Lyapunov Exponent Determination in the Stadium Billiards Problem

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Abstract

The motion of a billiard ball on an ideal stadium shaped table is analyzed. In order to investigate the chaotic nature of the system described in ref. 1, the system is evolved in time with varying initial conditions in order to obtain the so-called Lyapunov exponent. For a table with an elongation factor $\alpha = 0.02$, the Lyapunov exponent was found to be $0.25(6)s^{-1}$. Interesting features of the dynamics and the physical principles involved are also discussed.

Introduction

The formulation of this problem is straightforward. A billiard ball is on an ideal stadium-shaped table and has some initial velocity and heading. Upon colliding with one of the walls, it undergoes specular reflection. Mathematically that means that the initial angle that the velocity vector makes with the vector normal to the wall at the collision point is equal to the final, reflected angle between the final velocity vector and the normal vector. Yet another description of this behavior, which was used herein, is that the component of the velocity which is perpendicular to the wall before the collision is reversed and the parallel component is unchanged.[1]

At first glance this problem may seem uninteresting, and indeed it does have a somewhat trivial character for highly symmetrical geometries of the table such as circles and rectangles. The interesting feature is that for some table shapes the system is chaotic. One such shape is the so-called stadium shape. Depicted in figure 1 is such a shape; it consists of a circle which has had a rectangular region of height 2r and width $2\alpha r$ placed through its center, where r is the radius of the circle.

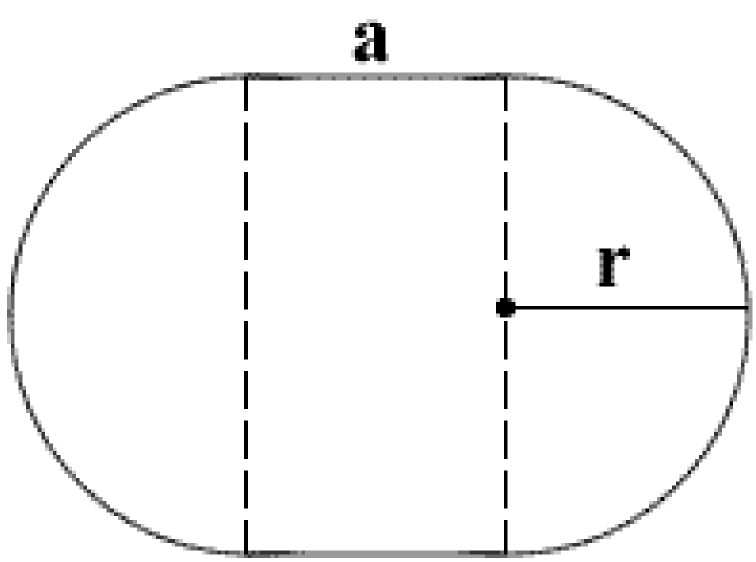

Figure 1

Stadium shape consisting of two halves of a circle of radius r on the ends, and a rectangular region of width $a = 2\alpha r$ and height 2r.[2]

Physical chaos is simply defined by systems that are extremely sensitive to small changes in initial conditions. It so happens that this system is chaotic for even very small values of α. This means that it is likely that any real system such as this would be difficult if not impossible to model accurately. This is because it is extremely difficult to produce a perfectly circular table and the range for which the initial conditions of the system (i.e. position, velocity, heading) are chaotic may well be within the measurement uncertainty of the instruments used to determine them. Moreover since this model neglects non-idealistic effects such as friction and air resistance, there is little hope to be able to reliably predict a real, physical system such as this.

Fortunately it is not completely fruitless to model the physics of the stadium billiard problem. It is possible to determine just how chaotic a system is though a Lyapunov exponent.

$$\textit{separation} \approx \exp(\lambda t) \tag{1}$$

This can give information as to which parameters of the system produce chaotic behavior and how rapidly trajectories with slightly different initial conditions will diverge. As in equation 1, the separation of two trajectories for a chaotic system increases exponentially, where λ is the Lyapunov exponent. Also of note is that this particular type of modeling can be useful to thermodynamics and statistical mechanics when modeling, for instance, the entropy of a gas in a container.[1]

Results and Discussion

Simulations were conducted with the following parameters:

Parameter	Value
v_0	1
x_0	0.1 (varied by increments of 10^{-5})
y_0	-0.1
θ_0	56.5°
r	1
α	0.02

Table 1: Simulation Parameters

These were chosen simply for convenience, and the simulation program is designed such that any of these parameters can be modified very easily in order to conduct different simulations. All subsequent results used the parameters found in Table 1.

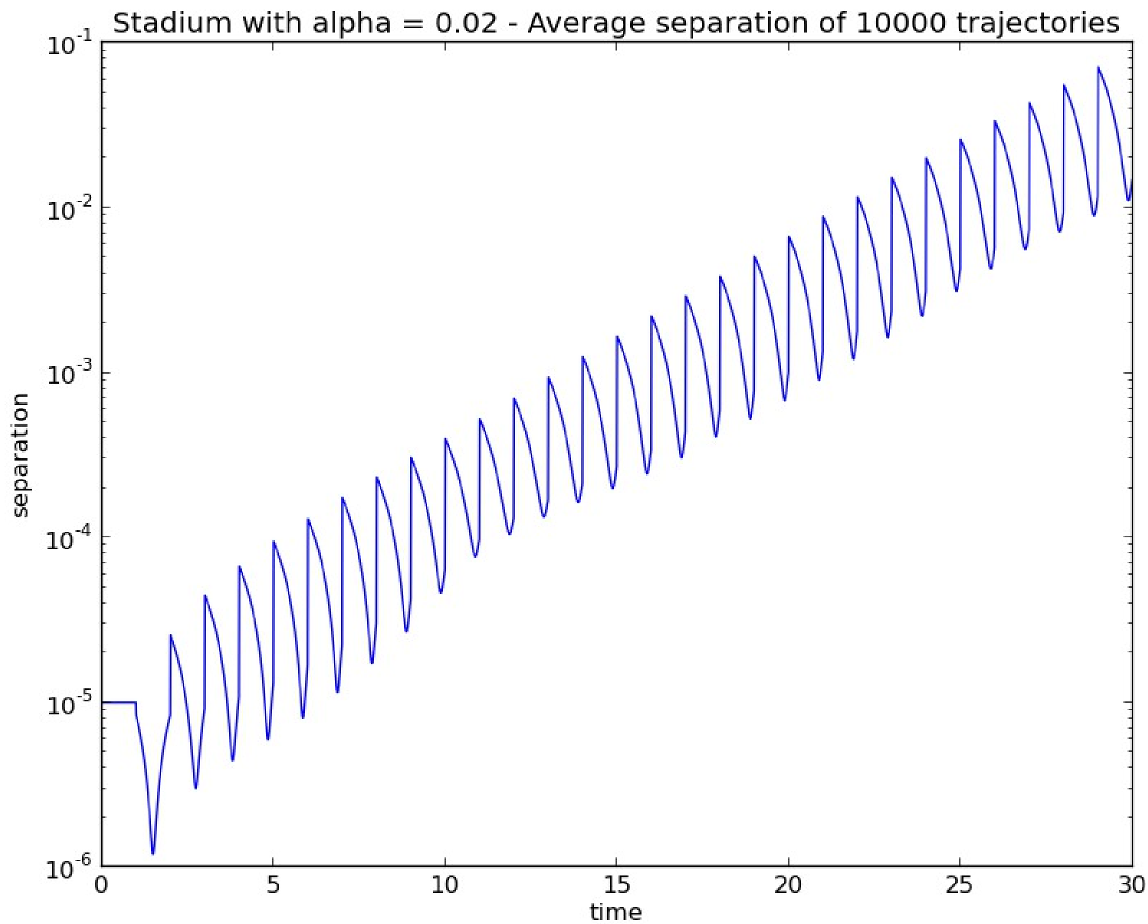

Figure 2

Separation of trajectories for a billiard ball on a stadium-shaped table with α=0.02. Averaged over 10,000 trajectories, each mutually 10^{-5} units apart.

Figure 2 is a plot of the average separation between 10,000 trajectories that are 10^{-5} units apart. Note that the separation starts at 10^{-5} as should be expected and then diverges exponentially (note the logarithmic y-axis). There is a sharp dip approximately every 1 time unit; this corresponds to a point where the billiard balls are both colliding with the walls. This simulation was done for 30 collisions. A Lyapunov exponent $\lambda = 0.25(6)\mathrm{s}^{-1}$ was obtained from this graph.

Figure 3 serves to illustrate the utility of averaging over many trajectories in order to obtain the Lyapunov exponent. Note the irregularities in the plot which are largely smoothed out when averaging over many trajectories.

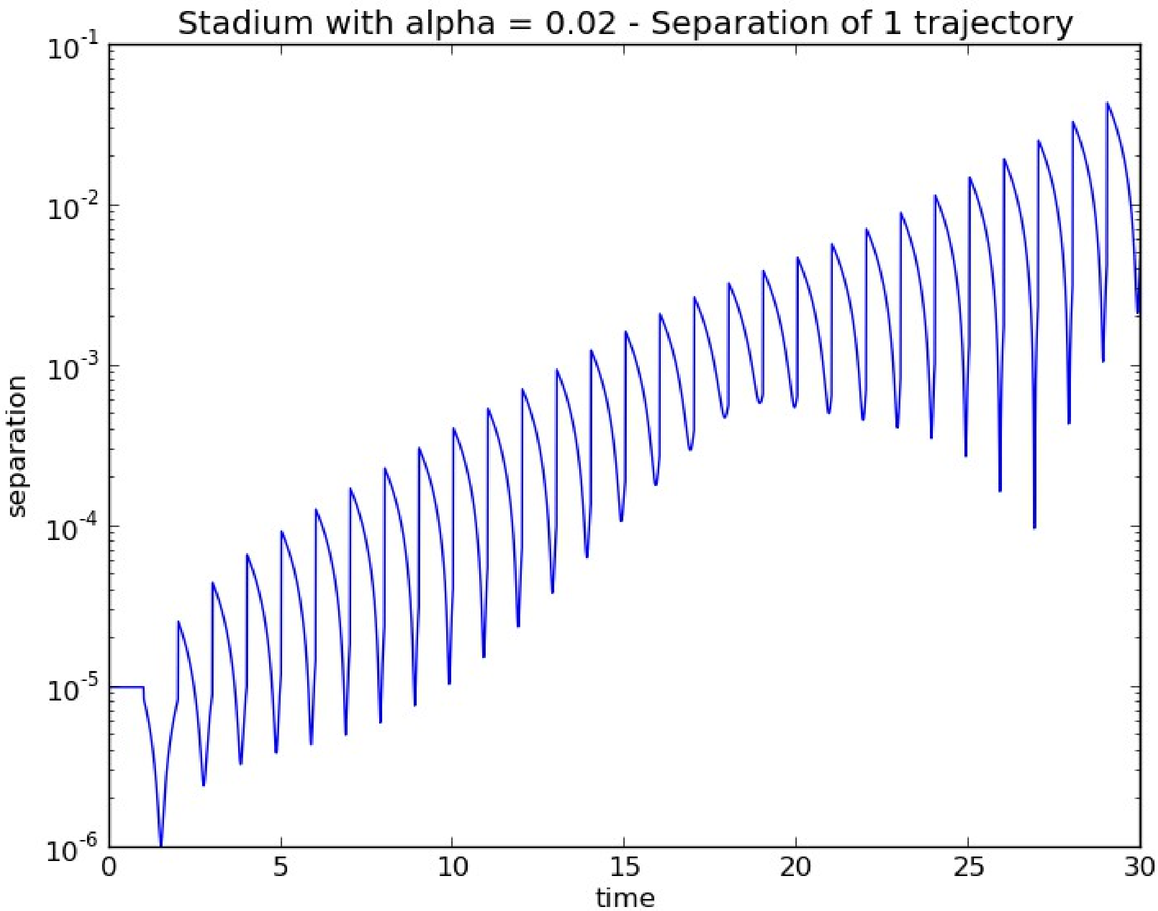

Figure 3
A single trajectory used in the average taken to obtain the plot in figure 2.
Several irregular sections are apparent.

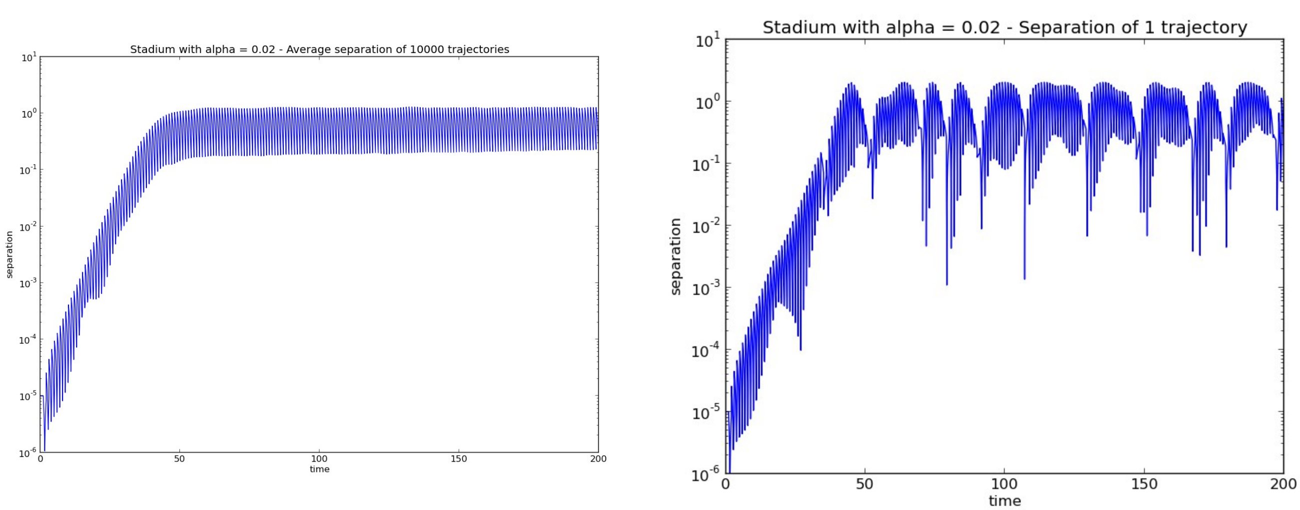

Figure 4
The simulation in figure 1 allowed to run for 200 collisions (left) and a single trajectory (right).

The irregular behavior of a single trajectory can be seen much more obviously for simulations that run for more collisions such as in figure 4. For the average, there is a smooth exponential increase in the separation ranging from the initial separation 10^{-5} up to the maximum possible separation which is obviously limited by the size of the table. For the single trajectory, roughly the same behavior is exhibited, but there is clearly some erratic behavior as well.

A couple of special cases were briefly investigated. In particular, in the large alpha limit, the system approximates a rectangular table and the behavior seems to be non-chaotic. This means that the separation of two trajectories in time remains roughly on the order of the initial separation. Also, there are a variety of trivial initial positions which result in very simple behavior. Such conditions include an initial velocity along one of the axes of symmetry of the table such that parallel component upon collision equals 0 and perpendicular component just flips direction on collisions; this results in the trajectory of the billiard ball being restricted to a line.

Conclusion

A numerical simulation was conducted to examine the dynamics of a billiard ball on an ideal stadium-shaped table. A Lyapunov exponent of $\lambda = 0.25(6)s^{-1}$ was found for a table with $\alpha = 0.02$. A robust application was developed for performing numerical simulations of this type which has proven to function properly even in the extrema of the parameters of the system. A potential improvement upon this technique would be to exploit the embarrassingly parallel nature of the structure of the problem in the averaging regime, making it a strong candidate for heterogeneous computing methods – in particular implementation on a general-purpose graphics processing unit (GPGPU).

References

1. *Computational Physics* 2nd Edition, N. Giordano, H. Nakanishi, 2006.
2. Stadium image, http://www.calculatorsoup.com/calculators/geometry-plane/geometricshapes.php