Douglas Wei Jing Allwood (A0183939L)

ST4234

Q1

Note: I will use $\vec{\ }$ to denote vectors, e.g. $\vec{\theta} = (\theta_1, \dots, \theta_J)$

(a) $$p(\vec{\theta}, \mu, \tau \mid \vec{y}, \vec{\sigma}) \propto p(\vec{y} \mid \vec{\theta}, \vec{\sigma}) \cdot p(\vec{\theta} \mid \mu, \tau) \cdot p(\mu, \tau)$$

$$\propto \prod_{j=1}^{J} \sigma_j^{-1} \exp\left\{-\frac{(y_j-\theta_j)^2}{2\sigma_j^2}\right\} \tau^{-1} \exp\left\{-\frac{(\theta_j-\mu)^2}{2\tau^2}\right\}$$

$$= \tau^{-J} \prod_{j=1}^{J} \sigma_j^{-1} \exp\left\{-\left[\frac{(y_j-\theta_j)^2}{2\sigma_j^2} + \frac{(\theta_j-\mu)^2}{2\tau^2}\right]\right\}$$

(b) $$p(\theta_j \mid \mu, \tau, \vec{y}, \vec{\sigma}) \propto \tau^{-1} \sigma_j^{-1} \exp\left\{-\left[\frac{(y_j-\theta_j)^2}{2\sigma_j^2} + \frac{(\theta_j-\mu)^2}{2\tau^2}\right]\right\}$$

$$\propto \exp\left\{-\left[\frac{y_j^2 - 2\theta_j y_j + \theta_j^2}{2\sigma_j^2} + \frac{\theta_j^2 - 2\mu\theta_j + \mu^2}{2\tau^2}\right]\right\}$$

$$\propto \exp\left\{-\left[\frac{\tau^2 y_j^2 - 2\tau^2\theta_j y_j + \tau^2\theta_j^2 + \sigma_j^2\theta_j^2 - 2\mu\sigma_j^2\theta_j + \sigma_j^2\mu^2}{2\sigma_j^2\tau^2}\right]\right\}$$

$$\propto \exp\left\{-\left[\frac{(\tau^2+\sigma_j)^2\theta_j^2 - 2(\tau^2 y_j + \mu\sigma_j^2)\theta_j}{2\sigma_j^2\tau^2}\right]\right\}$$

$$\propto \exp\left\{-\left[\frac{\theta_j^2 - \frac{2(\tau^2 y_j + \mu\sigma_j^2)}{\tau^2+\sigma_j^2}\theta_j}{2\frac{\sigma_j^2\tau^2}{\tau^2+\sigma_j^2}}\right]\right\}$$

$$\propto \exp\left\{-\frac{\left(\theta_j - \frac{\tau^2 y_j + \mu\sigma_j^2}{\tau^2+\sigma_j^2}\right)^2}{2\frac{\sigma_j^2\tau^2}{\tau^2+\sigma_j^2}}\right\}$$

$$\sim N\left(\frac{\tau^2 y_j + \mu\sigma_j^2}{\tau^2+\sigma_j^2}, \frac{\sigma_j^2\tau^2}{\tau^2+\sigma_j^2}\right)$$

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$$P(\theta_j \mid \mu, \tau, \vec{y}, \vec{\sigma}) \sim N\left(\frac{\tau^2 y_j + \mu\sigma_j^2}{\tau^2+\sigma_j^2}, \frac{\sigma_j^2\tau^2}{\tau^2+\sigma_j^2}\right)$$

Let $$\hat{\theta}_j = \frac{\frac{y_j}{\sigma_j^2} + \frac{\mu}{\tau^2}}{\frac{1}{\sigma_j^2}+\frac{1}{\tau^2}} = \frac{\tau^2 y_j + \mu\sigma_j^2}{\sigma_j^2\tau^2} \times \frac{\tau^2\sigma_j^2}{\tau^2+\sigma_j^2}$$

$$= \frac{\tau^2 y_j + \mu\sigma_j^2}{\tau^2+\sigma_j^2}$$

Let $$V_j = \frac{1}{\left(\frac{1}{\sigma_j^2}+\frac{1}{\tau^2}\right)} = \frac{1}{\left(\frac{\tau^2+\sigma_j^2}{\tau^2\sigma_j^2}\right)} = \frac{\sigma_j^2\tau^2}{\tau^2+\sigma_j^2}$$

$$\therefore P(\theta_j \mid \mu, \tau, \vec{y}, \vec{\sigma}) \sim N(\hat{\theta}_j, V_j)$$

$$P(\mu, \tau \mid \vec{y}, \vec{\sigma}) \propto \int P(\theta_j, \mu, \tau \mid \vec{y}, \vec{\sigma})\, d\theta_j$$

First find $P(\theta_j, \mu, \tau \mid \vec{y}, \vec{\sigma})$:

$$P(\theta_j, \mu, \tau \mid \vec{y}, \vec{\sigma}) \propto \tau^{-1} \exp\left\{-\frac{1}{2}\left[\frac{(y_j-\theta_j)^2}{\sigma_j^2} + \frac{(\theta_j-\mu)^2}{\tau^2}\right]\right\}$$

$$= \tau^{-1} \exp\left\{-\frac{1}{2}\left[\frac{y_j^2 - 2\theta_j y_j + \theta_j^2}{\sigma_j^2} + \frac{\theta_j^2 - 2\mu\theta_j + \mu^2}{\tau^2}\right]\right\}$$

$$= \tau^{-1} \exp\left\{-\frac{1}{2}\left[\frac{(\tau^2+\sigma_j^2)\theta_j^2 - \frac{2(y_j\tau^2+\mu\sigma_j^2)}{\tau^2+\sigma_j^2}\theta_j + \frac{\mu^2\sigma_j^2+\tau^2 y_j^2}{\tau^2+\sigma_j^2}}{\left(\frac{\sigma_j^2\tau^2}{\tau^2+\sigma_j^2}\right)}\right]\right\}$$

$$= \tau^{-1} \exp\left\{-\frac{1}{2}\left[\frac{\left(\theta_j - \frac{y_j\tau^2+\mu\sigma_j^2}{\tau^2+\sigma_j^2}\right)^2}{\left(\frac{\sigma_j^2\tau^2}{\tau^2+\sigma_j^2}\right)}\right]\right\} \exp\left\{-\frac{1}{2}\left[\frac{\mu^2\sigma_j^2+\tau^2 y_j^2}{\tau^2+\sigma_j^2} - \left(\frac{y_j\tau^2+\mu\sigma_j^2}{\tau^2+\sigma_j^2}\right)^2\right]\right\}$$

$$\int P(\theta_j, \mu, \tau \mid \vec{y}, \vec{\sigma})\, d\theta_j \propto \tau^{-1}\left(\frac{\sigma_j^2\tau^2}{\tau^2+\sigma_j^2}\right)^{\frac{1}{2}} \exp\left\{-\frac{1}{2}\left[\frac{\mu^2\sigma_j^2+\tau^2 y_j^2}{\tau^2+\sigma_j^2} - \left(\frac{y_j\tau^2+\mu\sigma_j^2}{\tau^2+\sigma_j^2}\right)^2\right]\right\}$$

In integrating over this, the result is sqrt(var) $\forall j$

$$\therefore P(\mu, \tau \mid \vec{y}, \vec{\sigma}) \propto \prod_{j=1}^{J} \tau^{-1}\left(\frac{\sigma_j^2\tau^2}{\tau^2+\sigma_j^2}\right)^{\frac{1}{2}} \exp\left\{-\frac{1}{2}\left[\frac{\mu^2\sigma_j^2+\tau^2 y_j^2}{\tau^2+\sigma_j^2} - \left(\frac{y_j\tau^2+\mu\sigma_j^2}{\tau^2+\sigma_j^2}\right)^2\right]\right\}$$

$$\propto \prod_{j=1}^{J} \phi\left(y_j \mid \mu, \sqrt{\sigma_j^2+\tau^2}\right)$$

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Q1(b) Part 2

$$p(\mu,\tau|\vec{y},\vec{\sigma}) \propto \int\int\int\int\int\int\int\int p(\vec{\theta},\mu,\tau|\vec{y},\vec{\sigma})\, d\theta_1\, d\theta_2\, d\theta_3\, d\theta_4\, d\theta_5\, d\theta_6\, d\theta_7\, d\theta_8$$

I will shorten the 8 integrals over θ_1 to θ_8 to just an integral over $\vec{\theta}$

$$p(\mu,\tau|\vec{y},\vec{\sigma}) \propto \int p(\vec{\theta},\mu,\tau|\vec{y},\vec{\sigma})\, d\vec{\theta}$$

$$\propto \int p(\vec{y}|\vec{\theta},\vec{\sigma}).\, p(\vec{\theta}|\mu,\tau).\, p(\mu,\tau)\, d\vec{\theta}$$

$$\propto \int \prod_{j=1}^{J} \sigma_j^{-1}\tau^{-1} \exp\left\{-\frac{1}{2}\left[\frac{(y_j-\theta_j)^2}{\sigma_j^2} + \frac{(\theta_j-\mu)^2}{\tau^2}\right]\right\} d\vec{\theta}$$

$$\propto \int \prod_{j=1}^{J} \sigma_j \tau^{-1} \exp\left\{-\frac{1}{2}\left[\frac{\tau^2 y_j^2 - 2\tau^2 y_j\theta_j + \tau^2\theta_j^2 + \sigma_j^2\theta_j^2 - 2\sigma_j^2\theta_j\mu + \sigma_j^2\mu^2}{\sigma_j^2\tau^2}\right]\right\} d\vec{\theta}$$

$$\propto \int \prod_{j=1} \sigma_j \tau^{-1} \exp\left\{-\frac{1}{2}\left[\frac{(\tau^2+\sigma_j^2)\theta_j^2 - 2(\tau^2 y_j + \sigma_j^2\mu)\theta_j + \tau^2 y_j^2 + \sigma_j^2\mu^2}{\sigma_j^2\tau^2}\right]\right\} d\vec{\theta}$$

$$= \int \prod_{j=1}^{J} \sigma_j \tau^{-1} \exp\left\{-\frac{1}{2}\left[\frac{\theta_j^2 - 2\frac{(\tau^2 y_j + \sigma_j^2\mu)}{\tau^2+\sigma_j^2}\theta_j + \frac{\tau^2 y_j^2 + \sigma_j^2\mu^2}{\tau^2+\sigma_j^2}}{\left(\frac{\sigma_j^2\tau^2}{\tau^2+\sigma_j^2}\right)}\right]\right\} d\vec{\theta}$$

$$= \int \prod_{j=1}^{J} \sigma_j \tau^{-1} \exp\left\{-\frac{1}{2}\left[\frac{\left(\theta_j - \frac{\tau^2 y_j + \sigma_j^2\mu}{\tau^2+\sigma_j^2}\right)^2 - \left(\frac{\tau^2 y_j + \sigma_j^2\mu}{\tau^2+\sigma_j^2}\right)^2 + \frac{\tau^2 y_j^2 + \sigma_j^2\mu^2}{\tau^2+\sigma_j^2}}{\left(\frac{\sigma_j^2\tau^2}{\tau^2+\sigma_j^2}\right)}\right]\right\} d\vec{\theta}$$

$$= \int \prod_{j=1}^{J} \sigma_j \tau^{-1} \exp\left\{-\frac{1}{2}\left[\frac{\left(\theta_j - \frac{\tau^2 y_j + \sigma_j^2\mu}{\tau^2+\sigma_j^2}\right)^2}{\left(\frac{\sigma_j^2\tau^2}{\tau^2+\sigma_j^2}\right)}\right]\right\}.\exp\left\{-\frac{1}{2}\left[\frac{\frac{\tau^2 y_j^2 + \sigma_j^2\mu^2}{\tau^2+\sigma_j^2} - \left(\frac{\tau^2 y_j + \sigma_j^2\mu}{\tau^2+\sigma_j^2}\right)^2}{\left(\frac{\sigma_j^2\tau^2}{\tau^2+\sigma_j^2}\right)}\right]\right\} d\vec{\theta}$$

$$\propto \int \prod_{j=1}^{J} \frac{1}{\sqrt{2\pi\sigma_j^2\tau^2}} \exp\left\{-\frac{1}{2}\left[\frac{(\theta_j-\hat{\theta}_j)^2}{V_j}\right]\right\} \exp\left\{-\frac{1}{2}\left[\frac{\frac{\tau^2 y_j^2 + \mu^2\sigma_j^2}{\tau_j^2+\sigma_j^2} - \left(\frac{\tau^2 y_j + \sigma_j^2\mu}{\tau^2+\sigma_j^2}\right)^2}{\left(\frac{\sigma_j^2\tau^2}{\tau^2+\sigma_j^2}\right)}\right]\right\} d\vec{\theta}$$

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Q1(b) part 3

$$p(\mu, \tau \mid \vec{y}, \vec{\sigma})$$

$$\propto \left[\prod_{j=1}^{J} \left[\frac{1}{\sqrt{2\pi(\tau^2+\sigma_j^2)}} \exp\left\{ -\frac{1}{2} \left[\frac{\frac{\tau^2 y_j^2 + \mu^2 \sigma_j^2}{\tau_j^2 + \sigma_j^2} - \frac{\tau^4 y_j^2 + 2\tau^2 y_j \sigma_j^2 \mu + \sigma_j^4 \mu^2}{(\tau_j^2+\sigma_j^2)^2}}{\left(\frac{\sigma_j^2 \tau^2}{\tau^2+\sigma_j^2}\right)} \right] \right\} \right. \right.$$

$$\left. \times \int \frac{1}{\sqrt{2\pi\left(\frac{\tau^2\sigma_j^2}{\tau^2+\sigma_j^2}\right)}} \exp\left\{ -\frac{1}{2} \frac{(\theta_j - \hat{\theta}_j)^2}{V_j} \right\} d\theta \right]$$

$$\propto \prod_{j=1}^{J} \frac{1}{\sqrt{2\pi(\tau^2+\sigma_j^2)}} \exp\left\{ -\frac{1}{2} \left[\frac{\tau^2\mu^2\sigma_j^2 + \sigma_j^2\tau^2 y_j^2 - 2\tau^2 y_j \mu \sigma_j^2}{\sigma_j^2(\tau^2+\sigma_j^2)\tau^2} \right] \right\}$$

$$\propto \prod_{j=1}^{J} \frac{1}{\sqrt{2\pi(\tau^2+\sigma_j^2)}} \exp\left\{ -\frac{1}{2} \left[\frac{\mu^2 + y_j^2 - 2\mu y_j}{\tau^2+\sigma_j^2} \right] \right\}$$

$$\propto \prod_{j=1}^{J} \frac{1}{\sqrt{2\pi(\tau^2+\sigma_j^2)}} \exp\left\{ -\frac{1}{2} \frac{(y_j-\mu)^2}{\tau^2+\sigma_j^2} \right\}$$

$$\propto \prod_{j=1}^{J} \phi\left(y_j \mid \mu, \sqrt{\tau^2+\sigma_j^2}\right)$$

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Q2

(a) $p(\vec{\theta}, \mu, \tau^2, \sigma^2 | \vec{y}) \propto p(\vec{y}|\vec{\theta}, \sigma^2, \mu, \tau^2) \cdot p(\vec{\theta}, \sigma^2, \mu, \tau^2)$

$$\propto p(\vec{y}|\vec{\theta}, \sigma^2) \cdot p(\vec{\theta}|\mu, \tau^2) \cdot p(\sigma^2) \cdot p(\mu) \cdot p(\tau^2)$$

$$\propto (\sigma^2)^{-\frac{a_2}{2}-1} e^{-\frac{b_2}{2(\sigma^2)}} \cdot (\tau^2)^{-\frac{a_1}{2}-1} e^{-\frac{b_1}{2(\tau^2)}} \cdot \exp\left\{-\frac{(\mu-\mu_0)^2}{2\sigma_0^2}\right\}$$
$$\times \prod_{j=1}^{J}\left[\tau^{-1}\exp\left\{-\frac{(\theta_j-\mu)^2}{2\tau^2}\right\}\right] \times \prod_{j=1}^{J}\prod_{i=1}^{n_j}\left[\sigma^{-1}\exp\left\{-\frac{(y_{i,j}-\theta_j)^2}{2\sigma^2}\right\}\right]$$

$$\propto (\sigma^2)^{-\frac{a_2}{2}-1}(\tau^2)^{-\frac{a_1}{2}-1}\exp\left\{-\left(\frac{b_2}{2\sigma^2}+\frac{b_1}{2\tau^2}+\frac{(\mu-\mu_0)^2}{2\sigma_0^2}\right)\right\}$$
$$\times \prod_{j=1}^{J}\left[\tau^{-1}\exp\left\{\frac{(\theta_j-\mu)^2}{2\tau^2}\right\}\prod_{i=1}^{n_j}\left[\sigma^{-1}\exp\left\{-\frac{(y_{i,j}-\theta_j)^2}{2\sigma^2}\right\}\right]\right]$$

For clarity:

$$p(\vec{\theta}, \mu, \tau^2, \sigma^2|\vec{y}) \propto (\sigma^2)^{-\frac{a_2}{2}-1}(\tau^2)^{-\frac{a_1}{2}-1}\exp\left\{-\left(\frac{b_2}{2\sigma^2}+\frac{b_1}{2\tau^2}+\frac{(\mu-\mu_0)^2}{2\sigma_0^2}\right)\right\}$$
$$\times\prod_{j=1}^{J}\left[\tau^{-1}\exp\left\{-\frac{(\theta_j-\mu)^2}{2\tau^2}\right\}\prod_{i=1}^{n_j}\sigma^{-1}\exp\left\{-\frac{(y_{i,j}-\theta_j)^2}{2\sigma^2}\right\}\right]$$

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Q2(b) 1st part

(b) $P(\theta_j \mid \mu, \tau^2, \sigma^2, \vec{y}) \propto \exp\left\{-\frac{(\theta_j-\mu)^2}{2\tau^2}\right\} \prod_{i=1}^{n_j} \exp\left\{-\frac{(y_{i,j}-\theta_j)^2}{2\sigma^2}\right\}$

$$\propto \exp\left\{-\frac{(\theta_j-\mu)^2}{2\tau^2}\right\} \exp\left\{-\frac{1}{2\sigma^2}\sum_{i=1}^{n_j}(y_{i,j}-\theta_j)^2\right\}$$

$$= \exp\left\{-\frac{1}{2}\left[\frac{(\theta_j-\mu)^2}{\tau^2} + \frac{\sum_{i=1}^{n_j}(y_{i,j}-\theta_j)^2}{\sigma^2}\right]\right\}$$

$$= \exp\left\{-\frac{1}{2}\left[\frac{\theta_j^2 - 2\mu\theta_j + \mu^2}{\tau^2} + \frac{\sum_{i=1}^{n_j}\left[y_{i,j}^2 - 2\theta_j y_{i,j} + \theta_j^2\right]}{\sigma^2}\right]\right\}$$

$$\propto \exp\left\{-\frac{1}{2}\left[\frac{\theta_j^2 - 2\mu\theta_j}{\tau^2} + \frac{n_j\theta_j^2 - 2\theta_j n_j \bar{y}_j}{\sigma^2}\right]\right\}$$

$$\propto \exp\left\{-\frac{1}{2}\left[\frac{\sigma^2\theta_j^2 - 2\mu\sigma^2\theta_j + n_j\tau^2\theta_j^2 - 2\theta_j n_j \bar{y}_j \tau^2}{\tau^2\sigma^2}\right]\right\}$$

$$\propto \exp\left\{-\frac{1}{2}\left[\frac{(\sigma^2 + n_j\tau^2)\theta_j^2 - 2(\mu\sigma^2 + n_j\bar{y}_j\tau^2)\theta_j}{\tau^2\sigma^2}\right]\right\}$$

$$\propto \exp\left\{-\frac{1}{2}\left[\frac{\left(\theta_j - \frac{\mu\sigma^2 + n_j\bar{y}_j\tau^2}{\sigma^2 + n_j\tau^2}\right)^2}{\left(\frac{\tau^2\sigma^2}{\sigma^2 + n_j\tau^2}\right)}\right]\right\}$$

$$\therefore \theta_j \mid \mu, \tau^2, \sigma^2, \vec{y} \sim N\left(\frac{n_j\tau^2}{n_j\tau^2 + \sigma^2}\bar{y}_j + \frac{\sigma^2}{n_j\tau^2 + \sigma^2}\mu, \frac{\tau^2\sigma^2}{\sigma^2 + n_j\tau^2}\right)$$

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Q2(b). 2nd part

$$p(\mu \mid \vec{\theta}, \tau^2, \sigma^2, \vec{y}) \propto \exp\left\{-\frac{(\mu-\mu_0)^2}{2\sigma_0^2}\right\} \prod_{j=1}^{J} \exp\left\{-\frac{(\theta_j-\mu)^2}{2\tau^2}\right\}$$

$$\propto \exp\left\{-\frac{1}{2}\left[\frac{\mu^2-2\mu\mu_0}{\sigma_0^2} + \frac{\sum_{j=1}^{J}(\mu^2-2\mu\theta_j)}{\tau^2}\right\}\right.$$

$$\propto \exp\left\{-\frac{1}{2}\left[\frac{\tau^2\mu^2-2\mu\mu_0\tau^2+\sigma_0^2J\mu^2-2\mu\sigma_0^2\bar{\theta}J}{\sigma_0\tau^2}\right]\right\}$$

$$\propto \exp\left\{-\frac{1}{2}\left[\frac{(\tau^2+\sigma_0^2J)\mu^2-2(\mu_0\tau^2+\sigma_0^2\bar{\theta}J)\mu}{\sigma_0\tau^2}\right]\right\}$$

$$\propto \exp\left\{-\frac{1}{2}\left[\frac{\left(\mu-\frac{\mu_0\tau^2+\sigma_0^2\bar{\theta}J}{\tau^2+\sigma_0^2J}\right)^2}{\left(\frac{\sigma_0\tau^2}{\tau^2+\sigma_0^2J}\right)}\right]\right\}$$

$$\therefore \mu \mid \vec{\theta}, \tau^2, \sigma^2, \vec{y} \sim N\left(\frac{\mu_0\tau^2+\sigma_0^2\bar{\theta}J}{\tau^2+\sigma_0^2J}, \frac{\sigma_0\tau^2}{\tau^2+\sigma_0^2J}\right)$$

$$p(\tau^2 \mid \vec{\theta}, \mu, \sigma^2, \vec{y}) \propto (\tau^2)^{-\frac{a_1}{2}-1} \exp\left\{-\frac{b_1}{2\tau^2}\right\} \times \prod_{j=1}^{J} \tau^{-1}\exp\left\{-\frac{(\theta_j-\mu)^2}{2\tau^2}\right\}$$

$$\propto (\tau^2)^{-\left(\frac{a_1}{2}+\frac{J}{2}\right)-1} \exp\left\{-\frac{1}{2}\left[\frac{b_1}{\tau^2} + \frac{\sum_{j=1}^{J}(\theta_j-\mu)^2}{\tau^2}\right]\right.$$

$$\propto (\tau^2)^{-\left(\frac{a_1}{2}+\frac{J}{2}\right)-1} \exp\left\{-\frac{1}{2}\left[\frac{b_1+\sum_{j=1}^{J}(\theta_j-\mu)^2}{\tau^2}\right]\right.$$

$$\therefore \tau^2 \mid \vec{\theta}, \mu, \sigma^2, \vec{y} \sim \text{Inv-Gamma}\left(\frac{a_1+J}{2}, \frac{b_1+\sum_{j=1}^{J}(\theta_j-\mu)^2}{2}\right)$$

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Q2(b) 3rd part

$$p(\sigma^2|\vec{\theta},\mu,\tau^2,\vec{y}) \propto (\sigma^2)^{-\frac{a_2}{2}-1} \exp\left\{-\frac{b_2}{2\sigma^2}\right\} \prod_{j=1}^{J}\prod_{i=1}^{n_j} \sigma^{-1}\exp\left\{-\frac{(y_{i,j}-\theta_j)^2}{2\sigma^2}\right\}$$

$$\propto (\sigma^2)^{-\left(\frac{a_2}{2}+\frac{\sum_{j=1}^{J} n_j}{2}\right)-1} \exp\left\{-\frac{1}{2}\left[\frac{b_2}{\sigma^2}+\frac{\sum_{j=1}^{J}\sum_{i=1}^{n_j}(y_{i,j}-\theta_j)^2}{\sigma^2}\right]\right\}$$

$$\propto (\sigma^2)^{-\left(\frac{a_2+\sum_{j=1}^{J} n_j}{2}\right)-1} \exp\left\{-\left[\frac{b_2+\sum_{j=1}^{J}\sum_{i=1}^{n_j}(y_{i,j}-\theta_j)^2}{2}\right](\sigma^2)^{-1}\right\}$$

$$\therefore\ \sigma^2|\vec{\theta},\mu,\tau^2,\vec{y} \sim \text{Inv-gamma}\left(\frac{a_2+\sum_{j=1}^{J} n_j}{2}, \frac{b_2+\sum_{j=1}^{J}\sum_{i=1}^{n_j}(y_{i,j}-\theta_j)^2}{2}\right)$$