

McGILL UNIVERSITY

MATRIX COMPUTATIONS

COMP 540

# Assignment 5

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November 21, 2015

# Question 1

## Part 1

Letting  $a_1(t) = 1$  and  $a_2(t) = t$ , the single precision condition number of  $A$  is  $\kappa_2(A) = 8.6798010\text{E}+02$ . The norm of the residual of the least-square solution by the QR factorization method is  $\|b - Ax_{qr}\|_2 = 8.0615806\text{E}-05$ .

The condition number of the tranpose of  $A$  with itself is  $\kappa_2(A^T A) = 7.5613331\text{E}+05$ . The norm of the residual of the least-square solution by the normal equation is  $\|b - Ax_{ne}\|_2 = 1.1265145\text{E}-04$ .

Using double precision for the  $A$  and the  $b$  matrix least allows to compare the previous solutions with the true solution. All the solutions have been compiled in Table 1.

|       | Single Precision QR | Single Precision NE | Double Precision QR |
|-------|---------------------|---------------------|---------------------|
| $x_1$ | 1.0167466           | 1.028004            | 1.016737195131018   |
| $x_2$ | 0.9849972           | 0.9748591           | 0.985005503165425   |

Table 1: Results for  $a_1(t) = 1$  and  $a_2(t) = t$

The single precision QR factorization is expected to give a better least-square solution than solving the normal equations. The condition number of  $A^T A$  is three orders of magnitude bigger than the condition number of  $A$ . The QR factorization allows to retrieve up to 4 digits of accuracy, whereas the normal equation only retrieves 2 digits of the real solution.

## Part 2

Now, let  $a_1(t) = 1$  and  $a_2(t) = (t - 1.11) * 100$ . The single precision condition number of  $A$  is  $\kappa_2(A) = 4.5301147$ . The norm of the residual of the least-square solution by the QR factorization method is  $\|b - Ax_{qr}\|_2 = 8.0615806\text{E}-05$ .

The condition number of the tranpose of  $A$  with itself is  $\kappa_2(A^T A) = 20.5219326$ . The norm of the residual of the least-square solution by the normal equation is  $\|b - Ax_{ne}\|_2 = 8.0527971\text{E}-05$ .

The single precision and double precision results have been compiled in Table 2.

|       | Single Precision QR | Single Precision NE | Double Precision QR |
|-------|---------------------|---------------------|---------------------|
| $x_1$ | 2.1100941           | 2.1100929           | 2.110093286276619   |
| $x_2$ | 0.0098492           | 0.0098508           | 0.009850104587809   |

Table 2: Results for  $a_1(t) = 1$  and  $a_2(t) = (t - 1.11) * 100$

The change of basis has improved the condition number of both  $A$  and  $A^T A$  by a few orders of magnitude. Therefore, both  $x_{qr}$  and  $x_{ne}$  are much closer to the real solution  $x_{qr}$ . Note that the resulting  $x_2$  in this new basis is just a multiple of the  $x_2$  from the first basis.

## Question 2

Handwritten derivations are attached.

## Question 3

The singular values and vectors of  $B$  are shown below.

$$\lambda_B = \{-0.026463896961605, 2.404909433839755, -35.134394975254480\}$$

$$U_B = \begin{bmatrix} 0.6482 & 0.6737 & 0.3550 & -0.0000 & 0.0000 \\ 0.6237 & -0.3900 & -0.3986 & -0.5140 & 0.1893 \\ -0.3149 & 0.0696 & 0.4427 & -0.5442 & 0.6355 \\ 0.0060 & 0.2508 & -0.4869 & 0.4535 & 0.7031 \\ 0.3028 & -0.5712 & 0.5310 & 0.4838 & 0.2569 \end{bmatrix}$$

$$V_B = \begin{bmatrix} 0.4183 & 0.8854 & 0.2026 \\ -0.8137 & 0.2662 & 0.5167 \\ 0.4036 & -0.3810 & 0.8318 \end{bmatrix}$$

In order to test for stability, every entry of  $B$  has been perturbed by 1E-05. The error of the singular value and vectors are given by  $\|S_{pert} - S_B\|_2 = 6.8766\text{E-}06$ ,  $\|U_{pert} - U_B\|_2 = 1.9715\text{E-}05$ , and  $\|V_{pert} - V_B\|_2 = 1.0605\text{E-}07$ . This test does not prove stability of the

algorithm since only mathematical derivations can, but it does support that the algorithm is *probably* numerically stable since a small error in the input lead to small error in the solution.

The SVD of  $A$  computed by MATLAB gives the following three singular values:

$$\lambda_A = \{35.127223335747, 2.46539669691652, 2.57621344955340\text{E-}16\}$$

This shows that the  $A$  matrix numerically has rank 3. However, it is easy to show that the columns of  $A$  are not linearly independent. Therefore, matrix  $A$  mathematically has a rank of 2.

To test the validity of the Moore-Penrose matrix, the following properties are checked:

$$\|AGA - A\| = 0 \quad \|GAG - G\| = 0 \quad \|(AG)^T - AG\| = 0 \quad \|(GA)^T - GA\| = 0$$

If we decide that the matrix 3 has full column rank, it is obvious that  $\Sigma_1^{-1}$  blows up due to the small singular value. Therefore, none of the above test are satisfied and the resulting Moore-Penrose matrix is invalid.

Instead, if we say that  $A$  has rank 2 instead of 3, the Moore-Penrose matrix given by  $G = V_1 \Sigma_1^{-1} U_1^T$  satisfies the above tests. Moreover, the `pinv` MATLAB command confirms the results obtained.

## Codes

All codes are available on my GitHub:

<https://github.com/dougshidong/comp540/tree/master/a5>