A Nonparametric Test Procedure for Paired Data Sampling, in the Presence of Considerable Number of Ties

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Abstract

This paper proposes a method to improve the power of the Sign test by modifying its treatment of zero difference between observations, and thereby increasing its utilisation of the sample information. We shall apply the tests to pairwise ordinal data samples since the non-parametric tests are more commonly applicable to such data sets. The test procedure to be developed is named the Trinomial test and will be compared with the ordinary Sign test. Trinomial test developed is found to have better performance compared to the ordinary Sign test which either ignores or equally shares the information obtainable from the observations resulting in ties.

1. Introduction

Methods based on signs and ranks form a substantial body of statistical techniques that provide alternatives to the classical parametric methods. The feature of non-parametric methods mainly responsible for their great popularity is the weak set of assumptions required for their validity. It is not necessary to postulate a population from which the subjects in a study have been obtained by random sampling, but only that the treatments being compared have been assigned to the subjects at random. A bibliography of non-parametric statistics by Savage [1962] lists about 3,000 items. If brought up-to-date, it probably would contain twice that many entries. The sign test is widely regarded as the oldest of all the non-parametric test procedures. The test found its application as early as 1710 in an article by Arbuthnott. It derived its name from the procedure of converting the data in the process of analysis into plus and minus signs.

Dixon and Mood [1946], Mackinnon [1964] have each published tables of critical values for the Sign test. Wilcoxon [1945] indicated, for the first time, the possibility of using ranking methods in order to obtain a rapid approximation of the significance of the differences in experiments containing both paired and unpaired data. Dixon and Mood [1953] and Walse [1946] each had published short notes commenting on the power function of the Sign test.

A few attempts have been made to modify the Sign test in order to increase its power. One such attempt to include the zero observations into the analysis is the randomised treatment of the zero observations. That is, the zero observations are randomly distributed into plus and minus signs. However, Putter [1955] and Hemelrijk [1952] have each proven that the non-randomised treatment of the zero observations is always better than the randomisation method for the ordinary Sign test.

However, this does not necessarily mean that the inclusion of zero observations in the Sign test will always result in a less powerful test. It is shown in this study that by incorporating the zeros in the Sign test we could improve its performance significantly. This newly proposed Trinomial test is found to be more powerful than the Sign test. The improvement becomes more obvious when the number of ties is large.

2. The Trinomial Test

In spite of the fact that the Sign test is so simple and is easy to apply, it does not usually compare favourably with other non-parametric test procedures. An obvious reason is that the Sign test utilises relatively less information about the testing samples where we have significant number of zeros and tied observations. Greater the number of zero or tied observations, the more the loss of information due to a smaller sample size. It is with the aim of reducing this loss of information that we introduce the Trinomial test. Thus the Trinomial test is an attempt to utilise the zero or tied observations effectively so that the power of the Sign test can be improved. In this sense the Trinomial test is actually a modification of the original Sign test.

Consider a random sample of n pairs $(X_1\ Y_1), (X_2\ Y_2), \ldots, (X_n\ Y_n)$. Let $D_i = (X_i - Y_i)$ for $i = 1, 2, \ldots, n$. The random variable D_i can be partitioned into three different outcomes D_+, D_0 and D_-. The outcome D_+ is defined as the event when D_i is positive, D_0 is the event when D_i is zero and D_- is the event when D_i is negative. Let n_k denote the number of trials which resulted in outcome D_k; and $p_k = P(D_k)$; $k = +, 0, -$. We then have,

$$P[N_+ = n_+,\ N_0 = n_0,\ N_- = n_-] = \frac{n!}{n_+!\, n_0!\, n_-!}\, p_+^{n_+}\, p_0^{n_0}\, p_-^{n_-};$$

where $(n_+ + n_0 + n_-) = n$, and $(p_+ + p_0 + p_-) = 1$.

It is intuitive that N_+ and N_- should be negatively related. Their covariance is proved to be $\operatorname{cov}(N_+, N_-) = -\, np_+\, p_-$.

Suppose we want to test,

$$\begin{aligned} H_0&: \quad p_+ = p_- \\ H_1&: \quad p_+ > p_- \end{aligned} \tag{2.1}$$

The construction of a test statistic here involves observing, in a sample of n pairs of observations, the value n_d, a particular realisation of the random variable $(N_+ - N_-)$. The moments of this random variable are given by,

$$E(N_+ - N_-) = n(p_+ - p_-)$$

and since $\operatorname{cov}(N_+, N_-) = -\, np_+\, p_-$

$$V(N_+ - N_-) = np_+(1 - p_+) + np_-(1 - p_-) + 2np_+p_-$$

Therefore under H_0;

$$E(N_+ - N_-) = 0$$

$$V(N_+ - N_-) = 2np$$

where $p_+ = p_- = (1 - p_0)/2 = p$ (say).

Now the test statistic is given by

$$n_d = (n_+ - n_-)$$

where n_+ and n_- are the number of positive and negative differences observed in a random sample of n pairs and H_0 is rejected if $n_d > C_\alpha$, where C_α is the critical value.

The probability distribution of $N_d = (N_+ - N_-)$ is given by

$$P(N_D = n_d) = \sum_{k=0}^{\left[\frac{n - n_d}{2}\right]} \frac{n!}{(n_d + k)!\, k!\, (n - n_d - 2k)!} \left(\frac{1 - p_0}{2}\right)^{n_d - 2k} (p_0)^{n - n_d - 2k}$$

Hence, the critical values C_α can be calculated. As an illustration the case n = 10, $\alpha = 0.05$ is considered and the critical values are displayed in the following Table 1.

Table 1: Critical Values for the Proposed Trinomial Test

p_0	0	.1	.2	.3	.4	.5	.6	.7	.8	.9
C_α	6	5	5	4	4	4	3	3	2	2
$P(n_d>C_\alpha)$	.011	.034	.025	.044	.032	.021	.038	.021	.036	.008
$P(n_d \geq C_\alpha)$	.055	.064	.055	.093	.076	.057	.104	.071	.135	.059

Since p_0 is unknown in practice, we use the estimate (n_0/n) in place of p_0 to perform the test procedure.

3. Power function for the Trinomial test

The Trinomial test regards the number of zero differences if any exist as a random variable in itself. We therefore arrive at the following Trinomial distribution.

$$(N_+, N_0, N_-) \sim \text{Trinomial}\ (n, p_+, p_0, p_-).$$

Consider the following test,

$$H_0: p_+ = p_- \text{ vs } H_1: p_+ > p_-$$

let $p_+ - p_- = \delta > 0$. As noted in Section 2 the test statistic is given by

$$n_d = (n_+ - n_-)$$

observed in the sample of n-pairs.

When the sample size n is reasonably large, we can use the normal approximation to the binomial distribution. Hence we have for a level of significance α, the power of the trinomial test is given by

$$\text{Power (trinomial test)} = 1 - \Phi\left\{ \frac{Z_\alpha - \frac{\sqrt{n}}{\sqrt{1 - p_0}}\delta}{\sqrt{\frac{(1 - p_0 - \delta^2)}{1 - p_0}}} \right\}$$

(4.1)

where $\delta = p_+ - p_-$ and consequently $p_+ = (1 - p_0 + \delta)/2$ and $p_- = (1 - p_0 - \delta)/2$.

4. Power function of the Sign test in the presence of ties

In usual practice we ignore the number of zero differences when performing the Sign test. Since our objective here is to compare its performance with the proposed Trinomial test where the existence of the zero differences is considered while power calculation we should consider the number of zero differences here as well while computing the power of the Sign test.

Let $n^* = (n_+ \, n_0 \, n_-)$, we have

$$f(n^*) = \binom{n}{n_+ \, n_0 \, n_-} p_+^{n_+} p_0^{n_0} p_-^{n_-}.$$

It can be easily shown that

$$n_+ | n_0 \sim B\left(n - n_0, \frac{p_+}{1 - p_0}\right)$$

Now consider the Sign test for the hypothesis

$$H_0: \; p_+^1 = p_-^1 = \frac{1}{2} \text{ or } p_+^1 = p_-^1$$

$$\text{vs } H_1: \; p_+^1 - p_-^1 = \Delta > 0$$

where

$$p_+^1 = P(n_+ | n_0) = \left(\frac{p_+}{1 - p_0}\right)$$

$$p_-^1 = P(n_- | n_0) = \left(\frac{p_-}{1 - p_0}\right)$$

Under H_0,

$$E(n_+ | n_0) = (n - n_0) p_+^1 = \frac{1}{2}(n - n_0)$$

$$V(n_+ | n_0) = (n - n_0) p_+^1 (1 - p_+^1) = \frac{1}{4}(n - n_0).$$

Under H_1,

$$E(n_+ | n_0) = (n - n_0) \, p_+^1 = (n - n_0) \frac{1 + \Delta}{2}$$

$$V(n_+ | n_0) = (n - n_0) \frac{1 - \Delta^2}{4}$$

Hence assuming the sample size is large, for a level of significance α, the power of the Sign test is given by

$$\text{Power (sign test)} = 1 - \Phi\left[\frac{Z_\alpha - (\sqrt{n - n_0})\,\Delta}{\sqrt{(1 - \Delta^2)}}\right] \tag{5.1}$$

Now we can compare the power of the Trinomial test with that of the Sign test by varying

$$\Delta = p_+^1 - p_-^1 = \frac{p_+}{1 - p_0} - \frac{p_-}{1 - p_0} = \frac{\delta}{1 - p_0}.$$

For the case when there is no zero observation (difference) we will have $p_0 = 0$ and $\delta = \Delta$.

Therefore when $p_0 = 0$, $n_0 = 0$, it follows from (5.1) and (4.1).

$$\text{Power of the Trinomial test} = 1 - \Phi\left[\frac{Z_\alpha - \sqrt{n}\,\Delta}{\sqrt{(1 - \Delta^2)}}\right] = \text{Power of the Sign test}$$

Table 2: Power of Sign Test vs Trinomial Test

p^+	Sign Test	Trinomial Test ($p_0 = n_0/n$)
	($p_0 = 0.1$)	
0.450	0.019	0.022
0.500	0.039	0.044
0.550	0.076	0.084
0.600	0.135	0.146
0.650	0.222	0.238
0.700	0.352	0.372
0.750	0.517	0.540
0.800	0.708	0.730
0.850	0.896	0.912
	($p_0 = 0.2$)	
0.400	0.021	0.033
0.450	0.045	0.066
0.500	0.088	0.121
0.550	0.158	0.208
0.600	0.268	0.332
0.650	0.416	0.494
0.700	0.608	0.691
0.750	0.818	0.881
	($p_0 = 0.3$)	
0.350	0.020	0.036
0.400	0.044	0.075
0.450	0.090	0.142
0.500	0.170	0.250
0.550	0.291	0.400
0.600	0.468	0.595
0.650	0.694	0.807
	($p_0 = 0.5$)	
0.250	0.013	0.033
0.300	0.038	0.079
0.350	0.089	0.167
0.400	0.185	0.312
0.450	0.353	0.532
0.470	0.448	0.643
0.490	0.563	0.765

5. The power comparison of Sign test and the Trinomial test with small sample size

Associated with any statistical test procedure is the natural question of how to assess its performance in detecting the correct alternative hypothesis value. Here we assess the performance by computing the power of the two tests. Such a comparison would usually depend on, (i) the sample size n, (ii) the value of the alternative, (iii) the chosen significance level α.

When the sample size is large one can use the binomial approximation and employ the usual Sign test, even in the presence of considerable number of ties. But in the case of small sample, for example a sample of size $n = 10$, where we have say 4 ties, the usual Sign test is not of much help. However, the proposed Trinomial test is found to be useful in such a situation. We compare the power of the Trinomial test against the Sign test based on 100,000 simulated samples of size 10. Here the value of p_0 is estimated by the ratio (n_0/n) and a significance level $\alpha = 0.05$ is considered. The simulation reuslts are displayed in the Table 2.

It can be seen very clearly from Table 2 that the performance of the Trinomial test is much better than that of the Sign test which usually ignores the presence of ties.

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