

A. Video 1.3

1. Trigonometric interpretation

Trigonometric interpretation is naturally connected to the geometric one. Its construction is achieved by switching the coordinate system to the polar one and represent z by $|z|$ and φ . The polar angle φ counted from the positive direction of the real axis is called argument of a complex number z .

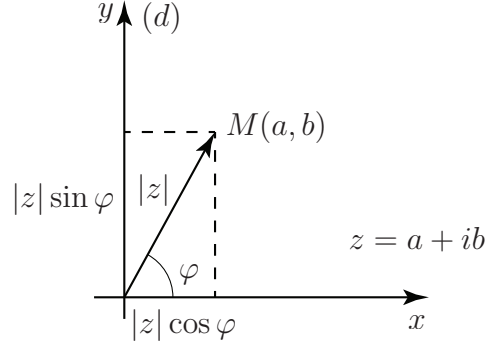


FIG. 1: Trigonometric form of a complex number

$$\varphi = \arg z. \quad (1)$$

A sort of ambiguity is present here, as this angle is defined up to addition of integer factors of 2π : this fact will be very important in what follows later. Now, to summarize our discussion of trigonometric representation, we wrap it up by the following equation:

$$z = a + ib = |z| \cos \varphi + i|z| \sin \varphi = |z|(\cos \varphi + i \sin \varphi). \quad (2)$$

Thus, a complex number is uniquely recovered from its modulus and argument.

Complex conjugation keeps the modulus intact and an argument can be considered as changing sign:

$$\begin{aligned} z^* &= a - ib = |z| \cos \varphi - i|z| \sin \varphi = |z|(\cos \varphi - i \sin \varphi) \\ &= |z|(\cos(-\varphi) + i \sin(-\varphi)). \end{aligned}$$

Let us inspect the effect of multiplication and division on the modulus and argument.

$$\begin{aligned} z_1 &= |z_1|(\cos \varphi_1 + i \sin \varphi_1), \quad z_2 = |z_2|(\cos \varphi_2 + i \sin \varphi_2), \\ z_3 &= z_1 z_2 = |z_1||z_2|(\cos \varphi_1 + i \sin \varphi_1)(\cos \varphi_2 + i \sin \varphi_2) = \\ &= |z_1||z_2|(\cos \varphi_1 \cos \varphi_2 - \sin \varphi_1 \sin \varphi_2 + i(\cos \varphi_1 \sin \varphi_2 + \cos \varphi_2 \sin \varphi_1)) \\ &= |z_1||z_2|(\cos(\varphi_1 + \varphi_2) + i \sin(\varphi_1 + \varphi_2)) = |z|(\cos \varphi + i \sin \varphi). \end{aligned}$$

Quite nicely, the modulus multiply under the product and arguments – add up:

$$|z_1 z_2| = |z_1||z_2|, \quad \arg(z_1 z_2) = \arg z_1 + \arg z_2 \quad (3)$$

For division operation, we find:

$$\begin{aligned} z_1 &= |z_1|(\cos \varphi_1 + i \sin \varphi_1), \quad z_2 = |z_2|(\cos \varphi_2 + i \sin \varphi_2), \\ z_3 &= z_1 / z_2 = \frac{z_1 z_2^*}{|z_2|^2} = \frac{|z_1||z_2|}{|z_2|^2}(\cos \varphi_1 + i \sin \varphi_1)(\cos \varphi_2 - i \sin \varphi_2) = \\ &= \frac{|z_1|}{|z_2|}(\cos \varphi_1 \cos \varphi_2 + \sin \varphi_1 \sin \varphi_2 + i(\cos \varphi_1 \sin \varphi_2 - \cos \varphi_2 \sin \varphi_1)) \\ &= \frac{|z_1|}{|z_2|}(\cos(\varphi_1 - \varphi_2) + i \sin(\varphi_1 - \varphi_2)) = |z|(\cos \varphi + i \sin \varphi). \end{aligned}$$

Thus, the modulus of the result of division is just a result of division of modulus and the arguments should be subtracted:

$$\left| \frac{z_1}{z_2} \right| = \frac{|z_1|}{|z_2|}, \quad \arg(z_1 - z_2) = \arg z_1 - \arg z_2 \quad (4)$$

This allows for a very simple interpretation of the power function. Indeed, raising the complex number z to the power n multiplies the argument by a factor of n times and the modulus (a real number) is simply raised to the same n -th power.

Consider a number of unit modulus:

$$z_0 = \cos \theta + i \sin \theta, \quad |z_0| = 1 \quad (5)$$

Then:

$$z_0^n = (\cos \theta + i \sin \theta)^n = \cos n\theta + i \sin n\theta \quad (6)$$

Let us consider an example. Compute argument and modulus of the complex number $z = -1 - i$ and write down the trigonometric representation. Reduce the argument to the interval

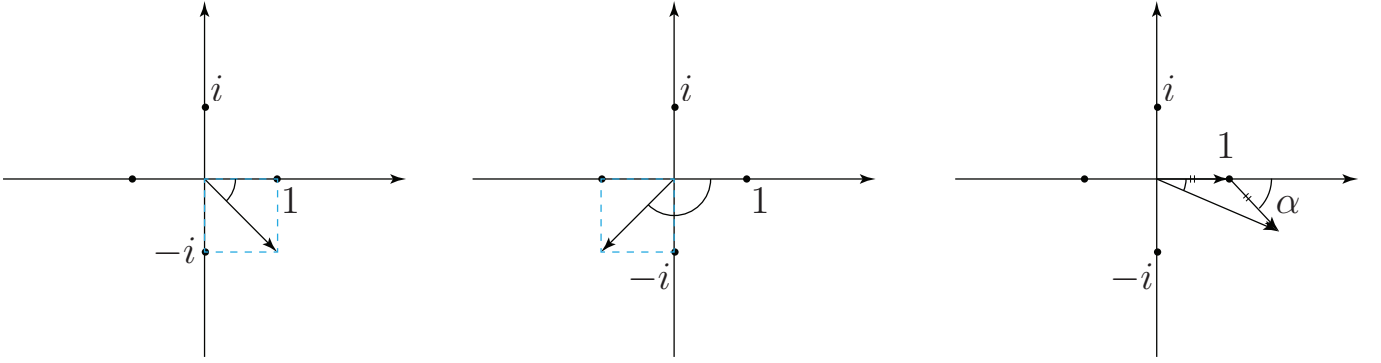


FIG. 2: Computation of arguments of complex numbers

We find $|z| = \sqrt{2}$

$$z = |z|(\cos \theta + i \sin \theta) = \sqrt{2} \left(-\frac{1}{\sqrt{2}} - i \frac{1}{\sqrt{2}} \right) \quad (7)$$

and hence:

$$\cos \theta = -\frac{1}{\sqrt{2}}, \rightarrow \theta = \pm \frac{3\pi}{4}, \quad \sin \theta = -\frac{1}{\sqrt{2}} \Rightarrow \theta = -\frac{3\pi}{4} \quad (8)$$

So, $\arg z = -3\pi/4$, which can be also easily read off the Fig. 2(b)

$$z = \sqrt{2} \left(\cos \left(-\frac{3\pi}{4} \right) + i \sin \left(-\frac{3\pi}{4} \right) \right) \quad \text{trigonometric form} \quad (9)$$