I. (VIDEO 1.6)

A. 1.6 Differentiation of a complex function. Cauchy-Riemann conditions

So far we dealt with complex algebra only. Now we will advance our knowledge and start discussing the infinitesimal properties of complex numbers.

First if all let us define a complex function. The function of complex variable is called a mapping f which for any point z of some domain assign some complex value. We can say that we define the function as a mapping between regions of two complex planes. The complex plane of z and the complex plane of f, values of our complex function.

Now we'd like to build differential analysis in the complex plane. Before we start though we need to define the notion of limit and convergence in the complex plane.

Fortunately, all these notions are carried over from real analysis with minimal modification.

For example,

The limit of sequence

$\{z_n\}$ is defined as the complex number a such that for any $\varepsilon > 0$ there exist number n_0 such that for any point inside a punctured circle of radius ε centered at a and for any $n > n_0$, $|z_n - a| < \varepsilon$.

In the same manner, the limit of the function is defined.

B. Cauchy-Riemann conditions

However we will face some difficulty when defining the derivative of the complex function.

Let us split the function into its real and imaginary parts.

$$f = u + iv$$

The question now is how to differentiate the function with respect to the change of complex variable z. Here is the issue. We have to define the change of the function as we shift the coordinate (x, y) in 2D. But, obviously, the change of the function will depend on a particular direction in which we move from point (x, y) to point $(x + \Delta x, y + \Delta y)$. Just imagine, say the, relief of the function $u(x, y)$. Then you get the geometric idea.

Let us write down the differential of the function f:

$$\Delta f = \Delta u + i\Delta v = \underset{I}{\frac{\partial u}{\partial x}}\Delta x + \underset{II}{\frac{\partial u}{\partial y}}\Delta y + i\Big(\underset{III}{\frac{\partial v}{\partial x}}\Delta x + \underset{IV}{\frac{\partial v}{\partial y}}\Delta y\Big)$$

What we'd like to introduce is a not a directional derivative but a different type of derivative, defined as the ratio of complex numbers.

Definition The function is called differentiable at point $z = z_0$ if exists a limit:

$$f'(z_0) = \frac{f(z) - f(z_0)}{z - z_0}\Big|_{z\to z_0} = \frac{\Delta f}{\Delta z}\Big|_{z\to 0} = A \tag{1}$$

which is called a derivative of a function at point z_0.

And we want this limit to be independent of the direction Δx, Δy. That means we need to require that Δf to be proportional $\Delta x + i\Delta y$.

Let us combine term I and IV and II and III. Let us look at the first combination. We have Δx and $i\Delta y$ with different factors. But if we require that

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \tag{2}$$

then these terms will be combined into the combination we seek for:

$$\Delta f_{I,iV} = \frac{\partial u}{\partial x}(\Delta x + i\Delta y). \tag{3}$$

The same goes for terms II and III if we require:

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}, \tag{4}$$

And we have:

$$\Delta f_{II,III} = i\frac{\partial v}{\partial x}(\Delta x + i\Delta y). \tag{5}$$

This way we obtain the following expression for the differential of the function:

$$\Delta f = \Big(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\Big)(\Delta x + i\Delta y). \tag{6}$$

And we get the desired limit which is independent of the direction Δz:

$$\frac{\Delta f}{\Delta z} = \Big(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\Big). \tag{7}$$

The conditions (2) and (4) are called Cauchy-Riemann's conditions. Cauchy-Riemann equations are necessary and sufficient conditions of the existance of the limit defining the derivative of the function at point z_0.

We have just proved the sufficiency. The necessity is even easier to prove. We just differentiated the function f in x and iy directions:

$$\frac{df}{dx} = \frac{\partial u}{\partial x} + i\frac{\partial v}{dy}, \quad \frac{df}{idy} = -i\frac{\partial u}{\partial y} + \frac{\partial v}{dy}. \tag{8}$$

This way, they establish a criteria of differentiability of the function at point $z = z_0$.

These are the first relations Riemann introduces in his doctorial thesis creating the field of Complex Analysis.

Definition

The function which is differential at every point of some region D is called analytic, or holomorphic (or regular) in this region.

C. The consequences of Cauchy-Riemann conditions.

1. The real and imaginary parts of a complex function are harmonic functions in a complex plane.

$$\Delta u = 0, \quad \Delta v = 0 \tag{9}$$

This is easily proven as follows:

$$\partial^2_{xx} u = \partial^2 v_{xy}, \quad \partial^2_{yy} u = -\partial^2 v_{yx}. \tag{10}$$

summing up these identities and recalling that for any smooth function $\partial^2_{xy} v = \partial^2_{yx} v$ we obtain the necessary equation. The same goes for v function.

2. The only critical points of the real and imaginary parts of a complex function are saddle points.

First of all, the point where $f'(z_0) = 0$ is called a critical point:

$$f'(z_0) = 0 \quad \textit{critical points}. \tag{11}$$

In this point $\partial_x u = 0, \quad \partial_y u = 0$. (Indeed, since $\partial_x u = 0$ and $\partial_x v = 0$ then according to Caucy-Riemann conditions we have $\partial_x v = -\partial_y u = 0$)

Therefore, the critical point of the function of complex variable is simultaneously the critical point of u -function as a function of (x,y).

Now, this critical point may be of three possible types, defined by the main curvatures of the relief of the function u: maximum, minimum or saddle.

So you are always able to find orthogonal directions such that in rotated coordinate system: $x = x_0 + \alpha_{11}\xi + \alpha_{12}\eta, \quad y = y_0 + \alpha_{21}\xi + \alpha_{22}\eta$ the behavior of the function in the vicinity of the critical point will be:

$$u(x + \Delta x, y + \Delta y) = u(x, y) + \lambda_1 \xi^2 + \lambda_2 \eta^2 \tag{12}$$

If both lambdas are positive we deal with minimum, if negative, we deal with maximum, and if they are of different sign, we have a saddle (maximum in one direction and minimum in the other).

The curvatures of the function is obtained from its second differential:

$$u(x + \Delta x, y + \Delta y) = u(x, y) + \frac{1}{2}u_{xx}\Delta x^2 + \frac{1}{2}u_{xy}\Delta x\Delta y + \frac{1}{2}u_{yy}\Delta y^2 \tag{13}$$

the second term is better to recast in the matrix form:

$$u(x+\Delta x, y+\Delta y) = u(x,y) + \frac{1}{2}(\Delta x, \Delta y)\begin{pmatrix} u_{xx} & u_{xy} \\ u_{xy} & u_{yy} \end{pmatrix}\begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix} \tag{14}$$

To see the type of the criticality we rotate the coordinate system so the matrix in new coordinates becomes diagonal. But we don't even need to do it. We know that the determinant of the matrix stays unchanged under the orthogonal transformation of the matrix.

We want to prove that the eigenvalues of this matrix have different signs. That means its determinant (which is the product of its eigenvalues) ought to be negative.

So let us compute the determinant:

$$u_{xx}u_{yy} - u_{xy}^2 = -v_{xy}^2 - u_{xy}^2 < 0 \tag{15}$$

In the second half we used Cauchy-Riemann conditions.

This simple fact, that a complex function has no maximums or minimums but only saddles has tremendous consequences for the whole field of complex analysis. It gives the name to significant part of asymptotic analysis called saddle-point approximations.

And it is all the consequence of differentiability of the function.