HOMEWORK 4 AND 5

MATH 578 NUMERICAL ANALYSIS
DUE IN CLASS TUES NOV 17

This is a double HW so you have extra time

Engineering students: do (1) and (2) and choose 4 problems out of 3 - 8 from the theory section. Math students: do all theory problems.

Theory. References for this section are Boyd, Convex Optimization, and Barvinok, A course in convexity

(1) (Reference: Strang Linear Algebra p209-210) Find the formula for the projection of a point b in $\mathbb{R}^n$ onto the subspace $c^t x = 0$. More generally, find the formula for the projection of a point b on the subspace spanned by the columns of an $n \times m$ matrix M. Relate this formula to the least squares solution of $Mx = b$.

(2) (Reference Cheney, Approximation Theory). Consider the inconsistent equations $x = 5, x = -5, x = 1, x = -1$, write these equations as $Mx = b$ where M is 4 by 1, and b is 4 by 1. Define $r = Mx - b$. Find by hand, the solution which minimizes $|r|_2$ and $|r|_\infty$.

(3) Prove that the permutation matrices are extreme points of the Birkhoff Polytope, which is the set of all double stochastic matrices, $M = (m_{i,j})$ where the rows and columns of M sum to 1. (This is the easy part of the Birkhoff-von Neumann Theorem, p57 of Barvinok).

(4) Write the matrix M as a convex combination of permutation matrices (you can do this by hand or use CVX).

$$M = \frac{1}{4}\begin{bmatrix} 2 & 2 & 0 \\ 1 & 0 & 3 \\ 1 & 2 & 1 \end{bmatrix}$$

(5) Show that the set of continuous functions $f : [0,1] \to \mathbb{R}$ with $|f(x)| \leq 1$ for all $x \in [0,1]$ is a convex set.

(6) Suppose $A \subset \mathbb{R}^d$ is a convex set, and $T : \mathbb{R}^d \to \mathbb{R}^m$ is an affine transformation, $T(x) = Mx + b$. Prove that $T(A)$ is a convex set as well.

(7) Recall that the d-cube, is defined to be the set $C = \{x \in \mathbb{R}^d \mid |x|_\infty \leq 1\}$ and the hyper-octahedron is defined by $O = \{x \in \mathbb{R}^d \mid |x|_1 \leq 1\}$. Write each of these sets in two ways. (i) as a a polyhedron $Mx \leq b$, for some M, b, (ii) as a polytope, the convex hull of $V = v_1, \ldots v_N$

(8) Recall the definition of a the polar of a set A,

$$A^\circ = \{y \mid y^T x \leq 1 \text{ for all } x \in A\}$$

(i) Prove that when A is a polytope, we can restrict to $A^\circ = \{y \mid y^T v_i \leq 1 \text{ for } i = 1, \ldots, N\}$ (ii) Find the polar of C, O.

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(9) Prove the easy half of the bipolar theorem, $A \subset (A^\circ)^\circ$. (see Barvinok Ch IV).

Numerical Problems. Use the example code provided to solve these problems in MATLAB, using CVX. Download CVX from `http://cvxr.com/`, and type `cvx_setup` in the MATLAB command line (making sure the CVX directory is in the path) to install CVX.

(10) Use the forward Euler, backward Euler, and Crank-Nicolson methods to solve the following ODEs: (i) $(x'(t), y'(t)) = (-y(t), x(t))$. (ii) $(x'(t), y'(t)) = (-y(t), x(t)))/r(t)$, where $r(t) = \sqrt{x(t)^2 + y(t)^2}$. Plot the trajectories. What happens numerically to $r(t)$ in each case?

(11) Run the code `ProblemPoly.m`˜to find best fit polynomials in the norms $p = 1, 2, \infty$. Modify the code to also consider the cases $p = 1.5, 2.5, 4, 8$. What is the difference in the maximum norm between the interpolated function values using these values of p and the other cases?

(12) Modify the code `ProblemLines.m` to solve problem (2) above.

(13) Write your own CVX code to solve the facility location problem. Given points $(x_1, y_1), \dots (x_n, y_n)$ in the plane, find the point (x, y) which minimizes the norm of the vectors of the distances to the points, using the (i) 1-norm, (ii) 2-norm (iii) ∞ norm. Plot some interesting looking examples with small values of n, and compare the solutions in the different norms.

(14) Write your own CVX code to the linear discrimination problem. (Ref Boyd 8.6.1). Use the data given in the file `discrim.mat`. Find a linear function $p(x) = c^T x - b$ which discriminates the data.