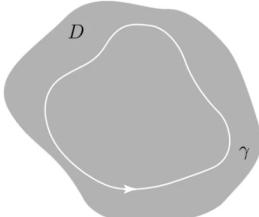


Complex analysis, Week 2, Part 1

Cauchy theorem

Cauchy's integral theorem



$f(z)$ analytic in D

$$\oint_{\gamma} f(z) dz = 0$$

$u = P, -v = Q$

$$\oint_{\gamma} u dx - v dy = \int_D \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) dxdy$$

$v = P, u = Q$

$$\oint_{\gamma} u dy + v dx = \int_D \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} \right) dxdy$$

$$f = u + iv, dz = dx + idy$$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

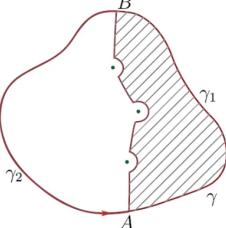
$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Green's formula:

$$\oint_{\gamma} P dx + Q dy = \int_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dxdy$$

Cauchy-Riemann conditions. That's how we complete the proof of Cauchy integral formula. Now

Cauchy theorem



$$I = \oint_{\gamma} f(z) dz$$

Cauchy theorem

$\oint_{\gamma} f(z) dz = \oint_{\gamma_1} f(z) dz + \oint_{\gamma_2} f(z) dz$

Cauchy theorem

$\oint_{\gamma} f(z) dz = \oint_{\gamma_1} f(z) dz + \oint_{\gamma_2} f(z) dz$

Cauchy theorem

$f(a) = \frac{1}{2\pi i} \oint_{\gamma} \frac{f(z)}{z-a} dz$

$\gamma_1 = \gamma + l_2 + \varepsilon + l_1$

$$\oint_{\gamma_1} \frac{f(z)}{z-a} dz = 0$$

$\oint_{\gamma} f(z) dz = \oint_{l_2} + \oint_{l_1} + \oint_{\varepsilon} + \oint_{\gamma}$

$\oint_{\gamma} f(z) dz = - \int_{\varepsilon}^a z = a + \varepsilon e^{i\varphi}, \varphi \in [0, -2\pi]$

$$dz = \varepsilon e^{i\varphi} id\varphi \quad \frac{dz}{z-a} = id\varphi$$

$$\int_{\varepsilon}^{-2\pi} id\varphi f(a + \varepsilon e^{i\varphi}) \Big|_{\varepsilon=0} = \int_0^{-2\pi} id\varphi f(a) = -2\pi i f(a)$$

Cauchy theorem

$f(a) = \frac{1}{2\pi i} \oint_{\gamma_1} \frac{f(z)}{z-a} dz - \frac{1}{2\pi i} \oint_{\gamma_2} \frac{f(z)}{z-a} dz$

$\gamma_1 + l_1 + l_2 + l_3 + l_4 - \gamma_2 + \varepsilon: \oint_{\gamma_1 + l_1 + l_2 + l_3 + l_4 - \gamma_2 + \varepsilon} \frac{f(z)}{z-a} dz = 0$

$$\oint_{\gamma_1} \frac{f(z)}{z-a} dz - \oint_{\gamma_2} \frac{f(z)}{z-a} dz = -2\pi i f(a)$$

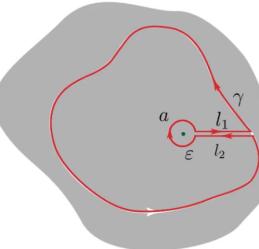
Cauchy theorem

$f(a), and this basically completes our proof even for this more complicated case.$

Complex analysis, Week 2, Part 2

Cauchy theorem

Cauchy theorem



$f(a) = \frac{1}{2\pi i} \oint_{\gamma} \frac{f(z)}{z-a} dz$

$\gamma_1 = \gamma + l_2 + \varepsilon + l_1$

$$\oint_{\gamma_1} \frac{f(z)}{z-a} dz = 0$$

$\oint_{\gamma} f(z) dz = \oint_{l_2} + \oint_{l_1} + \oint_{\varepsilon} + \oint_{\gamma}$

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Cauchy theorem

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$\gamma_1 + l_1 + l_2 + l_3 + l_4 - \gamma_2 + \varepsilon: \oint_{\gamma_1 + l_1 + l_2 + l_3 + l_4 - \gamma_2 + \varepsilon} \frac{f(z)}{z-a} dz = 0$

$$\oint_{\gamma_1} \frac{f(z)}{z-a} dz - \oint_{\gamma_2} \frac{f(z)}{z-a} dz = -2\pi i f(a)$$

Cauchy theorem

$f(a), and this basically completes our proof even for this more complicated case.$

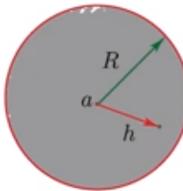
Taylor_expansion_1

Taylor expansion

$$f(z) \\ f(a) = \frac{1}{2\pi i} \oint_R \frac{f(z)}{z-a} dz$$

$$f'(a) = \frac{1}{2\pi i} \oint_R \frac{f(z)}{(z-a)^2} dz$$

$$f^{(n)}(a) = \frac{n!}{2\pi i} \oint_R \frac{f(z)}{(z-a)^{n+1}} dz$$



$$f(a+h) = \sum_{n=0}^{\infty} h^n \frac{1}{2\pi i} \oint_R \frac{f(z)}{(z-a)^{n+1}} dz = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} h^n$$

$$c_n = \frac{1}{2\pi i} \oint_R \frac{f(z)}{(z-a)^{n+1}} dz$$



Taylor expansion

$$f(z)$$

$$f(a) = \frac{1}{2\pi i} \oint_R \frac{f(z)}{z-a} dz$$

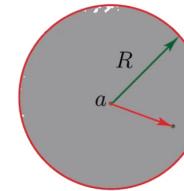
$$f'(a) = \frac{1}{2\pi i} \oint_R \frac{f(z)}{(z-a)^2} dz$$

$$f^{(n)}(a) = \frac{n!}{2\pi i} \oint_R \frac{f(z)}{(z-a)^{n+1}} dz$$

$$z = a + h, \quad |h| < R$$

$$\frac{1}{1-q} = 1 + q + q^2 + \dots + q^n + \frac{q^{n+1}}{1-q}$$

$$f(a+h) = \sum_{l=0}^n h^l \frac{1}{2\pi i} \oint_R \frac{f(z)}{(z-a)^{l+1}} dz + r_n(h)$$



Taylor_expansion_2

$$f(a+h) = \sum_{l=0}^n \frac{f^{(n)}(a)}{n!} h^n + r_n(h) \quad r_n(h) = \frac{1}{2\pi i} \oint_R f(z) \frac{h^{n+1}}{(z-a)^{n+1}(z-a-h)} dz$$

$$\left| \int \dots dz \right| \leq \int | \dots | |dz| \quad |z-a-h| \geq |z-a|-|h| \quad 0 < k < 1$$

$$|f(z)| \leq M \quad |r_n(h)| \leq \frac{1}{2\pi} |h|^{n+1} \int_0^{2\pi} \frac{R d\varphi}{R(1-k)R^{n+1}} M = M \frac{k^{n+1}}{1-k} \rightarrow 0$$



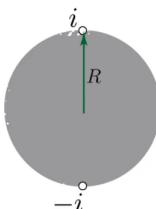
Taylor_expansion_3

Taylor expansion

$$f(z) = \frac{1}{1+z^2} = 1 - z^2 + z^4 - z^6 + \dots$$

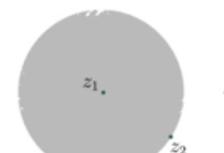
$$R = 1$$

Singularities: $z^2 = -1$, $\rightarrow z = \pm i$



Laurent expansion

$$R = |z_2 - z_1|$$



Laurent_expansion_1



of the corresponding Taylor series. And the answer is: yes, we can, but at the price of incorporating

Complex analysis, Week 2, Part 4

Laurent expansion_2

We write down Cauchy formula for some arbitrary point $z=a+h$ inside the ring.

$$f(a+h) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z-a-h} dz - \frac{1}{2\pi i} \oint_{C'} \frac{f(z)}{z-a-h} dz$$

$$\downarrow$$

$$\sum_{n=0}^{\infty} a_n h^n + \frac{1}{2\pi i h} \oint_{C'} f(z) \left(\frac{z-a}{h}\right)^n dz$$

$$a_n = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z-a)^{n+1}} dz$$

$$\frac{1}{z-a-h} = -\frac{1}{h} \frac{1}{1-\frac{z-a}{h}} \quad \left| \frac{z-a}{h} \right| < 1$$

Laurent expansion

$$= \sum_{n=0}^{\infty} a_n h^n + \sum_{n=1}^{\infty} b_n \frac{1}{h^n}$$

$$b_n = \frac{1}{2\pi i} \oint_C f(z) (z-a)^{n-1} dz$$

Laurent expansion

MISIS

Laurent expansion

$$f(z) = e^{z(z-1/z)}$$

$$z=0$$

$$z=e^{i\varphi}, \varphi \in [-\pi, \pi]$$

$$\frac{dz}{z} = id\varphi$$

$$a_n = \frac{1}{2\pi i} \int_{-\pi}^{\pi} e^{iz\sin\varphi - in\varphi} id\varphi = \frac{1}{2\pi} \int_{-\pi}^{\pi} \cos(x\sin\varphi - n\varphi) d\varphi = J_n(x)$$

$$a_n = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z-a)^{n+1}} dz$$

$$b_n = \frac{1}{2\pi i} \oint_C f(z) (z-a)^{n-1} dz$$

Laurent expansion_3

$$f(z) = \sum_{n=0}^{\infty} J_n(x) z^n + \sum_{n=1}^{\infty} (-1)^n J_n(x) z^{-n}$$

$$b_n = \frac{1}{2\pi i} \int_{-\pi}^{\pi} e^{ix\sin\varphi + in\varphi} id\varphi = (-1)^n a_n$$

$$\varphi \rightarrow \pi - \varphi$$

MISIS

times z to the power of negative n. And that's it: that completes our first discussion of the

Complex analysis, Week 2, Part 5

Laurent_exercises_1

Laurent expansion

$f(z) = \frac{1}{z^2 - z - 2}$ Center: $z = 0, \frac{3i}{2} \in D$

$f(z) = -\frac{1}{6} \sum_{n=0}^{\infty} \frac{z^n}{2^n} - \frac{1}{3} \sum_{n=0}^{\infty} \frac{(-1)^n}{z^{n+1}}$ $1 < |z| < 2$

$z = 2, z = -1 \rightarrow f(z) = \frac{1}{(z-2)(z+1)} = \left(\frac{1}{z-2} - \frac{1}{z+1}\right) \frac{1}{3}$

$\frac{1}{z+1} = 1 - z + z^2 + \dots \quad R = 1$

$\frac{1}{z+1} = \frac{1}{z(1 + \frac{1}{z})} = \frac{1}{z} \sum_{n=0}^{\infty} \frac{(-1)^n}{z^n}$

$\left| \frac{1}{z} \right| < 1 \rightarrow |z| > 1$

$\frac{1}{z-2} = -\frac{1}{2(1 - \frac{z}{2})} = -\frac{1}{2} \sum_{n=0}^{\infty} \frac{z^n}{2^n}$

$\left| \frac{z}{2} \right| < 1 \rightarrow |z| < 2$

And this completes our task. And of course not many functions have this suitable algebraic



Complex analysis, Week 2, Part 5

Laurent_exercises_2




Laurent expansion

$$f(z) = \frac{1}{(\sin z)^3} \quad z = \pi n, \quad n \in \mathbb{Z}$$

$$R = \pi$$

$$z = \pi n + \varepsilon$$

$$f(z) = \frac{1}{[\sin(\pi n + \varepsilon)]^3} = \frac{(-1)^n}{[\sin \varepsilon]^3} = \frac{(-1)^n}{(\varepsilon - \varepsilon^3/6)^3} = \frac{(-1)^n}{\varepsilon^3(1 - \varepsilon^2/6)^3}$$

$$\sin \varepsilon = \varepsilon - \frac{\varepsilon^3}{3!} + \frac{\varepsilon^5}{5!} + \dots$$

$$(1+x)^\alpha = 1 + \alpha x + \dots \quad \alpha = -3, \quad x = -\frac{\varepsilon^2}{6}$$

$$\frac{1}{(1 - \varepsilon^2/6)^3} = 1 - 3\left(-\frac{\varepsilon^2}{6}\right) + \dots$$

$$f(\pi n + \varepsilon) = (-1)^n \left(\frac{1}{\varepsilon^3} + \frac{1}{2\varepsilon} \right) + \dots$$

$$f(z) = \frac{(-1)^n}{(z - \pi n)^3} + \frac{(-1)^n}{2(z - \pi n)} + \dots$$

And this completes our discussion of the third method of the Laurent expansions.




singularities_1

Types of singularities

1. Isolated singularity



$f(z)$, z_0 , U : $0 < |z - z_0| < r$, $f(z)$ is analytic on U

$$\text{III. } f(z) = \dots + \frac{c_{-1}}{z - z_0} + c_0 + c_1(z - z_0) + \dots$$

z_0 – essential singularity

$$f(z) = e^{1/z} = \sum_{n=0}^{\infty} \frac{1}{n! z^n}$$

Es. sing.: $\lim_{z \rightarrow z_0} f(z)$



singularities_2

1. Isolated singularity

$$f(z_n) = e^{-n}, \quad \lim_{n \rightarrow \infty} f(z_n) = 0$$

Es. sing.: $\lim_{z \rightarrow z_0} f(z)$ does not exist

Definition of the limit by Heine:

$$\lim_{z \rightarrow z_0} f(z) = M$$

for any $\{z_n\} \xrightarrow{n \rightarrow \infty} z_0$

$$\{f(z_n)\} \xrightarrow{n \rightarrow \infty} M$$

$$f(z) = e^{1/z} \quad z_0 = 0$$

$$z_n = \frac{1}{2\pi i n}, \quad \lim_{n \rightarrow \infty} z_n = 0$$

$$f(z_n) = e^{2\pi i n} = 1, \quad \lim_{n \rightarrow \infty} f(z_n) = 1$$

$$z_n = -\frac{1}{n}, \quad \lim_{n \rightarrow \infty} z_n = 0$$

Z equals 0 would be an essential singularity. Why? because sine is a combination of exponentials.



singularities_3

Types of singularities

1. Isolated singularity

2. Non-isolated singularity

$$f(z) = \tan \frac{1}{z}$$

$$\frac{1}{z} = \frac{\pi}{2} + \pi n, \rightarrow z_n = \frac{1}{\frac{\pi}{2} + \pi n}$$

$z = 0$ – non-isolated singularity

3. Branch point

$$f(z) = \sqrt{z}, \ln z$$

$z = 0$ – branch point

