ON DERIVED GALOIS DEFORMATION RINGS IN CHARACTERISTIC 0 AND A CONJECTURE OF VENKATESH

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ABSTRACT. Let Π be a regular algebraic cuspidal automorphic representation of GL_n over a CM field F, and ρ its associated Galois representation in characteristic 0. With mild assumptions, we use the Taylor–Wiles method to define a free action of the potentially semistable derived deformation ring of ρ on the Π-part of the p-adic cohomology of GL_n/F, as predicted by a conjecture of Venkatesh. As part of the proof, we establish an equivalence between the derived deformation ring of ρ with the completion of the derived deformation ring of the reduction $\overline{\rho}$ at the point corresponding to ρ. This is a draft still undergoing revisions.

CONTENTS

1. INTRODUCTION

Langlands philosophy posits a correspondence between suitable subclasses of automorphic representations and Galois representations. This correspondence is expected to be compatible with p-adic deformations in the sense that there should exist an isomorphism of rings $R \to T$ where R is a deformation ring parametrising p-adic deformations of a Galois representation and T is a p-adic Hecke algebra which acts on p-adic automorphic forms. Restriction of scalars along $R \to T$ defines an action of R on p-adic automorphic forms. This article is concerned with defining a derived enhancement of this action.

To explain the motivation for this question, let us describe what we will mean by p-adic automorphic forms. Fix a prime p, a CM field F with maximal totally real subfield F^+ and an isomorphism $\iota\colon \mathbb{C} \cong \overline{\mathbb{Q}}_p$. Let $\mathbf{G} = \mathrm{GL}_n/F$ and suppose Π is a regular algebraic cuspidal automorphic representation of $\mathbf{G}(\mathbb{A}_F)$ of weight λ. Fix an open compact subgroup $K \subset \mathbf{G}(\mathbb{A}_F^\infty)$ (assumed small enough) and consider the locally symmetric space X_K of level K, which is a smooth manifold of dimension $\dim X_K = [F^+ \colon \mathbb{Q}](n^2-1)$ equipped with a locally constant sheaf of $\mathbb{Q}_p$-vector spaces $\mathcal{V}_\lambda$ determined by the weight λ. The cohomology group $H^*(X_K, \mathcal{V}_\lambda)$ is a finite-dimensional $\mathbb{Q}_p$-vector space equipped with an action of Hecke operators and serves as our space of p-adic automorphic forms of level K and weight λ. The system of eigenvalues associated to Π defines an eigenspace

$$H^*(X_K, \mathcal{V}_\lambda)_\Pi \subset H^*(X_K, \mathcal{V}_\lambda),$$

which is non-zero in the degrees $[\frac{1}{2}(d-l_0), \frac{1}{2}(d+l_0)]$, where $d = \dim X_K$ and $l_0 = [F^+\colon \mathbb{Q}](n-1) \in \mathbb{N}$ is the defect of $\mathbf{G}$. Disregarding a multiplicity factor depending on the prime-to-p level K^p, the dimension in degree $q_0 + i$ is $\binom{l_0}{i}$. This suggestive numerology indicates that the Π-contribution is a free graded module over an exterior algebra $\wedge^* V_\Pi$, where V_Π is a vector space of dimension l_0. A conjecture of Venkatesh – or rather, the p-adic realisation thereof – predicts that this does indeed hold in a natural way, where V_Π is a certain Selmer group defined in terms of the Galois representation attached to Π.

By work of Harris–Lan–Taylor–Thorne [HLTT16] and Scholze [Sch15], one can attach to Π a continuous Galois representation

$$\rho \coloneqq \rho_{\Pi,\iota}\colon \Gamma_{F,S} \to \mathrm{GL}_n(E)$$

whose values at Frobenius elements are related to the Hecke eigenvalues of Π in a precise way. Here, S is a finite set of finite places of F containing the ramified places of Π, $\Gamma_S = \mathrm{Gal}(\overline{F}_S/F)$ is the Galois group of the maximal extension of F which is unramified outside of S and $E/\mathbb{Q}_p$ is a finite extension. We will always assume that assume that the restrictions $\rho|_{\Gamma_v}$ have generic associated Weil–Deligne representations, equivalently that the local characteristic 0 deformation rings are formally smooth E-algebras. To simplify the exposition in the introduction, we ignore the places in S not above p.

The representation ρ and its Hodge–Tate weights $\mathrm{HT}(\rho)$ define a deformation functor $D_\rho^{\mathrm{ss}}\colon \mathrm{Art}_E \to \mathrm{Set}$ given by

$$B \mapsto \{r_B\colon \Gamma_{F,S} \to \mathrm{GL}_n(B) \mid r_B \text{ unramified outside } S, \text{ potentially semistable, } \mathrm{HT}(r_B) = \mathrm{HT}(\rho)\}/\sim$$

where Art_E is the category of artinian E-algebras and $\sim$ denotes strict equivalence. When ρ is irreducible, D_ρ^{ss} is pro-represented by a complete Noetherian local E-algebra R_ρ^{ss}, whose spectrum we view as an intersection

$$\mathrm{Spec}(R_\rho^{\mathrm{ss}}) \cong \mathrm{Spec}(R_\rho \otimes_{R_\rho^{\mathrm{loc}}} R_\rho^{\mathrm{loc,ss}}),$$

namely the intersection inside local deformations between the loci of global deformations and potentially semistable local deformations with preserved Hodge–Tate weights. The tangent space of R_ρ^{ss} is given by the geometric Bloch–Kato Selmer group $H^1_g(\Gamma_{F,S}, \mathrm{ad}\,\rho)$, which is known to vanish (as the Bloch–Kato conjecture predicts) in many cases by the main result of [A'C24]. In other words, $\mathrm{Spec}\, R_\rho^{\mathrm{ss}}$ is a point. However, tangent-obstruction theory predicts the dimension of R_ρ^{ss} to be given by the Euler characteristic, which comes out to $-l_0 < 0$. This suggests that the intersection above is non-transverse and that one ought to look for a derived version of R_ρ^{ss} which accounts for redundant obstructions. A natural candidate is given by replacing $\otimes$ with $\otimes^{\mathbf{L}}$, but the resulting object only has good formal properties when the rings involved are complete intersections of the expected dimension. In general, a more flexible framework is necessary.

A derived enhancement of classical deformation rings has been introduced for $\mathrm{mod}\, p$ representations by Galatius–Venkatesh [GV18], and for characteristic 0 representations a construction has been given by Zhu [Zhu21]. In both setups, derived deformation rings are simplicial (or animated) rings whose underlying static ring is canonically isomorphic to a classical deformation ring. For example, we can define a derived version of the functor D_ρ^{ss} above and obtain a *derived deformation ring* $\mathcal{R}_\rho^{\mathrm{ss}}$ such that

$$\pi_0(\mathcal{R}_\rho^{\mathrm{ss}}) \cong R_\rho^{\mathrm{ss}}.$$

The key benefit of considering $\mathcal{R}_\rho^{\mathrm{ss}}$ is best explained in terms of tangent complexes. Whereas R_ρ^{ss} is predicted to be a field and therefore to have a trivial tangent complex $\mathrm{T}_E^\bullet(R_\rho^{\mathrm{ss}}) \cong H^0(\mathrm{T}_E^\bullet(R_\rho^{\mathrm{ss}})) = 0$, the tangent complex of $\mathcal{R}_\rho^{\mathrm{ss}}$ has $H^1(\mathrm{T}_E^\bullet(\mathcal{R}_\rho^{\mathrm{ss}})) \cong H^1_g(\Gamma_{F,S}, \operatorname{ad}\rho(1))$, which is l_0-dimensional! In this way, the derived deformation ring witnesses obstructions not captured by the underlying static ring.

Thus, in hope of proving the dimension formula for the Π-contribution, we formulate two goals: define a free graded action

$$(\star) \qquad \pi_*(\mathcal{R}_\rho^{\mathrm{ss}}) \circlearrowright H^*(X_K, \mathcal{V}_\lambda)_\Pi,$$

and prove that $\pi_*(\mathcal{R}_\rho^{\mathrm{ss}})$ is an exterior algebra on a vector space V_Π of dimension l_0, a natural candidate being the l_0-dimensional dual adjoint Selmer group $H^1_g(\Gamma_{F,S}, \operatorname{ad}\rho(1))$. Without further ado, let us state a compact version of our main result.

Theorem 1.0.1 (Theorem 4.0.1)**.** *With notation as above, and under various assumptions on F and ρ, there exists a free graded action of $\pi_*(\mathcal{R}_\rho^{\mathrm{ss}})$ on $H^*(X_K, \mathcal{V}_\lambda)_\Pi$.*

Although Theorem 1.0.1 is a purely characteristic 0 statement, to apply the Taylor–Wiles method we will need to consider deformation rings of the reduction $\bar{\rho}\colon \Gamma_{F,S} \to \mathrm{GL}_n(k)$ where k is the residue field of E. To relate the two, we prove derived analogues of results by Kisin ([Kis09, Lem. 2.3.3, Prop. 2.3.5]) on classical deformation rings. For example, we obtain the following result as a consequence of Proposition 3.7.2 and 3.7.4, which are applicable in broad generality.

Corollary 1.0.2. *Let $\mathcal{R}_{\bar{\rho}} \in \mathcal{CNL}_{/k}$ denote the derived deformation ring (in the sense of* [GV18] *or* [Zhu21]*), and let $\xi\colon \mathcal{R}_{\bar{\rho}} \to \mathcal{O}$ be the point corresponding to an $\mathcal{O}$-lattice $\xi \subset \rho$. Then the derived completion*

$$(\mathcal{R}_{\bar{\rho}} \otimes_{\mathcal{O}} E)^\wedge_\xi := \lim_r (\mathcal{R}_{\bar{\rho}} \otimes_{\mathcal{O}} E)/^{\mathbf{L}} I_\xi^r$$

of $\mathcal{R}_{\bar{\rho}} \otimes_{\mathcal{O}} E$ at the ideal $I_\xi = \ker(\pi_0(\mathcal{R}_{\bar{\rho}}) \otimes_{\mathcal{O}} E \to E)$ represents the derived deformation ring of ρ in the sense of [Zhu21]*.*

Using the same results, we identify the potentially semistable deformation ring with the completion

$$\mathcal{R}_\rho^{\mathrm{ss}} \simeq (\mathcal{R}_{\mathbf{S}} \otimes_{\mathcal{O}} E)^\wedge_\xi$$

of a derived deformation ring representing a $\bmod p$ deformation problem $\mathbf{S}$ (see Corollary 3.7.5). In [GV18], an action of the crystalline version of $\pi_*(\mathcal{R}_{\mathbf{S}})$ on the *integral* cohomology is defined, in the Fontaine–Laffaille setting and assuming the Calegari–Geraghty vanishing conjecture for $\bmod p$ cohomology [CG18, Conj. B]. Inverting p then yields the desired action. However, beyond the Fontaine-Laffaille setting, the naive generalisation of the integral result is not expected to hold in general.

Let us outline the proof of Theorem 1.0.1. The first step, following [GV18], is to approximate the $\bmod p$ deformation ring $\mathcal{R} := \mathcal{R}_{\mathbf{S}}$ by a derived tensor product of static rings using the Taylor–Wiles method. For every $N \geq 1$, one has an augmented deformation problem $\mathbf{S}_N$ of Taylor–Wiles level N and a corresponding derived deformation ring $\mathcal{R}_N := \mathcal{R}_{\mathbf{S}_N}$ with underlying static ring $R_N := \pi_0(\mathcal{R}_N)$. Descending from Taylor–Wiles level yields a weak equivalence

$$\mathcal{R} \simeq \mathcal{R}_N \otimes^{\mathbf{L}}_{S_N} \mathcal{O},$$

where S_N is the $\mathcal{O}$-group ring of a finite p-group. Via this equivalence, we construct a morphism

$$\mathcal{R} \simeq \mathcal{R}_N \otimes^{\mathbf{L}}_{S_N} \mathcal{O} \to R_N \otimes^{\mathbf{L}}_{S_N} \mathcal{O} \to \overline{R}_N \otimes^{\mathbf{L}}_{\overline{S}_N} \mathcal{O}_N =: \mathcal{C}_N$$

where $R_N \twoheadrightarrow \overline{R}_N$ and $S_N \twoheadrightarrow \overline{S}_N$ are Artinian quotients and $\mathcal{O}_N := \mathcal{O}/\varpi^N$. A compactness argument yields a morphism to the limit

$$g\colon \mathcal{R}_{\mathbf{S}} \to \mathcal{C}_\infty := \lim_N \mathcal{C}_N,$$

which induces a morphism on localised and completed rings

$$(\mathcal{R}_{\mathbf{S}} \otimes_{\mathcal{O}} E)^\wedge_\xi \simeq \mathcal{R}_\rho^{\mathrm{ss}} \to (\mathcal{C}_\infty)^\wedge_\xi$$

Carrying out this non-constructive argument while retaining sufficient control over the induced morphism of tangent complexes is the main technical difficulty in the proof. Applying an '$R = T$' theorem of A'Campo [A'C24], we prove an equivalence

$$(\mathcal{C}_\infty)^\wedge_\xi \simeq R^\wedge_{\infty,\mathfrak{p}} \otimes^{\mathbf{L}}_{S^\wedge_{\infty,\mathfrak{a}}} E,$$

where $\mathfrak{a}$ is the augmentation ideal of S_∞. By A'Campo's result, the formally smooth E-algebra $R^\wedge_{\infty,\mathfrak{p}}$ acts freely on a patched module $M^\wedge_{\infty,\mathfrak{p}}$, and descending from Taylor–Wiles level induces a free action

$$\pi_*(\mathcal{R}^{\mathrm{ss}}_\rho) \circlearrowleft M^\wedge_{\infty,\mathfrak{p}} \otimes^{\mathbf{L}}_{S^\wedge_{\infty,\mathfrak{a}}} E \cong H^*(X_K, \mathcal{V}_\lambda)_\Pi.$$

A priori, this action might depend on the non-canonical choice made in defining g. In [GV18], this is ruled out by proving a compatibility with the action of the derived Hecke algebra introduced in [Ven19]. Thus, so far the naturality of the action is only known in the setting of [GV18], though it is expected to hold in general.

Let us describe the content of the sections below. In Section 2, we introduce some notation and recall some basic facts concerning ∞-categories, the cohomology of locally symmetric spaces and animated rings. Section 3 contains the needed facts about derived deformation theory in general as developed by Lurie, and Zhu's derived deformation rings of Galois representations. Subsection 3.7 contains results connecting derived deformations of a representation ρ and its reduction $\bar{\rho}$. In Section 4, we prove the main result (Theorem 4.0.1).

2. Preliminaries

We fix the following notation: F is a CM field; E is a finite extension of $\mathbb{Q}_p$ with ring of integers $\mathcal{O}$, uniformiser ϖ and residue field $\mathcal{O}/\varpi = k$. We will tacitly assume that E is large enough, so that all of our Galois representations are defined over E and E contains all embeddings $F \hookrightarrow \overline{\mathbb{Q}}_p$. When S is a set of places of F, we use the notation S_p for $\{v \in S \colon v \mid p\}$ and $S^p = S \setminus S_p$ For any $r \geq 1$, we let $\mathcal{O}_r = \mathcal{O}/\varpi^r$. Generally, we use calligraphic letters to denote ∞-categories, so that $\mathcal{A}lg_{\mathcal{O}}$ is the animation of $\mathrm{Alg}_{\mathcal{O}}$ (see 2.2). When we are interested in both an animated object and its underlying static object, we distinguish them by fonts; thus $R = \pi_0(\mathcal{R})$. Unless otherwise specified, limits and colimits are taken in the ∞-categorical sense. Throughout the paper, we use homological indexing.

This article uses the language of quasicategories (∞-categories). If Y is an object of an ∞-category $\mathcal{C}$, there is an over-∞-category $\mathcal{C}_{/Y}$ whose objects are given by morphisms $(X \to Y)$. The object $(1_Y \colon Y \to Y)$ is final in $\mathcal{C}_{/Y}$, i.e. for any $(X \to Y) \in \mathcal{C}_{/Y}$, the mapping space $\mathrm{Map}_{\mathcal{C}_{/Y}}((X \to Y), (Y \to Y))$ is contractible [Lur09, §1.2.9]. Given an ∞-category $\mathcal{C}$ and a full subcategory $h'\mathcal{C} \hookrightarrow h\mathcal{C}$ of its homotopy category, we define an ∞-category $\mathcal{C}'$ by forming the pullback of $N(h'\mathcal{C}) \to N(h\mathcal{C}) \leftarrow \mathcal{C}$. Equivalently, $\mathcal{C}'$ is the simplicial subset of $\mathcal{C}$ spanned by the vertices lying in $h'\mathcal{C}$ [Cis19, Ex. 3.9.5].

2.1. **Locally symmetric spaces.** In this section, we recall how automorphic representations of $\mathbf{G} := \mathrm{GL}_n/F$ contribute to the cohomology of locally symmetric spaces, following [Hev24, §2.1] and [KT17, §6.2]. We let $X^{\mathbf{G}}$ be the symmetric space of $\mathrm{Res}_{F/\mathbb{Q}}\,\mathbf{G}$ as in [BS73, §2] (a homogeneous $\mathbf{G}(F \otimes_{\mathbb{Q}} \mathbb{R})$-space).

An open compact subgroup $K \subset \mathbf{G}(\mathbb{A}_F^\infty)$ is *good* if $K = \prod_v K_v$ and K is neat. Given such a K, we define the locally symmetric space of level K as

$$X_K = \mathbf{G}(F)\backslash(X^{\mathbf{G}} \times \mathbf{G}(\mathbb{A}_F^\infty)/K),$$

a smooth manifold of dimension $[F^+ \colon \mathbb{Q}](n^2 - 1)$. The defect of $\mathbf{G}$ is the positive integer

$$l_0 := [F^+ \colon \mathbb{Q}](n-1).$$

Let $\lambda = (\lambda_\iota)_{\iota \in \mathrm{Hom}(F,E)} \in (\mathbb{Z}^n_+)^{\mathrm{Hom}(F,E)}$ be a dominant weight and suppose given representations $\tau_v \colon I_{F_v} \to \mathrm{GL}_n(E)$ with open kernel for $v \mid p$. Then [SZ99] associate to each τ_v an irreducible smooth $\mathcal{O}$-representation $\sigma^\circ(\tau_v)$ of $\mathrm{GL}_n(\mathcal{O}_{F_v})$, and we let $\sigma^\circ(\tau) = \bigotimes_{v|p} \sigma^\circ(\tau_v)$. To the pair (λ, τ) one then associates a locally constant sheaf of $\mathcal{O}$-modules $\mathcal{V}^\circ_{\lambda,\tau} = \mathcal{V}_\lambda \otimes \sigma(\tau) \in \mathrm{Sh}(X_K)$ where $\mathcal{V}^\circ_\lambda$ is a certain $\mathcal{O}$-lattice in the highest weight representation of $(\mathrm{GL}_n)_E^{\mathrm{Hom}(F,E)}$ of highest weight λ. The complex

$$R\Gamma(X_K, \mathcal{V}^\circ_{\lambda,\tau}) \in \mathcal{D}(\mathcal{O})$$

carries an action of the abstract Hecke algebra $\mathbb{T}^S := \mathcal{H}(\mathbf{G}(\mathbf{A}_F^{\infty,S}), K^S)$ where S is a finite set of places such that $K_v = \mathrm{GL}_n(\mathcal{O}_{F_v})$ for $v \notin S$ and $K^S = \prod_{v\notin S} K_v$. We define $\sigma(\tau) := \sigma^\circ(\tau) \otimes_{\mathcal{O}} E$ and $\mathcal{V}_{\lambda,\tau} := \mathcal{V}^\circ_{\lambda,\tau} \otimes_{\mathcal{O}} E$. If Π is a regular algebraic cuspidal representation of $\mathbf{G}$, we let $\mathfrak{p}_{\Pi,\iota} = \ker(\mathbb{T}^S \to \mathrm{End}_{\mathbb{C}}(\otimes'_{v\notin S} \Pi_v^{K_v}))$ and define the Π-part as the localisation

$$H^*(X_K, \mathcal{V}_{\lambda,\tau})_\Pi := H^*(X_K, \mathcal{V}_{\lambda,\tau})_{\mathfrak{p}_{\Pi,\iota}},$$

i.e. the eigenspace (over E) corresponding to the system of Hecke eigenvalues of Π. We write

$$\mathbb{T}^S\big(R\Gamma(X_K, \mathcal{V}^\circ_{\lambda,\tau})\big) \subset \mathrm{End}_{\mathcal{D}(\mathcal{O})}\left(R\Gamma(X_K, \mathcal{V}^\circ_{\lambda,\tau})\right)$$

for the corresponding faithful Hecke algebra. There exists an open compact subgroup $K_\tau \subset K$ such that

$$R\Gamma(X_K, \mathcal{V}_{\lambda,\tau})_\Pi = R\Gamma\big(K/K_\tau, R\Gamma(X_{K_\tau}, \mathcal{V}_\lambda) \otimes_E \sigma(\tau)\big),$$

and since taking invariants is an exact functor in characteristic 0, we have

$$H^*(X_K, \mathcal{V}_{\lambda,\tau})_\Pi \cong \mathrm{Hom}_{K/K_\tau}(\sigma(\tau), H^*(X_{K_\tau}, \mathcal{V}_\lambda)_\Pi).$$

Proposition 2.1.1. [ACC+23, Thm. 2.4.10] *Let $2q_0 = \dim X_K - l_0$, and fix a regular algebraic cuspidal representation Π of weight λ and inertial type $\tau = (\tau_v)_{v|p} = (\mathrm{LL}(\Pi_v)|_{I_{F_v}})_{v|p}$. Then $H^i(X_K, \mathcal{V}_{\lambda,\tau})_\Pi = 0$ for $i \notin [q_0, q_0 + l_0]$ and if $m_\Pi = \dim_{\mathbb{C}}(\Pi^\infty)^{K^p}$,*

$$\dim H^{q_0+i}(X_K, \mathcal{V}_{\lambda,\tau})_\Pi = m_\Pi \binom{l_0}{i}, \textit{for } i = 0, \dots, l_0.$$

2.2. **Animated algebras.** We begin by recalling the definition of animated algebras and its relation to simplicial rings, following the presentation in [ČS24, §5.1]. Let $\mathrm{Alg}_{\mathcal{O}}$ denote the ordinary category of commutative $\mathcal{O}$-algebras and let

$$\mathcal{A}lg_{\mathcal{O}} = \mathcal{A}ni(\mathrm{Alg}_{\mathcal{O}})$$

denote the ∞-category of animated $\mathcal{O}$-algebras, i.e. the ∞-category generated under sifted colimits by the subcategory $\mathrm{Alg}_{\mathcal{O}}^{\mathrm{ffg}}$ of finite free $\mathcal{O}$-algebras. It is the ∞-category underlying the simplicial model category

$$\mathrm{sAlg}_{\mathcal{O}} := \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathrm{Alg}_{\mathcal{O}})$$

of simplicial $\mathcal{O}$-algebras. That is, we have an equivalence of ∞-categories

$$\mathcal{A}lg_{\mathcal{O}} \simeq N\big((\mathrm{sAlg}_{\mathcal{O}})^\circ\big)[W^{-1}]$$

where N is the nerve functor, $(\mathrm{sAlg}_{\mathcal{O}})^\circ$ the full subcategory of bifibrant objects and W the class of weak equivalences [Lur09, Cor. 5.5.9.3]. We will freely alternate between the simplicial and animated viewpoints.

If $A \in \mathcal{A}lg_{\mathcal{O}}$ is an animated $\mathcal{O}$-algebra, an animated A-module is a connective module over the underlying $\mathbb{E}_1$-ring of A (see [Lur04, p. 19]). When $A = \pi_0(A)$ is static, the category

$$\mathcal{M}od_A := \mathcal{A}ni(\mathrm{Mod}_A)$$

of animated A-modules is equivalent to the derived category $D^{\geq 0}(A)$ of connective complexes of A-modules, which is equivalent to the localisation of the simplicial model category $\mathrm{sMod}_A = \mathcal{F}un(\Delta^{\mathrm{op}}, \mathrm{Mod}_A)$ of simplicial A-modules at weak equivalences.

Definition 2.2.1. [Lur04, Def. 2.5.9] An animated $\mathcal{O}$-algebra $A \in \mathcal{A}lg_{\mathcal{O}}$ is:

(1) Noetherian if $\pi_0(A)$ is Noetherian and $\pi_i(A)$ is a finitely generated $\pi_0(A)$-module for every $i \geq 0$.
(2) Local if $\pi_0(A)$ is local.

A local Noetherian $\mathcal{O}$-algebra A is complete if A is derived $\mathfrak{m}_{\pi_0(A)}$-adically complete. We let $\mathcal{CNL}_{/k} \subset (\mathcal{A}lg_{\mathcal{O}})_{/k}$ and $\mathcal{CNL}_{/E} \subset (\mathcal{A}lg_{\mathcal{O}})_{/E}$ denote the full subcategories of complete Noetherian local $\mathcal{O}$-algebras with residue fields $\pi_0(A)/\mathfrak{m}_{\pi_0(A)}$ equal to k and E, respectively.

Proposition 2.2.2. [Lur04, p. 32] *Let $A \in \mathcal{A}lg_{\mathcal{O}}$ and suppose B lies in the smallest subcategory of $\mathcal{A}lg_A$ containing $A[X]$ which is closed under finite colimits. Then $\mathrm{Map}_A(B, -)$ commutes with filtered colimits.*

If the assumption of Proposition 2.2.2 holds, we say that B is a *finitely presented A-algebra.*

Proposition 2.2.3. [Lur04, Prop. 3.1.5] *Let $A \in \mathcal{A}lg_{\mathcal{O}}$ be Noetherian, $j \geq 0$ and suppose B is an A-algebra. The following are equivalent:*

(1) *The functor $\mathrm{Map}_A(B, -)$ commutes with filtered colimits when restricted to the subcategory $\tau_{\leq j}(\mathcal{A}lg_A)$ of j-truncated A-algebras.*
(2) *The A-algebra $\tau_{\leq j} B$ is Noetherian and $\pi_0(B)$ is a finitely presented $\pi_0(A)$-algebra.*

If the equivalent conditions in Proposition 2.2.3 hold, we say that B is *of finite presentation to order j as an A-algebra*. If the conditions hold for every $j \geq 0$, we say that B is *almost of finite presentation over* A. From (2), we deduce that any Noetherian static ring $B \in \mathrm{Alg}_{\mathcal{O}}$ is almost of finite presentation.

Derived quotients. Fix an animated $\mathcal{O}$-algebra $A \in \mathcal{A}lg_{\mathcal{O}}$ and an element $a \in \pi_0(A)$. We define a morphism of $\mathcal{O}$-algebras $\mathcal{O}[t] \to A$ by choosing a lift $\tilde{a} \in A_0$, and let

$$A/^{\mathbf{L}}a \coloneqq A \otimes^{\mathbf{L}}_{\mathcal{O}[t]} \mathcal{O},$$

where t acts as 0 on $\mathcal{O}$. The algebra $A/^{\mathbf{L}}a$ is well-defined up to weak equivalence, and for every $r \geq 1$ we have a natural map

$$A/^{\mathbf{L}}a^{r+1} \to A/^{\mathbf{L}}a^r.$$

More generally, given $a_1, \ldots, a_m \in \pi_0(A)$ we define

$$A/^{\mathbf{L}}(a_1, \ldots, a_m) = A \otimes^{\mathbf{L}}_{\mathcal{O}[t_1,\ldots,t_m]} \mathcal{O} \simeq A/^{\mathbf{L}}a_1/^{\mathbf{L}} \ldots /^{\mathbf{L}}a_m.$$

There are natural maps

$$A/^{\mathbf{L}}(a_1^{r+1}, \ldots, a_m^{r+1}) \to A/^{\mathbf{L}}(a_1^r, \ldots, a_m^r)$$

forming an inverse system of animated $\mathcal{O}$-algebras and the limit depends only on the closed subspace of $\mathrm{Spec}\, \pi_0(A)$ defined by $(a_1, \ldots, a_m)$, as the following result shows.

Proposition 2.2.4. [Lur04, Prop. 6.1.1] *Let $A \in \mathcal{A}lg_{\mathcal{O}}$ and let $J \subseteq \pi_0(A)$ be a finitely generated ideal. Fix a set of generators $\{a_1, \ldots, a_m\}$ for J. For any $B \in \mathcal{A}lg_{\mathcal{O}}$, the subspace*

$$\operatorname*{colim}_r \mathrm{Map}_{\mathcal{A}lg_{\mathcal{O}}}(A/^{\mathbf{L}}(a_1^r, \ldots a_m^r), B) \to \mathrm{Map}_{\mathcal{A}lg_{\mathcal{O}}}(A, B)$$

is the space spanned by morphisms $f\colon A \to B$ such that the induced morphism $\pi_0(A) \to \pi_0(B)$ factors through $\pi_0(A)/J^N$ for some $N \geq 1$.

Definition 2.2.5. [ČS24, 5.6.1] Let $A \in \mathcal{A}lg_{\mathcal{O}}$ be Noetherian and let $I = (a_1, \ldots, a_m) \subset \pi_0(A)$. Then A is derived I-adically complete if the natural map

$$A \to \lim_r A/^{\mathbf{L}}(a_1^r, \ldots, a_m^r)$$

is a weak equivalence.

Lemma 2.2.6. [BS15, Lem. 3.4.13] *Let $A \in \mathcal{M}od_{\mathcal{O}}$ and suppose $\pi_i(A)$ is ϖ-adically complete in the usual sense for every $i \geq 0$. Then A is derived ϖ-adically complete.*

3. Derived deformation theory

3.1. **Artinian algebras.** In this section, we recall the notion of Artinian animated algebras following [Lur04].

Definition 3.1.1. The category $\mathcal{A}rt_{/k}$ (resp. $\mathcal{A}rt_{/E}$) of Artinian animated $\mathcal{O}$-algebras over k (resp. over E) is defined as the sub-∞-category of $(\mathcal{A}lg_{\mathcal{O}})_{/k}$ (resp. $(\mathcal{A}lg_{\mathcal{O}})_{/E}$) spanned by $(B \to k)$ (resp. $(B \to E)$) such that

(1) $\pi_0(B)$ is an Artinian ring with residue field k (resp. E).
(2) $\pi_*(B)$ is a finitely generated $\pi_0(B)$-module.

Any Artinian animated $\mathcal{O}$-algebra (over k or E) is truncated. Note that if $B \in \mathcal{A}rt_{/E}$ then $\pi_0(B)$ is an E-algebra, and B is truncated since $\pi_*(B)$ is a finite-dimensional E-vector space. A useful description of Artinian algebras allowing for proofs by induction is given in terms of small extensions.

Definition 3.1.2. [Lur04, Def. 3.3.1] Let $B \in \mathcal{A}lg_{\mathcal{O}}$ and $M \in \mathcal{M}od_B$. A small extension of B by M is a pullback

$$\begin{tikzcd} B' \arrow[r] \arrow[d] & B \arrow[d, "s"] \\ B \arrow[r, "(\mathrm{id},0)"'] & B \oplus M[1] \end{tikzcd}$$

where $B \oplus M[1]$ is the trivial square-zero extension [Lur04, p.32] and $s\colon B \to B \oplus M[1]$ is a section of the projection.

Lemma 3.1.3. [Lur04, Lem. 6.2.6] *Let $(B \to E) \in \mathcal{A}rt_{/E}$. Then there exists a factorisation*

$$B \simeq B_m \to B_{m-1} \to \cdots \to B_0 \simeq E$$

such that B_j is a small extension of B_{j-1} by $E[n_j]$ for an increasing sequence $n_j \geq 0$ and $\dim_E(\pi_*(B_j)) = j+1$. The analogous statement is true for $(B \to k) \in \mathcal{A}rt_{/k}$.

3.2. **Integral models of Artinian E-algebras.** In this section, we introduce a notion of integral models of Artinian E-algebras. The definition is made in analogy with [Kis09, 2.3.5] and will be used in Definition 3.7.1.

Note that we have a canonical equivalence of ∞-categories

$$\left((\mathcal{A}lg_{\mathcal{O}})_{/E}\right)_{/(B\to E)} \simeq (\mathcal{A}lg_{\mathcal{O}})_{/B}.$$

Definition 3.2.1. Let $(B \to E) \in \mathcal{A}rt_{/E}$. The ∞-category $\mathcal{I}nt B$ is defined as the sub-∞-category of $(\mathcal{A}lg_{\mathcal{O}})_{/B}$ spanned by all $(A \to B)$ such that:

(1) $\pi_*(A)$ is a finitely generated $\mathcal{O}$-module.
(2) The induced map $\pi_*(A) \to \pi_*(B)$ is injective.
(3) The induced map $A \otimes^{\mathbf{L}}_{\mathcal{O}} E \to B \otimes^{\mathbf{L}}_{\mathcal{O}} E$ is a weak equivalence.

The central result about the category $\mathcal{I}nt B$ that we will need is the following.

Proposition 3.2.2. *For any $B \in \mathcal{A}rt_{/E}$, the ∞-category $\mathcal{I}nt B$ is filtered.*

By Proposition 2.2.3(2) and conditon (1) in the definition, any $A \in \mathcal{I}nt B$ is finitely presented over $\mathcal{O}$. When B is clear from context, we sometimes denote objects of $\mathcal{I}nt B$ simply by A. Note that, if $A, A' \in \mathcal{I}nt B$ then by definition

$$\mathrm{Map}_{\mathcal{I}nt B}(A, A') = \mathrm{Map}_{(\mathcal{A}lg_{\mathcal{O}})_{/B}}(A, A').$$

Lemma 3.2.3. *Let $(B \to E) \in \mathcal{A}rt_{/E}$ and $(A \to B) \in \mathcal{I}nt B$. The natural map*

$$\mathrm{Map}_{\mathcal{A}lg_{\mathcal{O}}}(A, B \times_E \mathcal{O}) \to \mathrm{Map}_{\mathcal{A}lg_{\mathcal{O}}}(A, B)$$

is a weak equivalence.

Proof. Since $\mathcal{O}$ and E are static and $\pi_0(A)$ is a finitely generated $\mathcal{O}$-module, we have equivalences of mapping spaces

$$\mathrm{Map}_{\mathcal{A}lg_{\mathcal{O}}}(A, E) \simeq \mathrm{Map}_{\mathcal{A}lg_{\mathcal{O}}}(\pi_0(A), E) \simeq \mathrm{Map}_{\mathcal{A}lg_{\mathcal{O}}}(\pi_0(A), \mathcal{O}) \simeq \mathrm{Map}_{\mathcal{A}lg_{\mathcal{O}}}(A, \mathcal{O}).$$

Consequently,

$$\mathrm{Map}_{\mathcal{A}lg_{\mathcal{O}}}(A, B \times_E \mathcal{O}) \simeq \mathrm{Map}_{\mathcal{A}lg_{\mathcal{O}}}(A, B) \times_{\mathrm{Map}_{\mathcal{A}lg_{\mathcal{O}}}(A,E)} \mathrm{Map}_{\mathcal{A}lg_{\mathcal{O}}}(A, \mathcal{O}) \simeq \mathrm{Map}_{\mathcal{A}lg_{\mathcal{O}}}(A, B).$$

□

To prove Proposition 3.2.2, we will prove an analogue of Lemma 3.1.3 for the category $\mathcal{I}nt B$.

Lemma 3.2.4. *Let $B \in \mathcal{A}rt_{/E}$ be given by a sequence of small extensions*

$$B \simeq B_m \to B_{m-1} \to \cdots \to B_0 \simeq E$$

as in Lemma 3.1.3. Then, for any $A \in \mathcal{I}nt B$, the morphism $A \to \mathcal{O}$ factors as

$$A \simeq A_m \to A_{m-1} \to \cdots \to A_0 \simeq \mathcal{O},$$

where, for every $j \geq 1$, $A_j \in \mathcal{I}nt B_j$ is a small extension of A_{j-1} by $\varpi^{-r_j}\mathcal{O}[n_j]$ for a non-decreasing sequence $0 \leq n_1 \leq \cdots \leq n_m$ and some $r_j \geq 0$.

Before embarking on the proof of the Lemma, let us describe the homotopy groups of a small extension. Let $A \in \mathcal{A}lg_{\mathcal{O}}$ be n-truncated and suppose $M \in \mathcal{A}ni(\mathrm{Mod}\, A)$ has homotopy concentrated in degree $n-1$. Then $M[1]$ has homotopy concentrated in degree n, and any small extension $A' = A \times_{A \oplus M[1]} A$ gives rise to a long exact sequence

$$\cdots \to \pi_{i+1}(A) \to \pi_{i-1}(M) \to \pi_i(A') \to \pi_i(A) \to \pi_{i-2}(M) \to \ldots$$

which implies that $\pi_i(A') \cong \pi_i(A)$ for every $i \neq n$ and that

$$\pi_{n-1}(M) \cong \ker(\pi_n(A') \to \pi_n(A)).$$

Thus, a single small extension of the form considered in Lemma 3.2.4 only changes the homotopy groups in the top degree by adding a module which is free of rank 1 over $\mathcal{O}$.

Proof of Lemma 3.2.4. We prove the lemma by induction on the length m of the sequence $B_m \to \cdots \to B_0$. Let $n = n_m$ denote the largest integer such that $\pi_n(B) \neq 0$. The case $m = 0$ is trivial since $\mathcal{I}nt E \simeq \{\mathcal{O}\}$. For the induction step, suppose the statement holds for any length $m-1$ sequence $B'_{m-1} \to \cdots \to B'_0$.

The kernel

$$I = \ker(\pi_*(A) \hookrightarrow \pi_*(B_m) \to \pi_*(B_{m-1})) \subset \pi_n(A)$$

is a $\pi_0(A)$-submodule of $\pi_*(A)$, which is free of rank 1 as an $\mathcal{O}$-module. Using [Lur04, Prop. 3.3.3, Prop. 3.3.5], we construct a morphism in $\mathcal{A}lg_{\mathcal{O}}$

$$A \to A_{m-1}$$

where A_{m-1} ('A/I' in the notation of loc. cit.) satisfies the following properties:

(1) For any $Y \in \mathcal{A}lg_{\mathcal{O}}$,

$$\mathrm{Map}_{\mathcal{A}lg_{\mathcal{O}}}(A_{m-1}, Y) \hookrightarrow \mathrm{Map}_{\mathcal{A}lg_{\mathcal{O}}}(A, Y)$$

is the subspace spanned by maps for which the induced map on homotopy groups maps I to $0 \in \pi_*(Y)$.

(2) The induced map $\pi_*(A) \to \pi_*(A_{m-1})$ is an isomorphism in degree $i \neq n$ and for $i = n$ is the map $\pi_n(A) \to \pi_n(A)/I$.

(3) A is a small extension of A_{m-1} by $I[n]$, i.e. there is an equivalence

$$A \simeq A_m = A_{m-1} \times_{A_{m-1} \oplus I[n+1]} A_{m-1}.$$

From (1)-(3), we deduce that the map $A_{m-1} \to B_{m-1}$ defines an object of $\mathcal{I}nt B_{m-1}$ and that the square

$$\begin{tikzcd} A \arrow[r] \arrow[d] & A_{m-1} \arrow[d] \\ B \arrow[r] & B_{m-1} \end{tikzcd}$$

commutes up to homotopy. By the induction hypothesis, $A_{m-1} \to \mathcal{O}$ factors as a series of small extensions of the required form, and we are done. $\square$

Lemma 3.2.5. *Let $B \in \mathcal{A}rt_{/E}$. Then $B \times_E \mathcal{O}$ is equivalent in $(\mathcal{A}lg_{\mathcal{O}})_{/\mathcal{O}}$ to a filtered colimit of algebras of the form of A_m in Lemma 3.2.4.*

Proof. To prove the second claim, we use induction on m again. The case $m = 0$ is trivial. Suppose the statement is true for any B' given by a length $m-1$ sequence of small extensions. Let $n = n_m + 1$ and note that

$$E[n] = \operatorname*{colim}_{r\geq 0} \varpi^{-r}\mathcal{O}[n].$$

Since filtered colimits commute with finite limits, under the induction hypothesis we have

$$\begin{aligned} B_m \times_E \mathcal{O} &\simeq (B_{m-1} \times_{B_{m-1}\oplus E[n]} B_{m-1}) \times_E \mathcal{O} \\ &\simeq (B_{m-1} \times_E \mathcal{O}) \times_{(B_{m-1}\times_E \mathcal{O})\oplus E[n]} (B_{m-1} \times_E \mathcal{O}) \\ &\simeq \operatorname*{colim}_{A_{m-1}} (A_{m-1} \times_{A_{m-1}\oplus E[n]} A_{m-1}) \end{aligned}$$

where the colimit runs over a filtered diagram of $A_{m-1} \in \mathcal{I}nt B_{m-1}$ of the sought form. We have equivalences

$$\begin{aligned} A_{m-1} \oplus E[n] &\simeq A_{m-1} \oplus \operatorname*{colim}_{r\geq 0} \varpi^{-r}\mathcal{O}[n] \\ &\simeq \operatorname*{colim}_{r\geq 0} A_{m-1} \oplus \varpi^{-r}\mathcal{O}[n]. \end{aligned}$$

Indeed, A_{m-1} preserves $\mathcal{O}[n]$ inside $E[n]$, since if $\mathcal{E}nd$ denotes the derived endormorphism ring, we have

$$A_{m-1} \to \mathcal{E}nd_{\mathcal{O}}(E[n]) \simeq \pi_0\mathcal{E}nd_{\mathcal{O}}(E) \cong E,$$

and hence the action map $A_{m-1} \to E$ factors through $\pi_0(A_{m-1})$ and lands in $\mathcal{O}$. It follows that

$$\operatorname*{colim}_{A_{m-1}} (A_{m-1} \times_{A_{m-1}\oplus E[n]} A_{m-1}) \cong \operatorname*{colim}_{A_{m-1}} \operatorname*{colim}_{r\geq 0} (A_{m-1} \times_{A_{m-1}\oplus \varpi^{-r}\mathcal{O}[n]} A_{m-1}),$$

and thus B_m is a filtered colimit of the sought form. ◻

We are now in a position to prove the main result of this section, Proposition 3.2.2.

Proof of Proposition 3.2.2. By the local characterisation of filtered ∞-categories [Lur18, Tag 02PS], it suffices to prove the following statement: Given $A, A' \in \mathcal{I}nt B$ and a morphism of simplicial sets

$$\gamma\colon \partial\Delta^n \to \operatorname{Map}_{/B}(A, A')$$

for some $n \geq 0$, there exists an $A'' \in \mathcal{I}nt B$ and a morphism $g\colon A' \to A''$ such that the composition

$$g \circ \gamma\colon \partial\Delta^n \to \operatorname{Map}_{/B}(A, A') \to \operatorname{Map}_{/B}(A, A'')$$

is null-homotopic. In other words, we would like to define the dotted arrow in the diagram

$$\begin{tikzcd} \partial\Delta^n \arrow[r, "\gamma"] \arrow[d] & \operatorname{Map}_{/B}(A, A') \arrow[d, "g_*"] \\ \Delta^n \arrow[r, dotted] & \operatorname{Map}_{/B}(A, A'') \end{tikzcd}$$

Since B is final in $(\mathcal{A}lg_{\mathcal{O}})_{/B}$ and by Lemma 3.2.5, we have

$$\{*\} \simeq \operatorname{Map}_{/B}(A, B) \simeq \operatorname{Map}_{/B}(A, B \times_E \mathcal{O}) \simeq \operatorname{Map}_{/B}(A, \operatorname*{colim}_{A''} A''),$$

where A'' runs over a filtered subcategory of $\mathcal{I}nt B$. Thus, the composite

$$\partial\Delta^n \to \operatorname{Map}_{/B}(A, A') \to \operatorname{Map}_{/B}(A, \operatorname*{colim}_{A''} A'')$$

is null-homotopic. Now, since $A' \in \mathcal{I}nt B$ is a finitely presented $\mathcal{O}$-algebra by Proposition 2.2.3, the map $A' \to \operatorname{colim} A''$ factors through some A''. It follows that there exists a $g\colon A' \to A''$ such that the composition $g \circ \gamma$ is null-homotopic. ◻

Lemma 3.2.6. *Let $B \in \mathcal{A}rt_{/E}$. We have the following:*

(1) *There is a natural weak equivalence* $\operatorname*{colim}_{A\in\mathcal{I}nt B} A \to B \times_E \mathcal{O}$.
(2) *There is a canonical isomorphism of graded rings* $\pi_*(B \times_E \mathcal{O}) \cong \pi_*(B) \times_E \mathcal{O}$.
(3) *Any $A \in \mathcal{I}nt B$ is derived ϖ-adically complete.*
(4) *For any $A \in \mathcal{I}nt B$ and $r \geq 1$, $(A/^{\mathbf{L}}\varpi^r \to k) \in \mathcal{A}rt_{/k}$.*

Proof. (1) By Lemma 3.2.3, any $(A \to B) \in \mathcal{I}nt B$ defines a map $A \to B \times_E \mathcal{O}$. Taking the colimit over $\mathcal{I}nt B$, we produce

$$\operatorname*{colim}_{A \in \mathcal{I}nt B} A \to B \times_E \mathcal{O},$$

and we claim that this map is a weak equivalence. Since $\mathcal{I}nt B$ is filtered and filtered colimits commute with π_*, it is enough to prove that the map

$$\operatorname*{colim}_{A \in \mathcal{I}nt B} \pi_*(A) \to \pi_*(B \times_E \mathcal{O})$$

is an isomorphism. Injectivity follows from condition (3) in the definition of $\mathcal{I}nt B$. To see that the map is surjective, we use the isomorphism provided by Lemma 3.2.5, i.e.

$$\pi_*(B \times_E \mathcal{O}) \cong \operatorname*{colim}_{A''} \pi_*(A''),$$

a filtered colimit over objects $A'' \in \mathcal{I}nt B$. Surjectivity follows.

(2) There is a Mayer-Vietoris sequence

$$\cdots \to \pi_n(B \times_E \mathcal{O}) \to \pi_n(B) \times \pi_n(\mathcal{O}) \to \pi_n(E) \to \dots,$$

and hence for every $n \geq 1$, we have

$$\pi_n(B \times_E \mathcal{O}) \cong \pi_n(B) \times \pi_n(\mathcal{O}) \cong \pi_n(B)$$

since $\pi_n(B) \to E$ is the zero map. For $n = 0$, the sequence implies

$$\pi_0(B \times_E \mathcal{O}) \cong \ker(\pi_0(B) \times \mathcal{O} \to E) \cong \pi_0(B) \times_E \mathcal{O}.$$

(3) By definition, $\pi_*(A)$ is a finitely generated $\mathcal{O}$-module and hence A is ϖ-adically complete by Lemma 2.2.6.

(4) Let $A \in \mathcal{I}nt B$ and $r \geq 1$. Since $\pi_*(A)$ is a finitely generated and free $\mathcal{O}$-module, Quillen's spectral sequence [Qui67, Thm. 2.6]

$$E^2_{i,j} \colon \operatorname{Tor}_i^{\mathcal{O}[t]}\big(\pi_j(A), \mathcal{O}\big) \implies \pi_{i+j}(A/^{\mathbf{L}}\varpi^r)$$

amounts to an isomorphism

$$\pi_*(A/^{\mathbf{L}}\varpi^r) \cong \pi_*(A) \otimes_{\mathcal{O}} \mathcal{O}/\varpi^r.$$

It follows that $\pi_0(A/^{\mathbf{L}}\varpi^r)$ is Artinian and that $\pi_*(A/^{\mathbf{L}}\varpi^r)$ is finitely generated over $\pi_0(A/^{\mathbf{L}}\varpi^r)$. □

Lemma 3.2.7. *Let $(A \to B) \in \mathcal{I}nt B$. Then the map $A \to E$ factors through $\mathcal{O} \to E$, and we have a lifting*

$$\begin{tikzcd}
\operatorname{Int} B \arrow[r, dashed] \arrow[d] & (\mathcal{A}lg_{\mathcal{O}})_{/\mathcal{O}} \arrow[d] \\
(\mathcal{A}lg_{\mathcal{O}})_{/B} \arrow[r] & (\mathcal{A}lg_{\mathcal{O}})_{/E}
\end{tikzcd}$$

which reflects colimits.

Proof. The functor $\mathcal{I}nt B \to (\mathcal{A}lg_{\mathcal{O}})_{/B}$ factors through

$$(\mathcal{A}lg_{\mathcal{O}})_{/\operatorname*{colim}_{A \in \mathcal{I}nt B} A} \simeq (\mathcal{A}lg_{\mathcal{O}})_{/B \times_E \mathcal{O}} \simeq (\mathcal{A}lg_{\mathcal{O}})_{/B} \times_{(\mathcal{A}lg_{\mathcal{O}})_{/E}} (\mathcal{A}lg_{\mathcal{O}})_{/\mathcal{O}},$$

where we have used Lemma 3.2.6(1) in the first equivalence. Thus, we obtain the sought lifting. The second part follows from that colimits in an over-∞-category $\mathcal{C}_{/X}$ are computed in $\mathcal{C}$ [Lur09, Prop. 1.2.13.8]. □

3.3. **Topologies on animated algebras.** In this section, we discuss some notions of continuity (or topology) in the derived setting which we, following [Zhu21], will use later on to define derived moduli stacks of continuous representations.

Given a static $\mathcal{O}$-module M, we can equip it with an ind-ϖ-adic topology via the isomorphism

$$M \cong \varinjlim_{M' \subset M} \varprojlim_{r \geq 1} M'/\varpi^r M',$$

where the filtered colimit runs over the finitely generated submodules M' of M. In general, the ind-ϖ-adic topology is finer than the ϖ-adic, and if M is finitely generated, they coincide. As a topological space, M equipped with the ind-ϖ-adic topology is the image of M under the functor $\mathrm{Mod}_{\mathcal{O}} \to \mathrm{Ind}(\mathrm{Pro}(\mathrm{Set}^{\mathrm{cp}})) \hookrightarrow \mathrm{Top}$ where $\mathrm{Set}^{\mathrm{cp}}$ is the category of finite sets.

Lemma 3.3.1. [Zhu21, 2.4.18] *Let Γ be a profinite set and define*

$$\begin{aligned} C_{\mathrm{sc}}(\Gamma, -) \colon \mathrm{Mod}_{\mathcal{O}} &\to \mathrm{Mod}_{\mathcal{O}} \\ M &\mapsto \mathrm{Map}_{\mathrm{Ind}(\mathrm{Pro}(\mathrm{Set}^{\mathrm{cp}}))}(\Gamma, M) \end{aligned}$$

Then $C_{\mathrm{sc}}(\Gamma, -)$ is exact and lax symmetric monoidal and therefore extends to a t-exact functor $\mathcal{A}ni\,\mathrm{Mod}_{\mathcal{O}} \to \mathcal{A}ni\,\mathrm{Mod}_{\mathcal{O}}$ which preserves colimits. Moreover, $C_{\mathrm{sc}}(\Gamma, -)$ lifts to a nilcomplete functor $\mathcal{A}lg_{\mathcal{O}} \to \mathcal{A}lg_{\mathcal{O}}$ which preserves finite limits and sifted colimits.

Lemma 3.3.2. *Let $A \in \mathcal{M}od_{\mathcal{O}}^{\mathrm{perf}}$ be a perfect animated $\mathcal{O}$-module. Then there is a natural weak equivalence*

$$C_{\mathrm{sc}}(\Gamma, A) \to \lim_{r \geq 1} C_{\mathrm{sc}}(\Gamma, A/^{\mathbf{L}}\varpi^r)$$

which respects the monoidal structure. Hence, if $A \in \mathcal{A}lg_{\mathcal{O}}$ then the equivalence holds in $\mathcal{A}lg_{\mathcal{O}}$.

Proof. For any $A \in \mathcal{M}od_{\mathcal{O}} \simeq \mathcal{D}^{\geq 0}(\mathcal{O})$, we have natural morphisms (of $\mathcal{O}$-algebras, if $A \in \mathcal{A}lg_{\mathcal{O}}$)

$$C_{\mathrm{sc}}(\Gamma, A) \to C_{\mathrm{sc}}(\Gamma, \lim A/^{\mathbf{L}}\varpi^r) \to \lim C_{\mathrm{sc}}(\Gamma, A/^{\mathbf{L}}\varpi^r)$$

If A is derived ϖ-adically complete – in particular, if $\pi_*(A)$ is finitely generated over $\mathcal{O}$ (Lemma 2.2.6) – the first map is an equivalence. Let $P(A)$ be the property that the second map is a weak equivalence. Then $P(\mathcal{O}[0])$ holds, $P(A)$ implies $P(A[1])$ and for any fibre sequence $A \to B \to C \to A[1]$, if P holds for two of A, B, C then it holds for the third by the five lemma. Moreover, for any A, B, $P(A \oplus B)$ is implied by $P(A)$ and $P(B)$ since $C_{\mathrm{sc}}(\Gamma, -)$ commutes with colimits. Finally, any retract $A \to B \to A$ induces a commutative diagram

$$\begin{array}{ccccc} A & \longrightarrow & B & \longrightarrow & A \\ \downarrow & & \downarrow & & \downarrow \\ \lim_r C_{\mathrm{sc}}(\Gamma, A/^{\mathbf{L}}\varpi^r) & \longrightarrow & \lim_r C_{\mathrm{sc}}(\Gamma, B/^{\mathbf{L}}\varpi^r) & \longrightarrow & \lim_r C_{\mathrm{sc}}(\Gamma, A/^{\mathbf{L}}\varpi^r) \end{array}$$

where the composition of horizontal arrows is the identity, so that if $P(B)$ holds then the induced map $\pi_*(A) \to \pi_*(\lim_r C_{\mathrm{sc}}(\Gamma, B/^{\mathbf{L}}\varpi^r))$ is both injective and surjective, i.e. $P(A)$ holds. It follows that $P(A)$ holds for any perfect animated $\mathcal{O}$-module, since the category $\mathcal{D}^{\geq 0}_{\mathrm{perf}}(\mathcal{O})$ of perfect complexes is the smallest strictly full, saturated, triangulated subcategory of $\mathcal{D}^{\geq 0}(\mathcal{O})$ containing $\mathcal{O}[0]$ ([Sta23, Tag 0ATI]). $\square$

3.4. **Moduli stacks of Galois representations.** In this section, we discuss moduli stacks of representations of a profinite group Γ valued in a smooth affine group scheme $\mathbf{G}/\mathcal{O}$ of finite type, following [Zhu21]. We will only consider what is referred to as strongly continuous representations in loc. cit.

For any $n \in \mathbb{N}$, we let $\mathcal{O}[\mathbf{G}^n]$ denote the coordinate ring of $\mathbf{G}^n$. The functor of points $\mathbf{G} \colon \mathrm{Alg}_{\mathcal{O}} \to \mathrm{Grp}$ extends via animation to a functor

$$\mathbf{G} \colon \mathcal{A}lg_{\mathcal{O}} \to \mathcal{A}ni(\mathrm{Grp}).$$

As a simplicial space, $\mathbf{G}(A)$ is the functor $[n] \mapsto \mathrm{Map}_{\mathcal{A}lg_{\mathcal{O}}}(\mathcal{O}[\mathbf{G}^n], A)$ with boundary maps defined using the group law on $\mathbf{G}$ and degeneracies defined by inclusions $\mathbf{G}^{n_1} \times \{\mathrm{id}\} \times \mathbf{G}^{n_2} \to \mathbf{G}^{n_1+n_2+1}$. These maps define a cosimplicial animated $\mathcal{O}$-algebra $\mathcal{O}[\mathbf{G}^\bullet] \in \mathcal{F}un(\Delta, \mathcal{A}lg_{\mathcal{O}}) = \mathcal{F}un(\Delta, \mathcal{A}lg_{\mathcal{O}})$ which is finitely presented in the following sense.

Lemma 3.4.1. *The functor* $\mathrm{Map}_{\mathcal{F}un(\Delta,\mathcal{A}lg_{\mathcal{O}})}(\mathcal{O}[\mathbf{G}^\bullet], -)$ *commutes with filtered colimits.*

Proof. Since $\mathcal{O}[\mathbf{G}^n] \cong \mathcal{O}[\mathbf{G}] \otimes_{\mathcal{O}} \cdots \otimes_{\mathcal{O}} \mathcal{O}[\mathbf{G}]$ and $\mathcal{O}[\mathbf{G}]$ is a finitely presented $\mathcal{O}$-algebra, the statement follows from Proposition 2.2.2 and [Lur04, Prop. 5.3.4.13]. $\square$

Given $A \in \mathcal{A}lg_{\mathcal{O}}$, Zhu [Zhu21, Remark 2.2.2] proves the equivalence

$$\mathrm{Map}_{\mathcal{A}ni(\mathrm{Grp})}(\Gamma, \mathbf{G}(A)) \simeq \mathrm{Map}_{\mathcal{F}un(\Delta,\mathcal{A}lg_{\mathcal{O}})}(\mathcal{O}[\mathbf{G}^\bullet], C(\Gamma^\bullet, A))$$

where for any $[n] \in \Delta$, $C(\Gamma^n, -)$ is the animation of the functor which maps a static $\mathcal{O}$-module M to the set of all functions $\Gamma^n \to M$ (cf. the definition of $C_{\mathrm{sc}}(\Gamma^\bullet, -)$ in Lemma 3.3.1). Replacing $C(\Gamma^\bullet, -)$ by $C_{\mathrm{sc}}(\Gamma^\bullet, -)$, we obtain the following definition.

Definition 3.4.2. [Zhu21, Def. 2.4.21] The derived moduli stack of framed $\mathbf{G}$-valued (strongly continuous) Γ-representations is the functor

$$\begin{aligned} \mathcal{X}^{\square}_{\Gamma,\mathbf{G}} \colon \mathcal{A}lg_{\mathcal{O}} &\to \mathcal{A}ni \\ A &\mapsto \mathrm{Map}_{\mathcal{F}un(\Delta,\mathcal{A}lg_{\mathcal{O}})}\big(\mathcal{O}[\mathbf{G}^\bullet], C_{\mathrm{sc}}(\Gamma^\bullet, A)\big). \end{aligned}$$

Example 3.4.3. [Zhu21, Lem. 2.4.22] If $A \in \mathcal{A}lg_{\mathcal{O}}$ is static, then $\mathcal{X}^{\square}_{\Gamma,\mathbf{G}}(A)$ is the set of homomorphisms $\rho_A \colon \Gamma \to \mathrm{GL}_n(A)$ such that A^n is a filtered colimit of Γ-stable finitely generated $\mathcal{O}$-modules with continuous Γ-action.

Definition 3.4.4. Let $\mathcal{F} \colon \mathcal{A}lg_{\mathcal{O}} \to \mathcal{A}ni$ be a functor and fix $\bar{\rho} \in \mathcal{F}(k)$. We define

$$\begin{aligned} \mathcal{F}_{\bar{\rho}} \colon \mathcal{A}rt_{/k} &\to \mathcal{A}ni \\ (B \to k) &\mapsto \mathcal{F}(B) \times_{\mathcal{F}(k)} \{\bar{\rho}\}. \end{aligned}$$

Similarly, if $\rho \in \mathcal{F}(E)$, we obtain a functor $\mathcal{F}_\rho \colon \mathcal{A}rt_{/E} \to \mathcal{A}ni$ given by $\mathcal{F}(B \to E) = \mathcal{F}(B) \times_{\mathcal{F}(E)} \{\rho\}$.

The functor in Definition 3.4.2 is the functor of framed deformations. To construct the 'unframed' versions, let $\mathbf{Z} \subset \mathbf{G}$ be the center of $\mathbf{G}$ and consider the conjugation action of $\mathbf{PG} = \mathbf{G}/\mathbf{Z}$ on $\mathcal{X}^{\square}_{\Gamma,\mathbf{G}}$, which amounts to a simplicial presheaf

$$(\mathbf{PG}^\bullet \times \mathcal{X}^{\square}_{\Gamma,\mathbf{G}}) \colon (\ldots \Rrightarrow \mathbf{PG} \times \mathcal{X}^{\square}_{\Gamma,\mathbf{G}} \rightrightarrows \mathcal{X}^{\square}_{\Gamma,\mathbf{G}}).$$

Definition 3.4.5. [Zhu21, Def 2.2.14] The derived moduli stack of unframed $\mathbf{G}$-valued (strongly continuous) Γ-representations is the functor

$$\mathcal{X}_{\Gamma,\mathbf{G}} = |\mathbf{PG}^\bullet \times \mathcal{X}^{\square}_{\Gamma,\mathbf{G}}|$$

where $|\cdot|$ denotes geometric realisation in $\mathcal{F}un(\mathcal{A}lg_{\mathcal{O}}, \mathcal{A}ni)$.

Remark 3.4.6. Zhu [Zhu21] defines $\mathcal{X}_{\Gamma,\mathbf{G}}$ as the geometric realisation in the category of étale sheaves on $(\mathcal{A}lg_{\mathcal{O}})^{\mathrm{op}}$. We simplify the definition in order to prove the final statement in the following lemma. The difference between presheaves and sheaves is immaterial for our purposes since we are ultimately interested in only the local geometry of $\mathcal{X}_{\Gamma,\mathbf{G}}$.

Remark 3.4.7. Given $\bar{\rho} \in \mathcal{X}^{\square}_{\Gamma,\mathbf{G}}(k)$, the functor $(\mathcal{X}^{\square}_{\Gamma,\mathbf{G}})_{\bar{\rho}}$ is the same as the one defined by Galatius–Venkatesh in [GV18, Def. 5.4] (see [Zhu21, p.20]).

Lemma 3.4.8. *The functors* $\mathcal{X}^{\square}_{\Gamma,\mathbf{G}}$ *and* $\mathcal{X}_{\Gamma,\mathbf{G}}$ *of Definitions 3.4.2 and 3.4.5 commute with filtered colimits and finite limits. If* $A \in \mathcal{A}lg_{\mathcal{O}}$ *is perfect as an animated* $\mathcal{O}$*-module, then*

$$\mathcal{X}^{\square}_{\Gamma,\mathbf{G}}(\lim A/^{\mathbf{L}}\varpi^r) = \lim \mathcal{X}^{\square}_{\Gamma,\mathbf{G}}(A/^{\mathbf{L}}\varpi^r), \quad \mathcal{X}_{\Gamma,\mathbf{G}}(\lim A/^{\mathbf{L}}\varpi^r) = \lim \mathcal{X}_{\Gamma,\mathbf{G}}(A/^{\mathbf{L}}\varpi^r).$$

Proof. By definition,

$$\mathcal{X}^{\square}_{\Gamma,\mathbf{G}} = \mathrm{Hom}_{\mathcal{F}un(\Delta,\mathcal{A}lg_{\mathcal{O}})}(\mathcal{O}[\mathbf{G}^{\bullet}],-) \circ C_{\mathrm{sc}}(\Gamma,-).$$

The functor $C_{\mathrm{sc}}(\Gamma,-)$ preserves filtered colimits and finite limits by Lemma 3.3.1, and limits of the form $\lim A/^{\mathbf{L}}\varpi^r$ for A as in the statement by Lemma 3.3.1. The functor $\mathrm{Hom}_{\mathcal{F}un(\Delta,\mathcal{A}lg_{\mathcal{O}})}(\mathcal{O}[\mathbf{G}^{\bullet}],-)$ commutes with all limits and filtered colimits since the cosimplicial algebra $\mathcal{O}[\mathbf{G}^{\bullet}]$ is a compact object in $\mathcal{F}un(\Delta,\mathcal{A}lg_{\mathcal{O}})$ by Lemma 3.4.1. This completes the proof for the $\mathcal{X}^{\square}_{\Gamma,\mathbf{G}}$. Since geometric realisation of simplicial sets is a left Quillen equivalence, it commutes with all colimits and homotopy limits. Thus, the same holds for presheaves of simplicial sets and the statement for $\mathcal{X}_{\Gamma,\mathbf{G}}$ follows. □

3.5. **Derived deformation problems.** In this section, we discuss the notion of derived Galois deformation problems and T-framed deformations, defined in terms of the stacks introduced in the preceding section.

Definition 3.5.1. Let $\Gamma_{F,S} := \mathrm{Gal}(\overline{F}_S/F)$ with decomposition groups Γ_{F_v}, and $\mathbf{G} = \mathrm{GL}_n/\mathcal{O}$. Suppose $\bar\rho \in \mathcal{X}^{\square}_{\Gamma,\mathbf{G}}(k)$ with restriction $\bar\rho_v := \bar\rho|_{\Gamma_{F_v}} \in \mathcal{X}^{\square}_{\Gamma_{F_v},\mathbf{G}}$. We define functors $\mathcal{A}rt_{/k} \to \mathcal{A}ni$

$$\mathcal{D}^{\square}_{\bar\rho} = (\mathcal{X}^{\square}_{\Gamma_{F,S},\mathbf{G}})_{\bar\rho}, \quad \mathcal{D}^{\square}_{\bar\rho_v} = (\mathcal{X}^{\square}_{\Gamma_{F_v},\mathbf{G}})_{\bar\rho_v},$$

which we call the unrestricted derived deformation functors of $\bar\rho$ and $\bar\rho_v$, respectively.

Definition 3.5.2. A $(\mathrm{mod}\, p)$ derived deformation problem is a tuple

$$\mathbf{S} = (\bar\rho, S, \{\mathcal{D}_v\}_{v\in S})$$

where $\bar\rho\colon \Gamma \to \mathrm{GL}_n(k)$, S is a finite set of finite places of F and $\{\mathcal{D}_v\}_{v\in S}$ is a set of functors $\mathcal{D}_v\colon \mathcal{A}rt_{/k} \to \mathcal{A}ni$ equipped with actions of $\mathbf{PG}$ and equivariant maps $\mathcal{D}_v \to \mathcal{D}^{\square}_{\bar\rho_v}$.

Definition 3.5.3. Let $\mathbf{S} = (\bar\rho, S, \{\mathcal{D}_v\}_{v\in S})$ be a deformation problem. We define a functor $\mathcal{A}rt_{/k} \to \mathcal{A}ni$

$$\mathcal{D}^{\square}_{\mathbf{S}} = \mathcal{D}^{\square}_{\bar\rho} \times_{\prod_{v\in S}\mathcal{D}^{\square}_{\bar\rho_v}} \prod_{v\in S} \mathcal{D}_v$$

which we call the functor of derived framed deformations of type $\mathbf{S}$. Since the morphisms $\mathcal{D}_v \to \mathcal{D}^{\square}_{\bar\rho_v}$ are equivariant under conjugation, we may pass to the quotient as in Definition 3.4.5 to obtain the functor of derived deformations of type $\mathbf{S}$, denoted

$$\mathcal{D}_{\mathbf{S}} = |\mathcal{D}^{\square}_{\mathbf{S}} \times \mathbf{PG}^{\bullet}|.$$

We let $\hat{\mathbf{G}}\colon \mathcal{A}rt_{/k} \to \mathcal{A}ni$ be the functor $\mathbf{G}(B \to k) = \mathbf{G}(B) \times_{\mathbf{G}(k)} \{1\}$.

Definition 3.5.4. Let $\mathbf{S} = (\bar\rho, S, \{\mathcal{D}_v\}_{v\in S})$ be a deformation problem and suppose $T \subseteq S$ is a subset. We define the functor of T-framed liftings of type $\mathbf{S}$ as

$$\mathcal{D}^{\square,\square_T}_{\mathbf{S}} = \mathcal{D}^{\square}_{\mathbf{S}} \times \prod_{v\in T} \hat{\mathbf{G}}$$

There exists an action of $\hat{\mathbf{G}}$ on $\mathcal{D}^{\square,\square_T}_{\mathbf{S}}$ given by

$$\gamma\colon (\rho_A, \{\alpha_v\}_{v\in T}) \mapsto (\gamma\rho_A\gamma^{-1}, \{\gamma\alpha_v\}_{v\in T}).$$

We define the functor of derived T-framed deformations of type $\mathbf{S}$ as the geometric realisation (in $\mathcal{F}un(\mathcal{A}rt_{/k},\mathcal{A}ni)$)

$$\mathcal{D}^{T}_{\mathbf{S}} = |\mathcal{D}^{\square,\square_T}_{\mathbf{S}} \times \hat{\mathbf{G}}^{\bullet}|.$$

Proposition 3.5.5. *Let $\mathbf{S} = (\bar\rho, S, \{\mathcal{D}_v\}_{v\in S})$ be a deformation problem and $T \subset S$. Then the natural morphism $\mathcal{D}^{T}_{\mathbf{S}} \to \mathcal{D}_{\mathbf{S}}$ is formally smooth of relative dimension $n^2|T| - 1$.*

Example 3.5.6. Although Definition 3.5.3 is quite general, we will only consider local conditions arising from classical deformation rings. Let $v \in S_p$ and denote by $R^{\square}_{\bar\rho_v}$ the usual framed deformation ring of a representation $\bar\rho_v\colon \Gamma_{F_v} \to \mathrm{GL}_n(k)$. Then $\mathcal{D}^{\square}_{\bar\rho_v}$ is represented by $R^{\square}_{\bar\rho_v}$ by [BIPs23, Thm. 1.1] and [GV18, Lem. 7.5]. Any conjugation-invariant quotient $R^{\square}_{\bar\rho_v} \twoheadrightarrow R^{\square,D}_{\bar\rho_v}$ defines a local condition $\mathcal{D}_v$ via

$$\mathcal{D}_v \simeq \mathrm{Map}_{/k}(R^{\square,D}_{\bar\rho_v},-) \to \mathrm{Map}_{/k}(R^{\square}_{\bar\rho_v},-) \simeq \mathcal{D}^{\square}_{\bar\rho_v}.$$

At places not dividing p, we will have trivial local conditions $\mathcal{D}^{\square}_{\bar{\rho}_v}$ which are represented by the usual local deformation rings $R^{\square}_{\bar{\rho}_v}$.

Proposition 3.5.7. [All16, Prop. 1.2.2] *Suppose $v \nmid p$. Then*

$$\dim R^{\square}_{\bar{\rho}_v}[1/\varpi] = n^2.$$

At the places dividing p, our local conditions are provided by a variant of Kisin's [Kis08] potentially semistable deformation rings introduced in [Hev24, §5.1], defined by imposing conditions on Hodge–Tate weights and the action of inertia.

Definition 3.5.8. Let $v \mid p$. A Weil–Deligne inertial type of F_v (over E) is an isomorphism class of pairs $\tau = (\rho_\tau, N_\tau)$ where $\rho_\tau \colon I_{F_v} \to \mathrm{GL}_n(E)$ is a representation with open kernel and $N_\tau \in M_n(E)$ is a nilpotent matrix such that τ is isomorphic to the restriction $(r, N)|_{I_{F_v}}$ of a Weil–Deligne representation of Γ_{F_v}.

Hevesi defines a partial order $\preceq$ on Weil–Deligne inertial types, where $(\rho_\tau, N_\tau) \preceq (\rho_{\tau'}, N_{\tau'})$ if and only if $\rho_\tau \cong \rho_{\tau'}$ and, roughly, the nilpotence of N_τ is bounded from below by that of $N_{\tau'}$; see [Hev24, p.21] for the precise definition.

Theorem 3.5.9. *Let $v \mid p$. Fix a dominant weight $\lambda \in (\mathbb{Z}^n)^{\mathrm{Hom}(F,E)}$ and a Weil–Deligne inertial type τ of F_v. Suppose $\bar{\rho}_v \colon \Gamma_{F_v} \to \mathrm{GL}_n(k)$. Then we have the following:*

1. [Hev24, Thm. 5.2] *Then there exists a local condition $\mathcal{D}^{\lambda_v, \preceq\tau_v}_{\bar{\rho}_v} \to \mathcal{D}^{\square}_{\bar{\rho}_v} \simeq \mathrm{Map}_{/k}(R^{\square}_{\bar{\rho}_v}, -)$ represented by a static $\mathcal{O}$-flat reduced quotient $R^{\square}_{\bar{\rho}_v} \twoheadrightarrow R^{\lambda_v, \preceq\tau_v}_{\bar{\rho}_v}$ such that $R^{\square}_{\bar{\rho}_v} \to E$ factors through $R^{\lambda_v, \preceq\tau_v}_{\bar{\rho}_v}$ if and only if the corresponding representation is potentially semistable with labelled Hodge–Tate weights $(\lambda_{\iota,1} + n - 1 > \cdots > \lambda_{\iota,n})_{\iota \in \mathrm{Hom}(F,E)}$ and $\mathrm{WD}(\bar{\rho}_v)^{\mathrm{ss}}|_{I_{F_v}} \preceq \tau_v$.*
2. [Kis08, Thm. 3.3.8] $\dim R^{\lambda_v, \preceq\tau_v}_{\bar{\rho}_v}[1/\varpi] \leq n^2 + \frac{1}{2}n(n-1)[F_v : \mathbb{Q}_p]$
3. [All16, Prop. 1.3.12] *If ρ_v is a characteristic 0 lift of $\bar{\rho}_v$ such that the attached Weil–Deligne representation $\mathrm{WD}(\rho_v)$ is generic, the corresponding point of $\operatorname{Spec} R^{\lambda_v, \preceq\tau_v}_{\bar{\rho}_v}[1/\varpi]$ is smooth with tangent space canonically isomorphic to the local geometric Selmer group $H^1_g(\Gamma_{F_v}, \operatorname{ad}\rho_v)$.*

3.6. Tangent complexes and representability.

Definition 3.6.1. Let $\mathcal{X} \colon \mathcal{A}lg_{\mathcal{O}} \to \mathcal{A}ni$ be a functor and fix $x \in \mathcal{X}(A)$ for some static $A \in \mathcal{A}lg_{\mathcal{O}}$. The tangent complex of $\mathcal{X}$ at x is the unique A-module $\mathrm{T}^{\bullet}_A(\mathcal{X})$ such that for every connective A-module $M \in \mathcal{M}od_A$,

$$\mathrm{Map}_{/A}(\mathcal{R}, A \oplus M) \simeq \mathrm{Map}_{\mathcal{M}od(A)}\left(\mathrm{T}^{\bullet}_A(\mathcal{X}), M\right)$$

If $\mathcal{X}$ is represented by an algebra $\mathcal{R}$ and $x \colon \mathcal{R} \to A$, we write $\mathrm{T}^{\bullet}_A(\mathcal{R}) \coloneqq \mathrm{T}^{\bullet}_A(\mathcal{X})$.

Definition 3.6.2. Let $\mathcal{R} \in \mathcal{A}lg_{\mathcal{O}}$ and let $x \colon \mathcal{R} \to A$ be a morphism. The tangent complex of $\mathcal{R}$ at x is defined as

$$\mathrm{T}^{\bullet}_A(\mathcal{R}) = \mathrm{Map}_{\mathcal{M}od_{\mathcal{R}}}(\mathrm{L}_{\mathcal{R}/\mathcal{O}}, A) \cong \mathrm{Map}_{\mathcal{M}od\,A}(\mathrm{L}_{\mathcal{R}/\mathcal{O}} \otimes^{\mathbf{L}}_{\mathcal{R}} A, A)$$

where $\mathrm{L}_{\mathcal{R}/\mathcal{O}}$ is the algebraic cotangent complex of $\mathcal{R}$ over $\mathcal{O}$.

Definition 3.6.3. [Lur04, Def. 6.2.1] A functor $\mathcal{F} \colon \mathcal{A}rt_{/k} \to \mathcal{A}ni$ (resp. $\mathcal{F} \colon \mathcal{A}rt_{/E} \to \mathcal{A}ni$) is formally cohesive if:

1. $\mathcal{F}(k)$ is contractible (resp. $\mathcal{F}(E)$ is contractible).
2. If $B' \to B$ and $B'' \to B$ are surjective (on π_0) morphisms in $\mathcal{A}rt_{/k}$ (resp. $\mathcal{A}rt_{/E}$), then

$$\mathcal{F}(B' \times_B B) \simeq \mathcal{F}(B') \times_{\mathcal{F}(B)} \mathcal{F}(B'').$$

Lemma 3.6.4. *The derived deformation functors introduced in Section 3.5 are formally cohesive.*

Proof. Property (1) follows by definition, and (2) follows from commutation with finite limits. ☐

Proposition 3.6.5. *Let $(\mathcal{C}_N)$ be an inverse system in $\mathcal{A}lg_{\mathcal{O}}$ and $\mathcal{C}_\infty = \lim_N \mathcal{C}_N$. Then*

$$\mathrm{T}^{\bullet}_{\mathcal{O}_r}(\mathcal{C}_\infty) = \operatorname*{colim}_N \mathrm{T}^{\bullet}_{\mathcal{O}_r}(\mathcal{C}_N).$$

Proof. We have

$$\begin{aligned}\mathrm{T}^i_{\mathcal{O}_r}(\mathcal{C}_\infty) &= \pi_0\big(\mathrm{Map}_{/\mathcal{O}_r}(\mathcal{C}_\infty, \mathcal{O}_r \oplus \mathcal{O}_r[i])\big)\\ &\simeq \operatorname*{colim}_N \pi_0\big(\mathrm{Map}_{/\mathcal{O}_r}(\mathcal{C}_N, \mathcal{O}_r \oplus \mathcal{O}_r[i])\big)\\ &\simeq \operatorname*{colim}_N \mathrm{T}^i_{\mathcal{O}_r}(\mathcal{C}_N).\end{aligned}$$

□

Theorem 3.6.6. [Lur04, 6.2.14] *Let $\mathcal{F}\colon \mathcal{A}rt_{/k} \to \mathcal{A}ni$ be a formally cohesive functor. The following are equivalent:*

(1) *There exists an $\mathcal{R} \in \mathcal{CNL}_{/k}$ and an equivalence of functors*

$$\mathcal{F} \simeq \mathrm{Hom}_{(\mathcal{A}lg_{\mathcal{O}})_{/k}}(\mathcal{R}, -)$$

(2) *The k-vector spaces $\mathrm{T}^i_k(\mathcal{F})$ vanish for $i < 0$ and are finite dimensional for $i \geq 0$.*

The analogous statement holds for a formally cohesive functor $\mathcal{F}\colon \mathcal{A}rt_{/E} \to \mathcal{A}ni$.

Given $\mathcal{R} \in \mathcal{CNL}_{/k}$ with $\mathfrak{m} \subset \pi_0(\mathcal{R})$ its maximal ideal and an arbitrary $B \in \mathcal{A}rt_{/k}$, we have natural equivalences

$$\mathrm{Map}_{/k}(\mathcal{R}, B) \simeq \mathrm{Map}_{/k}(\lim_r \mathcal{R}/^{\mathbf{L}}\mathfrak{m}^r, B) \simeq \operatorname*{colim}_r \mathrm{Map}_{/k}(\mathcal{R}/^{\mathbf{L}}\mathfrak{m}^r, B).$$

The first isomorphism holds because $\mathcal{R}$ is derived $\mathfrak{m}$-adically complete and the second holds by Prop. 2.2.4 and the assumption on B. Since $\mathcal{R}$ is Noetherian, by [Lur04, Prop. 6.1.8] we have an isomorphism

$$\pi_0(\lim \mathcal{R}/^{\mathbf{L}}\mathfrak{m}^r) \cong \varprojlim \pi_0(\mathcal{R}).$$

The algebras $\mathcal{R}/^{\mathbf{L}}\mathfrak{m}^r$ might not lie in $\mathcal{A}rt_{/k}$ as they need not be truncated, but we can associate a pro-object $\{\tau_{\leq i}(\mathcal{R}/^{\mathbf{L}}/\mathfrak{m}^r)\} \in \mathrm{Pro}(\mathcal{A}rt_{/k})$. In this way, we can rephrase statement (1) of Theorem 3.6.6 in terms of pro-representability. This is the viewpoint chosen throughout [GV18], but we will use complete Noetherian local rings.

Proposition 3.6.7. *Let $\mathcal{R} \in \mathcal{CNL}_{/k}$ and let $\mathcal{R} \to \mathcal{O}_r$ be a morphism over k. Then for every $i \geq 0$, $\mathrm{T}^i_{\mathcal{O}_r}(\mathcal{R})$ is a finitely generated $\mathcal{O}_r$-module.*

Proof. The case $r = 1$ is Theorem 3.6.6. By considering the short exact sequence

$$0 \to \mathcal{O}_r \to \mathcal{O}_{r+1} \to k \to 0$$

we deduce the analogous statement for $\mathcal{O}_r$-coefficients using induction. □

Corollary 3.6.8. *Let $\mathcal{R} \in \mathcal{CNL}_{/k}$ and let $\mathcal{R} \to \mathcal{O}$ be a morphism over k. Then*

$$\mathrm{T}^i_{\mathcal{O}}(\mathcal{R}) \cong \varprojlim_r \mathrm{T}^i_{\mathcal{O}_r}(\mathcal{R})$$

Proof. By definition,

$$\mathrm{T}^\bullet_{\mathcal{O}}(\mathcal{R}) = \mathrm{Map}_{\mathrm{Mod}\,\mathcal{R}}(\mathrm{L}_{\mathcal{R}/\mathcal{O}}, \mathcal{O}) \simeq \lim_r \mathrm{Map}_{\mathrm{Mod}\,\mathcal{R}}(\mathrm{L}_{\mathcal{R}/\mathcal{O}}, \mathcal{O}_r).$$

By Prop. 3.6.7, the derived functors of $\varprojlim_r$ vanish and hence $\mathrm{T}^*_{\mathcal{O}}(\mathcal{R}) = \varprojlim_r \mathrm{T}^*_r(\mathcal{R})$. □

The tangent complexes of the moduli functors introduced above are given by Galois cohomology; see [Zhu21, §2.2] for details. For example, let A be a static $\mathcal{O}$-algebra and take $x \in \mathcal{X}_{\Gamma_{F,S},\mathbf{G}}(A)$ corresponding to an equivalence class of representations $\Gamma_{F,S} \to \mathrm{GL}_n(A)$. Regarding $\Gamma_{F,S}$ as a category with one object, we obtain a corresponding functor $N(\Gamma_{F,S}) \to \mathcal{M}od_A$, where N is the nerve; let

$$C^\bullet_{\mathrm{sc}}(\Gamma_{F,S}, \mathrm{ad}\, x) := \mathrm{colim}(N(\Gamma_{F,S}) \to \mathcal{M}od_A)$$

denote its colimit. Its homotopy $\pi_*(C^\bullet_{\mathrm{sc}}(\Gamma_{F,S}, \mathrm{ad}\, x))$ is given by the continuous cohomology $H^*(\Gamma_{F,S}, \mathrm{ad}\, x)$ of the profinite group $\Gamma_{F,S}$ in the usual sense.

Proposition 3.6.9. [Zhu21, Prop. 2.4.24] *Let $\bar\rho \in \mathcal{X}^{\square}_{\Gamma_{F,S},\mathbf{G}}(k)$. The tangent complex of $\mathcal{X}^{\square}_{\Gamma_{F,S},\mathbf{G}}$ at $\bar\rho$ is given by the cofibre of the natural map $C^\bullet_{\mathrm{sc}}(\Gamma_{F,S}, \operatorname{ad}\bar\rho) \to \operatorname{ad}\bar\rho$, denoted $\overline{C}^\bullet_{\mathrm{sc}}(\Gamma_{F,S}, \operatorname{ad}\bar\rho)[1]$. In particular,*

$$\mathrm{T}^i_k(\mathcal{X}^{\square}_{\Gamma_{F,S},\mathbf{G}}) \cong \pi_{-i}\big(\overline{C}^\bullet_{\mathrm{sc}}(\Gamma_{F,S}, \operatorname{ad}\bar\rho)[1]\big) \cong \begin{cases} Z^1(\Gamma_{F,S}, \operatorname{ad}\bar\rho) & \text{if } i = 0,\\ H^2(\Gamma_{F,S}, \operatorname{ad}\bar\rho) & \text{if } i = 1,\\ 0 & \text{otherwise.}\end{cases}$$

The tangent complex of $\mathcal{X}_{\Gamma_{F,S},\mathbf{G}}$ at the point corresponding to $\bar\rho$ is given by $C^\bullet_{\mathrm{sc}}(\Gamma_{F,S}, \operatorname{ad}\bar\rho)[1]$, i.e.

$$\mathrm{T}^i_k(\mathcal{X}_{\Gamma_{F,S},\mathbf{G}}) \cong \pi_{-i}\big(C^\bullet_{\mathrm{sc}}(\Gamma_{F,S}, \operatorname{ad}\bar\rho)[1]\big) \cong \begin{cases} H^0(\Gamma_{F,S}, \operatorname{ad}^0\bar\rho) & \text{if } i = -1,\\ H^1(\Gamma_{F,S}, \operatorname{ad}\bar\rho) & \text{if } i = 0,\\ H^2(\Gamma_{F,S}, \operatorname{ad}\bar\rho) & \text{if } i = 1,\\ 0 & \text{otherwise.}\end{cases}$$

In particular, if $\bar\rho$ is absolutely irreducible then

$$\mathrm{T}^i_k(\mathcal{X}^{\square}_{\Gamma,\mathbf{G}}) \cong H^{i+1}(\Gamma_{F,S}, \operatorname{ad}\bar\rho).$$

The analogous statements are true with $\bar\xi_r \in \mathcal{X}^{\square}_{\Gamma,\mathbf{G}}(\mathcal{O}_r)$ or $\rho \in \mathcal{X}^{\square}_{\Gamma,\mathbf{G}}(E)$ in place of $\bar\rho$.

Corollary 3.6.10. *Let $\Gamma = \operatorname{Gal}(F_S/F)$ or $\operatorname{Gal}(\overline{F}_v/F_v)$ and fix $\bar\rho \in \mathcal{X}^{\square}_{\Gamma,\mathbf{G}}(k)$. The framed derived deformation functor $\mathcal{D}^{\square}_{\bar\rho} = (\mathcal{X}^{\square}_{\Gamma,\mathbf{G}})_{\bar\rho}$ is representable and if $\bar\rho$ is Schur, the derived deformation functor $\mathcal{D}_{\bar\rho}$ is representable. The same statement is true for $\rho \in \mathcal{X}^{\square}_{\Gamma,\mathbf{G}}(E)$.*

Proof. This follows from Proposition 3.6.9 and Theorem 3.6.6. □

Lemma 3.6.11. *Suppose we have morphisms $A \to B \to \mathcal{O}$ in $\mathcal{A}lg_{\mathcal{O}}$ and that B is finitely presented over A. Then, if $\mathrm{T}^i_{\mathcal{O}}(A)$ is a finitely generated $\mathcal{O}$-module for every $i \geq 0$, the same holds for $\mathrm{T}^i_{\mathcal{O}}(B)$.*

Proof. By [Lur04, Prop. 3.2.14], the relative cotangent complex $\mathrm{L}_{B/A}$ is an almost perfect animated B-module, i.e. $\pi_i(\mathrm{L}_{B/A})$ is a finitely generated $\pi_0(B)$-module for every $i \geq 0$. It follows that $\mathrm{L}_{B/A} \otimes^{\mathbf{L}}_B \mathcal{O}$ is a perfect animated $\mathcal{O}$-module. Dualising the fibre sequence

$$\mathrm{L}_{A/\mathcal{O}} \otimes^{\mathbf{L}}_A \mathcal{O} \to \mathrm{L}_{B/\mathcal{O}} \otimes^{\mathbf{L}}_B \mathcal{O} \to \mathrm{L}_{B/A} \otimes^{\mathbf{L}}_B \mathcal{O} \xrightarrow{+1}$$

we obtain a fibre sequence of tangent complexes with $\mathcal{O}$-coefficients

$$\mathrm{T}^\bullet_{\mathcal{O}}(B/A) \to \mathrm{T}^\bullet_{\mathcal{O}}(B) \to \mathrm{T}^\bullet_{\mathcal{O}}(A) \xrightarrow{+1}.$$

By assumption, $\mathrm{T}^i_{\mathcal{O}}(A)$ is finitely generated over $\mathcal{O}$ for every i. The same now follows for $\mathrm{T}^\bullet_{\mathcal{O}}(B)$. □

3.7. **Localisation at a characteristic 0 point.** In this section, we prove two results which generalise results of Kisin [Kis09, Lem. 2.3.3, Prop. 2.3.5] relating the deformation ring of a characteristic 0 representation to that of its reduction mod ϖ.

In the following definition, we use the functor $\mathcal{I}nt B \to (\mathcal{A}lg_{\mathcal{O}})_{/\mathcal{O}}$ of Lemma 3.2.7 to view $A \in \mathcal{I}nt B$ as an object of $(\mathcal{A}lg_{\mathcal{O}})_{/\mathcal{O}}$.

Definition 3.7.1. Let $\mathcal{F}_{\bar\rho}\colon \mathcal{A}rt_{/k} \to \mathcal{A}ni$ be a functor and $\xi \in \mathcal{F}_{\bar\rho}(\mathcal{O}) = \lim \mathcal{F}_{\bar\rho}(\mathcal{O}_r)$. We define

$$\begin{aligned} \mathcal{F}_{\bar\rho,(\xi)}\colon \mathcal{A}rt_{/E} &\to \mathcal{A}ni \\ (B \to E) &\mapsto \operatorname*{colim}_{A \in \mathcal{I}nt B} \mathcal{F}(A) \times_{\mathcal{F}(\mathcal{O})} \{\xi\} \end{aligned}$$

Proposition 3.7.2. *Let $\mathcal{F}\colon \mathcal{A}lg_{\mathcal{O}} \to \mathcal{A}ni$ be a functor commuting with filtered colimits, finite limits and limits of the form $\lim A/^{\mathbf{L}}\varpi^r$ when A is perfect as an animated $\mathcal{O}$-module. Suppose $\bar\rho \in \mathcal{F}(\bar\rho)$, $\xi \in \mathcal{F}_{\bar\rho}(\mathcal{O})$ and let ρ be the image of ξ in $\mathcal{F}(E)$. Then there is a natural weak equivalence*

$$\mathcal{F}_{\bar\rho,(\xi)} \simeq \mathcal{F}_{\rho}.$$

Proof. Let $(B \to E) \in \mathcal{A}rt_{/E}$. From the assumptions and the fact that filtered colimits commute with finite limits, we see that

$$\begin{aligned}\mathcal{F}_{\bar\rho,(\xi)}(B) &= \operatorname*{colim}_{A\in\mathcal{I}nt B}(\mathcal{F}_{\bar\rho}(A)\times_{\mathcal{F}_{\bar\rho}(\mathcal{O})}\{\xi\})\\ &\simeq \operatorname*{colim}_{A\in\mathcal{I}nt B}(\mathcal{F}(A)\times_{\mathcal{F}(\mathcal{O})}\{\xi\})\\ &\simeq (\mathcal{F}(\operatorname*{colim}_{A\in\mathcal{I}nt B} A)\times_{\mathcal{F}(\mathcal{O})}\{\xi\})\\ &\simeq \mathcal{F}(B\times_E\mathcal{O})\times_{\mathcal{F}(\mathcal{O})}\{\xi\}\\ &\simeq \mathcal{F}(B)\times_{\mathcal{F}(E)}\{\rho\}.\end{aligned}$$

□

Corollary 3.7.3. *Let $\mathcal{X}^{\square}_{\Gamma,\mathbf{G}}$ and $\mathcal{X}_{\Gamma,\mathbf{G}}$ be the derived moduli stacks introduced in Definitions 3.4.2 and 3.4.5. Fix $\bar\rho\in\mathcal{X}^{\square}_{\Gamma,\mathbf{G}}(k)$, $\xi\in(\mathcal{X}^{\square}_{\Gamma,\mathbf{G}})_{\bar\rho}(\mathcal{O})$ and $\rho=\xi[1/\varpi]\in\mathcal{X}^{\square}_{\Gamma,\mathbf{G}}(E)$. Then there are natural weak equivalences of functors*

$$(\mathcal{X}^{\square}_{\Gamma,\mathbf{G}})_{\bar\rho,(\xi)}\simeq(\mathcal{X}^{\square}_{\Gamma,\mathbf{G}})_{\rho}\ \text{and}\ (\mathcal{X}_{\Gamma,\mathbf{G}})_{\bar\rho,(\xi)}\simeq(\mathcal{X}_{\Gamma,\mathbf{G}})_{\rho}.$$

Proof. This follows from Proposition 3.7.2, Lemma 3.2.6 and Lemma 3.4.8. □

Theorem 3.7.4. *Let $\mathcal{F}_{\bar\rho}\colon\mathcal{A}rt_{/k}\to\mathcal{A}ni$ be a formally cohesive functor and $\xi\in\mathcal{X}_{\bar\rho}(\mathcal{O})$. Suppose $\mathcal{F}_{\bar\rho}$ commutes with filtered colimits, finite limits and limits of the form $\lim_r A/^{\mathbf{L}}\varpi^r$ where A is perfect as an animated $\mathcal{O}$-module. Then, if $\mathcal{F}_{\bar\rho}$ is represented by $\mathcal{R}$, $\mathcal{X}_{\bar\rho,(\xi)}$ is represented by*

$$(\mathcal{R}\otimes_{\mathcal{O}}E)^{\wedge}_{\xi}\coloneqq\lim_r(\mathcal{R}\otimes_{\mathcal{O}}E)/^{\mathbf{L}}I_{\xi}^r,$$

the derived completion of $\mathcal{R}\otimes_{\mathcal{O}}E$ at the maximal ideal of $\pi_0(\mathcal{R}\otimes_{\mathcal{O}}E)$ defined by $\xi\colon\mathcal{R}\to\mathcal{O}$.

Proof. Let $(B\to E)\in\mathcal{A}rt_{/E}$ and let $\xi_r\in\mathcal{X}_{\bar\rho}(\mathcal{O}_r)$ denote the mod ϖ^r reduction of ξ. By definition,

$$\mathcal{X}_{\bar\rho,(\xi)}(B)=\operatorname*{colim}_{A\in\mathcal{I}nt B}(\mathcal{X}_{\bar\rho}(A)\times_{\mathcal{X}_{\bar\rho}(\mathcal{O})}\{\xi\}).$$

Now, by Lemmas 3.2.6 and 3.4.8,

$$\begin{aligned}\mathcal{X}_{\bar\rho}(A)\times_{\mathcal{X}_{\bar\rho}(\mathcal{O})}\{\xi\}&\simeq\lim_r\big(\mathcal{X}_{\bar\rho}(A/^{\mathbf{L}}\varpi^r)\times_{\mathcal{X}_{\bar\rho}(\mathcal{O}/^{\mathbf{L}}\varpi^r)}\{\xi_r\}\big)\\ &\simeq\lim_r\big(\operatorname{Map}_{/k}(\mathcal{R}_{\bar\rho},A/^{\mathbf{L}}\varpi^r)\times_{\operatorname{Map}_{/k}(\mathcal{R}_{\bar\rho},\mathcal{O}/^{\mathbf{L}}\varpi^r)}\{\xi_r\}\big)\\ &\simeq\operatorname{Map}_{/k}(\mathcal{R}_{\bar\rho},A)\times_{\operatorname{Map}_{/k}(\mathcal{R}_{\bar\rho},\mathcal{O})}\{\xi\}\\ &\simeq\operatorname{Map}_{/\mathcal{O}}(\mathcal{R}_{\bar\rho},A)\end{aligned}$$

Thus, we have

$$\mathcal{X}_{\bar\rho,(\xi)}(B)=\operatorname*{colim}_{A\in\mathcal{I}nt B}\operatorname{Map}_{/\mathcal{O}}(\mathcal{R}_{\bar\rho},A).$$

Now, Proposition 2.2.4 and the injectivity of $\pi_0(A)\to\pi_0(B)$ for any $A\in\mathcal{I}nt B$ together imply that

$$\operatorname{Map}_{/\mathcal{O}}(\mathcal{R}_{\bar\rho},A)\simeq\operatorname*{colim}_r\operatorname{Map}_{/\mathcal{O}}(\mathcal{R}_{\bar\rho}/^{\mathbf{L}}(I_{\xi}^{\circ})^r,A)$$

where $I_{\xi}^{\circ}\subset\pi_0(\mathcal{R}_{\bar\rho})$ is the inverse image of $I_{\xi}=\ker(\pi_0(\mathcal{R}_{\bar\rho})\otimes_{\mathcal{O}}E\to E)$. Since every $A\in\mathcal{I}nt B$ is n-truncated for n sufficiently big and depending only on B, and $\mathcal{R}_{\bar\rho}/^{\mathbf{L}}(I_{\xi}^{\circ})^r$ is almost of finite presentation in $\mathcal{A}lg_{\mathcal{O}}$ we therefore have (using Lemma 3.2.6)

$$\begin{aligned}\operatorname*{colim}_{A\in\mathcal{I}nt B}\operatorname{Map}_{/\mathcal{O}}(\mathcal{R}_{\bar\rho},A)&\simeq\operatorname*{colim}_r\operatorname{Map}_{/\mathcal{O}}(\mathcal{R}_{\bar\rho}/^{\mathbf{L}}(I_{\xi}^{\circ})^r,\operatorname*{colim}_{A\in\mathcal{I}nt B}A)\\ &\simeq\operatorname*{colim}_r\operatorname{Map}_{/\mathcal{O}}(\mathcal{R}_{\bar\rho}/^{\mathbf{L}}(I_{\xi}^{\circ})^r,B\times_E\mathcal{O})\\ &\simeq\operatorname*{colim}_r\operatorname{Map}_{/E}((\mathcal{R}_{\bar\rho}\otimes_{\mathcal{O}}E)/^{\mathbf{L}}I_{\xi}^r,B).\end{aligned}$$

Thus,

$$\mathcal{X}_{\bar\rho,(\xi)}(B\to E)\simeq\operatorname{Map}_{/E}\big((\mathcal{R}\otimes_{\mathcal{O}}E)^{\wedge}_{\xi},B\big)$$

as required. □

Corollary 3.7.5. *Let $\mathcal{D}_{\mathbf{S}}$ be the deformation functor of the deformation problem*

$$\mathbf{S} = (\bar{\rho}, S, \{\mathcal{D}_{\bar{\rho}_v}^{\lambda_v, \preceq\tau_v}\}_{v\in S_p} \cup \{\mathcal{D}_{\bar{\rho}_v}^{\square}\}_{v\in S^p})$$

where $\bar{\rho}$ is absolutely irreducible, and suppose $\xi \in \mathcal{D}_{\mathbf{S}}(\mathcal{O})$ is a point corresponding to a representation $\rho\colon \Gamma_{F,S} \to \mathrm{GL}_n(E)$ such that the Weil–Deligne representations $\mathrm{WD}(\rho|_{\Gamma_{F_v}})$ are generic for every $v \in S$. Then we have a weak equivalence

$$(\mathcal{D}_{\mathbf{S}})_{(\xi)} \simeq \mathcal{D}_{S,\rho}^{\mathrm{ss}},$$

where $\mathcal{D}_{S,\rho}^{\mathrm{ss}}$ is the deformation functor of the characteristic 0 deformation problem

$$(\rho, S, \{\mathcal{D}_{\rho_v}^{\mathrm{ss}}\}_{v\in S_p} \cup \{\mathcal{D}_{\rho_v}^{\square}\}_{v\in S^p}).$$

In particular, the tangent space of $(\mathcal{D}_{\mathbf{S}})_{(\xi)}$ is given by the geometric Selmer group $H^1_g(\Gamma_{F,S}, \mathrm{ad}\,\rho)$.

Proof. Since filtered colimits commute with finite limits and geometric realisation of simplicial sets is a Quillen equivalence, we have an equivalence in $\mathcal{F}un(\mathcal{A}rt_{/E}, \mathcal{A}ni)$

$$(\mathcal{D}_{\mathbf{S}})_{(\xi)} \simeq |(\mathcal{D}_{\mathbf{S}}^{\square})_{(\xi)} \times \mathbf{PG}^{\bullet}|.$$

Let $(B \to E) \in \mathcal{A}rt_{/E}$ and write $\xi_v := \xi|_{\Gamma_v}$. Then

$$(\mathcal{D}_{\mathbf{S}}^{\square})_{(\xi)} = (\mathcal{D}_{\bar{\rho}}^{\square})_{(\xi)} \times_{\prod_{v\in S}(\mathcal{D}_{\bar{\rho}_v}^{\square})_{(\xi_v)}} \prod_{v\in S} (\mathcal{D}_v)_{(\xi_v)},$$

and hence $(\mathcal{D}_{\mathbf{S}}^{\square})_{(\xi)}$ is the deformation functor corresponding to the characteristic 0 deformation problem

$$(\rho, S, \{(\mathcal{D}_v^{ss,r})_{(\xi_v)}\}_{v\in S_p} \cup \{\mathcal{D}_{\rho_v}^{\square}\}_{v\in S^p}).$$

By Theorem 3.5.9, Proposition 3.7.2 and the smoothness of the points $\xi_v \in \operatorname{Spec} R_{\bar{\rho}_v}^{\lambda_v, \preceq\tau_v}$, the local conditions $(\mathcal{D}_{\bar{\rho}_v}^{\lambda_v, \preceq\tau_v})_{(\xi_v)}$ for $v \in S_p$ are represented by the characteristic 0 potentially semistable fixed-weight local deformation rings $R_{\rho_v}^{\mathrm{ss},\mathrm{HT}(\rho)}$. This completes the proof. □

Corollary 3.7.6. *Let $\mathcal{X}_{\Gamma,\mathbf{G}}^{\square}$ be the derived moduli stack introduced in Definition 3.4.2. Suppose $\bar{\rho} \in \mathcal{X}_{\Gamma,\mathbf{G}}^{\square}(k)$ and $\xi \in (\mathcal{X}_{\Gamma,\mathbf{G}}^{\square})_{\bar{\rho}}(\mathcal{O})$ with $\rho = \xi[1/\varpi] \in \mathcal{X}_{\Gamma,\mathbf{G}}^{\square}(E)$. Then*

$$(\mathcal{X}_{\Gamma,\mathbf{G}}^{\square})_{\bar{\rho},(\xi)} \simeq (\mathcal{X}_{\Gamma,\mathbf{G}}^{\square})_{\rho},$$

and similarly for the unframed functor $\mathcal{X}_{\Gamma,\mathbf{G}}$ of Definition 3.4.5.

Proposition 3.7.7. *Let $F\colon \mathcal{A}rt_{/k} \to \mathcal{A}ni$ be a formally cohesive functor and let $\xi \in F(\mathcal{O})$. Then $F_{(\xi)}\colon \mathcal{A}rt_{/E} \to \mathcal{A}ni$ has tangent complex*

$$\big(\lim_r \mathrm{T}^{\bullet}_{\mathcal{O}_r}(F_{(\xi)})\big) \otimes_{\mathcal{O}} E$$

In particular, if $F = \mathrm{Map}_{/k}(\mathcal{R}, -)$ for $\mathcal{R} \in \mathcal{CNL}_{/k}$ then, with notation as in Proposition 3.7.4,

$$\mathrm{T}^{\bullet}_E\big((\mathcal{R} \otimes_{\mathcal{O}} E)^{\wedge}_{\xi}\big) = \big(\lim_r \mathrm{T}^{\bullet}_{\mathcal{O}_r}(\mathcal{R})\big) \otimes_{\mathcal{O}} E$$

Proof. Let $L = L_{F_{(\xi)}}$ denote the cotangent complex of $F_{(\xi)}$ over $\mathcal{O}$. Then

$$\begin{aligned}
T^{\bullet}_E(F_{(\xi)}) &= \mathrm{Map}_E(L_{F_{(\xi)}} \otimes^{\mathbf{L}} E, E) \\
&= \mathrm{Map}_{\mathcal{O}}(L_{F_{(\xi)}} \otimes^{\mathbf{L}} E, \mathcal{O}) \otimes_{\mathcal{O}} E \\
&= \mathrm{Map}(L_{F_{(\xi)}}, \mathcal{O}) \otimes_{\mathcal{O}} E \\
&= \big(\lim_r \mathrm{Map}(L_{F_{(\xi)}}, \mathcal{O}_r)\big) \otimes_{\mathcal{O}} E \\
&= (\lim_r \mathrm{T}^{\bullet}_{\mathcal{O}_r}(F_{(\xi)})) \otimes_{\mathcal{O}} E.
\end{aligned}$$

□

3.8. **A computation in Galois cohomology.** In Proposition 4.3.1 below, a statement about a certain tangent complex will boil down to a statement about Bloch–Kato Selmer groups, recorded here. We use lowercase letters to denote the dimension of cohomology groups or subspaces thereof, e.g.

$$h^1_g(\Gamma_{F,S}, \operatorname{ad}\rho) := \dim_E H^1_g(\Gamma_{F,S}, \operatorname{ad}\rho).$$

Proposition 3.8.1. *Let $\rho\colon \Gamma_{F,S} \to \mathrm{GL}_n(E)$ be a continuous representation, such that:*

1. *ρ is irreducible.*
2. *For every $v \in S \smallsetminus S_p$, the Weil-Deligne representation $\mathrm{WD}(\rho_v)$ is generic.*
3. *For every $v \in S_p$, ρ_v is potentially semistable with distinct Hodge-Tate weights.*

Then

$$h^1_g(\Gamma_{F,S}, \operatorname{ad}\rho) = h^1_g(\Gamma_{F,S}, \operatorname{ad}\rho(1)) - l_0.$$

Proof. By the Greenberg–Wiles formula and assumptions (1) and (2),

$$\begin{aligned} &h^1_g(\Gamma_{F,S}, \operatorname{ad}\rho) - h^1_g(\Gamma_{F,S}, \operatorname{ad}\rho(1)) = \\ &\quad \sum_{v\in S}\left(h^1_g(\Gamma_v, \operatorname{ad}\rho) - h^0(\Gamma_v, \operatorname{ad}\rho)\right) + h^0(\Gamma, \operatorname{ad}\rho) - h^0(\Gamma, \operatorname{ad}\rho(1)) - \sum_{v\mid\infty} h^0(\Gamma_v, \operatorname{ad}\rho) \\ &\quad = \sum_{v\in S}\left(h^1_g(\Gamma_v, \operatorname{ad}\rho) - h^0(\Gamma_v, \operatorname{ad}\rho)\right) - \sum_{v\mid\infty} h^0(\Gamma_v, \operatorname{ad}\rho) \end{aligned}$$

Assumptions (1) and (2) imply the vanishing of the global terms and terms in the first sum corresponding to $v \in S \smallsetminus S_p$, respectively. Hence, we have

$$h^1_g(\Gamma_{F,S}, \operatorname{ad}\rho) - h^1_g(\Gamma_{F,S}, \operatorname{ad}\rho(1)) = \sum_{v\in S_p}\left(h^1_g(\Gamma_v, \operatorname{ad}\rho) - h^0(\Gamma_v, \operatorname{ad}\rho)\right) - \sum_{v\mid\infty} h^0(\Gamma_v, \operatorname{ad}\rho).$$

Now, the difference $h^1_g(\Gamma_v, \operatorname{ad}\rho) - h^0(\Gamma_v, \operatorname{ad}\rho)$ equals the number of negative Hodge-Tate weights of $\operatorname{ad}\rho|_{\Gamma_v}$, and by assumption (3) this number is $n(n+1)/2$ for every $v \in S_p$. The result now follows from the formula for l_0 (see §2.2). □

4. Patching and Venkatesh's conjecture

In this section, we prove the following result.

Theorem 4.0.1. *Let Π be a regular algebraic cuspidal automorphic representation of $\mathrm{GL}_n(\mathbb{A}_F)$, cohomological with respect to an algebraic representation $\mathcal{V}_\lambda$ where $\lambda \in (\mathbb{Z}^n)^{\mathrm{Hom}(F,E)}$, and let $\rho = \rho_{\Pi,\iota}\colon \Gamma_F \to \mathrm{GL}_n(E)$ denote the Galois representation attached to (Π,ι) where $\iota\colon \overline{\mathbb{Q}}_p \xrightarrow{\sim} \mathbb{C}$ and $E/\mathbb{Q}_p$ is a finite extension containing all embeddings $F \to \overline{\mathbb{Q}}_p$. Suppose the following:*

1. *$p \nmid 2n$.*
2. *The reduction $\bar\rho$ is absolutely irreducible and decomposed generic* [ACC+23, Def 4.3.1].
3. *$\zeta_p \notin F$ and $\bar\rho|_{\Gamma_{F(\zeta_p)}}$ has enormous image* [ACC+23, Def 6.2.28].
4. *For every $v \in S^p$, the Weil-Deligne representation $\mathrm{WD}(\rho|_{\Gamma_{F_v}})$ is generic.*
5. *For every $v \in S_p$, $\rho|_{\Gamma_{F_v}}$ is de Rham and the Weil-Deligne representation $\mathrm{WD}(\rho|_{\Gamma_{F_v}})$ is generic.*
6. *F contains an imaginary quadratic subfield in which p splits.*
7. *For every pair $\bar v \neq \bar v'$ of places of F^+ above p, we have (with S_p^+ the places of F^+ above p)*

$$\sum_{\bar w \neq \bar v, \bar v' \in S_p^+} [F^+_{\bar w} : \mathbb{Q}_p] > \frac{1}{2}[F^+ : \mathbb{Q}].$$

8. *Π is tempered at all finite places.*

Let $\mathcal{R}^{\mathrm{ss}}_\rho$ denote the derived deformation ring representing the characteristic 0 deformation problem $(\rho, S, \{\mathcal{D}^{\lambda_v}_{\rho_v}\}_{v\in S_p} \cup \{\mathcal{D}^{\square}_{\rho_v}\}_{v\in S^p})$. Then

$$\pi_*(\mathcal{R}^{\mathrm{ss}}_\rho) \cong \bigwedge^{*} H^1_g(\Gamma_{F,S}, \operatorname{ad}\rho(1))^\vee$$

and there is an action of $\mathcal{R}_\rho^{\mathrm{ss}}$ on $R\Gamma(K,\mathcal{V}_\lambda)_{\mathfrak{p}}$ inducing a free action

$$\pi_*(\mathcal{R}_\rho^{\mathrm{ss}}) \circlearrowright H^*(X_K,\mathcal{V}_\lambda)_{\mathfrak{p}}.$$

Remark 4.0.2. A priori, the action of $\mathcal{R}_\rho^{\mathrm{ss}}$ depends on non-canonical choices made in the Taylor–Wiles argument below. With stronger local assumptions on $\bar\rho$, the naturality is proven in [GV18, §15] by identifying the characteristic p derived deformation ring $\mathcal{R}_{\mathbf{S}}$ with the derived Hecke algebra. It would be interesting to prove the naturality in characteristic 0 without additional assumptions.

Remark 4.0.3. Conditions (1)-(7) of the theorem are required to apply the main results of [A'C24] and [Hev24]. We require the purity assumption (8) to prove Proposition 4.1.2 below; the Ramanujan conjecture predicts that (8) always holds and the case $n=2$ is known [BCG$^+$25].

From now on, we fix Π and ρ as in the theorem, and an $\mathcal{O}$-lattice $\xi\colon \Gamma_{F,S}\to \mathrm{GL}_n(\mathcal{O})$ of ρ with reductions $\xi_r\colon \Gamma_{F,S}\to\mathrm{GL}_n(\mathcal{O}_r)$. The main result of [Hev24] then implies that

$$\mathbf{S} := (\bar\rho, S, \{\mathcal{D}_{\bar\rho_v}^{\lambda_v,\preceq\tau_v}\}_{v\in S_p}\cup\{\mathcal{D}_{\bar\rho_v}^{\square}\}_{v\in S^p})$$

is a well-defined deformation problem.

4.1. **Taylor–Wiles primes.**

The basis of our patching argument later on is a delicate choice of Taylor–Wiles places, detailed in this subsection.

Definition 4.1.1. Let $\mathbf{S}=(\bar\rho,S,\{\mathcal{D}_v\}_{v\in S})$ be a deformation problem. A Taylor–Wiles datum of level N for $\mathbf{S}$ is a tuple $(Q_N,\{(\alpha_{v,1},\dots,\alpha_{v,n})\}_{v\in Q_N})$ where:

(1) Q_N is a finite set of places of F, disjoint from S and such that for every $v\in Q_N$, $v\equiv 1 \bmod p^N$.
(2) For every $v\in Q_N$, $(\alpha_{v,1},\dots,\alpha_{v,n})\in k^n$ are the eigenvalues of $\bar\rho(\mathrm{Frob}_v)$, assumed to be pairwise distinct and k-rational.

Our main result Theorem 4.0.1 contains the assumption (3) that $\bar\rho\mid_{\Gamma_{F(\zeta_{p^\infty})}}$ has enormous image. In [NT23, Def. 2.23], a notion of 'enormous image' for characteristic 0 representations is defined. It is weaker in the sense that a characteristic 0 representation ρ has enormous image if its reduction $\bar\rho$ has enormous image.

Proposition 4.1.2. *Let $\mathbf{S}'=(\bar\rho,S,\{\mathcal{D}_v^{\square}\}_{v\in S})$ be the deformation problem with no conditions at $v\in S$, and suppose that*

- $p\nmid 2n$
- $\zeta_p\notin F$ *and* $\rho|_{\Gamma_{F(\zeta_p)}}$ *is enormous* [NT23, Def. 2.23].
- Π *is tempered at all finite places.*

Then there exists $q\geq 0$ and $l\geq 0$ such that for any $N\geq 1$, there exists a Taylor–Wiles datum $(Q_N,\{(\alpha_{v,1},\dots,\alpha_{v,n})\}_{v\in Q_N})$ for $\mathbf{S}'$ such that

(1) $|Q_N|=q$
(2) *For every $v\in Q_N$, $v\equiv 1 \bmod p^N$ and the rational prime below v splits in F_0.*
(3) *There is a local $\mathcal{O}$-algebra surjection $R_{\mathbf{S}}^{S,\mathrm{loc}}[[X_1,\dots,X_g]]\twoheadrightarrow R^S_{\mathbf{S}_{Q_N}}$.*
(4) *For every $r\leq N$,*

$$\mathrm{length}_{\mathcal{O}} H^1(\Gamma_{SQ_N},\mathrm{ad}\,\xi_r(1))\leq l,$$

hence if $\mathcal{L}_S=\{\mathcal{L}_{v,r}\}_{v\in S}$ is a Selmer system for $\mathrm{ad}\,\xi_r$ and $\mathcal{L}_{SQ_N}=\mathcal{L}_S\cup\{H^1(\Gamma_v,\mathrm{ad}\,\xi_r)\}_{v\in Q_N}$,

$$\mathrm{length}_{\mathcal{O}} H^1_{\mathcal{L}^{\perp}_{SQ_N}}(\Gamma_{SQ_N},\mathrm{ad}\,\xi_r(1))\leq l.$$

Proof. By [A'C24, Prop. 6.2.32], for q large enough we can find, for every $N\geq 1$, a Taylor–Wiles datum $(Q_N,\{(\alpha_{v,1},\dots,\alpha_{v,n})\}_{v\in Q_N})$ such that (1)-(3) hold. To achieve (4), we will enlarge the sets Q_N by adding suitable Taylor–Wiles primes, following the proof of [NT23, Lem. 2.30]. Thus, in (1) we replace if necessary q by a larger number. The splitting condition in (2) defines a set of Dirichlet density 1, so we may restrict our attention to places satisfying the condition. Enlarging Q_N does not affect (3), as the statement is equivalent to the vanishing of

$$H^1(\Gamma_{SQ_N}, \operatorname{ad}\bar\rho(1)) = \ker\left(H^1(\Gamma_S, \operatorname{ad}\bar\rho(1)) \to \prod_{v\in Q_N} H^1(\Gamma_v, \operatorname{ad}\bar\rho(1))\right)$$

and adding more factors to the product over v preserves the vanishing of this kernel.

The second statement in (4) follows from the first, since we have an inclusion

$$H^1_{\mathcal{L}^\perp_{SQ_N}}(\Gamma_{SQ_N}, \operatorname{ad}\xi_r(1)) \hookrightarrow H^1(\Gamma_{SQ_N}, \operatorname{ad}\xi_r(1)).$$

Let

$$\operatorname{ad}_{E/\mathcal{O}} = \operatorname{ad}\xi \otimes_{\mathcal{O}} E/\mathcal{O}$$

and similarly for $\operatorname{ad}_{\mathcal{O}}, \operatorname{ad}_E, \operatorname{ad}_{\mathcal{O}_r}$ so that $\operatorname{ad}_{E/\mathcal{O}} = \operatorname{colim}\operatorname{ad}_{\mathcal{O}_r}$. Fix an integer

$$s \geq \operatorname{corank}_{\mathcal{O}} H^1(\Gamma_{F,S}, \operatorname{ad}_{E/\mathcal{O}}(1)) \coloneqq \operatorname{rank}_{\mathcal{O}} \operatorname{Hom}_{\mathcal{O}}\left(H^1(\Gamma_{F,S}, \operatorname{ad}_{E/\mathcal{O}}(1)), E/\mathcal{O}\right).$$

Our goal is to find Frobenius elements $\sigma_1, \dots, \sigma_s \in G_{F(\zeta_{p^\infty})}$ such that:

(i) $\rho(\sigma_i)$ has n distinct eigenvalues in E.

(ii) The kernel of the map

$$H^1(\Gamma_S, \operatorname{ad}_{E/\mathcal{O}}(1)) \to \bigoplus_{i=1}^s H^1(\langle\sigma_i\rangle, \operatorname{ad}_{E/\mathcal{O}}(1)) \cong \bigoplus_{i=1}^s \operatorname{ad}_{E/\mathcal{O}}/(\sigma_i - 1)\operatorname{ad}_{E/\mathcal{O}}$$

is of finite length as an $\mathcal{O}$-module.

Then, for every $r \leq N$, we have a diagram

$$\begin{tikzcd}
0 \arrow[r] & H^0(\Gamma_S, \operatorname{ad}_{E/\mathcal{O}}(1))/\varpi^r \arrow[r] \arrow[d] & H^1(\Gamma_S, \operatorname{ad}_{\mathcal{O}_r}(1)) \arrow[r] \arrow[d] & H^1(\Gamma_S, \operatorname{ad}_{E/\mathcal{O}}(1))[\varpi^r] \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \oplus_{i=1}^s H^0(\langle\sigma_i\rangle, \operatorname{ad}_{E/\mathcal{O}}(1))/\varpi^r \arrow[r] & \oplus_{i=1}^s H^1(\langle\sigma_i\rangle, \operatorname{ad}_{\mathcal{O}_r}(1)) \arrow[r] & \oplus_{i=1}^s H^0(\langle\sigma_i\rangle, \operatorname{ad}_{E/\mathcal{O}}(1))[\varpi^r] \arrow[r] & 0
\end{tikzcd}$$

where the rows are exact. Note that $H^0(\Gamma_{F,S}, \operatorname{ad}_{E/\mathcal{O}}(1))$ has finite length by irreducibility of ρ. It follows from the snake lemma that the kernel of

$$H^1(\Gamma_{F,S}, \operatorname{ad}_{\mathcal{O}_r}(1)) \to \bigoplus_{i=1}^s H^1(\langle\sigma_i\rangle, \operatorname{ad}_{\mathcal{O}_r}(1))$$

has (finite) length bounded independently of r and N. By the Chebotarev density theorem, we can then find places $v_1, \dots v_s$ with Frobenius elements $\operatorname{Frob}_{v_1}, \dots, \operatorname{Frob}_{v_s}$ approximating $\sigma_1, \dots, \sigma_s$, and thus we obtain statement (4) of the proposition. Now, we have an isomorphism

$$H^1(\Gamma_{F,S}, \operatorname{ad}_{E/\mathcal{O}}(1)) \cong (E/\mathcal{O})^{s_0} \oplus M$$

where M is a finite length $\mathcal{O}$-module and $s_0 \leq s$ is the corank defined above. Thus, the problem reduces to finding, for every non-zero

$$f\colon E/\mathcal{O} \to H^1(\Gamma_{F,S}, \operatorname{ad}_{E/\mathcal{O}}(1))$$

an element $\sigma \in G_{F(\zeta_{p^\infty})}$ such that $\rho(\sigma)$ has n distinct eigenvalues and the composition

$$E/\mathcal{O} \to H^1(\Gamma_{F,S}, \operatorname{ad}_{E/\mathcal{O}}(1)) \to H^1(\langle\sigma_i\rangle, \operatorname{ad}_{E/\mathcal{O}}(1)) \cong \frac{\operatorname{ad}_{E/\mathcal{O}}(1)}{(\sigma-1)\operatorname{ad}_{E/\mathcal{O}}(1)}$$

is non-zero. Let $F_\infty = F(\zeta_{p^\infty})$ and $L_\infty = \overline{F}^{\ker \operatorname{ad}_E(1)}$. Then $F_\infty \subseteq L_\infty$ since $E(1) \subset \operatorname{ad}_E(1)$. The short exact sequencne

$$0 \to \operatorname{ad}_{\mathcal{O}} \to \operatorname{ad}_E \to \operatorname{ad}_{E/\mathcal{O}} \to 0$$

induces an exact sequence (with $\Gamma(L_\infty/F) = \operatorname{Gal}(L_\infty/F)$)

$$H^1\big(\Gamma(L_\infty/F), \operatorname{ad}_E(1)\big) \to H^1\big(\Gamma(L_\infty/F), \operatorname{ad}_{E/\mathcal{O}}(1)\big) \to H^2\big(\Gamma(L_\infty/F), \operatorname{ad}_{\mathcal{O}}(1)\big).$$

The first term vanishes by [Kis04, Lem. 6.2] and the assumption that Π is tempered at all finite places. The third term is a finitely generated $\mathcal{O}$-module. It follows that the middle term is annihilated by a power of p. Thus, the inflation-restriction sequence

$$0 \to H^1\big(\Gamma(L_\infty/F), \mathrm{ad}_{E/\mathcal{O}}(1)\big) \to H^1\big(\Gamma_{F,S}, \mathrm{ad}_{E/\mathcal{O}}(1)\big) \to H^1\big(\Gamma(F_S/L_\infty), \mathrm{ad}_{E/\mathcal{O}}(1)\big)^{\Gamma(L_\infty/F)}$$

implies that the composition

$$\begin{aligned} E/\mathcal{O} \xrightarrow{f} H^1\big(\Gamma_{F,S}, \mathrm{ad}_{E/\mathcal{O}}(1)\big) &\to H^1\big(\Gamma(F_S/L_\infty), \mathrm{ad}_{E/\mathcal{O}}(1)\big)^{\Gamma(L_\infty/F)} \\ &\cong \mathrm{Hom}_{\Gamma(F_S/F)}\big(\Gamma(L_{\infty,S_{L_\infty}}/L_\infty), \mathrm{ad}_{E/\mathcal{O}}(1)\big) \end{aligned}$$

is non-zero. Let $W \subset \mathrm{ad}_{E/\mathcal{O}}(1)$ be the $\mathcal{O}$-submodule spanned by $\{f(x)(\sigma) \mid x \in E/\mathcal{O}, \sigma \in G_{L_\infty}\}$. Then W is non-zero and divisible. By [NT23, Lem. 2.22] there exists $\sigma \in G_{F_\infty}$ such that $\rho(\sigma)$ has n distinct eigenvalues in E and $W \not\subset (\sigma - 1)\,\mathrm{ad}_{E/\mathcal{O}}(1)$. Thus, for some $m \geq 0$ and $\tau \in G_{L_\infty}$ we have

$$f(1/\varpi^m)(\tau) \notin (\sigma - 1)(\mathrm{ad}_{E/\mathcal{O}}(1)).$$

If $f(1/\varpi^m)(\sigma) \notin (\sigma - 1)(\mathrm{ad}_{E/\mathcal{O}}(1))$ then we are done. If not, then $\tau\sigma$ has the sought property, i.e.

$$E/\mathcal{O} \to H^1(F_S, \mathrm{ad}_{E/\mathcal{O}}(1)) \to \frac{\mathrm{ad}_{E/\mathcal{O}}(1)}{(\tau\sigma - 1)\,\mathrm{ad}_{E/\mathcal{O}}(1)}$$

is non-zero, since τ acts trivially on $\mathrm{ad}_E(1)$.

$\square$

4.2. **Patching.** In this section, we set up the patching argument following [A'C24] and [GV18]. Define deformation problems

$$\begin{aligned} \mathbf{S} &\coloneqq (\bar\rho, S, \{\mathcal{D}_{\bar\rho_v}^{\lambda_v, \preceq\tau_v}\}_{v\in S_p} \cup \{\mathcal{D}_{\bar\rho_v}^{\square}\}_{v\in S^p}) \\ \mathbf{S}' &\coloneqq (\bar\rho, S, \{\mathcal{D}_{\bar\rho_v}^{\square}\}_{v\in S}) \end{aligned}$$

We let

$$\mathcal{R} = \mathcal{R}_0 \coloneqq \mathcal{R}_{\mathbf{S}} \simeq \mathcal{R}_{\mathbf{S}'} \otimes^{\mathbf{L}}_{R^{S,\mathrm{loc}}_{\mathbf{S}'}} R^{S,\mathrm{loc}}_{\mathbf{S}}$$

be the derived deformation ring of $\mathbf{S}$. For a Taylor–Wiles-datum Q, we define an augmented deformation problem

$$\mathbf{S}_Q = (\bar\rho, S \cup Q, \{\mathcal{D}_{\bar\rho_v}^{\lambda_v, \preceq\tau_v}\}_{v\in S_p} \cup \{\mathcal{D}_{\bar\rho_v}^{\square}\}_{v\in S^p} \cup \{\mathcal{D}_{\bar\rho_v}^{\square}\}_{v\in Q}).$$

and similarly for $\mathbf{S}'_Q$. Applying Proposition 4.1.2, we obtain $q, l \geq 0$ and for every $N \geq 1$ a Taylor–Wiles datum Q_N satisfying the conclusions of the proposition. We let

$$\mathcal{R}_N \coloneqq \mathcal{R}^S_{\mathbf{S}_{Q_N}} \text{ and } \mathcal{R}'_N \coloneqq \mathcal{R}^S_{\mathbf{S}'_{Q_N}}$$

be the S-framed type-$\mathbf{S}$ and type-$\mathbf{S}'$ derived deformation rings, respectively, and set for $N \geq 0$

$$R_N \coloneqq \pi_0(\mathcal{R}_N).$$

In [GV18], the authors consider deformations of $\bar\rho_v$ for $v \in Q_N$ valued in the diagonal torus $\mathbf{T} \subset \mathbf{G}$. The $\mathbf{T}$-valued deformation functors are proved in [GV18, Lem. 8.3] to be equivalent to their $\mathbf{G}$-valued counterparts. We let $\mathcal{T} \coloneqq \mathcal{O}[[T_1, \ldots, T_{n^2|S|-1}]]$ be the power series ring in the 'S-frame variables' and define for every $v \in Q_N$

$$\begin{aligned} N_v &\coloneqq \max\{r \in \mathbb{Z} \mid v \equiv 1 \bmod p^r\} \\ \mathcal{S}_v &\coloneqq \mathcal{T}[[Y_1, \ldots, Y_n]]/\langle (1+Y_i)^{p^{N_v}} - 1\rangle \\ \mathcal{S}_N &\coloneqq \widehat{\bigotimes}_{v\in Q_N} \mathcal{S}_v \end{aligned}$$

with the completed tensor product taken over $\mathcal{T}$. Note that $N_v \geq N$. The ring $\mathcal{S}_N$ is static; to emphasise this we sometimes write

$$S_N \coloneqq \pi_0(\mathcal{S}_N) \simeq \mathcal{S}_N.$$

By [GV18, (8.14)], the static ring $\mathcal{S}_N \otimes^{\mathbf{L}}_{\mathcal{T}} \mathcal{O}$ is the framed deformation ring of the trivial representation of the product of inertia subgroups $\prod_{v\in Q_N} I_v$.

Lemma 4.2.1. *Let $r \leq N$. Then*

$$\mathrm{T}^1_{\mathcal{O}_r}(\mathcal{S}_N) \cong \prod_{v \in Q_N} H^2(\Gamma_v, \operatorname{ad}\xi_r)$$

Proof. This follows from the fact that $\mathcal{S}_N \otimes^{\mathbf{L}}_{\mathcal{T}} \mathcal{O}$ represents $\prod_{v\in Q_N} \mathcal{X}_{\Gamma_v,\mathbf{G}}$ and Proposition 3.6.9. □

For every N, we have natural maps

$$\mathcal{S}_N \to \mathcal{R}'_N \to \mathcal{R}_N$$

and R'_N and R_N are local S_N-algebras.

Lemma 4.2.2. [GV18, (11.4)] *For every $N \geq 1$, there are weak equivalences*

$$\mathcal{R} \simeq \mathcal{R}_N \otimes^{\mathbf{L}}_{\mathcal{S}_N} \mathcal{O}$$
$$\mathcal{R}' \simeq \mathcal{R}'_N \otimes^{\mathbf{L}}_{\mathcal{S}_N} \mathcal{O}.$$

Let

$$R_\infty := R^{S,\mathrm{loc}}_{\mathbf{S}}[[X_1, \dots, X_{nq-n^2[F^+:\mathbb{Q}]}]]$$
$$S_\infty := \mathcal{T}[[Y_1, \dots, Y_{qn}]]$$
$$\mathfrak{a} = (T_1, \dots, T_{n^2|S|-1}, Y_1, \dots, Y_{qn}) \subset S_\infty.$$

Given a Taylor–Wiles datum Q_N, we write

$$\mathfrak{a}_N = \langle (1+Y_i)^{p^{Nv_i}} - 1, (1+T_i)^{p^{Nv_i}} - 1 \rangle \subset S_\infty$$
$$\mathfrak{c}_N = \left(p^N, \mathfrak{a}_{Q_N}\right) \subset S_\infty,$$

so that $S_N \cong S_\infty/\mathfrak{a}_N$. These ideals depend on the specific Q_N, but since we will consider a single Q_N for every N, the simplified notation is justified. Then, for every $N \geq 1$, R_N is a local S_N-algebra and we have $R_N/\mathfrak{a} \cong \pi_0(\mathcal{R}_{\mathbf{S}})$, and by Proposition 4.1.2 there is a surjection $R_\infty \twoheadrightarrow R_N$. Note that $\mathfrak{c}_N \subset S_\infty$ is an open ideal and

$$\mathfrak{m}^{p^N}_{S_\infty} \subset \mathfrak{c}_N \subset \mathfrak{m}^N_{S_\infty},$$

so that $S_\infty \cong \varprojlim_N S_\infty/\mathfrak{c}_N$ as topological $\mathcal{O}$-algebras. We define Artinian quotients

$$\overline{R}_N := R^S_{\mathbf{S}_{Q_N}}/\mathfrak{m}^N_{R_\infty},$$
$$\overline{S}_N := S_\infty/\mathfrak{c}_N \cong S_N/\varpi^N.$$

Then $\overline{R}_N$ is a local $\overline{S}_N$-algebra.

Theorem 4.2.3. [A'C24] *With the same notation and assumptions as Theorem 4.0.1, we have:*

(i) *There exists an infinite subset $\mathbb{N}^* \subseteq \mathbb{N}$ and, for every $M \leq N$ in $\mathbb{N}^*$, isomorphisms of R_∞-algebras $\overline{R}_N/\mathfrak{m}^M_{R_\infty} \cong \overline{R}_M$ such that we have an isomorphism*

$$R^\wedge_{\infty,\mathfrak{p}} \to (\varprojlim_{\mathbb{N}^*} \overline{R}_N)^\wedge_{\mathfrak{p}},$$

where the transition maps in the inverse system are given by the compositions

$$\overline{R}_N \twoheadrightarrow \overline{R}_N/\mathfrak{m}^M_{R_\infty} \xrightarrow{\sim} \overline{R}_M$$

and $\mathfrak{p} = \ker(R_\infty \to \mathcal{O})$ corresponds to ξ.

(ii) *There is an R_∞-module M_∞ such that the completed localisation $M^\wedge_{\infty,\mathfrak{p}}$ is free over $R^\wedge_{\infty,\mathfrak{p}}$, and*

$$M^\wedge_{\infty,\mathfrak{p}} \otimes^{\mathbf{L}}_{S^\wedge_{\infty,\mathfrak{a}}} E \simeq R\Gamma(K, \mathcal{V}_\lambda)_{\mathfrak{p}}.$$

Consequently, the graded ring

$$\pi_*\left(R^\wedge_{\infty,\mathfrak{p}} \otimes^{\mathbf{L}}_{S^\wedge_{\infty,\mathfrak{a}}} E\right) \cong \mathrm{Tor}^{S^\wedge_{\infty,\mathfrak{a}}}_*\left(R^\wedge_{\infty,\mathfrak{p}}, E\right)$$

acts freely on the graded module

$$\pi_*\left(M^\wedge_{\infty,\mathfrak{p}} \otimes^{\mathbf{L}}_{S^\wedge_{\infty,\mathfrak{a}}} E\right) \cong H^*(K, \mathcal{V}_\lambda)$$

(iii) *The geometric Bloch–Kato Selmer group vanishes:*
$$H^1_g(\Gamma_{F,S}, \operatorname{ad}\rho) = 0.$$

Proof. All three essentially follow from the main result of [A'C24]. Since our local conditions at $v \mid p$ differ slightly, we spell out the proof which is a straightforward application of the abstract patching lemma [A'C24, Lem. 2.6.4]. For $N \geq 1$, one defines the following (see [A'C24, Prop. 5.3.3] for details):

- An open compact subgroup $K_0(Q_N) = \prod_v K_{0,v}(Q_N) \subset K$ where $K_{0,v}(Q_N) = K_v$ for $v \notin Q_N$ and $K_{0,v}(Q_N) = \mathrm{Iw}_v$ is the matrices which are upper triangular modulo ϖ_v for $v \in Q_N$.
- An open compact subgroup $K_1(Q_N) = \prod_v K_{1,v}(Q_N) \subset K_0(Q_N) \subset K$ where $K_{1,v}(Q_N) = K_v$ for $v \notin Q_N$ and $K_{1,v}(Q_N) = \mathrm{Iw}\, 1, v$ is the matrices which are upper triangular and unipotent modulo ϖ_v for $v \in Q_N$.
- For any object M with a $\mathbb{T}^S$-action, an enlarged (faithful) Hecke algebra $\mathbb{T}^{S\cup Q_N}_{Q_N}(M)$ obtained by adjoining Hecke operators at the places $v \in Q_N$.
- Maximal ideals $\mathfrak{m}_1, \mathfrak{m}_0$ such that, if $\mathfrak{m} \subset \mathbb{T}^S(R\Gamma(X_K, \mathcal{V}_{\lambda,\tau}))$ is the ideal corresponding to Π, we have morphisms

$$\mathbb{T}^{S\cup Q_N}_{Q_N}\big(R\Gamma(X_{K_1(Q_N)}, \mathcal{V}_{\lambda,\tau})\big)_{\mathfrak{m}_1} \to \mathbb{T}^{S\cup Q_N}\big(R\Gamma(X_{K_0(Q_N)}, \mathcal{V}_{\lambda,\tau})\big)_{\mathfrak{m}_0} \xrightarrow{\sim} \mathbb{T}^S\big(R\Gamma(X_K, \mathcal{V}_{\lambda,\tau})\big)_{\mathfrak{m}}.$$

where the first is given by taking $K_0(Q_N)/K_1(Q_N)$-coinvariants, and corresponding quasi-isomorphisms of complexes

$$R\Gamma\big(K_0(Q_N)/K_1(Q_N), R\Gamma(X_{K_1(Q_N)}, \mathcal{V}_{\lambda,\tau})_{\mathfrak{m}_1}\big) \xrightarrow{\sim} R\Gamma(X_{K_0(Q_N)}, \mathcal{V}_{\lambda,\tau})_{\mathfrak{m}_0} \xrightarrow{\sim} R\Gamma(X_K, \mathcal{V}_{\lambda,\tau})_{\mathfrak{m}}.$$

In the numbering of the abstract patching lemma, we have 7 statements to verify.

(1) R_∞ is a complete Noetherian local $\mathcal{O}$-algebra.
(2) R_N is a local S_N-algebra and a quotient of R_∞.
(3) Define complexes (or animated modules)
$$\begin{aligned} C_0 &:= R\operatorname{Hom}_{\mathcal{O}}\left(R\Gamma(X_K, \mathcal{V}_{\lambda,\tau})_{\mathfrak{m}}, \mathcal{O}\right) \\ C_N &:= R\operatorname{Hom}_{\mathcal{O}}\left(R\Gamma(X_{K_1(Q_N)}, \mathcal{V}_{\lambda,\tau})_{\mathfrak{m}_1}, \mathcal{T}\right)/\mathfrak{a}_N \end{aligned}$$
where the right-hand sides denote minimal representatives in the derived category of $\mathcal{O}$ and S_N, respectively. Then $C_N^\bullet/\mathfrak{a} \simeq C_0^\bullet$ by ref above.
(4) By [ACC+23, Thm. 2.4.10], $C_0[1/\varpi]$ has homotopy concentrated in degrees $[q_0, q_0 + l_0]$ and is not exact. Moreover, $\dim(S_\infty)_{\mathfrak{a}} = n^2|S| - 1 + qn$ and using Proposition 3.5.7 and Theorem 3.5.9(2) we deduce
$$\dim R_\infty[1/\varpi] \leq n^2|S| + qn - n[F^+ : \mathbb{Q}] = \dim(S_\infty)_{\mathfrak{a}} - l_0.$$
(5) For $N \geq 1$, one lets
$$T_N := \mathbb{T}^{S\cup Q_N}_{Q_N}(C_N)_{\mathfrak{m}_1} \cdot \mathcal{T} \subset \operatorname{End}_{\mathbf{D}(S_N)}(C_N),$$
where $\mathfrak{m}_1, \mathfrak{m}_0$ are maximal ideals as in [A'C24, Prop. 5.3.3].The image of T_N in $\operatorname{End}_{\mathbf{D}(S_0)}(C_0)$ equals
$$\mathbb{T}^{S\cup Q_N}(C_0) \subset \mathbb{T}^S(C_0) =: T_0.$$
(6) Using Lemma 3.5.5 and [Hev24, Thm. 2.24], we obtain for every N an S_N-algebra surjection (after taking a tensor product with $\mathcal{T}$)
$$R_N \to T_N/I_N$$
where $I_N^\delta = 0$ for some δ independent of N, obtaining a commutative square
$$\begin{tikzcd} R_N \arrow[r, two heads] \arrow[d] & T_N/I_N \arrow[d] \\ R_0 \arrow[r, two heads] & T_0/I_0 \end{tikzcd}$$
(7) Since the Weil–Deligne representations $\mathrm{WD}(\rho_v)$ are generic for every $v \in S$, the representation ρ defines a smooth point of $R_\infty[1/\varpi] \cong R^{S,\mathrm{loc}}_{\mathbf{S}}[[X_1, \ldots, X_{nq-n^2[F^+:\mathbb{Q}]}]]$.

Now, [A'C24, Lem. 2.6.4, Lem. 2.3.3] provide a subsequence $\mathbb{N}^* \subseteq \mathbb{N}$, morphisms $\overline{R}_N \to \overline{R}_M$ for every $M \leq N$ in $\mathbb{N}^*$ as in the statement, and an isomorphism as in (i):

$$R^{\wedge}_{\infty,\mathfrak{p}} \cong \mathbf{P}_{R_\infty}(R_N)^{\wedge}_{\mathfrak{p}} \cong (\varprojlim_N \overline{R}_N)^{\wedge}_{\mathfrak{p}}.$$

Statement (ii) follows from [A'C24, Thm. 5.3.5], and statement (iii) follows from [A'C24, Lem. 2.6.4]. $\square$

Remark 4.2.4. The subset $\mathbb{N}^* \subseteq \mathbb{N}$ and the maps $\overline{R}_N \to \overline{R}_M$ depends on a choice of non-principal ultrafilter on $\mathbb{N}$. As such, the isomorphism provided by (1) is non-canonical. This ambiguity will not play a role in the sequel; we fix from now on a subset $\mathbb{N}^*$ and maps as in the theorem.

In light of part (2) of Theorem 4.2.3, we see that to prove Theorem 4.0.1 it suffices to prove a weak equivalence

$$(\mathcal{R}_{\mathbf{S}} \otimes_{\mathcal{O}} E)^{\wedge}_{\xi} \simeq R^{\wedge}_{\infty,\mathfrak{p}} \otimes^{\mathbf{L}}_{S^{\wedge}_{\infty,\mathfrak{a}}} E.$$

4.3. **Tangent complex calculations.** Our proof of the main theorem will rely on a comparison of tangent complexes. In this section, we collect some preliminary calculations with tangent complexes.

Proposition 4.3.1. *With notation and assumptions as above, the tangent complex of $(\mathcal{R}_{\mathbf{S}} \otimes_{\mathcal{O}} E)^{\wedge}_{\xi}$ is concentrated in degrees $[0,1]$ and satisfies*

$$\dim_E \mathrm{T}^i_E\left((\mathcal{R}_{\mathbf{S}} \otimes_{\mathcal{O}} E)^{\wedge}_{\xi}\right) = \begin{cases} 0 & \text{if } i = 0 \\ l_0 & \text{if } i = 1. \end{cases}$$

Proof. Recall that we have an isomorphism

$$\mathrm{T}^0_E\left((\mathcal{R}_{\mathbf{S}} \otimes_{\mathcal{O}} E)^{\wedge}_{\xi}\right) \cong \mathrm{T}^0_E\left(\pi_0\big((\mathcal{R}_{\mathbf{S}} \otimes_{\mathcal{O}} E)^{\wedge}_{\xi}\big)\right).$$

Since $(\mathcal{R}_{\mathbf{S}} \otimes_{\mathcal{O}} E)^{\wedge}_{\xi}$ is Noetherian, [Lur04, Prop. 6.1.8] implies

$$\pi_0\big((\mathcal{R}_{\mathbf{S}} \otimes_{\mathcal{O}} E)^{\wedge}_{\xi}\big) \cong \big(\pi_0(\mathcal{R}_{\mathbf{S}}) \otimes_{\mathcal{O}} E\big)^{\wedge}_{\xi},$$

whose tangent space is given by the Bloch–Kato Selmer group

$$H^1_g(\Gamma_{F,S}, \operatorname{ad}\rho)$$

which vanishes under our assumptions by the main theorem of [A'C24]. This vanishing together with the fibre sequence of $\mathcal{R}_{\mathbf{S}}$ and Proposition 3.7.7 yield an exact sequence (recall that both $\mathrm{T}^1_E(R^{S,\mathrm{loc}}_{\mathbf{S}})$ and $\mathrm{T}^1_E(R^{S,\mathrm{loc}}_{\mathbf{S}'})$ vanish by genericity of $\mathrm{WD}(\rho_v)$ at $v \in S$ and irreducibility of ρ_v at $v \in S_p$)

$$0 \to \mathrm{T}^0_E(\mathcal{R}_{\mathbf{S}'}) \oplus \mathrm{T}^0_E(R^{S,\mathrm{loc}}_{\mathbf{S}}) \to \mathrm{T}^0_E(R^{S,\mathrm{loc}}_{\mathbf{S}'}) \to \mathrm{T}^1_E\left((\mathcal{R}_{\mathbf{S}} \otimes_{\mathcal{O}} E)^{\wedge}_{\xi}\right) \to \mathrm{T}^1_E(\mathcal{R}_{\mathbf{S}'}) \to \mathrm{T}^1_E(R^{S,\mathrm{loc}}_{\mathbf{S}'}).$$

Comparing the sequence to the Poitou–Tate sequence associated to the Selmer system corresponding to $\mathbf{S}$ gives

$$\dim_E \mathrm{T}^1_E\left((\mathcal{R}_{\mathbf{S}} \otimes_{\mathcal{O}} E)^{\wedge}_{\xi}\right) = \dim_E H^1(\Gamma_{F,S}, \operatorname{ad}\rho(1))^\vee = l_0$$

by Proposition 3.8.1. $\square$

For any $N \geq 1$, we let

$$\mathcal{C}_N = \overline{R}_N \otimes^{\mathbf{L}}_{\overline{S}_N} \mathcal{O}_N.$$

For every $M \leq N$ in $\mathbb{N}^*$, Theorem 4.2.3(1) provides a commutative diagram

$$\begin{tikzcd} \overline{R}_N \arrow[d] & \overline{S}_N \arrow[l] \arrow[r] \arrow[d] & \mathcal{O}_N \arrow[d] \\ \overline{R}_M & \overline{S}_M \arrow[l] \arrow[r] & \mathcal{O}_M \end{tikzcd}$$

and we denote by

$$e_{N,M} \colon \mathcal{C}_N \to \mathcal{C}_M$$

the induced map on derived tensor products.

Lemma 4.3.2. *With notation as above, we have the following:*

(1) *The limit* $\mathcal{C}_\infty = \lim_N \mathcal{C}_N$ *is Noetherian.*
(2) *The tangent complex of the completion* $(\mathcal{C}_\infty \otimes_{\mathcal{O}} E)^\wedge_\xi \cong \mathcal{C}^\wedge_{\infty,\mathfrak{p}}$ *has cohomology concentrated in degrees* $[0,1]$ *and*

$$\dim_E \mathrm{T}^0_E(\mathcal{C}^\wedge_{\infty,\mathfrak{p}}) - \dim_E \mathrm{T}^1_E(\mathcal{C}^\wedge_{\infty,\mathfrak{p}}) = -\ell_0.$$

Proof. (1) By Theorem 3.6.6, it suffices to prove that $\mathrm{T}^i_k(\mathcal{C}_\infty) = \mathrm{colim}\, \mathrm{T}^i_k(\mathcal{C}_N)$ is finite-dimensional for every $i \geq 0$. Consider the fibre sequence

$$\mathrm{T}^\bullet_k(\mathcal{C}_N) \to \mathrm{T}^\bullet_k(\overline{R}_N) \oplus \mathrm{T}^\bullet_k(\mathcal{O}_N) \to \mathrm{T}^\bullet_k(\overline{S}_N) \xrightarrow{+1} .$$

Filtered colimits are exact, whence we have a sequence

$$\mathrm{T}^\bullet_k(\mathcal{C}_\infty) \to \mathrm{T}^\bullet_k(\lim_N \overline{R}_N) \to \mathrm{T}^\bullet_k(S_\infty).$$

For every $N \geq 0$, we have a surjection $R_\infty \twoheadrightarrow \overline{R}_N$, and thus $R_\infty \twoheadrightarrow \lim_N \overline{R}_N \cong \varprojlim_N \overline{R}_N$. Since R_∞ and S_∞ are Noetherian, it follows from Theorem 3.6.6 and the fibre sequence above that $\mathrm{T}^i_k(\mathcal{C}_\infty)$ is finite-dimensional for every $i \geq 0$.
(2) For any $r \leq N$, we have a fibre sequence

$$\mathrm{T}^\bullet_{\mathcal{O}_r}(\mathcal{C}_N) \to \mathrm{T}^\bullet_{\mathcal{O}_r}(\overline{R}_N) \oplus \mathrm{T}^\bullet_{\mathcal{O}_r}(\mathcal{O}_N) \to \mathrm{T}^\bullet_{\mathcal{O}_r}(\overline{S}_N) \xrightarrow{+1} .$$

Passing to the colimit over $N \geq r$ and limit over r, then inverting p yields a fibre sequence

$$\big(\lim_r \operatorname*{colim}_{N\geq r} \mathrm{T}^\bullet_{\mathcal{O}_r}(\mathcal{C}_N)\big) \otimes_{\mathcal{O}} E \to \mathrm{T}^\bullet_E(R^\wedge_{\infty,\mathfrak{p}}) \to \mathrm{T}^\bullet_E(S^\wedge_{\infty,\mathfrak{a}}) \xrightarrow{+1},$$

The first term is identified with $\mathrm{T}^\bullet_E(\mathcal{C}^\wedge_{\infty,\mathfrak{p}})$ by Proposition 3.7.7. Now, $R^\wedge_{\infty,\mathfrak{p}}$ and $S^\wedge_{\infty,\mathfrak{a}}$ are smooth E-algebras and therefore upon taking cohomology we obtain an exact sequence of E-vector spaces

$$0 \to \mathrm{T}^0_E(\mathcal{C}^\wedge_{\infty,\mathfrak{p}}) \to \mathrm{T}^0_E(R^\wedge_{\infty,\mathfrak{p}}) \to \mathrm{T}^0_E(S^\wedge_{\infty,\mathfrak{a}}) \to \mathrm{T}^1_E(\mathcal{C}^\wedge_{\infty,\mathfrak{p}}) \to 0.$$

Using the smoothness of $R^\wedge_{\infty,\mathfrak{p}}$ and $S^\wedge_{\infty,\mathfrak{a}}$ again, we deduce that

$$\dim_E \mathrm{T}^0_E(\mathcal{C}^\wedge_{\infty,\mathfrak{p}}) - \dim_E \mathrm{T}^1_E(\mathcal{C}^\wedge_{\infty,\mathfrak{p}}) = \dim R^\wedge_{\infty,\mathfrak{p}} - \dim S^\wedge_{\infty,\mathfrak{a}} = -l_0.$$

□

Proposition 4.3.3. *With assumptions and notation as above and* $r \leq N$*, the kernel*

$$\ker\big(\mathrm{T}^1_{\mathcal{O}_r}(\mathcal{R}'_N) \to \mathrm{T}^1_{\mathcal{O}_r}(\mathcal{S}_N)\big)$$

is a finite length $\mathcal{O}$*-module with length independent of* r *and* N*.*

Proof. Let $r \leq N$. By Proposition 3.6.9 and Lemma 4.2.1(4), we have isomorphisms

$$\mathrm{T}^1_{\mathcal{O}_r}(\mathcal{R}'_N) \cong H^2(\Gamma_{SQ_N}, \mathrm{ad}\,\xi_r)$$
$$\mathrm{T}^1_{\mathcal{O}_r}(\mathcal{S}_N) \cong \prod_{v\in Q_N} H^2(\Gamma_v, \mathrm{ad}\,\xi_r)$$

We factor the map $\gamma\colon\ \mathrm{T}^1_{\mathcal{O}_r}(\mathcal{R}'_N) \to \mathrm{T}^1_{\mathcal{O}_r}(\mathcal{S}_N)$ as $\gamma = \beta \circ \alpha$ where

$$H^2(\Gamma_{SQ_N}, \mathrm{ad}\,\xi_r) \xrightarrow{\alpha} \prod_{v\in SQ_N} H^2(\Gamma_v, \mathrm{ad}\,\xi_r) \xrightarrow{\beta} \prod_{v\in Q_N} H^2(\Gamma_v, \mathrm{ad}\,\xi_r)$$

to obtain an exact sequence

$$0 \to \ker(\alpha) \to \ker(\gamma) \to \ker(\beta).$$

To prove the proposition, it suffices to show that $\ker(\alpha)$ and $\ker(\beta)$ have finite length bounded independently of r and N. To see this, consider the Poitou–Tate sequence for the Selmer system

$$\{\mathcal{L}_{v,r}\}_{v\in SQ_N} = \{H^1(\Gamma_v, \mathrm{ad}\,\xi_r)\}_{v\in SQ_N} \cup \{H^1_{\mathrm{ur}}(\Gamma_v, \mathrm{ad}\,\xi_r)\}_{v\notin SQ_N},$$

part of which reads

$$H^1(\Gamma_{SQ_N}, \mathrm{ad}\,\xi_r(1)) \to H^2(\Gamma_{SQ_N}, \mathrm{ad}\,\xi_r) \xrightarrow{\alpha} \prod_{v\in SQ_N} H^2(\Gamma_v, \mathrm{ad}\,\xi_r).$$

By Proposition 4.1.2(4), we have

$$\mathrm{length}_{\mathcal{O}} \ker(\alpha) \leq \mathrm{length}_{\mathcal{O}} H^1(\Gamma_{SQ_N}, \mathrm{ad}\,\xi_r(1)) \leq l.$$

Now,

$$\ker(\beta) = \prod_{v \in S} H^2(\Gamma_v, \mathrm{ad}\,\xi_r) = \mathrm{T}^1_{\mathcal{O}_r}(R^{S,\mathrm{loc}}_{\mathbf{S}'})$$

and since the characteristic 0 point of Spec $R^{S,\mathrm{loc}}_{\mathbf{S}'}$ corresponding to ξ is smooth by assumptions , the $\mathcal{O}$-module $\ker(\beta)$ has finite length bounded independently of r (and N). This completes the proof. □

4.4. **Proof of the main theorem.**

Theorem 4.4.1. *There exists a morphism $\mathcal{R} \to \mathcal{C}_\infty$ such that $\xi\colon \mathcal{R} \to \mathcal{O}$ factors through $\mathcal{C}_\infty$ and the induced morphism*

$$(\mathcal{R} \otimes_{\mathcal{O}} E)^\wedge_\xi \to (\mathcal{C}_\infty \otimes_{\mathcal{O}} E)^\wedge_\xi$$

is a weak equivalence in $\mathcal{CNL}_{/E}$.

Remark 4.4.2. Recall that the existence of the inverse system $(\mathcal{C}_N)_{N\in\mathbb{N}^*}$ is not canonically defined. The proof of the existence of $g\colon \mathcal{R} \to \mathcal{C}_\infty$ is also non-constructive and as such, the weak equivalence is a priori non-canonical and depends on a choice of non-principal ultrafilter on $\mathbb{N}$. In [GV18, §15], the authors are able to prove, under their assumptions, that the equivalence is canonical and independent of any choices in the Taylor–Wiles process. In the future, we hope to prove a similar result in characteristic 0 without introducing extra assumptions.

Proof of Theorem 4.0.1: Let

$$[\mathcal{R}, -] = \pi_0(\mathrm{Map}_{/k}(\mathcal{R}, -))$$

and denote by $\tau_{\leq j}$ the j-truncation functor. We claim that the natural maps

$$[\mathcal{R}, \mathcal{C}_\infty] \to \varprojlim_N [\mathcal{R}, \mathcal{C}_N] \to \varprojlim_{j,N} [\mathcal{R}, \tau_{\leq j}\mathcal{C}_N]$$

are bijections. Indeed, by [GV18, Lem. A.9] we have a short exact sequence of pointed sets (for any choice of basepoint)

$$0 \to \varprojlim^1_{j,N} \pi_1 \mathrm{Map}_{/k}(\mathcal{R}, \tau_{\leq j}\mathcal{C}_N) \to [\mathcal{R}, \mathcal{C}_\infty] \to \varprojlim_{j,N} [\mathcal{R}, \tau_{\leq j}\mathcal{C}_N] \to 0.$$

We claim that $\mathrm{Map}_{/k}(\mathcal{R}, \tau_{\leq j}\mathcal{C}_N)$ has finite homotopy groups (and in particular, $[\mathcal{R}, \tau_{\leq j}\mathcal{C}_N]$ is finite for every j, N). Indeed, since $\tau_{\leq j}\mathcal{C}_N \in \mathcal{A}rt_{/k}$, using Lemma 3.1.3 we may write $\tau_{\leq j}\mathcal{C}_N$ as an iterated series of small extensions

$$\tau_{\leq j}\mathcal{C}_N \simeq B_m \to \cdots \to B_0 \simeq k$$

for some $m \geq 0$. Since $\mathcal{R}$ is Noetherian and

$$\pi_i(\mathrm{Map}_{/k}(\mathcal{R}, k \oplus k[n])) \cong \pi_0(\mathrm{Map}_{/k}(\mathcal{R}, k \oplus k[n-i])) \cong \mathrm{T}^{n-i}_k(\mathcal{R})$$

is finite, the claim follows. Hence, the $\varprojlim^1_j$-term in the short exact sequence vanishes and we have proved

$$[\mathcal{R}, \mathcal{C}_\infty] \cong \varprojlim_{j,N} [\mathcal{R}, \tau_{\leq j}\mathcal{C}_N].$$

It follows that $[\mathcal{R}, \mathcal{C}_\infty] \to \varprojlim_N [\mathcal{R}, \mathcal{C}_N]$ is injective. It is automatically surjective by the existence of the short exact sequence (using [GV18, Lem. A.9] again)

$$0 \to \varprojlim^1_N \pi_1 \mathrm{Map}_{/k}(\mathcal{R}, \mathcal{C}_N) \to [\mathcal{R}, \mathcal{C}_\infty] \to \varprojlim_N [\mathcal{R}, \mathcal{C}_N] \to 0.$$

From the above, we see that defining a morphism $\mathcal{R} \to \mathcal{C}_\infty$ amounts up to homotopy to defining a compatible system of maps $\mathcal{R} \to \mathcal{C}_N$. For $a \geq 0$, we let

$$X_N(a) = \{[f] \in [\mathcal{R}, \mathcal{C}_N] \mid \forall r \leq M \leq N\colon\ \mathrm{T}^0_{\mathcal{O}_r}(e_{N,M} \circ f) \text{ isomorphism}, p^a \operatorname{coker}(\mathrm{T}^1_{\mathcal{O}_r}(e_{N,M} \circ f)) = 0\}.$$

Here, $\mathrm{T}^i_{\mathcal{O}_r}(e_{N,M} \circ f)$ denotes the homomorphism of $\mathcal{O}_r$-modules

$$\mathrm{T}^i_{\mathcal{O}_r}(\mathcal{C}_M) \to \mathrm{T}^i_{\mathcal{O}_r}(\mathcal{R})$$

induced by the composition $e_{N,M} \circ f$. We will prove that for a sufficiently large, $(X_N(a))_{N\in\mathbb{N}^*}$ is an inverse system of non-empty profinite sets with continuous transition maps. In particular, $\varprojlim X_N(a) \neq \emptyset$ and hence there exists a class

$$[g] = ([g_N])_{N\in\mathbb{N}^*} \in \varprojlim_N X_N(a) \subset [\mathcal{R}, \mathcal{C}_\infty].$$

$(X_N(a))$ is an inverse system: Let $[f] \in X_N(a)$. Then, for any $L \le M \le N$, we have

$$[e_{M,L} \circ e_{N,M} \circ f] = [e_{N,L} \circ f]$$

and hence $[e_{N,M} \circ f] \in X_M(a)$ and the transition maps are compatible, i.e. we have a commutative triangle

$$\begin{tikzcd} X_N(a) \arrow[rr] \arrow[dr] & & X_L(a) \\ & X_M(a) \arrow[ur] & \end{tikzcd}$$

Hence, $(X_N(a))_{N\in\mathbb{N}^*}$ is an inverse system.
$X_N(a)$ is nonempty: Fix N. We prove that there exists $a \ge 0$ such that

$$f = f_N \colon \mathcal{R} \simeq \mathcal{R}_N \otimes^{\mathbf{L}}_{\mathcal{S}_N} \mathcal{O} \to \overline{R}_N \otimes^{\mathbf{L}}_{\overline{S}_N} \mathcal{O}_N = \mathcal{C}_N$$

lies in $X_N(a)$. Note that we do *not* prove that the sequence $[f_N]$ forms a compatible system; we merely use it to deduce the existence of g, which a priori has no relation to the $[f_N]$ beyond satisfying the same conditions on the level of tangent complexes. For any $r \le M \le N$, the map $e_{N,M} \circ f$ induces a commutative diagram with exact rows

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{T}^0_r(\mathcal{C}_M) \arrow[r] \arrow[d] & \mathrm{T}^0_r(\overline{R}_M) \arrow[r] \arrow[d] & \mathrm{T}^0_r(\overline{S}_M) \arrow[d] \\ 0 \arrow[r] & \mathrm{T}^0_r(\mathcal{R}) \arrow[r] & \mathrm{T}^0_r(\mathcal{R}_N) \arrow[r] & \mathrm{T}^0_r(\mathcal{S}_N) \end{tikzcd}$$

The two rightmost vertical maps are isomorphisms, since $\mathfrak{c}_M$ is generated by elements whose $\bmod p^r$ reductions lie in the square of the maximal ideal, whereby

$$\mathrm{T}^0_r(\mathcal{R}_N) \cong \mathrm{T}^0_r(R_N) \cong \mathrm{T}^0_r(R_N/\mathfrak{c}_M) = \mathrm{T}^0_r(\overline{R}_M)$$

and similarly for $\mathcal{S}_N$ and $\mathcal{S}_N^{\mathrm{ur}}$. It follows from the five lemma that the left vertical map in the diagram is also an isomorphism, as required (this part does not depend on a).

To prove the remaining statement, we argue as before to obtain a commutative diagram with exact rows

$$\begin{tikzcd} \mathrm{T}^0_r(\overline{S}_M) \arrow[r] \arrow[d, "\cong"] & \mathrm{T}^1_{\mathcal{O}_r}(\mathcal{C}_M) \arrow[r] \arrow[d, "\alpha"] & \mathrm{T}^1_{\mathcal{O}_r}(\overline{R}_M) \arrow[r] \arrow[d] & \mathrm{T}^1_{\mathcal{O}_r}(\overline{S}_M) \arrow[d] \\ \mathrm{T}^0_r(\mathcal{S}_N) \arrow[r, "\beta"] & \mathrm{T}^1_{\mathcal{O}_r}(\mathcal{R}) \arrow[r] & \mathrm{T}^1_{\mathcal{O}_r}(\mathcal{R}_N) \arrow[r, "\gamma"] & \mathrm{T}^1_{\mathcal{O}_r}(\mathcal{S}_N) \end{tikzcd}$$

Note that $\mathrm{T}^1_{\mathcal{O}_r}(\mathcal{S}_N^{\mathrm{ur}}) = 0$ since $\mathcal{S}_N^{\mathrm{ur}}$ is formally smooth. The claim to be proven is that $p^a \operatorname{coker}(\alpha) = 0$ for an $a \ge 0$ which is independent of r, M and N. By the diagram, we have

$$\operatorname{coker}(\alpha) \twoheadleftarrow \operatorname{coker}(\beta) \cong \ker(\gamma),$$

so it suffices to prove that $\ker(\gamma)$ is annihilated by a sufficiently large power of p not depending on r, M, N. To analyse $\ker(\gamma)$, recall that $\mathcal{R}'_N$ represents the deformation problem without the $\mathcal{D}_v^{ss,r}$-conditions at $v \in S_p$. We have a weak equivalence

$$\mathcal{R}_N \simeq \mathcal{R}'_N \otimes^{\mathbf{L}}_{R^{S,\mathrm{loc}}_{\mathbf{S}'}} R^{S,\mathrm{loc}}_{\mathbf{S}},$$

and if we let

$$\mathrm{cok}_{r,N} = \operatorname{coker}\left(\mathrm{T}^0_{\mathcal{O}_r}(\mathcal{R}'_N) \oplus \mathrm{T}^0_{\mathcal{O}_r}(R^{S,\mathrm{loc}}_{\mathbf{S}}) \to \mathrm{T}^0_{\mathcal{O}_r}(R^{S,\mathrm{loc}}_{\mathbf{S}'}) \right),$$

then the fibre sequence of $\mathcal{R}'_N \otimes^{\mathbf{L}}_{R^{S,\mathrm{loc}}_{\mathbf{S}'}} R^{S,\mathrm{loc}}_{\mathbf{S}}$ induces a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{cok}_{r,N} \arrow[r] & \mathrm{T}^1_{\mathcal{O}_r}(\mathcal{R}_N) \arrow[r] \arrow[d, "\gamma"] & \mathrm{T}^1_{\mathcal{O}_r}(\mathcal{R}'_N) \oplus \mathrm{T}^1_{\mathcal{O}_r}(R^{S,\mathrm{loc}}_{\mathbf{S}}) \arrow[d, two heads] \\
& & \mathrm{T}^1_{\mathcal{O}_r}(\mathcal{S}_N) & \mathrm{T}^1_{\mathcal{O}_r}(\mathcal{R}'_N) \arrow[l]
\end{tikzcd}$$

where the top row is exact and the right vertical map is the projection onto the first coordinate. By exactness, we see that

$$\mathrm{cok}_{r,N} \hookrightarrow \ker(\gamma)$$

so that γ factors as

$$\mathrm{T}^1_{\mathcal{O}_r}(\mathcal{R}_N) \twoheadrightarrow \mathrm{T}^1_{\mathcal{O}_r}(\mathcal{R}_N)/\,\mathrm{cok}_{r,N} \xrightarrow{\tilde{\gamma}} \mathrm{T}^1_{\mathcal{O}_r}(\mathcal{S}_N).$$

and we have an exact sequence

$$0 \to \mathrm{cok}_{r,N} \to \ker(\gamma) \to \ker(\tilde{\gamma}) \to 0.$$

By Proposition 4.1.2(4), $\mathrm{cok}_{r,N}$ is annihilated by a power of p which is independent of r and N. Thus, it suffices to prove the same holds for $\ker(\tilde{\gamma})$. Now, by the first diagram we have

$$\ker(\tilde{\gamma}) \cong \ker\left(\mathrm{T}^1_{\mathcal{O}_r}(\mathcal{R}'_N) \oplus \mathrm{T}^1_{\mathcal{O}_r}(R^{S,\mathrm{loc}}_{\mathbf{S}}) \twoheadrightarrow \mathrm{T}^1_{\mathcal{O}_r}(\mathcal{R}'_N) \to \mathrm{T}^1_{\mathcal{O}_r}(\mathcal{S}_N)\right),$$

whereby it suffices to show that $\mathrm{T}^1_{\mathcal{O}_r}(R^{S,\mathrm{loc}}_{\mathbf{S}})$ and $\ker(\mathrm{T}^1_{\mathcal{O}_r}(\mathcal{R}'_N) \to \mathrm{T}^1_{\mathcal{O}_r}(\mathcal{S}_N))$ are annihilated by powers of p that are independent of r and N. The latter statement is contained in Proposition 4.3.3. To prove the statement for $\mathrm{T}^1_{\mathcal{O}_r}(R^{S,\mathrm{loc}}_{\mathbf{S}})$, consider the short exact sequence

$$0 \to \mathcal{O} \to \mathcal{O} \to \mathcal{O}_r \to 0.$$

It induces an exact sequence

$$\mathrm{T}^1_{\mathcal{O}}(R^{S,\mathrm{loc}}_{\mathbf{S}}) \to \mathrm{T}^1_{\mathcal{O}_r}(R^{S,\mathrm{loc}}_{\mathbf{S}}) \to \mathrm{T}^2_{\mathcal{O}}(R^{S,\mathrm{loc}}_{\mathbf{S}}).$$

By Lemma 3.6.11 and Prop. 3.6.7, these are finitely generated $\mathcal{O}$-modules. Moreover, since ξ defines a smooth point of $R^{S,\mathrm{loc}}_{\mathbf{S}}$ by genericity of the associated Weil–Deligne representations, $\mathrm{T}^i_{\mathcal{O}}(R^{S,\mathrm{loc}}_{\mathbf{S}})$ is torsion for $i \geq 1$. It follows that

$$\mathrm{length}_{\mathcal{O}}\left(\mathrm{T}^1_{\mathcal{O}_r}(R^{S,\mathrm{loc}}_{\mathbf{S}})\right) \leq \mathrm{length}_{\mathcal{O}}\left(\mathrm{T}^1_{\mathcal{O}}(R^{S,\mathrm{loc}}_{\mathbf{S}})\right) + \mathrm{length}_{\mathcal{O}}\left(\mathrm{T}^2_{\mathcal{O}}(R^{S,\mathrm{loc}}_{\mathbf{S}})\right)$$

which is finite and independent of r, M, N as required. Thus, we have shown that there exists an absolute constant a such that $[f_N] \in X_N(a)$ and in particular, $X_N(a) \neq \emptyset$.

$X_N(a)$ is profinite: Since $[\mathcal{R}, \mathcal{C}_N] = \varprojlim_j [\mathcal{R}, \tau_{\leq j}\mathcal{C}_N]$ and $\tau_{\leq j}\mathcal{C}_N \in \mathcal{A}rt_{/k}$, to prove that $X_N(a)$ is profinite it suffices to prove that for any $B \in \mathcal{A}rt_{/k}$, the set $[\mathcal{R}, B]$ is finite. By Lemma 3.1.3, any $(B \to k) \in \mathcal{A}rt_{/k}$ is equivalent to an iterated series of small extensions

$$B \simeq B_m \to \cdots \to B_0 \simeq k,$$

and we prove the claim by induction on m. The case $m = 0$ is trivially true, and from the equalities

$$[\mathcal{R}, B_m] = [\mathcal{R}, B_{m-1} \times_{B_{m-1} \oplus k[n_m+1]} B_{m-1}] = [\mathcal{R}, B_{m-1}] \times_{[\mathcal{R}, B_{m-1} \oplus k[n_m+1]]} [\mathcal{R}, B_{m-1}]$$

we see that to carry out the induction step it suffices to prove that $[\mathcal{R}, k \oplus k[n]]$ is finite for every $n \geq 1$ (since we are considering morphisms over k, $[\mathcal{R}, k] = \{*\}$). In this case, we have

$$[\mathcal{R}, k \oplus k[n]] = \pi_0(\mathrm{Map}_{/k}(\mathcal{R}, k \oplus k[n])) = \mathrm{T}^n_k(\mathcal{R})$$

which is finite-dimensional by Theorem 3.6.6. Thus, $[\mathcal{R}, \mathcal{C}_N] = \varprojlim_j [\mathcal{R}, \mathcal{C}^j_N]$ is profinite, and to prove that $X_N(a)$ is profinite we show that $X_N(a)$ is the inverse limit of a a pro-subset of $([\mathcal{R}, \mathcal{C}^j_N])_{j \geq 0}$.

To see this, note that $\mathrm{T}^1_{\mathcal{O}_r}(\mathcal{C}_M) = \mathrm{colim}_j\, \mathrm{T}^1_{\mathcal{O}_r}(\tau_{\leq j}\mathcal{C}_M)$ and therefore

$$\mathrm{coker}\left(\mathrm{T}^1_{\mathcal{O}_r}(e_{N,M} \circ f)\right) = \mathrm{colim}_j\, \mathrm{coker}\left(\mathrm{T}^1_{\mathcal{O}_r}(\tau_{\leq j}\mathcal{C}_M) \to \mathrm{T}^1_{\mathcal{O}_r}(\mathcal{R})\right) =: \mathrm{colim}_j\, Q_{\leq j}$$

We can formulate the condition defining $X_N(a)$ degree-wise in j using the equivalence

$$p^a \operatorname{colim}_j Q_{\leq j} = 0 \iff \forall j \geq 0: \; p^a Q_{\leq j} \subseteq \ker\left(Q_{\leq j} \to \operatorname{colim}_j Q_{\leq j}\right)$$

Thus, we define finite subsets $X_N^j(a) \subseteq [\mathcal{R}, \mathcal{C}_N^j]$ using the expression above and conclude that $X_N = \varprojlim X_N^j(a)$ is profinite.

$X_N(a) \to X_M(a)$ is continuous: Let $M \leq N$ and consider $e_{N,M}\colon \mathcal{C}_N \to \mathcal{C}_M$. For any j, we have a commutative diagram

$$\begin{tikzcd} \mathcal{C}_N \arrow[r] \arrow[d] & \mathcal{C}_M \arrow[d] \\ \tau_{\leq j}\mathcal{C}_N \arrow[r] & \tau_{\leq j}\mathcal{C}_M \end{tikzcd}$$

and hence

$$\begin{tikzcd} {[\mathcal{R}, \mathcal{C}_N]} \arrow[r] \arrow[d] & {[\mathcal{R}, \mathcal{C}_M]} \arrow[d] \\ {[\mathcal{R}, \tau_{\leq j}\mathcal{C}_N]} \arrow[r] & {[\mathcal{R}, \tau_{\leq j}\mathcal{C}_M]}. \end{tikzcd}$$

It follows that the top horizontal map is continuous, and hence the restriction $X_N(a) \to X_M(a)$ is also continuous.

Passage to characteristic 0: We now have a constant $a \geq 0$ and a map

$$g\colon \mathcal{R} \to \lim_N \mathcal{C}_N$$

such that for every N, $\mathrm{T}_r^0(g)$ is an isomorphism and the cokernel

$$Q_{r,N}(g) = \operatorname{coker}\left(\mathrm{T}^1_{\mathcal{O}_r}(\mathcal{C}_N) \to \mathrm{T}^1_{\mathcal{O}_r}(\mathcal{R})\right)$$

is p^a-torsion. We claim that the induced map $\mathcal{R}_{\mathfrak{p}}^\wedge \to (\lim_N \mathcal{C}_N)_{\mathfrak{p}}^\wedge$ induces an isomorphism on T_E^0 and an epimorphism on T_E^1. The first statement is clear from the definition of X_N. To prove the second statement, we note that p^a-torsion modules are preserved under filtered colimits and inverse limits, so that

$$p^a Q_{r,N}(g) = 0 \implies (\varprojlim_r \operatorname{colim}_N Q_{r,N}(g)) \otimes_{\mathcal{O}} E = 0.$$

Now, $\lim_N \mathcal{C}_N$ is Noetherian by Lemma 4.3.2(2). Thus, $\operatorname{colim}_N \mathrm{T}^1_{\mathcal{O}_r}(\mathcal{C}_N)$ is finite, and we have

$$\begin{aligned} (\varprojlim_r \operatorname{colim}_N Q_{r,N}(g)) \otimes_{\mathcal{O}} E &\cong \operatorname{coker}\Big((\varprojlim_r \operatorname{colim}_N \mathrm{T}^1_{\mathcal{O}_r}(\mathcal{C}_N)) \otimes_{\mathcal{O}} E \to (\varprojlim_r \operatorname{colim}_N \mathrm{T}^1_{\mathcal{O}_r}(\mathcal{R})) \otimes_{\mathcal{O}} E\Big) \\ &\cong \operatorname{coker}\Big(\mathrm{T}^1_E\left((\lim_N \mathcal{C}_N)_{\mathfrak{p}}^\wedge\right) \to \mathrm{T}^1_E(\mathcal{R}_{\mathfrak{p}}^\wedge)\Big) \end{aligned}$$

since colimits commute and $\varprojlim$ is exact on finite abelian groups. We have now proved that the map

$$\mathrm{T}^*_E\left((\lim_N \mathcal{C}_N)_{\mathfrak{p}}^\wedge\right) \to \mathrm{T}^*_E(\mathcal{R}_{\mathfrak{p}}^\wedge)$$

induced by $\mathcal{R}_{\mathfrak{p}}^\wedge \to (\lim_N \mathcal{C}_N)_{\mathfrak{p}}^\wedge$ is an isomorphism in degree 0 and an epimorphism in degree 1. Now, these graded E-vector spaces are concentrated in degrees $[0,1]$ and have the same dimensions. It follows that $\mathcal{R}_{\mathfrak{p}}^\wedge \to (\lim_N \mathcal{C}_N)_{\mathfrak{p}}^\wedge$ is a weak equivalence. $\square$

Lemma 4.4.3. *With the same notation as above, there is a weak equivalence*

$$(\mathcal{C}_\infty)_{\mathfrak{p}}^\wedge \simeq R_{\infty,\mathfrak{p}}^\wedge \otimes^{\mathbf{L}}_{S_{\infty,\mathfrak{a}}^\wedge} E,$$

Proof. By Proposition 3.7.7, we have an equivalence of fibre sequences

$$\begin{tikzcd} \mathrm{T}^\bullet_E\left((\mathcal{C}_\infty)_{\mathfrak{p}}^\wedge\right) \arrow[r] \arrow[d] & \mathrm{T}^\bullet_E\left(R_{\infty,\mathfrak{p}}^\wedge\right) \arrow[r] \arrow[d] & \mathrm{T}^\bullet_E\left(S_{\infty,\mathfrak{a}}^\wedge\right) \arrow[r, "+1"] \arrow[d] & {} \\ \lim_r \operatorname{colim}_{N \geq r} \mathrm{T}^\bullet_{\mathcal{O}_r}(\mathcal{C}_N) \arrow[r] & \lim_r \operatorname{colim}_{N \geq r} \mathrm{T}^\bullet_{\mathcal{O}_r}(\overline{R}_N) \arrow[r] & \lim_r \operatorname{colim}_{N \geq r} \mathrm{T}^\bullet_{\mathcal{O}_r}(\overline{S}_N) \arrow[r, "+1"] & {} \end{tikzcd}$$

Now, the tangent complex of $R^{\wedge}_{\infty,\mathfrak{p}} \otimes^{\mathbf{L}}_{S^{\wedge}_{\infty,\mathfrak{a}}} E$ is given by the same fibre:

$$\mathrm{T}^{\bullet}_{E}\left(R^{\wedge}_{\infty,\mathfrak{p}} \otimes^{\mathbf{L}}_{S^{\wedge}_{\infty,\mathfrak{a}}} E\right) \to \mathrm{T}^{\bullet}_{E}\left(R^{\wedge}_{\infty,\mathfrak{p}}\right) \to \mathrm{T}^{\bullet}_{E}\left(S^{\wedge}_{\infty,\mathfrak{a}}\right) \xrightarrow{+1}.$$

Hence, by comparing tangent complexes we see that the natural morphism

$$R^{\wedge}_{\infty,\mathfrak{p}} \otimes^{\mathbf{L}}_{S^{\wedge}_{\infty,\mathfrak{a}}} E \to \lim_{r}(R_{\infty}[1/\varpi]/\mathfrak{p}^{r} \otimes^{\mathbf{L}}_{S_{\infty}[1/\varpi]^{\wedge}_{\mathfrak{a}}/\mathfrak{a}^{r}} E)$$

induces an equivalence on tangent complexes, and the same is true for

$$(R_{\infty} \otimes^{\mathbf{L}}_{S_{\infty}} \mathcal{O})^{\wedge}_{\mathfrak{p}} \simeq (R_{\infty}[1/\varpi] \otimes^{\mathbf{L}}_{S_{\infty}[1/\varpi]} E)^{\wedge}_{\mathfrak{p}} \to \lim_{r}(R_{\infty}[1/\varpi]/\mathfrak{p}^{r} \otimes^{\mathbf{L}}_{S_{\infty}[1/\varpi]^{\wedge}_{\mathfrak{a}}/\mathfrak{a}^{r}} E).$$

Thus, these are weak equivalences of E-algebras and we obtain

$$(R_{\infty} \otimes^{\mathbf{L}}_{S_{\infty}} \mathcal{O})^{\wedge}_{\mathfrak{p}} \simeq R^{\wedge}_{\infty,\mathfrak{p}} \otimes^{\mathbf{L}}_{S^{\wedge}_{\infty,\mathfrak{a}}} E,$$

as needed. □

Proof. The first isomorphism is Corollary 3.7.5. For the second isomorphism, the idea is the same as that of [GV18, §15.3]. We have surjections

$$S^{\wedge}_{\infty,\mathfrak{a}} \to R^{\wedge}_{\infty,\mathfrak{p}} \to E.$$

Indeed, the first map is surjective since □

We are now in a position to deduce our main result Theorem 4.0.1. By Theorem 4.2.3, there exists a free $R^{\wedge}_{\infty,\mathfrak{p}}$-module $M^{\wedge}_{\infty,\mathfrak{p}}$ such that

$$M^{\wedge}_{\infty,\mathfrak{p}} \otimes^{\mathbf{L}}_{S^{\wedge}_{\infty,\mathfrak{a}}} E \simeq R\Gamma(K, \mathcal{V}_{\lambda})_{\mathfrak{p}}.$$

Theorem 4.4.1 together with Lemma 4.4.3 serve to define an action of $(\mathcal{R} \otimes^{\mathbf{L}}_{\mathcal{O}} E)^{\wedge}_{\xi} \simeq \mathcal{R}^{\mathrm{ss}}_{\rho}$ on this complex, and in particular we obtain a free action

$$\pi_{*}\left(\mathcal{R}^{\mathrm{ss}}_{\rho}\right) \circlearrowright H^{*}(K, \mathcal{V}_{\lambda})_{\mathfrak{p}}.$$

Now, since $\mathrm{T}^{i}_{E}(\mathcal{R}^{\mathrm{ss}}_{\rho}) = 0$ for $i \notin [0,1]$, $\mathcal{R}^{\mathrm{ss}}_{\rho}$ is quasi-smooth and hence by [AG15, Cor. 2.1.6], we have an isomorphism

$$\pi_{*}(\mathcal{R}^{\mathrm{ss}}_{\rho}) \cong \bigwedge^{*} \mathrm{T}^{1}_{E}(\mathcal{R}^{\mathrm{ss}}_{\rho}) \cong \bigwedge^{*} H^{1}(\Gamma_{F,S}, \mathrm{ad}\,\rho(1))^{\vee}.$$

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