Doing Physics with Quaternions

A Forgotten Path to New Physics

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Overview

Unifying Two Views of Events

An experimentalist collects events about a physical system. A theorists builds a model to describe what patterns of events within a system might generate the experimentalist's data set. With hard work and luck, the two will agree!

Events are handled mathematically as 4-vectors. They can be added or subtracted from another, or multiplied by a scalar. Nothing else can be done. A theorist can import very powerful tools to generate patterns, like metrics and group theory. Theorists in physics have been able to construct the most accurate models of nature in all of science.

I hope to bring the full power of mathematics down to the level of the events themselves. This may be done by representing events as the mathematical division algebra of quaternions. All the standard tools for creating mathematical patterns - multiplication, trigonometric functions, transcendental functions, infinite series, the special functions of physics - should be available for quaternions. Now a theorist can create patterns of events with events. This may lead to a better unification between the work of a theorist and the work of an experimentalist.

An Overview of Doing Physics with Quaternions

It has been said that one reason physics succeeds is because all the terms in an equation are tensors of the same rank. This work challenges that assumption, proposing instead an integrated set of equations which are all based on the same 4-dimensional mathematical division algebra of quaternions. Mostly this document shows in cookbook style how quaternion equations are equivalent to approaches already in use. As Feynman pointed out, "whatever we are allowed to imagine in science must be consistent with everything else we know." Fresh perspectives arise because, in essence, tensors of different rank can mix within the same equation. The four Maxwell equations become one nonhomogeneous quaternion wave equation, and the Klein-Gordon equation is part of a quaternion simple harmonic oscillator. There is hope for a new approach to gravity which puts terms that have been ignored to good use. Since all of the tools used are woven from the same mathematical fabric, the interrelationships become more clear to my eye. Hope you enjoy.

A Brief History of Quaternions

Complex numbers were a hot subject for research in the early eighteen hundreds. An obvious question was that if a rule for multiplying two numbers together was known, what about multiplying three numbers? For over a decade, this simple question had bothered Hamilton, the big mathematician of his day. The pressure to find a solution was not merely from within. Hamilton wrote to his son:

Every morning in the early part of the above-cited month [Oct. 1843] on my coming down to breakfast, your brother William Edwin and yourself used to ask me, 'Well, Papa, can you multiply triplets?' Whereto I was always obliged to reply, with a sad shake of the head, 'No, I can only add and subtract them.'

We can guess how Hollywood would handle the Brougham Bridge scene in Dublin. Strolling along the Royal Canal with Mrs. H-, he realizes the solution to the problem, jots it down in a notebook. So excited, he took out a knife and carved the answer in the stone of the bridge.

Hamilton had found a long sought-after solution, but it was weird, very weird, it was 4D. One of the first things Hamilton did was get rid of the fourth dimension, setting it equal to zero, and calling the result a "proper quaternion." He spent the rest of his life trying to find a use for quaternions. By the end of the nineteenth century, quaternions were viewed as an oversold novelty.

In the early years of this century, Prof. Gibbs of Yale found a use for proper quaternions by reducing the extra fluid surrounding Hamilton's work and adding key ingredients from Rodrigues concerning the application to the rotation of spheres. He ended up with the vector dot product and cross product we know today. This was a useful and potent brew. Our investment in vectors is enormous, eclipsing their place of birth (Harvard had >1000 references under "vector", about 20 under "quaternions", most of those written before the turn of the century).

In the early years of this century, Albert Einstein found a use for four dimensions. In order to make the speed of light constant for all inertial observers, space and time had to be united. Here was a topic tailor-made for a 4D tool, but Albert was not a math buff, and built a machine that worked from locally available parts. We can say now that Einstein discovered Minkowski spacetime and the Lorentz transformation, the tools required to solve problems in special relativity.

Today, quaternions are of interest to historians of mathematics. Vector analysis performs the daily mathematical routine that could also be done with quaternions. I personally think that there may be 4D roads in physics that can be efficiently traveled only by quaternions, and that is the path which is laid out in these web pages.

In a longer history, Gauss would get the credit for seeing quaternions first in one of his notebooks. Rodrigues developed 3D rotations all on his own also in the 1840's. The Pauli spin matrices and Penrose's spinors are reinventions of the wheel that miss out on division. Although I believe that is a major omission and cause of subtle flaws at the foundations of modern physics, spin matrices and spinors have many more adherents today than quaternions.

Multiplying Quaternions the Easy Way

Multiplying two complex numbers a + b I and c + d I is straightforward.

$$(a, b)\ (c, d) = (ac - bd,\ ad + bc)$$

Figure 1: (a, b) times (c, d) = (ac - bd, ad + bc)

For two quaternions, b I and d I become the 3-vectors B and D, where B = x I + y J + z K and similarly for D. Multiplication of quaternions is like complex numbers, but with the addition of the cross product.

$$(a, \vec{B})\ (c, \vec{D}) = (ac - \vec{B}.\vec{D},\ a\,\vec{D} + \vec{B}\,c + \vec{B} \times \vec{D})$$

Figure 2: (a, B) times (c, D) = (ac - B dot D, a D + B c + B cross D)

Note that the last term, the cross product, would change its sign if the order of multiplication were reversed (unlike all the other terms). That is why quaternions in general do not commute.

If a is the operator d/dt, and B is the del operator, or d/dx I + d/dy J + d/dz K (all partial derivatives), then these operators act on the scalar function c and the 3-vector function D in the following manner:

$$\left(\frac{d}{dt}, \vec{\nabla}\right)\ (c, \vec{D}) = \left(\frac{dc}{dt} - \vec{\nabla}.\vec{D},\ \frac{d\,\vec{D}}{dt} + \vec{\nabla}\,c + \vec{\nabla} \times \vec{D}\right)$$

Figure 3: (d/dt, Del) times (c, D) = (c dot - div D, D dot + Del c + Curl D)

This one quaternion contains the time derivatives of the scalar and 3-vector functions, along with the divergence, the gradient and the curl. Dense notation :-)

Inner and Outer Products of Quaternions

A good friend of mine has wondered what is means to multiply two quaternions together (this question was a hot topic in the nineteenth century). I care more about what multiplying two quaternions together can do. There are two basic ways to do this: just multiply one quaternion by another, or first take the transpose of one then multiply it with the other. Each of these products can be separated into two parts: a symmetric (inner product) and an antisymmetric (outer product) components. The symmetric component will remain unchanged by exchanging the places of the quaternions, while the antisymmetric component will change its sign. Together they add up to the product. In this section, both types of inner and outer products will be formed and then related to physics.

The Grassman Inner and Outer Products

There are two basic ways to multiply quaternions together. There is the direct approach.

$$(t, \vec{X}) \left(t', \vec{X'}\right) = \left(t\, t' - \vec{X}.\vec{X'},\ t\, \vec{X'} + \vec{X}\, t' + \vec{X} \times \vec{X'}\right)$$

Figure 4: (t, X) times (t prime , X prime) = (t t prime - X dot X prime , t X prime + X t prime + X cross X prime)

I call this the Grassman product (I do not know if anyone else does, but I need a label). The inner product can also be called the symmetric product, because it does not change signs if the terms are reversed.

$$\mathrm{even}\left((t, \vec{X}),\ \left(t', \vec{X'}\right)\right) \equiv$$

Figure 5: even((t, X),(t prime , X)) is defined to be

$$\equiv \frac{(t, \vec{X}) \left(t', \vec{X'}\right) + \left(t', \vec{X'}\right) (t, \vec{X})}{2} = \left(t\, t' - \vec{X}.\vec{X'},\ t\, \vec{X'} + \vec{X}\, t'\right)$$

Figure 6: ((t, X) times (t prime , X) + (t prime , X) times (t, X)) over 2 = (t t prime -X dot X, X + t prime)

I have defined the anticommutator (the bold curly braces) in a non-standard way, including a factor of two so I do not have to keep remembering to write it. The first term would be the Lorentz invariant interval if the two quaternions represented the same difference between two events in spacetime (i.e. t=t'=delta t,...). The invariant interval plays a central role in special relativity. The vector terms are a frame-dependent, symmetric product of space with time and does not appear on the stage of physics, but is still a valid measurement.

The Grassman outer product is antisymmetric and is formed with a commutator.

$$\mathrm{odd}\left((t, \vec{X}),\ \left(t', \vec{X}'\right)\right) \equiv$$

Figure 7: odd((t, X),(t prime , X prime)) is defined to be

$$\equiv \frac{(t, \vec{X})\left(t', \vec{X}'\right) - \left(t', \vec{X}'\right)(t, \vec{X})}{2} = \left(0, \vec{X} \times \vec{X}'\right)$$

Figure 8: ((t, X) times (t prime , X prime) - (t prime , X prime) times (t, X)) over 2 = (0, X cross X prime)

This is the cross product defined for two 3-vectors. It is unchanged for quaternions.

The Euclidean Inner and Outer Products

Another important way to multiply a pair of quaternions involves first taking the conjugate of one of the quaternions. For a real-valued matrix representation, this is equivalent to multiplication by the transpose which involves flipping the sign of the 3-vector.

$$(t, \vec{X})^{*}\left(t', \vec{X}'\right) = (t, -\vec{X})\left(t', \vec{X}'\right)$$

Figure 9: (t, X) times (t prime , X prime) conjugated = (t, -X) times (t prime , X prime) =

Form the Euclidean inner product.

The first term is the Euclidean norm if the two quaternions are the same (this was the reason for using the adjective "Euclidean"). The Euclidean inner product is also the standard definition of a dot product.

Form the Euclidean outer product.

The first term is zero. The vector terms are an antisymmetric product of space with time and the negative of the cross product.

Implications

When multiplying vectors in physics, one normally only considers the Euclidean inner product, or dot product, and the Grassman outer product, or cross product. Yet, the Grassman inner product, because it naturally generates the invariant interval, appears to play a role in special relativity. What is interesting to

$$= \left(t\,t' + \vec{X}.\vec{X'},\ t\,\vec{X'} - \vec{X}\,t' - \vec{X} \times \vec{X'} \right)$$

Figure 10: = (t t prime +X dot X prime , t X prime -t prime X - X cross X prime)

$$\frac{(t, \vec{X})^{*} \left(t', \vec{X'}\right) + \left(t', \vec{X'}\right)^{*} (t, \vec{X})}{2} = \left(t\,t' + \vec{X}.\vec{X'},\ \vec{0} \right)$$

Figure 11: ((t, X) conjugated times (t prime , X prime) + (t prime , X prime) conjugated (t, X)) over 2 = (t t prime + X dot X prime , 0)

speculate about is the role of the Euclidean outer product. It is possible that the antisymmetric, vector nature of the space/time product could be related to spin. Whatever the interpretation, the Grassman and Euclidean inner and outer products seem destined to do useful work in physics.

$$\frac{(t, \vec{X})^{*} \left(t', \vec{X'}\right) - \left(t', \vec{X'}\right)^{*} (t, \vec{X})}{2} = \left(0,\ t\,\vec{X'} - \vec{X}\,t' - \vec{X} \times \vec{X'} \right)$$

Figure 12: (t, X) conjugated times (t prime , X prime) - (t prime , X prime) conjugated times (t, X) over 2 = (0, t X prime - X t prime - X cross X prime)

Scalars, Vectors, Tensors and All That

According to my math dictionary, a tensor is …

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An abstract object having a definitely specified system of components in every
coordinate system under consideration and such that, under transformation of
coordinates, the components of the object undergoes a transformation of a
certain nature.
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To make this introduction less abstract, I will confine the discussion to the simplest tensors under rotational transformations. A rank-0 tensor is known as a scalar. It does not change at all under a rotation. It contains exactly one number, never more or less. There is a zero index for a scalar. A rank-1 tensor is a vector. A vector does change under rotation. Vectors have one index which can run from 1 to the number of dimensions of the field, so there is no way to know a priori how many numbers (or operators, or …) are in a vector. n-rank tensors have n indices. The number of numbers needed is the number of dimensions in the vector space raised by the rank. Symmetry can often simplify the number of numbers actually needed to describe a tensor.

There are a variety of important spin-offs of a standard vector. Dual vectors, when multiplied by its corresponding vector, generate a real number, by systematically multiplying each component from the dual vector and the vector together and summing the total. If the space a vector lives in is shrunk, a contravariant vector shrinks, but a covariant vector gets larger. A tangent vector is, well, tangent to a vector function.

Physics equations involve tensors of the same rank. There are scalar equations, polar vector equations, axial vector equations, and equations for higher rank tensors. Since the same rank tensors are on both sides, the identity is preserved under a rotational transformation. One could decide to arbitrarily combine tensor equations of different rank, and they would still be valid under the transformation.

There are ways to switch ranks. If there are two vectors and one wants a result that is a scalar, that requires the intervention of a metric to broker the transaction. This process in known as an inner tensor product or a contraction. The vectors in question must have the same number of dimensions. The metric defines how to form a scalar as the indices are examined one-by- one. Metrics in math can be anything, but nature imposes constraints on which ones are important in physics. An aside: mathematicians require that the distance is non-negative, but physicists do not. I will be using the physics notion of a metric. In looking at events in spacetime (a 4-dimensional vector), the axioms of special relativity require the Minkowski metric, which is a 4x4 real matrix that has (1, -1, -1, -1) down the diagonal and zeros elsewhere. Some people prefer the signs to be flipped, but to be consistent with everything else on this site, I choose this convention. Another popular choice is the Euclidean metric, which is the same as an identity matrix. The result of general relativity for a spherically symmetric, non-rotating mass is the Schwarzschild metric, which has "non-one" terms down the diagonal, zeros elsewhere, and becomes the Minkowski metric in the limit of the mass going to zero or the radius going to infinity.

An outer tensor product is a way to increase the rank of tensors. The tensor product of two vectors will be a 2-rank tensor. A vector can be viewed as the tensor product of a set of basis vectors.

What Are Quaternions?

Quaternions could be viewed as the outer tensor product of a scalar and a 3-vector. Under rotation for an event in spacetime represented by a quaternion, time is unchanged, but the 3-vector for space would be rotated. The treatment of scalars is the same as above, but the notion of vectors is far more restrictive, as restrictive as the notion of scalars. Quaternions can only handle 3-vectors. To those familiar to playing with higher dimensions, this may appear too restrictive to be of interest. Yet physics on both the quantum and cosmological scales is confined to 3-spatial dimensions. Note that the infinite Hilbert spaces in quantum mechanics a function of the principle quantum number n, not the spatial dimensions. An infinite collection of quaternions of the form (En, Pn) could represent a quantum state. The Hilbert space is formed using the Euclidean product (q* q').

A dual quaternion is formed by taking the conjugate, because q* q = (t^2 + X.X, 0). A tangent quaternion is created by having an operator act on a quaternion-valued function

$$\left(\frac{\partial}{\partial t}, \vec{\nabla}\right)(f(q), \vec{F}(q)) = \left(\frac{\partial f}{\partial t} - \vec{\nabla}.\vec{F}, \ \frac{\partial \vec{F}}{\partial t} + \vec{\nabla} f + \vec{\nabla} X \vec{F}\right)$$

Figure 13: (d/dt, Del) acting on (f(q), F(q)) = (f dot - div F, F dot + Grad f + Curl F)

What would happen to these five terms if space were shrunk? The 3-vector F would get shrunk, as would the divisors in the Del operator, making functions acted on by Del get larger. The scalar terms are completely unaffected by shrinking space, because df/dt has nothing to shrink, and the Del and F cancel each other. The time derivative of the 3-vector is a contravariant vector, because F would get smaller. The gradient of the scalar field is a covariant vector, because of the work of the Del operator in the divisor makes it larger. The curl at first glance might appear as a draw, but it is a covariant vector capacity because of the right-angle nature of the cross product. Note that if time where to shrink exactly as much as space, nothing in the tangent quaternion would change.

A quaternion equation must generate the same collection of tensors on both sides. Consider the product of two events, q and q':

$$(t, \vec{X})\left(t', \vec{X'}\right) = \left(t\,t' - \vec{X}.\vec{X'},\ t\,\vec{X'} + \vec{X}\,t' + \vec{X} \times \vec{X'}\right)$$
$$\text{scalars}: t,\ t',\ tt' - \vec{X}.\vec{X'}$$
$$\text{polar vectors}: \vec{X},\ \vec{X'},\ t\,\vec{X'} + \vec{X}\,t'$$
$$\text{axial vectors}: \vec{X} \times \vec{X'}$$

Figure 14: (t, X) times (t prime , X prime) = (t t prime - X dot X prime , t X prime + X t prime + X cross X prime)axial vectors: X cross X prime

Where is the axial vector for the left hand side? It is imbedded in the multiplication operation, honest :-)

$$\left(t', \vec{X'}\right)(t, \vec{X}) = \left(t'\,t - \vec{X'}.\vec{X},\ t'\,\vec{X} + \vec{X'}\,t + \vec{X'} \times \vec{X}\right)$$
$$= \left(t\,t' - \vec{X}.\vec{X'},\ t\,\vec{X'} + \vec{X}\,t' - \vec{X} \times \vec{X'}\right)$$

Figure 15: (t prime , X prime) times (t, X) = (t prime t - X prime dot X, t prime X + X prime t + X prime cross X) = (t t prime - X dot X prime , t X prime + X t prime - X cross X prime)

The axial vector is the one that flips signs if the order is reversed.

Terms can continue to get more complicated. In a quaternion triple product, there will be terms of the form (XxX').X". This is called a pseudo-scalar, because it does not change under a rotation, but it will change signs under a reflection, due to the cross product. You can convince yourself of this by noting that the cross product involves the sine of an angle and the dot product involves the cosine of an angle. Neither of these will change under a rotation, and an even function times an odd function is odd. If the order of quaternion triple product is changed, this scalar will change signs for at each step in the permutation.

It has been my experience that any tensor in physics can be expressed using quaternions. Sometimes it takes a bit of effort, but it can be done.

Individual parts can be isolated if one chooses. Combinations of conjugation operators which flip the sign of a vector, and symmetric and antisymmetric products can isolate any particular term. Here are all the terms of the example from above

$$(t, \vec{X})\left(t', \vec{X'}\right) = \left(t\,t' - \vec{X}.\vec{X'},\ t\,\vec{X'} + \vec{X}\,t' + \vec{X} \times \vec{X'}\right)$$

Figure 16: (t,X) times (t prime ,X prime) = (t t prime - X dot X prime , t X prime + X t prime + X cross X prime)

$$\text{scalars}: t = \frac{q + q^*}{2}, \quad t' = \frac{q' + q'^*}{2}, \; tt' - \vec{X}.\vec{X'} = \frac{qq' + (qq')^*}{2}$$

Figure 17: scalars: t = (q + q conjugated) over 2, t prime = (q prime + q prime conjugated) over 2, t t prime - X dot X prime = (q q prime + (q q prime) conjugated)/2

$$\text{polar vectors}: \vec{X} = \frac{q - q^*}{2}, \quad \vec{X'} = \frac{q' - q'^*}{2},$$

$$t\vec{X'} + \vec{X}t' = \frac{(qq' + (q'q)) - (qq' + (q'q))^*}{4}$$

Figure 18: polar vectors: X = (q - q conjugated) over 2, X prime = (q prime - q prime conjugated) over 2, t X prime + X t prime = ((q q prime +(q prime q)) - (qq prime +(q prime q)) conjugated) over 4

$$\text{axial vectors}: \vec{X} \times \vec{X'} = \frac{qq' - (q'q)}{2}$$

Figure 19: axial vectors: X cross X prime = (q q prime -(q prime q)) over 2

$$\vec{i}^2 = \vec{j}^2 = \vec{k}^2 = \vec{i}\,\vec{j}\,\vec{k} = -1$$

Figure 20: i squared = j squared = k squared = i j k = - 1

The metric for quaternions is imbedded in Hamilton’s rule for the field.

This looks like a way to generate scalars from vectors, but it is more than that. It also says implicitly that i j = k, j k = i, and i, j, k must have inverses. This is an important observation, because it means that inner and outer tensor products can occur in the same operation. When two quaternions are multiplied together, a new scalar (inner tensor product) and vector (outer tensor product) are formed.

How can the metric be generalized for arbitrary transformations? The traditional approach would involve playing with Hamilton’s rules for the field. I think that would be a mistake, since that rule involves the fundamental definition of a quaternion. Change the rule of what a quaternion is in one context and it will not be possible to compare it to a quaternion in another context. Instead, consider an arbitrary transformation T which takes q into q’

$$q \longrightarrow q' = T\, q$$

Figure 21: q transform to q prime = T times q

T is also a quaternion, in fact it is equal to q’ q^-1. This is guaranteed to work locally, within neighborhoods of q and q’. There is no promise that it will work globally, that one T will work for any q. Under certain circumstances, T will work for any q. The important thing to know is that a transformation T necessarily exists because quaternions are a field. The two most important theories in physics, general relativity and the standard model, involve local transformations (but the technical definition of local transformation is different than the idea presented here because it involves groups).

This quaternion definition of a transformation creates an interesting relationship between the Minkowski and Euclidean metrics.

$$\text{Let } T = I, \text{ the identity matrix}$$

Figure 22: Let T = I, the identity matrix

$$\frac{I\,q\,I\,q + (I\,q\,I\,q)^*}{2} = (t^2 - \vec{X}.\vec{X},\ 0)$$

Figure 23: (I q I q + (I q I q) conjugated) over 2 = (t squared - X dot X, 0), the Minkowski interval

In order to change from wrist watch time (the interval in spacetime) to the norm of a Hilbert space does not require any change in the transformation quaternion, only a change in the multiplication step. Therefore a transformation

$$(I\,q)^*\, I\,q = (t^2 + \vec{X}.\vec{X},\ 0)$$

Figure 24: (I q) conjugated times I q = (t squared + X dot X, 0), the Euclidean norm

which generates the Schwarzschild interval of general relativity should be easily portable to a Hilbert space, and that might be the start of a quantum theory of gravity.

So What Is the Difference?

I think it is subtle but significant. It goes back to something I learned in a graduate level class on the foundations of calculus. To make calculus rigorous requires that it is defined over a mathematical field. Physicists do this be saying that the scalars, vectors and tensors they work with are defined over the field of real or complex numbers.

What are the numbers used by nature? There are events, which consist of the scalar time and the 3-vector of space. There is mass, which is defined by the scalar energy and the 3-vector of momentum. There is the electromagnetic potential, which has a scalar field phi and a 3-vector potential A.

To do calculus with only information contained in events requires that a scalar and a 3-vector form a field. According to a theorem by Frobenius on finite dimensional fields, the only fields that fit are isomorphic to the quaternions (isomorphic is a sophisticated notion of equality, whose subtleties are appreciated only by people with a deep understanding of mathematics). To do calculus with a mass or an electromagnetic potential has an identical requirement and an identical solution. This is the logical foundation for doing physics with quaternions.

Can physics be done without quaternions? Of course it can! Events can be defined over the field of real numbers, and then the Minkowski metric and the Lorentz group can be deployed to get every result ever confirmed by experiment. Quantum mechanics can be defined using a Hilbert space defined over the field of complex numbers and return with every result measured to date.

Doing physics with quaternions is unnecessary, unless physics runs into a compatibility issue. Constraining general relativity and quantum mechanics to work within the same division may be the way to unite these two separately successful areas.

Update Nope, not going to work according to my current efforts. An new symmetry is at the heart of gravity, basically the great Minkowski light cone rotated by 45 degrees. Quantum mechanics need quaternion series which is neither normed or a division algebra (it is a semi-group instead).

Quaternion Analysis

Complex numbers are a subfield of quaternions. My hypothesis is that complex analysis should be self-evident within the structure of quaternion analysis.

The challenge is to define the derivative in a non-singular way, so that a left derivative always equals a right derivative. If quaternions would only commute… Well, the scalar part of a quaternion does commute. If, in the limit, the differential element converged to a scalar, then it would commute. This idea can be defined precisely. All that is required is that the magnitude of the vector goes to zero faster than the scalar. This might initially appears as an unreasonable constraint. However, there is an important application in physics. Consider a set of quaternions that represent events in spacetime. If the magnitude of the 3-space vector is less than the time scalar, events are separated by a timelike interval. It requires a speed less than the speed of light to connect the events. This is true no matter what coordinate system is chosen.

Defining a Quaternion

A quaternion has 4 degrees of freedom, so it needs 4 real-valued variables to be defined:

$$q = (a_0, a_1, a_2, a_3)$$

Figure 25: q = (a0, a1, a2, a3)

Imagine we want to do a simple binary operation such as subtraction, without having to specify the coordinate system chosen. Subtraction will only work if the coordinate systems are the same, whether it is Cartesian, spherical or otherwise. Let e_0, e_1, e_2, and e_3 be the shared, but unspecified, basis. Now we can define the difference between two quaternion q and q' that is independent of the coordinate system used for the measurement.

$$dq = q' - q = ((a_0' - a_0)\, e_0, (a_1' - a_1)\, e_1 / 3, (a_2' - a_2)\, e_2 / 3, (a_3' - a_3)\, e_3 / 3)$$

Figure 26: dq = q prime - q = ((a0 prime - a0) e0, (a1 prime - a1) e1/3, (a2 prime - a2) e2/3, (a3 prime - a3) e3/3)

What is unusual about this definition are the factors of a third. They will be necessary later in order to define a holonomic equation later in this section. Hamilton gave each element parity with the others, a very reasonable approach. I have found that it is important to give the scalar and the sum of the 3-vector parity. Without this “scale” factor on the 3-vector, change in the scalar is not given its proper weight.

If dq is squared, the scalar part of the resulting quaternion forms a metric.

$$dq\hat{\ }2 = \left(da_0{}^2 e_0{}^2 + da_1{}^2 \frac{e_1{}^2}{9} + da_2{}^2 \frac{e_2{}^2}{9} + da_3{}^2 \frac{e_3{}^2}{9}, 2\, da_0\, da_1\, e_0 \frac{e_1}{3}, 2\, da_0\, da_2\, e_0 \frac{e_2}{3}, 2\, da_0\, da_3\, e_0 \frac{e_3}{3}\right)$$

Figure 27: dq squared = (da0 squared e0 squared + da1 squared e1 squared over 9 + da2 squared e2 squared over 9 + da3 squared e3 squared over 9, 2 da0 da1 e0 e1 over 3, 2 da0 da2 e0 e2 over 3, 2 da0 da3 e0 e3 over 3)

What should the connection be between the squares of the basis vectors? The amount of intrinsic curvature should be equal, so that a transformation between two basis 3-vectors does not contain a hidden bump. Should time be treated exactly like space? The Schwarzschild metric of general relativity suggests otherwise. Let e_1, e_2, and e_3 form an independent, dimensionless, orthogonal basis for the 3-vector such that:

$$-\frac{1}{e_1{}^2} = -\frac{1}{e_2{}^2} = -\frac{1}{e_3{}^2} = e_0{}^2$$

Figure 28: - 1 over e1 squared = - 1 over e2 squared = - 1 over e3 squared = e0 squared

This unusual relationship between the basis vectors is consistent with Hamilton's choice of 1, i, j, k if $e_0{}^2 = 1$. For that case, calculate the square of dq:

$$dq^2 = \left(da_0{}^2 e_0{}^2 - \frac{da_1{}^2}{9\, e_0{}^2} - \frac{da_2{}^2}{9\, e_0{}^2} - \frac{da_3{}^2}{9\, e_0{}^2}, 2\, da_0 \frac{da_1}{3}, 2\, da_0 \frac{da_2}{3}, 2\, da_0 \frac{da_3}{3}\right)$$

Figure 29: dq squared = (da0 squared e0 squared - da1 squared over 9 e0 squared - da2 squared over 9 e0 squared - da3 squared over 9 e0 squared, 2 da0 da1 over 3, 2 da0 da2 over 3, 2 da0 da3 over 3)

The scalar part is known in physics as the Minkowski interval between two events in flat spacetime. If $e_0{}^2$ does not equal one, then the metric would apply to a non-flat spacetime. A metric that has been measured experimentally is the Schwarzchild metric of general relativity. Set $e_0{}^2 = (1 - 2\ GM/c^2\ R)$, and calculate the square of dq:

This is the Schwarzchild metric of general relativity. Notice that the 3-vector is unchanged (this may be a defining characteristic). There are very few opportunities for freedom in basic mathematical definitions. I have chosen this unusual relationships between the squares of the basis vectors to make a result from physics easy to express. Physics guides my choices in mathematical definitions :-)

$$dq^2 = \left(da_0{}^2 \left(1 - \frac{2\,G\,M}{c^2\,R}\right) - \frac{dA.dA}{9\,(1 - \frac{2\,G\,M}{c^2\,R})},\ 2\,da_0\,\frac{da_1}{3},\ 2\,da_0\,\frac{da_2}{3},\ 2\,da_0\,\frac{da_3}{3}\right)$$

Figure 30: dq squared = (da0 squared (1 - 2 G M over c squared r) - dA dot dA over 9 (1 - 2 G M over c squared r) , 2 da0 da1 over 3, 2 da0 da2 over 3, 2 da0 da3 over 3)

An Automorphic Basis for Quaternion Analysis

A quaternion has 4 degrees of freedom. To completely specify a quaternion function, it must also have four degrees of freedom. Three other linearly- independent variables involving q can be defined using conjugates combined with rotations:

$$q^* = (a_0\,e_0,\ -a_1\,e_1 / 3,\ -a_2\,e_2 / 3,\ -a_3\,e_3 / 3)$$

Figure 31: q conjugated = (a0 e0, - a1 e1 over 3, - a2 e2 over 3, - a3 e3 over 3)

$$q^{*1} = (-\,a_0\,e_0,\ a_1\,e_1 / 3,\ -a_2\,e_2 / 3,\ -a_3\,e_3 / 3) = (e_1\,q\,e_1)^*$$

Figure 32: q conjugated first is defined to be (- a0 e0, a1 e1 over 3, - a2 e2 over 3, - a3 e3 over 3) = (e1 q e1) conjugated

The conjugate as it is usually defined (q$^{\frown}$) *flips the sign of all but the scalar. The q$^{\frown}$1* flips the signs of all but the e_1 term, and q$^{2\frown}$ *all but the e_2 term. The set q, q$^{\frown}$*, q$^{1\frown}$, *q$^{\frown}$2* form the basis for quaternion analysis. The conjugate of a conjugate should give back the original quaternion.

Something subtle but perhaps directly related to spin happens looking at how the conjugates effect products:

The conjugate applied to a product brings the result directly back to the reverse order of the elements. The first and second conjugates point things in exactly the opposite way. The property of going "half way around" is reminiscent of spin. A tighter link will need to be examined.

Future Timelike Derivative

Instead of the standard approach to quaternion analysis which focuses on left versus right derivatives, I concentrate on the ratio of scalars to 3-vectors. This is natural when thinking about the structure of Minkowski spacetime, where the ratio of the change in time to the change in 3-space defines five separate regions: timelike past, timelike future, lightlike past, lightlike future, and space-like. There are no continuous Lorentz transformations to link these regions. Each region will require a separate definition of the derivative, and they will

$$q^{*2} \equiv (-a_0 e_0, -a_1 e_1 / 3, +a_2 e_2 / 3, -a_3 e_3 / 3) = (e_2 q e_2)^*$$

Figure 33: q conjugated second is defined to be (- a0 e0, - a1 e1 over 3, a2 e2 over 3, - a3 e3 over 3) = (e2 q e2) conjugated

$$(q^*)^* = q, \quad (q^{*1})^{*1} = q, \quad (q^{*2})^{*2} = q$$

Figure 34: (q conj) conj = q, (q conjugated first) conjugated first = q, (q conjugated second) conjugated second = q

$$(q\, q')^* = q'^* q^*$$

Figure 35: (q times q prime) conjugated first = q prime conjugated first times q conjugated first

$$(q\, q')^{*1} = -q'^{*1} q^{*1}, \quad (q\, q')^{*2} = -q'^{*2} q^{*2}$$

Figure 36: (q times q prime) conjugated first = - q prime conjugated first times q conjugated first, (q times q prime) conjugated second = - q prime conjugated second times q conjugated second

$$(q\, q'\, q\, q')^{*1} = q'^{*1} q^{*1} q'^{*1} q^{*1}$$

Figure 37: (q times q prime times q times q prime) conjugated first = q prime conjugated first times q conjugated first times q prime conjugated first times q conjugated first

each have distinct properties. I will start with the simplest case, and look at a series of examples in detail.

Definition: The future timelike derivative:

Consider a covariant quaternion function f with a domain of H and a range of H. For a future timelike derivative to be defined, the 3-vector must approach zero faster than the positive scalar. If this is not the case, then this definition cannot be used. Implementing these requirements involves two limit processes applied sequentially to a differential quaternion D. First the limit of the three vector is taken as it goes to zero, (D - D^)/2 -> 0. *Second, the limit of the scalar is taken, (D + D^*)/2 -> +0 (the plus zero indicates that it must be approached with a time greater than zero, in other words, from the future). The net effect of these two limit processes is that D->0.

$$\frac{\partial f(q, q^*, q^{*1}, q^{*2})}{\partial q} =$$

Figure 38: d f(q, q conjugated, q conjugated first, q conjugated second) over d q =

$$= \text{limit as } (d, \vec{0}) \to +0 \left(\text{limit as } (d, \vec{D}) \to (d, \vec{0}) \left(f(q + (d, \vec{D}), q^*, q^{*1}, q^{*2}) - f(q, q^*, q^{*1}, q^{*2})\right) (d, \vec{D})^{-1}\right)$$

Figure 39: = the limit as the scalar d approaches zero from the positive direction of (the limit as (d, D) approaches the scalar (d, 0) of (f(q + (d, D), q conjugated, q conjugated first, q conjugated second) - f(q, q conjugated, q conjugated first, q conjugated second) times (d, D) inverted))

The definition is invariant under a passive transformation of the basis.

The 4 real variables a0, a1, a2, a3 can be represented by functions using the conjugates as a basis.

$$f(q, q^*, q^{*1}, q^{*2}) = a_0 = \frac{e_0 (q + q^*)}{2}$$

Figure 40: f(q, q conjugated, q conjugated first, q conjugated second) = a0 = e0 (q + q conjugated) over 2

Begin with a simple example:

The definition gives the expected result.

$$f = a_1 = \frac{e_1 \left(q + q^{*1}\right)}{(-2/3)} = \frac{\left(q + q^{*1}\right) e_1}{(-2/3)}$$

Figure 41: f = a1 = e1 (q + q conj 1) over (-2 over 3) = (q + q conjugated first) e1 over (-2 over 3)

$$f = a_2 = \frac{e_2 \left(q + q^{*2}\right)}{(-2/3)} = \frac{\left(q + q^{*2}\right) e_2}{(-2/3)}$$

Figure 42: f = a2 = e2 (q + q conjugated second) over (-2 over 3) = (q + q conjugated second) e2 over (-2 over 3)

$$f = a_3 = \frac{e_3 \left(q + q^* + q^{*1} + q^{*2}\right)}{(2/3)} = \frac{\left(q + q^* + q^{*1} + q^{*2}\right) e_3}{(2/3)}$$

Figure 43: f = a3 = e3 times (q + q conjugated + q conjugated first + the second conjugated of q) over (2 over 3) = (q + q conjugated + q conjugated first + the second conjugated of q) times e3 over (2 over 3)

$$f\left(q, q^*, q^{*1}, q^{*2}\right) = a_0 = \frac{e_0 \left(q + q^*\right)}{2}$$

Figure 44: f = a0 = (q + q conjugated) over 2

$$\frac{\partial a_0}{\partial q} = \frac{\partial a_0}{\partial q^*} = \lim \left(\lim \left(\left(e_0 \left(\left(q + (d, \vec{D}) + q^*\right) - \left(q + q^*\right)\right)\right) \left(2 (d, \vec{D})\right)^{-1}\right)\right) = \frac{e_0}{2}$$

Figure 45: d a0 by d q = da0 by d q conjugated = the limit of (the limit of((e0(q + (d, D) + q conjugated) - (q + q conjugated)) times the inverse of (2(d, D)))) = e0 over 2

$$\frac{\partial a_0}{\partial q^{*1}} = \frac{\partial a_0}{\partial q^{*2}} = 0$$

Figure 46: d a0 by d q conjugated first = da0 by d q conjugated second = 0

$$f = a_1 = \frac{e_1 \left(q + q^{*1}\right)}{(-2/3)}$$

Figure 47: f = a1 = e1 (q + q conjugated first) over (- 2 over 3)

$$\frac{\partial a_1}{\partial q} = \frac{\partial a_1}{\partial q^{*1}} = \lim \left(\lim \left((e_1 ((q + (d, \vec{D}) + q^{*1}) - (q + q^{*1}))) ((-2/3)(d, \vec{D}))^{-1} \right)\right) = -\frac{3\, e_1}{2}$$

Figure 48: d a1 by d q = d a1 by d q conjugated = the limit of (the limit of(e1 times ((q + (d, D) + q conjugated first) - (q + q conjugated first)) times the inverse of (-2 over 3 (d, D)))) = - 3 e1 over 2

$$\frac{\partial a_1}{\partial q^*} = \frac{\partial a_1}{\partial q^{*2}} = 0$$

Figure 49: d a1 over d q conjugated = d a1 over d q conjugated second = 0

A simple approach to a trickier example:

So far, the fancy double limit process has been irrelevant for these identity functions, because the differential element has been eliminated. That changes with the following example, a tricky approach to the same result.

$$f\,(q,\, q^*,\, q^{*1},\, q^{*2}) = a_1 = \frac{(q + q^{*1})\, e_1}{(-2/3)}$$

Figure 50: f = a1 = (q + q conjugated first) times e1 over (-2 over 3)

Because the 3-vector goes to zero faster than the scalar for the differential element, after the first limit process, the remaining differential is a scalar so it commutes with any quaternion. This is what is required to dance around the e_1 and lead to the cancellation.

The initial hypothesis was that complex analysis should be a self-evident subset of quaternion analysis. So this quaternion derivative should match up with the complex case, which is:

These are the same result up to two subedits. Quaternions have three imaginary axes, which creates the factor of three. The conjugate of a complex number is really doing the work of the first quaternion conjugate q^*1^ (which equals -z^), because z^ flips the sign of the first 3-vector component, but no others.

The derivative of a quaternion applies equally well to polynomials.

This is the expected result for this polynomial. It would be straightforward to show that all polynomials gave the expected results.

Mathematicians might be concerned by this result, because if the 3-vector D goes to -D nothing will change about the quaternion derivative. This is actually consistent with principles of special relativity. For timelike separated events,

$$\frac{\partial\ a_1}{\partial\ q} = \frac{\partial\ a_1}{\partial\ q^{*1}} =$$

Figure 51: d a1 by d q = d a1 by d q conjugated first =

$$= \lim \left(\lim \left(((q + (d, \vec{D}) + q^{*1}) - (q + q^{*1}))\ e_1\ ((-2\ /\ 3)\ (d, \vec{D}))^{-1}\right)\right) =$$

Figure 52: = the limit of(the limit of(((q + (d, D) + q conjugated first) - (q + q conjugated first)) times e1 (-2 over 3 (d, D)) inverted)) =

$$= \lim \left(\lim \left((d, \vec{D})\ e_1\ ((-2\ /\ 3)\ (d, \vec{D}))^{-1}\right)\right) =$$

Figure 53: = the limit of(the limit of((d, D) times e1 (-2 over 3 (d, D)) inverted)) =

$$= \lim \left(\left(d, \vec{0}\right)\ e_1\ \left((-2\ /\ 3)\ \left(d, \vec{0}\right)\right)^{-1}\right) = -\frac{3\ e_1}{2}$$

Figure 54: = the limit of ((d, 0) times e1 times (-2 over 3 (d, 0)) inverted) = -3 over 2 e1

$$z = a + b\,i,\ b = (Z - Z^*)\ /\ 2\ i$$

Figure 55: z = a + b i, y = (Z - Z conjugated) over 2 i

$$\frac{\partial\ b}{\partial\ z} = -\frac{i}{2} = -\frac{\partial\ b}{\partial\ z^*}$$

Figure 56: d b by d z = - i over 2 = - d b by d Z conjugated

$$\text{let } f = q^2$$

Figure 57: let f = q squared

$$\frac{\partial\ f}{\partial\ q} = \lim \left(\lim \left(\left((q + (d, \vec{D}))^2 - q^2\right)\ (d, \vec{D})^{-1}\right)\right) =$$

Figure 58: d f by d q = the limit of(the limit of (((q + (d, D)) squared - q squared) times (d, D) inverted)) =

$$= \lim \left(\lim \left(\left(q^2 + q\,(d, \vec{D}) + (d, \vec{D})\,q + (d, \vec{D})^2 - q^2 \right) (d, \vec{D})^{-1} \right) \right) =$$

Figure 59: = the limit of (the limit of ((q squared + q times (d, D) + (d, D) times q + (d, D) squared - q squared) times (d, D) inverted)) =

$$= \lim \left(\lim \left(q + (d, \vec{D})\,q\,(d, \vec{D})^{-1} + (d, \vec{D}) \right) \right) =$$

Figure 60: = the limit of (the limit of ((q + (d, D) times q times (d, D) inverted + (d, D))) =

right and left depend on the inertial reference frame, so a timelike derivative should not depend on the direction of the 3-vector.

Analytic Functions

There are 4 types of quaternion derivatives and 4 component functions. The following table describes the 16 derivatives for this set

This table will be used extensively to evaluate if a function is analytic using the chain rule. Let's see if the identity function w = q is analytic.

Use the chain rule to calculate the derivative will respect to each term:

Use combinations of these terms to calculate the four quaternion derivatives using the chain rule.

This has the derivatives expected if w=q is analytic in q.

Another test involves the Cauchy-Riemann equations. The presence of the three basis vectors changes things slightly.

This also solves a holonomic equation.

There are no off diagonal terms to compare.

This exercise can be repeated for the other identity functions. One noticeable change is that the role that the conjugate must play. Consider the identity function w = q^*1^. To show that this is analytic in q$^{1\hat{}\ \textit{requires that one always works with basis vectors of the } q\hat{}1}$ variety.

This also solves a first conjugate holonomic equation.

$$= \lim \left(2\,q + \left(d, \vec{0} \right) \right) = 2\,q$$

Figure 61: = the limit of (2 q + (d, 0)) = 2 q

\	a_0	a_1	a_2	a_3
$\frac{\partial}{\partial q}$	$\frac{e_0}{2}$	$\frac{e_1}{-2/3}$	$\frac{e_2}{-2/3}$	$\frac{e_3}{2/3}$
$\frac{\partial}{\partial q^*}$	$\frac{e_0}{2}$	0	0	$\frac{e_3}{2/3}$
$\frac{\partial}{\partial q^{*1}}$	0	$\frac{e_1}{-2/3}$	0	$\frac{e_3}{2/3}$
$\frac{\partial}{\partial q^{*2}}$	0	0	$\frac{e_2}{-2/3}$	$\frac{e_3}{2/3}$

Figure 62: Reading each derivative for the 4 components a0, a1, a2, a3 in a row, the derivatives with respect to q are $e_0/2$, - 3 $e_1/2$, - 3 $e_2/3$, 3 $e_3/3$; the derivatives with respect to q conj are $e_0/2$, 0, 0, 3 $e_3/3$;the derivatives with respect to q conj1 are 0, - 3 $e_1/2$, 0, 3 $e_3/3$;the derivatives with respect to q conj2 are 0, 0, - 3 $e_2/2$, 3 $e_3/3$;

$$\text{Let } w = q = \left(a_0 e_0,\ a_1 \frac{e_1}{3},\ a_2 \frac{e_2}{3},\ a_3 \frac{e_3}{3}\right)$$

Figure 63: Let w = q = (a0 e0, a1 e_1 over 3, a2 e2 over 3, a3 e3 over 3)

$$\frac{\partial w}{\partial a_0} \frac{\partial a_0}{\partial q} = e_0 \frac{e_0}{2} = \frac{1}{2}$$

Figure 64: d w by d a0 times d a0 by d q = e0 e0 over 2 = 1 half

$$\frac{\partial w}{\partial a_1} \frac{\partial a_1}{\partial q} = \frac{e_1}{3} \frac{e_1}{(-2/3)} = \frac{1}{2}$$

Figure 65: d w by d a1 time d a1 by d q = e1 over 3 times e1 over (-2 over 3) = 1 half

$$\frac{\partial w}{\partial a_2} \frac{\partial a_2}{\partial q} = \frac{e_2}{3} \frac{e_2}{(-2/3)} = \frac{1}{2}$$

Figure 66: d w by d a2 times d a2 by d q = e2 over 3 times e2 over (-2 over 3) = 1 half

$$\frac{\partial w}{\partial a_3} \frac{\partial a_3}{\partial q} = \frac{e_3}{3} \frac{e_3}{(2/3)} = -\frac{1}{2}$$

Figure 67: d w by d a3 times d a3 by d q = e3 over 3 times d e3 over (2 over 3) = - 1 half

$$\frac{\partial w}{\partial q} = \frac{\partial w}{\partial a_0} \frac{\partial a_0}{\partial q} + \frac{\partial w}{\partial a_1} \frac{\partial a_1}{\partial q} + \frac{\partial w}{\partial a_2} \frac{\partial a_2}{\partial q} + \frac{\partial w}{\partial a_3} \frac{\partial a_3}{\partial q} = \frac{1}{2} + \frac{1}{2} + \frac{1}{2} - \frac{1}{2} = 1$$

Figure 68: d w by d q = d w by d a0 times d a0 by d q + d w by d a1 times d a1 by d q + d w by d a2 times d a2 by d q + d w by d a3 times d a3 by d q = 1 half + 1 half + 1 half - 1 half = 1

$$\frac{\partial w}{\partial q^*} = \frac{\partial w}{\partial a_0} \frac{\partial a_0}{\partial q^*} + \frac{\partial w}{\partial a_3} \frac{\partial a_3}{\partial q^*} = \frac{1}{2} - \frac{1}{2} = 0$$

Figure 69: d w by d q conj =d w by d a0 times d a0 by d q conj +d w by d a3 times d a3 by d q conj = 1 half - 1 half = 0

$$\frac{\partial w}{\partial q^{*1}} = \frac{\partial w}{\partial a_1} \frac{\partial a_1}{\partial q^{*1}} + \frac{\partial w}{\partial a_3} \frac{\partial a_3}{\partial q^{*1}} = \frac{1}{2} - \frac{1}{2} = 0$$

Figure 70: d w by d q conjugated first = d w by d a1 times d a1 by d q conjugated first + d w by d a3 times d a3 by d q conjugated first = 1 half - 1 half = 0

$$\frac{\partial w}{\partial q^{*2}} = \frac{\partial w}{\partial a_2} \frac{\partial a_2}{\partial q^{*2}} + \frac{\partial w}{\partial a_3} \frac{\partial a_3}{\partial q^{*2}} = \frac{1}{2} - \frac{1}{2} = 0$$

Figure 71: d w by d q conjugated second = d w by d a2 times d a2 by d q conjugated second + d w by d a3 times d a3 by d q conjugated second = 1 half - 1 half = 0

$$\text{Let } u = (a_0 e_0, 0, 0, 0), \ \vec{\nabla} = \left(0, a_1 \frac{e_1}{3}, \ a_2 \frac{e_2}{3}, \ a_3 \frac{e_3}{3}\right)$$

Figure 72: Let u = (a0 e0, 0, 0, 0) , V = (0, a1 e1 over 3, a2 e2 over 3, a3 e3 over 3)

$$\frac{\partial u}{\partial a_0} \frac{e_1}{3} = \frac{\partial \vec{\nabla}}{\partial a_1} e_0, \ \frac{\partial u}{\partial a_0} \frac{e_2}{3} = \frac{\partial \vec{\nabla}}{\partial a_2} e_0, \ \frac{\partial u}{\partial a_0} \frac{e_3}{3} = \frac{\partial \vec{\nabla}}{\partial a_3} e_0$$

Figure 73: d u by d a0 times e1 over 3 = d V by d a1 e0, d u by d a0 times e2 over 3 = d V by d a2 times e0, d u by d a0 times e3 over 3 = d V by d a3 times e0

$$\mathrm{Scalar}\left(\left(\frac{\partial u}{\partial a_0}, \frac{\partial \vec{\nabla}}{\partial a_1}, \frac{\partial \vec{\nabla}}{\partial a_2}, \frac{\partial \vec{\nabla}}{\partial a_3}\right)(e_0, e_1, e_2, e_3)\right) = e_0\, e_0 + \frac{e_1}{3}\, e_1 + \frac{e_2}{3}\, e_2 + \frac{e_3}{3}\, e_3 = 0$$

Figure 74: the scalar part of (d u by d a0, d V by d a1, d V by d a2, d V by d a3) times (e0, e1, e2, e3)) = e0 times e0 + e1 times e1 over 3 + e2 times e2 over 3 + e3 times e3 over 3 = 0

$$\mathrm{Let}\ u = (-a_0\, e_0, 0, 0, 0),\ \vec{\nabla} = \left(0,\ a_1 \frac{e_1}{3},\ -a_2 \frac{e_2}{3},\ -a_3 \frac{e_3}{3}\right)$$

Figure 75: Let u = (- a0 e0, 0, 0, 0), V = (0, a1 e1 over 3, - a2 e2 over 3, - a3 e3 over 3)

$$\frac{\partial u}{\partial a_0}\left(-\frac{e_1}{3}\right) = \frac{\partial \vec{\nabla}}{\partial a_1}\, e_0,\ \frac{\partial u}{\partial a_0}\, \frac{e_2}{3} = \frac{\partial \vec{\nabla}}{\partial a_2}\, e_0,\ \frac{\partial u}{\partial a_0}\, \frac{e_3}{3} = \frac{\partial \vec{\nabla}}{\partial a_3}\, e_0$$

Figure 76: d u by d a0 times (-e1 over 3)= d V by d a1 times e0, d u by d a0 times e2 over 3 = d V by d a2 times e0, d u by d a0 times e3 over 3 = d V by d a3 times e0

$$\mathrm{Scalar}\left(\left(\frac{\partial u}{\partial a_0}, \frac{\partial \vec{\nabla}}{\partial a_1}, \frac{\partial \vec{\nabla}}{\partial a_2}, \frac{\partial \vec{\nabla}}{\partial a_3}\right)(e_0, e_1, e_2, e_3)^{*1}\right) = -e_0\,(-e_0) + \frac{e_1}{3}\, e_1 - \frac{-e_2}{3}\, e_2 - \frac{-e_3}{3}\, e_3 = 0$$

Figure 77: (d u by d a0 + d V by d a1 + d V by d a2 + dV by d a3) dot (e0 + e1 + e2 + e3) = e0 e0 + e1 over 3 times e1 + e2 over 3 time e2 + e3 over 3 times e3 = 0

Power functions can be analyzed in exactly the same way:

$$\text{Let } w = q^2 = \left(a_0{}^2\, e_0{}^2 + a_1{}^2\, \frac{e_1{}^2}{9} + a_2{}^2\, \frac{e_2{}^2}{9} + a_3{}^2\, \frac{e_3{}^2}{9},\ 2\, a_0\, a_1\, e_0\, \frac{e_1}{3},\ 2\, a_0\, a_2\, e_0\, \frac{e_2}{3},\ 2\, a_0\, a_3\, e_0\, \frac{e_3}{3}\right)$$

Figure 78: Let w = q squared = (a0 squared times e0 squared + a1 squared times e1 squared over 9 + a2 squared times e2 squared over 9 + a3 squared times e3 squared over 9, 2 a0 a1 e0 e1 over 3, 2 a0 a2 e0 e2 over 3, 2 a0 a3 e0 e3 over 3)

$$u = \left(a_0{}^2\, e_0{}^2 + a_1{}^2\, \frac{e_1{}^2}{9} + a_2{}^2\, \frac{e_2{}^2}{9} + a_3{}^2\, \frac{e_3{}^2}{9},\ 0,\ 0,\ 0\right)$$

Figure 79: u = (a0 squared e0 squared + a1 squared e1 squared over 9 + a2 squared e2 squared over 9 + a3 squared e3 squared over 9, 0, 0, 0)

$$\vec{V} = \left(0,\ 2\, a_0\, a_1\, e_0\, \frac{e_1}{3},\ 2\, a_0\, a_2\, e_0\, \frac{e_2}{3},\ 2\, a_0\, a_3\, e_0\, \frac{e_3}{3}\right)$$

Figure 80: V = (0, 2 a0 a1 e0 e1 over 3, 2 a0 a2 e0 e2 over 3, 2 a0 a3 e0 e3 over 3)

This time there are cross terms involved.

At first glance, one might think these are incorrect, since the signs of the derivatives are suppose to be opposite. Actually they are, but it is hidden in an accounting trick :-) For example, the derivative of u with respect to a1 has a factor of $e_1{}^2$, which makes it negative. The derivative of the first component of V with respect to a0 is positive. Keeping all the information about signs in the e's makes things look non-standard, but they are not.

Note that these are three scalar equalities. The other Cauchy-Riemann equations evaluate to a single 3-vector equation. This represents four constraints on the four degrees of freedom found in quaternions to find out if a function happens to be analytic.

This also solves a holonomic equation.

Since power series can be analytic, this should open the door to all forms of analysis. (I have done the case for the cube of q, and it too is analytic in q).

4 Other Derivatives

So far, this work has only involved future timelike derivatives. There are five other regions of spacetime to cover. The simplest next case is for past timelike derivatives. The only change is in the limit, where the scalar approaches zero from below. This will make many derivatives look time symmetric, which is the case for most laws of physics.

$$\frac{\partial u}{\partial a_0}\ \frac{e_1}{3} = \frac{2\ a_0\ e_0{}^2\ e_1}{3} = \frac{\partial \vec{V}}{\partial a_1}\ e_0$$

Figure 81: d u by d a0 e1 over 3 = 2 a0 e0 squared e1 over 3 = d V by d a1 e0

$$\frac{\partial u}{\partial a_0}\ \frac{e_2}{3} = \frac{2\ a_0\ e_0{}^2\ e_2}{3} = \frac{\partial \vec{V}}{\partial a_2}\ e_0$$

Figure 82: d u by d a0 e2 over 3 = 2 a0 e0 squared e2 over 3 = d V by d a2 e0

$$\frac{\partial u}{\partial a_0}\ \frac{e_3}{3} = \frac{2\ a_0\ e_3{}^2}{3} = \frac{\partial \vec{V}}{\partial a_3}$$

Figure 83: d u by d a0 e3 over 3 = 2 a0 e0 e3 squared over 3 = d V by d a3 e0

$$\frac{\partial u}{\partial a_1}\ e_0 = \frac{2\ a_1\ e_0\ e_1{}^2}{9} = \frac{\partial \vec{V}_1}{\partial a_0}\ \frac{e_1}{3}$$

Figure 84: - d u by d a1 3 times e0 = - 2 a1 e0 e1 squared over 9 = d V1 by d a0 e1 over 3

$$\frac{\partial u}{\partial a_2}\ e_0 = \frac{2\ a_2\ e_0\ e_2{}^2}{9} = \frac{\partial \vec{V}_2}{\partial a_0}\ \frac{e_2}{3}$$

Figure 85: - d u by d a2 3 e0 = - 2 a2 e0 e2 squared over 9 = d V2 by d a0 e2 over 3

$$\frac{\partial u}{\partial a_3}\ e_0 = \frac{2\ a_3\ e_0\ e_3{}^2}{9} = \frac{\partial \vec{V}_3}{\partial a_0}\ \frac{e_3}{3}$$

Figure 86: d u by d a3 e0 = - 2 a3 e0 e3 squared over 9 = - d V3 by d a0 e3 over 3

$$\text{Scalar}\left(\left(\frac{\partial u}{\partial a_0}, \frac{\partial \vec{V}}{\partial a_1}, \frac{\partial \vec{V}}{\partial a_2}, \frac{\partial \vec{V}}{\partial a_3}\right)(e_0, e_1, e_2, e_3)\right) =$$

Figure 87: (d u by d a0 + d V over a1 + d V by d a2 +d V by d a3) dot (e0 +e1 + e2 + e3) =

$$= 2\, a_0\, e_0{}^3 + \frac{2\, a_0\, e_0\, e_1}{3}\, e_1 + \frac{2\, a_0\, e_0\, e_2}{3}\, e_2 + \frac{2\, a_0\, e_0\, e_3}{3}\, e_3 = 0$$

Figure 88: = 2 a0 e0 cubed + 2 a0 e1 over 3 e1 + 2 a0 e2 over 3 e2 + 2 a0 e3 over 3 e3 = 0

A more complicated case involves spacelike derivatives. In the spacelike region, changes in time go to zero faster than the absolute value of the 3-vector. Therefore the order of the limit processes is reversed. This time the scalar approaches zero, then the 3-vector. This creates a problem, because after the first limit process, the differential element is (0, D), which will not commute with most quaternions. That will lead to the differential element not cancelling. The way around this is to take its norm, which is a scalar.

A spacelike differential element is defined by taking the ratio of a differential quaternion element D to its 3-vector, D - D^. *Let the norm of D approach zero. To be defined, the three vector must approach*
- D^)*(D - D^)*))* approaches (1, 0), a scalar.

$$\frac{\partial f(q, q^*, q^{*1}, q^{*2})}{\partial q}\ \frac{\partial f(q, q^*, q^{*1}, q^{*2})^*}{\partial q} =$$

Figure 89: the norm of the partial derivative of f with respect to q =

To make this concrete, consider a simple example, f = q^2. Apply the definition:

The second and fifth terms are unitary rotations of the 3-vector B. Since the differential element D could be pointed anywhere, this is an arbitrary rotation. Define:

Substitute, and continue:

Look at how wonderfully strange this is! The arbitrary rotation of the 3-vector B means that this derivative is bound by an inequality. If D is in direction of B, then it will be an equality, but D could also be in the opposite direction, leading to a destruction of a contribution from the 3-vector. The spacelike derivative can therefore interfere with itself. This is quite a natural thing to do in quantum mechanics. The spacelike derivative is positive definite, and could be used to define a Banach space.

Defining the lightlike derivative, where the change in time is equal to the change in space, will require more study. It may turn out that this derivative is singular

$$= \text{limit as } (0, \vec{D}) \to 0 \left(\text{limit as } (d, \vec{D}) \to (0, \vec{D}) \left((f(q + (d, \vec{D}), q^*, q^{*1}, q^{*2}) - f(q, q^*, q^{*1}, q^{*2}))(d, \vec{D})^{-1} (f(q + (d, \vec{D}), q^*, q^{*1}, q^{*2}) - f(q, q^*, q^{*1}, q^{*2}))^* (d, \vec{D})^{-1*}\right)\right)$$

Figure 90: = the limit as the 3-vector (0, D) goes to 0 of (the limit as the quaternion differential element (d, D) goes to the 3-vector (0, D) of (f(q + (d, D), q conjugated, q conjugated first, q conjugated second) - f(q, q conjugated, q conjugated first, q conjugated second) times (d, D) inverted)) times the preceding difference conjugated))

$$\text{Norm}\left(\frac{\partial q^2}{\partial q}\right) = \text{limit } ((0, \vec{D}) \to 0 \; (\text{limit as } (d, \vec{D}) \to (0, \vec{D})$$

Figure 91: the norm of d q squared by d q = the limit as (0, D) goes to 0 of (the limit as (d, D) goes to (0, D)

$$\left(\left(((a, \vec{B}) + (d, \vec{D}))^2 - (a, \vec{B})^2\right)(d, \vec{D})^{-1} \left(((a, \vec{B}) + (d, \vec{D}))^2 - (a, \vec{B})^2\right)^* (d, \vec{D})^{-1*}\right)\right) =$$

Figure 92: ((((a, B) + (d, D)) squared - (a, B) squared) times (d, D) inverted times the conjugate of the preceding difference =

$$= \lim \left(((a, \vec{B}) + (0, \vec{D})(a, \vec{B})(0, -\vec{D}) / \text{norm}((0, \vec{D})) + (0, \vec{D})) ((a, \vec{B}) + (0, \vec{D})(a, \vec{B})(0, -\vec{D}) / \text{norm}((0, \vec{D})) + (0, \vec{D}))^*\right) =$$

Figure 93: = the limit of (((a, B) + (0, D) times (a, B) times (0, D) conjugated over the norm of (0, D) + (0, D)) times the preceding sum conjugated) =

$$(a, \vec{B}') = (0, \vec{D})(a, \vec{B})(0, -\vec{D}) / \text{norm}((0, \vec{D}))$$

Figure 94: (a, B prime) = (0, D) times (a, B) times (0, -D) over the norm of (0, D)

$$= \lim \left(((a, \vec{B}) + (a, \vec{B}') + (0, \vec{D}))((a, \vec{B}) + (a, \vec{B}') + (0, \vec{D}))^*\right) =$$

Figure 95: = the limit of (((a, B) + (a, B prime) + (0, D)) times ((a, B) + (a, B prime) + (0, D)) conjugated) =

$$= \lim \left(4\,a\hat{\ }2 + 2\,\vec{B}.\vec{B} + 2\,\vec{B}.\vec{B}' + 2\,\vec{D}.\vec{B} + 2\,\vec{D}.\vec{B}', \vec{0}\right)$$

Figure 96: = the limit of (4 times a squared + 2 B squared + 2 B dot B prime + 2 D dot B + 2 D dot B prime, 0)

$$= \left(4\, a\,\hat{}\,2 + 2\, \vec{B}.\vec{B} + 2\, \vec{B}.\vec{B}', \vec{0}\right) <= \left| 2\, q \right|^2$$

Figure 97: = (4 a squared + 2 B squared + 2 B dot B prime, 0) is less than 2 q absolute value squared

everywhere, but it will require some skill to find a technically viable compromise between the spacelike and timelike derivative to synthesis the lightlike derivative.

Topological Properties of Quaternions

I have not taken a topology class, so no doubt this particular section contains errors.

Mathematician are so much more precise than I will ever be.

Topological Space

If we choose to work systematically through Wald's "General Relativity", the starting point is "Appendix A, Topological Spaces". Roughly, topology is the structure of relationships that do not change if a space is distorted. Some of the results of topology are required to make calculus rigorous.

In this section, I will work consistently with the set of quaternions, H^1, or just H for short. The difference between the real numbers R and H is that H is not a totally ordered set and multiplication is not commutative. These differences are not important for basic topological properties, so statements and proofs involving H are often identical to those for R.

First an open ball of quaternions needs to be defined to set the stage for an open set. Define an open ball in H of radius (r, 0) centered around a point (y, Y) [note: small letters are scalars, capital letters are 3-vectors] consisting of points (x, X) such that

$$\sqrt{((x - y, X - Y)^* (x - y, X - Y))} < (r, 0)$$

Figure 98: The square root of ((x - y, X - Y) conjugated times (x - y, X - Y)) is less than (r, 0)

An open set in H is any set which can be expressed as a union of open balls. [p. 423 translated] A quaternion topological space (H,T) consists of the set H together with a collection T of subsets of H with these properties:

1.The union of an arbitrary collection of subsets, each in T, is in T
2.The intersection of a finite number of subsets of T is in T
3.The entire set H and the empty set are in T

T is the topology on H. The subsets of H in T are open sets. Quaternions form a topology because they are what mathematicians call a metric space, since q* q evaluates to a real positive number or equals zero only if q is zero. Note: this is not the meaning of metric used by physicists. For example, the Minkowski metric can be negative or zero even if a point is not zero. To keep the same word with two meanings distinct, I will refer to one as the topological metric, the other as an interval metric. These descriptive labels are not used in general since context usually determines which one is in play.

An important component to standard approaches to general relativity is product spaces. This is how a topology for is created. Events in spacetime require R4, one place for time, three for space. Mathematicians get to make choices: what would change if work was done in R2, R3, or R5? The precision of this notion, together with the freedom to make choices, makes exploring these decisions fun (for those few who can understand what is going on :-)

By working with H, product spaces are unnecessary. Events in spacetime can be members of an open set in H. Time is the scalar, space the 3-vector. There is no choice to be made.

Open Sets

The edges of sets will be examined by defining boundaries, open and closed sets, and the interior and closure of a set.

I am a practical guy who likes pragmatic definitions. Let the real numbers L and U represent arbitrary lower and upper bounds respectively such that L < U. For the quaternion topological space (H, T), consider an arbitrary induced topology (A, t) where x and a are elements of A. Use inequalities to define:

$$\text{an open set}: \ (L, 0) < (x - a)^* (x - a) < (U, 0)$$

Figure 99: an open set: the lower bound (L, 0) is less than (x - a) conjugated times (x - a) is less than the upper bound (U, 0)

$$\text{a closed set}: \ (L, 0) \leq (x - a)^* (x - a) \leq (U, 0)$$

Figure 100: a closed set: (L, 0) is less than or equal to (x - a) conjugated times (x - a) is less than or equal to the upper bound (U, 0)

$$\text{a half open set}: \ (L, 0) \leq (x - a)^* (x - a) < (U, 0)$$

Figure 101: a half open set (L, 0) is less than or equal to (x - a) conjugated times (x - a) is less than the upper bound (U, 0)

The union of an arbitrary collection of open sets is open.
The intersection of a finite number of open sets is open.
The union of a finite number of closed sets is closed.
The intersection of an arbitrary number of closed sets is closed.

Clearly there are connections between the above definitions

$$\text{open set union boundary} \rightarrow \text{closed set}$$

This creates complementary ideas. [Wald, p.424]
The interior of A is the union of all open sets contained within A.

$$\text{or } (L, 0) < (x - a)^* (x - a) \leq (U, 0)$$

Figure 102: or (L, 0) is less than (x - a) conjugated times (x - a) is less than or equal to the conjugate of (U, 0)

$$\text{a boundary}: (L, 0) = (x - a)^* (x - a)$$

Figure 103: a boundary: the lower bound (L, 0) equals (x - a) conjugated times (x - a)

The interior equals A if and only if A is open.
The closure of A is the intersection of all closed sets containing A.
The closure of A equals A if and only if A is closed.

Define a point set as the set where the lower bound equals the upper bound. The only open set that is a point set is the null set. The closed point set is H. A point set for the real numbers has only one element which is identical to the boundary. A point set for quaternions has an infinite number of elements, one of them identical to the boundary.

What are the implications for physics?

With quaternions, the existence an open set of events has nothing to do with the causality of that collection of events.

$$\text{an open set}: (L, 0) < (x - a)^* (x - a) < (U, 0)$$

Figure 104: an open set: the lower bound (L, 0) is less than (x - a) conjugated times (x - a) is less than the upper bound (U, 0)

A proper time can have exactly the same absolute value as a pure spacelike separation, so these two will be included in the same sets, whether open, closed or on a boundary.

There is no correlation the reverse way either. Take for example a collection of lightlike events. Even though they all share exactly the same interval - namely zero - their absolute value can vary all over the map, not staying within limits.

Although independent, these two ideas can be combined synergistically. Consider an open set S of timelike intervals.

The set S could depict a classical world history since they are causally linked and have good topological properties. A closed set of lightlike events could be a focus of quantum electrodynamics. Topology plus causality could be the key for subdividing different regions of physics.

Hausdorff Topology

$$\text{timelike events} : \text{scalar}\left((x - a)^2\right) > (0,\ 0)$$

Figure 105: timelike events: the scalar of ((x - a) squared) is greater than 0

$$\text{lightlike events} : \text{scalar}\left((x - a)^2\right) = (0,\ 0)$$

Figure 106: lightlike events: the scalar of ((x - a) squared) = 0

This property is used to analyze compactness, something vital for rigorously establishing differentiation and integration.

[Wald p424] The quaternion topological space (H, T) is Hausdorff because for each pair of distinct points a, b E H, a not equal to b, one can find open sets Oa, Ob E T such that a E Oa, b I Ob and the intersection of Oa and Ob is the null set.

For example, find the half-way point between a and b. Let that be the radius of an open ball around the points a and b:

Neither set quite reaches the other, so their intersection is null.

Compact Sets

In this section, I will begin an investigation of compact sets of quaternions. I hope to share some of my insights into this subtle but significant topic.

First we need the definition of a compact set of quaternions.

[Translation of Wald p. 424] Let A be a subset of the quaternions H. Set A could be opened, closed or neither. An open cover of A is the union of open sets {Oa} that contains A. A union of open sets is open and could have an infinite number of members. A subset of {Oa} that still covers A is called a subcover. If the subcover has a finite number of elements it is called a finite subcover. The set A subset of H is compact if every open cover of A has a finite subcover.

Let's find an example of a compact set of quaternions. Consider a set S composed of points with a finite number of absolute values:

The set S has an infinite number of members, since for any of the equalities, specifying the absolute value still leaves three degrees of freedom (if the domain had been x E R, then S would have had a finite number of elements). The set S can be covered by an open set {O} which could have an infinite number of

$$\text{spacelike events} : \text{scalar}\left((x - a)^2\right) < (0,\ 0)$$

Figure 107: spacelike events: the scalar of ((x - a) squared) is less than 0

S = {x, a E H, a fixed; U, L E R | (L, 0) < (x - a)* (x - a) < (U, 0), and scalar ((x - a)^2) > 0}

Figure 108: the set S = {x, a are elements of H, a is fixed; U, L are elements of R | (L, 0) is less than (x - a) conjugated times (x - a) is less than (U, 0), and scalar of ((x - a) squared) is greater than 0}

let (r, 0) = (a - b)* (a - b) / 4

Figure 109: let (r, 0) = (a - b) conjugated times (a - b) over 4

Oa = {a, x E H, a is fixed, r E R | (a - x)* (a - x) < r}

Figure 110: the set Oa = {a, x are elements of H, a is fixed, r is an element of R | (a - x) conjugated times (a - x) is less than r}

Ob = {b, x E H, b is fixed, r E R | (b - x)* (b - x) < r}

Figure 111: the set Ob = {b, x are elements of H, b is fixed, r is an element of R | (b - x) conjugated times (b - x) is less than r}

S = {x1, x2, ..., xn E H; a1, a2, ..., an E R, n is finite | (x1 * x1) ^ .5 = (a1, 0),
(x2 * x2) ^ .5 = (a2, 0), ...}

Figure 112: the set S = {x1, x2, ..., xn are elements of H; a1, a2, ..., an are elements of R, n is finite | the square root of x1 conjugated times x1 = (a1, 0), the square root of x2 conjugated times x2 = (a2, 0), ...}

members. There exists a subset {C} of {O} that is finite and still covers S. The subset {C} would have one member for each absolute value.

$$C = \left\{y \text{ E } \{O\},\ e \text{ E } R,\ e > 0 \;\middle|\; (a1 - e) < \sqrt{y^* y} < (a1 + e,\ 0),\ (a2 - e) < \sqrt{y^* y} < (a2 + e,\ 0),\ \ldots,\ \text{one y exists for each inequality}\right\}$$

Figure 113: the set C = {y E {O}, e E R, e is greater than 0 | (a1 - e) is less than the square root of y conjugated times y is less than (a1 + e, 0), (a2 - e) is less than the square root of y conjugated times y is less than (a2 + e, 0), ..., one y exists for each inequality}

Every set of quaternions composed of a finite number of absolute values like the set S is compact.

Notice that the set S is closed because it consists of a boundary without an interior. The link between compact, closed and bound set is important, and will be examined next

A compact set is a statement about the ability to find a finite number of open sets that cover a set, given any open cover. A closed set is the interior of a set plus the boundary of that set. A set is bound if there exists a real number M such that the distance between a point and any member of the set is less than M.

For quaternions with the standard topology, in order to have a finite number of open sets that cover the set, the set must necessarily include its boundary and be bound. In other words, to be compact is to be closed and bound, to be closed and bound is to be compact.

[Wald p. 425] Theorem 1 (Heine-Borel). A closed interval of quaternions S:

$$S = \left\{x \text{ E } H,\ a,\ b \text{ E } R,\ a < b \;\middle|\; (a,\ 0) \le \sqrt{x^* x} \le (b,\ 0)\right\}$$

Figure 114: the set S = {x an element of H, a, b elements of R, a less than b | (a, 0) is less than or equal to the square root of (x conjugated times x) is less than or equal to (b, 0)}

with the standard topology on H is compact.

Wald does not provide a proof since it appears in many books on analysis. Invariably the Heine-Borel Theorem employs the domain of the real numbers, x E R. However, nothing in that proof changes by using quaternions as the domain.

[Wald p. 425] Theorem 2. Let the topology (H, T) be Hausdorff and let the set A subset of H be compact. Then A is closed.

Theorem 3. Let the topology (H, T) be compact and let the set A subset of H be closed. Then A is compact.

Combine these theorems to create a stronger statement on the compactness of subsets of quaternions H.

Theorem 4. A subset A of quaternions is compact if and only if it is closed and bounded.

The property of compactness is easily proved to be preserved under continuous maps.

Theorem 5. Let (H, T) and (H', T') be topological spaces. Suppose (H, T) is compact and the function f: H -> H' is continuous. The f[H] = {h' E H' | h' = f(h)} is compact. This creates a corollary by theorem 4.

Theorem 6. A continuous function from a compact topological space into H is bound and its absolute value attains a maximum and minimum values.

[end translation of Wald]

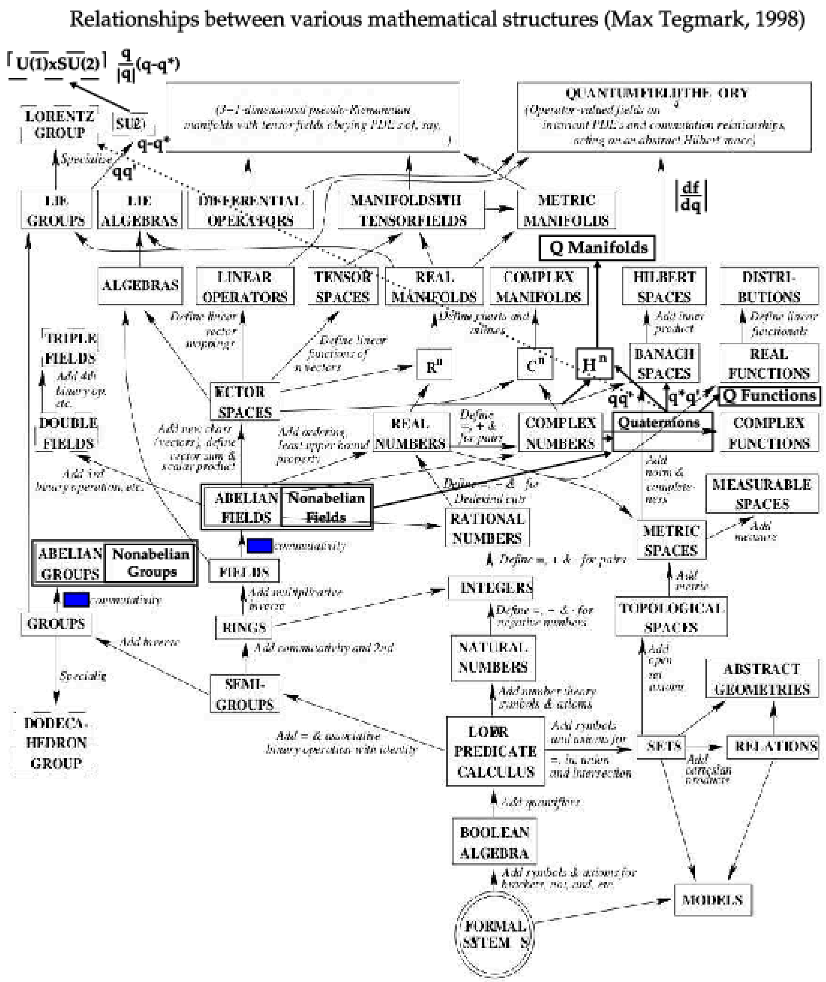
R1 versus Rn

It is important to note that these theorems for quaternions are build directly on top of theorems for real numbers, R1. Only the domain needs to be changed to H1. Wald continues with theorems on product spaces, specifically Tychonoff's Theorem, so that the above theorems can be extended to Rn. In particular, the product space R4 should have the same topology as the quaternions.

Hopefully, subtlety matters in the discussion of the logical foundations of general relativity. Both R1 and H1 have a rule for multiplication, but H1 has an antisymmetric component. This is a description of a difference. R4 does not come equipped with a rule for multiplication, so it is qualitatively different, even if topologically similar to the quaternions.

Where do quaternions fit in with math?

Adapted from a figure by Max Tegmark, 1998.



A Quaternion Algebra Tool Set

Here is a compilation of basic algebra for quaternions. It should look very similar to complex algebra, since it contains three sets of complex numbers, t + x i, t + y j, and t + z k. To strengthen the link, and keep things looking simpler, all quaternions have been written as a pair of a scalar t and a 3-vector V, as in (t, V). All these relations have been tested in a C library and a Java quaternion calculator.

Technical note: it is vital that every tool in this set can be expressed as working with a whole quaternion q. This will make doing quaternion analysis with automorphic functions fruitful.

Parts

$$\mathrm{scalar}\,(q) = (q + q^*) / 2 = (t, 0)$$

Figure 115: The scalar of q equals q plus its conjugate over two equals (t, zero)

$$\mathrm{vector}\,(q) = (q - q^*) / 2 = (0, V)$$

Figure 116: The vector of q equals q minus its conjugate over two equals (zero, V)

Simple algebra

$$|\, q \,| = \sqrt{(q q^*)} = \left(\sqrt{t^2 + V.V},\ 0\right)$$

Figure 117: The absolute value of q equals the square root of q times its conjugate equals (the square root of t squared plus V dot V, 0)

Multiplication

The Grassman product as defined here uses the same rule Hamilton developed. The Euclidean product takes the conjugate of the first of the two elements (following a tradition from quantum mechanics).

Trigonometry

Note: since the unit vectors of sine and cosine are the same, these two commute so the order is irrelevant.

$$\mathrm{norm}(q) = qq^* = (t^2 + V.V,\ 0)$$

Figure 118: The norm of q equals q times its conjugate equals (t squared plus V dot V, 0)

$$\det(q) = (qq^*)^2 = \left((t^2 + V.V)^2,\ 0\right)$$

Figure 119: The determinant of q equals q times its conjugate squared equals (the square of t squared plus V dot V, 0)

$$\mathrm{sum}(q, q') = (t + t',\ V + V')$$

Figure 120: The sum of q and q prime equals (t plus t', V plus V')

$$\mathrm{dif}(q, q') = (t - t',\ V - V')$$

Figure 121: The difference of q and q prime equals (t minus t', V minus V')

$$\mathrm{conj}(q) = q^* = (t,\ -V)$$

Figure 122: The conjugate of q equals q* equals (t, minus V)

$$\mathrm{inv}(q) = q^* / (qq^*) = (t,\ -V) / (t^2 + V.V)$$

Figure 123: The inverse of q equal q conjugated over its norm equals (t, -V) over (t squared plus V dot V).

$$\mathrm{adj}(q) = q^*(qq^*) = (t,\ -V)\ \mathrm{norm}(q)$$

Figure 124: The adjoint of q equals q conjugated times its norm equals (t, -V) times (t squared plus V dot V).

$$\mathrm{Grassman\_product}(q, q') = qq' = (tt' - V.V',\ tV' + Vt' + V \times V')$$

Figure 125: The Grassman product of q and q' equals q times q prime equals (t t prime minus V dot V prime, t V prime plus V t prime plus V cross V prime).

$$\mathrm{Grassman\_even\_product}(q, q') = \frac{qq' + q'q}{2} = (tt' - V.V',\ tV' + Vt')$$

Figure 126: The Grassman even product of q and q' equals q times q prime plus q prime q over two equals (t t prime minus V dot V prime, t V prime plus V t prime).

$$\text{Grassman\_odd\_product}(q, q') = \frac{qq' - q'q}{2} = (0, V \times V')$$

Figure 127: The Grassman odd product of q and q' equals q times q prime minus q prime q over two equals (zero, V cross V prime).

$$\text{Euclidean\_product}(q, q') = q^* q' = (tt' + V.V', tV' - Vt' - V \times V')$$

Figure 128: The Euclidean product of q and q' equals q conjugated times q prime equals (t t prime plus V dot V prime, t V prime minus V t prime minus V cross V prime).

$$\text{Euclidean\_even\_product}(q, q') = \frac{q^* q' + q' q^*}{2} = (tt' + V.V', 0)$$

Figure 129: The Euclidean even product of q and q' equals q conjugated times q prime plus q prime q conjugated over two equals (t t prime plus V dot V prime, zero).

$$\text{Euclidean\_odd\_product}(q, q') = \frac{q^* q' - q' q^*}{2} = (0, tV' - Vt' - V \times V')$$

Figure 130: The Euclidean odd product of q and q' equals q conjugated times q prime minus q prime q conjugated over two equals (zero, t V prime minus V t prime minus V cross V prime).

$$\sin(q) = (\sin(t)\cosh(|V|), \cos(t)\sinh(|V|)V/|V|)$$

Figure 131: The sine of q equals (sin t hyperbolic cosine absolute value of V, cosine t hyperbolic sine of the absolute value of V times V normalized to V)

$$\cos(q) = (\cos(t)\cosh(|V|), -\sin(t)\sinh(|V|)V/|V|)$$

Figure 132: The cosine of q equals (cos t hyperbolic cosine absolute value of V, minus sine t hyperbolic sine of the absolute value of V times V normalized to V)

$$\tan(q) = \sin(q)/\cos(q)$$

Figure 133: The tangent of q equals the sine of q times the inverse of the cosine of q

$$\text{asin}(q) = -V/|V|\,\text{asinh}(qV/|V|)$$

Figure 134: The arcsine of q equals minus V normalized to V times the hyperbolic arcsine of q times V normalized to V.

$$acos(q) = -V/|V|\, acosh(q)$$

Figure 135: The arccosine of q equals minus V normalized to V times the hyperbolic arccosine of q.

$$atan(q) = -V/|V|\, atanh(q V/|V|)$$

Figure 136: The arctangent of q equals minus V normalized to V times the hyperbolic arctangent of q times V normalized to V.

$$sinh(q) = (sinh(t) \cos(|V|), cosh(t) \sin(|V|) V/|V|)$$

Figure 137: The hyperbolic sine of q equals (hyperbolic sin t cosine absolute value of V, hyperbolic cosine t sine of the absolute value of V times V normalized to V)

$$cosh(q) = (cosh(t) \cos(|V|), sinh(t) \sin(|V|) V/|V|)$$

Figure 138: The hyperbolic cosine of q equals (hyperbolic cos t cosine absolute value of V, hyperbolic sine t sine of the absolute value of V times V normalized to V)

$$tanh(q) = sinh(q) / cosh(q)$$

Figure 139: The hyperbolic tangent of q equals the hyperbolic sine of q times the inverse of the hyperbolic cosine of q

$$asinh(q) = ln(q + (q\hat{}2 + 1)\hat{}.5)$$

Figure 140: The hyperbolic arcsine of q equals the natural log of (q plus the square root of q squared plus q).

$$acosh(q) = ln(q + /-(q\hat{}2 - 1)\hat{}.5)$$

Figure 141: The hyperbolic arccosine of q equals the natural log of (q plus or minus the square root of q squared minus one).

$$atanh(q) = .5\, ln((1 + q) / (1 - q))$$

Figure 142: The hyperbolic arctangent of q equals one half times the natural log of (one plus q over one minus q).

Powers

```
exp (q) = (exp (t) cos ( | V | ), exp (t) sin ( | V | ) V/ | V | )
```

Figure 143: The exponential of q equals (e to the t cosine absolute value of V, e to the t sine of the absolute value of V times V normalized to V)

```
q^q' = exp (ln (q) x q')
```

Figure 144: q raised to the q prime equals the exponential of the natural log of q time q prime.

Logs

```
ln (q) = (0.5 ln (t^2 + V.V), atan2 ( | V |, t) V/ | V | )
```

Figure 145: The natural log of q equals (one half times the natural log of t squared plus V dot V, the arctan of absolute value of V, angle t time V normalized to V.

Quaternion exponential multiplication

Andrew Millard suggested the result for the Grassman product.

```
log (q) = ln (q) / ln (10)
```

Figure 146: The log base 10 equals the natural log of q over the natural log of 10.

```
q q' = {q, q'} + | [q, q'] | exp (pi[q, q'] / 2 | [q, q'] |)
```

Figure 147: The Grassman product of two exponentials q and q' equals the even Grassman product times the absolute value of the odd Grassman product times the exponential of pi over 2 times the odd Grassman product normalized to itself.

Newton's Second Law

The form of Newton's second law for three separate cases will be generated using quaternion operators acting on position quaternions. In classical mechanics, time and space are decoupled. One way that can be achieved algebraically is by having a time operator cat only on space, or by space operator only act on a scalar function. I call this the "2 zero" rule: if there are two zeros in the generator of a law in physics, the law is classical.

Newton's 2nd Law for an Inertial Reference Frame in Cartesian

Coordinates

Define a position quaternion as a function of time.

Operate on this once with the differential operator to get the velocity quaternion.

Operate on the velocity to get the classical inertial acceleration quaternion.

This is the standard form for acceleration in Newton's second law in an inertial reference frame. Because the reference frame is inertial, the first term is zero.

Newton's 2nd Law in Polar Coordinates for a Central Force in a Plane

Repeat this process, but this time start with polar coordinates.

```
q*q' = {q*, q'} + | [q*, q'] | exp (pi[q*, q'] / 2 | [q*, q'] |)
```

Figure 148: The Euclidean product of two exponentials q and q' equals the even Euclidean product times the absolute value of the odd Euclidean product times the exponential of pi over 2 times the odd Euclidean product normalized to itself.

$$R = (t, \vec{R})$$

Figure 149: R = (t, R)

$$V = \left(\frac{d}{dt}, \vec{0}\right) (t, \vec{R}) = \left(1, \dot{\vec{R}}\right)$$

Figure 150: V = (d by dt, 0) acting on R = (1, R dot)

The velocity in a plane.

Acceleration in a plane.

Not a pretty sight. For a central force, $\dot{\theta} = L/mr^2$, and $\ddot{\theta} = 0$. Make these substitution and rotate the quaternion to get rid of the theta dependence.

The second term is the acceleration in the radial direction, the third is acceleration in the theta direction for a central force in polar coordinates.

Newton's 2nd Law in a Noninertial, Rotating Frame

Consider the "noninertial" case, with the frame rotating at an angular speed omega. The differential time operator is put into the first term of the quaternion, and the three directions for the angular speed are put in the next terms. This quaternion is then multiplied by the position quaternion to get the velocity in a rotating reference frame. Unlike the previous examples where t did not interfere with the calculations, this time it must be set explicitly to zero (I wonder what that means?).

Operate on the velocity quaternion with the same operator.

The first three terms of the 3-vector are the translational, coriolis, and azimuthal alterations respectively. The last term of the 3-vector may not look like the centrifugal force, but using a vector identity it can be rewritten:

If the angular velocity an the radius are orthogonal, then

The scalar term is not zero. What this implies is not yet clear, but it may be

$$A = \left(\frac{d}{dt}, \vec{0}\right) \left(1, \dot{\vec{R}}\right) = \left(0, \ddot{\vec{R}}\right)$$

Figure 151: A = (d by dt, 0) squared acting on R = (0, R double dot)

$$R = (t,\ r\,\mathrm{Cos}[\theta],\ r\,\mathrm{Sin}[\theta],\ 0)$$

Figure 152: R = (t, r cosine theta, r sine theta, 0)

$$V = \left(\frac{d}{dt},\ \vec{0}\right)\ (t,\ r\,\mathrm{Cos}[\theta],\ r\,\mathrm{Sin}[\theta],\ 0) =$$

Figure 153: V = (d by dt, 0) acting on (t, r cosine (theta), r sine (theta, 0) =

$$= (1,\ \dot{r}\,\mathrm{Cos}[\theta] - r\,\mathrm{Sin}[\theta]\,\dot{\theta},\ \dot{r}\,\mathrm{Sin}[\theta] + r\,\mathrm{Cos}[\theta]\,\dot{\theta},\ 0)$$

Figure 154: = (1, r dot cosine theta - r sine theta theta dot, r dot sine theta + r cosine theta theta dot, 0)

$$A = \left(\frac{d}{dt},\ \vec{0}\right)\ (1,\ \dot{r}\,\mathrm{Cos}[\theta] - r\,\mathrm{Sin}[\theta]\,\dot{\theta},\ \dot{r}\,\mathrm{Sin}[\theta] + r\,\mathrm{Cos}[\theta]\,\dot{\theta},\ 0) =$$

Figure 155: A = (d by dt, 0) acting on (1, r dot cosine theta - r sine theta theta dot, r dot sine theta + r cosine theta theta dot, 0) =

$$= \left(0,\ -2\,\dot{r}\,\mathrm{Sin}[\theta]\,\dot{\theta} - r\,\mathrm{Cos}[\theta]\,\dot{\theta}^2 + \ddot{r}\,\mathrm{Cos}[\theta] - r\,\mathrm{Sin}[\theta]\,\ddot{\theta},\ 2\,\dot{r}\,\mathrm{Cos}[\theta]\,\dot{\theta} - r\,\mathrm{Sin}[\theta]\,\dot{\theta}^2 + \ddot{r}\,\mathrm{Sin}[\theta] + r\,\mathrm{Cos}[\theta]\,\ddot{\theta},\ 0\right)$$

Figure 156: = (0, - 2 sine theta R dot theta dot - cosine theta R (theta dot)squared + cosine theta R double dot - R sine theta theta double dot,2 cosine theta R dot theta dot -R sine theta (theta dot)squared + sine theta R double dot + cosine theta R theta double dot, 0)

$$A = (\mathrm{Cos}[\theta],\ 0,\ 0,\ -\mathrm{Sin}[\theta])\ \left(\frac{d}{dt},\ \vec{0}\right)^2\ (t,\ r\,\mathrm{Cos}[\theta],\ r\,\mathrm{Sin}[\theta],\ 0) =$$

Figure 157: A = (cosine theta, 0, 0, - sine theta) times (d by dt,0) squared acting on R =

$$= \left(0,\ \frac{L^2}{m^2\,r^3} + \ddot{r},\ \frac{2\,L\,\dot{r}}{m\,r^2},\ 0\right)$$

Figure 158: = (0, -L squared over m squared R cubed + R double dot, 2 L R dot over m R squared , 0)

$$V = \left(\frac{d}{dt},\ \vec{\omega}\right)\ (0,\ \vec{R}) = \left(-\ \vec{\omega}.\vec{R}\ ,\ \dot{\vec{R}}\ +\ \vec{\omega}\times\vec{R}\right)$$

Figure 159: V = (d by dt, Omega) acting on (0,R) = (- omega dot R, R dot + Omega Cross R)

$$A = \left(\frac{d}{dt} , \vec{\omega} \right) \left(- \vec{\omega} . \vec{R} , \dot{\vec{R}} + \vec{\omega} \times \vec{R} \right) =$$

Figure 160: A = (d by dt, Omega) acting on (- omega dot R, R dot + Omega Cross R)

$$= \left(- \dot{\vec{\omega}} . \vec{R} , \ddot{\vec{R}} + 2 \vec{\omega} \times \dot{\vec{R}} + \dot{\vec{\omega}} \times \vec{R} - \vec{\omega} . \vec{R} \vec{\omega} \right)$$

Figure 161: = (- Omega dot dot R, R double dot + 2 Omega Cross R dot + Omega dot Cross R - Omega dot R Omega)

$$- \vec{\omega} . \vec{R} \vec{\omega} = - \vec{\omega} \times (\vec{\omega} \times \vec{R}) + \vec{\omega}^2 \vec{R}$$

Figure 162: - Omega dot R Omega = - Omega Cross (Omega Cross R) + Omega squared R

$$\vec{\omega} \times (\vec{\omega} \times \vec{R}) = \vec{\omega}^2 \vec{R} \text{ iff } \vec{\omega} . \vec{R} = 0$$

Figure 163: Omega Cross (Omega Cross R) = Omega squared R if and only if Omega dot R = 0

related to the fact that the frame is not inertial.

Implications

Three forms of Newton's second law were generated by choosing appropriate operator quaternions acting on position quaternions. The differential time operator was decoupled from any differential space operators. This may be viewed as an operational definition of "classical" physics.

Oscillators and Waves

A professor of mine once said that everything in physics is a simple harmonic oscillator. Therefore it is necessary to get a handle on everything.

The Simple Harmonic Oscillator (SHO)

The differential equation for a simple harmonic oscillator in one dimension can be express with quaternion operators.

$$\left(\frac{d}{dt}, \vec{0}\right)^2 (0,\ x,\ 0,\ 0) + \left(0,\ \frac{k}{m}x,\ 0,\ 0\right) = \left(0,\ \frac{d^2 x}{dt^2} + \frac{k\,x}{m},\ 0,\ 0\right) = 0$$

Figure 164: (d by dt, 0) squared acting on (0, x, 0, 0) + (0, k x over m, 0, 0) = (0,x double dot + k x/m, 0, 0) = 0

This equation can be solved directly.

$$x \to C[2]\ \mathrm{Cos}\left[\frac{\sqrt{k}\ t}{\sqrt{m}}\right] + C[1]\ \mathrm{Sin}\left[\frac{\sqrt{k}\ t}{\sqrt{m}}\right]$$

Figure 165: x = C(2) cosine (the square root of k over m times t) + C(1) sine (the square root of k over m times t)

Find the velocity by taking the derivative with respect to time.

$$\dot{x} \to \frac{\sqrt{k}\ C[1]\ \mathrm{Cos}\left[\frac{\sqrt{k}\ t}{\sqrt{m}}\right]}{\sqrt{m}} - \frac{\sqrt{k}\ C[2]\ \mathrm{Sin}\left[\frac{\sqrt{k}\ t}{\sqrt{m}}\right]}{\sqrt{m}}$$

Figure 166: x dot = the square root of k over m times C(1) cosine (the square root of k over m times t) - the square root of k over m times times C(2) sine (the square root of k over m times t))

The Damped Simple Harmonic Oscillator

Generate the differential equation for a damped simple harmonic oscillator as done above.

Solve the equation.

$$\left(\frac{d}{dt}, \vec{0}\right)^2 (0, x, 0, 0) + \left(\frac{d}{dt}, \vec{0}\right) (0, b x, 0, 0) + \left(0, \frac{k}{m} x, 0, 0\right) =$$

Figure 167: (d by dt,0) squared acting on (0,x, 0, 0) + (d by dt,0) acting on (0, b x, 0, 0) + (0, k x over m,0,0) =

$$= \left(0, \frac{d^2 x}{dt^2} + \frac{b\, d x}{dt} + \frac{k x}{m}, 0, 0\right) = 0$$

Figure 168: = (0, x double dot + b x dot + k x over m, 0, 0) = 0

The Wave Equation

Consider a wave traveling along the x direction. The equation which governs its motion is given by

The third term is the one dimensional wave equation. The forth term is the instantaneous power transmitted by the wave.

Implications

Using the appropriate combinations of quaternion operators, the classical simple harmonic oscillator and wave equation were written out and solved. The functional definition of classical physics employed here is that the time operator is decoupled from any space operator. There is no reason why a similar combination of operators cannot be used when time and space operators are not decoupled. In fact, the four Maxwell equations appear to be one nonhomogeneous quaternion wave equation, and the structure of the simple harmonic oscillator appears in the Klein-Gordon equation.

$$x \to C[1]\, E^{\frac{\left(-b m-\sqrt{-4 k m+b^2 m^2}\right) t}{2 m}} + C[2]\, E^{\frac{\left(-b m+\sqrt{-4 k m+b^2 m^2}\right) t}{2 m}}$$

Figure 169: x = c1 times e to the - b - the square root of b squared - 4 k over m over 2 times t + c2 times e to the - b + the square root of b squared - 4 k over m over 2 times t

$$\left(\frac{d}{v\,dt},\ \frac{d}{dx},\ 0,\ 0\right)^2 (0,\ 0,\ f[t\,v + x],\ 0) =$$

Figure 170: (1/v d by dt,d by dx, 0, 0) squared acting on (0, 0, f(t v + x), 0) =

$$= \left(0,\ 0,\ \left(-\frac{d^2}{dx^2} + \frac{d^2}{dt^2\,v^2}\right) f[t\,v+x],\ \frac{2\,d^2\,f[t\,v+x]}{dt\,dx\,v}\right)$$

Figure 171: =(0, 0, (-d squared by d x squared +1/v squared d squared by d t squared) f(t v+x), 2 d squared f(t v + x)/v dt dx)

Four Tests for a Conservative Force

There are four well-known, equivalent tests to determine if a force is conservative: the curl is zero, a potential function whose gradient is the force exists, all closed path integrals are zero, and the path integral between any two points is the same no matter what the path chosen. In this notebook, quaternion operators perform these tests on quaternion-valued forces.

1. The Curl Is Zero

To make the discussion concrete, define a force quaternion F.

$$F = (0,\ -k\,x,\ -k\,y,\ 0)$$

Figure 172: F = (0, - k x, - k y, 0)

The curl is the commutator of the differential operator and the force. If this is zero, the force is conservative.

Let the differential operator quaternion act on the force, and test if the vector components equal zero.

2. There Exists a Potential Function for the Force

Operate on force quaternion using integration. Take the negative of the gradient of the first component. If the field quaternion is the same, the force is conservative.

This is the same force as we started with, so the scalar inside the integral is the scalar potential of this vector field. The vector terms inside the integral arise as constants of integration. They are zero if t=z=0. What role these vector terms in the potential quaternion may play, if any, is unknown to me.

$$\left[\left(\frac{d}{dt}, \vec{\nabla}\right), \vec{F}\right] = 0$$

Figure 173: The commutator of the operator(d by dt,Del) acting on a function F = 0

$$\left(\frac{d}{dt}, \nabla\right) F = (2k, 0, 0, 0)$$

Figure 174: (d by dt, Del) acting on F = (2 k, 0, 0, 0)

$$F = \int F\,(dt, dx, dy, dz) =$$

Figure 175: F = integral F times (dt, dx, dy, dz)

$$= \int (k\,x\,dx + k\,y\,dy, -k\,x\,dt + k\,y\,dz, -k\,y\,dt - k\,x\,dz, 0) =$$

Figure 176: = integral(k x dx +k y dy, - k x dt + k y dz, - k y dt - k x dz, 0)

$$= \left(\frac{k\,x^2}{2} + \frac{k\,y^2}{2}, -k\,t\,x + k\,y\,z, -k\,t\,y - k\,x\,z, 0\right) =$$

Figure 177: =(k x squared over 2 + k y squared over 2, -k t x + k y z, - k t y - k x z, 0)

$$\left(\frac{d}{dt}, \vec{\nabla}\right)\left(\frac{k\,x^2}{2} + \frac{k\,y^2}{2}, \vec{0}\right) = (0, -k\,x, -k\,y, 0)$$

Figure 178: (d by dt,Del) acting on (k x squared over 2 + k y squared over 2, 0) = (0, - k x, - k y, 0)

3. The Line Integral of Any Closed Loop Is Zero

Use any parameterization in the line integral, making sure it comes back to go.

$$\text{path} = (0,\ r\,\text{Cos}(t),\ r\,\text{Sin}(t),\ 0)$$

Figure 179: path = (0, r cosine (t), r sine (t), 0)

$$\int_0^{2\pi} F\,dt = 0$$

Figure 180: integral from 0 to 2 pi F dt = 0

4. The Line Integral Along Different Paths Is the Same

Choose any two parameterizations from A to B, and test that they are the same. These paths are from (0, r, 0, 0) to (0, -r, 2 r, 0).

$$\text{path1} = \left(0,\ r\,\text{Cos}(t),\ 2\,r\,\text{Sin}\left(\frac{t}{2}\right),\ 0\right)$$

Figure 181: path 1 = (0, r cosine (t), 2 r sine (t over 2), 0)

The same!

Implications

The four standard tests for a conservative force can be done with operator quaternions. One new avenue opened up is for doing path integrals. It would be interesting to attempt four dimensional path integrals to see where that might lead!

$$\int_0^{2\pi} dt = -2\, k\, r^2$$

Figure 182: integral from 0 to 2 pi along path 1 F dt = - 2 k r squared

$$\text{path2} = (0,\ -t\, r + r,\ t\, r,\ 0)$$

Figure 183: path 2 = (0, - t r + r, t r, 0)

Problem set answers for MITs 8.033 using real valued quaternions

Problem set 1

Problem sets 2-6 are at another site, Doing Special Relativity with Quaternions

The Back Story

In early 1997, I had a meeting with a famous physicist to discuss my research project. Actually, I sat outside his door and talked with him to and from the chinese food truck. He thought my project was "Interesting, but not very interesting." The reason was that I had a few math widgets, but no theory. I thanked him for his time.

Now I had to find a theory. This is a crazy assignment. I decided to begin my search by posing a question: define a brief definition of time that must be two sentences or less and only be about phyiscs or math, not philosophy. My runon sentence answer used quaternions as a definition of events in space-time.

To test the hypothesis, I asked the professor who taught the class if I could audit 8.033, Classical and Relativitic Mechanics (it is now online). He approved. As a test of quaternions as an essential tool for physics, I had three ground rules for the assigned problems:

1. Each problem had to be solved the standard way
2. Each problem had to be solved using real-valued quaternions

$$\int_0^{2} F\, dt = -2\, k\, r^2$$

Figure 184: integral from 0 to 2 pi along path 1 F dt = - 2 k r squared

3. If any problem could not be solved with real-valued quaternions, there would be no need to look further into quaternions

It turned out that all 53 assigned problems were solved using real-valued quaternions. That was the start of my ongoing study. To avoid the book being overrun with the problem set answers, they were moved to a separate book.

8.033 Problem Set 1, Kinematic Effects of Relativity

Preamble: Initiation functions

There are a few tools required to solve problems in special relativity using quaternions to characterize events in spacetime. The most basic are gamma and a round value for c.

$$\gamma[\beta\_] := \frac{1}{\sqrt{1 - \beta^2}}$$

Figure 185: [Graphics:ps1gr1. gif]

$$c = 3\ 10^8\, m / s;$$

Figure 186: [Graphics:ps1gr3. gif]

Define a function for quaternions using its matrix representation.

$$q[t\_,\ x\_,\ y\_,\ z\_] := \begin{pmatrix} t & -x & -y & -z \\ x & t & -z & y \\ y & z & t & -x \\ z & -y & x & t \end{pmatrix}$$

Figure 187: [Graphics:ps1gr4. gif]

A quaternion L that perform a transform on a quaternion -
L q[**x**] = q[**x'**] - identical to how the Lorentz transformation acts on 4-vectors - Lambda **x** = **x'** - should exist. These are described in detail in the notebook "A different algebra for boosts." For boosts along the x axis with y = z = 0, the general function for L is

Most of the problems here involve much simpler cases for L, where t or x is zero, or t is equal to x.

If t = 0, then

If x = 0, then

If t = x, then

Note: this is for blueshifts. Redshifts have a plus instead of the minus.

$$L[t\_,\ x\_,\ \beta\_] := \frac{1}{t^2 + x^2}\, q[\gamma[\beta]\, t^2 - 2\,\gamma[\beta]\,\beta\, t\, x + \gamma[\beta]\, x^2,\ -\beta\,\gamma[\beta]\,(t^2 - x^2),\ 0,\ 0]$$

Figure 188: [Graphics:ps1gr5. gif]

$$L[0,\ x,\ \beta]\ .\ \{1,\ 0,\ 0,\ 0\}$$

Figure 189: [Graphics:ps1gr6.gif]

$$\left\{\frac{1}{\sqrt{1-\beta^2}},\ \frac{\beta}{\sqrt{1-\beta^2}},\ 0,\ 0\right\}$$

Figure 190: [Graphics:ps1gr7.gif]

$$L[t,\ 0,\ \beta]\ .\ \{1,\ 0,\ 0,\ 0\}$$

Figure 191: [Graphics:ps1gr8.gif]

$$\left\{\frac{1}{\sqrt{1-\beta^2}},\ -\frac{\beta}{\sqrt{1-\beta^2}},\ 0,\ 0\right\}$$

Figure 192: [Graphics:ps1gr9.gif]

$$\mathrm{Simplify}[L[t,\ t,\ \beta]\ .\ \{1,\ 0,\ 0,\ 0\}]$$

Figure 193: [Graphics:ps1gr10 .gif]

$$\left\{\frac{1-\beta}{\sqrt{1-\beta^2}},\ 0,\ 0,\ 0\right\}$$

Figure 194: [Graphics:ps1gr11.gif]

The problems are from “Basic Concepts in Relativity” by Resnick and Halliday, 1992 by Macmillian Publishing, “Special Relativity” by A. P. French, 1966, 1968 by MIT, and Prof. M. Baranger of MIT.

R&H 2-9: A moving clock

Q: A clock moves along the x axis at a speed of 0.6c and reads zero as it passes the origin. What time does it read as it passes the 180 m mark on the x axis?

A: A clock measures an interval between two events. The first event occurs at the origin. The second event happens at 180 m in a time of 180 m/v. Calculate the interval by squaring the difference quaternion and then taking the square root of the first term.

$$\sqrt{(q[180\,m/(.6\,c),\ 180\,m/c,\ 0,\ 0]\ .\ q[180\,m/(.6\,c),\ 180\,m/c,\ 0,\ 0])[[1,\ 1]]}$$

Figure 195: [Graphics:ps1gr12 .gif]

$$8.\times 10^{-7}\sqrt{s^2}$$

Figure 196: [Graphics:ps1gr13.gif]

The moving clock reads 8 x 10^{-7} seconds.

R&H 2-10: A moving rocket

Q: A rod lies parallel to the x axis of reference frame S, moving along this axis at a speed of 0.6c. Its rest length is 1.0 m. What will be its measured length in frame S?

A: Consider the meter stick at rest in a frame S’, one end at the origin, the other at q[0, 1 m, 0, 0]. We want to boost the stick end quaternion to frame S. The boost quaternion when t=y=z=0 is $L = q[\gamma,\ -\gamma\beta,\ 0,\ 0]$. In frame S’, frame S is moving at -0.6c.

$$stick_{end} = q[\gamma[-0.6],\ -0.6\,\gamma[-0.6],\ 0,\ 0]\ .\ q[0,\ 1\,m,\ 0,\ 0];$$

Figure 197: [Graphics:ps1gr16 .gif]

The start of the stick will move for a time equal to the first term of the boosted quaternion, and moved by a distance x = vt/c.

```
stick_start = q[stick_end[[1, 1]],
  0.6 stick_end[[1, 1]], 0, 0];
```

Figure 198: [Graphics:ps1gr17 .gif]

The meter stick's length in frame S will be the difference at the same instant in this frame between the boosted stick end and translocated stick start.

```
stick_length = (stick_end - stick_start) . {1, 0, 0, 0
```

Figure 199: [Graphics:ps1gr18 .gif]

```
{0. m, 0.8 m, 0, 0}
```

Figure 200: [Graphics:ps1gr19.gif]

The meter stick is length contracted to 0.8 meters in frame S.

R&H 2-13: A fast spaceship

Q: The length of a spaceship is measured to be exactly half its rest length. (a) What is the speed of the spaceship relative to the observer's frame? (b) By what factor does the spaceship's clocks run slow, compared to clocks in the observer's frame?

A: (a) Consider the spaceship at rest, one end at the origin, the other at q[0, d, 0, 0]. We want to boost the ship end quaternion to the observer's frame. The boost quaternion when t=y=z=0 is $L = q[\gamma, \gamma\beta, 0, 0]$. In the ship's frame, the observer is moving at -v/c.

The start of the ship will move for a time equal to the first term of the boosted quaternion, and moved by a distance x = vt/c.

The ship's length in the observer's frame will be the difference at the same instant in this frame between the boosted ship end and translocated ship start.

Solve for beta setting this distance to d/2.

Beta is $\sqrt{3}$ /2 = 0.866.

(b) The factor that the clocks appear to run at different rates is gamma.

$ship_{end} =$
$q[\gamma[-\beta], -\beta\gamma[-\beta], 0, 0] . q[0, d, 0, 0];$

Figure 201: [Graphics:ps1gr21 .gif]

$ship_{start} =$
$q[ship_{end}[[1, 1]], \beta\ ship_{end}[[1, 1]], 0, 0];$

Figure 202: [Graphics:ps1gr22 .gif]

$ship_{length} = (ship_{end} - ship_{start}) . \{1, 0, 0, 0\}$

Figure 203: [Graphics:ps1gr23 .gif]

$$\left\{0, \frac{d}{\sqrt{1-\beta^2}} - \frac{d\,\beta^2}{\sqrt{1-\beta^2}}, 0, 0\right\}$$

Figure 204: [Graphics:ps1gr24.gif]

$Solve[ship_{length}[[2]] == d/2, \beta]$

Figure 205: [Graphics:ps1gr25 .gif]

$$\left\{\left\{\beta \to -\frac{\sqrt{3}}{2}\right\}, \left\{\beta \to \frac{\sqrt{3}}{2}\right\}\right\}$$

Figure 206: [Graphics:ps1gr26.gif]

$$\gamma\left[\frac{\sqrt{3}}{2}\right]$$

Figure 207: [Graphics:ps1gr28 .gif]

$$2$$

Figure 208: [Graphics:ps1gr29.gif]

Totally-ordered and Disordered Sets in Space-time under Lorentz Transformations

In Einstein's first paper on special relativity in 1905, he shows how simultaneity is not an invariant under a Lorentz transformation. While there may be an observer that says events A and B happened at the same moment in time, a second observer may say A happened before B while a third reports B happened before A. Is there something that all three observers can agree upon about events A and B, that is invariant under a proper Lorentz transformation? For space-like separated events A and B, so long as the observers have put in the effort to agree about their choices in coordinates and the origin, then the three observers will all agree on the ordering in space of these events in space. If the first observer says A is left of B, then so do observers two and three. If events A and B were in the same location for one of the dimensions, they would remain together. A totally-ordered set means one can say exactly one of three things about any pair of numbers: one is bigger than the other, one is less than the other, or both have the same value. An axis on a graph is a totally-ordered set. Here we are thinking about pairs of space-like events that can be connected by a straight line that runs through the origin. Under a Lorentz transformation, the time for these pairs can switch order depending on the inertial observer chosen. I will call this a disordered in time set of events for space-like separated events under proper Lorentz transformations. The measurements of space will form three totally-ordered sets.

The same exercise can be repeated for all time-like pairs events that fall on a line running through the origin. These pairs of events will be totally-ordered in time: event A did happen before event B and all observers agree to that. If event A was simultaneous to event B, that will remain true for all possible inertial observers. To be time-like, simultaneous, and be on a line through the origin requires that the spatial location of A is identical to B. What is disordered are measurements of space. If event A is located at the same place as event B for one observer, a different observer could put A left of B. The third observer may see event A to the right of B. There are three disordered in space sets of events for time-like separated events in a straight line through the origin under Lorentz transformations.

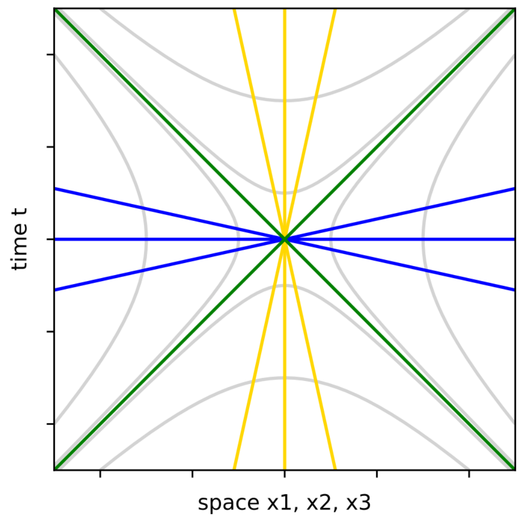
Pairs of light-like events remain totally-ordered in both time and the three directions of space. Since everything in the light-cone travels at the same speed, there is no way for one event to "catch up" and pass another. Using a proper Lorentz transformation, an event is always stuck in a particular quadrant of a space-time plot. Light-like and space-like events on hyperbolas can use Lorentz transformations to cross one axis (the disordered one), and not the other (the totally-ordered one).

Here is a summary table:

Relation to origin	Space-time component	Ordering
space-like	t	disordered
”	X1	totally-ordered
”	X2	totally-ordered
”	X2	totally-ordered
time-like	t	totally-ordered
”	X1	disordered
”	X2	disordered
”	X3	disordered
light-like	t	totally-ordered
”	X1	totally-ordered
”	X2	totally-ordered
”	X3	totally-ordered

For this summary to be valid, for all the events in the set there must exist an inertial reference frame where all the events are on the space axis for space-like events, on the time axis for time-like events or on the light cone for light-like events. For any other collection of events, this analysis does not apply. It is relevant to Einstein's initial observation about simultaneous space-like events which fall on the space axis. I find it interesting that events on the light cone retain their relative ordering under Lorentz transformations. For different inertial observers, there will be relativistic Doppler shifting which is well-understood.

Another way to understand these observations is with a space-time diagram:



Under a Lorentz transformation, events on
blue lines are: disordered in time,
totally-ordered in space
gold lines are: totally-ordered in time,
disordered in space
green lines are: totally-ordered in time,
totally-ordered in space

Axes are totally-ordered sets. A Lorentz transformation does not alter that quality. Instead, it is the other axis that becomes disordered under a Lorentz transformation. It is interesting that the light cone itself remains totally-ordered in both time and space under Lorentz transformations. Values will change, but ordering will not.

Is the requirement of this analysis that for all event there exists straight line so narrow as to make this observation uninteresting? I would argue that every material object's world line to the itself is a straight world line. While I may bike around suburbia, I am at the center of my observable world. I can never ride fast enough to separate from my own eyes. Everyone does agree about which moment of my life came first, second and last. Being North or South, East or West, and up or down would depend on the inertial observer.

The subject of all totally ordered sets in space-time came up while thinking about causality in quantum mechanics. Bell's Future Quantum Mechanics is a page devoted to the new interpretation. See the bottom of the page for other presentations of the idea.

Three Roads to Quaternion Gravity

D. B. Sweetser

Three roads merge to create a different approach to gravity. Our deepest insights into nature use symmetries because symmetries remain unchanged. Using quaternion algebra instead of tensor calculus, the conservation of space-times-time is the symmetry underlying the quaternion gravity proposal for non-inertial observers in a gravitational field. Where there is a symmetry, there need also be a transformation law to detail how change is permitted to happen. The notion of relaxed relativity holds that in a gravitational field, one observer looking at another observer measuring the speed of light will find the product of wavelength and frequency differs from the speed of light in a precise way (). Lorentz invariance remains for inertial observers, but non-inertial observers are governed by different symmetries. Gravity is different everywhere, so a field theory is also necessary using escape velocities. With some reasonable guesses constrained by observations, one can form a quaternion gravity proposal that is consistent with weak field gravity tests. No gravitons are required for this technical variant of special relativity.

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Conserving Changes in Space-times-time

The views of space and time which I wish to lay before you have sprung from the soil of experimental physics, and therein lies their strength. They are radical. Henceforth space by itself, and time by itself, are doomed to fade away into mere shadows, and only a kind of union of the two will preserve an independent reality. *Herman Minkowski, 1908*

To enforce that prediction, I have made a study of writing physics equations in terms of the division algebra known as quaternions as a "kind of union of the two" (results at http://quaternions.com). Such an approach builds into all subsequent operations the unification of space and time. Using quaternions to this extent makes me an honorary member of the Quaternion Society, a long disbanded group of as many as sixty mathematicians and physicists who worked for almost twenty years promoting the utility of quaternions at the turn of the twentieth century. They considered quaternions to be a part of static geometry. The advent of special relativity opened an unseen door to the dynamic geometry of space-time.

A corollary is that we should think about all possible combinations of space and time. Velocity is a change in space over a change in time. A differential area is a change in space times a change in space. A differential angle is a change in space divided by change in space. But what is a change in space times a change in time? Under what situation would this simple yet odd product appear?

Consider two events measured that are arbitrarily close to one another. To make life simpler, use Euclidean coordinates, assuming space-time is flat. Because quaternions are numbers, the difference between the two numbers can be squared without using a metric tensor:

The scalar term (aka first term) is the Lorentz invariant interval squared of spatial relativity for two inertial observers. The 3-vector term (aka the next three terms) I call space-times-time. For inertial observers, space-times-time is Lorentz *variant*: how they change under a rotation or a boost is understood. Some might find this equation an abomination - a universally useful Lorentz invariant term sitting right next to these three amigos that vary. To my eye, this looks like a complete story, one that provides more useful information from the same starting information. Relativity is a game of what one observer says about another's observations. Let our primary observer be called observer A, and the one moving away is observer B. Observer A makes a measurement between two events, recording it as . Observer A is able to say that the measurement made by observer B looks to be . If all one reports is the Lorentz invariant interval, then the interval squared, , will be the same according to special relativity. If instead one squares these two as quaternions, again the first terms are identical. The space-times-time terms are different and can be use to calculate how observer B is moving relative to observer A. From an information point of view, if one starts with four pieces of information, one should end with four pieces of information as happens with using quaternions to do the calculation.

Given the long history of doing calculations with 4-vectors and metrics, I expect no one to be persuaded. The space-times-time terms are the off-diagonal elements of a contraction with a metric tensor. I argue that a scalar times a vector element should point in the direction of the vector element since all that happens to the vector is that its scale changes. There is an opportunity to do new physics. For special relativity, the interval will be the same for two inertial observers while the space-times-time is Lorentz variant. Ask the opposite question: what sort of physics results from when observer A looks at the measurements happening for observer B and finds that the space-times-time values are identical but the intervals are not? The two observers are non-inertial based on standard special relativity.

General relativity is a relativity theory: it describes what observer A says about observer B's measurements in a gravitational field. In the simple case of an electrically neutral, uncharged, non-rotating source, the changes in measurements of time accidentally almost cancel the changes in space. The details depend on coordinate choice. For those well-schooled in general relativity, coordinate-dependent statements are of nominal interest. Quaternion gravity proposes that space-times-time is an invariant in a gravitational field.

At this point, some might ask if a flat space-time metric is being assumed due to the inflexible rules of quaternion multiplication? Recall that real numbers are rank zero tensors. As such, the rules for multiplying them are the same in all expressions, wether the topic is special relativity, general relativity, or quantum mechanics. The same is true for complex numbers as rank zero tensors. Complex numbers contain as a subgroup the real numbers and the rules for multiplying them are the same in all physics expressions. For the sake of logical consistency, quaternions in this proposal are rank zero tensors. Quaternions contain real and complex numbers as subgroups and the rules for multiplying them are the same for all physics expressions. That translates to the rules for multiplying quaternions are the same with or without gravity. A metric tensor, dynamic or not, has no place in the quaternion gravity proposal.

Quaternion expressions can always be written in a coordinate-free way by using the one-dimensional quaternion manifold . There is never an exception to this rule. To be honest, this author does always default to a Cartesian calculation-world view on the manifold . But for the sake of good practice, write space-times-time invariance in a manifestly coordinate-free form:

The quaternion gravity proposal is that the second term above will be constant in a simple gravitational field.

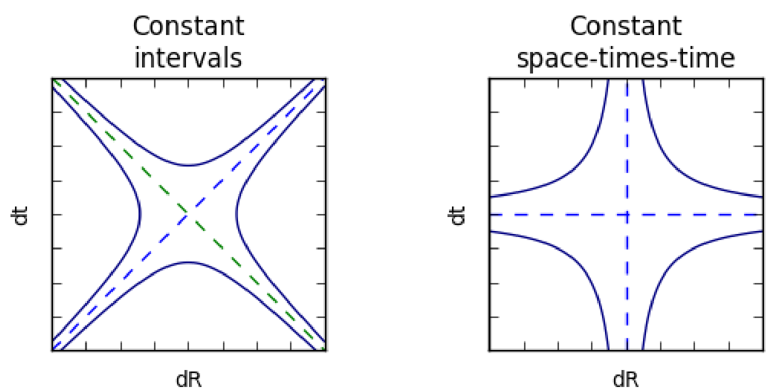
Relaxed Relativity Transformations

In special relativity, all observers agree on the speed of light . I ask the reader's patience because I am going to be pedantic about how this is discussed. Observer A measures the speed of light locally as . Observer B likewise measures the speed of light locally as . This is not what special relativity is about. Nature is graciously consistent. Special relativity shows how observer A looking at how observer B is measuring the speed of light will say that observer B's measurements also show the speed of light to be . Special relativity provides the relativistic Doppler shift for both the wavelength and the frequency such that the product is constant: , , and . This well-establish result can serve as a test that observer B is a relatively inertial observer for observer A. "Relatively inertial"? Everywhere in the Universe is changed by the presence of gravity. What relatively inertial means is that the effect of gravity is the same for both observers. Effectively it means that two observers at sea level on the Earth experience the same pull of gravity but may be moving at a steady speed relative to each other. The sea level observer A will find that sea level observer B has measured the speed of light and the product of the wavelength and frequency remains the speed of light in this weak gravitational field.

Relaxed relativity is defined operationally as observer A looking at how observer B is measuring the speed of light and seeing that . This covers every situation not covered by special relativity, so is huge. This paper will only focus on transformations that conserve space-times-time as discussed in the first section. How does that play out for observer A? It means . Wavelength *divided* by frequency? It is simple but strange. Perhaps it is not that strange. When the path of a wave is bent going though a medium, the wave travels at a different speed. In a vacuum with gravity, there is no medium. The path of light is bent. The consequence of observer A seeing the product of observer B's wavelength and frequency does not equal is indistinguishable from the path of light bending.

Many of the ideas carefully crafted for special relativity still apply to relaxed relativity. The time measured by observer A is what observer A sees on her wristwatch. The length of any bar is measured as a time between two events emitted simultaneously from the ends of the bar, treating the term “simultaneously” with care.

Since this study is restricted to observer A transforming measurements made by observer B such that the three space-times-time values are equal, that applies to events that are simultaneous for observer A, . The product of anything with zero is zero, thus when , . Observer A will find that simultaneous for observer A appears to be simultaneous for observer B. The same applies to changes in space at the spatial origin, where , . The non-zero constant space-times-time values will be parabolas that approach each axis:



The familiar constant intervals of the light cone are rotated 45 degrees to form the constant space-times-time graph. The mathematical perspective is that both are about the same function, a quaternion squared. The difference is that for special relativity, what is constant is a real value, . The focus of this paper is on three constant imaginary values, .

The form of transformations in relaxed relativity must start from the same form as for special relativity.[1]

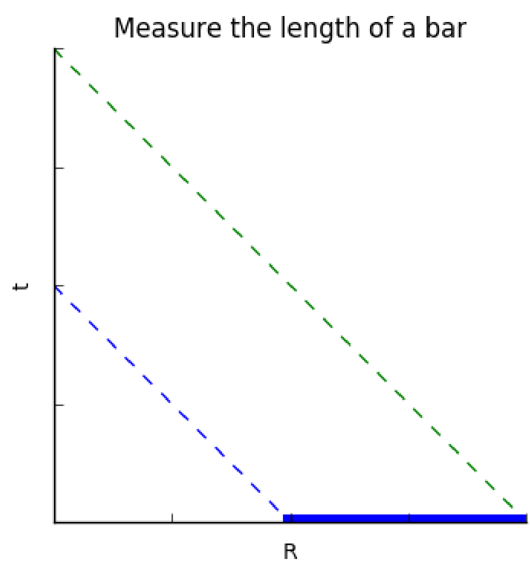
[1]A driver of traffic to my sign is a claim on the Internet that the first edition of “A Treatise on Electricity and Magnetism” had 200 equations written with quaternions that were deleted by the second edition by Heaviside. Finding the first version was a struggle, but I did find it. It had two sections with “Quaternion” in the title. It was clear that this grand master of old

In relaxed relativity, one can explore other types of transformations that are not allowed under special relativity. It has been established that when events are simultaneous, , there is a space-times-time constant independent of changes in space, . Figure out a transformation of the first expression that eliminates the spatial term. Any constant linear function can be added to the primed frame since it will be subtracted away.

See how this coordinate choice effects the transformation:

This coordinate choice eliminates the dependence on changes in space, . Events that are simultaneous in the unprimed frame will be simultaneous in the double primed frame.

The next task is to figure out how changes in space happen going from the unprimed to the doubly primed reference frame. Consider a bar of length L in the unprimed frame emitting two photons from the ends of that bar simultaneously in the unprimed frame.



The two events will be simultaneous in the doubly primed frame. Say gamma was 2. Then it would appear to take twice as long to appear in the double primed frame based on the time transformation described above. As such, the length would be twice as long in the double primed frame. The transformation is thus:

With these two transformations in hand, look at what happens for space-times-

was only using the 3-vector part of a quaternion. This is not where the fun is in my opinion. Since I have derived and rederived and rewritten my derivations of the Maxwell equations using only quaternions many times, I can assure you, nothing is missing.

time, velocities, and a differential quaternion squared:

This coordinate transformation does preserve space-times-time. This is not special relativity since an observer in the double primed frame will report that the observer in the unprimed frame who is trying to measure the speed of light is not , but changes by a factor of gamma squared, . The square alters the interval but leaves the space-times-time constant. To this point, this is just a math exercise.

Teaching Newtonian Gravity New Tricks

Newton's theory of gravity produces a field where every point in space-time is assigned a force. Both space and time are absolute. Special relativity demonstrates that time and space can mix via boosts, so space and time are not absolute. General relativity showed how to view gravity not as a force, but instead as the easiest path through space-time. Observer A looking at observer B making a measurement would determine that the interval was different depending on exactly where observer B was in a gravitational field.

In *Principia*, Newton pictured a cannon ball being fired off of a mountain top. Without being concerned with any engineering details, he calculated the escape velocity for the cannon ball, how fast it would need to go to escape the gravity field and come to rest at infinity. The escape velocity can be calculated for every point in space-time, so is an escape-velocity field. Use this escape velocity field as the velocity needed to calculate the square interval, eq. ([eq:dR_squared]).

This is an improvement on Newton's original proposal for gravity. Notice how measurements of space change. This is enough for the three classical tests that general relativity first passed by 1919: light bending around the Sun during an eclipse, the precession of the perihelion of Mercury, and the gravitational redshift of light.

Unfortunately, this proposal is not good enough to match the today's data. There have been theoretical and experimental developments used to characterize alternative approaches to general relativity, in particular the Parameterized Post-Newtonian (PPN) formalism developed by Nordvedt and Will. Multiple experiments have confirmed that there is a second order term for time with a coefficient of +2 (the actual parameter is with a multiplier of 2).

Now we get to have some real fun, to challenge ideas deeply locked into our notions of what gravity is. Newton's work got close, but not close enough. That means that a search is afoot for a new, better gravitational potential, one that can match to first order PPN accuracy...

Actually, that is not the right direction to head. What must be done is embrace the lesson of the first section: this proposal is a different class of symmetries of space-time made possible by quaternion algebra. In special relativity applied to inertial observers, one has a velocity between observers that has been created by the history of observer A compared to the history of observer B. No one asks why do they have that exact velocity relationship? In exactly that way, I argue that there are different escape velocities for observers created by the history of observer A compared to the history of observer B.

Newton's escape velocity is a truncated series of the truth. How should we go about finding the full series? Look to special relativity. Both rotations and boosts can be represented with exponential functions.[2] Therefore is seams natural to propose that the geometric length of a source mass, , be used in an exponential function since it will generate the five terms found in the first order PPN formalism.

The first term of this expression has the form of the contraction of two rank 1 4-vectors using a flat Minkowski metric. Mathematically, it is no such animal. The input is two rank zero tensors. The output is not one value, but four in a rank 0 tensor. The rules of multiplying two quaternion do not change under any circumstance which has similarities to a fix background tensor. The off diagonal terms that can appear for a tensor contraction end up in the space-times-time terms.

The first term of eq. ([eq:exp_squared]) will be consistent with all weak field tests of gravity to first order PPN accuracy. This covers spherically symmetric, non-rotating, uncharged sources. This "exponential metric" - in form only - is not a solution to the field equations of general relativity. In the literature, it is known as the Rosen metric. That particular proposal assumed there was a second, always constant background metric. Unfortunately the Rosen bi-metric proposal had a dipole mode for gravitational wave emission, so was not consistent with gravity wave energy loss of binary pulsar data. Notice that the quaternion gravity does not allow any off-diagonal contributions to the interval, thus eliminating effects for preferred-location, preferred-frame, or violations of total momentum conservation. The same is true of general relativity but not other proposals for gravity.

What happens when things get more complicated, like the source spins, or there are multiple sources? For every point in space-time, there will be an escape velocity such that going that precise speed means an object will stop when it reaches infinity. That is not too practical. There is an experimental approach. For observer A, set up many other observers, each measuring the speed of light. Each of these observers measures the speed of light locally is c. Have observer A determine for each of these many other observers what the wavelength and frequency look like to observer A. The result of the product will be . The escape velocity field will always exist and can be measured.

At second order PPN accuracy, the interval found in eq.([eq:exp_squared]) will not agree with the Schwarzschild solution of general relativity. The difference for light bending around the Sun is about 6% and under a micro-arcsecond.

[2]Deeper insights can sometimes be found working on a complex manifold, C^1 using a complex number and its conjugate instead of R^2.

Our current accuracy for measuring light deflection is on the order of 100 micro-arcseconds. Even if the state of the art were improved, effects like the rotation of mass in the Sun and the Sun's quadrapole moment would be on the same order. Getting a direct experimental conformation of the difference between this proposal and general relativity will depend on future developments.

Using the same approach that appears in the analysis of the Schwarzschild solution of general relativity, one could rewrite the first term of the square in spherical coordinates and see no depedence on either time or angles. As such, the expression will conserve both energy and angular momentum. If one only keeps the lowest order terms of the exponential equation, the equations of motion are:

These are exactly the same equations of motion as the Schwarzschild solution of general relativity since the first order expressions have an identical form. This form of the proposal may be more convenient for some calculations.

Extreme Gravity

Black holes, quantum gravity, and gravitational waves are three topics of current research. How does quaternion gravity deal with these topics?

It used to be that when a physics theory had a mathematical singularity, that was considered a sign that the theory had to be replaced in the domain of the singularity. Einstein considered the "frozen stars" solution of general relativity to be serious given the historical pattern of progress in physics. It was part of his motivation in his unsuccessful quest for a unified field theory. While the event horizon of a black hole can be eliminated by a different choice in coordinates, work by Penrose and Hawking showed the mathematical singularity could not be transformed away so the singularity was essential. It is common opinion that a quantum theory for gravity will eliminate this issue. In this way, black hole physics and quantum gravity are linked.

There are few exact solutions to general relativity. Numerical solutions are also hard to craft. Because general relativity has ten non-linear second-order differential equations, one expects an enormous challenge. In contrast, any function that can be inverted could be used to preserve space-times-time while changing how observer A sees an interval measured by observer B. This may sound too easy, but is consistent with the logic of general relativity. What contributes to bending of space-time in general relativity? Absolutely everything, no exceptions. In quaternion gravity, the only question is what does it take to escape your shared history with the particles around you, no matter what every particles are doing? Everything can and does contribute to inching the escape velocity up.

The exponential factor in eq. ([eq:exp_squared]) does have a singularity at . Since the distance is positive definite, the singularity will only be approached for positive. The real part will approach zero. The imaginary part is a pole that goes to positive infinity. It should be possible to remove the singularity, so the expression for the exponential quaternion gravity is well-behaved.

What happens when more matter than can be supported by the nuclear forces falls into a neutron star? I don't know.

General relativity has resisted all efforts at renormalizable quantization. It is a reasonable hypothesis that a spin 2 graviton would be the particle to mediate the force. There is some internal tension in the phrase "particle to mediate the force" since the strong equivalence principle asserts that geodesics are the easiest of all possible paths through space-time, requiring no force what-so-ever. There is no particle for special relativity. Instead, observations of all particles by relatively inertial observers must obey the rules set by special relativity. There is no particle for quaternion gravity. All particles must obey the rules of quaternion gravity. In some ways, this is a minor loss: there are no efforts now or in the near future to detect gravitons since they beyond our ability to generate. Quaternion gravity proposed there is no graviton. In other ways, this is huge given the ongoing efforts to find quantum gravity. All relativity theories are about exactly what one observer says about another observers measurements. Stated that way, there should be no particles doing this work.

The physics community is excited by the observations of gravitational waves. The waves matched models of a ring down event for a binary system to a black hole. The space-times-time symmetry requirement for the quaternion gravity proposal is dynamic in the sense that the distance used in the exponential can be a function of time. Yet it is far to early to claim the proposal can be consistent with the gravity wave data collected to date.

Quaternion Gravity as a Simpler Competitor to General Relativity

Some have argued that there is no simpler metric theory possible than general relativity. In a vacuum, the Lagrangian is only composed of the Ricci scalar . The radical simplicity of general relativity has two different implications for tests of gravity. First is that the path of light is changed. Since light has no rest mass but does have equal parts energy and momentum, a proposal that centers on rest mass will fail. A second consequence has to do with gravity wave emissions. Observations of pulsar data support the conclusion that the lowest mode of gravity wave emission is a quadrapole (think wobbling water balloon). Add any new field and the new proposal will likely have a dipole mode of emission and thus be in conflict with experimental data.

The only possible way to have a simpler Lagrangian in a vacuum than general relativity is one with no letters at all. At first and second glance, that does not sound like a theory. Careful reflection is needed on what special relativity is as a physics theory. Special relativity is a set of algebraic constraints on all physics theories. There is no "particle" for special relativity. Rather all particles must obey its rules. Those rules are not about how things move. The rules are about observers observing other observers. The rules only cover inertial observers.

Quaternion gravity is profoundly similar to special relativity given it is merely a rotation of the light-cone by 45 degrees. There is no particle needed for differences in observers observing other observers. Both special relativity and quaternion gravity will necessarily apply to all particles whether they have a rest mass or not.

There is no extra field where energy could be stored. For an isolated mass to conserve both energy and momentum, a system would have to wobble like a water balloon. Could a system lose energy? It is vital to recall that a gravity wave is not something done via particles like electromagnetic waves and the photon. Masses in cyclic motion will create cyclic variations in the escape velocity field because an observer will be different distances away from the source at different times. Anything varying cyclically can be characterized by a wave. Waves have both energy and momentum.

Why then does gravity work? Actually, gravity does no work, just like special relativity. Both quaternion gravity and special relativity are about the consequences of differences in space-time history of two observers. If observer A is moving at a steady velocity relative to observer B who is measuring the speed of light, observer A will say both the frequency and the wavelength measured at B are different, but the product, the speed of light, is c. If observer A is not moving relative to observer B, but is closer to a gravitational source, a different situation arises. Since observer B is farther away from the gravitational source than observer A, observer A will see the measurement made by B as easier - there is less stuff - meaning observer A measurers observer B's clock as ticking freer and faster, and observer B's ruler looks bigger. This means observer A will say the speed of light looks greater than c at B. One could argue that such an effect should be crazy small, yet that is what gravity is, a crazy small effect. If gravity is treated as a force, then it is more than forty orders of magnitude smaller than electromagnetism. The disparity between gravity and the three other fundamental forces is an open mystery.

What about Newton's apple falling from a tree? The apple is trying to oscillate around the center of the Earth, reach its antipode in 45 minutes, and return in 90 minutes (approximately). The rest of the mass of the Earth is in its way in a traffic jam so that does not happen. If the tree were uprooted and put into deep space with no gravitational mass of any consequence near by, the apple would not fall anywhere. Herein lies the appeal of the exponential solution. In deep space, the exponent is zero. On the surface of the Earth, the exponent is so tiny, only the first term of the Taylor series comes into play. That first term is Newton's proposal for gravity cast in a different format, eq. [eq:newt_squared]. Quaternion gravity may be the only comprise between Newtonian gravity, the space-time curvature of general relativity, and an alternative view on the math of special relativity.

Measure the difference between space-time events

Two stars go supernova while four kids watch.

There is a difference in time (dt).

There is a difference in space (dR).

Together they make a difference in space-time.

Each kid measures a different values for time and space.

Yet the walkers agree on something they can calculate called the interval. This is the standard physics of Special Relativity, the physics of moving.

This site promotes a new proposal for gravity called Quaternion Gravity where the kids above or below agree on a different value they calculate, space-times-time. In this standard approach to gravity known as General Relativity, this is almost, but not quite true.

Page-cast

A brief explantion of this page.

For nerds

Special relativity is special because it is restricted to inertial observers. To cover more cases requires the machinery of differential geometry. A tensor can be added to another tensor or multiplied by a scalar. An interval is formed by contracting two rank 1 contra-variant tensors with a symmetric, rank-2 metric tensor. A connection is needed to describe how the metric changes in space-time. There are many technical choices one makes along the way to calculating an interval in curved space-time.

The site issues a formal challenge to the algebraic standards of differential geometry used today by physics. In place of tensors, metrics, and connections, only quaternions will be used. For those trained in the craft of differential geometry, that should sound wildly inadequate. It is always a great challenge to do more with less.

Special relativity

Walking changes how one measures deadly supernovae.

As long as the kids move at a constant rate, special relativity comes into play.

Relativity is not an arbitrary change, but increadibly precise change.

And there is that interval that is *exactly* the same size.

Note: the numbers are **far too big** (off by 16 orders of magnitude, I just didn't want to write *lots* of zeros).

Page-cast

A brief explantion of this page.

For nerds

Take measurements made by two observers written as quaternions and square them. If the first terms are the same, then the two observers are in reference frames that are moving in a steady way relative to each other. Note that the observers can be in a gravity field which is a non-inertial reference frame, but that does not cause an issue here.

The observers do not have to make sure their coordinate systems are the same. So long as both agree to compare so called natural units (a way to consistently be dimensionless), then the numbers will be the same. As long as the kids move at a constant rate, special relativity comes into play.

Quaternion gravity

Looking down or looking from below changes time and space measurements in opposite ways.

If one kid is at a different height in a gravity field to another, then time measurements get smaller while spatial ones get bigger. That is standard physics (general relativity, GR).

The Quaternion Gravity (QG) proposal says the space-times-time values are precisely the same. With general relativity, this space-times-time is not mentioned, but it is almost - but not quite - the same.

Note: the numbers are **far too big** (off by 16 orders of magnitude, I just didn't want to write *lots* of zeros).

Page-cast

A brief explantion of this page.

For nerds

Special relativity could have been called "special invariance" because it is the invariant interval that all inertial observers argee upon. Inviarnce principles are deep insights into how Nature works. They are truths that do not change.

The quaternion gravity proposal postulates a new invariance principle: that different observers making measurements in different locations in a gravitational field will agree on an invariant value for space-times-time. Some care is required to say this in a coordinate-independent way. All observers are free to pick their coordinate system. There then exists a norm-preserving rotation in space such that the space-times-time measurement of one observer is exactly equal to another.

But how precisely does the interval change? For a spherically symmetric, non-rotating, unchanged source, only one dimensionless ratio comes into play: GM/c^2R. Orbital systems are harmonic systems, suggesting that one use exponential of the dimensionless ratio. The requirment that the space-times-time term is invariant means the time term is the inverse of the exponential experienced by the space term, like so:

The resulting exponential interval has the same form as the Rosen metric. The Rosen matric makes the same experimental predictions as the Schwarzschild metric for all weak field gravity tests to first order Parameterize Post-Newtonian (PPN) accuracy. At second order PPN accuracy, the exponential interval predicts 12% more bending of light around the Sun. We have yet to acheive the precision to decide this issue on experimental data.

Special relativity is not a field field theory. It is a constraint on all field theories. The quaternion gravity proposal is also not a field theory. Like special relativity,

$$
\begin{aligned}
\left(e^{-z}dt, e^{z}dR_i/c\right)^2 &= \left(e^{-2z}dt^2 - e^{2z}dR_i^2/c^2, 2\,dt\,dR_i/c\right) \\
&= \left(e^{-\frac{2GM}{c^2R}}dt^2 - e^{\frac{2GM}{c^2R}}dR_i^2/c^2, 2\,dt\,dR_i/c\right) \\
\text{if} \quad z &= \frac{GM}{c^2R}, \quad i = 1, 2, 3
\end{aligned}
$$

Figure 209: e to the -z dt, e to the z dR sub i over c squared equals e to the -2 z dt squared - e to the 2 z dR sub i squared, 2 dt dR sub i over c equals e to the - 2 G M c squared R dt squared - e to the 2 G M over c squared R dR sub i squared, 2 dt dR sub i if z = G M over c squared R

it is a constraint on all field theories. As such, there is no need for a graviton. Quaternion gravity makes the search for quantum gravity moot.

SR + QG - ***S**pecial **R**elativity and **Q**uaternion **G**ravity*

This graphic says most of it...

Start with the reference square which has an interval of 16 and a space-times time of 30.

Compare the reference square with the walkers. They all have the same interval of 16 because that in what is invariant for inertial observers, folks moving at a constant speed compared to the reference.

Compare the reference square with the girl above and boy below. Because they are in a gravitational field, they are not inertial observers. The quaternion gravity proposal says the space-times-time value are exactly the same at 30. The interval will be of different sizes.

If one compares a walker to one of the kids above or below, there is no overlap between them.

Page-cast

A brief explantion of this page.

For nerds

Since there is a gravitational field everywhere, there are no inertial observers. Working with the squares of quaternions, things are a little easier. Just compare the reference square with any other square. For the walkers, since they travel at a constant speed and are at the same location in the gravitational field as the reference square, they will have the same interval.

The kids above and below are not moving compared to the reference square. By the quaternion gravity proposal, the space-times-time is an invariant. All agree on the value of 30. What then has to be different is the interval. But how different, and how does that depend on the gravitional source mass?

Fortunately, there is no choice in answering the question if one is to be consistent with current experimental tests of gravity. For a spherically symmetric, non-rotating, uncharged source, gravity depends on the ratio of the graviational source mass over the distance to the center of that mass. Whatever function is used to make the time measurement smaller must be the exact inverse of the one that makes a spatial measurment larger. Since gravitational systems follow simple harmonic patterns for billions of years, an exponential and its inverse that depens on the M/R ratio is an obvious thing to propose.

The interval looks just like the Rosen bi-metric proposal, even though quaternion gravity uses no metrics. The Rosen metric is known to be consistent with current tests of weak field gravity up to first-order Parametrized Post-Newtonian

$$
\begin{aligned}
\left(e^{-z}dt, e^{z}dR_i/c\right)^2 &= \left(e^{-2z}dt^2 - e^{2z}dR_i^2/c^2, 2\,dt\,dR_i/c\right)\\
&= \left(e^{-\frac{2GM}{c^2R}}dt^2 - e^{\frac{2GM}{c^2R}}dR_i^2/c^2, 2\,dt\,dR_i/c\right)\\
\text{if} \quad z &= \frac{GM}{c^2R}, \quad i = 1, 2, 3
\end{aligned}
$$

Figure 210: e to the -z dt, e to the z dR sub i over c squared equals e to the -2 z dt squared - e to the 2 z dR sub i squared, 2 dt dR sub i over c equals e to the - 2 G M c squared R dt squared - e to the 2 G M over c squared R dR sub i squared, 2 dt dR sub i if z = G M over c squared R

accuracy. The extra metric creates a problem for Rosen's proposal since gravity waves would have a dipole moment and lose entry faster than observed. The simplicity of the quaternion gravity proposal would require for an isolated mass in space that the lowest mode of emission is a quadrapole, consistent with what is seen. Yet there is no graviton with quaternion gravity. The energy could be carried away with photons that happen to have a quadrapole moment, but no a dipole one.

My entry to the *2015 Awards for Essays on Gravitation* is a more formal presentation of this research effort, available as a pdf.

Quaternion Space-times-time Invariance as Gravity*

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*Essay written for the Gravity Research Foundation 2015 Awards for Essays on Gravitation

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Abstract

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The square of a quaternion luckily has the Lorentz invariant interval of
special relativity as its first term. The other three space-times-time
terms are commonly ignored. Ways to vary a quaternion with a continuous
function that leave the intraval in the square invariant are discussed.
One method uses exponentials, leading to the hyperbolic functions found
useful in special relativity. Using the same approach to keep the
space-times-time invariantleads to a dynamic interval term. By preserving
the space-times-time terms using an exponential function and the geometric
source mass, an interval term is found that is similar but experimentally
distinct from the Schwarzschild metric applied to space-time 4-vectors.
Space-times-time invariance is not a field theory, so gravitons are not
necessary and quantization is moot.
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General relativity, Einstein's elegant theory of gravity, is a field theory like the three other fundamental forces of Nature: electromagnetism, the weak force, and the strong force. The cause of gravity is any form of energy or momentum. The field equations dictate the motion of particles with energy, thus applying to all particles, even light.

Special relativity is not a special case of general relativity, despite the name. Special relativity is about an invariant quantity in Nature that all inertial observers agree upon: the interval, a difference of squares in measurements of space and time. Special relativity applies to all measurements, even those involving the fundamental forces. The product of a measurement in space and one in time, referred to hereafter as space-times- time, will change in known ways for different inertial observers.[3]

This essay explores the opposite situation: what if two observers find their space-times-time was an invariant, but their intervals were different? It is sug-

[3]A driver of traffic to my sign is a claim on the Internet that the first edition of "A Treatise on Electricity and Magnetism" had 200 equations written with quaternions that were deleted by the second edition by Heaviside. Finding the first version was a struggle, but I did find it. It had two sections with "Quaternion" in the title. It was clear that this grand master of old was only using the 3-vector part of a quaternion. This is not where the fun is in my opinion. Since I have derived and rederived and rewritten my derivations of the Maxwell equations using only quaternions many times, I can assure you, nothing is missing.

gestive that a defining characteristic of general relativity is that intervals vary at different places in a gravitational field.

The invariant interval of special relativity in flat spacetime is generated by contracting a 4-vector using the Minkowski metric. With such a simple system, other products are omitted, namely, any with the space-times-time form, dtdxi. Such terms could appear if one used a metric with non-diagonal components which are unnecessary for flat spacetime.

There is a type of math that naturally embraces space-times-time terms. All are familiar with real numbers, a mathematical field that allows for addition, subtraction, multiplication, and division. The complex numbers are also a mathematical field, but now one has two degrees of freedom, often represented by a pair of numbers.[4] Complex numbers are no longer a totally ordered set. The next sort of numbers has four-part harmonies, with a real bass and three imaginary tenors. Known as the quaternions, they do not commute, so live with the label of a division algebra. Quaternions still retain addition, subtraction, multiplication, and division. The rules are similar for the complex numbers, with the imaginary i replaced by an imaginary 3-vector and the inclusion of the anti-symmetric cross product. Quaternions play a minor technical role as the best way to do rotations in three dimensions.[8] A unit quaternion SU(2) sits in the center of the standard model gauge symmetries. Despite that central role, quaternions have historically been vilified to a comic degree.

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"Quaternions came from Hamilton after his really good work had been done; and
though beautifully ingenious, have been an unmixed evil to those who have
touched them in any way, including Maxwell."

Lord Kelvin[10, See vol. II, p. 1070.]
```

There are published claims that one cannot write the Maxwell equations or represent the Lorentz group using real-valued quaternions.[2, 4, 6, 9] Neither happens to be true. See the appendix for details if interested.

The square of a space-time measurement represented with quaternions is:

$$(dt, dx_1, dx_2, dx_3)^2 = (dt^2 - dx_1^2 - dx_2^2 - dx_3^2, 2\,dt\,dx_1, 2\,dt\,dx_2, 2\,dt\,dx_3)$$

Figure 211: dt, dx sub 1, dx sub 2, dx sub 3 squared equals dt squared minus dx sub 1 squared minus dx sub 2 squared minus dx sub 3 squared, 2 dt dx sub 1, 2 dt dx sub 2, 2 dt dx sub 3

The first term of the square is the Lorentz invariant interval. It is followed by the three space-times-time terms. There are a few advantages to having these three extra bits of information. Say two inertial observers Alice and Bob saw a collection of events. The first term of the square of the quaternion would be the

[4]Deeper insights can sometimes be found working on a complex manifold, C^1 using a complex number and its conjugate instead of R^2.

same. An analysis of the space-times-time value would let us know how Alice was moving relative to Bob. If they also calculate the product of two different events in both orders, then we would know something about the angle between the events and the observer. If ab = ba, then they are in a straight line. If for the three space-times-time terms, ab = -ba , they are at a right angle. Anything between those extremes is in between.

What happens in curved space-time? With the standard machinery of differential geometry, a simple subtraction is not allowed. Instead, one has to parallel transport one event to another along a geodesic using a known connection. Then the subtraction can be done properly.

Quaternions don't have a metric. Without a metric, there is no connection. Maybe quaternions are an "unmixed evil." Let's explore anyway.

Construct a quaternion out of space-time functions that can be varied, yet the first term of the square is invariant as required by special relativity:

$$(f, g_1, g_2, g_3)^2 = (g^2 - g_1^2 - g_2^2 - g_3^2, 2\,dt\,g_1, 2\,dt\,g_2, 2\,dt\,g_3)$$

Figure 212: dt, dg sub 1, dg sub 2, dg sub 3 squared equals dt squared minus dg sub 1 squared minus dg sub 2 squared minus dg sub 3 squared, 2 dt dg sub 1, 2 dt dg sub 2, 2 dt dg sub 3

If the function f was exactly the same as each normalized gi, then the first term in the square, the interval, would always be zero.[5] This is an important case: it is light. Changes in time are exactly equal to the magnitude of changes in space.

With zero covered, find a way so the first term in the square is equal to one for all inertial observers. The square of f must cancel out the square of g, but leave unity behind. Work with a third function h and its inverse:

So long as the function h has an inverse, this will always work.[6] Exponential functions play an important role in theoretical physics. If the exponent is zero, unity is the result and nothing is changed. For tiny exponents, the result may contain a simple harmonic oscillator which are ubiquitous in Nature. For the case in hand, the function f is a hyperbolic cosine which is the stretch factor gamma of special relativity. The function g is the hyperbolic sine, the gamma beta factor that also appears in special relativity.

Repeat these two simple math exercises for space-times-time. Find a general way to make the square of a measurement have either three zeroes or three ones - times the factor of two that is from the sum of two identical terms. Generating three zero space-times-time factors is easy: take the norm of any quaternion.

[5] The normalization depends on the count of non-zero g factors, 1 over the square root of 3 if none are zero.

[6] Adjusting the normalization factor as needed.

$$f = \frac{1}{2}\left(h + \frac{1}{h}\right)$$
$$g_{1,2,3} = \frac{1}{2\sqrt{3}}\left(h - \frac{1}{h}\right)$$
$$(f, g_1, g_2, g_3)^2 = \left(1, \frac{1}{2\sqrt{3}}\left(h^2 - \frac{1}{h^2}\right), \frac{1}{2\sqrt{3}}\left(h^2 - \frac{1}{h^2}\right), \frac{1}{2\sqrt{3}}\left(h^2 - \frac{1}{h^2}\right)\right)$$

Figure 213: f equals one half h + the inverse of h, g 1, 2, 3 equals one over 2 the square root of 3 times h minus the inverse of h. The square of f g 1, 2, 3 equals one, then three timer of one over 2 the sqaure root of 3 times h squared minus the inverse of h squared

Some effort has gone into quantum mechanics that uses quaternions in place of complex numbers.[1] That topic is beyond the scope of this short essay. The general way to generate three factors of two is also not difficult:

$$f = \frac{1}{h}$$
$$g_{1,2,3} = h$$
$$(f, g_1, g_2, g_3)^2 = \left(\frac{1}{h^2} - 3h^2, 2, 2, 2\right)$$

Figure 214: f equals the inverse of h. g 1, 2, 3 equals h. f, g 1, 2, 3 squared equals the inverse of h squared minus 3 h squared, 2, 2, 2

An exponential function could be plugged in as before. This moves from a pure math exercise to one with physics content if the exponential is chosen to be related to gravity by using the geometric length of a mass[7]:

Let's pause to discuss this expression. No metric was used to get here. No field equation was solved. Instead a new invariance of Nature has been proposed as it applies to products of quaternions in a weak gravitational system characterized by one length. Algebraically, the first term is the same as the Rosen exponential metric applied to an event 4-vector.[7]

Experimental tests of weak gravity fields use the first three terms of the Taylor series expansion in z for the change in time, and the first two for changes in space. Those terms are identical for the space-times-time invariant expression

[7] The coordinate-independent formulation is that the product of time and the norm of space is invariant in a gravitational field.

$$
\begin{aligned}
\left(e^{-z}dt, e^{z}dR_i/c\right)^2 &= \left(e^{-2z}dt^2 - e^{2z}dR_i^2/c^2, 2\,dt\,dR_i/c\right) \\
&= \left(e^{-\frac{2GM}{c^2R}}dt^2 - e^{\frac{2GM}{c^2R}}dR_i^2/c^2, 2\,dt\,dR_i/c\right) \\
\text{if} \quad z &= \frac{GM}{c^2R}, \quad i = 1, 2, 3
\end{aligned}
$$

Figure 215: e to the minus z dt, e to the z dR sub i over c squared equals e to the minus 2 z dt squared minus e to the 2 z dR sub i squared, 2 dt dR sub i over c equals e to minus twice the geometric length of the source mass times dt squared minus e to twice the geometric length of the source mass times dR sub i squared over c squared, 2 dt dR sub i over c if z equals the geometric source mass or Newton's gravitational constant times big M over c squared and i goes from 1 to 3

and the Schwarzschild metric in Cartesian coordinates. There is no way to distinguish these two at what is called first-order Parameterized Post- Newtonian (PPN) accuracy. At second order, the new invariance proposal predicts 6% more bending of light around a gravitational source.[3] Since the effect is smaller than a micro-arcsecond, that is beyond our reach today.

Massless light is bent by gravity. That can be accounted for in general relativity because the coupling is to energy density which light has. With the space-times-time invariance, there is no coupling term nor any field equations. The same thing happens in special relativity: there is no coupling, nor field equations. The space-times-time invariance may be the correct variation on the invariant interval of special relativity, thus being the simplest pure geometry approach to gravity, but not too simple.

With no graviton to quantize, there is no issue of quantizing a gravitational field. What about energy loss by a binary pulsar? The Rosen metric allows for a dipole mode of gravity wave emission, so is ruled out by the data which requires a lower rate of gravity wave emissions.[11, See section 12.3(b).] In the space-times-time invariant proposal, the exponential function and its inverse applied to gravity above was static. Make it dynamic by including a time factor in a way consistent with how we see the metric change in time for a binary pulsar.

Is a graviton required to carry away the energy? The system in question is an isolated binary pulsar that conserves both energy and momentum. It does not have a dipole moment like a magnet, but does have a quadruple moment, like a wobbling water balloon. The energy could be carried away by an electromagnetic field that had a quadruple as its lowest moment. While unusual, it is possible.

Does gravity as a space-times-time invariance play nicely with the three other fundamental forces of physics? Given the stellar record of special relativity, there is reason to hope.

Appendix: Maxwell equations and the Lorentz group using real-valued

quaternions

The homogeneous Maxwell equations are vector identities. They hold when written using quaternions. The Lagrange density used to derive the Maxwell source equations is the difference of the squared magnetic and electric fields[5]:

$$\mathcal{L} = \frac{1}{2}(B^2 - E^2)$$

Figure 216: The Lagrange density of electromagnetism is one half B squared minus E squared

The difference of two squares is the product of their sums and difference. The simplest product of a quaternion differential operator and potential generates the difference of the magnetic and electric fields:

$$\nabla A = \left(\frac{\partial}{\partial t}, \vec{\nabla}\right)\left(\phi, \vec{A}\right) = \left(\frac{\partial \phi}{\partial t} - \nabla \cdot A, \frac{\partial \vec{A}}{\partial t} + \vec{\nabla}\phi + \vec{\nabla} \times \vec{A}\right)$$
$$= \left(g, \vec{B} - \vec{E}\right)$$

Figure 217: The differential quaternion operation action on a quaternion potential phi, A is the time derivative of phi minus the divergence of A, the time derivative of A plus the gradient of phi plus the curl of A which equals a gauge field g, B minus E

This also has a gauge field g which can easily be eliminated by subtracting the conjugate of this product. The sum of these two fields - times a factor of minus one - is formed by reversing the order of the differential with the potential:

$$\frac{1}{8}(\nabla A - (\nabla A)^*)(A\nabla - (A\nabla)^*) = \frac{1}{2}\left(B^2 - E^2, 2\,\vec{E} \times \vec{B}\right)$$

Figure 218: one eighth Del A minus the conjugate of Del A times A Del minus the conjugate of A Del equations one half B squared minus E squared, 2 E cross B

The first term drops into the Euler-Lagrange equations to generate the Gauss and Ampere laws of electromagnetism. As a bonus, there is the Poynting vector, the directional energy flux density of an electromagnetic field.

Representing the compact Lie group needed to do spatial rotations is itself compact when using quaternions:

$$\begin{aligned}
R &= (0, x_1, x_2, x_3)\\
U &= (\cos(\theta), \sin(\theta), 0, 0)\\
R \to R' &= URU^*\\
&= (0, x_1, x_2\cos(2\theta) - x_3\sin(2\theta), x_3\cos(2\theta) + x_2\sin(2\theta))
\end{aligned}$$

Figure 219: R equals 0, x sub 1, x sub 2, x sub 3. U equals cosine theta, sine theta, 0, 0. R goes to R prime equals U times R times the conjugate of U which equals 0, x sub 1 cosine 2 theta minus x sub 3 sine 2 theta, x sub 3 cosine 2 theta plus x sub 2 sine 2 theta

If one tries to simply change from the cosine and sine function to the hyperbolic cosine and sine function, a member of the Lorentz group is not generated. This should not be a surprise since that group is not compact, a non-trivial change. Other terms are required to pull off the trick:

$$\begin{aligned}
B &= (t, x_1, x_2, x_3)\\
H &= (\cosh(\alpha), \sin(\alpha), 0, 0)\\
B \to B' &= BHB^* + ((HHB)^* - (H^*H^*B)^*)/2\\
&= (\cosh(2\alpha)t - \sinh(2\alpha)x_1, \cosh(2\alpha)x_1 - \sinh(2\alpha)t, x_2, x_3)\\
&= (\gamma t - \gamma\beta x_1, \gamma x_1 - \gamma\beta t, x_2, x_3).
\end{aligned}$$

Figure 220: B equals t, x sub 1, x sub 2, x sub 3. H equals the hyperbolic cosine of alpha, the hyperbolic sine of alpha, 0, 0. B goes to B prime equals H B the conjugate of H plus one half the difference of the congugate of H H B and the conjugate of the conjugate of H the conjugate of H times B

Quaternions provide another way to write these expressions. Nothing new is learned, other than to be skeptical of claims about the limitations of quaternions.

Bibliography

[1] S. L. Adler. Quaternionic Quantum Mechanics and Quantum Fields. Oxford University Press, 1995.

[2] A. W. Conway. On the application of quaternions to some recent developments of electrical theory. Proc. Roy. Irish Acad., 29:1-9, 1911.

[3] R. Epstein and I. I. Shapiro. Post-post-newtonian deflection of light by the sun. Phys. Rev. D, 22(12):2947-2949, 1980.

[4] P. R. Girard. The quaternion group and modern physics. Eur. J. Phys., 5:25-32, 1984.

[5] J. D. Jackson. Classical Electrodynamics, Second Edition. John Wiley and Sons, 1975.

[6] J. Lambek. If hamilton had prevailed: quaternions in physics. Mathematical Intelligencer, 17(4), 1995.

[7] N. Rosen. A bi-metric theory of gravitation. General Relativity Gravitation, 4(6):435-447, 1973.

[8] E. Salamin. Application of quaternions to computation with rotations. Internal Working Paper, Santford Artificial Intelligence Lab, pages 1-9, 1979.

[9] L. Silberstein. Quaternionic form of relativity. Phil. Mag., 23(137):790-809, 1912.

[10] S. P. Thompson. The Life of William Thomson, Baron Kelvin of Largs, volume II. Macmillan, 1910.

[11] C. M. Will. Theory and experiment in gravitational physics: Revised edition. Cambridge University Press, 1993.

Footnotes

Essays on Gravitation contest

The essay, “Quaternion space-times-time invariance as gravity” was submitted to the *2015 Essays on Gravitation* contest run by the Gravity Research Foundation.

By comparison with typical technical physics papers, an effort was made to make the math simpler and have less jargon. It is seven pages long. It has five equations in the body of the paper, and five more in the appendix. There are eleven references.

The results are in… The winner was an essay by Gerard ’t Hooft, a Nobel Prize winner and all around smart guy. My essay did not win second, or third, or forth, or fifth, or honerable mention. The contest gets plenty of submissions from fringe physicists. I suspect the word “quaternion” in the title and my non-academic address meant the paper was quickly dismissed.

Hard core stuff

I did learn something about my own efforts from reading the paper. His paper was about conformal symmetry breaking. Roughly speaking, that has to do with the scale of measurements in time and space. Maxwell’s theory for how light works has conformal symmetry. If we use light to measure things, then the absolute size of *anything* cannot be known. Cool. We can figure out relative sizes and times.

This cannot be the entire story since we can figure out the the absolute sizes of things. He writes that something about quantum gravity is going to break conformal symmetry. You would have to read the paper to struggle to see his point.

In my own effort, there are two numbers that enter in to make measurements different. One is the relativistic velocity of an observer. That is plain old special relativity. The other is the dimensionless gravitational length of a graviational source. It would be this value that breaks conformal symmetry in quaternion gravity.

Derive the Euler-Lagrange equations

The Euler-Lagrange equations are used to generate field equations from a Lagrange density. Think of a Lagrange density as every way energy can be traded inside of a box. The action S integrates the Lagrange density (mass per volume) over space and *time*, resulting in t mass times time.

$$S = \int \mathcal{L} \; d\vec{R} dt$$

Figure 221: The action S equals the integral of the Lagrange density over space-time

Notice that the action could be just about any value by integrating over different amounts of time, from a nano-second to a billion years.

The approach is to *vary something* in the action S so this integral *does not change*. This means that the “something” is a symmetry of the action. Where there is a symmetry, there is necessarily a conserved quantity.

$$\delta S = 0 = \int \mathcal{L}(\delta t) d\vec{R} \, dt$$

Figure 222: The variaton of the action S equals 0 equals the integral of the Lagrange density varied with respect to a variable and posibly the derivatives of the variable integrated over space-time

This is a minimization problem, or more formally, the calculus of variations. the first types of minimization problems one learns are about the minimum value of something like a velocity at a point in space-time. this is about a minimization of a function over all of space-time. the mechanics are the same - take a derivative, set it to zero - but the thing that gets plugged in is different.

Examples

- If the lagrange density is not a function of **time**, then time is a symmetry and **energy** is conserved.
- If the lagrange density is not a function of **space**, then space is a symmetry and **linear momentum** is conserved.

- If the lagrange density is not a function of **angle**, then rotation is a symmetry and **angular momentum** is conserved.

Counter example

- If a lagrange equation **is a function of space and time**, then **energy and momentum are not conserved**. this happens for systems that have friction. the energy and momentum go into waste heat. those terms usually are not included in the lagrange density.

Deriving the euler-lagrange equations

If a lagrange density depends on a 4-potential a and the derivatives of a, then vary these and find a minimum. this is the heart of the euler-lagrange equations.

$$0 = \int \mathcal{L}(\delta A, \delta \nabla A)\ d\vec{R}dt$$

Figure 223: the integral of the lagrange density varied with respect to a and the derivative of a integrated over space-time equals 0

This is a mimnum problem with the potential A and its derivative, A'.

1: Start with a Lagrange density that is a function of the potential and its derivatives.

$$\mathcal{L} = f(A, \nabla A)$$

Figure 224: The Lagrange density is a function of the potential and the derivatives of the potential

Note that one is not allowed to vary position or speed. If we were to do the reverse - fix the potential and its derivative, but vary position and velocity - then we would be deriving the force equation from the same Lagrange density.

2: For the action by integrating over a volume of space-time.

3: Vary the action.

4: The problem is with the variation in A versus the variantion is the derivative of A. Use the product rule to get two variations in A.

5: A theorem of Gauss says:

so:

$$S = \int \mathcal{L}(A, \nabla A)\; d\vec{R}dt$$

Figure 225: The action S equations the integral over space-time of the Lagrange density that is a function of the potential A and the derivatives of the potential A'

$$\delta S = \int (\frac{\partial \mathcal{L}}{\partial A}\delta A + \frac{\partial \mathcal{L}}{\partial \nabla A}\delta \nabla A) d\vec{R}dt$$

Figure 226: Vary the action S which equals the integral over space-time of the partial derivative of the Lagrange density with repect to A while varying A plus the derivative of the Lagrange density with repect to the derivative of A while varying the derivative of A

$$\nabla(\frac{\partial \mathcal{L}}{\partial \nabla A}\delta A) = \nabla \frac{\partial \mathcal{L}}{\partial \nabla A}\delta A + \frac{\partial \mathcal{L}}{\partial \nabla A}\delta \nabla A$$

Figure 227: The derivative of the product of the patial derivative of the Lagrange density with respect to Del A times the variation in A equals Del the partial derivative of the Lagrangian with respect to Del A while varying A plus the partial derivative of the Lagrangian with repect to Del A while varying Del A

$$\nabla(\frac{\partial \mathcal{L}}{\partial \nabla A}\delta A) = 0$$

Figure 228: The gradient of the partial derivative of the Lagrangian with respec to Del A while varying A equals zero

$$-\nabla \frac{\partial \mathcal{L}}{\partial \nabla A} \delta A = \frac{\partial \mathcal{L}}{\partial \nabla A} \delta \nabla A$$

6: Subsitute 5 into the variation in 3:

$$\delta S = \int (\frac{\partial \mathcal{L}}{\partial A} \delta A + \frac{\partial \mathcal{L}}{\partial \nabla A} \delta \nabla A) d\vec{R} dt$$

Figure 229: Vary the action S which equals the integral over space-time of the partial derivative of the Lagrange density with repect to A minus the derivative of the Lagrange density with repect to Del A while varying with repect to A

7: The variation will be at the minimum if the variation in the action S is zero, which happends if the integrand is zero:

$$\frac{\partial \mathcal{L}}{\partial A} = \nabla \frac{\partial \mathcal{L}}{\partial \nabla A}$$

Figure 230: The derivative of the Lagrange density with repect to A is equal to the derivative of the partial derivative of the Lagrange density with respect to Del A

QED

There are so many partial differential equations when using Euler-Lagrange, people with thin you are brilliant.

$$\frac{\partial \mathcal{L}}{\partial A} = \nabla \frac{\partial \mathcal{L}}{\partial \nabla A}$$

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial \phi} &= \frac{\partial}{\partial t}\frac{\partial \mathcal{L}}{\partial(\frac{\partial \phi}{\partial t})} + \frac{\partial}{\partial x}\frac{\partial \mathcal{L}}{\partial(\frac{\partial \phi}{\partial x})} + \frac{\partial}{\partial y}\frac{\partial \mathcal{L}}{\partial(\frac{\partial \phi}{\partial y})} + \frac{\partial}{\partial z}\frac{\partial \mathcal{L}}{\partial(\frac{\partial \phi}{\partial z})} \\
\frac{\partial \mathcal{L}}{\partial Ax} &= \frac{\partial}{\partial t}\frac{\partial \mathcal{L}}{\partial(\frac{\partial Ax}{\partial t})} + \frac{\partial}{\partial x}\frac{\partial \mathcal{L}}{\partial(\frac{\partial Ax}{\partial x})} + \frac{\partial}{\partial y}\frac{\partial \mathcal{L}}{\partial(\frac{\partial Ax}{\partial y})} + \frac{\partial}{\partial z}\frac{\partial \mathcal{L}}{\partial(\frac{\partial Ax}{\partial z})} \\
\frac{\partial \mathcal{L}}{\partial Ay} &= \frac{\partial}{\partial t}\frac{\partial \mathcal{L}}{\partial(\frac{\partial Ay}{\partial t})} + \frac{\partial}{\partial x}\frac{\partial \mathcal{L}}{\partial(\frac{\partial Ay}{\partial x})} + \frac{\partial}{\partial y}\frac{\partial \mathcal{L}}{\partial(\frac{\partial Ay}{\partial y})} + \frac{\partial}{\partial z}\frac{\partial \mathcal{L}}{\partial(\frac{\partial Ay}{\partial z})} \\
\frac{\partial \mathcal{L}}{\partial Az} &= \frac{\partial}{\partial t}\frac{\partial \mathcal{L}}{\partial(\frac{\partial Az}{\partial t})} + \frac{\partial}{\partial x}\frac{\partial \mathcal{L}}{\partial(\frac{\partial Az}{\partial x})} + \frac{\partial}{\partial y}\frac{\partial \mathcal{L}}{\partial(\frac{\partial Az}{\partial y})} + \frac{\partial}{\partial z}\frac{\partial \mathcal{L}}{\partial(\frac{\partial Az}{\partial z})}
\end{aligned}$$

EM invariants

To derive the field equations of electromagnetism (EM), we need to find Lorentz invariants that use the electric (E) and magnetic (B) fields of EM that then get plugged into the Euler-Lagrange equation.

An invariant is something all observers can agree on. The Egyptions fiture out one long ago:

$$3^2 + 4^2 = 5^2$$

Figure 231: 3 squared plus 4 squared equals 5 squared

This was the basis for surveying in the flood plains of the Nile. It remains true today. The 3D Egyptions know we could also include a third spacial dimension:

$$a^2 + b^2 + c^2 = R^2$$

Figure 232: a squared plut b squared plus c squared equals R squared

Einstein showed that Egyptions in rockets could only agree on the inverval between events.

$$d\tau^2 = dt^2 - d\vec{R}^2/c^2$$

Figure 233: d tau squared equals d t squared minus d R squared over the speed of light c squared

Accelerating or spinning is alright for observing events is you can figure out the right sort of functions to put into the interval.

Masters of general relativity can figure out the dynamic functions for f and g in only a few special cases because the math remains so difficult. The proposal described in this site, quaternion gravity, should make this issue tractable since then one has an algebra problem instead of ten nonliner differential equations to solve. For the rest of the discussion of EM, it is assumed f and g are equal to one since it makes the math simple.

The first term of a quaternion product is a Lorentz invariant scalar. Couple the current with the potential by multiplying them together:

$$d\tau^2 = f dt^2 - g d\vec{R}^2/c^2$$

Figure 234: d tau squared equals the function f times d t squared minus the function g times d R squared over the speed of light c squared

$$\begin{aligned} JA &= (\rho, \vec{J})(\phi, \vec{A}) \\ &= (\rho\phi - \vec{J} \cdot \vec{A}, \phi\vec{A} + \vec{J}\phi + \vec{J} \times \vec{A}) \end{aligned}$$

Figure 235: J times A equals rho, J times phi, A equals r phi minus J dot A, phi A plus J phi plus the cross product of J and A

The electric and magentic fields can be written in terms of differential operators acting on a potential. Form the product:

$$\begin{aligned} DA &= \left(\frac{\partial}{\partial t}, \vec{\nabla}\right)\left(\phi, \vec{A}\right) \\ &= \left(\frac{\partial \phi}{\partial t} - \nabla \cdot A, \frac{\partial \vec{A}}{\partial t} + \vec{\nabla}\phi + \vec{\nabla} \times \vec{A}\right) \end{aligned}$$

Figure 236: The differential times the potential A equals the time derivative, Del times phi A equals the time derivative of phi minus the divergence of A, the time derivative of A plus the gradient of phi plus the curl of A

The first term is a gauge term. EM has gauge symmetry. Set this to zero in a way that assures that no matter what gauge we pick - terms involving the time derivative of phi or divergence of A - the other terms are unchanged.

There are two times of 3-vectors. An axial vector will not change if the the order of the product is reverse. A polar vector will flip signs by changing the order. That is a property of cross products and curls. Here are the two possibilities:

These should both be as “long” as each other, but will point in a different direction so long as the magnetic field B is not zero. Zero is an invariant, so take the difference of the norms of both of these, and that will always, necssarily, be equal to zero.

The dot product of the electric and magnetic field will be used to derive the homogenous Maxwell equations, the no monopoles and Faraday’s law.

$$\frac{1}{2}(DA - (DA)^*) = \left(0, \frac{\partial \vec{A}}{\partial t} + \vec{\nabla}\phi + \vec{\nabla} \times \vec{A}\right)$$

Figure 237: One half Del times A minus the conjugate of Del times A equals zero the time derivative of A plus the gradient of phi plus the curl of A

$$\frac{1}{2}(DA - (DA)^*) = \left(0, \frac{\partial \vec{A}}{\partial t} + \vec{\nabla}\phi + \vec{\nabla} \times \vec{A}\right)$$

Figure 238: One half Del times A minus the conjugate of Del times A equals zero the time derivative of A plus the gradient of phi plus the curl of A

$$\frac{1}{2}(AD - (AD)^*) = \left(0, \frac{\partial \vec{A}}{\partial t} + \vec{\nabla}\phi - \vec{\nabla} \times \vec{A}\right)$$

Figure 239: One half A times Del minus the conjugate of A times Del equals zero the time derivative of A plus the gradient of phi minus the curl of A

$$\begin{aligned}
||(0, -E + B)|| &= E^2 - 2E \cdot B + B^2 \\
||(0, -E - B)|| &= E^2 + 2E \cdot B + B^2 \\
||(0, -E - B)|| - ||(0, -E + B)|| &= 4E \cdot B
\end{aligned}$$

Figure 240: The norm of minus E plus B equals E squared minus 2 E dot B plus B squared. The norm of minus E minus B equals E squared plus 2 E dot B plus B squared. The second norm minus the first is 4 E dot B

The product of the two ways to multiply a differential and a potential also form an invariant:

$$(0, -E + B)(0, -E - B) = (B^2 - E^2, 2E \times B)$$

Figure 241: 0, minus E plus B times 0, minus E minus B equals B squared minus E squared, 2 E cross B

This Lorentz invariant quantity will be used when deriving the Maxwell source equations, Gauss' and Ampere's laws.

Derive the Maxwell homogeneous equations

The easy way

There are several different roads to the same results, the no monopoles law and Faraday's law, known together as the homogenous equations. The quickest path is to show how they are vector idenities. The divergence of a curl is zero, so if the magnetic field is the curl of the potential A, then there are no magnetic monopoles.

One down, one to go.

Plug the potential definitions of a B and E field into Faraday's law, and watch all the terms drop.

QED.

The path followed here is *considerably* longer. Everyone uses the Euler-Lagrange equations to derive the source equations, Gauss' law and Ampere's law. For the sake of logical consistency, and to get practice with the details of Euler-Lagrange, the same machinery will be used to derive all four Maxwell equations.

Maxwell defined

The Maxwell equations are the pinnacle of classical physics, the way all light, electric charges, and magnets play with each other gracefully. Here is a one sentence definition:

```
The Maxwell equations define how a current density is the source of all the
changes in space-time of changes in space-time of a space-time potential
that travels at the speed of light.
```

The Maxwell equations are a complete set of second order differential equations along with the vector identities. Nothing is missing.[8]

The fields defined

A quaternion derivative has a time derivative and three spatial derivatives. A quaternion potential has a scalar potential and three others for space no matter what one's choice of coordinate systems. Construct the complete set of first order changes of a potential by taking the product.

Simple enough.

[8] A driver of traffic to my sign is a claim on the Internet that the first edition of "A Treatise on Electricity and Magnetism" had 200 equations written with quaternions that were deleted by the second edition by Heaviside. Finding the first version was a struggle, but I did find it. It had two sections with "Quaternion" in the title. It was clear that this grand master of old was only using the 3-vector part of a quaternion. This is not where the fun is in my opinion. Since I have derived and rederived and rewritten my derivations of the Maxwell equations using only quaternions many times, I can assure you, nothing is missing.

$$
\begin{aligned}
DA &= \left(\frac{\partial}{\partial t}, \vec{\nabla}\right)\left(\phi, \vec{A}\right) \\
&= \left(\frac{\partial \phi}{\partial t} - \nabla \cdot A, \frac{\partial \vec{A}}{\partial t} + \vec{\nabla}\phi + \vec{\nabla} \times \vec{A}\right) \\
&= \left(\frac{\partial \phi}{\partial t} - \nabla \cdot A, -\vec{E} + \vec{B}\right)
\end{aligned}
$$

Figure 242: The differential times the potential A equals the time derivative, Del times phi A; which equals the time derivative of phi minus the divergence of A, the time derivative of A plus the gradient of phi plus the curl of A; which equals a gauge term, minus the electric field E plus the magnetic field B

No, stop. This is amazing enough to repeat.

The most basic complete quaternion derivative of a potential is EM

$$
\begin{aligned}
DA &= \left(\frac{\partial}{\partial t}, \vec{\nabla}\right)\left(\phi, \vec{A}\right) \\
&= \left(\frac{\partial \phi}{\partial t} - \nabla \cdot A, \frac{\partial \vec{A}}{\partial t} + \vec{\nabla}\phi + \vec{\nabla} \times \vec{A}\right) \\
&= \left(\frac{\partial \phi}{\partial t} - \nabla \cdot A, -\vec{E} + \vec{B}\right)
\end{aligned}
$$

Figure 243: The differential times the potential A equals the time derivative, Del times phi A; which equals the time derivative of phi minus the divergence of A, the time derivative of A plus the gradient of phi plus the curl of A; which equals a gauge term, minus the electric field E plus the magnetic field B

But what about that first term? One of the defining characteristic of light is how its interval is zero. A photon cannot wear a watch. Photons are timeless. The way to implement that quality is to set this gauge term equal to zero like so:

This is a recurring technique. If something travels at the speed of light, there will be non of the the four gauge terms:

$$\frac{1}{2}(\nabla A - (\nabla A)^*) = (0, -\vec{E} + \vec{B})$$

Figure 244: One half Del times A minus the conjugate of Del times A equals 0, -E + B

$$\frac{\partial \phi}{\partial t} \quad \frac{\partial Ax}{\partial x} \quad \frac{\partial Ay}{\partial y} \quad \frac{\partial Az}{\partial z}$$

The electric and magnetic field are unchanged by changing the guage because the gauge terms are always subtracted away.

One enormous subject I have not looked into is what happens if one keeps this gauge term. The resulting physics must describe thing that do not travel at the speed of light. It is the subject of particles with a mass.

The plan

Here is how we will derive the no monopoles law.

- Start with 1 easy term, E_x
- Pair that with 1 B_x
- Multiply it out
- Clone lines, filling in E_y and E_z
- Look for patterns

Writing out the Lagrangian

The dot product of the electric and magnetic fields has 24 terms. It is scary, so start simple with one term only, E_x:

$$\vec{E} \cdot \vec{B} = (-\frac{\partial Ax}{\partial t} - \frac{\partial \phi}{\partial x})(B_{something}) + \ldots$$

Figure 245: E dot B equals minus the time derivative of A_x minux the derivative of phi with respect to x

The magnetic field B_x has everything not found in E_x, including both the potentials and derivatives.

$$\vec{E} \cdot \vec{B} = (-\frac{\partial Ax}{\partial t} - \frac{\partial \phi}{\partial x})(\frac{\partial Ay}{\partial z} - \frac{\partial Az}{\partial y}) + ...$$

Figure 246: E dot B equals minus the time derivative of A_x minux the derivative of phi with respect to x times the derivative of A_y with respect of z minus the derivative of A_z with respect to y

It is seeing details like all four potential terms and all four differentials in each line that makes the Maxwell equations feel so complete.

Multiply this out.

$$\vec{E} \cdot \vec{B} = (-\frac{\partial Ax}{\partial t} - \frac{\partial \phi}{\partial x})(\frac{\partial Ay}{\partial z} - \frac{\partial Az}{\partial y}) + ...$$
$$= -\frac{\partial Ax}{\partial t}\frac{\partial Ay}{\partial z} - \frac{\partial \phi}{\partial x}\frac{\partial Ay}{\partial z} + \frac{\partial Ax}{\partial t}\frac{\partial Az}{\partial y} + \frac{\partial \phi}{\partial x}\frac{\partial Az}{\partial y} + ...$$

Figure 247: E dot B equals minus the time derivative of A_x minux the derivative of phi with respect to x times the derivative of A_y with respect of z minus the derivative of A_z with respect to y; which equals minus the time derivative of A_x the derivative of A_y with respect to z minus the derivative of phi with respect to x the derivative of A_y with respect to z plus the time derivative of A_x the derivative of A_z with respect to y plus the derivative of phi with respect to x the derivative of A_z with respect to y

- Half the terms are positive, half are negative, setting up for cancellations.
- Each term has a t, x, y, z-ish part.
- 8 down, 16 to go.

Clone E_x to make E_y and E_z making all necessary subsitutions:

$$\vec{E}\cdot\vec{B} = (-\frac{\partial Ax}{\partial t} - \frac{\partial \phi}{\partial x})(\frac{\partial Ay}{\partial z} - \frac{\partial Az}{\partial y}) + \ldots$$

$$= -\frac{\partial Ax}{\partial t}\frac{\partial Ay}{\partial z} - \frac{\partial \phi}{\partial x}\frac{\partial Ay}{\partial z} + \frac{\partial Ax}{\partial t}\frac{\partial Az}{\partial y} + \frac{\partial \phi}{\partial x}\frac{\partial Az}{\partial y}$$

$$-\frac{\partial Ay}{\partial t}\frac{\partial A.}{\partial .} - \frac{\partial \phi}{\partial y}\frac{\partial A.}{\partial .} + \frac{\partial Ay}{\partial t}\frac{\partial A.}{\partial .} + \frac{\partial \phi}{\partial y}\frac{\partial A.}{\partial .}$$

$$-\frac{\partial Az}{\partial t}\frac{\partial A.}{\partial .} - \frac{\partial \phi}{\partial z}\frac{\partial A.}{\partial .} + \frac{\partial Az}{\partial t}\frac{\partial A.}{\partial .} + \frac{\partial \phi}{\partial z}\frac{\partial A.}{\partial .}$$

Look for patterns in the partial derivatives:

$$\vec{E}\cdot\vec{B} = (-\frac{\partial Ax}{\partial t} - \frac{\partial \phi}{\partial x})(\frac{\partial Ay}{\partial z} - \frac{\partial Az}{\partial y}) + \ldots$$

$$= -\frac{\partial Ax}{\partial t}\frac{\partial Ay}{\partial z} - \frac{\partial \phi}{\partial x}\frac{\partial Ay}{\partial z} + \frac{\partial Ax}{\partial t}\frac{\partial Az}{\partial y} + \frac{\partial \phi}{\partial x}\frac{\partial Az}{\partial y}$$

$$-\frac{\partial Ay}{\partial t}\frac{\partial A.}{\partial .} - \frac{\partial \phi}{\partial y}\frac{\partial A.}{\partial .} + \frac{\partial Ay}{\partial t}\frac{\partial A.}{\partial .} + \frac{\partial \phi}{\partial y}\frac{\partial A.}{\partial .}$$

$$-\frac{\partial Az}{\partial t}\frac{\partial A.}{\partial .} - \frac{\partial \phi}{\partial z}\frac{\partial A.}{\partial .} + \frac{\partial Az}{\partial t}\frac{\partial A.}{\partial .} + \frac{\partial \phi}{\partial z}\frac{\partial A.}{\partial .}$$

The electric field terms are in yellow. The top line has the magnetic field, B_x. None of these has an x, it is pairs of y's and z's.

The next part of the puzzle is to figure out where the rest of the derivative with respect to x go. That will dictate where the other partials go too.

$$\vec{E} \cdot \vec{B} = (-\frac{\partial Ax}{\partial t} - \frac{\partial \phi}{\partial x})(\frac{\partial Ay}{\partial z} - \frac{\partial Az}{\partial y}) + ...$$

$$= -\frac{\partial Ax}{\partial t}\frac{\partial Ay}{\partial z} - \frac{\partial \phi}{\partial x}\frac{\partial Ay}{\partial z} + \frac{\partial Ax}{\partial t}\frac{\partial Az}{\partial y} + \frac{\partial \phi}{\partial x}\frac{\partial Az}{\partial y}$$

$$-\frac{\partial Ay}{\partial t}\frac{\partial A.}{\partial x} - \frac{\partial \phi}{\partial y}\frac{\partial A.}{\partial x} + \frac{\partial Ay}{\partial t}\frac{\partial A.}{\partial z} + \frac{\partial \phi}{\partial y}\frac{\partial A.}{\partial z}$$

$$-\frac{\partial Az}{\partial t}\frac{\partial A.}{\partial y} - \frac{\partial \phi}{\partial z}\frac{\partial A.}{\partial y} + \frac{\partial Az}{\partial t}\frac{\partial A.}{\partial x} + \frac{\partial \phi}{\partial z}\frac{\partial A.}{\partial x}$$

This is the game of curl Sudoku. Fill in the missing values for x, y and z.

$$\vec{E} \cdot \vec{B} = (-\frac{\partial Ax}{\partial t} - \frac{\partial \phi}{\partial x})(\frac{\partial Ay}{\partial z} - \frac{\partial Az}{\partial y}) + ...$$

$$= -\frac{\partial Ax}{\partial t}\frac{\partial Ay}{\partial z} - \frac{\partial \phi}{\partial x}\frac{\partial Ay}{\partial z} + \frac{\partial Ax}{\partial t}\frac{\partial Az}{\partial y} + \frac{\partial \phi}{\partial x}\frac{\partial Az}{\partial y}$$

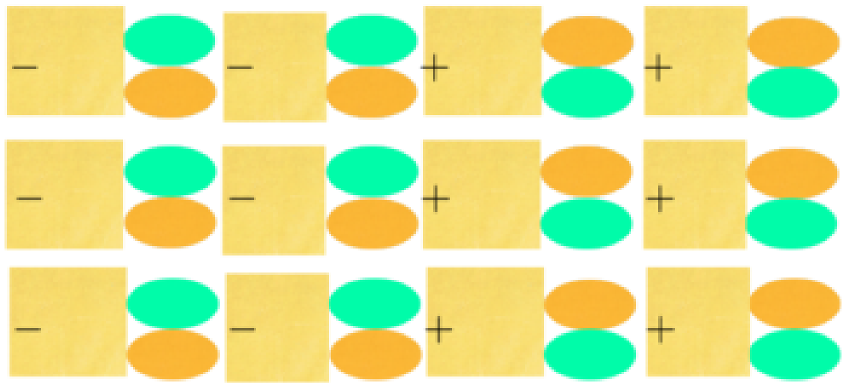
$$-\frac{\partial Ay}{\partial t}\frac{\partial Az}{\partial x} - \frac{\partial \phi}{\partial y}\frac{\partial Az}{\partial x} + \frac{\partial Ay}{\partial t}\frac{\partial Ax}{\partial z} + \frac{\partial \phi}{\partial y}\frac{\partial Ax}{\partial z}$$

$$-\frac{\partial Az}{\partial t}\frac{\partial Ax}{\partial y} - \frac{\partial \phi}{\partial z}\frac{\partial Ax}{\partial y} + \frac{\partial Az}{\partial t}\frac{\partial Ay}{\partial x} + \frac{\partial \phi}{\partial z}\frac{\partial Ay}{\partial x}$$

All the needed slots are filled in. The Soduko game is complete.

Now remove some of the details. The dot product of E and B is pretty:

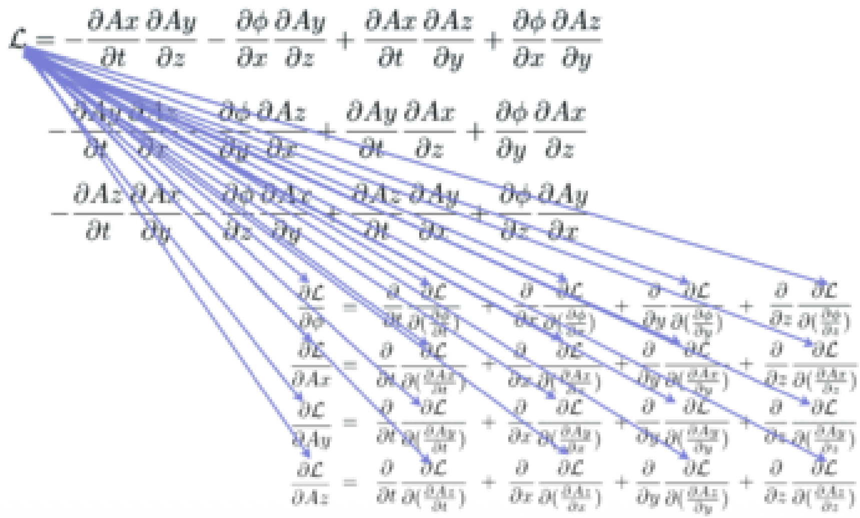
$$\vec{E} \cdot \vec{B} =$$



The Lorentz invariant Lagrange density is complete.

Derive the no monopoles law

Plug the 16 terms of the Lagrange density into 20 slots in the Euler-Lagrange equations:



This is a mountain of details. People are much better at spotting patterns.

Do simple things, one at a time. Here is the first Euler-Lagrange equation:

$$\frac{\partial \mathcal{L}}{\partial \phi} = \frac{\partial}{\partial t}\frac{\partial \mathcal{L}}{\partial(\frac{\partial \phi}{\partial t})} + \frac{\partial}{\partial x}\frac{\partial \mathcal{L}}{\partial(\frac{\partial \phi}{\partial x})} + \frac{\partial}{\partial y}\frac{\partial \mathcal{L}}{\partial(\frac{\partial \phi}{\partial y})} + \frac{\partial}{\partial z}\frac{\partial \mathcal{L}}{\partial(\frac{\partial \phi}{\partial z})}$$

- Only terms with phi matter.
- The derivative repeat.
- That's it.

Here are the terms in the Lagrangian that have a phi:

$$\mathcal{L} = -\frac{\partial Ax}{\partial t}\frac{\partial Ay}{\partial z} - \frac{\partial \phi}{\partial x}\frac{\partial Ay}{\partial z} + \frac{\partial Ax}{\partial t}\frac{\partial Az}{\partial y} + \frac{\partial \phi}{\partial x}\frac{\partial Az}{\partial y}$$
$$-\frac{\partial Ay}{\partial t}\frac{\partial Az}{\partial x} - \frac{\partial \phi}{\partial y}\frac{\partial Az}{\partial x} + \frac{\partial Ay}{\partial t}\frac{\partial Ax}{\partial z} + \frac{\partial \phi}{\partial y}\frac{\partial Ax}{\partial z}$$
$$-\frac{\partial Az}{\partial t}\frac{\partial Ax}{\partial y} - \frac{\partial \phi}{\partial z}\frac{\partial Ax}{\partial y} + \frac{\partial Az}{\partial t}\frac{\partial Ay}{\partial x} + \frac{\partial \phi}{\partial z}\frac{\partial Ay}{\partial x}$$

Every term with a phi is mixed. Derivatives of mixed terms is simple:

$$\textbf{Consider: } \frac{\partial xy}{\partial x} = y \quad \textbf{if: } x = \frac{\partial \phi}{\partial x}$$

$$-\frac{\partial}{\partial x}\left(\frac{\frac{\partial \phi}{\partial x}\frac{\partial Ay}{\partial z}}{\frac{\partial \phi}{\partial x}}\right) = -\frac{\partial^2 Ay}{\partial x\,\partial z}$$

Here's what happens: * After Euler-Lagrange is applied, there is no phi left. * There is 1 term in the numerator, and two partial derivatives. * All three spatial directions appear once. Using these three guides, you should be able to picture how the Lagrange density is changed by applying the Euler-Lagrange equation.

With minuses in one column, and pluses in the other, cancellations happen:

$$
\begin{aligned}
\mathcal{L} = &-\frac{\partial Ax}{\partial t}\frac{\partial Ay}{\partial z} - \frac{\partial \phi}{\partial x}\frac{\partial Ay}{\partial z} + \frac{\partial Ax}{\partial t}\frac{\partial Az}{\partial y} + \frac{\partial \phi}{\partial x}\frac{\partial Az}{\partial y} \\
&-\frac{\partial Ay}{\partial t}\frac{\partial Az}{\partial x} - \frac{\partial \phi}{\partial y}\frac{\partial Az}{\partial x} + \frac{\partial Ay}{\partial t}\frac{\partial Ax}{\partial z} + \frac{\partial \phi}{\partial y}\frac{\partial Ax}{\partial z} \\
&-\frac{\partial Az}{\partial t}\frac{\partial Ax}{\partial y} - \frac{\partial \phi}{\partial z}\frac{\partial Ax}{\partial y} + \frac{\partial Az}{\partial t}\frac{\partial Ay}{\partial x} + \frac{\partial \phi}{\partial z}\frac{\partial Ay}{\partial x}
\end{aligned}
$$

$$
\text{example} : -\frac{\partial^2 \mathrm{Ay}}{\partial \mathrm{x}\, \partial \mathrm{z}} + \frac{\partial^2 \mathrm{Ay}}{\partial \mathrm{z}\, \partial \mathrm{x}} = 0
$$

Focus on teh example. See how the phi drops, and one has mixed derivatives with opposite signs. Nice.

What is going on in terms of the E and B fields? Look at things row by row:

$$
\mathcal{L} = -\frac{\partial Ax}{\partial t}\frac{\partial Ay}{\partial z} - \frac{\partial \phi}{\partial x}\frac{\partial Ay}{\partial z} + \frac{\partial Ax}{\partial t}\frac{\partial Az}{\partial y} + \frac{\partial \phi}{\partial x}\frac{\partial Az}{\partial y}
$$

$$
\frac{\partial^2 Az}{\underline{\partial x}\, \partial y} - \frac{\partial^2 Ay}{\underline{\partial x}\, \partial z} = \frac{\partial}{\partial x}\left(\frac{\partial Az}{\partial y} - \frac{\partial Ay}{\partial z}\right) = -\frac{\partial Bx}{\partial x}
$$

The first line of the Lagrangian has two derivatives of phi with respect to x. After going through the Euler-Lagrange equation, one is left with a second order derivative which is the x derivative of the magnetic field.

Rinse and repeat for y and z:

$$
\begin{aligned}
\mathcal{L} = &-\frac{\partial Ax}{\partial t}\frac{\partial Ay}{\partial z} - \frac{\partial \phi}{\partial x}\frac{\partial Ay}{\partial z} + \frac{\partial Ax}{\partial t}\frac{\partial Az}{\partial y} + \frac{\partial \phi}{\partial x}\frac{\partial Az}{\partial y} \\
&-\frac{\partial Ay}{\partial t}\frac{\partial Az}{\partial x} - \frac{\partial \phi}{\partial y}\frac{\partial Az}{\partial x} + \frac{\partial Ay}{\partial t}\frac{\partial Ax}{\partial z} + \frac{\partial \phi}{\partial y}\frac{\partial Ax}{\partial z} \\
&-\frac{\partial Az}{\partial t}\frac{\partial Ax}{\partial y} - \frac{\partial \phi}{\partial z}\frac{\partial Ax}{\partial y} + \frac{\partial Az}{\partial t}\frac{\partial Ay}{\partial x} + \frac{\partial \phi}{\partial z}\frac{\partial Ay}{\partial x}
\end{aligned}
$$

$$0 = \vec{\nabla} \cdot \vec{B}$$

QED

These cancellations all happen because of a vector identity: the divergence of a curl is zero.

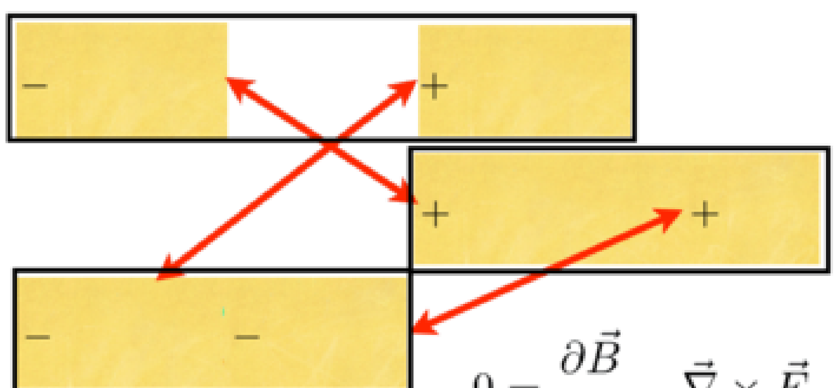
Derive Faraday's law

To continue down this longer road and arrive at Faraday's law, start from the same Lagrangian, but focus on the A_x terms:

$$
\begin{aligned}
\mathcal{L} = & -\frac{\partial Ax}{\partial t}\frac{\partial Ay}{\partial z} - \frac{\partial \phi}{\partial x}\frac{\partial Ay}{\partial z} + \frac{\partial Ax}{\partial t}\frac{\partial Az}{\partial y} + \frac{\partial \phi}{\partial x}\frac{\partial Az}{\partial y} \\
& -\frac{\partial Ay}{\partial t}\frac{\partial Az}{\partial x} - \frac{\partial \phi}{\partial y}\frac{\partial Az}{\partial x} + \frac{\partial Ay}{\partial t}\frac{\partial Ax}{\partial z} + \frac{\partial \phi}{\partial y}\frac{\partial Ax}{\partial z} \\
& -\frac{\partial Az}{\partial t}\frac{\partial Ax}{\partial y} - \frac{\partial \phi}{\partial z}\frac{\partial Ax}{\partial y} + \frac{\partial Az}{\partial t}\frac{\partial Ay}{\partial x} + \frac{\partial \phi}{\partial z}\frac{\partial Ay}{\partial x}
\end{aligned}
$$

- The Euler-Lagrange will wipe out the A_x's, leading to cancellations.
- The top line is a time derivative of B_x.
- The second and third lines together form the curl of E_x.

Here is the pattern:



$$0 = \frac{\partial \vec{B}}{\partial t} - \vec{\nabla} \times \vec{E}$$

QED

This is Faraday's law.

Derive the Maxwell source equations

The Maxwell source equations are Gauss' law and Ampere's law. A Lorentz invariant coupling of the current density to the potential is needed. That is simply the product of the current density with the potential:

$$\begin{aligned} JA &= (\rho, \vec{J})(\phi, \vec{A}) \\ &= (\rho\phi - \vec{J} \cdot \vec{A}, \phi\vec{A} + \vec{J}\phi + \vec{J} \times \vec{A}) \end{aligned}$$

Figure 248: J times A equals rho, J times phi, A equals r phi minus J dot A, phi A plus J phi plus the cross product of J and A

We also need a Lorentz invariant of the E and B fields. As discussed in EM invariants, the gauge-free derivative of a potential written in both orders does the trick:

$$\begin{aligned} &(0, -\vec{E} + \vec{B})(0, -\vec{E} - \vec{B}) \\ &\quad = (B^2 - E^2, 2\vec{E} \times \vec{B}) \end{aligned}$$

This is a difference of squares. We get the Poynting 3-vector created for free. This is not a minor addition. Neither of the fields E or B are changed if time is reversed. The square of the E field has two terms that flip signs, so effectively no sign flips, while the square of the B field has none. The same is not true of the Poynting vector. There is only one term with a time factor in the E field, so it would flip signs. One long standing riddle in physics is the perfect time symmetry in the Maxwell equations. Those equations start with only the difference of squares, not the Poynting vector. If, for the biggest complete vision, we need to consider both of these terms, a solution to the time symmetry riddle could be found. That is speculation, but worth pointing out.

The plan

Start by writing out the Lagrange density * Write out E_x and B_x. * Multiply it out. * Clones E_y, E_z, B_y, and B_z.

The Lagrange density

Start easy by writing out E_x and B_x:

$$B^2 - E^2 = (Bx^2 - Ex^2 + ...$$
$$= (\frac{\partial Ay}{\partial z} - \frac{\partial Az}{\partial y})^2 - (-\frac{\partial Ax}{\partial t} - \frac{\partial \phi}{\partial x})^2 + ...$$

Multiply it out.

$$B^2 - E^2 = (Bx^2 - Ex^2 + ...$$
$$= (\frac{\partial Ay}{\partial z} - \frac{\partial Az}{\partial y})^2 - (-\frac{\partial Ax}{\partial t} - \frac{\partial \phi}{\partial x})^2 + ...$$
$$= (\frac{\partial Ay}{\partial z})^2 - 2\frac{\partial Ay}{\partial z}\frac{\partial Az}{\partial y} + (\frac{\partial Az}{\partial y})^2$$
$$-(\frac{\partial Ax}{\partial t})^2 - 2\frac{\partial Ax}{\partial t}\frac{\partial \phi}{\partial x} - (\frac{\partial \phi}{\partial x})^2 + ...$$

Every term is negative except the squares of the B_x field. Continue the process for $_y$ and $_z$.

$$\mathcal{L} = B^2 - E^2$$
$$=$$
$$(\frac{\partial Ay}{\partial z})^2 - 2\frac{\partial Ay}{\partial z}\frac{\partial Az}{\partial y} + (\frac{\partial Az}{\partial y})^2 - (\frac{\partial Ax}{\partial t})^2 - 2\frac{\partial Ax}{\partial t}\frac{\partial \phi}{\partial x} - (\frac{\partial \phi}{\partial x})^2$$
$$+(\frac{\partial Az}{\partial x})^2 - 2\frac{\partial Az}{\partial x}\frac{\partial Ax}{\partial z} + (\frac{\partial Ax}{\partial z})^2 - (\frac{\partial Ay}{\partial t})^2 - 2\frac{\partial Ay}{\partial t}\frac{\partial \phi}{\partial y} - (\frac{\partial \phi}{\partial y})^2$$
$$+(\frac{\partial Ax}{\partial y})^2 - 2\frac{\partial Ax}{\partial y}\frac{\partial Ay}{\partial x} + (\frac{\partial Ay}{\partial x})^2 - (\frac{\partial Az}{\partial t})^2 - 2\frac{\partial Az}{\partial t}\frac{\partial \phi}{\partial z} - (\frac{\partial \phi}{\partial z})^2$$

The electric field terms in yellow are simple substitutions. The magnetic field involves the curl, so it can be intimidating. Once one term is done, the rest follow from the pattern set for_ B_x_ along with a “no redundant” requirement, much like a Sudoku puzzle.

The Lagrange density needs the current coupling and the difference of the square of the fields. One detail is that a factor of a half is needed to simplify derivative equations

$$
\begin{aligned}
\mathcal{L}_{EM} &= scalar(JA + \frac{1}{2}\left(B^2 - E^2\right)) \\
&= \rho\phi - Jx\,Ax - Jy\,Ay - Jz\,Az \\
&+\frac{1}{2}((\frac{\partial Ay}{\partial z})^2 - 2\frac{\partial Ay}{\partial z}\frac{\partial Az}{\partial y} + (\frac{\partial Az}{\partial y})^2 - (\frac{\partial Ax}{\partial t})^2 - 2\frac{\partial Ax}{\partial t}\frac{\partial \phi}{\partial x} - (\frac{\partial \phi}{\partial x})^2 \\
&\quad +(\frac{\partial Az}{\partial x})^2 - 2\frac{\partial Az}{\partial x}\frac{\partial Ax}{\partial z} + (\frac{\partial Ax}{\partial z})^2 - (\frac{\partial Ay}{\partial t})^2 - 2\frac{\partial Ay}{\partial t}\frac{\partial \phi}{\partial y} - (\frac{\partial \phi}{\partial y})^2 \\
&\quad +(\frac{\partial Ax}{\partial y})^2 - 2\frac{\partial Ax}{\partial y}\frac{\partial Ay}{\partial x} + (\frac{\partial Ay}{\partial x})^2 - (\frac{\partial Az}{\partial t})^2 - 2\frac{\partial Az}{\partial t}\frac{\partial \phi}{\partial z} - (\frac{\partial \phi}{\partial z})^2)
\end{aligned}
$$

There is a clean separation of electric fields (in yellow) and the magnetic field (in green and orange).

The Lagrange density for the Maxwell source equations is complete.

Derive Gauss's law.

Start applying the Euler-Lagrange equation by focusing on terms with a phi in them:

$$\begin{aligned}\mathcal{L}_{EM} &= scalar(JA + \frac{1}{2}(B^2 - E^2)) \\ &= \rho\phi - Jx\,Ax - Jy\,Ay - Jz\,Az \\ + \frac{1}{2}&((\frac{\partial Ay}{\partial z})^2 - 2\frac{\partial Ay}{\partial z}\frac{\partial Az}{\partial y} + (\frac{\partial Az}{\partial y})^2 - (\frac{\partial Ax}{\partial t})^2 - 2\frac{\partial Ax}{\partial t}\frac{\partial \phi}{\partial x} - (\frac{\partial \phi}{\partial x})^2 \\ &+ (\frac{\partial Az}{\partial x})^2 - 2\frac{\partial Az}{\partial x}\frac{\partial Ax}{\partial z} + (\frac{\partial Ax}{\partial z})^2 - (\frac{\partial Ay}{\partial t})^2 - 2\frac{\partial Ay}{\partial t}\frac{\partial \phi}{\partial y} - (\frac{\partial \phi}{\partial y})^2 \\ &+ (\frac{\partial Ax}{\partial y})^2 - 2\frac{\partial Ax}{\partial y}\frac{\partial Ay}{\partial x} + (\frac{\partial Ay}{\partial x})^2 - (\frac{\partial Az}{\partial t})^2 - 2\frac{\partial Az}{\partial t}\frac{\partial \phi}{\partial z} - (\frac{\partial \phi}{\partial z})^2)\end{aligned}$$

There is one current coupling term. There are six terms all coming from the electric field. This is reasonable because only the electric field has a phi, and the E and B fields are separated in the differences of squares EM Lagrangian.

We need to do calculus on two types of terms. One is a square, the other is a mixed derivative:

$$\frac{1}{2}\frac{\partial x^2}{\partial x} = x \quad \frac{1}{2}(2\frac{\partial xy}{\partial x}) = y \qquad \textbf{if: } x = \frac{\partial \phi}{\partial x}$$

$$\frac{1}{2}\frac{\partial}{\partial x}(\frac{(\frac{\partial \phi}{\partial x})^2}{\frac{\partial \phi}{\partial x}}) = \frac{\partial^2 \phi}{\partial x^2} \qquad \frac{1}{2}\frac{\partial}{\partial x}(\frac{\frac{\partial \phi}{\partial x}\frac{\partial Ax}{\partial t}}{\frac{\partial \phi}{\partial x}}) = \frac{\partial^2 Ax}{\partial x\,\partial t}$$

Even though these expressions look impressively complicated, one is only taking the derivatives of x^2 and xy, the first sorts of derivatives one learns in the study of calculus. It is x that is odd, being another derivative.

Apply the Euler-Lagrange equation to all the terms with a phi:

$$
\begin{aligned}
\mathcal{L}_{EM} &= scalar(JA + \frac{1}{2}\,(B^2 - E^2)) \\
&= \boxed{\rho\phi} - Jx\,Ax - Jy\,Ay - Jz\,Az \\
&\boxed{+\frac{1}{2}}((\frac{\partial Ay}{\partial z})^2 - 2\frac{\partial Ay}{\partial z}\frac{\partial Az}{\partial y} + (\frac{\partial Az}{\partial y})^2 - (\frac{\partial Ax}{\partial t})^2 - 2\frac{\partial Ax}{\partial t}\frac{\partial \phi}{\partial x} - (\frac{\partial \phi}{\partial x})^2 \\
&\quad + (\frac{\partial Az}{\partial x})^2 - 2\frac{\partial Az}{\partial x}\frac{\partial Ax}{\partial z} + (\frac{\partial Ax}{\partial z})^2 - (\frac{\partial Ay}{\partial t})^2 - 2\frac{\partial Ay}{\partial t}\frac{\partial \phi}{\partial y} - (\frac{\partial \phi}{\partial y})^2 \\
&\quad + (\frac{\partial Ax}{\partial y})^2 - 2\frac{\partial Ax}{\partial y}\frac{\partial Ay}{\partial x} + (\frac{\partial Ay}{\partial x})^2 - (\frac{\partial Az}{\partial t})^2 - 2\frac{\partial Az}{\partial t}\frac{\partial \phi}{\partial z} - (\frac{\partial \phi}{\partial z})^2)
\end{aligned}
$$

$$
\rho = \underline{-\frac{\partial^2 Ax}{\partial x\,\partial t} - \frac{\partial^2 \phi}{\partial x^2}} \underline{- \frac{\partial^2 Ay}{\partial y\,\partial t} - \frac{\partial^2 \phi}{\partial y^2}} \underline{- \frac{\partial^2 Az}{\partial z\,\partial t} - \frac{\partial^2 \phi}{\partial z^2}}
$$

Here is the pattern:

$$\mathcal{L}_{EM} = scalar(JA + \frac{1}{2}\,(B^2 - E^2))$$

$$\rho = \vec{\nabla} \cdot \vec{E}$$

QED

This is Gauss's law.

Derive Ampere's law

This time focus on terms on A_x in the Lagrange density:

$$\mathcal{L}_{EM} = scalar(JA + \frac{1}{2}(B^2 - E^2))$$

$$= \rho\phi - Jx\,Ax - Jy\,Ay - Jz\,Az$$

$$+\frac{1}{2}((\frac{\partial Ay}{\partial z})^2 - 2\frac{\partial Ay}{\partial z}\frac{\partial Az}{\partial y} + (\frac{\partial Az}{\partial y})^2 - (\frac{\partial Ax}{\partial t})^2 - 2\frac{\partial Ax}{\partial t}\frac{\partial \phi}{\partial x} - (\frac{\partial \phi}{\partial x})^2$$

$$+(\frac{\partial Az}{\partial x})^2 - 2\frac{\partial Az}{\partial x}\frac{\partial Ax}{\partial z} + (\frac{\partial Ax}{\partial z})^2 - (\frac{\partial Ay}{\partial t})^2 - 2\frac{\partial Ay}{\partial t}\frac{\partial \phi}{\partial y} - (\frac{\partial \phi}{\partial y})^2$$

$$+(\frac{\partial Ax}{\partial y})^2 - 2\frac{\partial Ax}{\partial y}\frac{\partial Ay}{\partial x} + (\frac{\partial Ay}{\partial x})^2 - (\frac{\partial Az}{\partial t})^2 - 2\frac{\partial Az}{\partial t}\frac{\partial \phi}{\partial z} - (\frac{\partial \phi}{\partial z})^2)$$

Write out the Euler-Lagrange derivatives:

$$-\frac{\partial^2 Ax}{\partial t^2} - \frac{\partial^2 \phi}{\partial t\,\partial x} = \frac{\partial Ex}{\partial t}$$

$$\mathcal{L}_{EM} = scalar(JA + \frac{1}{2}(B^2 - E^2))$$

$$= \rho\phi - Jx\,Ax - Jy\,Ay - Jz\,Az$$

$$+\frac{1}{2}((\frac{\partial Ay}{\partial z})^2 - 2\frac{\partial Ay}{\partial z}\frac{\partial Az}{\partial y} + (\frac{\partial Az}{\partial y})^2 - (\frac{\partial Ax}{\partial t})^2 - 2\frac{\partial Ax}{\partial t}\frac{\partial \phi}{\partial x} - (\frac{\partial \phi}{\partial x})^2$$

$$+(\frac{\partial Az}{\partial x})^2 - 2\frac{\partial Az}{\partial x}\frac{\partial Ax}{\partial z} + (\frac{\partial Ax}{\partial z})^2 - (\frac{\partial Ay}{\partial t})^2 - 2\frac{\partial Ay}{\partial t}\frac{\partial \phi}{\partial y} - (\frac{\partial \phi}{\partial y})^2$$

$$+(\frac{\partial Ax}{\partial y})^2 - 2\frac{\partial Ax}{\partial y}\frac{\partial Ay}{\partial x} + (\frac{\partial Ay}{\partial x})^2 - (\frac{\partial Az}{\partial t})^2 - 2\frac{\partial Az}{\partial t}\frac{\partial \phi}{\partial z} - (\frac{\partial \phi}{\partial z})^2)$$

$$+\frac{\partial^2 Ax}{\partial y^2} - \frac{\partial^2 Ax}{\partial y\,\partial x} - \frac{\partial^2 Az}{\partial x\,\partial z} + \frac{\partial^2 Ax}{\partial z^2} = -\vec{\nabla} \times (\vec{\nabla} \times \vec{A})|_x = -\vec{\nabla} \times \vec{B}|_x$$

Collect the terms generated by the Euler-Lagrange equations:

$$\mathcal{L}_{EM} = scalar(JA + \frac{1}{2}(B^2 - E^2))$$

$$= \rho\phi - Jx\,Ax - Jy\,Ay - Jz\,Az$$

$$+\frac{1}{2}((\frac{\partial Ay}{\partial z})^2 - 2\frac{\partial Ay}{\partial z}\frac{\partial Az}{\partial y} + (\frac{\partial Az}{\partial y})^2 - (\frac{\partial Ax}{\partial t})^2 - 2\frac{\partial Ax}{\partial t}\frac{\partial \phi}{\partial x} - (\frac{\partial \phi}{\partial x})^2$$

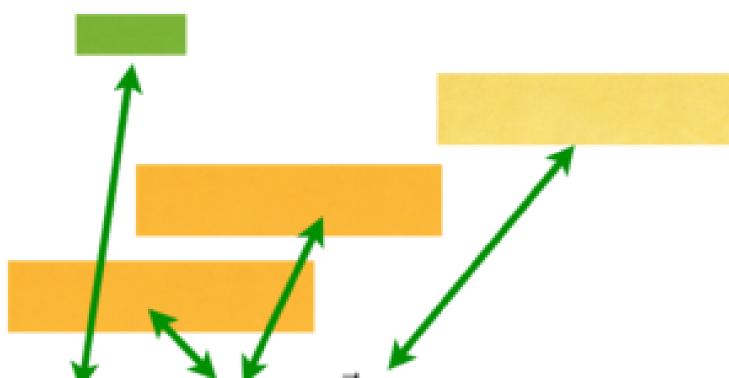
$$+(\frac{\partial Az}{\partial x})^2 - 2\frac{\partial Az}{\partial x}\frac{\partial Ax}{\partial z} + (\frac{\partial Ax}{\partial z})^2 - (\frac{\partial Ay}{\partial t})^2 - 2\frac{\partial Ay}{\partial t}\frac{\partial \phi}{\partial y} - (\frac{\partial \phi}{\partial y})^2$$

$$+(\frac{\partial Ax}{\partial y})^2 - 2\frac{\partial Ax}{\partial y}\frac{\partial Ay}{\partial x} + (\frac{\partial Ay}{\partial x})^2 - (\frac{\partial Az}{\partial t})^2 - 2\frac{\partial Az}{\partial t}\frac{\partial \phi}{\partial z} - (\frac{\partial \phi}{\partial z})^2)$$

$$-Jx = -\vec{\nabla} \times \vec{B}|_x + \frac{\partial Ex}{\partial t}$$

Look at the pattern:

$$\mathcal{L}_{EM} = scalar(JA + \frac{1}{2}(B^2 - E^2))$$



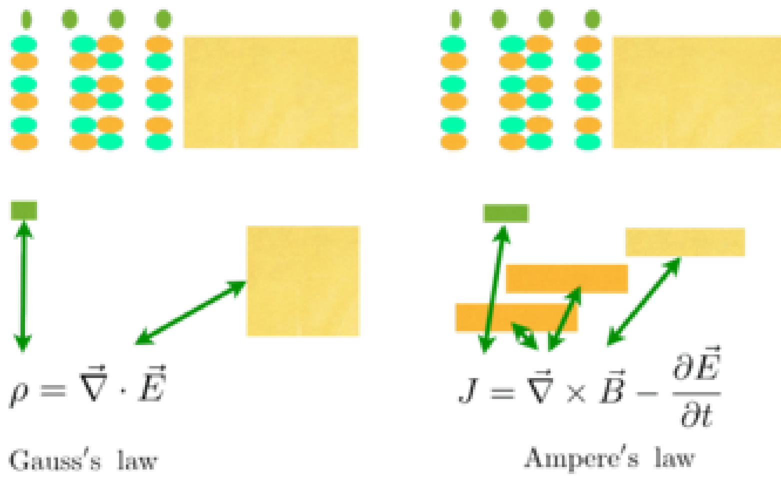
$$J = \vec{\nabla} \times \vec{B} - \frac{\partial \vec{E}}{\partial t}$$

QED

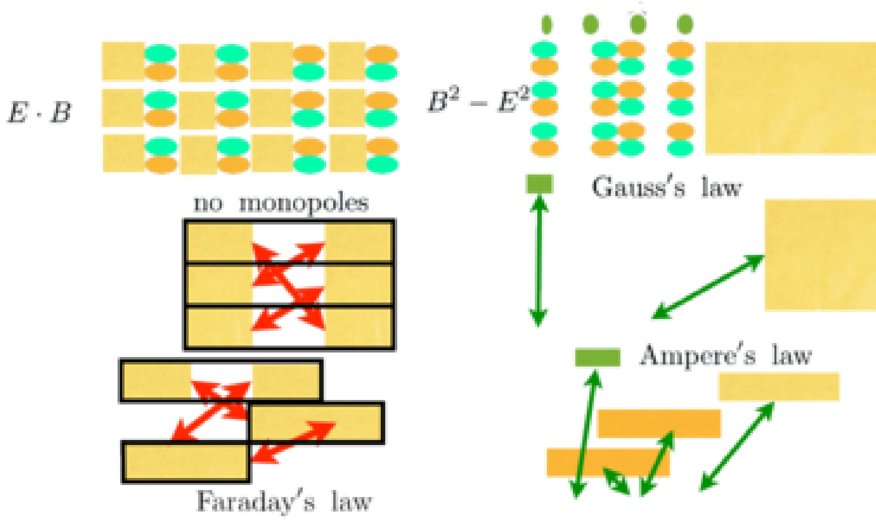
This is Ampere's law.

The big pictures

Gauss's law is only about the electric field. Ampere's law is about both.



The two Lagrangians and four Maxwell field equations together on one stage:



Maxwell by hand

To really lean anything, one needs to practice over and over, by hand. Here is a complete derivation of the Maxwell source equations using quaternions. Instead of picking the familiar Cartesian coordinates of t, x, y, and z, generalized coordiantes are used with the numbers 0-3.

Copy this over until you can do so without looking at a reference image. Leave your practice sheets around where you work or study. People will presume you are a genius.

The Maxwell Source Equations

using quaternions operators

Eq#

$$\frac{1}{4}\left(\nabla A - (\nabla A)^*\right)\left(A\nabla - (A\nabla)^*\right) = \left(0, \nabla_0 A + \nabla_u \phi + \nabla \times A\right)\left(0, \nabla_0 A + \nabla_u \phi - \nabla \times A\right) \qquad 1.1$$

$$= \left(0, -E + B\right)\left(0, -E - B\right) = \left(B^2 - E^2, 2E \times B\right) \qquad 1.2$$

$$\mathcal{L}_{EB} \equiv \frac{1}{4}\left(\left(B^2 - E^2, 2E \times B\right) + \left(B^2 - E^2, 2E \times B\right)^*\right) - \frac{1}{2}\left(\left(JA + (JA)^*\right)\right) \qquad 1.3$$

$$= \frac{1}{2}\Big(-(\nabla_1 \phi)^2 - (\nabla_2 \phi)^2 - (\nabla_3 \phi)^2 - (\nabla_0 A_1)^2 - (\nabla_0 A_2)^2 - (\nabla_0 A_3)^2$$

$$+ (\nabla_3 A_2)^2 + (\nabla_2 A_3)^2 + (\nabla_1 A_3)^2 + (\nabla_3 A_1)^2 + (\nabla_2 A_1)^2 + (\nabla_1 A_2)^2\Big) - \rho\phi + J_1 A_1 + J_2 A^2 + J_3 A_3$$

$$- (\nabla_3 A_2)(\nabla_2 A_3) - (\nabla_1 A_3)(\nabla_3 A_1) - (\nabla_1 A_2)(\nabla_2 A_1) - (\nabla_1 \phi)(\nabla_0 A_1) - (\nabla_2 \phi)(\nabla_0 A_2) - (\nabla_3 \phi)(\nabla_0 A_3)$$

Calculate the field equations

$$\nabla_u\left(\frac{\partial \mathcal{L}_{EB}}{\partial(\nabla_u \phi)}\right) = -\nabla_1^2 \phi - \nabla_2^2 \phi - \nabla_3^2 \phi - \nabla_0 \nabla_1 A_1 - \nabla_0 \nabla_2 A_2 - \nabla_0 \nabla_3 A_3 - \rho = \nabla \cdot E - \rho = 0 \qquad 1.4$$

$$\nabla_u\left(\frac{\partial \mathcal{L}_{EB}}{\partial(\nabla_u A_1)}\right) = -\nabla_0^2 A_1 + \nabla_3^2 A_1 + \nabla_2^2 A_1 - \nabla_1 \nabla_3 A_3 - \nabla_1 \nabla_2 A_2 - \nabla_0 \nabla_1 \phi - J_1 = \nabla_0 E_1 - (\nabla \times B)_1 + J_1 = 0$$

$$\nabla_u\left(\frac{\partial \mathcal{L}_{EB}}{\partial(\nabla_u A_2)}\right) = -\nabla_0^2 A_2 + \nabla_3^2 A_2 + \nabla_1^2 A_2 - \nabla_2 \nabla_3 A_3 - \nabla_1 \nabla_2 A_1 - \nabla_0 \nabla_2 \phi - J_2 = \nabla_0 E_2 - (\nabla \times B)_2 + J_2 = 0$$

$$\nabla_u\left(\frac{\partial \mathcal{L}_{EB}}{\partial(\nabla_u A_3)}\right) = -\nabla_0^2 A_3 + \nabla_2^2 A_3 + \nabla_1^2 A_3 - \nabla_2 \nabla_3 A_2 - \nabla_1 \nabla_3 A_1 - \nabla_0 \nabla_3 \phi - J_3 = \nabla_0 E_3 - (\nabla \times B)_3 + J_3 = 0$$

$$\boxed{\begin{gathered} \nabla \cdot E = \rho \\ \nabla \times B - \nabla_0 E = J \end{gathered}} \qquad \begin{gathered} 1.5 \\ 1.6 \end{gathered}$$

The Maxwell Source Equations

Figure 249: The Maxwell source equations derivation by hand

Bell's Future Quantum Mechanics - a Novel Interpretation

This essay provides an introduction to a new interpretation for quantum mechanics. Here it is in two sentences:

```
Bell's inequality backed by experimental evidence shows that
quantum mechanics must be non-local, thus the wave function
is space-like separated from the observer at the origin,
here-now. The product of the wave function and its conjugate
provide the odds for an interaction with the observer
happening here (0, 0, 0) in the future.
```

This novel interpretation is called Bell's future quantum mechanics.

New Views on Old Space-time

Start with a Minkowski space-time graph. All information that is local to the observer at the origin here-now is in the past lightcone. Quantum mechanics is non-local, ergo delete all local information - delete the past lightcone! The wave function has to reside in the space-like regions of space-time. The conjugate of the wave function goes on the other side. The product of the wave function and its conjugate is necessarily in the future at the spatial origin (here, or (0, 0, 0)). Quantum mechanics has always been about the future. What are the odds that an event will happen to an observer in the future? Bell's inequality is about the non-local nature of quantum mechanics. Deleting the past lightcone enforces non-locality. Space-like information can be used only to predict the future. This is the Bell's Future interpretation of quantum mechanics.

Historical Background

In 1935, Einstein, Polodsky, and Rosen (EPR) proposed that variables hidden in the past light cone could explain how quantum mechanics worked. The claimed inherent uncertainty of quantum mechanics could be traded for something more real, variables that are hidden. This was not an easy to dismiss proposal given Einstein's stature. It took until the 1960s when John Bell found an inequality that could test if variables are hidden in the past lightcone or the entangled states of quantum mechanics where somehow real because quantum information was non-local. If one asks the same question the same way, both models make identical predictions. If questions are asked at a different angle, the hidden variable hypothesis is unchanged. Quantum mechanics says correlations between measurements become stronger. A huge experimental effort from the 1980s until today has always confirmed the same result: quantum mechanics is non-local and hidden variable models are wrong.

My Beliefs About It All in 3D Space + Time

Einstein put Lorentz transformations to great use to solve difficult theoretical problems in physics. It was his math professor who recognized that Einstein was doing rotations not just in space, but in space-time. Here is a picture of all of space:

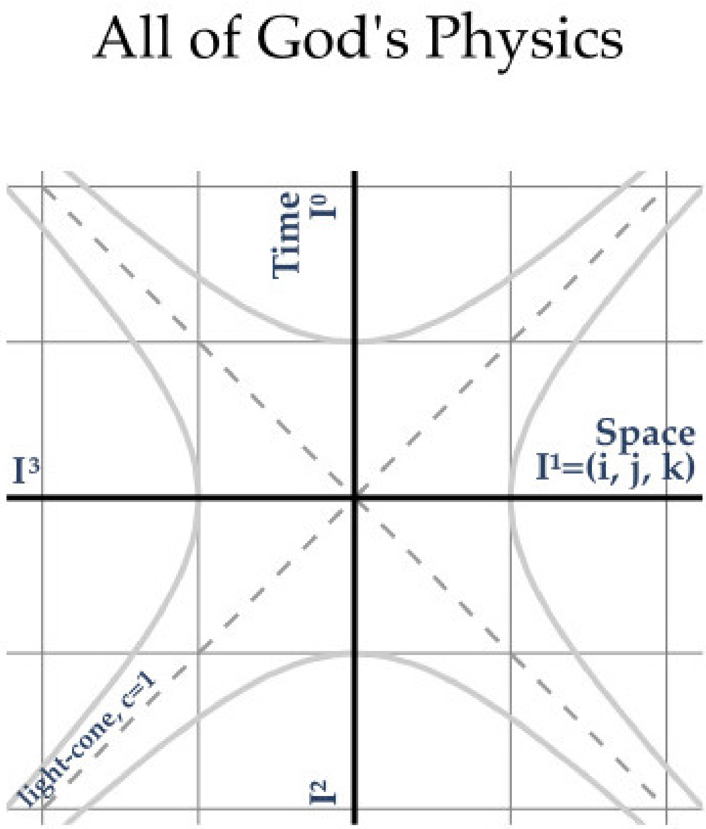


Figure 250: All of God's Physics

I hope the gentle reader is not offput by referencing a diety. The word choice was made because it is my belief that all of physics, both that that is currently known which is the vast majority, and that which remain unknown, must live only in 3D space + time, or space-time. I am more concerned with why parity is not conserved for beta decay than any biblical issue.

Noice how three spatial dimensions are written explicity in the space-time graph. Starting from studies done with five dimensions in the 1920s, research begun in the 1970s created a significant investigation into higher spatial dimensions. I believe all such work will have no lasting value. More recently, people have been championing the multi-verse. A multi-verse has multiple space-times. Again, I believe all such work witll have no lasting value.

I am radical conservative circa 1960s in regards to space-time.

Technical Tangent: quaternions

The graph reveals that I am a 1908 Minkowski radical. He wrote:

```
Henceforth space by itself, and time by itself,
are doomed to fade away into mere shadows,
and only a kind of union of the two
will preserve an independent reality.
```

If an event in space-time is just a bag of numbers (a vector with scant structure), it is OK to ask if the bag can be expanded as higher dimensional research does. If an event is just one number, the bag cannot be expanded. I study a kind of number with that property, quaternions. A breadcrumb appears on how the axes are labeled with power series of the 3-vector I (I^0 for the positive reals, I^1 for the imaginaries, I^2 for the negative reals, I^3 for the negative imaginaries). I like to algegraically enforce Minkowski's vision.

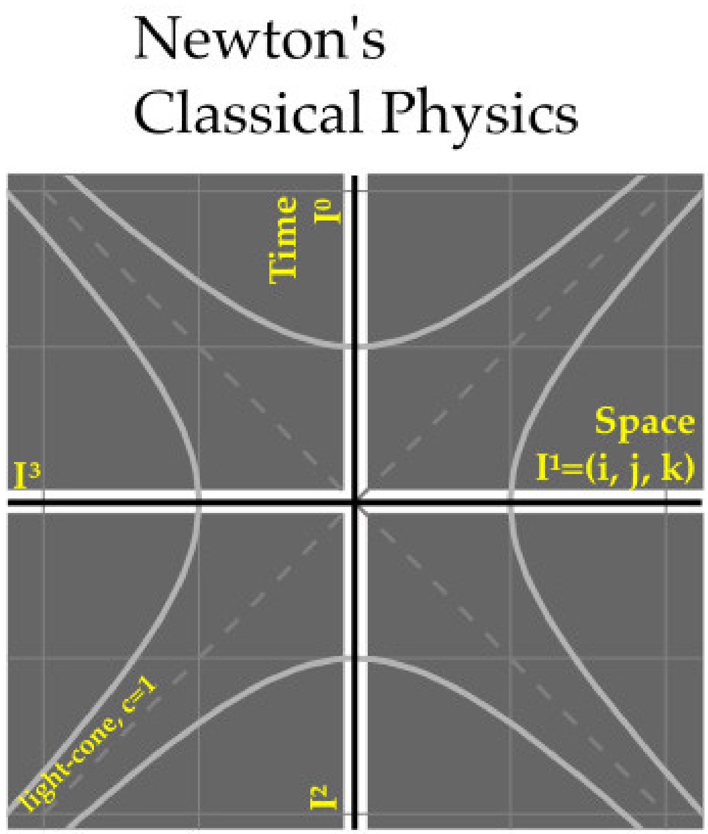
Quaternions are not central to Bell's future quantum mechanics. Still, while the car is in for repairs, one might as well consider a complete overhaul.

END of technical tangent.

In summary, space-time is everything we know, everything we do not know, all on the same stage.

Newton Through Subtraction

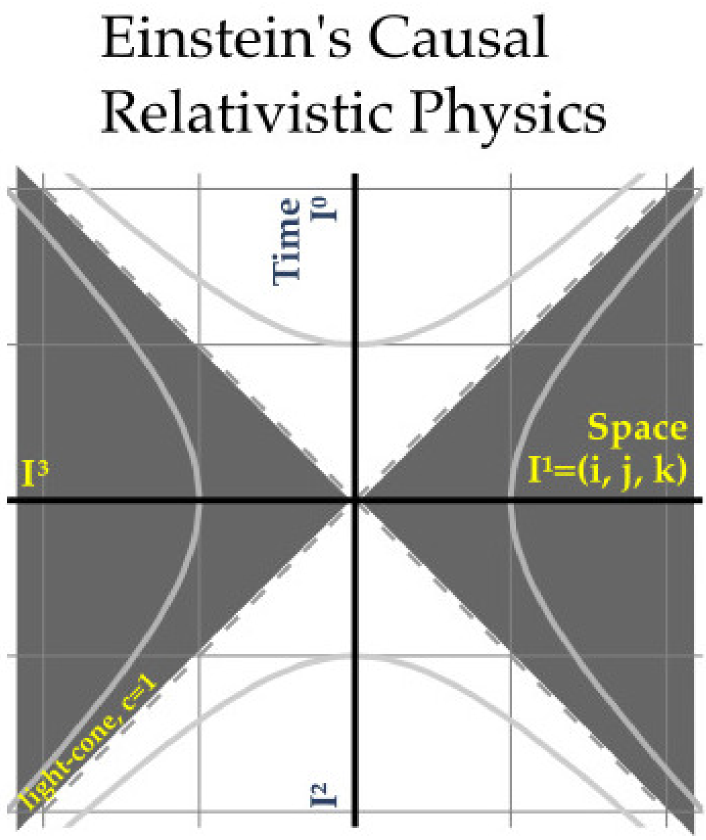
It is odd that most of space-time gets subtracted for Newtonian physics.



Space is abosulte. Time is absolute. There is no way in Newtonian physics to rotate space into time. This is the physics we experience everyday.

Einstein's Causal Relativistic Physics

The only kinds of events that can change an observer at the origin are events from the past lightcone.



Einstein was adroit at working with the space-like regions, realizing for example that events that are simultaneous in one reference frame will not be so in another. If one decides to restrict to the study causality, that is the reason to black out the space-time regions. Why did something happen? The answer is in the past lightcone, not the space-like region.

Bell's Future Quantum Mechanics

This is from a Wikipedia discussion of EPR paper:

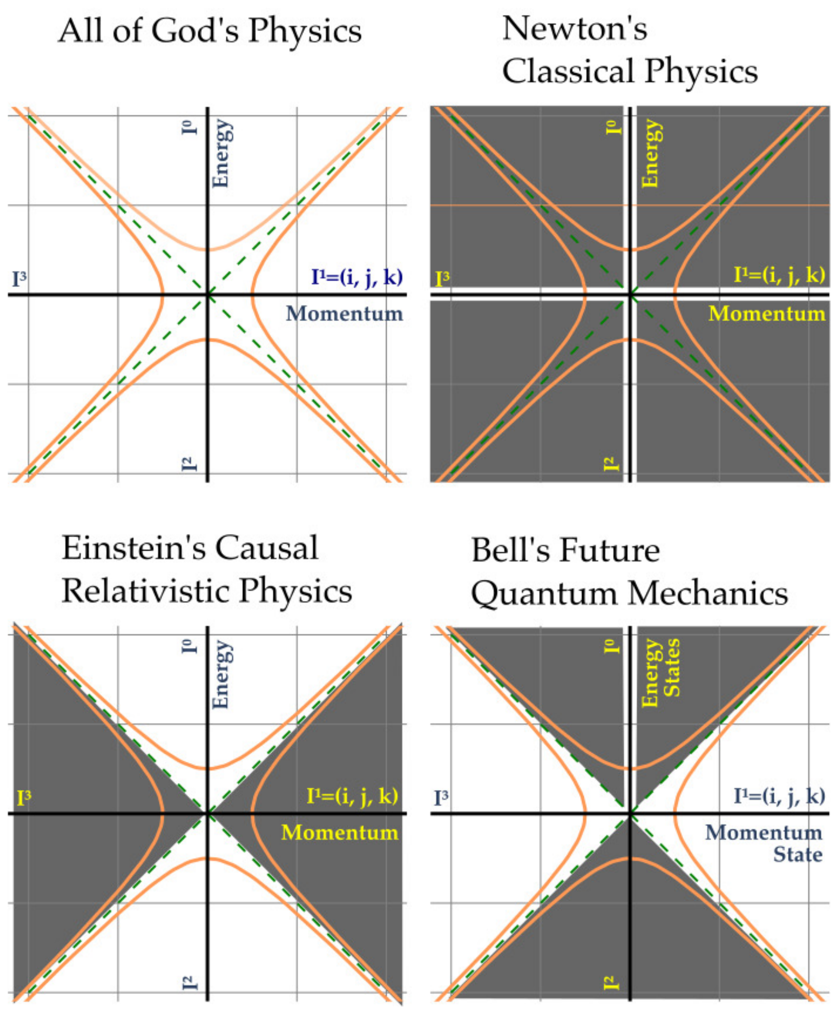
```
The 1935 EPR paper condensed the philosophical discussion into a physical argument. The auth
```

It was the clause "only be influenced by events which are located in the backward light cone" that caught my attention. If there are no hidden variables as shown by experiments, remove any possibility.

Notice how Einstein causality and Bell's future combine to cover all of space-time. Both relativity and quantum mechanics were born and matured in the same time window, and at for Einstein, in the same mind.

Repeat the Exercise for Tangent Spaces

Space-time records where-when things are: location, location, location. Space-time formally has no information about change. Change lives in tangent spaces. Which tangent space is used determines what change is under study. The most common one is energy-momentum. Tangent spaces can also be broken up into the same four types: all, classical, relativistic, and quantum:



Imagine one were to study the classical motion of a rock. There would be energy and momentum at the point in space-time where the rock happened to be. Everywhere-when else in space-time, the energy-momentum space would be zero. These zeroes are usually ignored, so the topic of tangent spaces only appears at the graduate-level. It makes the switch to continuous fields seem mystical. Instead, the difference between the two is more like discrete and continuous for energy-momentum.

Physicists have studied all combinations of the base space space-time with the tangent space energy-momentum. There is both classical and relativistic quantum mechanics.

Uncool Sidebar: The Base Space is the Base

The base space, space-time, cannot be changed by anything. I appreciate that this clear statement will be violently rejected by those who have made a serious study of Einstein's general relativity. Tens of thousands of times it has been repeated: gravity bends space-time. I am not in denial of the words. I do feel compelled to at least question the link to the math. Gravity alters a tangent space of space-time as seen by the dt and dR in metric solutions. Space-time has Lorentz symmetry and its origin. Gravity and energy-momentum have Poincare symmetry. Summing up all the changes in all the tangent spaces results in a curved path in space-time. All the change happens in the tangent space. We should be saying the tangent space is curved then summed, not that the base space is curved.

End sidebar

Momentum versus a Momentum State

Why use momentum in relativity, but momentum states in quantum mechanics? Momentum from the past lightcone can change the motion of the observer at the origin. The entire chain of events leading to that chang in the observer can be known. A space-like momentum state is different. Momentum states may never change the observer at the origin, here-now. The precise odds of a momentum state changing the momentum of an observer can be calculated. In the future, if an interaction does occur, it will change the path of the observer in the usual way. The entire chain of events leading to this momentum change cannot be known because they are space-like separated. The observer is necessarily blind-sided by a momentum state.

The same story applies to energy versus an energy state. Observers can absorb energy from the past lightcone and heat up. Observers cannot absorb energy from an energy state as it is too far away. The can in the futur absorb the energy to the same effect. We can calculate the odds.

The Wave Function

The wave function is a set of space-like energy-momentum states. Each state may not have a time-like relationship to the observer at the origin, here-now. Each state of the wave function can have a time-like relationship with other states in the wave function. Light-like relationships are not addressed for the moment as that is a refinement one will have to include with care later.

For a complex-valued wave function, the conjugate is simple to construct. The product of a wave function and its conjugate evaluates to a positive real number. If properly normalized, the postive number is the odds of an interation happening. Nothing unusual is happening under the Bell's future interpretation, all calculations will be the same.

Dull Quaterion Series Quantum Mechanics

Quaternion quantum mechanics has been studied and presented in a book-length form by Stephen Adler. The topic has been commented on in a December 2018 blog by Scott Aaronson where he came to the conclusion that quaternion quantum mechanics was a "complete dumpster fire" because it would allow superluminal transfer of information. I agree, any algebraic system that allows superluminal transformation is boring and deserves no futher study. I was suprised that this flaw was known to Adler as he admitted to Aaronson.

In a rapid exchange I had with Aaronson, I came up with the idea of "point-one-way" quaternions. Pick an arbitrary direction and stick with that for all calculations. Aaronson agreed it would work. He just thought it was so dull it did not even deserve a new name.

As I considered it more, a better name would have been "point-with-precision" quaternion series quantum mechanics. In the lab, physics experiments are re-known for their precision of the experimental apperatus. It is common to use tables that isolate the vibrations of the surroundings from the experiment. The precision of location known at the bench is apply to the math used. Quaternions that point in the same direction commute. A quaternion series is not a division algebra like the quaternions. Instead it is a semi-group with inverses. A semi-group has more than one inverse.

For quaternion-valued wave functions, the conjugate has a physical meaning: it is a mirror reflection in space. Why do so? The product of the wave function and its mirror reflection is a here-future value, (0, 0, 0) for the 3-vector and a positive real number. If properly normalized, the real value is the odds of seeing an interaction. It is the simple, physical interpretation of an otherwise abstraction notion of a complex-valued wave function that I see as a benefit worthy of exploration.

The Bell's future quantum mechanics interpretation in no way depends on quaternion series quantum mechanics being a viable algebric approach to doing calculations.

Interpretations of Quantum Mechanics

There are at least 20 intepretations of quantum mechanics. Nearly all of them make the same predictions as does this one. I have seen Sean Carroll take a poll of graduate students to find their favorite. This is not the was physics works. Physics is a contact sport with only one eventual winner.

Physics by subtraction defines areas of study. Newton's classical physics uses only the axes. Causality in special relativity uses only the past lightcone. By contrast, quantum mechancis uses nothing from the past lightcone. Quantum mechanics uses space-like states to calculate the odds of interactions in the future.

Bell's future quantum mechanics looks bright. I hope this idea goes viral in a good way.

Other Presentations

8 page PDF

You Tube, 3 minutes

You Tube, 23 minutes

A Complete Inner Product Space with Dirac's Bracket Notation

A mathematical connection between the bracket notation of quantum mechanics and quaternions is detailed. It will be argued that quaternions have the properties of a complete inner-product space (a Banach space for the field of quaternions). A central issue is the definition of the square of the norm. In quantum mechanics:

$$|| \varphi ||^2 = < \varphi | \varphi >$$

Figure 251: the norm of phi squared = <phi | phi>

In this notebook, the following assertion will be examined (star is the conjugate, so the vector flips signs):

$$|| (t, \vec{X}) ||^2 = (t, \vec{X})^* (t, \vec{X}) (t, \vec{X})^* (t, \vec{X})$$

Figure 252: the norm of (t, X) squared = (t, X) conjugated times (t, X) times (t, X) conjugated times (t, X)

The inner-product of two quaternions is defined here as the transpose (or conjugate) of the first quaternion multiplied by the second. The inner product of a function with itself is the norm.

The Positive Definite Norm of a Quaternion

The square of the norm of a quaternion can only be zero if every element is zero, otherwise it must have a positive value.

$$(t, \vec{X})^* (t, \vec{X}) = \left(t^2 + \vec{X}.\vec{X}, \vec{0}\right)$$

Figure 253: (t, X) conjugated times (t, X) = (t squared +X dot X, 0)

This is the standard Euclidean norm for a real 4-dimensional vector space.

The Euclidean inner-product of two quaternions can take on any value, as is the case in quantum mechanics for <phi|theta>. The adjective "Euclidean" is used to distinguish this product from the Grassman inner- product which plays a central role in special relativity (see alternative algebra for boosts).

Completeness

With the topology of a Euclidean norm for a real 4-dimensional vector space, quaternions are complete.

Quaternions are complete in a manner required to form a Banach space if there exists a neighborhood of any quaternion x such that there is a set of quaternions y

$$||x - y||^2 < \epsilon^4$$

Figure 254: the norm of x - y squared is less than epsilon to the fourth

for some fixed value of epsilon.

Construct such a neighborhood.

$$\left((t, \vec{X}) - \frac{\epsilon}{4}(t, \vec{X})\right)^* \left((t, \vec{X}) - \frac{\epsilon}{4}(t, \vec{X})\right) \left((t, \vec{X}) - \frac{\epsilon}{4}(t, \vec{X})\right)^* \left((t, \vec{X}) - \frac{\epsilon}{4}(t, \vec{X})\right) =$$

Figure 255: ((t, X) - epsilon (t, X) over 4)conjugated times ((t, X) - epsilon (t, X) over 4) times ((t, X) - epsilon (t, X) over 4)conjugated times ((t, X) - epsilon (t, X) over 4) =

$$= \left(\frac{\epsilon^4}{16}, 0, 0, 0\right) < (\epsilon^4, 0, 0, 0)$$

Figure 256: = (epsilon to the fourth over 4, 0, 0, 0) is less than (epsilon to the fourth, 0, 0, 0)

An infinite number of quaternions exist in the neighborhood.

Any polynomial equation with quaternion coefficients has a quaternion solution in x (a proof done by Eilenberg and Niven in 1944, cited in Birkhoff and Mac Lane's “A Survey of Modern Algebra.”)

Identities and Inequalities

The following identities and inequalities emanate from the properties of a Euclidean norm. They are worked out for quaternions here in detail to solidify the connection between the machinery of quantum mechanics and quaternions.

The conjugate of the square of the norm equals the square of the norm of the two terms reversed.

For quaternions,

$$< \phi \mid \varphi >^* = < \varphi \mid \phi >$$

Figure 257: <phi | phi> conjugated = <phi | phi>

$$\left((t, \vec{X})^* \left(t', \vec{X'}\right) \right)^* = \left(t\,t' + \vec{X}.\vec{X'}, -t\,\vec{X'} + \vec{X}\,t' + \vec{X} \times \vec{X'} \right)$$

Figure 258: ((t, X) conjugated times (t prime, X prime))conjugated = (t t prime+ X dot X prime, -t X prime + t prime X + X cross X prime)

These are identical, because the terms involving the cross produce will flip signs when their order changes.

For products of squares of norms in quantum mechanics,

This is also the case for quaternions.

The triangle inequality in quantum mechanics:

For quaternions,

If the signs of each pair of component are the same, the two sides will be equal. If the signs are different (a t and a -t for example), then the cross terms will cancel on the left hand side of the inequality, making it smaller than the right hand side where terms never cancel because there are only squared terms.

The Schwarz inequality in quantum mechanics is analogous to dot products and cosines in Euclidean space.

Let a third wave function, chi, be the sum of these two with an arbitrary parameter lambda.

The norm of chi will necessarily be greater than zero.

Choose the value for lambda that helps combine all the terms containing lambda.

Multiply through by the denominator, separate the two resulting terms and do some minor rearranging.

This is now the Schwarz inequality.

Another inequality:

Examine the square of the norm of the difference between two quaternions which is necessarily equal to or greater than zero.

$$\left(t', \vec{X'}\right)^* (t, \vec{X}) = \left(t'\,t + \vec{X'}.\vec{X}, t'\,\vec{X} - \vec{X'}\,t - \vec{X'} \times \vec{X} \right)$$

Figure 259: (t prime, X prime) conjugated times (t, X) = (t prime t + X prime dot X, t prime X - X prime t - X prime Cross X)

$$\langle \varphi\phi \mid \varphi\phi \rangle = \langle \varphi \mid \varphi \rangle \langle \phi \mid \phi \rangle$$

Figure 260: <psi phi | psi phi> = <psi | phi> times <psi | phi>

$$\langle (t, \vec{X}) \left(t', \overrightarrow{X'}\right) \mid (t, \vec{X}) \left(t', \overrightarrow{X'}\right) \rangle =$$

Figure 261: <(t, X)(t prime, X prime) | (t, X)(t prime, X prime)> =

$$= \left((t, \vec{X}) \left(t', \overrightarrow{X'}\right) \right)^* (t, \vec{X}) \left(t', \overrightarrow{X'}\right)$$

Figure 262: = ((t, X) times (t prime, X prime)) conjugated times (t, X) times (t prime, X prime) =

$$= \left(t', \overrightarrow{X'}\right)^* (t, \vec{X})^* (t, \vec{X}) \left(t', \overrightarrow{X'}\right)$$

Figure 263: = (t prime, X prime) conjugated times (t, X) conjugated (t, X) times (t prime, X prime) =

$$= \left(t', \overrightarrow{X'}\right)^* (t^2 + x^2 + y^2 + z^2, 0, 0, 0) \left(t', \overrightarrow{X'}\right)$$

Figure 264: = (t prime, X prime) conjugated times (t squared +x squared +y squared +z squared, 0, 0, 0) times (t prime, X prime) =

$$= (t^2 + x^2 + y^2 + z^2, 0, 0, 0) \left(t', \overrightarrow{X'}\right)^* \left(t', \overrightarrow{X'}\right)$$

Figure 265: = (t squared +x squared +y squared +z squared, 0, 0, 0) times (t prime, X prime) conjugated times (t prime, X prime) =

$$= (t, \vec{X})^* (t, \vec{X}) \left(t', \overrightarrow{X'}\right)^* \left(t', \overrightarrow{X'}\right)$$

Figure 266: = (t, X) conjugated times (t, X) times (t prime, X prime) conjugated times (t prime, X prime) =

$$= \langle (t, \vec{X}) \mid (t, \vec{X}) \rangle \langle \left(t', \overrightarrow{X'}\right) \mid \left(t', \overrightarrow{X'}\right) \rangle$$

Figure 267: =<(t, X) | (t, X)> times <(t prime, X prime) | (t prime, X prime)>

$$< \varphi + \phi \mid \phi + \varphi >^2 \leq \left(< \varphi \mid \varphi > + < \phi \mid \phi > \right)^2$$

Figure 268: <psi + phi | psi + phi> squared is less than (<psi | psi>+ <phi | phi>) squared

$$< (t, \vec{X}) + \left(t', \overrightarrow{X'}\right) \mid (t, \vec{X}) + \left(t', \overrightarrow{X'}\right) >^2 =$$

Figure 269: <(t, X)+(t prime, X prime) | (t, X)+(t prime, X prime)> squared =

$$= \left(\left(t + t', \vec{X} + \overrightarrow{X'}\right)^* \left(t + t', \vec{X} + \overrightarrow{X'}\right) \right)^2$$

Figure 270: = ((t+t prime, X+X prime) conjugated times(t+t prime, X+X prime)) squared

$$= \left(t^2 + t'^2 + \vec{X}^2 + \overrightarrow{X'}^2 + 2\,t\,t' + 2\,\vec{X}.\overrightarrow{X'},\ 0\right)^2$$

Figure 271: = (t squared + t prime squared + X squared + X prime squared + 2 t t prime + 2X dot X, 0) squared

$$\leq$$

Figure 272: is less than

$$\left(t^2 + \vec{X}^2 + t'^2 + \overrightarrow{X'}^2 + 2\sqrt{(t, \vec{X})^* (t, \vec{X})\ (t', \overrightarrow{X'})^* (t', \overrightarrow{X'})},\ 0\right)\hat{}\,2 =$$

Figure 273: = (t squared + t prime squared + X squared + X prime squared + 2 times the square root of the norm of (t, X) times (t prime, X prime), 0) squared =

$$\left(< (t, \vec{X}) \mid (t, \vec{X}) > + < \left(t', \overrightarrow{X'}\right) \mid \left(t', \overrightarrow{X'}\right) > \right)^2$$

Figure 274: (<(t, X) | (t, X)> + <(t prime, X prime) | (t prime, X prime)>) squared

$$\left| < \varphi \mid \phi > \right|^2 \leq < \varphi \mid \varphi > < \phi \mid \phi >$$

Figure 275: the absolute value of <psi | phi> squared is less than or equal to <psi | psi> times <phi | phi>

$$\chi \equiv \varphi + \lambda \phi$$

Figure 276: chi is defined to be psi + lambda phi

$$(\varphi + \lambda \phi)^* (\varphi + \lambda \phi) = \varphi^* \varphi + \lambda \varphi^* \phi + \lambda^* \phi^* \varphi + \lambda^* \lambda \phi^* \phi \geq 0$$

Figure 277: psi + lambda phi conjugated times psi + lambda phi = psi squared + lambda psi conjugated time phi + lambda conjugated times phi conjugated times psi + lambda conjugated times lambda times phi conjugated times phi is greater than or equal to zero

$$\lambda \to - \frac{\phi^* \varphi}{\phi^* \phi}$$

Figure 278: lambda goes to - phi conjugated time psi over phi conjugated times phi

$$\varphi^* \varphi - \frac{\phi^* \varphi \varphi^* \phi}{\phi^* \phi} \geq 0$$

Figure 279: psi conjugated time psi - phi conjugated times psi times psi conjugated times phi over phi conjugated times phi

$$(\varphi^* \phi)^* \varphi^* \phi \leq \varphi^* \varphi \ \phi^* \phi$$

Figure 280: (psi conjugated phi) conjugated times psi conjugated phi is less than or equal to psi conjugated psi times phi conjugated phi

$$2 \operatorname{Re} < \varphi \mid \phi > \leq < \varphi \mid \varphi > + < \phi \mid \phi >$$

Figure 281: 2 Re <phi | phi> <= <phi | phi> + <phi | phi>

$$0 \leq < (t, \vec{X}) - \left(t', \overrightarrow{X'}\right) \Big| (t, \vec{X}) - \left(t', \overrightarrow{X'}\right) >$$

Figure 282: 0 <= <(t, X) - (t prime, X prime) | (t, X)-(t prime, X prime)>

$$= \left((t - t')^2 + \left(\vec{X} - \overrightarrow{X'}\right) . \left(\vec{X} - \overrightarrow{X'}\right), \vec{0} \right)$$

Figure 283: = ((t - t prime) squared + (X-X prime).(X-X prime), 0)

The cross terms can be put on the other side of inequality, changing the sign, and leaving the sum of two norms behind.

$$\left(2\left(t\ t' + \vec{X}.\overrightarrow{X'}\right), \vec{0}\right) \leq \left(t^2 + \vec{X}^2 + t'^2 + \overrightarrow{X'}^2, \vec{0}\right)$$

Figure 284: (2 (t t prime + X dot X prime), 0) <= (t squared + x squared + t prime squared + X prime squared , 0)

$$2\,\mathrm{Re} < (t, \vec{X}) \mid \left(t', \overrightarrow{X'}\right) > \leq < (t, \vec{X}) \mid (t, \vec{X}) > + < \left(t', \overrightarrow{X'}\right) \mid \left(t', \overrightarrow{X'}\right) >$$

Figure 285: 2 Re <(t, X) | (t prime, X prime)> <= <(t, X) | (t, X)> + <(t prime, X prime) | (t prime, X prime)>

The inequality holds.

The parallelogram law:

$$< \varphi + \phi \mid \phi + \varphi > + < \varphi - \phi \mid \phi - \varphi > = 2 < \varphi \mid \varphi > + 2 < \phi \mid \phi >$$

Figure 286: <phi +phi | phi +phi> + <phi -phi | phi -phi> = 2 <phi | phi> + 2 <phi | phi>

Test the quaternion norm

This is twice the square of the norms of the two separate components.

Implications

In the case for special relativity, it was noticed that by simply squaring a quaternion, the resulting first term was the Lorentz invariant interval. From that solitary observation, the power of a mathematical field was harnessed to solve a wide range of problems in special relativity.

In a similar fashion, it is hoped that because the product of a transpose of a quaternion with a quaternion has the properties of a complete inner product space, the power of the mathematical field of quaternions can be used to solve a wide range of problems in quantum mechanics. This is an important area for further research.

Note: this goal is different from the one Stephen Adler sets out in “Quaternionic Quantum Mechanics and Quantum Fields.” He tries to substitute quaternions in the place of complex numbers in the standard Hilbert space formulation of quantum mechanics. The analytical properties of quaternions do not play a critical role. It is the properties of the Hilbert space over the field of quaternions that is harnessed to solve problems. It is my opinion that since the product of a transpose of a quaternion with a quaternion already has the properties of a

$$< (t, \vec{X}) + \left(t', \vec{X'}\right) \mid (t, \vec{X}) + \left(t', \vec{X'}\right) > + < (t, \vec{X}) - \left(t', \vec{X'}\right) \mid (t, \vec{X}) - \left(t', \vec{X'}\right) > =$$

Figure 287: <(t, X) + (t prime, X prime) | (t, X)+ (t prime, X prime)> + <(t, X) - (t prime, X prime) | (t, X) - (t prime, X prime)> =

$$= \left((t + t')^2 + \left(\vec{X} + \vec{X'}\right) . \left(\vec{X} + \vec{X'}\right), \vec{0}\right) + \left((t - t')^2 + \left(\vec{X} - \vec{X'}\right) . \left(\vec{X} - \vec{X'}\right), \vec{0}\right) =$$

Figure 288: = ((t + t prime) squared + (X+X prime).(X+X prime), 0) + ((t - t prime) squared + (X-X prime).(X-X prime), 0) =

norm in a Hilbert space, there is no need to imbed quaternions again within another Hilbert space. I like a close shave with Occam's razor.

$$= 2 \left(t^2 + \vec{X}^2 + t'^2 + \vec{X'}^2, \vec{0}\right) =$$

Figure 289: = 2 (t squared + X squared + t prime squared + X prime squared , 0) =

$$= 2 < (t, \vec{X}) \mid (t, \vec{X}) > + 2 < \left(t', \vec{X'}\right) \mid \left(t', \vec{X'}\right) >$$

Figure 290: = 2 <(t, X) | (t, X)> + 2 <(t prime, X prime) | (t prime, X prime)>

Multiplying Quaternions in Polar Coordinate Form

Any quaternion can be written in polar coordinate form, which involves a scalar magnitude and angle, and a 3-vector I (which in some cases can be the more familiar i).

$$q = || q || \exp[\theta \vec{I}] = \sqrt{q^* q}(\mathrm{Cos}[\theta] + \vec{I}\,\mathrm{Sin}[\theta])$$

Figure 291: q = the norm of q times e to the (theta i) = q conjugate times q times (cosine (theta) + I sine (theta))

This representation can be useful due to the properties of the exponential function, cosines and sines.

The absolute value of a quaternion is the square root of the norm, which is the transpose of a quaternion multiplied by itself.

$$|| q || = \sqrt{q^* q}$$

Figure 292: The absolute value of q = the square root of q conjugated times q

The angle is the arccosine of the ratio of the first component of a quaternion over the norm.

The vector component is generated by normalizing the pure quaternion (the final three terms) to the norm of the pure quaternion.

I^2 equals -1 just like i^2. Let (0, V) = (q - q*)/2.

It should be possible to do Fourier analysis with quaternions, and to form a Dirac delta function (or distribution). That is a project for the future. Those tools are necessary for solving problems in quantum mechanics.

New method for multiplying quaternion exponentials

Multiplying two exponentials is at the heart of modern analysis, whether one works with Fourier transforms or Lie groups. Given a Lie algebra of a Lie group in a sufficiently small area the identity, the product of two exponentials can be defined using the Campbell-Hausdorff formula:

$$\theta = \mathrm{ArcCos}\left(\frac{q + q^{*}}{2\,|\,q\,|}\right)$$

Figure 293: theta = arccos(q + q conjugated over 2 the absolute value of q)

$$I = \frac{q - q^{*}}{2\,|\,q - q^{*}\,|}$$

Figure 294: I = q - q conjugated over 2 the absolute value of q - q conjugated

$$I^{2} = \frac{(0,\ V)\ (0,\ V)}{|\ (0,\ V)\ ||\ (0,\ V)\ |} = \frac{(-V.V,\ VxV)}{(V^{2},\ 0)} = -1$$

Figure 295: I squared = (0, V) times (0, V) over the absolute value of V squared = (-V dot V, V Cross V) over (V squared , 0) = -1

$$\mathrm{Exp}[X]\ \mathrm{Exp}[y] = (X + Y) + \frac{1}{2}[X,\ Y]\ (X + Y)$$
$$+ \frac{1}{12}\ ([[X,\ Y],\ Y] - [[X,\ Y],\ X])\ (X + Y) + \ldots$$

Figure 296: e to the X times the e to the Y equals X + Y + 1/2 the commutator of X, Y times X + Y + 1/12 (the commutator of (the commutator of X, Y), Y - (the commutator of (the commutator of X, Y), X) times X + Y) + ...

This formula is not easy to use, and is only applicable in a small area around unity. Quaternion analysis that relies on this formula would be very limited.

I have developed (perhaps for the first time) a simpler and general way to express the product of two quaternion exponentials as the sum of two components. The product of two quaternions splits into a commuting and an anti-commuting part. The rules for multiplying commuting quaternions are identical to those for complex numbers. The anticommuting part needs to be purely imaginary. The Grassman product (q q') of two quaternion exponentials and the Euclidean product (q* q') should both have these properties. Together these define the needs for the product of two quaternion exponentials.

$$\mathrm{Let}\ q = \mathrm{Exp}[X] \quad q' = \mathrm{Exp}[Y]$$

Figure 297: Let q = e to the X, q prime = e to the Y

$$q\,q' = \frac{\{q,\ q'\}^*}{2} + \frac{\mathrm{Abs}[q,\ q']^*}{2}\,\mathrm{Exp}\left[\frac{\pi}{2}\,\frac{[q,\ q']^*}{\mathrm{Abs}[q,\ q']^*}\right]$$

$$\mathrm{where}\ \{q,\ q'\}^* \equiv q\,q' + q'^*\,q^* \ \mathrm{and}\ [q,\ q']^* \equiv q\,q' - q'^*\,q^*$$

Figure 298: q times q' = the even conjugator of q, q prime over 2 + the absolute value of the odd conjugator of q, q' over 2 times e to the pi over 2 times the odd conjugator of q, q prime over its absolute value

$$q^*\,q' = \mathrm{same\ as\ above}$$

$$\mathrm{where}\ \{q,\ q'\} = q^*\,q' + q'^*\,q \ \mathrm{and}\ [q,\ q'] = q^*\,q' - q'^*\,q$$

Figure 299: q conjugated time q prime = same as above but were the even conjugator q, q' equals q conjugated time q prime + q prime conjugated time q and the odd conjugator of q, q prime equals q conjugated time q prime minus q prim conjugated times q

I call these operators “conjugators” because they involve taking the conjugate of the two elements. Andrew Millard made the suggestion for the Grassman product that unifies these approaches nicely. What is happening here is that both commuting and anticommuting parts scale themselves appropriately. By using an exponential that has pi/2 multiplied by a normalized quaternion, this always has a zero scalar, as it must to accurately represent an anticommuting part.

Commutators and the Uncertainty Principle

Commutators and the uncertainty principle are central to quantum mechanics. Using quaternions in these roles has already been established by others (Horwitz and Biedenharn, Annals of Physics, 157:432, 1984). The first proof of the uncertainty principle I saw relied solely on the properties of complex numbers, not on physics! In this notebook I will repeat that analysis, showing how commutators and an uncertainty principle arise from the properties of quaternions (or their subfield the complex numbers).

Commutators

Any quaternion can be written in a polar form.

$$q = (s,\ V) = \sqrt{q^* q}\ \mathrm{Exp}\left[\frac{s}{\sqrt{q^* q}}\ \frac{V}{\sqrt{V^* V}} \right]$$

Figure 300: q = (s, V) = root q conjugated q times e to the (s over root q conjugated times q times V over root V conjugate times V)

This is identical to Euler's formula except that the imaginary unit vector i is replaced by the normalized 3-vector. The two are equivalent if j = k = 0. Any quaternion could be the limit of the sum of an infinite number of other quaternions expressed in a polar form. I hope to show that such a quaternion mathematically behaves like the wave function of quantum mechanics, even if the notation is different.

To simplify things, use a normalized quaternion, so that q* q = 1. Collect the normalized 3-vector together with I = V/(V* V)^.5.

The angle s/(q* q)^.5 is a real number. Any real number can be viewed as the product of two other real numbers. This seemingly irrelevant observation lends much of the flexibility seen in quantum mechanics :-) Here is the rewrite of q.

$$q = \mathrm{Exp}[a\ b\ I]$$

Figure 301: q = e to the (a b I)

$$\text{where} \quad q^* q = 1, \quad a\ b = \frac{s}{\sqrt{q^* q}}, \quad I = \frac{V}{\sqrt{V^* V}}$$

Figure 302: where q conjugate times q = 1, a b = s over root q conjugated times q, I = V over root V conjugated times V

The unit vector I could also be viewed as the product of two quaternions.For classical quantum mechanics, this additional complication is unnecessary. It may be required for relativistic quantum mechanics, so this should be kept in mind.

A point of clarification on notation:the same letter will be used 4 distinct ways.There are operators, A hat, which act on a quaternion wave function by multiplying by a quaternion, capital A. If the operator A hat is an observable, then it generates a real number, (a, 0), which commutes with all quaternions, whatever their form. There is also a variable with respect to a component of a quaternion, a_i, that can be used to form a differential operator.

Define a linear operator A hat that multiplies q by the quaternion A.

$$\hat{A}\, q = A\, q$$

Figure 303: A hat acts on q = A times q

If the operator A hat is an observable, then the quaternion A is a real number, (a, 0). This will commute with any quaternion. This equation is functionally equivalent to an eigenvalue equation, with A hat as an eigenvector of q and (a, 0) as the eigenvalue. However, all of the components of this equation are quaternions, not separate structures such as an operator belonging to a group and a vector. This might make a subtle but significant difference for the mathematical structure of the theory, a point that will not be investigated here.

Define a linear operator B hat that multiplies q by the quaternion B. If B hat is an observable, then this operator can be defined in terms of the scalar variable a.

$$\text{Let}\quad \hat{B} = -I\, \frac{d}{d\, a}$$

Figure 304: Let B hat = - I d by d a

$$\hat{B}\, q = -I\, \frac{d\, \mathrm{Exp}[a\, b\, I]}{d\, a} = b\, q$$

Figure 305: B hat acting on q = - I d e to the (a b I) over d a = b times q

Operators A and B are linear.

Calculate the commutator [A, B], which involves the scalar a and the derivative with respect to a.

$$(\hat{A} + \hat{B})\, q = \hat{A}\, q + \hat{B}\, q = a\, q + b\, q = (a + b)\, q$$

Figure 306: (A hat + B hat) acting on q = A hat acting on q + B hat acting on q = a times q + b times q = (a + b) times q

$$\hat{A}\,(q + q') = \hat{A}\, q + \hat{A}\, q' = a\, q + a'\, q'$$

Figure 307: A hat acting on (q + q prime) = A hat acting on q + A hat acting on q prime = a times q + a prime times q prime

The commutator acting on a quaternion is equivalent to multiplying that quaternion by the normalized 3-vector I.

The Uncertainty Principle

Use these operators to construct things that behave like averages (expectation values) and standard deviations.

The scalar a–generated by the observable operator A hat acting on the normalized q–can be calculated using the Euclidean product.

It is hard to shuffle quaternions or their operators around. Real scalars commute with any quaternion and are their own conjugates. Operators that generate such scalars can move around. Look at ways to express the expectation value of A.

Define a new operator A' based on A whose expectation value is always zero.

Define the square of the operator in a way designed to link up with the standard deviation.

An identical set of tools can be defined for B.

In the section on bracket notation, the Schwarz inequality for quaternions was shown.

The Schwarz inequality applies to quaternions, not quaternion operators. If the operators A' and B' are surrounded on both sides by q and q*, then they will behave like scalars.

The left-hand side of the Schwarz inequality can be rearranged to form a commutator.

$$[\hat{A}, \hat{B}]\, q = (\hat{A}\hat{B} - \hat{B}\hat{A})\, q = -a\, I\, \frac{d\, q}{d\, a} + I\, \frac{d\, a\, q}{d\, a}$$

Figure 308: The commutator of A hat, B hat acting on q = (A hat times B hat - B hat times A hat) acting on q = -a I d q by d a + I d a q by d a

$$= -a\,I\,\frac{d\,q}{d\,a} + a\,I\,\frac{d\,q}{d\,a} + I\,q\,\frac{d\,a}{d\,a} = I\,q$$

Figure 309: = -a I d q by d a + a I d q by d a + I q d a by d a = I q

$$q^{*}\,(A\,q) = q^{*}\,a\,q = a\,q^{*}\,q = a$$

Figure 310: q conjugated (A hat acting on q) = q conjugated a q = a q conjugated q = a

$$q^{*}\,(\hat{A}\,q) = q^{*}\,a\,q = a\,q^{*}\,q = a^{*}\,q^{*}\,q = (\hat{A}\,q)^{*}\,q = a$$

Figure 311: q conjugated (A hat acting on q) = q conjugated a q = a q conjugated q = A conjugated q conjugated q = (A hat acting on q) conjugated q = a

$$\text{Let } A' = A - q^{*}\,(A\,q)$$

Figure 312: Let A prime = A hat - q conjugated (A hat acting on q)

$$q^{*}\,(A'\,q) = q^{*}\,(A - q^{*}\,(A\,q))\,q) = a - a = 0$$

Figure 313: q conjugated (A prime q) = q conjugated (A - q conjugated (A q))q) = a - a = 0

$$\text{Let } DA'^{2} = q^{*}\,(A'^{2}\,q) - (q^{*}\,(A'\,q))^{2} = q^{*}\,(A'^{2}\,q)$$

Figure 314: Let DA prime squared = q conjugated (A prime acting twice on q) - (q conjugated (A prime q)) squared = q conjugated (A prime acting twice on q)

$$\frac{A'^{*}\,B' + B'^{*}\,A'}{2} \leq \left|A'\right|\,\left|B'\right|$$

Figure 315: (A prime conjugated times B prime + B prime conjugated times A prime) over 2 is less than the absolute value of A prime times the absolute value of B prime

$$q^* (A'^* B' + B'^* A') q = q^* A'^* B' q + q^* B'^* A' q = q^* a'^* B' q + q^* (-I)^* \frac{d}{da} A' q =$$

Figure 316: q conjugated (A prime conjugated B prime + B prime conjugated A prime) q = q A prime conjugated B prime q + q B prime conjugated A prime q = q conjugated A prime conjugated B prime q + q conjugated (-I)conjugated d by d a A prime q =

$$= q^* a' B' q - q^* (-I) \frac{d}{da} A' q = q^* (A' B' - B' A') q = q^* [A', B'] q$$

Figure 317: = q conjugated A prime B prime q - q conjugated (-I)d by d a A prime q = q conjugated(A prime B prime - B prime A prime)q = q conjugated times the conjugator of (A prime, B prime) q

The right-hand side of the Schwarz inequality can be rearranged to form the square of the standard deviation operators.

$$q^* | A' | \; | B' | q = q^* A^{*'} A' B^{*'} B' q = q^* A'^2 B'^2 q = q^* DA'^2 DB'^2 q$$

Figure 318: q conjugated times the absolute value of A prime times the absolute value of B prime times q = q conjugated A prime A primeB prime conjugated B prime q = q A prime squared B prime squared q = q conjugated DA prime squared DB prime squared q

Plug both of these back into the Schwarz inequality, stripping the primes and the q's which appear on both sides along the way.

This is the uncertainty principle for complementary observable operators.

Connections to Standard Notation

This quaternion exercise can be mapped to the standard notation used in physics

One subtlety to note is that a quaternion operator is anti-Hermitian only if the scalar is zero. This is probably the case for classical quantum mechanics, but quantum field theory may require full quaternion operators. The proof of the uncertainty principle shown here is independent of this issue. I do not yet understand the consequence of this point.

To get to the position-momentum uncertainty equation, make these specific maps

The product of the squares of the standard deviation for position and momentum in the x-direction has a lower bound equal to half the expectation value of the commutator of those operators. The proof is in the structure of quaternions.

$$\frac{[A,\ B]}{2} \le DA^2\, DB^2$$

Figure 319: The commutator of A, B over 2 is less than the standard deviation squared of the operator A times the standard deviation squared of the operator B

$$\mathrm{bra}:\ |\,\psi > \longrightarrow q$$

Figure 320: bra vector: |psi> maps to q

$$\mathrm{ket}:\ < \psi\,| \longrightarrow q^*$$

Figure 321: ket vector: <psi| maps to q conjugated

$$\mathrm{operator}:\ A \longrightarrow A$$

Figure 322: operator: A maps to A

$$\mathrm{imaginary}:\ i \longrightarrow I$$

Figure 323: imaginary: i maps to 3-vector I

$$\mathrm{commutator}:\ [A,\ B] \longrightarrow [A,\ B]$$

Figure 324: commutator: [A, B] maps to [A, B]

$$\mathrm{norm}:\ < \psi\,|\,\psi > \longrightarrow q^*\, q$$

Figure 325: norm: <psi|psi> maps to q conjugated times q

$$\mathrm{expectation\ of\ A}:\ < \psi\,|\,A\,\psi > \mathrm{\ maps\ to\ } q^*\, A\, q$$

Figure 326: expectation of A: <psi|A psi> maps to q conjugated A q

$$A \mathrm{\ is\ Hermitian} \longrightarrow (0,\ \vec{A}) \mathrm{\ is\ anti-Hermitian\ } q^*\,((0,\ \vec{A})\ q) = ((0,\ -\vec{A})\ q)^*\ q$$

Figure 327: A is Hermitian maps to (0, A) is anti- Hermitian

$$\mathrm{The\ square\ of\ the\ standard\ deviation}:\ \delta A^2 = < \psi\,|\,A^2\,\psi > - < \psi\,|\,A\,\psi >^2 \longrightarrow DA^2$$

Figure 328: The square of the standard deviation: dA squared = <psi | A squared psi> -<psi| A psi> squared maps to DA squared

$$A \longrightarrow X$$

Figure 329: A goes to X

$$B \longrightarrow P = i \hbar \frac{d}{dx}$$

Figure 330: B goes to P = i h bar d by dx

Implications

There are many interpretations of the uncertainty principle. I come away with two strange observations. First, the uncertainty principle is about quaternions of the form q = Exp[a b I]. With this insight, one can see by inspection that a plane wave Exp[((Et - P.X)/hbar I], or wave packets that are superpositions of plane waves, will have four uncertainty relations, one for the scalar Et and another three for the three-part scalar P.X. This perspective should be easy to generalize.

Second, the uncertainty principle and gravity are related to the same mathematical properties. This proof of the uncertainty relation involved the Schwarz inequality. It is fairly straightforward to convert that inequality to the triangle inequality. Finding geodesics with quaternions involves the triangle inequality. If a complete theory of gravity can be built from these geodesics (it hasn't yet been done :-) then the inequalities may open connections where none appeared before.

$$I = [A, B] \longrightarrow i \hbar [X, P]$$

Figure 331: I h bar = the commutator of A, B goes to i h bar the commutator of X, P

$$\frac{[A,\ B]}{2} = \frac{I}{2} \leq DA^2\,DB^2 \longrightarrow \frac{[X,\ P]}{2} = \frac{i\,\hbar}{2} \leq \delta X^2\,\delta P^2$$

Figure 332: The commutator of A, B over 2 = I over 2 is less than or equal to DA squared DB squared goes to the commutator of X, P over 2 = i h bar over 2 is less than or equal to the standard deviation of X squared times the standard deviation of P squared

Unifying the Representation of Spin and Angular Momentum

I will show how to represent both integral and half-integral spin within the same quaternion division algebra. This involves using quaternion automorphisms. First a sketch of why this might work will be provided. Second, small rotations in a plane around two axes will be used to show how the resulting vector points in an opposite way, depending on which involution is used to construct the infinitesimal rotation. Finally, a general identity will be used to look at what happens under exchange of two quaternions in a commutator.

Automorphism, Rotations, and Commutators

Quaternions are formed from the direct product of a scalar and a 3-vector. Rotational operators that act on each of the 3 components of the 3-vector act like integral angular momentum. I will show that a rotation operator that acts differently on two of the three components of the 3-vector acts like half-integral spin. What happens with the scalar is irrelevant to this dimensional counting. The same rotation matrix acting on the same quaternion behaves differently depending directly on what involutions are involved.

Quaternions have 4 degrees of freedom. If we want to represent quaternions with automorphisms, 4 are required: They are the identity automorphism, the conjugate anti-automorphism, the first conjugate anti-automorphism, and the second conjugate anti-automorphism:

$$I : q \rightarrow q$$

Figure 333: The identity automorphism I maps q into q

$$^{*} : q \rightarrow q^{*}$$

Figure 334: The conjugate anti-automorphism * maps q into q*

where

$$*^{1} : q \rightarrow q^{*1}$$

Figure 335: The first conjugate anti-automorphism *1 maps q into* q1

$$*^{2} : q \rightarrow q^{*2}$$

Figure 336: The second conjugate anti-automorphism *2 maps q into* q2

e1, e2, e3 are orthogonal basis vectors

The most important automorphism is the identity. Life is stable around small permutations of the identity:-) The conjugate flips the signs of the each component in the 3-vector. These two automorphisms, the identity and the conjugate, treat the 3-vector as a unit. The first and second conjugate flip the signs of all terms but the first and second terms, respectively. Therefore these operators act on only the two of the three components in the 3-vector. By acting on only two of three components, a commutator will behave differently. This small difference in behavior inside a commutator is what creates the ability to represent integral and half-integral spins.

Small Rotations

Small rotations about the origin will now be calculated. These will then be expressed in terms of the four automorphisms discussed above.

I will be following the approach used in J. J. Sakurai's book "Modern Quantum Mechanics", chapter 3, making modifications necessary to accommodate quaternions. First, consider rotations about the origin in the z axis. Define:

Two technical points. First, Sakurai considered rotations around any point along the z axis. This analysis is confined to the z axis at the origin, a significant but not unreasonable constraint. Second, these rotations are written with generalized coordinates instead of the very familiar and comfortable x, y, z. This extra effort will be useful when considering how rotations are effected by curved spacetime. This machinery is also necessary to do quaternion analysis (please see that section, it's great :-)

There are similar rotations around the first and second axes at the origin;

Consider an infinitesimal rotation for these three rotation operators. To second

$$q^{*1} \equiv (e_1 \, q \, e_1)^*$$

Figure 337: q*1 is defined to be (e1 q e1)*

$$q^{*2} \equiv (e_2\, q\, e_2)^*$$

Figure 338: q*2 === (e2 q e2)*

$$R_{e_3=0}(\theta) \equiv \left(\cos(\theta)\, e_0,\ 0,\ 0,\ \sin(\theta)\ \frac{e_3}{3}\right)$$

Figure 339: Rotation around e3 at zero by theta is defined to be (cosine theta e0, 0, 0, sine theta e3 over 3)

$$\text{if } q = \left(0,\ a_1\ \frac{e_1}{3},\ a_2\ \frac{e_2}{3},\ 0\right)$$

Figure 340: if q = (0, a1 e1/3, a2 e2/3, 0)

$$R_{e_3=0}(\theta)\ q = q' = \left(0,\ (a_1 \cos(\theta) - a_2 \sin(\theta))\ e_0\ \frac{e_1}{3},\ (a_2 \cos(\theta) + a_1 \sin(\theta))\ e_0\ \frac{e_2}{3},\ 0\right)$$

Figure 341: Rotation around e3 acting on q = q prime= (0, (a1 cosine theta-a2 sine theta) e0 e1 over 3, (a2 cosine theta+a1 sine theta) e0 e2 over 3, 0)

$$R_{e_1=0}(\theta) = \left(\cos(\theta)\, e_0,\ \sin(\theta)\ \frac{e_1}{3},\ 0,\ 0\right)$$

Figure 342: Rotation around e1 at zero by theta = (cosine theta e0, sine theta e1 over 3, 0, 0)

$$R_{e_2=0}(\theta) = \left(\cos(\theta)\, e_0,\ 0,\ \sin(\theta)\ \frac{e_2}{3},\ 0\right)$$

Figure 343: Rotation around e2 at zero by theta = (cosine theta e0, 0, sine theta e2 over 3, 0)

order in theta,

$$\sin(\theta) = \theta + O(\theta^3), \ \cos(\theta) = \left(1 - \frac{\theta^2}{2}\right) + O(\theta^3)$$

Figure 344: sine theta = theta + errors of the order of theta cubed, cosine theta = 1 - theta squared over 2 + errors of the order of theta cubed

$$R_{e_1=0}(\theta << 1) = \left(\left(1 - \frac{\theta^2}{2}\right) e_0, \ \theta \frac{e_1}{3}, \ 0, \ 0\right) + O(\theta^3)$$

Figure 345: Rotation around e1 at zero by theta much less than 1 = ((1 - theta squared over 2) e0, theta e1 over 3, 0, 0) + errors of the order of theta cubed

$$R_{e_2=0}(\theta << 1) = \left(\left(1 - \frac{\theta^2}{2}\right) e_0, \ 0, \ \theta \frac{e_2}{3}, \ 0\right) + O(\theta^3)$$

Figure 346: Rotation around e12at zero by theta much less than 1 = ((1 - theta squared over 2) e0, 0, theta e2 over 3, 0) + errors of the order of theta cubed

Calculate the commutator of the first two infinitesimal rotation operators to second order in theta:

To second order, the commutator of infinitesimal rotations of rotations about the first two axes equals twice one rotation about the third axis given the squared angle minus a zero rotation about an arbitrary axis (a fancy way to say the identity). Now I want to write this result using anti-automorphic involutions for the small rotation operators.

Nothing has changed. Repeat this exercise one last time for the first conjugate:

This points exactly the opposite way,even for an infinitesimal angle!

This is the kernel required to form a unified representation of integral and half integral spin. Imagine adding up a series of these small rotations, say 2 pi of these. No doubt the identity and conjugates will bring you back exactly where you started. The first and second conjugates in the commutator will point in the opposite direction. To get back on course will require another 2 pi, because the minus of a minus will generate a plus.

Automorphic Commutator Identities

This is a very specific example. Is there a general identity behind this work? Here it is:

$$R_{e_3=0}\ (\theta << 1) = \left(\left(1 - \frac{\theta^2}{2}\right) e_0,\ 0,\ 0,\ \theta\ \frac{e_3}{3}\right) + O\ (\theta^3)$$

Figure 347: Rotation around e3 at zero by theta much less than 1 = ((1 - theta squared over 2) e0, 0, 0, theta e3 over 3) + errors of the order of theta cubed

$$[R_{e_1=0},\ R_{e_2=0}] = \left(\left(1 - \frac{\theta^2}{2}\right) e_0,\ \theta\ \frac{e_1}{3},\ 0,\ 0\right) \left(\left(1 - \frac{\theta^2}{2}\right) e_0,\ 0,\ \theta\ \frac{e_2}{3},\ 0\right) -$$

Figure 348: The commutator of a rotation around e1, rotation around e2 = ((1 - theta squared over 2) e0, theta e1 over 3, 0, 0) times ((1 - theta squared over 2) e0, 0, theta e2 over 3, 0) -

$$- \left(\left(1 - \frac{\theta^2}{2}\right) e_0,\ 0,\ \theta\ \frac{e_2}{3},\ 0\right) \left(\left(1 - \frac{\theta^2}{2}\right) e_0,\ \theta\ \frac{e_1}{3},\ 0,\ 0\right) =$$

Figure 349: - ((1 - theta squared over 2) e0, 0, theta e2 over 3, 0) times ((1 - theta squared over 2) e0,theta e1 over 3, 0, 0) =

$$= \left((1 - \theta^2)\ e_0{}^2,\ \theta\ \frac{e_0\ e_1}{3},\ \theta\ \frac{e_0\ e_2}{3},\ \theta^2\ \frac{e_1\ e_2}{9}\right) - \left((1 - \theta^2)\ e_0{}^2,\ \theta\ \frac{e_0\ e_1}{3},\ \theta\ \frac{e_0\ e_2}{3},\ -\ \theta^2\ \frac{e_1\ e_2}{9}\right) =$$

Figure 350: = ((1 - theta squared) e0 squared, theta e0 e1 over 3, theta e0 e2 over 3, theta squared e1 e2 over 9) - ((1 - theta squared) e0 squared, theta e0 e1 over 3, theta e0 e2 over 3, - theta squared e1 e2 over 9)

$$= 2\ \left(0,\ 0,\ 0,\ \theta^2\ \frac{e_1\ e_2}{9}\right) = 2\ (R_{e_3=0}\ (\theta^2) - R\ (0))$$

Figure 351: = 2 (0, 0, 0, theta squared e1 e2 over 9) = 2 (Re3=0 (theta squared) - R(0))

$$[R^*{}_{e_1=0},\ R^*{}_{e_2=0}] = \left(\left(1 - \frac{\theta^2}{2}\right) e_0,\ -\ \theta\ \frac{e_1}{3},\ 0,\ 0\right) \left(\left(1 - \frac{\theta^2}{2}\right) e_0,\ 0,\ -\ \theta\ \frac{e_2}{3},\ 0\right) -$$

Figure 352: The commutator of a rotation around e1 conjugated, a rotation around e2 conjugated = ((1-theta squared over 2) e0, - theta e1 over 3, 0, 0) times ((1 - theta squared over 2) e0, 0, - theta e2 over 3, 0) -

$$- \left(\left(1 - \frac{\theta^2}{2}\right) e_0,\ 0,\ -\ \theta\ \frac{e_2}{3},\ 0\right) \left(\left(1 - \frac{\theta^2}{2}\right) e_0,\ -\ \theta\ \frac{e_1}{3},\ 0,\ 0\right) =$$

Figure 353: - ((1 - theta squared over 2) e0, 0, - theta e2 over 3, 0) times ((1 - theta squared over 2) e0, - theta e1 over 3, 0, 0) =

$$= \left((1-\theta^2)\, e_0{}^2,\ -\theta\,\frac{e_0\, e_1}{3},\ -\theta\,\frac{e_0\, e_2}{3},\ \theta^2\,\frac{e_1\, e_2}{9}\right) - \left((1-\theta^2)\, e_0{}^2,\ -\theta\,\frac{e_0\, e_1}{3},\ -\theta\,\frac{e_0\, e_2}{3},\ -\theta^2\,\frac{e_1\, e_2}{9}\right) =$$

Figure 354: = ((1 - theta squared) e0 squared, - theta e0 e1 over 3, - theta e0 e2 over 3, theta squared e1 e2 over 9) - ((1 - theta squared) e0 squared, - theta e0 e1 over 3, - theta e0 e2 over 3, - theta squared e1 e2 over 9) =

$$= 2\left(0,\ 0,\ 0,\ \theta^2\,\frac{e_1\, e_2}{9}\right) = 2\,(R_{e_3=0}\,(\theta^2) - R\,(0))$$

Figure 355: = 2 (0, 0, 0, theta squared e1 e2 over 9) = 2 (Re3=0 (theta squared) - R(0))

$$[R^{*1}{}_{e_1=0},\ R^{*1}{}_{e_2=0}] = \left(-\left(1-\frac{\theta^2}{2}\right) e_0,\ \theta\,\frac{e_1}{3},\ 0,\ 0\right)\left(-\left(1-\frac{\theta^2}{2}\right) e_0,\ 0,\ -\theta\,\frac{e_2}{3},\ 0\right) -$$

Figure 356: The commutator of a rotation around e1 first conjugated, a rotation around e2 first conjugated = (-(1 - theta squared over 2) e0, theta e1 over 3, 0, 0) times (-(1 - theta squared over 2) e0, 0, - theta e2 over 3, 0) -

$$-\left(-\left(1-\frac{\theta^2}{2}\right) e_0,\ 0,\ -\theta\,\frac{e_2}{3},\ 0\right)\left(-\left(1-\frac{\theta^2}{2}\right) e_0,\ \theta\,\frac{e_1}{3},\ 0,\ 0\right) =$$

Figure 357: - (-(1 - theta squared over 2) e0, 0, - theta e2 over 3, 0) times (-(1- theta squared over 2) e0, theta e1 over 3, 0, 0) =

$$= \left((1-\theta^2)\, e_0{}^2,\ -\theta\,\frac{e_0\, e_1}{3},\ -\theta\,\frac{e_0\, e_2}{3},\ \theta^2\,\frac{e_1\, e_2}{9}\right) - \left((1-\theta^2)\, e_0{}^2,\ -\theta\,\frac{e_0\, e_1}{3},\ -\theta\,\frac{e_0\, e_2}{3},\ -\theta^2\,\frac{e_1\, e_2}{9}\right) =$$

Figure 358: = ((1 - theta squared) e0 squared, - theta e0 e1 over 3, - theta e0 e2 over 3, theta squared e1 e2 over 9) - ((1 - theta squared) e0 squared, -theta e1 over 3, - theta e2 over 3, - theta squared e1 e2 over 9) =

$$= 2\left(0,\ 0,\ 0,\ \theta^2\,\frac{e_1\, e_2}{9}\right) = -\,2\,(R_{e_3=0}\,(\theta^2) - R\,(0))$$

Figure 359: = 2 (0, 0, 0, theta squared e1 e2 over 9) = 2 (Re3=0 (theta squared) - R(0))

$$[q,\ q'] = [q^*,\ q'^*] = [q^{*1},\ q'^{*1}]^{*1} = [q^{*2},\ q'^{*2}]^{*2}$$

Figure 360: The commutator of q, q prime = the commutator of q conjugate, q prime conjugated = the first conjugate of the commutator of q first conjugated, q prime first conjugated = the second conjugate of the commutator of q second conjugated, q prime second conjugated

It is usually a good sign if a proposal gets more subtle by generalization :-) In this case, the negative sign seen on the z axis for the first conjugate commutator is due to the action of an additional first conjugate. For the first conjugate, the first term will have the correct sign after a 2 pi journey, but the scalar, third and forth terms will point the opposite way. A similar, but not identical story applies for the second conjugate.

With the identity, we can see exactly what happens if q changes places with q' with a commutator. Notice, I stopped right at the commutator (not including any additional conjugator). In that case:

$$[q, q'] = -[q', q] = [q^*, q'^*] = -[q'^*, q^*] =$$

Figure 361: The commutator of q conjugate, q prime = - the commutator of q prime, q = the commutator of q conjugated, q prime conjugated = - the commutator of q prime conjugated, q conjugated =

$$= \left\{0,\ a_2\, a_3\, \frac{e_2\, e_3}{9} + a_3\, a_2\, \frac{e_3\, e_2}{9},\ a_3\, a_1\, \frac{e_3\, e_1}{9} + a_1\, a_3\, \frac{e_1\, e_3}{9},\ a_1\, a_2\, \frac{e_1\, e_2}{9} + a_2\, a_1\, \frac{e_2\, e_1}{9}\right\}$$

Figure 362: = (0, a2 a3 e2 e3 over 9 + a3 a2 e3 e2 over 9, a3 a1 e3 e1 over 9 + a1 a3 e1 e3 over 9, a1 a2 e1 e2 over 9 + a2 a1 e2 e1 over 9)

$$[q^{*1}, q'^{*1}] = -[q'^{*1}, q^{*1}] =$$

Figure 363: The commutator of q first conjugated, q prime first conjugated = - the commutator of q prime first conjugated, q first conjugated =

Under an exchange, the identity and conjugate commutators form a distinct group from the commutators formed with the first and second conjugates. The behavior in a commutator under exchange of the identity automorphism and the anti-automorphic conjugate are identical. The first and second conjugates are similar, but not identical.

There are also corresponding identities for the anti-commutator:

At this point, I don't know how to use them, but again, the identity and first conjugates appear to behave differently that the first and second conjugates.

Implications

This is not a super-symmetric proposal. For that work, there is a super- partner particle for every currently detected particle. At this time, not one of those particles has been detected, a serious omission.

Three different operators had to be blended together to perform this feat: commutators, conjugates and rotations. These involve issue of even/oddness, mir-

$$= \left(0,\ a_2 a_3 \frac{e_2 e_3}{9} + a_3 a_2 \frac{e_3 e_2}{9},\ - a_3 a_1 \frac{e_3 e_1}{9} - a_1 a_3 \frac{e_1 e_3}{9},\ - a_1 a_2 \frac{e_1 e_2}{9} - a_2 a_1 \frac{e_2 e_1}{9}\right)$$

Figure 364: = (0, a2 a3 e2 e3 over 9 + a3 a2 e3 e2 over 9, - a3 a1 e3 e1 over 9 - a1 a3 e1 e3 over 9, - a1 a2 e1 e2 over 9 - a2 a1 e2 e1 over 9)

$$[q^{*2}, q'^{*2}] = -[q'^{*2}, q^{*2}] =$$

Figure 365: The commutator of q first conjugated, q prime first conjugated = - the commutator of q prime first conjugated, q first conjugated =

rors, and rotations. In a commutator under exchange of two quaternions, the identity and the conjugate behave in a united way, while the first and second conjugates form a similar, but not identical set. Because this is a general quaternion identity of automorphisms, this should be very widely applicable.

$$= \left(0,\ - a_2 a_3 \frac{e_2 e_3}{9} - a_3 a_2 \frac{e_3 e_2}{9},\ a_3 a_1 \frac{e_3 e_1}{9} + a_1 a_3 \frac{e_1 e_3}{9},\ - a_1 a_2 \frac{e_1 e_2}{9} - a_2 a_1 \frac{e_2 e_1}{9}\right)$$

Figure 366: = (0, a2 a3 e2 e3 over 9 + a3 a2 e3 e2 over 9, - a3 a1 e3 e1 over 9 - a1 a3 e1 e3 over 9, - a1 a2 e1 e2 over 9 - a2 a1 e2 e1 over 9)

$$\{q,\ q'\} = \{q^*,\ q'^*\}^* = -\{q^{*1},\ q'^{*1}\}^{*1} = -\{q^{*2},\ q'^{*2}\}^{*2}$$

Figure 367: The anti-commutator of q, q prime = the conjugate of the anti-commutator of q conjugated, q prime conjugated = - the first conjugate of the anti-commutator of q first conjugated, q prime first conjugated = - the anti-commutator of q second conjugated, q prime second conjugated

Deriving A Quaternion Analog to the Schrödinger Equation

The Schrödinger equation gives the kinetic energy plus the potential (a sum also known as the Hamiltonian H) of the wave function psi, which contains all the dynamical information about a system. Psi is a scalar function with complex values.

$$H\psi = -i\hbar \frac{\partial \psi}{\partial t} = \frac{-\hbar^2}{2\,m} \nabla^2 \psi + V\,(0,\ X)\,\psi$$

Figure 368: The hamiltonian operator acting on psi = -i h bar phi dot = -h bar squared over 2 m Laplacian psi + the potential V(0, X) psi

For the time-independent case, energy is written at the operator -i hbar d/dt, and kinetic energy as the square of the momentum operator, i hbar Del, over 2m. Given the potential V(0, X) and suitable boundary conditions, solving this differential equation generates a wave function psi which contains all the properties of the system.

In this section, the quaternion analog to the Schrödinger equation will be derived from first principles. What is interesting are the constraint that are required for the quaternion analog. For example, there is a factor which might serve to damp runaway terms.

The Quaternion Wave Function

The derivation starts from a curious place :-) Write out classical angular momentum with quaternions.

$$(0,\ \vec{L}) = (0,\ \vec{R} \times \vec{P}) = \text{odd}\,((0,\ \vec{R})\ (0,\ \vec{P}))$$

Figure 369: (0, L) = (0, R Cross P) = the odd part of (0, R) times (0, P)

What makes this “classical” are the zeroes in the scalars. Make these into complete quaternions by bringing in time to go along with the space 3-vector R, and E with the 3-vector P.

$$(t, \vec{R})\ (E, \vec{P}) = (Et - \vec{R}.\vec{P},\ E\,\vec{R} + \vec{P}\,t + \vec{R} \times \vec{P})$$

Figure 370: (t, R) times (E, P) = (E t - R dot P, E R + P t + R Cross P)

Define a dimensionless quaternion psi that is this product over h bar.

$$\psi \equiv \frac{(t, \vec{R})\ (E, \vec{P})}{\hbar} = (Et - \vec{R}.\vec{P},\ E\,\vec{R} + \vec{P}\,t + \vec{R} \times \vec{P}) / \hbar$$

Figure 371: psi is defined to be (t, R) times (E, P) over hbar = (E t - R dot P, E R + P t + R Cross P) over h bar

The scalar part of psi is also seen in plane wave solutions of quantum mechanics. The complicated 3-vector is a new animal, but notice it is composed of all the parts seen in the scalar, just different permutations that evaluate to 3-vectors. One might argue that for completeness, all combinations of E, t, R and P should be involved in psi, as is the case here.

Any quaternion can be expressed in polar form:

$$q = \left| q \right| e^{\arccos\left(\frac{s}{|q|}\right) \frac{\vec{v}}{|\vec{v}|}}$$

Figure 372: q = the absolute value of q times e to the arc cosine of the scalar over the absolute value of q times the 3-vector over its absolute value

Express psi in polar form. To make things simpler, assume that psi is normalized, so |psi| = 1. The 3-vector of psi is quite complicated, so define one symbol to capture it:

Now rewrite psi in polar form with these simplifications:

This is what I call the quaternion wave function. Unlike previous work with quaternionic quantum mechanics (see S. Adler's book “Quaternionic Quantum Mechanics”), I see no need to define a vector space with right-hand operator multiplication. As was shown in the section on bracket notation, the Euclidean product of psi (psi* psi) will have all the properties required to form a Hilbert space. The advantage of keeping both operators and the wave function as quaternions is that it will make sense to form an interacting field directly using a product such as psi psi'. That will not be done here. Another advantage is that all the equations will necessarily be invertible.

$$I \equiv \frac{E\vec{R} + \vec{P}\,t + \vec{R} \times \vec{P}}{|\,E\vec{R} + \vec{P}\,t + \vec{R} \times \vec{P}\,|}$$

Figure 373: I is defined to be E R + P t + R cross P over the absolute value of the numerator

$$\psi = e^{\left(Et - \vec{R}.\vec{P}\right) I/\hbar}$$

Figure 374: psi = e to the E t - R dot P time I over h bar

Changes in the Quaternion Wave Function

We cannot derive the Schrödinger equation per se, since that involves Hermitian operators that acting on a complex vector space. Instead, the operators here will be anti-Hermitian quaternions acting on quaternions. Still it will look very similar, down to the last h bar :-) All that needs to be done is to study how the quaternion wave function psi changes. Make the following assumptions.

1. Energy and Momentum are conserved.

$$\frac{\partial E}{\partial t} = 0 \quad \text{and} \quad \frac{\partial \vec{P}}{\partial t} = 0$$

Figure 375: d E by d t = 0 and d P by d t = 0

2. Energy is evenly distributed in space

3. The system is isolated

4. The position 3-vector X is in the same direction as the momentum 3-vector P

The implications of this last assumption are not obvious but can be computed directly by taking the appropriate derivative. Here is a verbal explanation. If energy and momentum are conserved, they will not change in time. If the position 3-vector which does change is always in the same direction as the momentum 3-vector, then I will remain constant in time. Since I is in the direction of X, its curl will be zero.

This last constraint may initially appear too confining. Contrast this with the typical classical quantum mechanics. In that case, there is an imaginary factor i which contains no information about the system. It is a mathematical tool tossed in so that the equation has the correct properties. With quaternions, I is

$$\vec{\nabla} E = 0$$

Figure 376: The Gradient of E = 0

$$\vec{\nabla} \times \vec{P} = 0$$

Figure 377: The Curl of P = 0

determined directly from E, t, P and X. It must be richer in information content. This particular constraint is a reflection of that.

Now take the time derivative of psi.

The denominator must be at least 1, and can be greater that that. It can serve as a damper, a good thing to tame runaway terms. Unfortunately, it also makes solving explicitly for energy impossible unless Et - P.X equals zero. Since the goal is to make a direct connection to the Schrödinger equation, make one final assumption:

There are several important cases when this will be true. In a vacuum, E and P are zero. If this is used to study photons, then t = |R| and E = |P|. If this number happens to be constant in time, then this equation will apply to the wave front.

Now with these 5 assumptions in hand, energy can be defined with an operator.

The equivalence of the energy E and this operator is called the first quantization.

Take the spatial derivative of psi using the under the same assumptions:

Square this operator.

The Hamiltonian equals the kinetic energy plus the potential energy.

Typographically, this looks very similar to the Schrödinger equation. Capital I is a normalized 3-vector, and a very complicated one at that if you review the assumptions that got us here. phi is not a vector, but is a quaternion. This give the equation more, not less, analytical power. With all of the constraints in place, I expect that this equation will behave exactly like the Schrödinger equation. As the constraints are removed, this proposal becomes richer. There is

$$\frac{X.P}{|X||P|} = 1 \text{ which implies } \frac{de^{\vec{I}}}{dt} = 0 \text{ and } \vec{\nabla} \times e^{\vec{I}} = 0$$

Figure 378: X dot P over the absolute value of the two = 1 which implies d e to the I by d t = 0 and the Curl of e to the I = 0

$$\frac{\partial \psi}{\partial t} = \frac{E\,I}{\hbar}\;\frac{\psi}{\sqrt{1 + \left(\frac{Et - \vec{R}.\vec{P}}{\hbar}\right)^2}}$$

Figure 379: d psi by dt = E I over h bar times psi over the square root of (E t - R dot P over h bar) squared

$$Et - \vec{R}.\vec{P} = 0$$

Figure 380: E t - R dot P = 0

$$\text{if } \frac{\partial Et - \vec{R}.\vec{P}}{\partial t} = 0,\; E = \frac{\partial \vec{R}}{\partial t}.\vec{P} \;\text{ or }\; \frac{\partial \vec{R}}{\partial t} = \frac{E}{\vec{P}}$$

Figure 381: if d E t - R dot P by d t = 0, then E = d R by d t dot P or d R by d t = E over P

$$\frac{\partial \psi}{\partial t} = \frac{E\,I}{\hbar}\,\psi$$

Figure 382: d psi dt = E I over h bar psi

$$-I\hbar\,\frac{\partial \psi}{\partial t} = E\,\psi \;\text{ or }\; E = -I\hbar\,\frac{\partial}{\partial t}$$

Figure 383: - I h bar d psi d t = E psi or E = - I h bar d by dt

$$\vec{\nabla}\psi = -\frac{\vec{P}\,I}{\hbar}\;\frac{\psi}{\sqrt{1 + \left(\frac{Et - \vec{R}.\vec{P}}{\hbar}\right)^2}}$$

Figure 384: Del psi = - P I over h bar times psi over the square root of (E t - R dot P over h bar) squared

$$\vec{I}\,\hbar\,\vec{\nabla}\psi = \vec{P}\,\psi \quad \text{or}\ \vec{P} = I\hbar\,\vec{\nabla}$$

Figure 385: I h bar Del acting on psi = P acting on psi or P = I h bar Del

$$\vec{P}^2 = (mv)^2 = 2\,m\,\frac{mv^2}{2} = 2\,m\,KE = -\hbar^2\,\vec{\nabla}^2$$

Figure 386: P squared = m v squared = 2 m times m v squared over 2 = 2 m Kinetic Energy = - h bar squared Del squared

a damper to quench runaway terms. The 3-vector I becomes quite the nightmare to deal with, but it should be possible, given we are dealing with a division algebra.

Implications

Any attempt to shift the meaning of an equation as central to modern physics had first be able to regenerate all of its results. I believe that the quaternion analog to Schrödinger equation under the listed constraints will do the task. These is an immense amount of work needed to see as the constraints are relaxed, whether the quaternion differential equations will behave better. My sense at this time is that first quaternion analysis as discussed earlier must be made as mathematically solid as complex analysis. At that point, it will be worth pushing the envelope with this quaternion equation. If it stands on a foundation as robust as complex analysis, the profound problems seen in quantum field theory stand a chance of fading away into the background.

$$\vec{H}\,\psi = -\vec{I}\,\hbar\,\frac{\partial\psi}{\partial t} = -\hbar^2\,\vec{\nabla}^2\,\psi + V\,\psi$$

Figure 387: The Hamiltonian acting on psi = - I hbar d psi by d t = - h bar squared Del squared + the potential acting on psi

Introduction to Relativistic Quantum Mechanics

The relativistic quantum mechanic equation for a free particle is the Klein-Gordon equation (h=c=1)

$$\left(\frac{\partial^2}{\partial t^2} - \nabla^2 + m^2 \right) \Psi = 0$$

Figure 388: (The second time derivative - Laplacian + m squared) acting on psi = 0

The Schrödinger equation results from the non-relativistic limit of this equation. In this section, the machinery of the Klein-Gordon equation will be ported to quaternions.

The Wave Function

The wave function is the superposition of all possible states of a system. The product of the conjugate of a wave function with another wave function forms a complete inner product space. In the energy/momentum representation, this would involve all possible energy levels and momenta.

$$\Psi \equiv \text{the sum from n = 0 to infinity of } (E_n, \vec{P}_n)$$

Figure 389: psi is defined to be the sum from n = 0 to infinity of (E sub n, P sub n)

This infinite sum of quaternions should contain all the information about a system. The quaternion wave function can be normalized.

$$\sum_{n=0}^{\infty} (E_n, \vec{P}_n)^* (E_n, \vec{P}_n) = \sum_{n=0}^{\infty} \left(E_n^2 + \vec{P}_n^2, \quad 0 \right) = 1$$

Figure 390: The sum from n = 0 to infinity of (E sub n, P sub n) conjugated times (E sub n, P sub n) = the sum from n = 0 to infinity of (E sub n squared + P sub n squared, 0) = 1

The first quaternion is the conjugate or transpose of the second. Since the transpose of a quaternion wave function times a wave function creates a Euclidean norm, this representation of wave functions as an infinite sum of quaternions can form a complete, normed product space.

The Klein-Gordon Equation

The Klein-Gordon equation can be divided into two operators that act on the wave function: the D'Alembertian and the scalar m^2. The quaternion operator required to create the D'Alembertian, along with vector identities, has already been worked out for the Maxwell equations in the Lorenz gauge.

$$\sum_{n=0}^{\infty}\left(\left(\frac{\partial}{\partial t}, \vec{\nabla}\right)^2 + \left(\frac{\partial}{\partial t}, -\vec{\nabla}\right)^2\right)(E_n, \vec{P}_n) / 2 =$$

Figure 391: the sum from n = 0 to infinity of ((d by dt, Del) squared + (d by dt, - Del) squared) acting on (E sub n, P sub n) over 2 =

$$= \sum_{n=0}^{\infty}\left(\frac{\partial^2 E_n}{\partial t^2} - \vec{\nabla}\cdot\vec{\nabla} E_n - \vec{\nabla}\cdot\vec{\nabla} \times \vec{P}_n,\ \frac{\partial^2 \vec{P}_n}{\partial t^2} - \vec{\nabla}\,\vec{\nabla}\cdot\vec{P}_n + \vec{\nabla}\times\vec{\nabla}\times\vec{P}_n + \vec{\nabla}\times\vec{\nabla} E_n\right)$$

Figure 392: = the sum from n = 0 to infinity of (E sub n double dot - div Grad E sub n - div curl P sub n, P sub n double dot - Grad div P sub n + Del Cross Del Cross P sub n + Del Cross Grad E sub n)

The first term of the scalar, and the second term of the vector, are both equal to zero. What is left is the D'Alembertian operator acting on the quaternion wave function.

To generate the scalar multiplier m^2, substitute En and Pn for the operators d/dt and del respectively, and repeat. Since the structure of the operator is identical to the previous one, instead of the D'Alembertian times the wave function, there is En^2-Pn^2. The sum of all these terms becomes m^2.

Set the sum of these two operators equal to zero to form the Klein-Gordon equation.

$$\sum_{n=0}^{\infty}\left(\left(\frac{\partial}{\partial t}, \vec{\nabla}\right)^2 + \left(\frac{\partial}{\partial t}, -\vec{\nabla}\right)^2 + (E_n, \vec{P}_n)^2 + (E_n, -\vec{P}_n)^2\right)(E_n, \vec{P}_n) / 2 =$$

$$= \sum_{n=0}^{\infty}\left(-\vec{\nabla}\cdot(\vec{\nabla}\times\vec{P}_n) - \vec{\nabla}\cdot\vec{\nabla} E_n - \vec{P}_n\cdot(\vec{P}_n\times\vec{P}_n) - (\vec{P}_n\cdot\vec{P}_n) E_n + E_n^3 + \frac{\partial^2 E_n}{\partial t^2},\right.$$

$$\left.\vec{\nabla}\times(\vec{\nabla}\times\vec{P}_n) + \vec{\nabla}\times(\vec{\nabla} E_n) + \vec{P}_n\times(\vec{P}_n\times\vec{P}_n) + (\vec{P}_n\times\vec{P}_n) E_n - \vec{\nabla}(\vec{\nabla}\cdot\vec{P}_n) + \vec{P}_n E_n^2 - \vec{P}_n(\vec{P}_n\cdot\vec{P}_n) + \frac{\partial^2 \vec{P}_n}{\partial t^2}\right)$$

It takes some skilled staring to assure that this equation contains the Klein-Gordon equation along with vector identities.

Connection to the Maxwell Equations

If m=0, the quaternion operators of the Klein-Gordon equation simplifies to the operators used to generate the Maxwell equations in the Lorenz gauge. In the homogeneous case, the same operator acting on two different quaternions equals the same result. This implies that

$$(\varphi, \vec{A}) = \sum_{n=0}^{\infty} (E_n, \vec{P}_n)$$

Figure 393: (phi, A) = the sum from n = 0 to infinity of (E sub n, P sub n)

Under this interpretation, a nonzero mass changes the wave equation into a simple harmonic oscillator. The simple relationship between the quaternion potential and the wave function may hold for the nonhomogeneous case as well.

Implications

The Klein-Gordon equation is customarily viewed as a scalar equation (due to the scalar D'Alembertian operator) and the Maxwell equations are a vector equation (due to the potential four vector). In this notebook, the quaternion operator that generated the Maxwell equations was used to generate the Klein-Gordon equation. This also created several vector identities which are usually not mentioned in this context. A quaternion differential equation is needed to perform the work of the Dirac equation, but since quaternion operators are a field, an operator that does the task must exist.

An Introduction to the Standard Model

The Standard Model of physics was developed in the 1970's to explain the ~270 different types of particles seen in colliders (a general introduction is available on wikipedia, and a one page cheat sheet). The part we need to understand is the groups that describe the symmetry of the standard model.

What is a group? A group has an identity, an inverse, and a binary operation (multiplication). One member of the group times another member of the group generates yet another member of the same group. This is a case where the math name is accurate: once in a group, you are always in a group.

The standard model has three continuous groups that characterize three of the four known fundamental forces of nature. The simplest group is known as U(1) and governs electromagnetism via the photon. The reason there is one photon is that the Lie algebra u(1) - note that was a small u! - has one degree of freedom. This group is called the unary group, complex numbers with a norm of 1. The members of this group commute, so it does not matter the order things are written in. Quaternions have this property only when all point in the same or opposite directions, which is the case for when using one quaternion times itself.

The continuous group SU(2) rules the weak force, the stuff driving radioactive decay. Mathematically this is call unitary quaternions, quaternions with a norm of 1. The Lie algebra used to generate this group has three degrees of freedom. That is why the weak force is mediated by three particles, the W+, W-, and the Z.

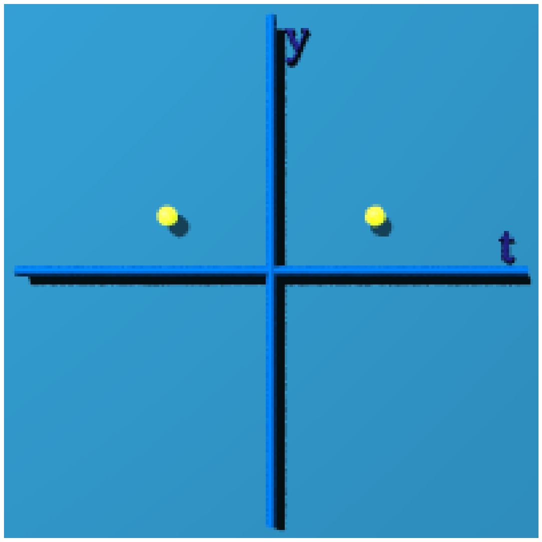
The group SU(3) is for the strong force whose residual interactions keep nuclei together. Its Lie algebra has eight members, and there are eight gluons.

Animations of Groups

Start with a simple picture, layer pictures together, and we will be able to see what the standard model of particle physics looks like.

S0 - So simple!

This is the symmetry of +/-R, one number. What was up/down | What is | What can be
—|—|—


y
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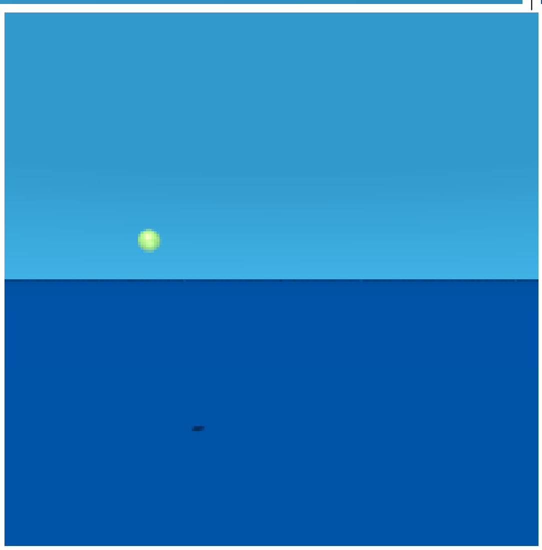

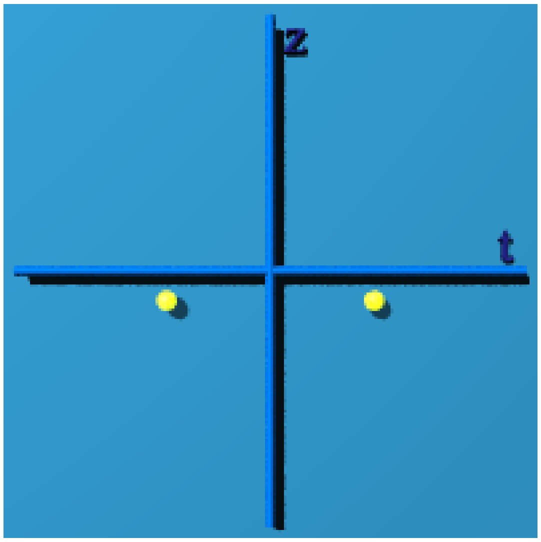

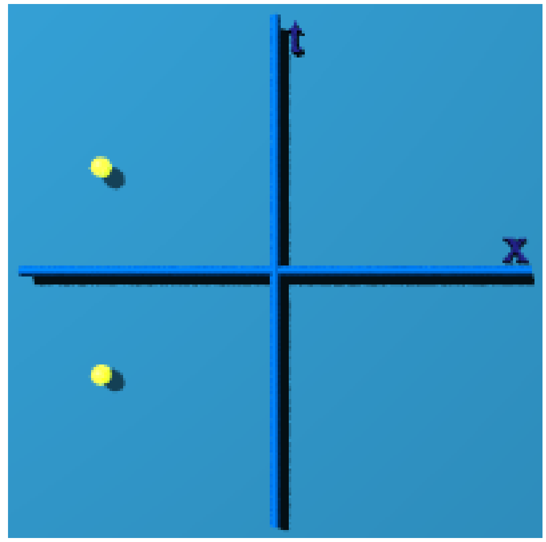

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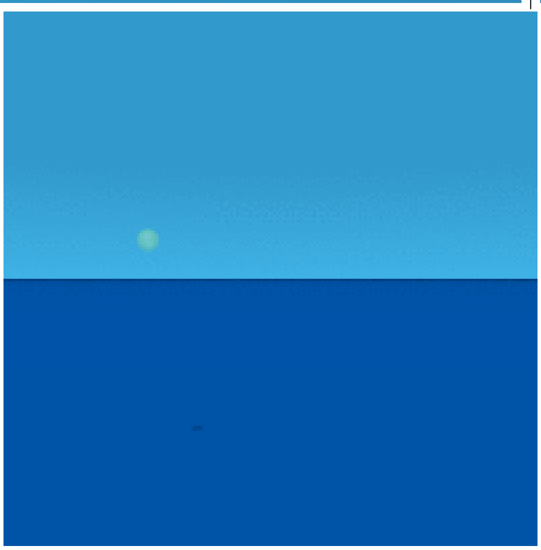
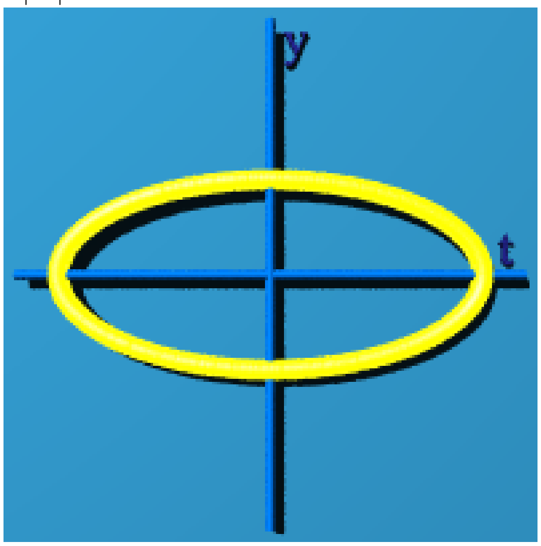
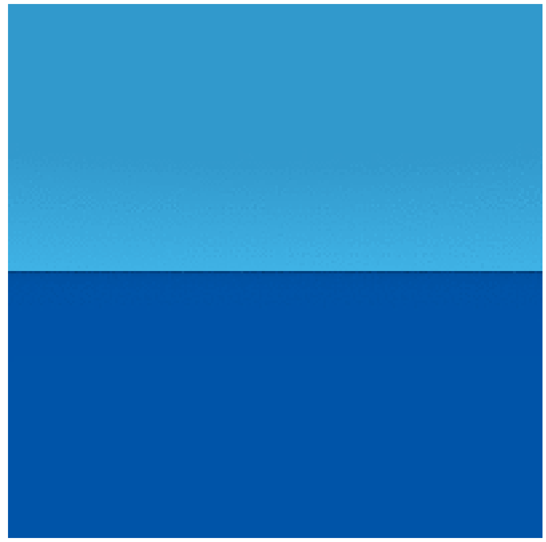
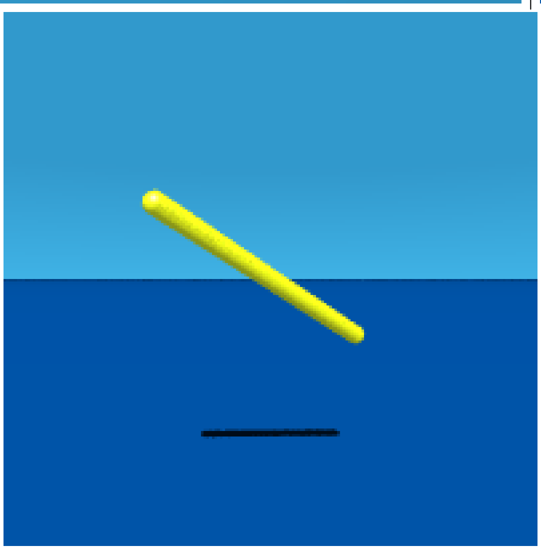

|
What was
near/far | What was
left/right | What can be
that is
This is (-1, 0, 0, 0) and (+1, 0, 0, 0). It sits in the center, as quaternions of the form (n, 0, 0, 0) like to do.

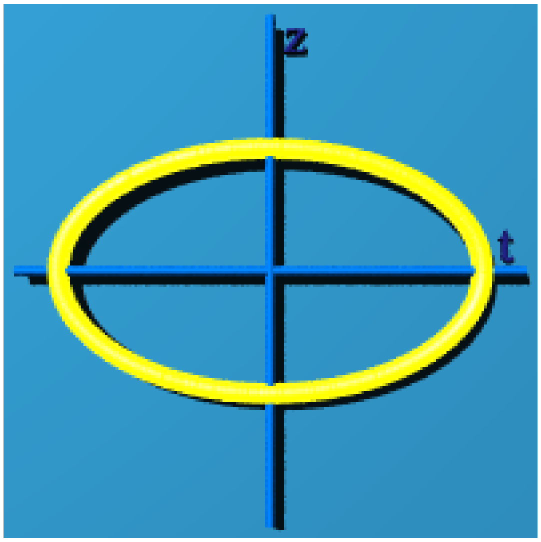
S1 - The Circle

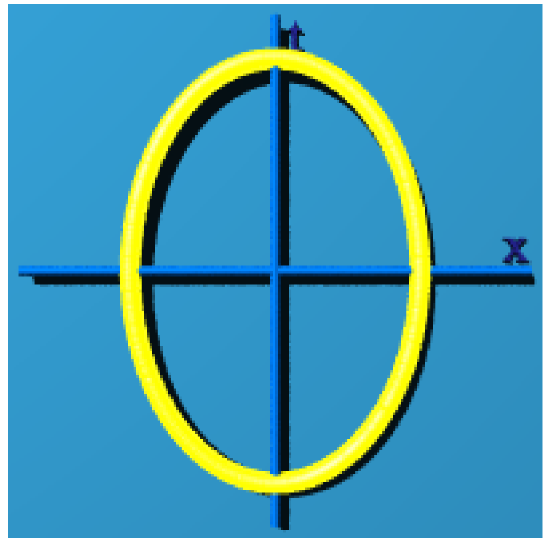
Now we let the sum of squares of 2 numbers equal 1. This creates a circle. What was
up/down | What is | What can be


y
t









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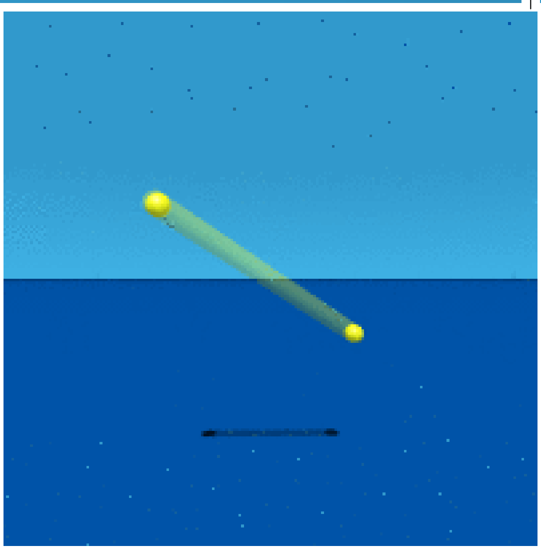
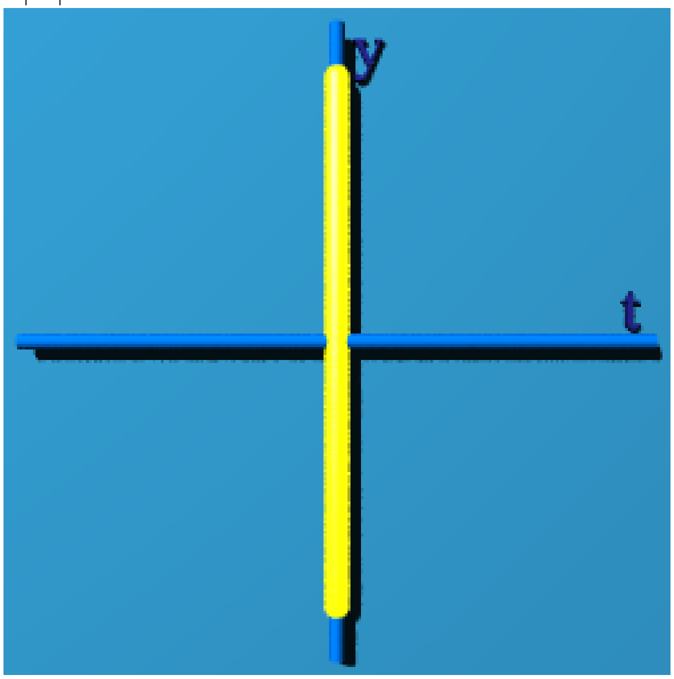
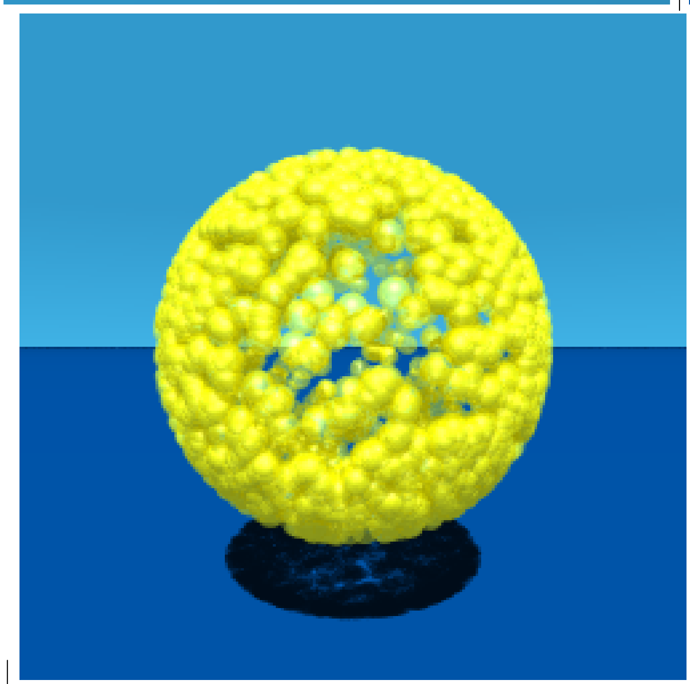
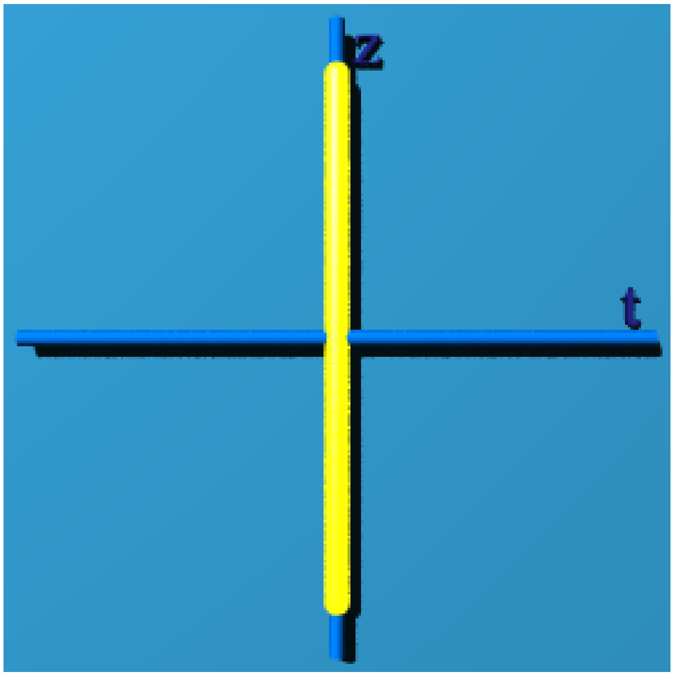
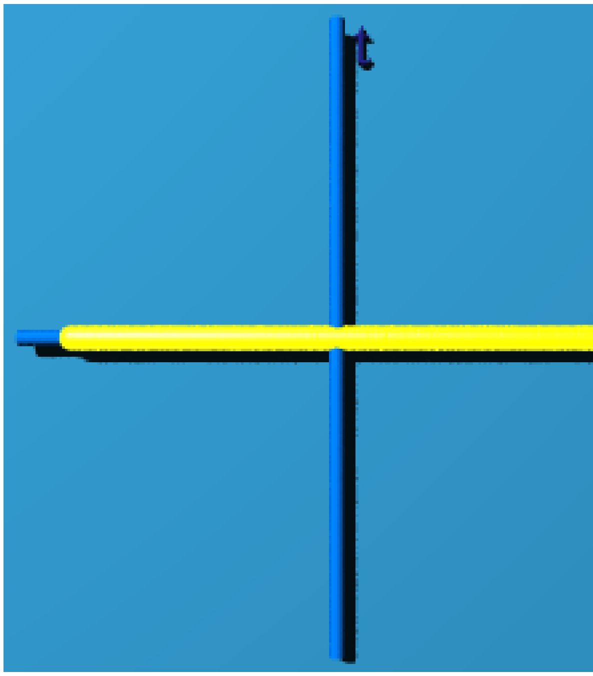

|
What was
near/far | What was
left/right | What can be
that is
The circle could have any orientation in 3D space. The program choose one at random.

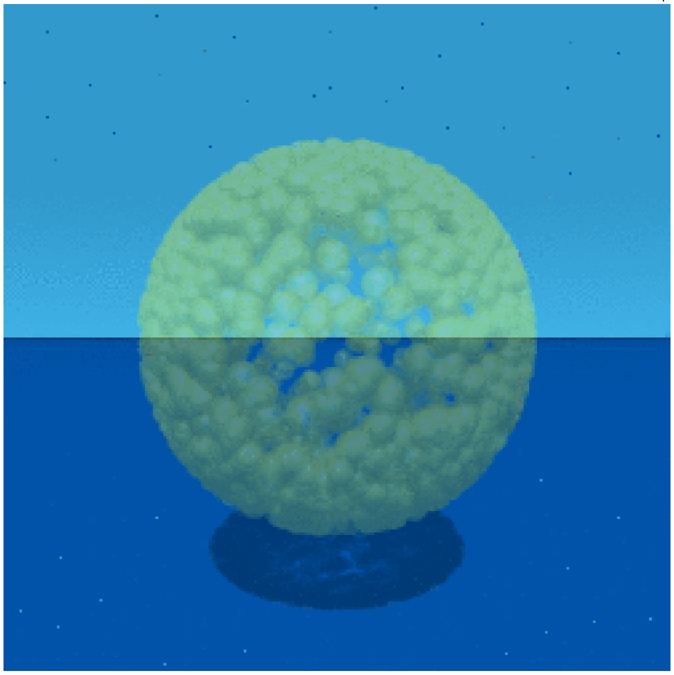
S2 - Slice of an Expanding/Contracting Sphere

The sum of squares of 3 numbers equal 1. A quaternion has four numbers. One approach to representing S2 is to set t=0. You get the standard sphere, but only at the instant of t=0. Blink! What was

up/down	What is	What can be
		



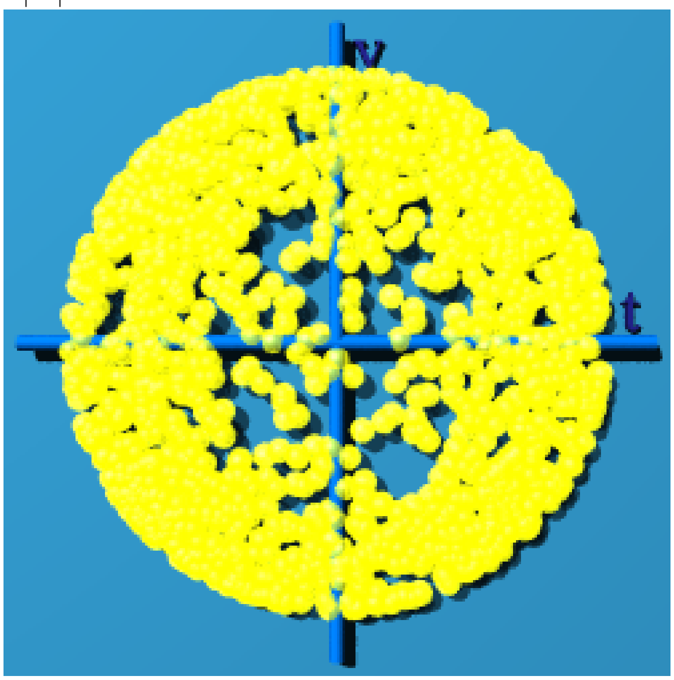
| 

| 

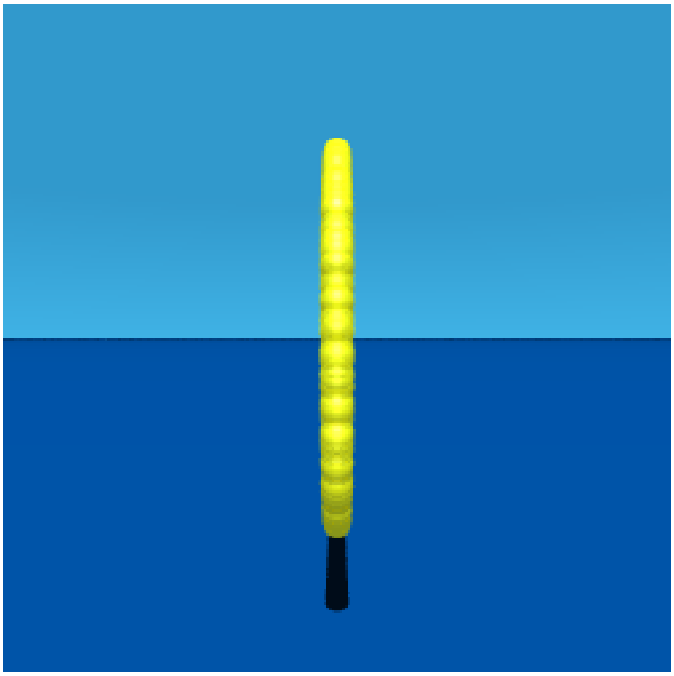
|
What was
near/far | What was
left/right | What can be

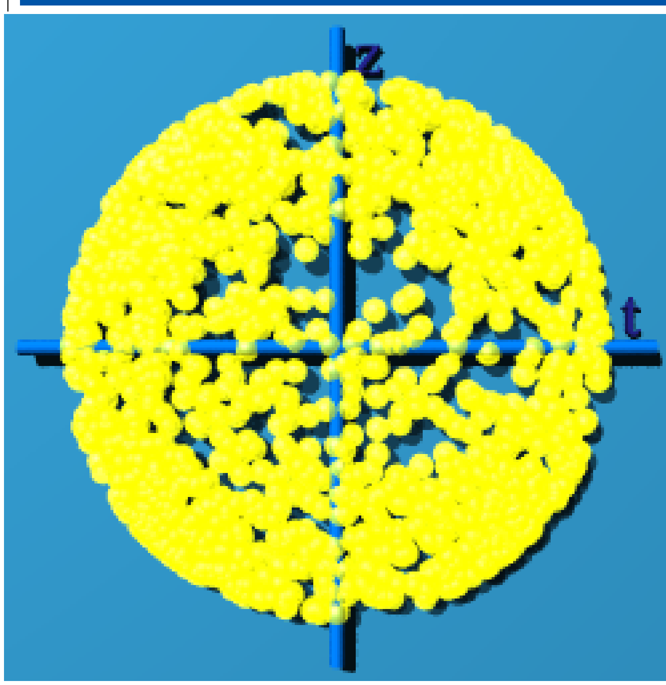
that is
That looks like a typical sphere, except it doesn't last long. Three straight lines appear in the "what was" graph because time is fixed.

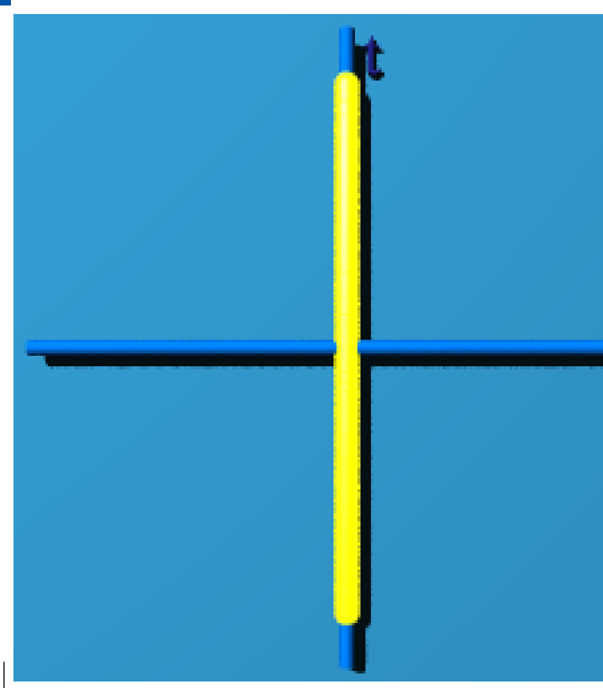
Another way to represent S2 is to set x=0. Then you have an edge view of an expanding circle.

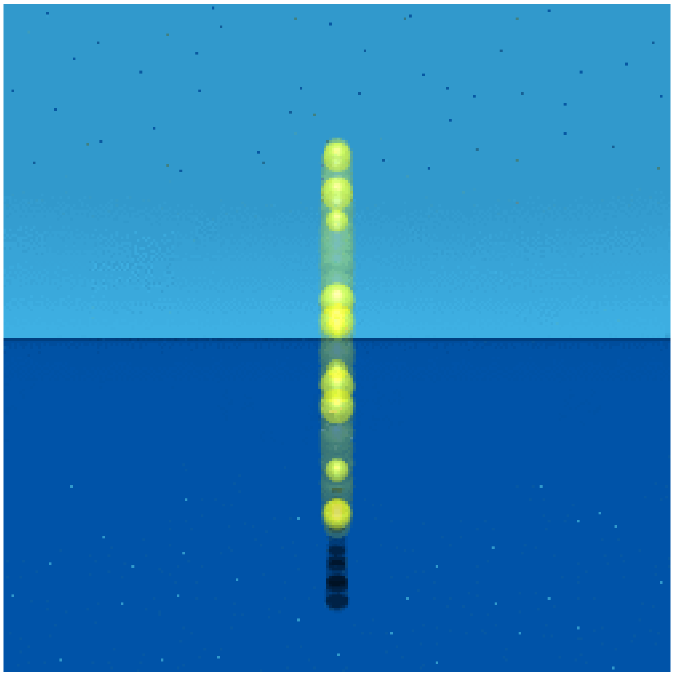
What was
up/down | What is | What can be
—|—|—


|

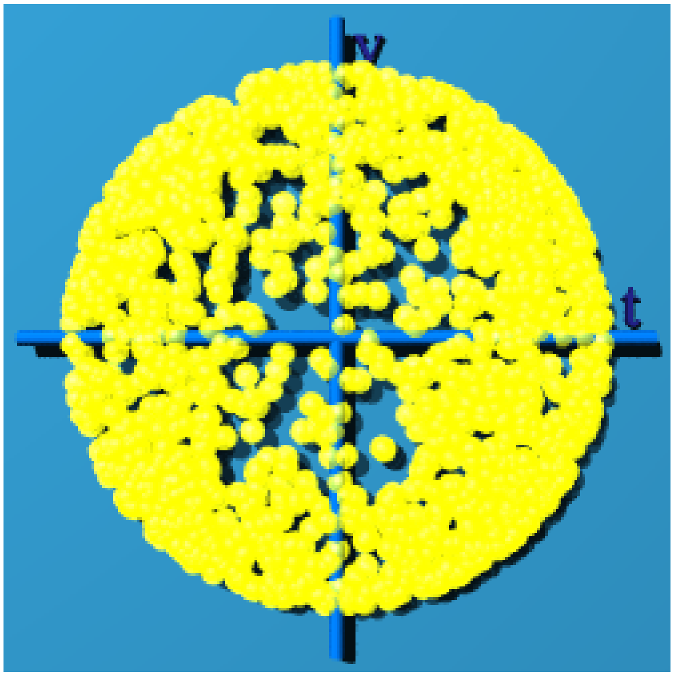

z
t

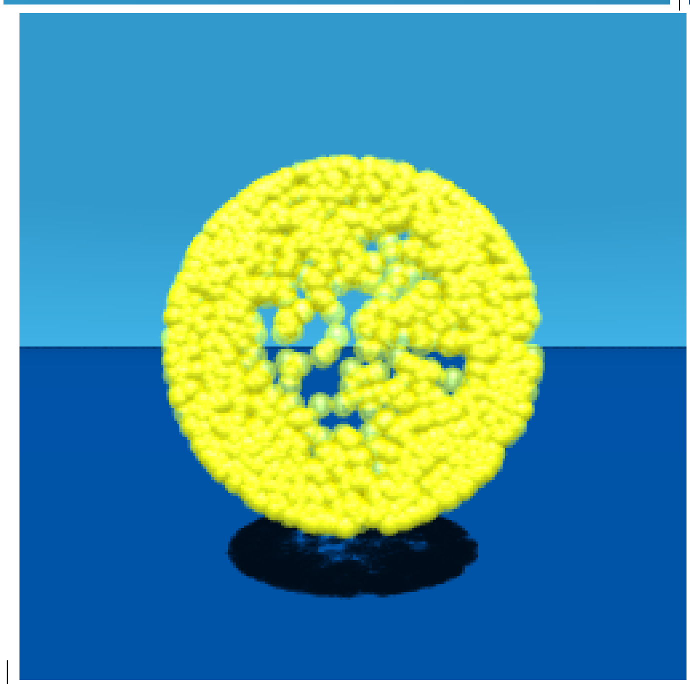

t

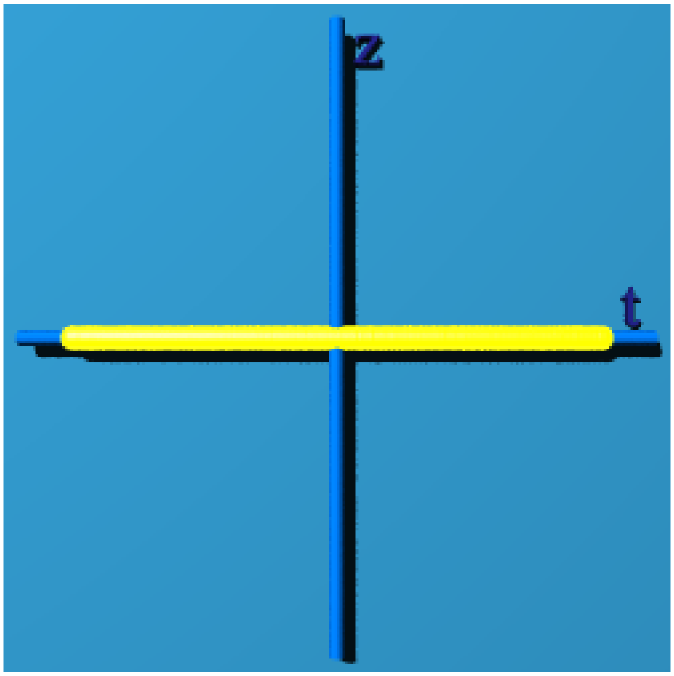
|
What was
near/far | What was
left/right | What can be
that is
Only the “what was left/right” graph has a fixed, straight line graph, because x=0.

If z=0, at least you can see the “circleness”

What was
up/down | What is | What can be
—|—|—


v
t





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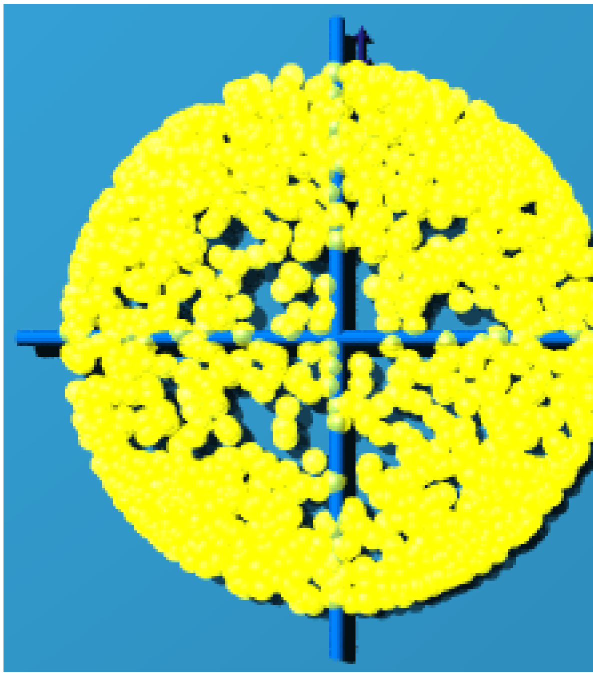
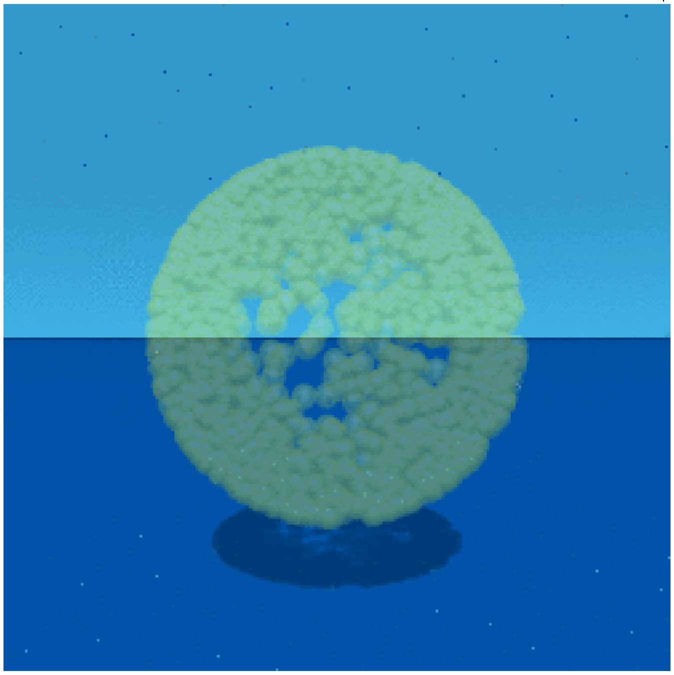
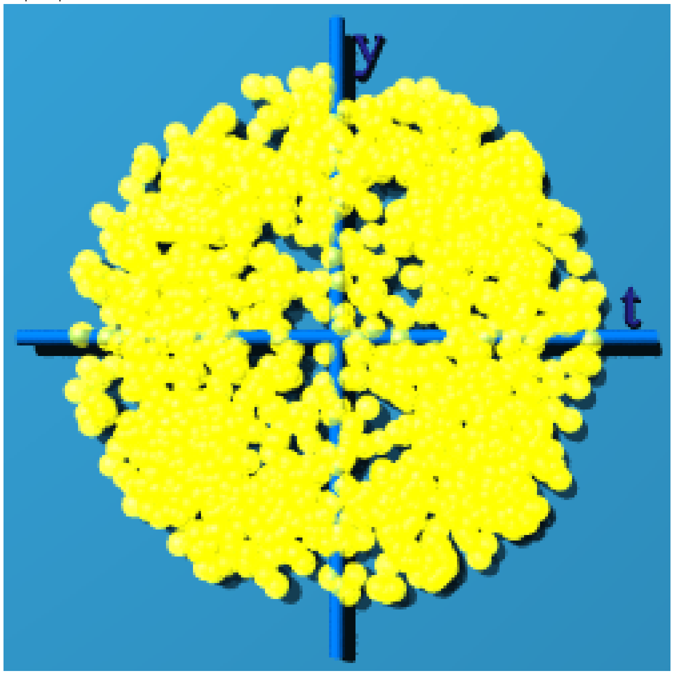



|
What was
near/far | What was
left/right | What can be

that is
In the “what was” graphs, it is easy to spot which dimension is set to zero: it is the straight line.

S3 - A Quaternion Sphere

Now use all 4 terms, and fill in the sphere in both time and space! What was up/down | What is | What can be
—|—|—


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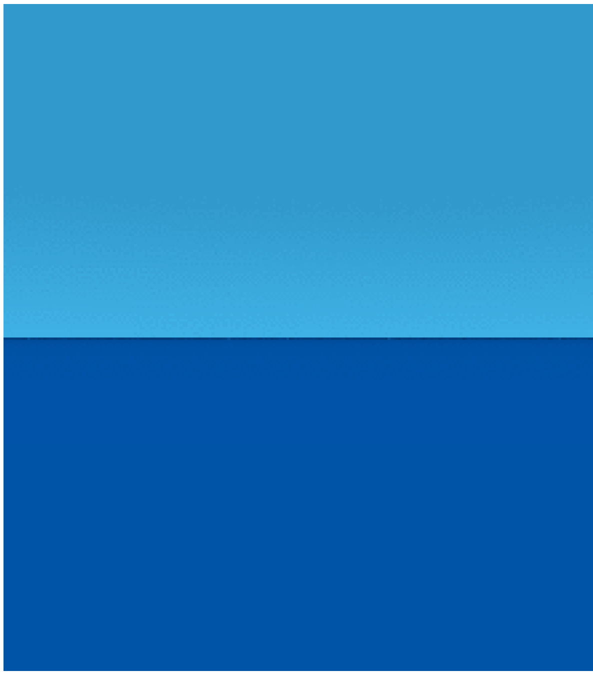

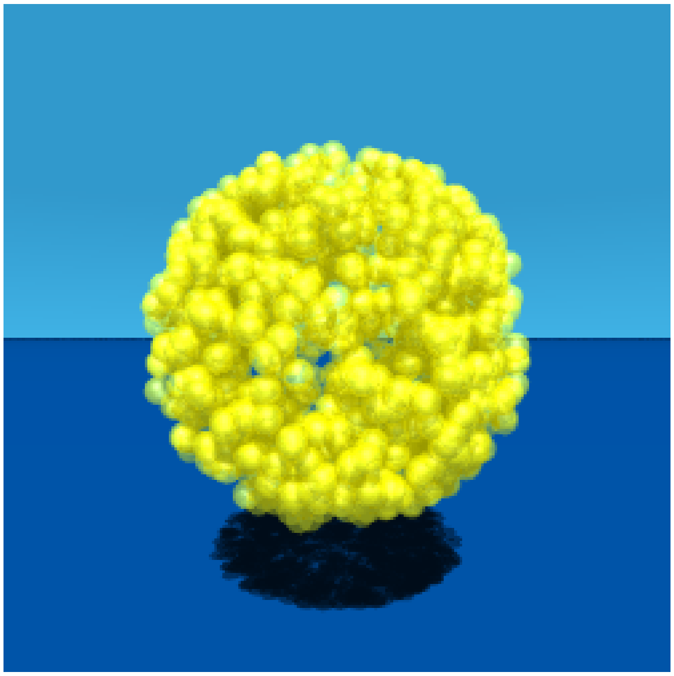

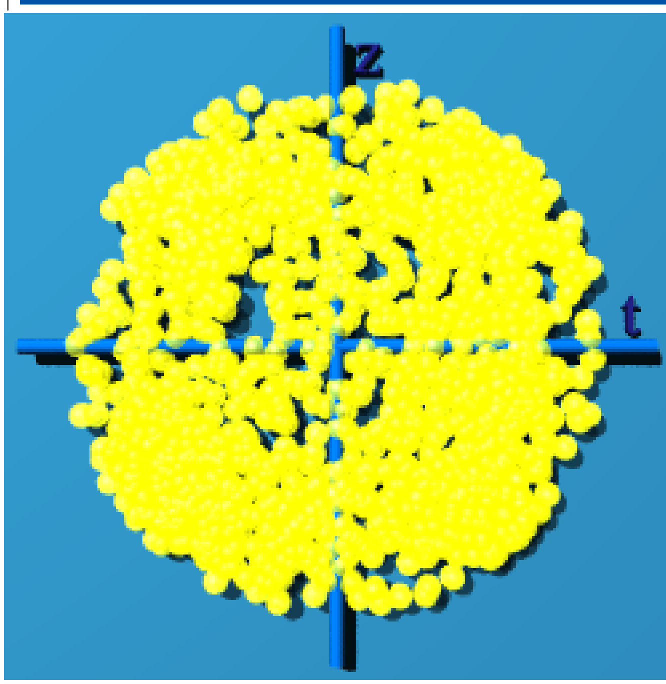
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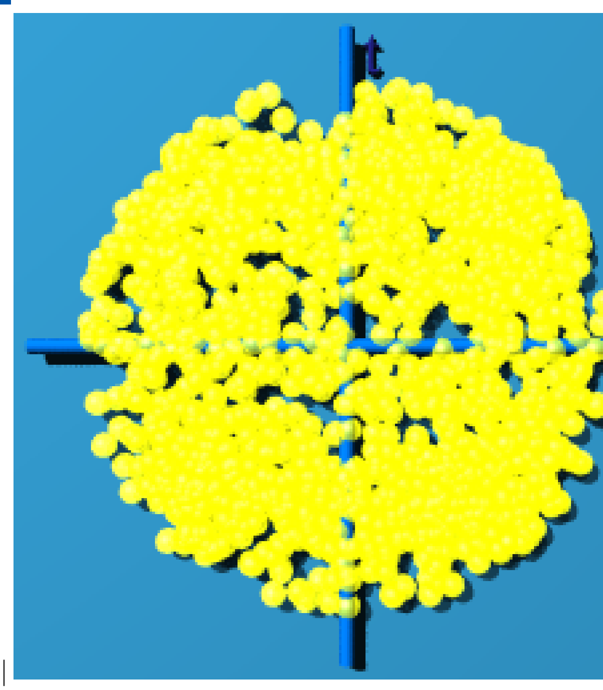
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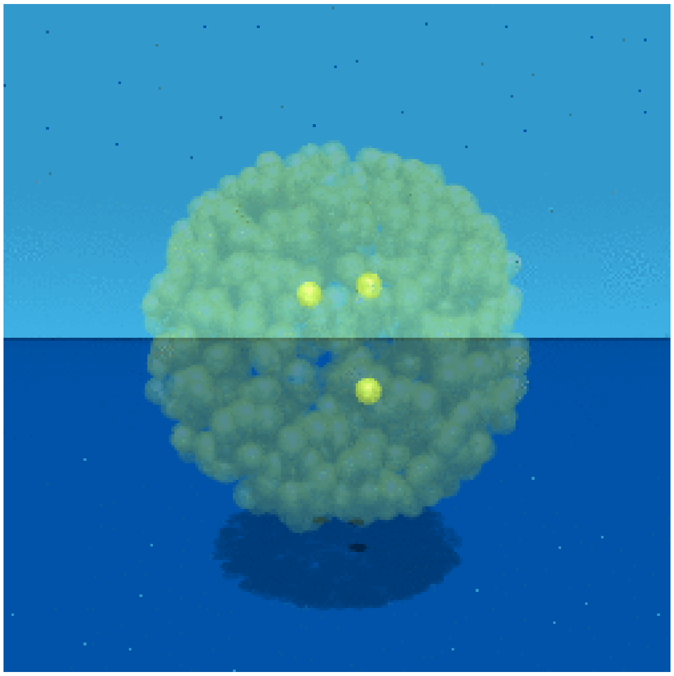
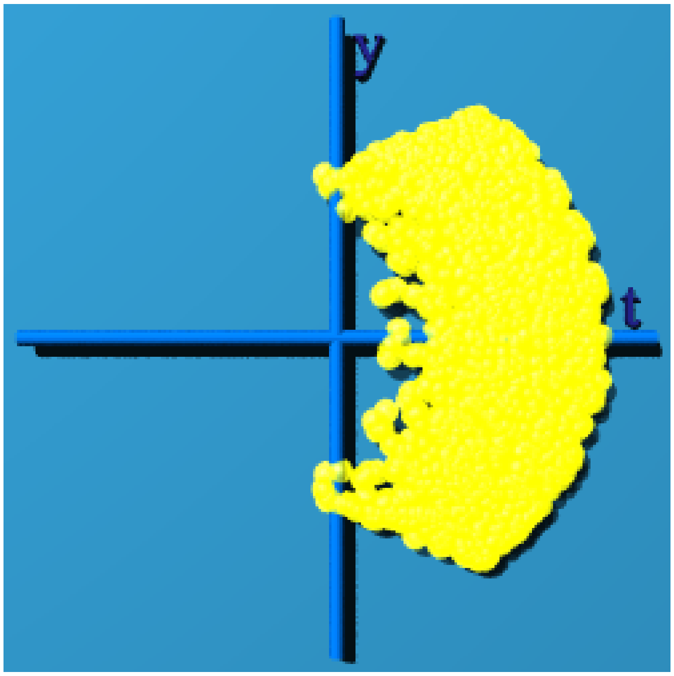


|
What was
near/far | What was
left/right | What can be
that is

SU(2) - Like S2, the Partial Sphere

The graphs of S2 were all paper thin or fleeting. They do not “fill up” spacetime. The next graph, generated by putting random quaternions into the expression exp(q-q*), fills up spacetime. What was
up/down | What is | What can be
—|—|—


y
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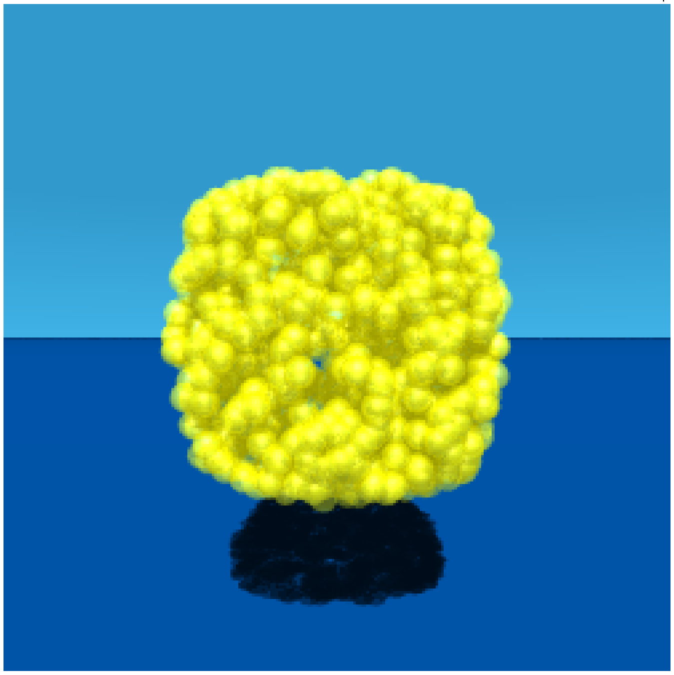

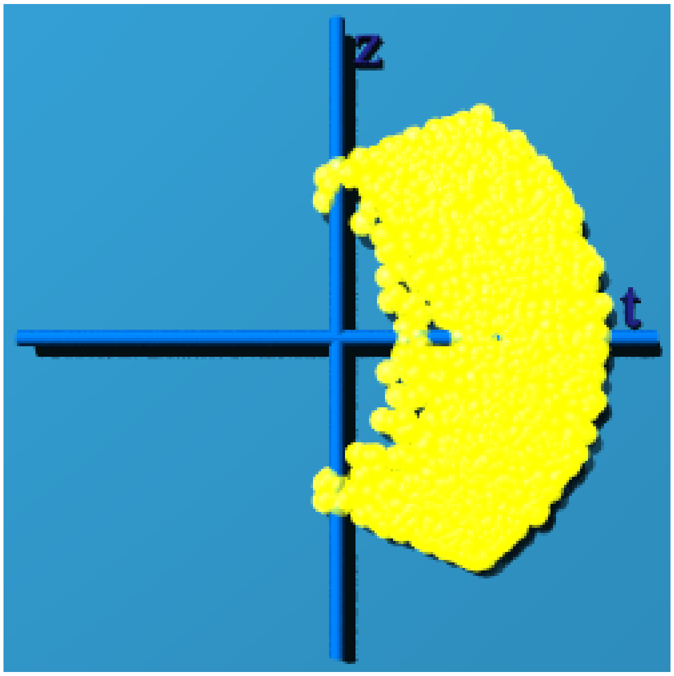

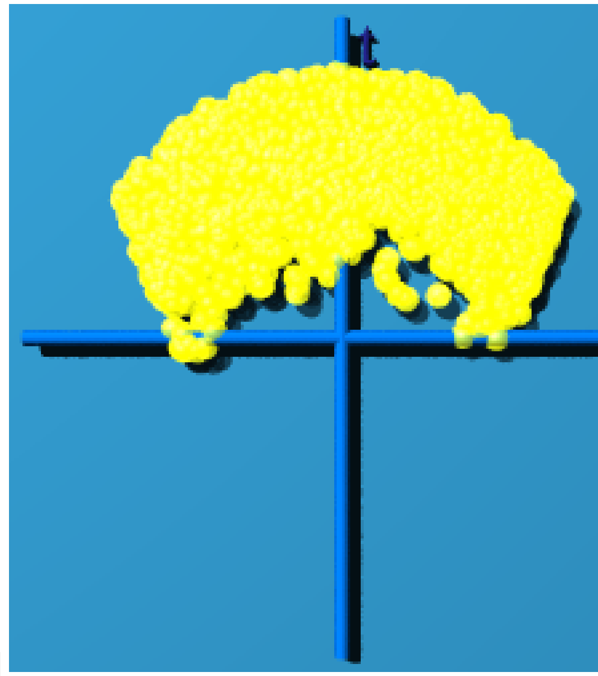

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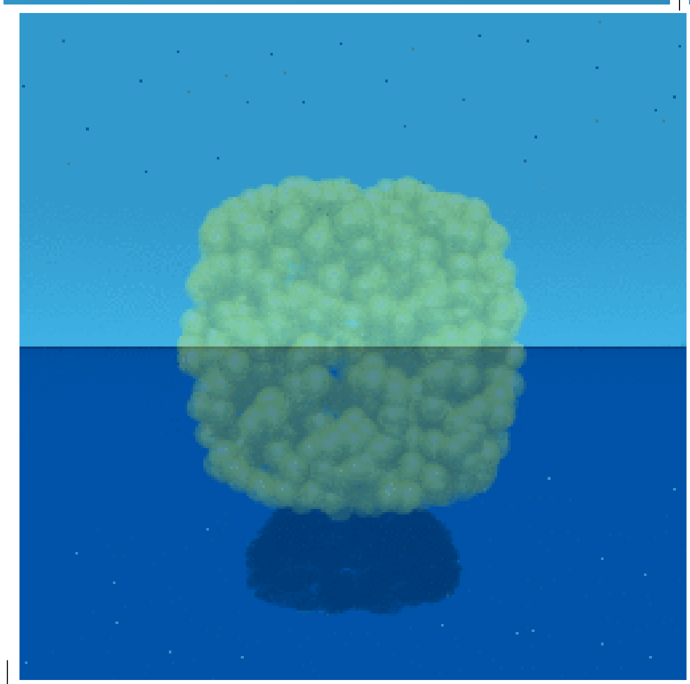

|
What was
near/far | What was
left/right | What can be

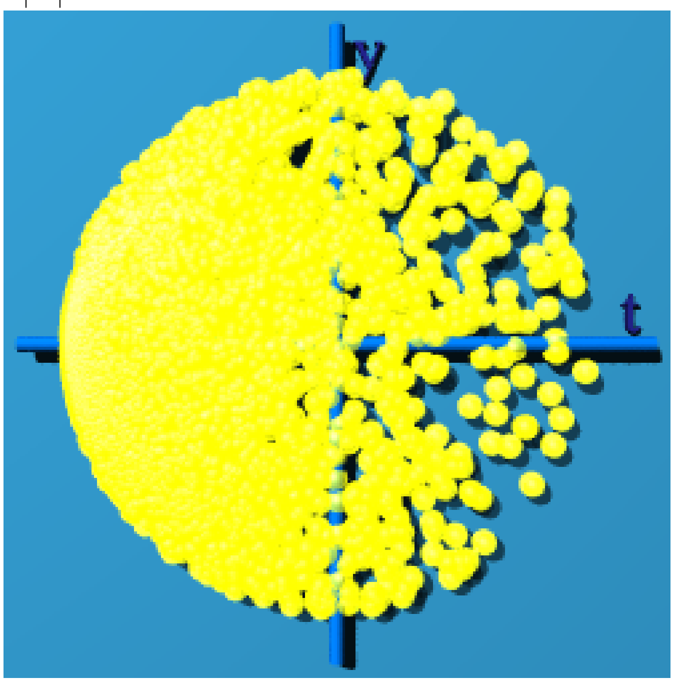
that is

$$\mathrm{SU}(2) \rightarrow \exp(q - q^*)$$

Although more of spacetime appears filled, only places were time is greater than zero have a chance to have an event.

U(1)xSU(2) - Like the Complete Quaternion Sphere S3

SU(2) has only three of the four degrees of freedom available to a quaternion. There is no way to fill up all of spacetime with just SU(2). Now fill spacetime in by multiplying by itself, or q/|q| exp(q-q*) What was up/down | What is | What can be
—|—|—



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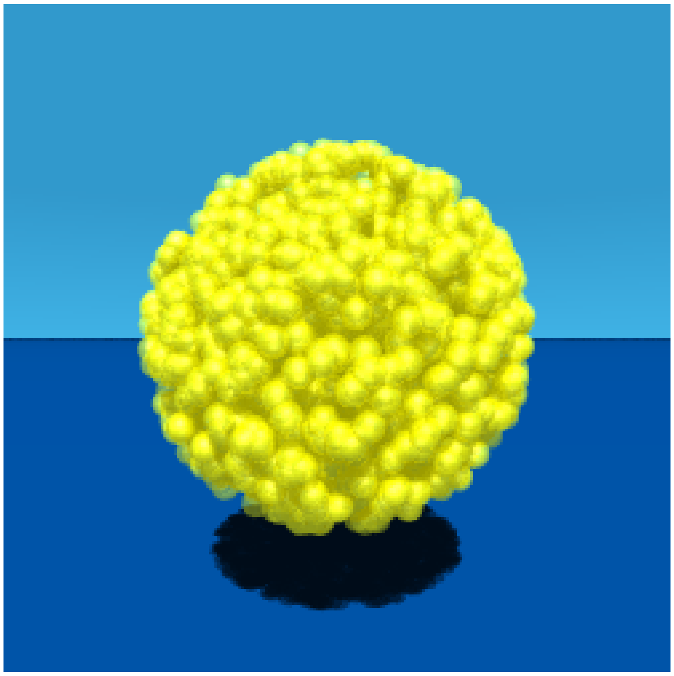

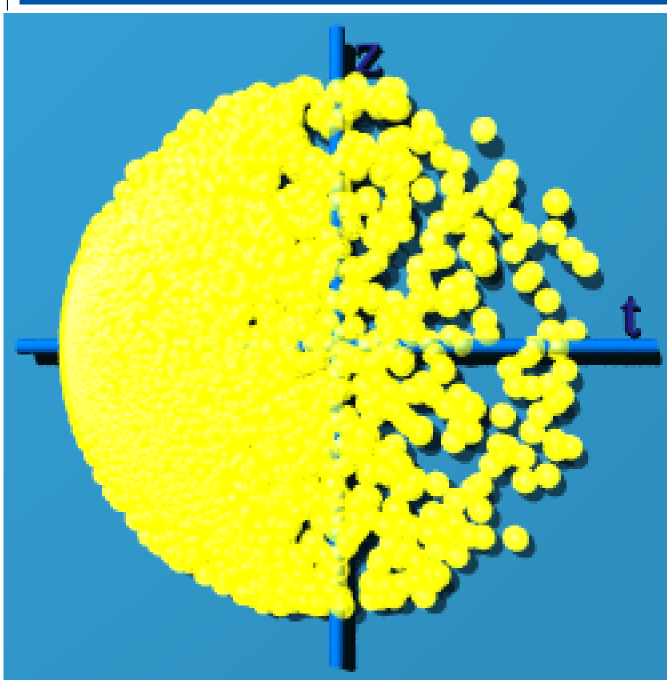
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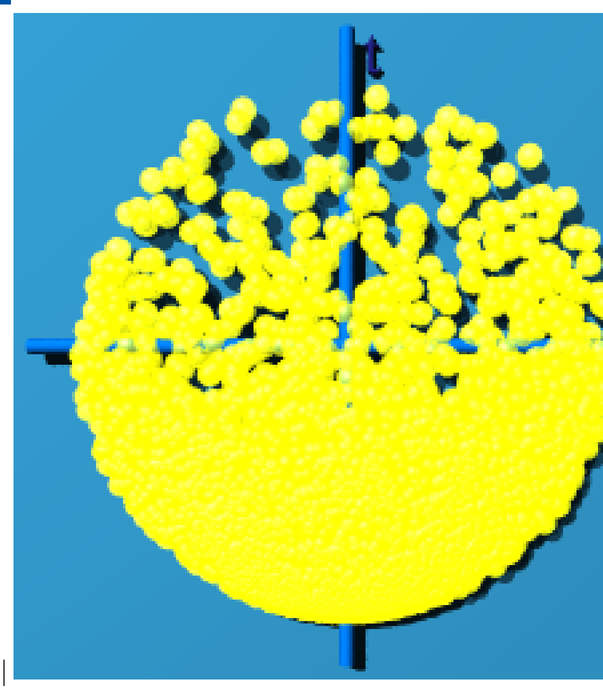
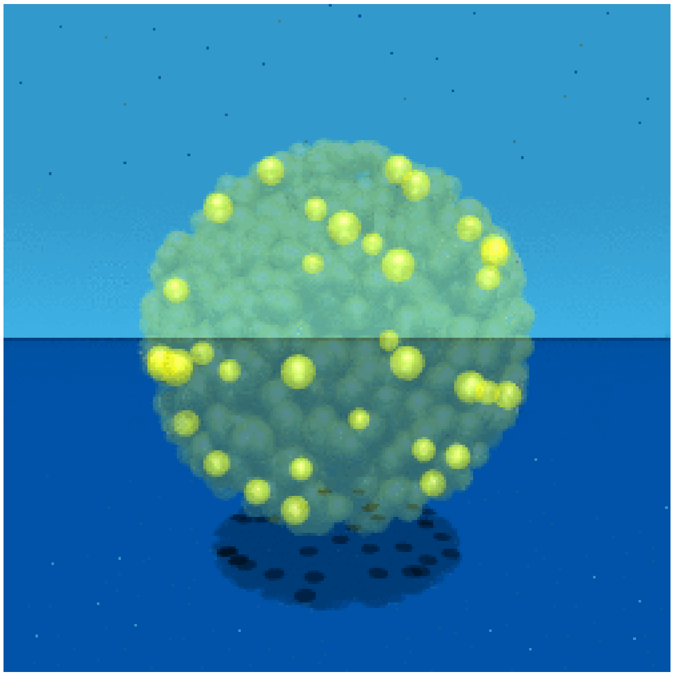

t

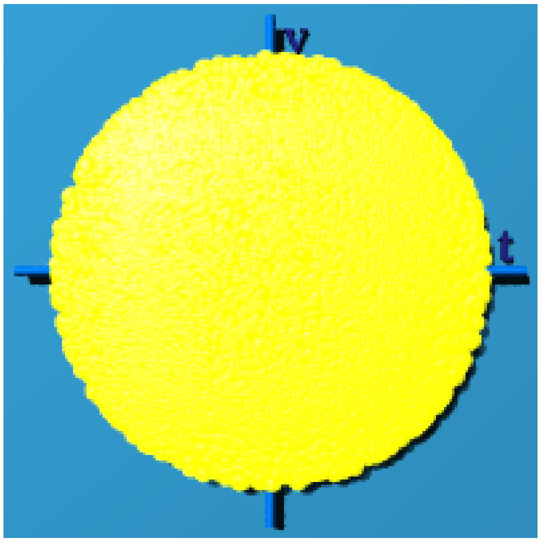

|
What was
near/far | What was
left/right | What can be
that is

$$U(1)xSU(2) \rightarrow \frac{q}{|q|} \exp(q - q^*) = \exp(q - q^*) \frac{q}{|q|}$$

Most of the points cluster on the negative side of the time line.

U(1)xSU(2)xSU(3) - The Standard Model

The question is how to generate SU(3)? It has a Lie algebra su(3) that has eight elements. Based on work done on quaternion quantum mechanics, it is clear I need to work with the conjugate of one quaternion times another, what I call the Euclidean product, because q* q generates the norm of a quaternion q, (t2 + x2 + y2 + z2, 0, 0, 0). If we have 2 different quaternions, q and q', we can write them as q* q' as U(1)xSU(2): (q/|q| exp(q-q*)) (q'/|q'| exp(q'-q'*)). *Here is its animation: What was*
up/down | What is | What can be
—|—|—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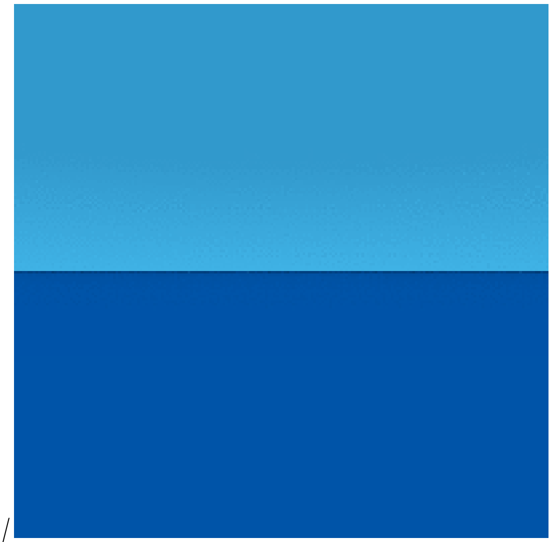

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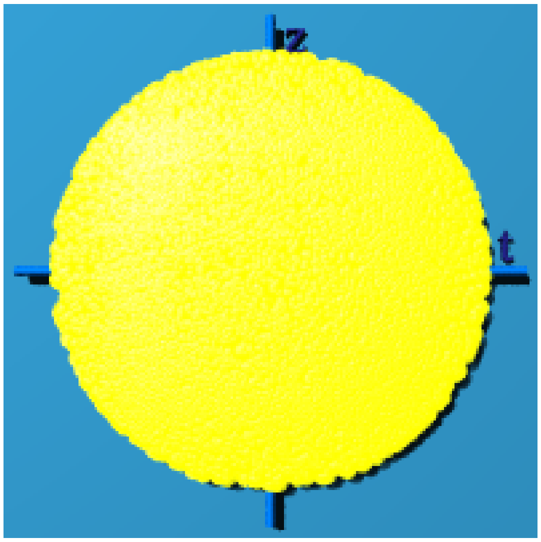

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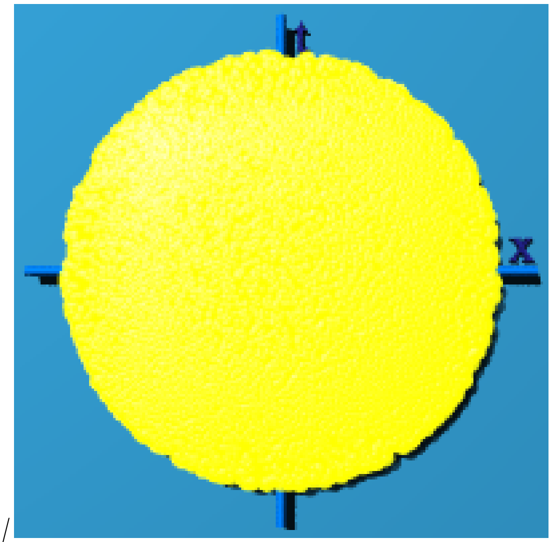

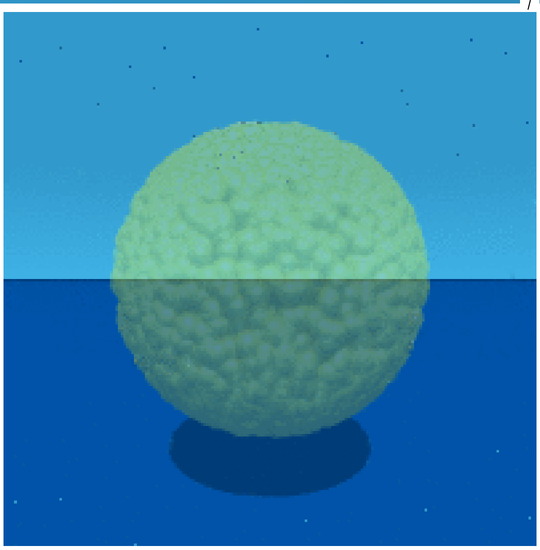
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What was near/far / What was left/right / What can be that is

$$\mathrm{U(1)xSU(2)xSU(3)} \rightarrow \left(\frac{q}{|q|}\exp(q-q^*)\right)^* \frac{q'}{|q'|}\exp(q'-q'^*)$$

*Notice how all of spacetime is filled evenly with events. A product of two quaternions that uses a conjugate different from a standard product because multiplication is no longer associative ((a b)*c does not equal a* (b c)). The norms are preserved, so the norm will remain 1. Eight independent number are used to make something with a norm of one. The identity is 1, and all elements have an inverse under what I call "Euclidean multiplication", q* q'. Based on the animation, the group is compact and simply connected. All of this traits contribute to the conclusion that the symmetry of the standard model can be represented by quaternions in this way.

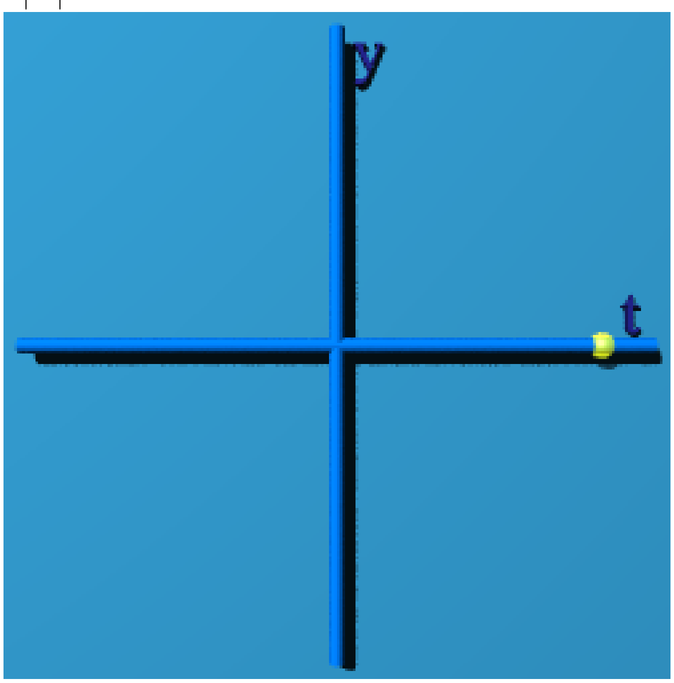
It would be great to include gravity, which is all about how measurements change as one moves around a differentiable 4D manifold. Include the metric as part of the calculation of a quaternion product.

$$\mathrm{Diff(M)xU(1)xSU(2)xSU(3)} \rightarrow g_{\mu\nu}(\frac{q}{|q|}\exp(q-q^*))^{*\mu}\frac{q'}{|q'|}\exp(q'-q'^*)^{\nu}$$

The group Diff(M) is all diffeomorphisms of a compact smooth manifold. It is at the heart of general relativity. One can imagine this spacetime filling sphere on any compact smooth manifold.

What happens if q=q'? That is shown below:

What was
up/down | What is | What can be
—|—|—



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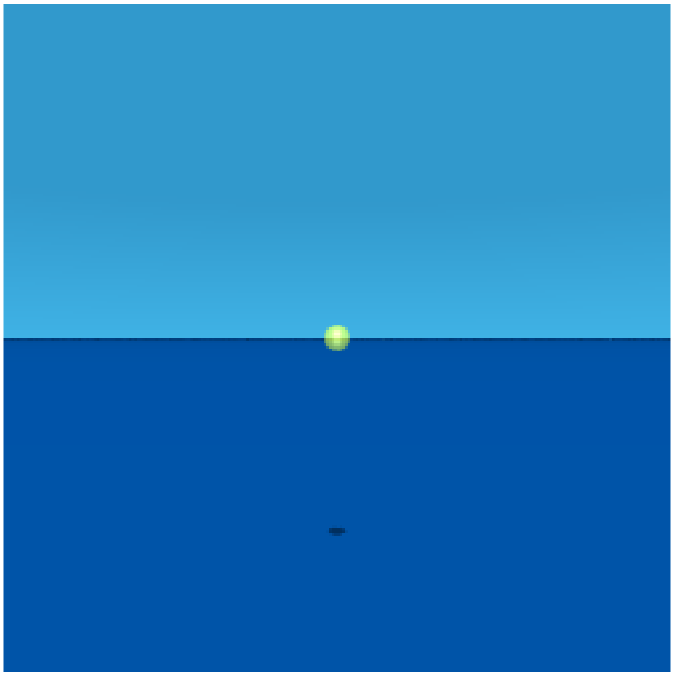

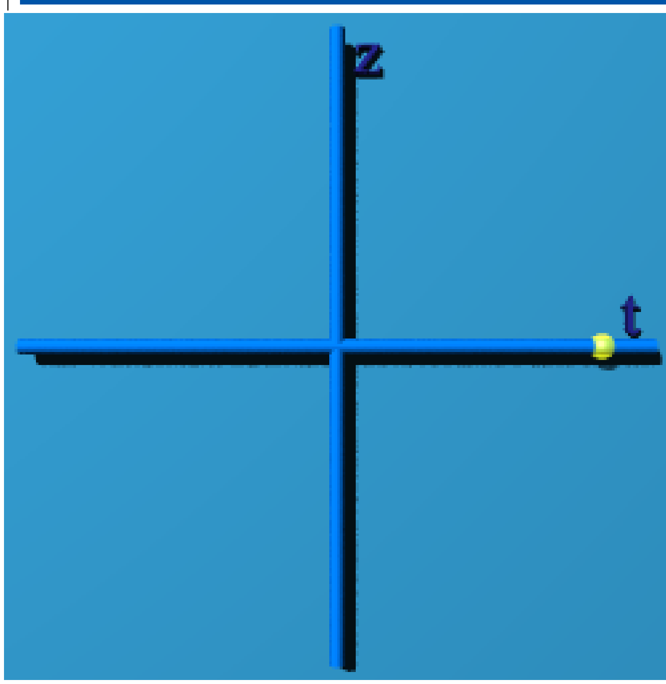
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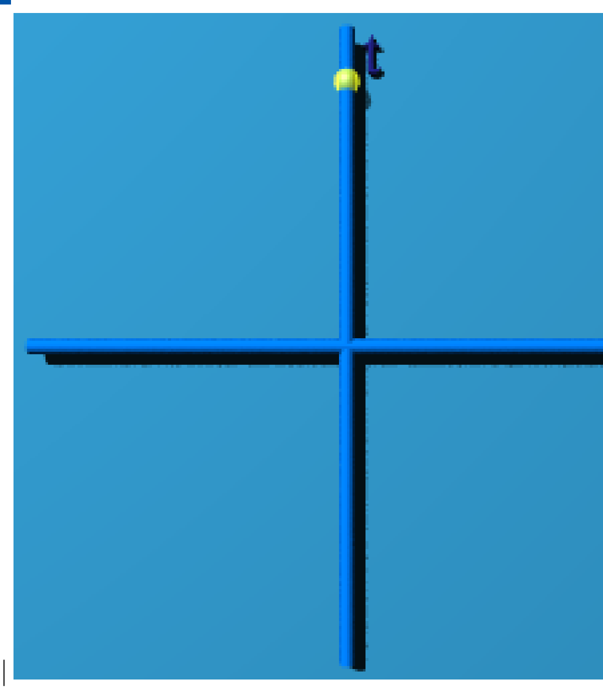
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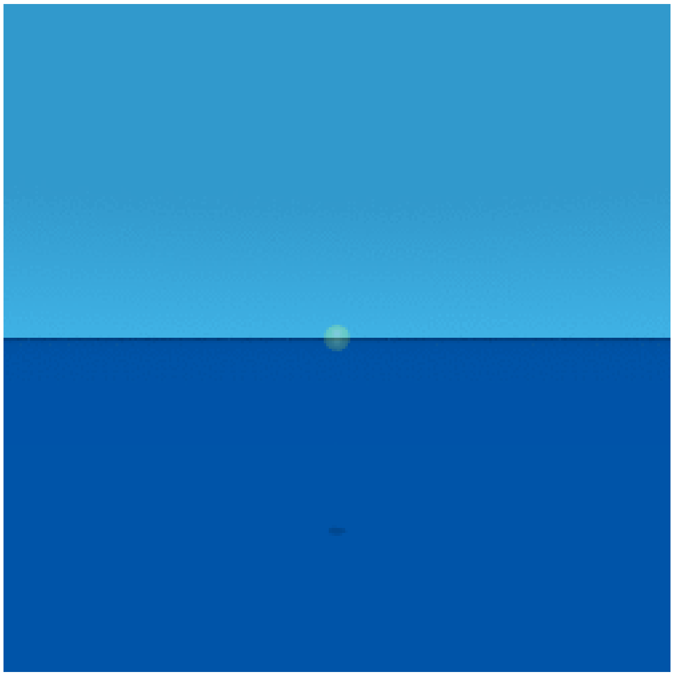


|
What was
near/far | What was
left/right | What can be
that is

The standard model is about the group symmetry of the quaternion multiplication identity in spacetime.

OK, but what does that mean? Here is my take. Observers sit at here-now in spacetime, or numerically at (0, 0, 0, 0). An observer sees something out there, and tries to characterize the "thingie". The basic bit of information it can classify is an event. Whatever set of events is collected, they are all tied up in describing this one thingie out there. Every event contributes to the description of the thingie, and so makes a group. The multiplicative identity of a quaternion, (1, 0, 0, 0) is a way to represent the thingie. Almost none of the events map to (1, 0, 0, 0). The events are scattered all around spacetime. U(1)xSU(2)xSU(3) is the way to cow-rope all the events and bring them home, while remaining part of the same group, the one thing being observed.

Time Reversal Transformations for Intervals

Classical Time Reversal
Relativistic Time Reversal
Implications

The following transformation R for quaternions reverses time:

$$(t, \vec{X}) \to (-t, \vec{X}) = R\,(t, \vec{X})$$

Figure 394: (t, X) goes to (-t, X) = R times (t, X)

The quaternion R exist because quaternions are a field.

R will equal (-t, X)(t, $X)^{-1}$. The inverse of quaternion is the transpose over the square of the norm, which is the scalar term of the transpose of a quaternion times itself.

$$R = (-t, \vec{X})\,(t, \vec{X})^{-1} = (-t^2 + \vec{X}.\vec{X},\ 2\,t\,\vec{X}) \,/\, (t^2 + \vec{X}.\vec{X})$$

Figure 395: R = (-t, X)times(t, X)inverse= (-t squared + X dot X, 2 t X) over (t squared + X dot X)

For any given time, R can be defined based on the above.

Classical Time Reversal

Examine the form of the quaternion which reverses time under two conditions. A interval normalized to the interval takes the form (1, beta), a scalar one and a 3-vector relativistic velocity beta . In the classical region, beta«<1. Calculate R in this limit to one order of magnitude in beta.

$$R = (-t, \vec{\beta})\,(t, \vec{\beta})^{-1} = (-t^2 + \vec{\beta}.\vec{\beta},\ 2\,t\,\vec{\beta}) \,/\, (t^2 + \vec{\beta}.\vec{\beta},\ 0)$$

Figure 396: R = (-t, Beta)times (t, Beta) inverse = (-t squared + Beta dot Beta, 2 t Beta) over (t squared + Beta dot Beta, 0)

The operator R is almost the negative identity, but the vector is non-zero, so it would not commute.

Relativistic Time Reversal

For a relativistic interval involving one axis, the interval could be characterized by the following:

$$\text{if } \beta << 1 \text{ then } R \approx (-1, 2\,t\,\vec{\beta})$$

Figure 397: if beta is much much less than 1 then R is approximately equal to (-1, 2 t Beta)

$$(T + \epsilon, T, 0, 0)$$

Figure 398: (T + e,T, 0, 0)

Find out what quaternion is required to reverse time for this relativistic interval to first order in epsilon.

$$R = \left(\frac{T^2 - (T+\epsilon)^2}{T^2 + (T+\epsilon)^2}, \frac{2\,T\,(T+\epsilon)}{T^2 + (T+\epsilon)^2}, 0, 0\right) = \left(-\frac{\epsilon}{T} + O[\epsilon]^2, 1 + O[\epsilon]^2, 0, 0\right)$$

Figure 399: R= (t squared -(T+ e) squared ,2 T (T+e), 0, 0) all over t squared + (T + e) squared

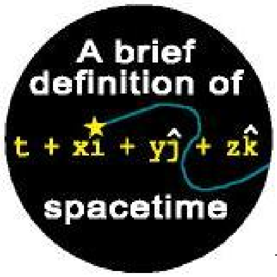
This approaches q[-e/T, 1, 0, 0], almost a pure vector, a result distinct from the classical case.

Implications

In special relativity, the interval between events is considered to be 4 vector are operated on by elements of the Lorentz group. The element of this group that reverses time has along its diagonal
{-1, 1, 1, 1}, zeroes elsewhere. There is no dependence on relative velocity. Therefore special relativity predicts the operation of time reversal should be indistinguishable for classical and relativistic intervals. Yet classically, time reversal appears to involve entropy, and relativistically, time reversal involves antiparticles.

In this notebook, a time reversal quaternion has been derived and shown to work. Time reversal for classical and relativistic intervals have distinct limits, but these transformations have not yet been tied explicitly to the laws of physics.

Buttons



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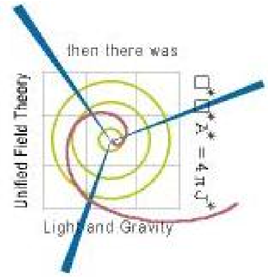

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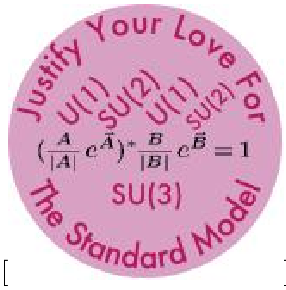

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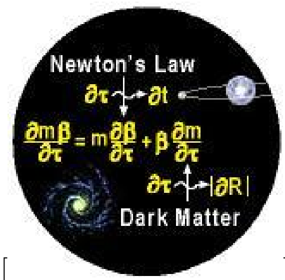

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Promote the Doing Physics with Quaternions Project, and look cool :-)

These 5 buttons represent a visual presentation of some of the key ideas behind my efforts to unify gravity and electromagnetism.

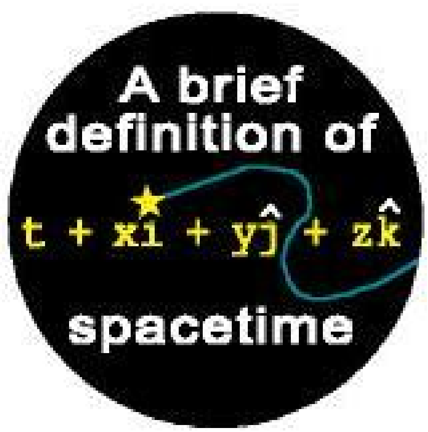

Figure 400: Definition of Spacetime

Fine cuisine depends on starting with the best ingredients. Great physics depends on using new powerful math. An event in space-time has one dimension for time and three dimensions for space, the most powerful tool would be a generalized number that can be added, subtracted, multiplied and divided, but has four parts to it. There is one such number called a quaternion, shown here. Time is t, and x, y, and z are the values the event may have in the three directions, i, j, and k. Quaternions are the mathematical foundation for all my efforts in theoretical physics.

Figure 401: The General Equivalence Principle

A person standing on a scale in a closed box could not tell the difference between the box sitting on the surface of the Earth or the box accelerating at the same rate as the Earth's gravity in a rocket ship as depicted here. Einstein called this the equivalence principle, but it only applied to mass, not charge. The General Equivalence Principle extends this idea of fooling observers to cover any measurement. The box mathematically means "all the changes in time and space." One can ask, what are all the possible changes in the direction rulers (I) and the potential (A), or Box IA? The usual answer is the rulers are fixed and it is the potential that changes, I Box A. However, the answer could also be the potential is the same, but the rulers are changing, A Box I.

A unified field theory describes light and gravity in the same equation. Einstein spent the last half of his life looking for one, but did not succeed. The equation on the right reads in mathlish, "The change in the change in the potential equals some constants times the source." The stars are involved in games with plus and minus signs. It takes work to see that the signs are correct. A unified field theory must be able to describe all forms of change. The graphic tries to depict this: changes in a square grid, changes moving straight out from the center, changes that are circular ripples, and changes that are spirals. The graphic does not show changes in time or the third dimension. It is only a graphic after all.

The standard model is used to quantify why there are this many of that kind of

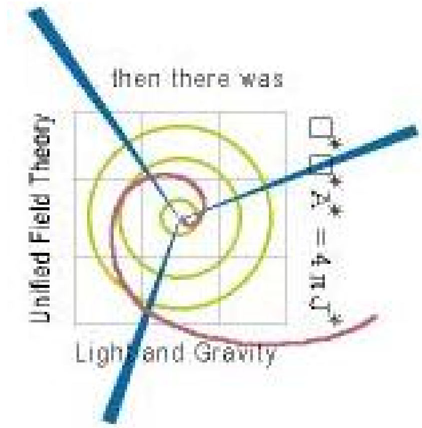

Figure 402: Unified Field Theory

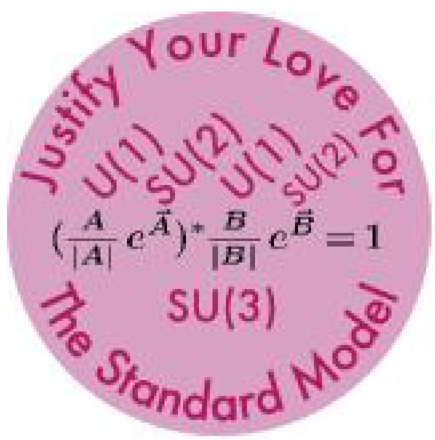

Figure 403: Justify Your Love for the Standard Model

subatomic particle, and that many of those particles. A vast amount of exquisite detail is explained. No one has justified why the standard model should be this way. The standard model is constructed from what are called symmetry groups which go by the names U(1), SU(2), and SU(3). The quaternion unified field theory can be written in a way that lines up perfectly with two of these symmetries. The connection to the SU(3) symmetry is unclear now, although technically it has the right number of fingers and toes (eight in this case).

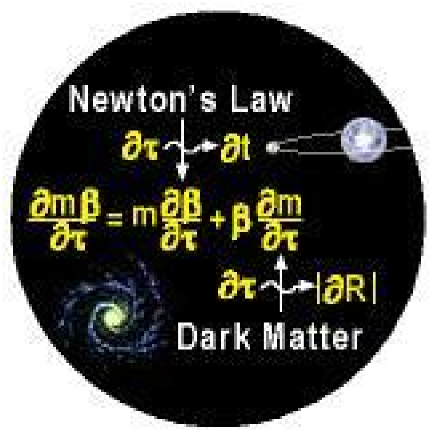

Figure 404: No Dark Matter Needed

Newton's law of gravity works to explain the rotation of the Moon around the Earth. Newton's law of gravity works to explain the velocity of stars near the center of a galaxy, but fails to explain why stars further from the center maintain that same velocity. One hypothesis is called dark matter, stuff that cannot be seen but makes the velocities work out just so. All physics laws must respect relativity and the rules of calculus. These rules create the second term which has a constant velocity and accurately describes the distribution of mass with respect to the radius, no dark matter required.

PDFs for further reading

This site

The entire site is available a number of ways.

Avaible for $10 is a pdf, epub, or mobi file.

Put the site on your own computer. At the command line, run:

git clone https://github.com/dougsweetser/Q.git

A collection of 6 problem sets solved using only real-valued quaternions is available for $1.99 as a pdf, epub, or mobi file

Bits of this site

1. Space-times-time invariance as gravity in 2 pages - 2 pages
2. Where quaternions fit - 1 page
3. Space-time equivalence classes - 10 pages
4. Space-time Geometries - 1 page
5. Quaternion space-times-time invariance as gravity - 7 pages
6. Deriving the Maxwell source equations by hand - 1 page
7. PGT - Personal Gravity Theory Homework Assignment - 1 page

Other people's efforts

1. Gauss's notebook - Written in 1819, published 1900, 6 pages. All calculations, no discussion.
2. Sudbery's first paper, "Quaternion Analysis", memo, 1977, 44 pages, on why quaternion analysis is no good.
3. Sudbery's second paper, same title, 1979, 28 pages on the topic. Please look to my work above on quaternion analysis for a much better alternative!.
4. C. A. Deavours paper, "The Quaternion Calculus". My critique is that using his definition of a quaternion derivative, if a function like f=q is analytic in q, f^2 is not. That indicates a better definition must be found before quaternion analysis can really begin.
5. Salamin's paper, "Application of quaternions to computation with rotations", (1979, 9 pages) on rotations. Howell and Lafon's paper, 1975, 13 pages, on the efficiency of quaternion multiplication.
6. Silberstein's paper, "Quaternionic Form of Relativity", 1912, 20 pages, on using biquaternions for quaternion special relativity. Biquaternions are NOT a division algebra, and are not used in any operations on this web site.

Gimble lock

Resources on gimble lock, a problem that arises from not using quaternions for 3D rotations, links provided by Alex Green:

1. Apollo
2. A fourth gimbol for Christmas
3. skylab
4. Gimble Lock - Explained.

Bottom line: always avoid Euler angles.

Personal Sketch of Doug Sweetser

My father went to Harvard University and Harvard Law School as his father did. He read voraciously. My mother went to Boston University getting a degree in Physical Therapy. They had one poor date in a canoe. While in Europe, my father tracked down my mother, and with nearly all the details lost to me, wooed her accross the pond.

I was born in Bronxville, New York, in 1962. We had both a stay at home mom and live in maid. My father worked at the New York Times a corporate lawyer. When I was five, he got a job at the Minneapolis Star and Tribune. The family moved into a house with almost 10 acres of land. I went to a The Blake School an all boys private school, until it became the plural Blake Schools and coed in seventh grade (great timing). My education continued at M.I.T. where I got degrees in Biology and Chemical engineering. Note: I did not get a degree in either mathematics or physics.

I am a solid core nerd. A hard core nerd has strong opinions about all the versions of Star Trek. A solid core nerd does not watch the show because the science is so impossibly wrong. In Star Trek, space is treated like Manhattan: go to the next stop, and a wonderfully different collection of people are there. One can easy go from Wall Street Suits, to Chinatown, to Uptown, to Harlem, Spanish or otherwise. Space in the Universe is really just space, with distances too far for people to travel.

Professional sketch of Doug Sweetser

Education

MIT, 1980-1984 SB: Biology SB: Chemical Engineering GPA: 4.0 (I should point out MIT is on a 5.0, so 4.0 sounds great)

MCD Biology - University of Colorado Boulder, 1987-1989 Ph.D. Program Left on own accord.

Math Department - University of Indiana, Bloomington, 1990 Graduate level courses Left on own accord.

Brandeis, 2000-2002 MS: Computer Software Engineering

##Work Experience

Biology

My first job was as a lab tech at the newly opened Whitehead Institute for Biomedical Research. On the first week on the job, Prof. Rick Young showed me how to clone the first genes ever from the Mycobacteria that causes leprosy. The work was featured on a NOVA documentary. Twelve technical papers were written in a three year paper in the most important journals in biology, including Nature, Cell, and the Proceedings of the National Academy of Sciences (PNAS).

Worked as a lab tech for Jac Nickoloff at the Harvard School of Public Health (8 years). My heart was elsewhere (which caused problems on the job).

Computers

The first Internet Service Provider (ISP) was started in Brookline, MA in 1989. They were my first ISP. After leaving the biology business, I worked the phones in a support role. I also helped maintain the billing software written in Perl. The shifting landscape of ISPs lead then to downsize in 2003.

I was the 9th employee of Black Duck Software. They initially wanted to help companies understand the licensing requirements behind open source software. I was hired as a "spider", crawling the Internet and harvesting open source software for our knowledgebase.

Physics

Initial interest

In the Christmas of 1988, both my mother and sister independently bought the book: "A Brief History of Time" by Stephen Hawking. I view that as my "born again" moment, one based on physics, not the Bible. For a month long period, my mind kept rearranging information, sure that everything somehow made sense. I was aware I was babbling - saying words that are part of the lexicon of physics, but would not make sense to an actual physicist. I decide to keep a day job, but work study physics in the background.

Boston was the perfect city in the 1990s to study physics on the side. I took a Harvard Extension class on Special Relativity taught by Edwin F. Taylor as he was writing the book “Spacetime Physics” with Johnathon Wheeler. Taylor would bring in a new chapter from Copy Copy, and we were assigned to critique it.
Thus became my respectful skeptical analysis of physics. I decide to keep a day job, but work study physics in the background.

I was able to show some of my earliest efforts to a famous physicist who worked at MIT. He said it was “interesting, but not very interesting”. The reason was I did not have a theory, a means to make many calculations. At the time, all I had were a few math widgets.

First work on quaternions

In the struggle to find a theory, I held a small contest on a moderated newsgroup, sci.physics.research in 1997. I asked readers to provide a brief definition of time. It had to be about math or physics, not philosophy, and needed to be two sentences or less. In one math book, I recalled some odd sort of number that was like a scalar and a vector. The contest motivated me to reread that passage, then go to an old-fashion library and read up about quaternions. This was the first equation I saw, the squaring of a quaternion:

Figure 405: A quaternion squared equals (t squared - x squared - y squared - z squared, 2 t times x, 2 t times y, 2 t times z)

To this day, this makes me excited because the first term is at the core of special relativity as taught to me by Edwin F. Taylor. It is there, for free.
That cannot be an accident. That observation drove me to purchase the domain quaternions.com.

Major failure at a unified field theory

I struggled to come up with a way to deal with gravity using only quaternions. One lesson from general relativity is that a theory of gravity must be a metric theory - a theory where the calculation of distance depends on where one happens to be in a gravitational field. Measuring distance is a symmetric operation, meaning there is no handed to doing the work. Quaternions do have a handedness. There is no way around that. So I reinvented hypercomplex numbers.
Those have all the same multiplication rules as quaternions, but no minus signs anywhere. As such, hypercomplex numbers could be used to characterize a metric.

I confess to having pride in the proposal, enough to print up t-shirts. It was difficult to find a professional to review the body of work. I began blogging on Science20.com. There I found a few technically skilled readers. When I finally got to presenting what I called the GEM proposal, several readers objected to the work. It took a little over a month for me to really see the proposal like they did. The flaw was deadly. A Lagrangian constructed from quaternions will not change under a rotation. It will therefore conserve angular momentum. A Lagrangian constructed from hypercomplex numbers will change under a rotation. It will not conserve angular momentum. Kepler's law of equal area in equal time is a statement that gravity conserves angular momentum. Thus a proposal for gravity using hypercomplex number in the Lagrangian is wrong. End of story.

Current research

Web sites:

- Quaternions.com, the mothership of my private research project.
- Numbers 101, a visual introduction to space-time numbers, aka quaternions.
- Measurement 101, a site dedicated to my new proposal for quaternion gravity
- VisualPhysics.org has a collection of analytic animations generated with a user-hostile set of tools I wrote.

Quaternion space-times-time invariance as gravity I went back to the equation that sparked my initial interest in quaternions:

Figure 406: A quaternion squared equals (t squared - x squared - y squared - z squared, 2 t times x, 2 t times y, 2 t times z)

The first term is called the interval. But what are the next three called? Physics doesn't have a name. That is a problem because the first term is one of the most important in physics because it is at the heart of special relativity.
I came up with a reasonable name for the three amigos: space-times-time. I asked a simple question: what if space-times-time was invariant, then what sort of physics results? That could be gravity, but a very different theory because it is not a field theory. There would be no graviton, nor any quantum gravity theory. The idea had enough promise that I submitted an 1500 word paper to the 2015 Gravitation Research Foundation Essay on Gravity. We will see if they like it.

Thanks to ...

To be productive in my many endeavors, I've needed plenty of help and encouragement, so I wish to acknowledge it explicitly.

The Physics

Prof. Michel Baranger, MIT. It was great fun recreating special relativity while taking special relativity.

Prof. Eric Carlson, Harvard.

Prof. Mitchell Golden, (formerly of) Harvard. The discussions while you were in gentle academia instead of the brutal real world of software designed increased my understanding significantly.

Prof. Alan Guth, MIT. We talked for a total of five minutes on two occasions, but each time that gave me directions for months.

Prof. Lisa Randall, MIT. Nothing quite as useful as a harsh critic, because nature is far tougher.

Dr. Vincent Robert, U Chicago. This stuff is still moving forward. I needed that special French translation, even if the artwork still doesn't make any sense.

Dr. Paul Romanelli. You tried to listen in the early days, and that mattered a lot.

Prof. Guido Sandri, BU, RIP. You were always FANTASTIC. Prof. Sandri was an Italian antidepressant.

Prof. Edwin F. Taylor, MIT, the world's best teacher of relativity.

Pop Science

Amanda Annis, clay sculptures. OK, there are no clay sculptures included in Pop Science, but thinking more that 2D is one of the key themes.

Paul Fata, wandering world artist. I love cheap art! (I also like some of the expensive stuff to, but I never actually buy it :)

Jennifer Hall, Do While Studios. I hope to build a chunk of software using some of the math in here that can live up to the standard of Do While.

Maureen Metzger, Mass. College of Art. The critiques in "Collage and Beyond..." helped mature the works presented. True to the title, we did go beyond...

Meredyth Moses, Clark Gallery. The comments on the portfolio were appreciated.

Mo Ramage, artist to the core.

Joan Shafran, Do While Studios. "Creative Seeing" was the course that got me seeing creatively as a young adult my key advantage in attacking the tough issues in physics.

Lynn Tallo. A core supporter of this small project!

John Yager, formerly of Creative Framing of Chestnut Hill. The work looks professional beyond the skills of its creator. The painting consultations made all the difference.

The Bike

Bob Barrett, the big man who inspired the project, and tolerated the design process as it consumed the dining room.

Bill Darby, Special Purpose Vehicles. Even more important than all the welding were your comments on what would make a workable machine. And it's still working today.

Jeffrey Ferris, Ferris Wheels. The class in bike repair got me thinking about the simple mechanical beauty of the bicycle.

Prof. Harold Washburn, Harvard. Now I think that "market research" is a useful enterprise. I just wish I had capital (a frequent lament :)

Lindy Hop

Darra Sweetser. Someday, you may get to stomp at the Savoy. That would make a cool road trip!

Tony and Aurelie Tye, Hop to the Beat Dance Studios. You cats know how to dance.

Friends...

Prof. Leonard Burrello. It was a humbling year for me in Indiana, but I kept my core vision.

Dr. Steve Chervitz, Stanford. The best work should be unrecognizable for a long time, but hopefully not too long.

Dr. Win Ping Deng. Remember to keep drawing. It can help your science and soul.

The Guild clan. Let's do the 4th of July together, again!

Dr. Derek Kane. Could you check the math? How about the metaphysics? ?Punctuation?

Doug Kuller. The physics here will NOT help with ping pong. It might have helped with 8.012, but I doubt it.

Dr. Don Olivier. It is sometimes difficult for someone how is an approximation to a mathematician to deal with someone who is a mathematician, but that just makes my approximation better in the long run.

Michael Phillips. Hope I can get some respect for my work like you do at the Wall Street Journal. Sorry, so far there is no economic angle for my various projects, otherwise I'd give you the inside scoop. (I hope you have finally "awakened to the fascination that is Harvard.")

Dean & Leslie Potashner. We will get on Letterman so day!

Prof. Richard Young, MIT. I've kept the "whatever it takes (spend freely) to answer the question" attitude with me while doing my own science and art projects. Peer review the results.

...and family*

Darra Sweetser and my daughter Elle.

Arthur, Cindy, Teddy, Grace, Asa (and...?) Sweetser.

Lydia, Billy, Allie, and Nickolaus Gollner.

Adrienne Sweetser.

Grandma may not be surfing to the site soon, but I informed her of the web on her 94th birthday.

And last, but most, Joan Sweetser.

Love is the creation and reflection of Life. I thank you for all of your love.