i=1}^s H^0(\langle\sigma_i\rangle, \operatorname{ad}_{E/\mathcal{O}}(1))/\varpi^r & \longrightarrow & \oplus_{i=1}^s H^1(\langle\sigma_i\rangle, \operatorname{ad}_{\mathcal{O}_r}(1)) & \longrightarrow & \oplus_{i=1}^s H^0(\langle\sigma_i\rangle, \operatorname{ad}_{E/\mathcal{O}}(1))[\varpi^r] & \longrightarrow & 0
\end{array}$$

where the rows are exact. Note that $H^0(\Gamma_S, \operatorname{ad}_{E/\mathcal{O}}(1))$ has finite length by irreducibility of ρ. It follows from the snake lemma that the kernel of

$$H^1(\Gamma_S, \operatorname{ad}_{\mathcal{O}_r}(1)) \to \bigoplus_{i=1}^s H^1(\langle\sigma_i\rangle, \operatorname{ad}_{\mathcal{O}_r}(1))$$

has (finite) length bounded independently of r and N. By the Chebotarev density theorem, we can then find places $v_1, \ldots v_s$ with Frobenius elements $\operatorname{Frob}_{v_1}, \ldots, \operatorname{Frob}_{v_s}$ approximating $\sigma_1, \ldots, \sigma_s$, and thus we obtain statement (4) of the proposition. Now, we have an isomorphism

$$H^1(\Gamma_S, \operatorname{ad}_{E/\mathcal{O}}(1)) \cong (E/\mathcal{O})^{s_0} \oplus M$$

where M is a finite length $\mathcal{O}$-module and $s_0 \leq s$ is the corank defined above. Thus, the problem reduces to finding, for every non-zero

$$f \colon E/\mathcal{O} \to H^1(\Gamma_S, \operatorname{ad}_{E/\mathcal{O}}(1))$$

an element $\sigma \in G_{F(\zeta_{p^\infty})}$ such that $\rho(\sigma)$ has n distinct eigenvalues and the composition

$$E/\mathcal{O} \to H^1(\Gamma_S, \operatorname{ad}_{E/\mathcal{O}}(1)) \to H^1(\langle\sigma_i\rangle, \operatorname{ad}_{E/\mathcal{O}}(1)) \cong \frac{\operatorname{ad}_{E/\mathcal{O}}(1)}{(\sigma-1)\operatorname{ad}_{E/\mathcal{O}}(1)}$$

is non-zero. Let $F_\infty = F(\zeta_{p^\infty})$ and $L_\infty = \overline{F}^{\ker \mathrm{ad}_E(1)}$. Then $F_\infty \subseteq L_\infty$ since $E(1) \subset \mathrm{ad}_E(1)$. The short exact sequencne

$$0 \to \mathrm{ad}_{\mathcal{O}} \to \mathrm{ad}_E \to \mathrm{ad}_{E/\mathcal{O}} \to 0$$

induces an exact sequence (with $\Gamma(L_\infty/F) = \mathrm{Gal}(L_\infty/F)$)

$$H^1\big(\Gamma(L_\infty/F), \mathrm{ad}_E(1)\big) \to H^1\big(\Gamma(L_\infty/F), \mathrm{ad}_{E/\mathcal{O}}(1)\big) \to H^2\big(\Gamma(L_\infty/F), \mathrm{ad}_{\mathcal{O}}(1)\big).$$

The first term vanishes by [Kis04, Lem. 6.2] and the assumption that Π is tempered at all finite places. The third term is a finitely generated $\mathcal{O}$-module. It follows that the middle term is annihilated by a power of p. Thus, the inflation-restriction sequence

$$0 \to H^1\big(\Gamma(L_\infty/F), \mathrm{ad}_{E/\mathcal{O}}(1)\big) \to H^1\big(\Gamma_S, \mathrm{ad}_{E/\mathcal{O}}(1)\big) \to H^1\big(\Gamma(F_S/L_\infty), \mathrm{ad}_{E/\mathcal{O}}(1)\big)^{\Gamma(L_\infty/F)}$$

implies that the composition

$$\begin{aligned} E/\mathcal{O} \xrightarrow{f} H^1\big(\Gamma_S, \mathrm{ad}_{E/\mathcal{O}}(1)\big) &\to H^1\big(\Gamma(F_S/L_\infty), \mathrm{ad}_{E/\mathcal{O}}(1)\big)^{\Gamma(L_\infty/F)} \\ &\cong \mathrm{Hom}_{\Gamma(F_S/F)}\big(\Gamma(L_{\infty,S_{L_\infty}}/L_\infty), \mathrm{ad}_{E/\mathcal{O}}(1)\big) \end{aligned}$$

is non-zero. Let $W \subset \mathrm{ad}_{E/\mathcal{O}}(1)$ be the $\mathcal{O}$-submodule spanned by $\{f(x)(\sigma) \mid x \in E/\mathcal{O}, \sigma \in G_{L_\infty}\}$. Then W is non-zero and divisible. By [NT23, Lem. 2.22] there exists $\sigma \in G_{F_\infty}$ such that $\rho(\sigma)$ has n distinct eigenvalues in E and $W \not\subset (\sigma - 1)\,\mathrm{ad}_{E/\mathcal{O}}(1)$. Thus, for some $m \geq 0$ and $\tau \in G_{L_\infty}$ we have

$$f(1/\varpi^m)(\tau) \notin (\sigma - 1)(\mathrm{ad}_{E/\mathcal{O}}(1)).$$

If $f(1/\varpi^m)(\sigma) \notin (\sigma - 1)(\mathrm{ad}_{E/\mathcal{O}}(1))$ then we are done. If not, then $\tau\sigma$ has the sought property, i.e.

$$E/\mathcal{O} \to H^1(F_S, \mathrm{ad}_{E/\mathcal{O}}(1)) \to \frac{\mathrm{ad}_{E/\mathcal{O}}(1)}{(\tau\sigma - 1)\,\mathrm{ad}_{E/\mathcal{O}}(1)}$$

is non-zero, since τ acts trivially on $\mathrm{ad}_E(1)$.

□

4.2 Patching

Let us recall the setup of the patching argument in [A'C24] and fix notation. Consider the deformation problems

$$\begin{aligned} \mathbf{S} &= (\bar{\rho}, S, \{\mathcal{D}_v^{ss,r}\}_{v \in S_p} \cup \{\mathcal{D}_v^{\square}\}_{v \in S \setminus S_p}) \\ \mathbf{S}' &= (\bar{\rho}, S, \{\mathcal{D}_v^{\square}\}_{v \in S}) \end{aligned}$$

Remark 4.6. As a preliminary step, to ensure goodness of the level, one should augment $\mathbf{S}$ with Taylor–Wiles primes as in [A'C24, p.51]. This does not change what happens in characteristic 0, so in the interest of simplifying notation, we elide this step.

We let

$$\mathcal{R} = \mathcal{R}_{\mathbf{S}}.$$

For a Taylor–Wiles-datum Q, we define an augmented deformation problem

$$\mathbf{S}_Q = (\bar{\rho}, S \cup Q, \{\mathcal{D}_v\}_{v \in S} \cup \{\mathcal{D}_v^{\square}\}_{v \in Q}).$$

and similarly for $\mathbf{S}'_Q$. Applying Proposition 4.5, we obtain $q, l \geq 0$ and for every $N \geq 1$ a Taylor–Wiles datum Q_N satisfying the conclusions of the proposition. We let

$$\mathcal{R}_N = \mathcal{R}^S_{\mathbf{S}_{Q_N}} \text{ and } \mathcal{R}'_N = \mathcal{R}^S_{\mathbf{S}'_{Q_N}}$$

be the S-framed type-$\mathbf{S}$ and type-$\mathbf{S}'$ derived deformation rings, respectively, and set

$$R_N = \pi_0(\mathcal{R}_N)$$

In [GV18], the authors also consider deformations of $\bar{\rho}_v$ for $v \in Q_N$ valued in the torus $\mathbf{T} \subset \mathbf{G}$ for Taylor–Wiles places $v \in Q$. The corresponding deformation functors are proved in [GV18, Lem. 8.3] to be equivalent to their $\mathbf{G}$-valued counterparts. We let $\mathcal{T} = \mathcal{O}[[T_1, \ldots, T_{n^2|S|-1}]]$ be the power series ring in the 'S-frame variables' and define

$$\begin{aligned} \mathcal{S}_N &= \mathcal{T}[[X_1, \ldots, X_{qn}, Y_1, \ldots, Y_{nq}]]/\langle (1+Y_i)^{p^N} - 1 \rangle, \\ \mathcal{S}_N^{\mathrm{ur}} &= \mathcal{S}_N / \langle T_i, Y_j \rangle \cong \mathcal{O}[[X_1, \ldots, X_{nq}]] \end{aligned}$$

Our notation deviates from that of [GV18] in that we add the S-frame variables to $\mathcal{S}_N$. We have:

$$\begin{aligned} \mathcal{S}_N \otimes^{\mathbf{L}}_{\mathcal{T}} \mathcal{O} &= \text{representing object for } \prod_{v \in Q_N} \mathcal{X}_{\Gamma_v, \mathbf{G}}, \\ \mathcal{S}_N^{\mathrm{ur}} &= \text{representing object for } \prod_{v \in Q_N} \mathcal{X}_{\Gamma_v/I_v, \mathbf{G}}. \end{aligned}$$

The rings $\mathcal{S}_N$ and $\mathcal{S}_N^{\mathrm{ur}}$ are static; to emphasise this we also write

$$S_N = \mathcal{S}_N, \quad S_N^{\mathrm{ur}} = \mathcal{S}_N^{\mathrm{ur}}.$$

Lemma 4.7. *Let $r \leq N$. Then*

$$\mathrm{T}^1_{\mathcal{O}_r}(\mathcal{S}_N) \cong \prod_{v \in Q_N} H^2(\Gamma_v, \operatorname{ad} \xi_r)$$

Proof. This follows from the fact that $\mathcal{S}_N \otimes^{\mathbf{L}}_{\mathcal{T}} \mathcal{O}$ represents $\prod_{v \in Q_N} \mathcal{X}_{\Gamma_v, \mathbf{G}}$ and Proposition 3.34. $\square$

For every N, we have natural maps

$$\mathcal{S}_N \to \mathcal{R}'_N \to \mathcal{R}_N.$$

Lemma 4.8. *[GV18, (11.4)] For every $N \geq 1$, there are weak equivalences*

$$\begin{aligned} \mathcal{R} &\simeq \mathcal{R}_N \otimes^{\mathbf{L}}_{\mathcal{S}_N} \mathcal{S}_N^{\mathrm{ur}} \\ \mathcal{R}' &\simeq \mathcal{R}'_N \otimes^{\mathbf{L}}_{\mathcal{S}_N} \mathcal{S}_N^{\mathrm{ur}}. \end{aligned}$$

We define

$$\begin{aligned} R_\infty &= R^{S,\mathrm{loc}}_{\mathbf{S}}[[X_1, \ldots, X_{nq - n^2[F^+:\,\mathbb{Q}]}]] \\ S_\infty^\circ &= \mathcal{T}[[Y_1, \ldots, Y_{nq}]] \\ \mathfrak{a} &= (T_1, \ldots, T_{n^2|S|-1}, Y_1, \ldots, Y_{nq}) \subset S_\infty^\circ \\ \mathfrak{a}_N &= \langle (1+Y_i)^{p^N} - 1 \rangle \subset S_\infty^\circ \\ \mathfrak{c}_N &= \left(p^N, (1+T_i)^{p^N} - 1, (1+Y_i)^{p^N} - 1 \right) \subset S_\infty^\circ \\ S_N^\circ &= S_\infty^\circ / \mathfrak{a}_N \cong \mathcal{T}[[Y_1, \ldots, Y_{nq}]]/\langle (1+Y_i)^{p^N} - 1 \rangle \end{aligned}$$

Then, for every $N \geq 1$, R_N is a local S_N°-algebra via the morphisms

$$S_N^\circ \to S_N \to R_N,$$

we have $R_N/\mathfrak{a} \cong \pi_0(\mathcal{R}_{\mathbf{S}})$, and by Proposition 4.5 there is a surjection $R_\infty \twoheadrightarrow R_N$. Note that $\mathfrak{c}_N \subset S_\infty^\circ$ is an open ideal and

$$\mathfrak{m}_{S_\infty^\circ}^{p^N} \subset \mathfrak{c}_N \subset \mathfrak{m}_{S_\infty^\circ}^{N},$$

so that $S_\infty^\circ \cong \varprojlim_N S_\infty^\circ/\mathfrak{c}_N$ as topological $\mathcal{O}$-algebras. We let

$$\mathfrak{b}_N = \langle p^N, (1+T_i)^{p^N} - 1, (1+X_j)^{p^N} - 1 \rangle \subset \mathcal{S}_N$$

and define Artinian quotients

$$\begin{aligned}
\overline{R}_N &= R^S_{\mathbf{S}_{Q_N}}/\mathfrak{m}^N_{R_\infty},\\
\overline{S}_N &= S_N/\mathfrak{b}_N\\
\overline{S}_N^{\mathrm{ur}} &= S_N^{\mathrm{ur}}/\mathfrak{b}_N\\
\overline{S}_N^{\circ} &= S_\infty^\circ/\mathfrak{c}_N.
\end{aligned}$$

Then $\overline{R}_N$ is an $\overline{S}_N$-algebra (and an $\overline{S}_N^\circ$-algebra).

Theorem 4.9. *[A'C24] With the same notation and assumptions as Theorem 4.1, we have:*

(1) There exists an infinite subset $\mathbb{N}^* \subseteq \mathbb{N}$ and, for every $M \leq N$ in $\mathbb{N}^*$, isomorphisms of R_∞-algebras $\overline{R}_N/\mathfrak{m}^M_{R_\infty} \cong \overline{R}_M$ such that we have an isomorphism

$$(R_\infty)^\wedge_{\mathfrak{p}} \to (\varprojlim_{\mathbb{N}^*} \overline{R}_N)^\wedge_{\mathfrak{p}},$$

where the transition maps in the inverse system are given by the compositions

$$\overline{R}_N \twoheadrightarrow \overline{R}_N/\mathfrak{m}^M_{R_\infty} \xrightarrow{\sim} \overline{R}_M$$

and $\mathfrak{p} = \ker(R_\infty \to \mathcal{O})$ corresponds to ξ.

(2) There is an R_∞-module M_∞ such that $(M_\infty)^\wedge_{\mathfrak{p}}$ is free over $(R_\infty)^\wedge_{\mathfrak{p}}$, and

$$(M_\infty)^\wedge_{\mathfrak{p}} \otimes^{\mathbf{L}}_{(S_\infty^\circ)^\wedge_{\mathfrak{a}}} E \simeq R\Gamma(K, \mathcal{V}_\lambda)_{\mathfrak{p}}$$

Consequently, the Tor*-algebra*

$$\pi_*\big((R_\infty)^\wedge_{\mathfrak{p}} \otimes^{\mathbf{L}}_{(S_\infty^\circ)^\wedge_{\mathfrak{a}}} E\big) \cong \mathrm{Tor}_*^{(S_\infty^\circ)^\wedge_{\mathfrak{a}}}\big((R_\infty)^\wedge_{\mathfrak{p}}, E\big)$$

acts freely on the graded module

$$\pi_*\big((M_\infty)^\wedge_{\mathfrak{p}} \otimes^{\mathbf{L}}_{(S_\infty^\circ)^\wedge_{\mathfrak{a}}} E\big) \cong H^*(K, \mathcal{V}_\lambda)$$

Proof. Both (1) and (2) are contained in (the proof of) the main theorem of [A'C24]. We provide some guiding remarks. For (1), the proof of the abstract patching result [A'C24, Lem. 2.6.4] proves an isomorphism

$$(R_\infty)^\wedge_{\mathfrak{p}} \cong \mathbf{P}_{R_\infty}(R_N)^\wedge_{\mathfrak{p}}$$

where $\mathbf{P}$ denotes A'Campo's patching functor [A'C24, §2.3]. From [A'C24, Lem. 2.3.3], we obtain a subsequence $\mathbb{N}^* \subseteq \mathbb{N}$, a morphism $\overline{R}_N \to \overline{R}_M$ as in the statement for every $M \leq N$ in $\mathbb{N}^*$ and an isomorphism

$$\mathbf{P}(R_N) \cong \varprojlim_N \overline{R}_N.$$

The second statement (2) is contained in [A'C24, Lem. 2.6.4, Thm. 5.3.5]. $\square$

Remark 4.10. The subset $\mathbb{N}^* \subseteq \mathbb{N}$ and the maps $\overline{R}_N \to \overline{R}_M$ depends on a choice of non-principal ultrafilter on $\mathbb{N}$. As such, the isomorphism provided by (1) is non-canonical. This ambiguity will not play a role in the sequel; we fix from now on a subset $\mathbb{N}^*$ and maps as in the theorem.

In light of part (2) of Theorem 4.9, we see that to prove Theorem 4.1 it suffices to prove a weak equivalence

$$(\mathcal{R}_{\mathbf{S}} \otimes_{\mathcal{O}} E)^\wedge_\xi \simeq (R_\infty)^\wedge_{\mathfrak{p}} \otimes^{\mathbf{L}}_{(S^\circ_\infty)^\wedge_{\mathfrak{a}}} E.$$

4.3 Tangent complex calculations

Our proof of the main theorem will rely on a comparison of tangent complexes. In this section, we collect some preliminary calculations with tangent complexes.

Proposition 4.11. *With notation and assumptions as above, the tangent complex of* $(\mathcal{R}_{\mathbf{S}} \otimes_{\mathcal{O}} E)^\wedge_\xi$ *is concentrated in degrees* $[0,1]$ *and satisfies*

$$\dim_E \mathrm{T}^i_E\left((\mathcal{R}_{\mathbf{S}} \otimes_{\mathcal{O}} E)^\wedge_\xi\right) = \begin{cases} 0 & \text{if } i = 0 \\ l_0 & \text{if } i = 1. \end{cases}$$

Proof. Recall that we have an isomorphism

$$\mathrm{T}^0_E\left((\mathcal{R}_{\mathbf{S}} \otimes_{\mathcal{O}} E)^\wedge_\xi\right) \cong \mathrm{T}^0_E\left(\pi_0\big((\mathcal{R}_{\mathbf{S}} \otimes_{\mathcal{O}} E)^\wedge_\xi\big)\right).$$

Since $(\mathcal{R}_{\mathbf{S}} \otimes_{\mathcal{O}} E)^\wedge_\xi$ is Noetherian, [Lur04, Prop. 6.1.8] implies

$$\pi_0\big((\mathcal{R}_{\mathbf{S}} \otimes_{\mathcal{O}} E)^\wedge_\xi\big) \cong \big(\pi_0(\mathcal{R}_{\mathbf{S}}) \otimes_{\mathcal{O}} E\big)^\wedge_\xi,$$

whose tangent space is given by the Bloch–Kato Selmer group (see [All16, Prop. 1.3.12])

$$H^1_g(\Gamma_S, \operatorname{ad}\rho)$$

which vanishes under our assumptions by the main theorem of [A'C24]. This vanishing together with the fibre sequence of $\mathcal{R}_{\mathbf{S}}$ and Proposition 3.41 yield an exact sequence (recall that both $\mathrm{T}^1_E(R^{S,\mathrm{loc}}_{\mathbf{S}})$ and $\mathrm{T}^1_E(R^{S,\mathrm{loc}}_{\mathbf{S}'})$ vanish by genericity of $\mathrm{WD}(\rho_v)$ at $v \in S$ and irreducibility of ρ_v at $v \in S_p$)

$$0 \to \mathrm{T}^0_E(\mathcal{R}_{\mathbf{S}'}) \oplus \mathrm{T}^0_E(R^{S,\mathrm{loc}}_{\mathbf{S}}) \to \mathrm{T}^0_E(R^{S,\mathrm{loc}}_{\mathbf{S}'}) \to \mathrm{T}^1_E\left((\mathcal{R}_{\mathbf{S}} \otimes_{\mathcal{O}} E)^\wedge_\xi\right) \to \mathrm{T}^1_E(\mathcal{R}_{\mathbf{S}'}) \to \mathrm{T}^1_E(R^{S,\mathrm{loc}}_{\mathbf{S}'}).$$

Comparing the sequence to the Poitou–Tate sequence associated to the Selmer system corresponding to $\mathbf{S}$ gives

$$\dim_E \mathrm{T}^1_E\left((\mathcal{R}_{\mathbf{S}} \otimes_{\mathcal{O}} E)^\wedge_\xi\right) = \dim_E H^1(\Gamma_S, \operatorname{ad}\rho(1))^\vee = l_0$$

by Proposition 3.37. $\square$

For any $N \geq 1$, we let

$$\mathcal{C}_N = \overline{R}_N \otimes^{\mathbf{L}}_{\overline{S}_N} \overline{S}^{\mathrm{ur}}_N.$$

For every $M \leq N$ in $\mathbb{N}^*$, Theorem 4.9(1) provides a commutative diagram

$$\begin{array}{ccccc} \overline{R}_N & \longleftarrow & \overline{S}_N & \longrightarrow & \overline{S}^{\mathrm{ur}}_N \\ \big\downarrow & & \big\downarrow & & \big\downarrow \\ \overline{R}_M & \longleftarrow & \overline{S}_M & \longrightarrow & \overline{S}^{\mathrm{ur}}_M \end{array}$$

and we denote by

$$e_{N,M} \colon \mathcal{C}_N \to \mathcal{C}_M$$

the induced map on derived tensor products.

Lemma 4.12. *With notation as above, we have the following:*

(1) There is a weak equivalence

$$\mathcal{C}_N = \overline{R}_N \otimes^{\mathbf{L}}_{\overline{S}_N} \overline{S}^{\mathrm{ur}}_N \simeq \overline{R}_N \otimes^{\mathbf{L}}_{\overline{S}^{\circ}_N} \mathcal{O}_N.$$

(2) The limit $\mathcal{C}_\infty = \lim_N \mathcal{C}_N$ is Noetherian.

(3) The tangent complex of the completion $(\mathcal{C}_\infty \otimes_{\mathcal{O}} E)^\wedge_{\xi}$ has cohomology concentrated in degrees $[0,1]$ and

$$\dim_E \mathrm{T}^0_E((\mathcal{C}_\infty \otimes_{\mathcal{O}} E)^\wedge_{\xi}) - \dim_E \mathrm{T}^1_E((\mathcal{C}_\infty \otimes_{\mathcal{O}} E)^\wedge_{\xi}) = -\ell_0$$

Proof. (1) This is [GV18, (8.16, 8.17)].
(2) By the derived Schlessinger criterion (Theorem 3.31), it suffices to prove that $\mathrm{T}^i_k(\mathcal{C}_\infty) = \operatorname*{colim}_N \mathrm{T}^i_k(\mathcal{C}_N)$ is finite-dimensional for every $i \geq 0$. Consider the fibre sequence

$$\mathrm{T}^\bullet_k(\mathcal{C}_N) \to \mathrm{T}^\bullet_k(\overline{R}_N) \oplus \mathrm{T}^\bullet_k(\mathcal{O}_N) \to \mathrm{T}^\bullet_k(\overline{S}^{\circ}_N) \xrightarrow{+1}.$$

Filtered colimits are exact, whence we have a sequence

$$\mathrm{T}^\bullet_k(\mathcal{C}_\infty) \to \mathrm{T}^\bullet_k(\lim_N \overline{R}_N) \to \mathrm{T}^\bullet_k(S^\circ_\infty).$$

For every $N \geq 0$, we have a surjection $R_\infty \twoheadrightarrow \overline{R}_N$, and thus $R_\infty \twoheadrightarrow \lim_N \overline{R}_N \cong \varprojlim_N \overline{R}_N$. Since R_∞ and S°_∞ are Noetherian, it follows that $\mathrm{T}^i_k(\mathcal{C}_\infty)$ is finite-dimensional for every $i \geq 0$.
(3) For any $r \leq N$, we have a fibre sequence

$$\mathrm{T}^\bullet_{\mathcal{O}_r}(\mathcal{C}_N) \to \mathrm{T}^\bullet_{\mathcal{O}_r}(\overline{R}_N) \oplus \mathrm{T}^\bullet_{\mathcal{O}_r}(\mathcal{O}_N) \to \mathrm{T}^\bullet_{\mathcal{O}_r}(\overline{S}^{\circ}_N) \xrightarrow{+1}.$$

Passing to the colimit over $N \geq r$ and limit over r, then inverting p yields a fibre sequence

$$\big(\lim_r \operatorname*{colim}_{N \geq r} \mathrm{T}^\bullet_{\mathcal{O}_r}(\mathcal{C}_N)\big) \otimes_{\mathcal{O}} E \to \mathrm{T}^\bullet_E((R_\infty)^\wedge_{\mathfrak{p}}) \to \mathrm{T}^\bullet_E((S^\circ_\infty)^\wedge_{\mathfrak{a}}) \xrightarrow{+1},$$

Now, $(R_\infty)^\wedge_{\mathfrak{p}}$ and $(S^\circ_\infty)^\wedge_{\mathfrak{a}}$ are smooth E-algebras and therefore by taking cohomology we obtain an exact sequence of E-vector spaces

$$0 \to \mathrm{T}^0_E((\mathcal{C}_\infty)^\wedge_{\mathfrak{p}}) \to \mathrm{T}^0_E((R_\infty)^\wedge_{\mathfrak{p}}) \to \mathrm{T}^0_E((S^\circ_\infty)^\wedge_{\mathfrak{a}}) \to \mathrm{T}^1_E((\mathcal{C}_\infty)^\wedge_{\mathfrak{p}}) \to 0.$$

It follows that

$$\dim_E \mathrm{T}^0_E((\mathcal{C}_\infty)^\wedge_{\mathfrak{p}}) - \dim_E \mathrm{T}^1_E((\mathcal{C}_\infty)^\wedge_{\mathfrak{p}}) = \dim(R_\infty)^\wedge_{\mathfrak{p}} - \dim(S^\circ_\infty)^\wedge_{\mathfrak{a}} = -l_0.$$

□

Proposition 4.13. *With assumptions and notation as above and $r \leq N$, the kernel*

$$\ker\left(\mathrm{T}^1_{\mathcal{O}_r}(\mathcal{R}'_N) \to \mathrm{T}^1_{\mathcal{O}_r}(\mathcal{S}_N)\right)$$

is a finite length $\mathcal{O}$-module with length independent of r and N.

Proof. Let $r \leq N$. By Proposition 3.34 and Lemma 4.7(4), we have isomorphisms

$$\mathrm{T}^1_{\mathcal{O}_r}(\mathcal{R}'_N) \cong H^2(\Gamma_{SQ_N}, \mathrm{ad}\,\xi_r)$$
$$\mathrm{T}^1_{\mathcal{O}_r}(S_N) \cong \prod_{v \in Q_N} H^2(\Gamma_v, \mathrm{ad}\,\xi_r)$$

We factor the map $\gamma\colon\ \mathrm{T}^1_{\mathcal{O}_r}(\mathcal{R}'_N) \to \mathrm{T}^1_{\mathcal{O}_r}(S_N)$ as $\gamma = \beta \circ \alpha$ where

$$H^2(\Gamma_{SQ_N}, \mathrm{ad}\,\xi_r) \xrightarrow{\alpha} \prod_{v \in SQ_N} H^2(\Gamma_v, \mathrm{ad}\,\xi_r) \xrightarrow{\beta} \prod_{v \in Q_N} H^2(\Gamma_v, \mathrm{ad}\,\xi_r)$$

to obtain an exact sequence

$$0 \to \ker(\alpha) \to \ker(\gamma) \to \ker(\beta).$$

To prove the proposition, it suffices to show that $\ker(\alpha)$ and $\ker(\beta)$ have finite length bounded independently of r and N. To see this, consider the Poitou–Tate sequence for the Selmer system

$$\{\mathcal{L}_{v,r}\}_{v \in SQ_N} = \{H^1(\Gamma_v, \mathrm{ad}\,\xi_r)\}_{v \in SQ_N} \cup \{H^1_{\mathrm{ur}}(\Gamma_v, \mathrm{ad}\,\xi_r)\}_{v \notin SQ_N},$$

part of which reads

$$H^1(\Gamma_{SQ_N}, \mathrm{ad}\,\xi_r(1)) \to H^2(\Gamma_{SQ_N}, \mathrm{ad}\,\xi_r) \xrightarrow{\alpha} \prod_{v \in SQ_N} H^2(\Gamma_v, \mathrm{ad}\,\xi_r).$$

By Proposition 4.5(4), we have

$$\mathrm{length}_{\mathcal{O}} \ker(\alpha) \leq \mathrm{length}_{\mathcal{O}} H^1(\Gamma_{SQ_N}, \mathrm{ad}\,\xi_r(1)) \leq l.$$

Now,

$$\ker(\beta) = \prod_{v \in S} H^2(\Gamma_v, \mathrm{ad}\,\xi_r) = \mathrm{T}^1_{\mathcal{O}_r}(R^{S,\mathrm{loc}}_{\mathbf{S}'})$$

and since the characteristic 0 point of $\operatorname{Spec} R^{S,\mathrm{loc}}_{\mathbf{S}'}$ corresponding to ξ is smooth by assumptions , the $\mathcal{O}$-module $\ker(\beta)$ has finite length bounded independently of r (and N). This completes the proof. □

4.4 Proof of the main theorem

Theorem 4.14. *There exists a morphism $\mathcal{R} \to \mathcal{C}_\infty$ such that $\xi\colon \mathcal{R} \to \mathcal{O}$ factors through $\mathcal{C}_\infty$ and the induced morphism*

$$(\mathcal{R} \otimes_{\mathcal{O}} E)^\wedge_\xi \to (\mathcal{C}_\infty \otimes_{\mathcal{O}} E)^\wedge_\xi$$

is a weak equivalence in $\mathcal{CNL}_E$.

Remark 4.15. Recall that the existence of the inverse system $(\mathcal{C}_N)_{N\in\mathbb{N}^*}$ is not canonically defined. The proof of the existence of $g\colon \mathcal{R} \to \mathcal{C}_\infty$ is also non-constructive and as such, the weak equivalence is a priori non-canonical and depends on a choice of non-principal ultrafilter on $\mathbb{N}$. In [GV18, §15], the authors are able to prove, under their assumptions, that the equivalence is canonical and independent of any choices in the Taylor–Wiles process. In the future, we hope to prove a similar result in characteristic 0 without introducing extra assumptions.

Proof of Theorem 4.1: Let

$$[\mathcal{R}, -] = \pi_0(\mathrm{Map}_{/k}(\mathcal{R}, -))$$

and denote by $\tau_{\leq j}$ the j-truncation functor. We claim that the natural maps

$$[\mathcal{R}, \mathcal{C}_\infty] \to \varprojlim_N [\mathcal{R}, \mathcal{C}_N] \to \varprojlim_{j,N} [\mathcal{R}, \tau_{\leq j}\mathcal{C}_N]$$

are bijections. Indeed, by [GV18, Lem. A.9] we have a short exact sequence of pointed sets (for any choice of basepoint)

$$0 \to \varprojlim^1_{j,N} \pi_1 \mathrm{Map}_{/k}(\mathcal{R}, \tau_{\leq j}\mathcal{C}_N) \to [\mathcal{R}, \mathcal{C}_\infty] \to \varprojlim_{j,N} [\mathcal{R}, \tau_{\leq j}\mathcal{C}_N] \to 0.$$

We claim that $\mathrm{Map}_{/k}(\mathcal{R}, \tau_{\leq j}\mathcal{C}_N)$ has finite homotopy groups (and in particular, $[\mathcal{R}, \tau_{\leq j}\mathcal{C}_N]$ is finite for every j, N). Indeed, since $\tau_{\leq j}\mathcal{C}_N \in \mathcal{A}rt_k$, using Lemma 3.3 we may write $\tau_{\leq j}\mathcal{C}_N$ as an iterated series of small extensions

$$\tau_{\leq j}\mathcal{C}_N \simeq B_m \to \cdots \to B_0 \simeq k$$

for some $m \geq 0$. Since $\mathcal{R}$ is Noetherian and

$$\pi_i(\mathrm{Map}_{/k}(\mathcal{R}, k \oplus k[n])) \cong \pi_0(\mathrm{Map}_{/k}(\mathcal{R}, k \oplus k[n-i])) \cong \mathrm{T}^{n-i}_k(\mathcal{R})$$

is finite, the claim follows. Hence, the $\varprojlim^1_j$-term in the short exact sequence vanishes and we have proved

$$[\mathcal{R}, \mathcal{C}_\infty] \cong \varprojlim_{j,N} [\mathcal{R}, \tau_{\leq j}\mathcal{C}_N].$$

It follows that $[\mathcal{R}, \mathcal{C}_\infty] \to \varprojlim_N [\mathcal{R}, \mathcal{C}_N]$ is injective. It is automatically surjective by the existence of the short exact sequence (using [GV18, Lem. A.9] again)

$$0 \to \varprojlim^1_N \pi_1 \mathrm{Map}_{/k}(\mathcal{R}, \mathcal{C}_N) \to [\mathcal{R}, \mathcal{C}_\infty] \to \varprojlim_N [\mathcal{R}, \mathcal{C}_N] \to 0.$$

From the above, we see that defining a morphism $\mathcal{R} \to \mathcal{C}_\infty$ amounts up to homotopy to defining a compatible system of maps $\mathcal{R} \to \mathcal{C}_N$. For $a \geq 0$, we let

$$X_N(a) = \{[f] \in [\mathcal{R}, \mathcal{C}_N] \mid \forall r \leq M \leq N\colon\ \mathrm{T}^0_{\mathcal{O}_r}(e_{N,M} \circ f) \text{ isomorphism, } p^a \operatorname{coker}(\mathrm{T}^1_{\mathcal{O}_r}(e_{N,M} \circ f)) = 0\}.$$

Here, $\mathrm{T}^i_{\mathcal{O}_r}(e_{N,M} \circ f)$ denotes the homomorphism of $\mathcal{O}_r$-modules

$$\mathrm{T}^i_{\mathcal{O}_r}(\mathcal{C}_M) \to \mathrm{T}^i_{\mathcal{O}_r}(\mathcal{R})$$

induced by the composition $e_{N,M} \circ f$. We will prove that for a sufficiently large, $(X_N(a))_{N\in\mathbb{N}^*}$ is an inverse system of non-empty profinite sets with continuous transition maps. In particular, $\varprojlim X_N(a) \neq \emptyset$ and hence there exists a class

$$[g] = ([g_N])_{N\in\mathbb{N}^*} \in \varprojlim_N X_N(a) \subset [\mathcal{R}, \mathcal{C}_\infty].$$

$(X_N(a))$ **is an inverse system:** Let $[f] \in X_N(a)$. Then, for any $L \leq M \leq N$, we have

$$[e_{M,L} \circ e_{N,M} \circ f] = [e_{N,L} \circ f]$$

and hence $[e_{N,M} \circ f] \in X_M(a)$ and the transition maps are compatible, i.e. we have a commutative triangle

$$\begin{tikzcd} X_N(a) \arrow[rr] \arrow[dr] & & X_L(a) \\ & X_M(a) \arrow[ur] & \end{tikzcd}$$

Hence, $(X_N(a))_{N\in\mathbb{N}^*}$ is an inverse system.

$X_N(a)$ **is nonempty:** Fix N. We prove that there exists $a \geq 0$ such that

$$f = f_N : \mathcal{R} \simeq \mathcal{R}_N \otimes^{\mathbf{L}}_{\mathcal{S}_N} \mathcal{S}_N^{\mathrm{ur}} \to \overline{R}_N \otimes^{\mathbf{L}}_{\overline{S}_N} \overline{S}_N^{\mathrm{ur}} = \mathcal{C}_N$$

lies in $X_N(a)$. Importantly, we do *not* prove that $[f_N]$ is a compatible system – in particular, we prove no relation between g and f. For any $r \leq M \leq N$, the map $e_{N,M} \circ f$ induces a commutative diagram with exact rows

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{T}^0_r(\mathcal{C}_M) \arrow[r] \arrow[d] & \mathrm{T}^0_r(\overline{R}_M) \oplus \mathrm{T}^0_r(\overline{S}^{\mathrm{ur}}_M) \arrow[r] \arrow[d] & \mathrm{T}^0_r(\overline{S}_M) \arrow[d] \\ 0 \arrow[r] & \mathrm{T}^0_r(\mathcal{R}) \arrow[r] & \mathrm{T}^0_r(\mathcal{R}_N) \oplus \mathrm{T}^0_r(\mathcal{S}^{\mathrm{ur}}_N) \arrow[r] & \mathrm{T}^0_r(\mathcal{S}_N) \end{tikzcd}$$

The two rightmost vertical maps are isomorphisms, since $\mathfrak{c}_M$ is generated by elements whose mod p^r reductions lie in the square of the maximal ideal, whereby

$$\mathrm{T}^0_r(\mathcal{R}_N) \cong \mathrm{T}^0_r(R_N) \cong \mathrm{T}^0_r(R_N/\mathfrak{c}_M) = \mathrm{T}^0_r(\overline{R}_M)$$

and similarly for $\mathcal{S}_N$ and $\mathcal{S}^{\mathrm{ur}}_N$. It follows from the five lemma that the left vertical map in the diagram is also an isomorphism, as required (this part does not depend on a).

To prove the remaining statement, we argue as before to obtain a commutative diagram with exact rows

$$\begin{tikzcd} \mathrm{T}^0_r(\overline{S}_M) \arrow[r] \arrow[d, "\cong"] & \mathrm{T}^1_{\mathcal{O}_r}(\mathcal{C}_M) \arrow[r] \arrow[d, "\alpha"] & \mathrm{T}^1_{\mathcal{O}_r}(\overline{R}_M) \oplus \mathrm{T}^1_{\mathcal{O}_r}(\overline{S}^{\mathrm{ur}}_M) \arrow[r] \arrow[d] & \mathrm{T}^1_{\mathcal{O}_r}(\overline{S}_M) \arrow[d] \\ \mathrm{T}^0_r(\mathcal{S}_N) \arrow[r, "\beta"] & \mathrm{T}^1_{\mathcal{O}_r}(\mathcal{R}) \arrow[r] & \mathrm{T}^1_{\mathcal{O}_r}(\mathcal{R}_N) \oplus \mathrm{T}^1_{\mathcal{O}_r}(\mathcal{S}^{\mathrm{ur}}_N) \arrow[r, "\gamma"] & \mathrm{T}^1_{\mathcal{O}_r}(\mathcal{S}_N) \end{tikzcd}$$

Note that $\mathrm{T}^1_{\mathcal{O}_r}(\mathcal{S}^{\mathrm{ur}}_N) = 0$ since $\mathcal{S}^{\mathrm{ur}}_N$ is formally smooth. The claim to be proven is that $p^a \operatorname{coker}(\alpha) = 0$ for an $a \geq 0$ which is independent of r, M and N. By the diagram, we have

$$\operatorname{coker}(\alpha) \twoheadleftarrow \operatorname{coker}(\beta) \cong \ker(\gamma),$$

so it suffices to prove that $\ker(\gamma)$ is annihilated by a sufficiently large power of p (independent of r, M, N).

To analyse $\ker(\gamma)$, recall that $\mathcal{R}'_N$ represents the deformation problem without the $\mathcal{D}_v^{ss,r}$-condition at $v \in S_p$. We have a weak equivalence

$$\mathcal{R}_N \simeq \mathcal{R}'_N \otimes^{\mathbf{L}}_{R^{S,\mathrm{loc}}_{\mathbf{S}'}} R^{S,\mathrm{loc}}_{\mathbf{S}},$$

and if we let

$$\mathrm{cok}_{r,N} = \mathrm{coker}\left(\mathrm{T}^0_{\mathcal{O}_r}(\mathcal{R}'_N) \oplus \mathrm{T}^0_{\mathcal{O}_r}(R^{S,\mathrm{loc}}_{\mathbf{S}}) \to \mathrm{T}^0_{\mathcal{O}_r}(R^{S,\mathrm{loc}}_{\mathbf{S}'})\right),$$

then the fibre sequence of $\mathcal{R}'_N \otimes^{\mathbf{L}}_{R^{S,\mathrm{loc}}_{\mathbf{S}'}} R^{S,\mathrm{loc}}_{\mathbf{S}}$ induces a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{cok}_{r,N} \arrow[r] & \mathrm{T}^1_{\mathcal{O}_r}(\mathcal{R}_N) \arrow[r] \arrow[d, "\gamma"] & \mathrm{T}^1_{\mathcal{O}_r}(\mathcal{R}'_N) \oplus \mathrm{T}^1_{\mathcal{O}_r}(R^{S,\mathrm{loc}}_{\mathbf{S}}) \arrow[d, two heads] \\
& & \mathrm{T}^1_{\mathcal{O}_r}(\mathcal{S}_N) & \mathrm{T}^1_{\mathcal{O}_r}(\mathcal{R}'_N) \arrow[l]
\end{tikzcd}$$

where the top row is exact and the right vertical map is the projection onto the first coordinate. By exactness, we see that

$$\mathrm{cok}_{r,N} \hookrightarrow \ker(\gamma)$$

so that γ factors as

$$\mathrm{T}^1_{\mathcal{O}_r}(\mathcal{R}_N) \twoheadrightarrow \mathrm{T}^1_{\mathcal{O}_r}(\mathcal{R}_N)/\,\mathrm{cok}_{r,N} \xrightarrow{\tilde{\gamma}} \mathrm{T}^1_{\mathcal{O}_r}(\mathcal{S}_N).$$

and we have an exact sequence

$$0 \to \mathrm{cok}_{r,N} \to \ker(\gamma) \to \ker(\tilde{\gamma}) \to 0.$$

By Proposition 4.5(4), $\mathrm{cok}_{r,N}$ is annihilated by a power of p which is independent of r and N. Thus, it suffices to prove the same holds for $\ker(\tilde{\gamma})$. Now, by the first diagram we have

$$\ker(\tilde{\gamma}) \cong \ker\left(\mathrm{T}^1_{\mathcal{O}_r}(\mathcal{R}'_N) \oplus \mathrm{T}^1_{\mathcal{O}_r}(R^{S,\mathrm{loc}}_{\mathbf{S}}) \twoheadrightarrow \mathrm{T}^1_{\mathcal{O}_r}(\mathcal{R}'_N) \to \mathrm{T}^1_{\mathcal{O}_r}(\mathcal{S}_N)\right),$$

whereby it suffices to show that $\mathrm{T}^1_{\mathcal{O}_r}(R^{S,\mathrm{loc}}_{\mathbf{S}})$ and $\ker(\mathrm{T}^1_{\mathcal{O}_r}(\mathcal{R}'_N) \to \mathrm{T}^1_{\mathcal{O}_r}(\mathcal{S}_N))$ are annihilated by powers of p that are independent of r and N. The latter statement is contained in Proposition 4.13. To prove the statement for $\mathrm{T}^1_{\mathcal{O}_r}(R^{S,\mathrm{loc}}_{\mathbf{S}})$, consider the short exact sequence

$$0 \to \mathcal{O} \to \mathcal{O} \to \mathcal{O}_r \to 0.$$

It induces an exact sequence

$$\mathrm{T}^1_{\mathcal{O}}(R^{S,\mathrm{loc}}_{\mathbf{S}}) \to \mathrm{T}^1_{\mathcal{O}_r}(R^{S,\mathrm{loc}}_{\mathbf{S}}) \to \mathrm{T}^2_{\mathcal{O}}(R^{S,\mathrm{loc}}_{\mathbf{S}}).$$

By Lemma 3.36 and Prop. 3.32, these are finitely generated $\mathcal{O}$-modules. Moreover, since ξ defines a smooth point of $R^{S,\mathrm{loc}}_{\mathbf{S}}$ by genericity of the associated Weil–Deligne representations, $\mathrm{T}^i_{\mathcal{O}}(R^{S,\mathrm{loc}}_{\mathbf{S}})$ is torsion for $i \geq 1$. It follows that

$$\mathrm{length}_{\mathcal{O}}\left(\mathrm{T}^1_{\mathcal{O}_r}(R^{S,\mathrm{loc}}_{\mathbf{S}})\right) \leq \mathrm{length}_{\mathcal{O}}\left(\mathrm{T}^1_{\mathcal{O}}(R^{S,\mathrm{loc}}_{\mathbf{S}})\right) + \mathrm{length}_{\mathcal{O}}\left(\mathrm{T}^2_{\mathcal{O}}(R^{S,\mathrm{loc}}_{\mathbf{S}})\right)$$

which is finite and independent of r, M, N as required. Thus, we have shown that for an absolute constant a, $[f] \in X_N(a)$ and in particular, $X_N(a) \neq \emptyset$.

$X_N(a)$ **is profinite:** Since $[\mathcal{R},\mathcal{C}_N] = \varprojlim_j [\mathcal{R},\tau_{\leq j}\mathcal{C}_N]$ and $\tau_{\leq j}\mathcal{C}_N \in \mathcal{A}rt_k$, to prove that $X_N(a)$ is profinite it suffices to prove that for any $B \in \mathcal{A}rt_k$, the set $[\mathcal{R},B]$ is finite. By Lemma 3.3, any $(B \to k) \in \mathcal{A}rt_k$ is equivalent to an iterated series of small extensions

$$B \simeq B_m \to \cdots \to B_0 \simeq k,$$

and we prove the claim by induction on m. The case $m=0$ is trivially true, and from the equalities

$$[\mathcal{R},B_m] = [\mathcal{R},B_{m-1}\times_{B_{m-1}\oplus k[n_m+1]} B_{m-1}] = [\mathcal{R},B_{m-1}]\times_{[\mathcal{R},B_{m-1}\oplus k[n_m+1]]}[\mathcal{R},B_{m-1}]$$

we see that to carry out the induction step it suffices to prove that $[\mathcal{R},k\oplus k[n]]$ is finite for every $n\geq 1$ (since we are considering morphisms over k, $[\mathcal{R},k]=\{*\}$). In this case, we have

$$[\mathcal{R},k\oplus k[n]] = \pi_0(\mathrm{Map}_{/k}(\mathcal{R},k\oplus k[n])) = \mathrm{T}^n_k(\mathcal{R})$$

which is finite-dimensional by Theorem 3.31. Thus, $[\mathcal{R},\mathcal{C}_N] = \varprojlim_j [\mathcal{R},\mathcal{C}^j_N]$ is profinite, and to prove that $X_N(a)$ is profinite we show that $X_N(a)$ is the inverse limit of a a pro-subset of $([\mathcal{R},\mathcal{C}^j_N])_{j\geq 0}$.

To see this, note that $\mathrm{T}^1_{\mathcal{O}_r}(\mathcal{C}_M) = \mathrm{colim}_j\,\mathrm{T}^1_{\mathcal{O}_r}(\tau_{\leq j}\mathcal{C}_M)$ and therefore

$$\mathrm{coker}\left(\mathrm{T}^1_{\mathcal{O}_r}(e_{N,M}\circ f)\right) = \mathrm{colim}_j\,\mathrm{coker}\left(\mathrm{T}^1_{\mathcal{O}_r}(\tau_{\leq j}\mathcal{C}_M)\to \mathrm{T}^1_{\mathcal{O}_r}(\mathcal{R})\right) =: \mathrm{colim}_j\, Q_{\leq j}$$

We can formulate the condition defining $X_N(a)$ degree-wise in j using the equivalence

$$p^a\,\mathrm{colim}_j\, Q_{\leq j} = 0 \iff \forall j\geq 0 :\ p^a Q_{\leq j}\subseteq \ker\left(Q_{\leq j}\to \mathrm{colim}_j\, Q_{\leq j}\right)$$

Thus, we define finite subsets $X^j_N(a)\subseteq [\mathcal{R},\mathcal{C}^j_N]$ using the expression above and conclude that $X_N = \varprojlim X^j_N(a)$ is profinite.

$X_N(a)\to X_M(a)$ **is continuous:** Let $M\leq N$ and consider $e_{N,M}\colon \mathcal{C}_N\to\mathcal{C}_M$. For any j, we have a commutative diagram

$$\begin{tikzcd} \mathcal{C}_N \arrow[r] \arrow[d] & \mathcal{C}_M \arrow[d] \\ \tau_{\leq j}\mathcal{C}_N \arrow[r] & \tau_{\leq j}\mathcal{C}_M \end{tikzcd}$$

and hence

$$\begin{tikzcd} {[\mathcal{R},\mathcal{C}_N]} \arrow[r] \arrow[d] & {[\mathcal{R},\mathcal{C}_M]} \arrow[d] \\ {[\mathcal{R},\tau_{\leq j}\mathcal{C}_N]} \arrow[r] & {[\mathcal{R},\tau_{\leq j}\mathcal{C}_M]}. \end{tikzcd}$$

It follows that the top horizontal map is continuous, and hence the restriction $X_N(a)\to X_M(a)$ is also continuous.

Passage to characteristic 0: We now have a constant $a\geq 0$ and a map

$$g\colon \mathcal{R}\to \lim_N \mathcal{C}_N$$

such that for every N, $\mathrm{T}^0_r(g)$ is an isomorphism and the cokernel

$$Q_{r,N}(g) = \operatorname{coker}\left(\mathrm{T}^1_{\mathcal{O}_r}(\mathcal{C}_N) \to \mathrm{T}^1_{\mathcal{O}_r}(\mathcal{R})\right)$$

is p^a-torsion. We claim that the induced map $\mathcal{R}^\wedge_{\mathfrak{p}} \to (\lim_N \mathcal{C}_N)^\wedge_{\mathfrak{p}}$ induces an isomorphism on T^0_E and an epimorphism on T^1_E. The first statement is clear from the definition of X_N. To prove the second statement, we note that p^a-torsion modules are preserved under filtered colimits and inverse limits, so that

$$p^a Q_{r,N}(g) = 0 \implies (\varprojlim_r \operatorname*{colim}_N Q_{r,N}(g)) \otimes_{\mathcal{O}} E = 0.$$

Now, $\lim_N \mathcal{C}_N$ is Noetherian by Lemma 4.12(2). Thus, $\operatorname*{colim}_N \mathrm{T}^1_{\mathcal{O}_r}(\mathcal{C}_N)$ is finite, and we have

$$\begin{aligned}(\varprojlim_r \operatorname*{colim}_N Q_{r,N}(g)) \otimes_{\mathcal{O}} E &\cong \operatorname{coker}\left((\lim_r \operatorname*{colim}_N \mathrm{T}^1_{\mathcal{O}_r}(\mathcal{C}_N)) \otimes_{\mathcal{O}} E \to (\lim_r \operatorname*{colim}_N \mathrm{T}^1_{\mathcal{O}_r}(\mathcal{R})) \otimes_{\mathcal{O}} E\right) \\ &\cong \operatorname{coker}\left(\mathrm{T}^1_E\left((\lim_N \mathcal{C}_N)^\wedge_{\mathfrak{p}}\right) \to \mathrm{T}^1_E(\mathcal{R}^\wedge_{\mathfrak{p}})\right)\end{aligned}$$

since colimits commute and $\varprojlim$ is exact on finite abelian groups. We have now proved that the map

$$\mathrm{T}^*_E\left((\lim_N \mathcal{C}_N)^\wedge_{\mathfrak{p}}\right) \to \mathrm{T}^*_E(\mathcal{R}^\wedge_{\mathfrak{p}})$$

induced by $\mathcal{R}^\wedge_{\mathfrak{p}} \to (\lim_N \mathcal{C}_N)^\wedge_{\mathfrak{p}}$ is an isomorphism in degree 0 and an epimorphism in degree 1. Now, these graded E-vector spaces are concentrated in degrees $[0,1]$ and have the same dimensions. It follows that $\mathcal{R}^\wedge_{\mathfrak{p}} \to (\lim_N \mathcal{C}_N)^\wedge_{\mathfrak{p}}$ is a weak equivalence. □

Lemma 4.16. *With the same notation as above, there is a weak equivalence*

$$(\mathcal{C}_\infty)^\wedge_{\mathfrak{p}} \simeq (R_\infty)^\wedge_{\mathfrak{p}} \otimes^{\mathbf{L}}_{(S^\circ_\infty)^\wedge_{\mathfrak{a}}} E,$$

Proof. By Proposition 3.41, we have an equivalence of fibre sequences

$$\begin{tikzcd}
\mathrm{T}^\bullet_E\left((\mathcal{C}_\infty)^\wedge_{\mathfrak{p}}\right) \arrow[r] \arrow[d] & \mathrm{T}^\bullet_E\left((R_\infty)^\wedge_{\mathfrak{p}}\right) \arrow[r] \arrow[d] & \mathrm{T}^\bullet_E\left((S^\circ_\infty)^\wedge_{\mathfrak{a}}\right) \arrow[r, "+1"] \arrow[d] & {} \\
\lim_r \operatorname*{colim}_{N\geq r} \mathrm{T}^\bullet_{\mathcal{O}_r}(\mathcal{C}_N) \arrow[r] & \lim_r \operatorname*{colim}_{N\geq r} \mathrm{T}^\bullet_{\mathcal{O}_r}(\overline{R}_N) \arrow[r] & \lim_r \operatorname*{colim}_{N\geq r} \mathrm{T}^\bullet_{\mathcal{O}_r}(\overline{S}^\circ_N) \arrow[r, "+1"] & {}
\end{tikzcd}$$

Now, the tangent complex of $(R_\infty)^\wedge_{\mathfrak{p}} \otimes^{\mathbf{L}}_{(S^\circ_\infty)^\wedge_{\mathfrak{a}}} E$ is given by the same fibre:

$$\mathrm{T}^\bullet_E\left((R_\infty)^\wedge_{\mathfrak{p}} \otimes^{\mathbf{L}}_{(S^\circ_\infty)^\wedge_{\mathfrak{a}}} E\right) \to \mathrm{T}^\bullet_E\left((R_\infty)^\wedge_{\mathfrak{p}}\right) \to \mathrm{T}^\bullet_E\left((S^\circ_\infty)^\wedge_{\mathfrak{a}}\right) \xrightarrow{+1}.$$

Hence, by comparing tangent complexes we see that the natural morphism

$$(R_\infty)^\wedge_{\mathfrak{p}} \otimes^{\mathbf{L}}_{(S^\circ_\infty)^\wedge_{\mathfrak{a}}} E \to \lim_r (R_\infty[1/\varpi]/\mathfrak{p}^r \otimes^{\mathbf{L}}_{S^\circ_\infty[1/\varpi]^\wedge_{\mathfrak{a}}/\mathfrak{a}^r} E)$$

induces an equivalence on tangent complexes, and the same is true for

$$(R_\infty \otimes^{\mathbf{L}}_{S^\circ_\infty} \mathcal{O})^\wedge_{\mathfrak{p}} \simeq (R_\infty[1/\varpi] \otimes^{\mathbf{L}}_{S^\circ_\infty[1/\varpi]} E)^\wedge_{\mathfrak{p}} \to \lim_r (R_\infty[1/\varpi]/\mathfrak{p}^r \otimes^{\mathbf{L}}_{S^\circ_\infty[1/\varpi]^\wedge_{\mathfrak{a}}/\mathfrak{a}^r} E).$$

Thus, these are weak equivalences of E-algebras and we obtain

$$(R_\infty \otimes^{\mathbf{L}}_{S^\circ_\infty} \mathcal{O})^\wedge_{\mathfrak{p}} \simeq (R_\infty)^\wedge_{\mathfrak{p}} \otimes^{\mathbf{L}}_{(S^\circ_\infty)^\wedge_{\mathfrak{a}}} E,$$

as needed. □

We are now in a position to deduce our main result Theorem 4.1. By Theorem 4.9, there exists a free $(R_\infty)^\wedge_{\mathfrak{p}}$-module $(M_\infty)^\wedge_{\mathfrak{p}}$ such that

$$(M_\infty)^\wedge_{\mathfrak{p}} \otimes^{\mathbf{L}}_{(S^\circ_\infty)^\wedge_{\mathfrak{a}}} E \simeq R\Gamma(K, \mathcal{V}_\lambda)_{\mathfrak{p}}.$$

Theorem 4.14 together with Lemma 4.16 serve to define an action of $(\mathcal{R} \otimes^{\mathbf{L}}_{\mathcal{O}} E)^\wedge_\xi$ on this complex, and in particular we obtain a free action

$$\pi_*\big((\mathcal{R} \otimes^{\mathbf{L}}_{\mathcal{O}} E)^\wedge_\xi\big) \circlearrowright H^*(K, \mathcal{V}_\lambda)_{\mathfrak{p}}.$$

References

[A'C24] Lambert A'Campo. Rigidity of automorphic Galois representations over CM fields. *Int. Math. Res. Not. IMRN*, (6):4541–4623, 2024.

[ACC+23] Patrick B. Allen, Frank Calegari, Ana Caraiani, Toby Gee, David Helm, Bao V. Le Hung, James Newton, Peter Scholze, Richard Taylor, and Jack A. Thorne. Potential automorphy over CM fields. *Ann. of Math. (2)*, 197(3):897–1113, 2023.

[All16] Patrick B. Allen. Deformations of polarized automorphic Galois representations and adjoint Selmer groups. *Duke Math. J.*, 165(13):2407–2460, 2016.

[BCG+25] George Boxer, Frank Calegari, Toby Gee, James Newton, and Jack A. Thorne. The Ramanujan and Sato-Tate conjectures for Bianchi modular forms. *Forum Math. Pi*, 13:Paper No. e10, 65, 2025.

[BIPs23] Gebhard Böckle, Ashwin Iyengar, and Vytautas Pa ˇ skūnas. On local Galois deformation rings. *Forum Math. Pi*, 11:Paper No. e30, 54, 2023.

[BS15] Bhargav Bhatt and Peter Scholze. The pro-étale topology for schemes. *Astérisque*, (369):99–201, 2015.

[Cis19] Denis-Charles Cisinski. *Higher categories and homotopical algebra*, volume 180 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2019.

[ČS24] Kestutis Česnavičius and Peter Scholze. Purity for flat cohomology. *Ann. of Math. (2)*, 199(1):51–180, 2024.

[GV18] S. Galatius and A. Venkatesh. Derived Galois deformation rings. *Adv. Math.*, 327:470–623, 2018.

[Hev24] Bence Hevesi. Ordinary parts and local-global compatibility at $\ell = p$, 2024.

[HLTT16] Michael Harris, Kai-Wen Lan, Richard Taylor, and Jack Thorne. On the rigid cohomology of certain Shimura varieties. *Res. Math. Sci.*, 3:Paper No. 37, 308, 2016.

[HT17] David Hansen and Jack A. Thorne. On the GL_n-eigenvariety and a conjecture of Venkatesh. *Selecta Math. (N.S.)*, 23(2):1205–1234, 2017.

[Kis04] Mark Kisin. Geometric deformations of modular Galois representations. *Invent. Math.*, 157(2):275–328, 2004.

[Kis09] Mark Kisin. Moduli of finite flat group schemes, and modularity. *Ann. of Math. (2)*, 170(3):1085–1180, 2009.

[KT17] Chandrashekhar B. Khare and Jack A. Thorne. Potential automorphy and the Leopoldt conjecture. *Amer. J. Math.*, 139(5):1205–1273, 2017.

[Liu07] Tong Liu. Torsion p-adic Galois representations and a conjecture of Fontaine. *Ann. Sci. École Norm. Sup. (4)*, 40(4):633–674, 2007.

[Lur04] Jacob Lurie. *Derived algebraic geometry*. ProQuest LLC, Ann Arbor, MI, 2004. Thesis (Ph.D.)–Massachusetts Institute of Technology.

[Lur09] Jacob Lurie. *Higher topos theory*, volume 170 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 2009.

[Lur18] Jacob Lurie. Kerodon. https://kerodon.net, 2018.

[NT23] James Newton and Jack A. Thorne. Adjoint Selmer groups of automorphic Galois representations of unitary type. *J. Eur. Math. Soc. (JEMS)*, 25(5):1919–1967, 2023.

[Qui67] Daniel G. Quillen. *Homotopical algebra*, volume No. 43 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin-New York, 1967.

[Sch15] Peter Scholze. On torsion in the cohomology of locally symmetric varieties. *Ann. of Math. (2)*, 182(3):945–1066, 2015.

[Sta23] The Stacks project authors. The Stacks project. https://stacks.math.columbia.edu, 2023.

[Ven19] Akshay Venkatesh. Derived Hecke algebra and cohomology of arithmetic groups. *Forum Math. Pi*, 7:e7, 119, 2019.

[Zhu21] Xinwen Zhu. Coherent sheaves on the stack of langlands parameters, 2021.