Jordans_lemma_example_1__1

Complex analysis, Week 3, Part 10

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Integration with residues

$$I=\int_0^\infty \frac{\cos x}{x^2+a^2}dx, \quad a>0$$

$$x \to -x$$

$$I=\frac{1}{2}\int_0^\infty \frac{e^{ix}}{x^2+a^2}dx+\frac{1}{2}\int_0^\infty \frac{e^{-ix}}{x^2+a^2}dx$$

$$\int_0^{-\infty} \frac{e^{ix}}{x^2+a^2}(-dx)=\int_{-\infty}^0 \frac{e^{ix}}{x^2+a^2}dx$$

$$I=\frac{1}{2}\int_{-\infty}^\infty \frac{e^{ix}}{x^2+a^2}dx$$

$$f(z)=\frac{e^{iz}}{z^2+a^2}$$

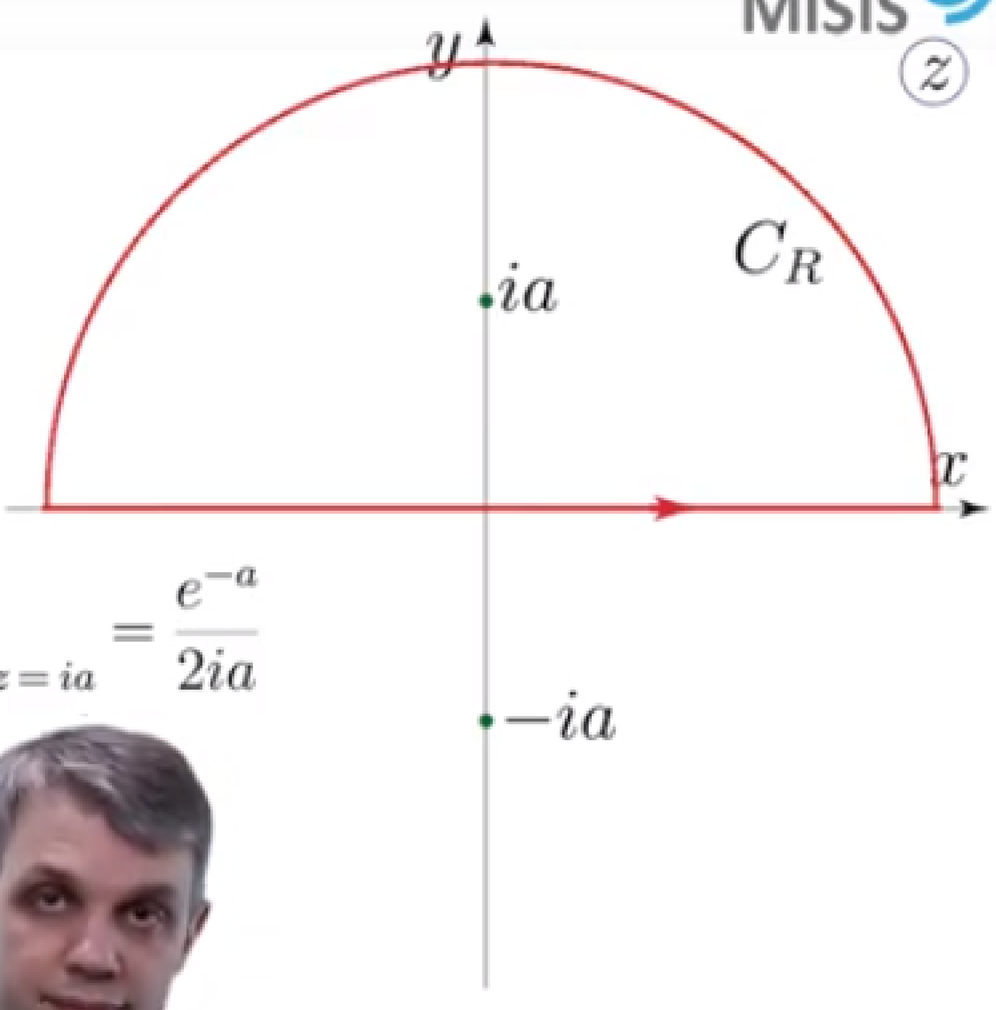
$$\lambda=1, \quad g(z)=\frac{1}{z^2+a^2}\underset{z\to\infty}{\to} 0$$

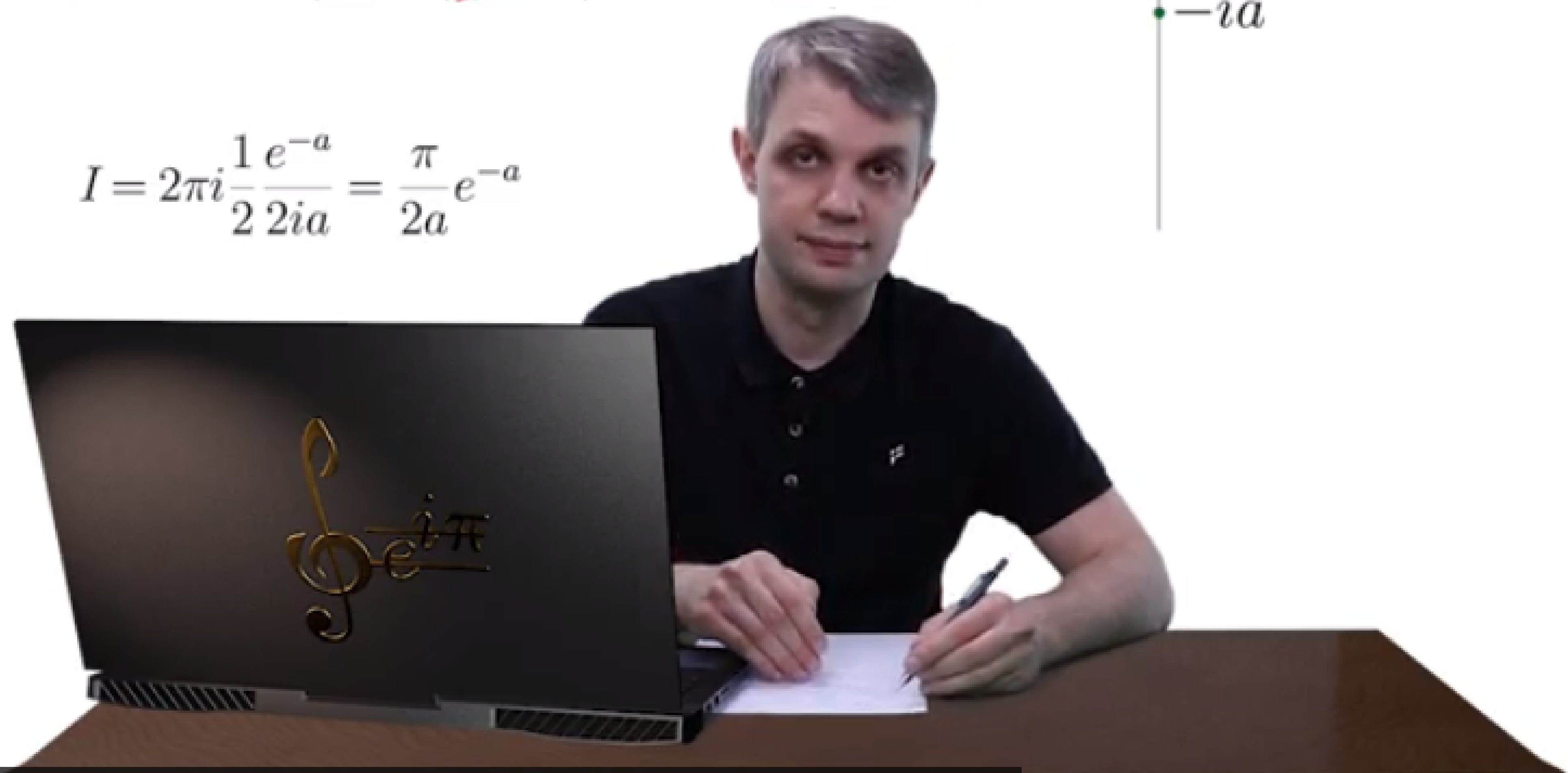
$$\oint f(z)dz=\int_{-\infty}^{\infty}+\cancel{\int_{C_R}}^{\,0}=I$$

$$z=\pm ia$$

$$\operatorname*{res}_{z=ia} f(z)=\frac{e^{iz}}{(z+ia)\cancel{(z-ia)}}\cancel{(z-ia)}\Big|_{z=ia}=\frac{e^{-a}}{2ia}$$

$$I=2\pi i\frac{1}{2}\frac{e^{-a}}{2ia}=\frac{\pi}{2a}e^{-a}$$





be able to close the contour with upper or lower semi-circles. Our next video will be dedicated to

integrand has two simple poles at points z = +- I a. Only one of these poles is positioned inside

our integration contour, namely I a, so let us compute the residue of our function at this point.

We decompose our denominator into z + I a times z - I a. It's the first order pole, so we multiplied

by z - I a and make a cancellation. And in the end we set z equals to I a so we obtain e^{- a} over 2

I a as a residue of the function. And finally we have an answer for our integral, it's equal to 2

pi I times 1/2, which comes as a prefactor in front of our integral times our residual

to obtain pi by 2a times e ^{-a}.

And this is how Jordan's lemma is applied to standard trigonometric integrals.

Basically, we have two steps here: first we need to decompose our trigonometric function cosine of

sine in such a way that the integral contains only single exponential, and second we need to

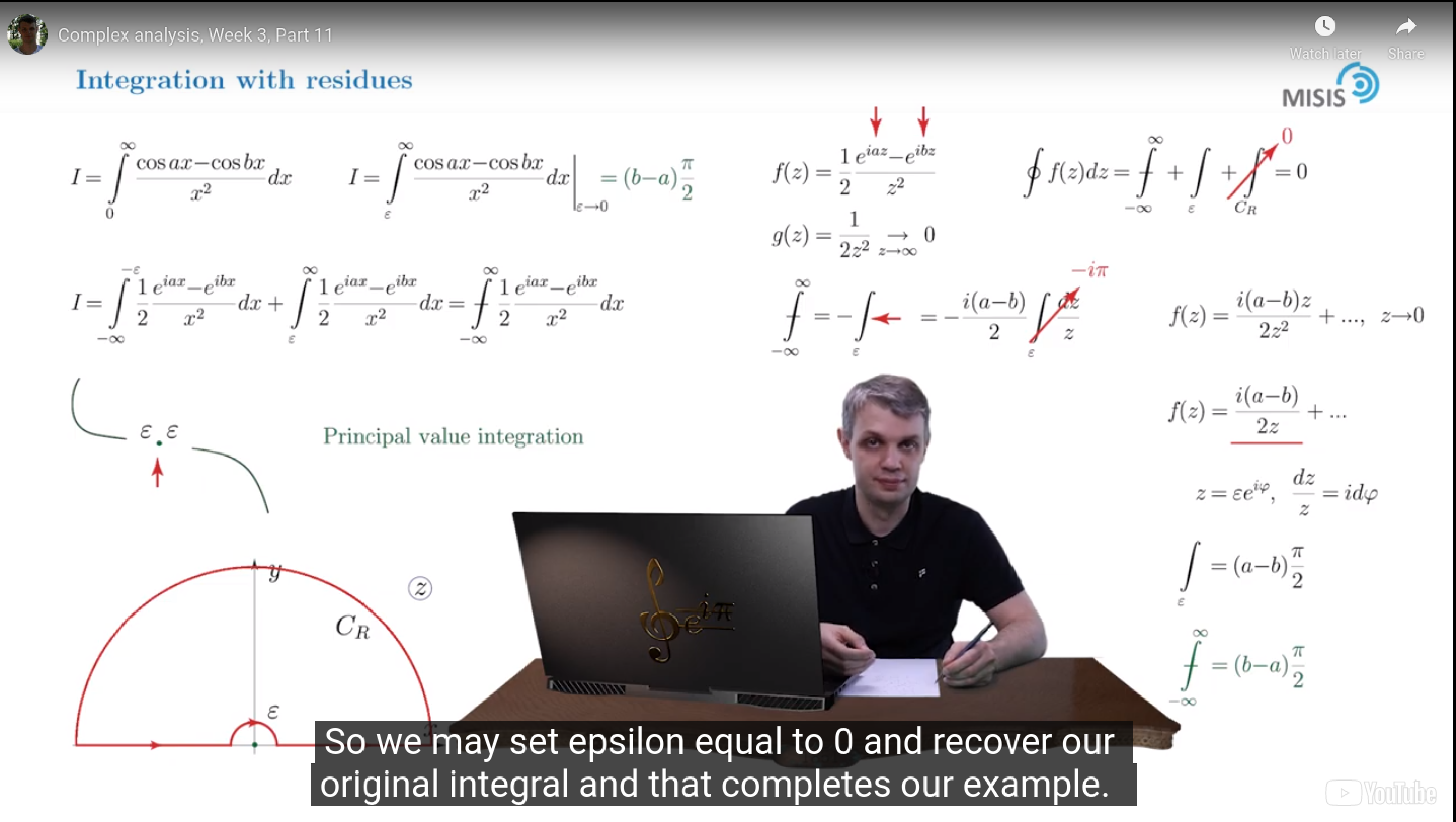
stretch the contour so it goes from minus infinity to plus infinity. Only under this condition we'll

be able to close the contour with upper or lower semi-circles. Our next video will be dedicated to

a more interesting example where we'll introduce the concept of principal value integration.

4:56 / 5:09 Speed 1.0x HD

Jordons_lemma_example_2__1



(a - b) z over z^2 and so

on. So our first term in the Laurent expansion of our function is I (a - b)/z and plus some regular

terms. Now, why don't I write these regular terms? well that's because when we integrate along this

infinitesimal circle, they will vanish as the radius of the circle tends to zero.

This way we only need singular terms, and there is only one of them. And it's what is written

here. Now let's plug in this expansion into our integral, and we'll obtain minus -I (a - b)/2

times the integral dz/z. And we introduce the standard parameterization z = epsilon e^{i phi},

and therefore dz/z is simply I dphi and since phi changes from pi to zero, this integral is equal

to -I pi. So in the end, for our semicircular integral we obtain the answer, which is (a - b)

times pi/2. And that's it, our principal value integral is now expressed as b - a times pi/2,

which is nothing but our initial integral. And the answer is luckily independent of epsilon.

So we may set epsilon equal to 0 and recover our original integral and that completes our example.

In our next video we'll consider slightly more interesting example principle value integration.

7:44 / 7:58 Speed 1.0x HD

principle_value_integration__1

Complex analysis, Week 3, Part 12

Watch later Share

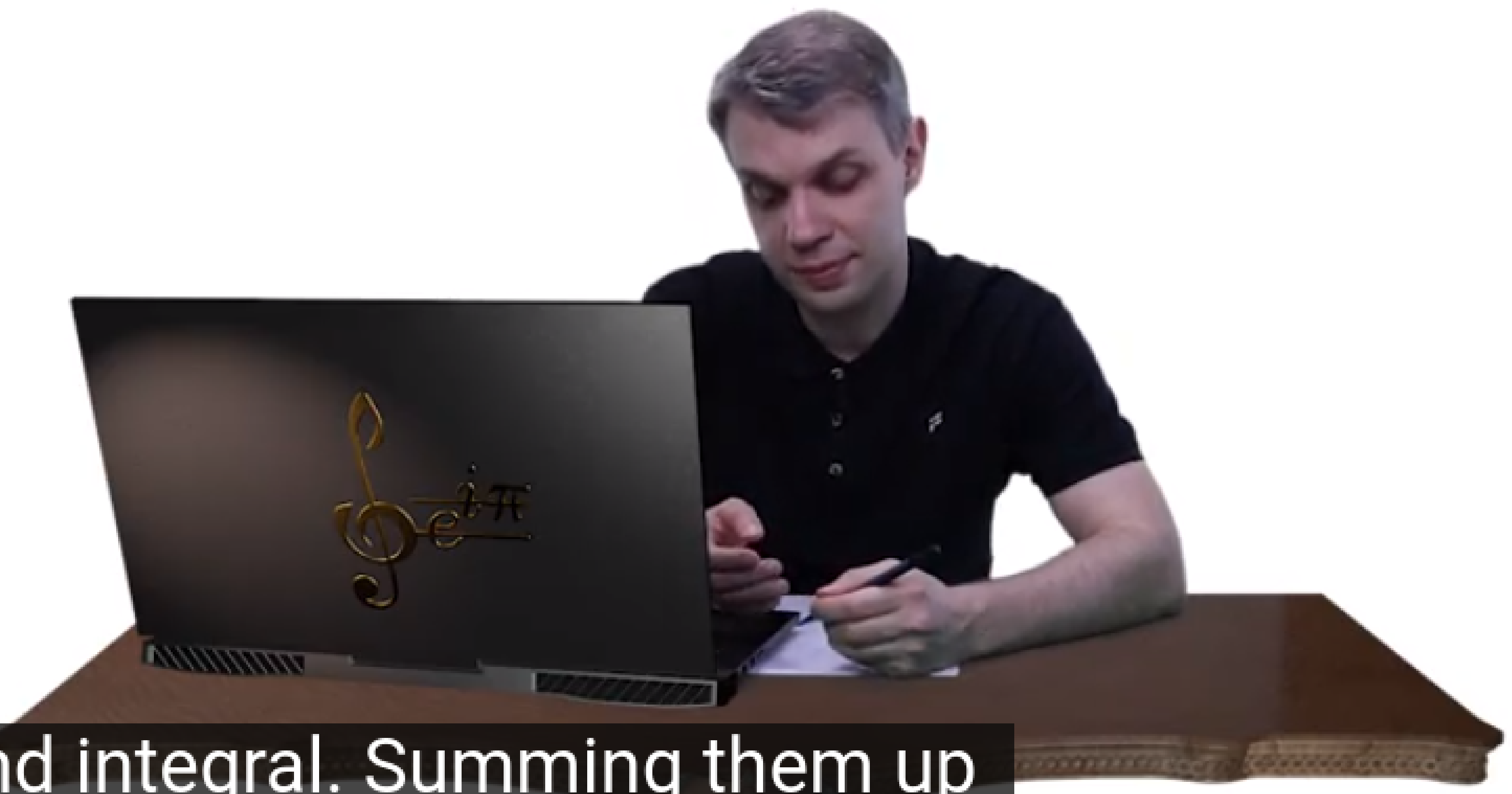
Integration with residues

MISIS

$$I = \int_{-1}^{1} \frac{dx}{x}$$

Cauchy's principal value of an integral

$$\int_{-1}^{1} \frac{dx}{x} = \int_{-1}^{-\varepsilon} \frac{dx}{x} + \int_{\varepsilon}^{1} \frac{dx}{x}\Big|_{\varepsilon\to 0} = \ln\varepsilon + \ln\frac{1}{\varepsilon} = 0$$



epsilon for the second integral. Summing them up
we are already obtained zero and of course setting

this expression is meaningless, because the integration contour passes

right through the first order pole of the integrand. But on the other hand,

the integrand is an odd function of x and the integration domain is symmetric,

so there is a temptation to prescribe a zero value to this expression.

That kind of examples provoked the introduction of the so-called Cauchy's principal value of the

integral. And it is introduced as follows: we simply split the contour and the singularity.

Then we insert an infinitesimal separation centered at this singularity so the principal

value of this integral is deciphered at the sum of the integrals from -1 to minus epsilon plus the

integral from epsilon to 1 dx/x, where epsilon is set to 0 and the end of the calculation.

And indeed one obtains logarithm of epsilon for the first integral and logarithm of one over

epsilon for the second integral. Summing them up we are already obtained zero and of course setting

epsilon to zero we obtain the zero answer. And now let's study a less trivial example.

Compute the principal value integral from minus infinity to plus infinity

e^{i b x} over (x^2 - 1) dx, where b is negative.

First of all, we see that our integration contour passes through two singularities of the integrand

which are simple poles at points x = +- 1. As a first step let us draw the contour.

It is split into three pieces and let us decipher the principle integration sign and the sum of the

integral from minus infinity to -1 minus epsilon plus the integral from minus one plus epsilon to

one minus epsilon and plus the integral from one plus epsilon to plus infinity. Naturally,

to employ Residual theorem, we need to close the contour somehow and minding the future application

of Jordan's lemma we connect the infinite edges of this contour by a lower infinite semicircle

C_r. Next, we connect the adjacent pieces of the contour by two infinitesimal lower semi-circles

1:22 / 6:03 Speed 1.0x HD

principle_value_integration__2

Complex analysis, Week 3, Part 12

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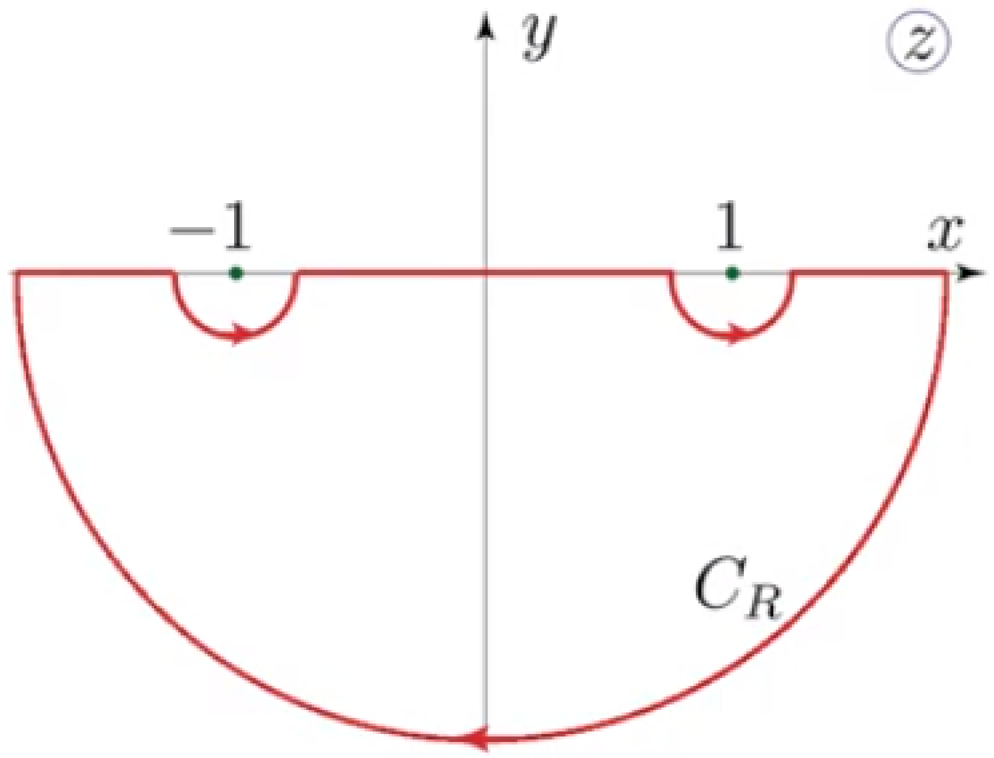
Integration with residues

MISIS

$$I=\int_{-\infty}^{\infty}\frac{e^{ibx}}{x^2-1}dx,\quad b<0$$

Simple poles: $x=\pm 1$



$$\int_{-\infty}^{\infty}=\int_{-\infty}^{-1-\varepsilon}+\int_{-1+\varepsilon}^{1-\varepsilon}+\int_{1+\varepsilon}^{\infty}$$

$$f(z)=\frac{e^{ibz}}{z^2-1}$$

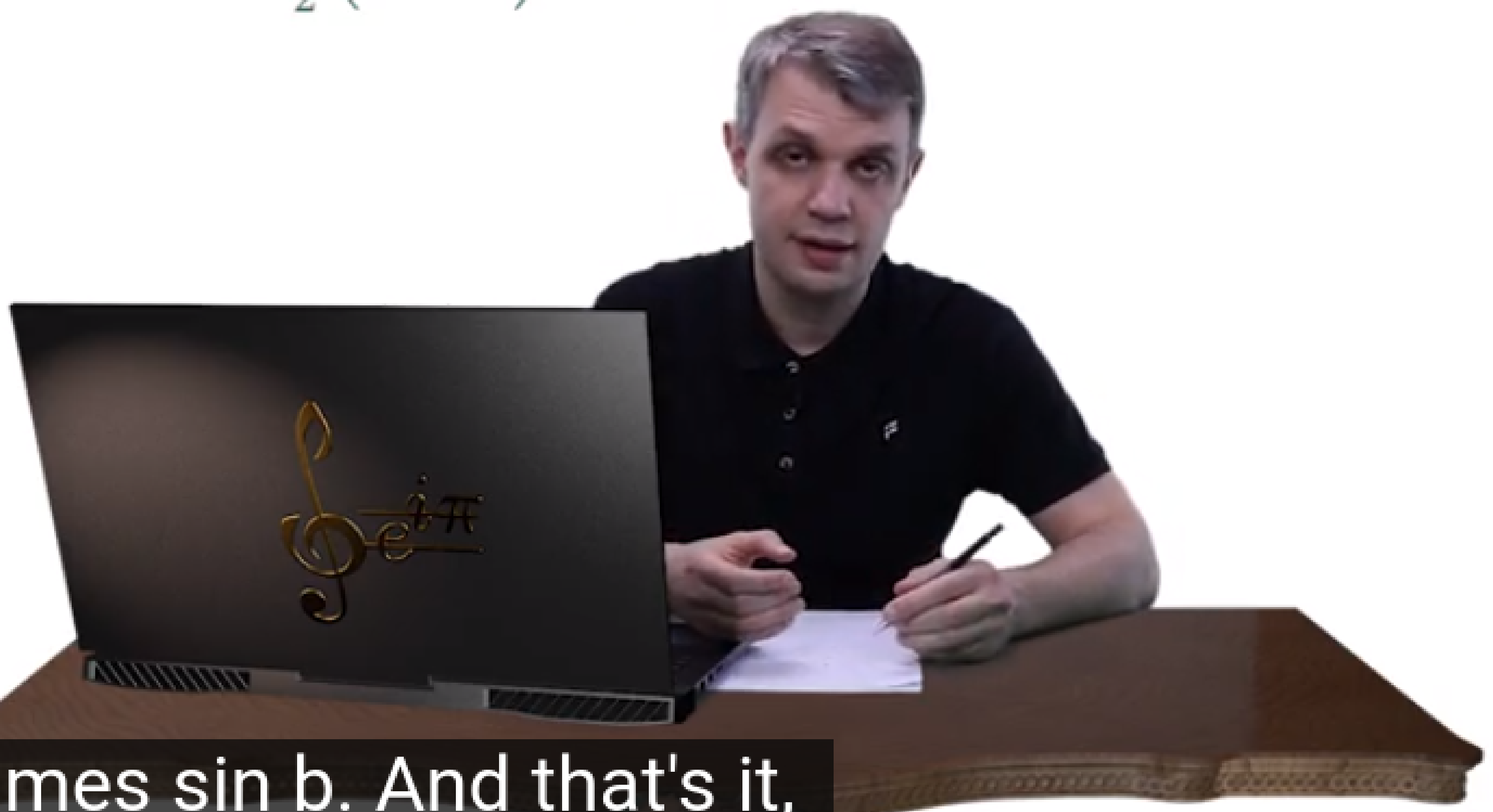
$$\oint f(z)dz=\int_{-\infty}^{\infty}+\int_{\varepsilon_{-1}}+\int_{\varepsilon_1}+\overbrace{\cancel{\int_{C_R}}}^{0}=0\qquad \int_{-\infty}^{\infty}=-\int_{\varepsilon_1}-\int_{\varepsilon_{-1}}$$

$$z=-1+\varepsilon\qquad f(-1+\varepsilon)=\frac{e^{-ib}}{-2\varepsilon}+\ldots$$

$$\int_{\varepsilon_{-1}}=-\frac{1}{2}e^{-ib}\overbrace{\cancel{\int_{\varepsilon_{-1}}\frac{d\varepsilon}{\varepsilon}}}^{i\pi}\qquad \varepsilon=|\varepsilon|e^{i\varphi},\quad \varphi\in[\pi,\,2\pi]\qquad \int_{\varepsilon_{-1}}=-\frac{i\pi}{2}e^{-ib}$$

$$z=1+\varepsilon,\quad f(1+\varepsilon)=\frac{e^{ib}}{2\varepsilon}+\ldots\qquad \int_{\varepsilon_1}=\frac{1}{2}e^{ib}\overbrace{\cancel{\int_{\varepsilon_1}\frac{d\varepsilon}{\varepsilon}}}^{i\pi}=\frac{i\pi}{2}e^{ib}$$

$$I=\frac{i\pi}{2}\left(e^{-ib}-e^{ib}\right)=\pi\sin b$$



and we obtain pi times sin b. And that's it,
that completes our initial discussion of the

plus the regular terms. Now the integral is equal to -1/2 times e^{ - ib} times the integral of d

epsilon over epsilon - this integral is computed by a standard parameterization epsilon equals

|epsilon| times e ^{i phi}, where phi changes from pi to 2 pi. The evaluation is straightforward

and we obtain I pi, so the integral along the left semicircle is equal to - I pi / 2 times e^{ - ib}.

Now the next integral. Again we perform Laurent expansion of our integrand in the vicinity of 1:

z equals 1 + epsilon and f(1 + epsilon) is equal to e^{ib} over 2 epsilon.

and we perform the integration: 1/2 times e^{ib} times the integral of d epsilon over epsilon. The

integral is taken with the same parameterization and naturally yields the same I pi. So the second

integral is equal to I pi/2 times e^{Ib}. And as a result, our principal value integral is equal to I

pi/2 times e^{-Ib} - e^{ib}. Now this combination of exponentials is reduced to a sine function,

and we obtain pi times sin b. And that's it, that completes our initial discussion of the

integration methods of complex analysis. And i hope to see you the next week.

5:51 / 6:03 Speed 1.0x HD

residue_theory_introduction__1

Complex analysis, Week 3, Part 1.

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MISIS

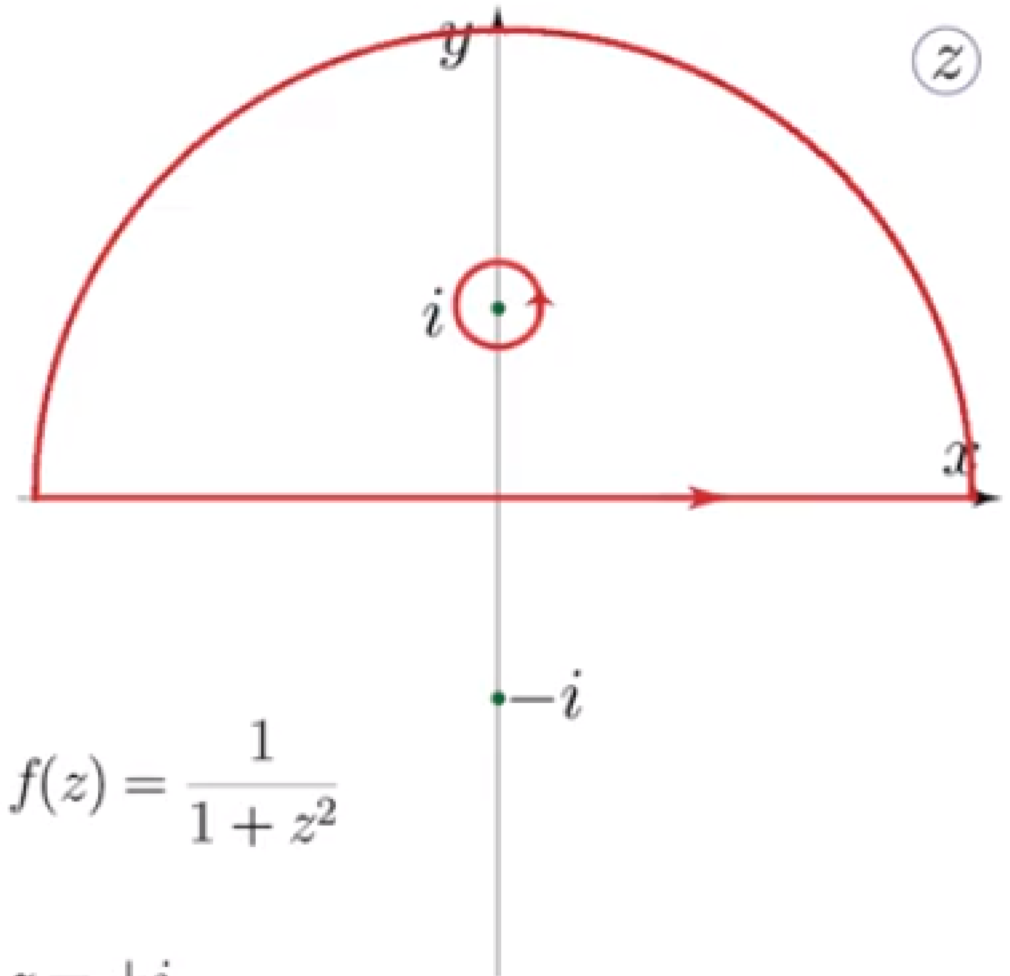
Integration with residues

$$I=\int_{-\infty}^{\infty}\frac{dx}{1+x^2}=\arctan x\Big|_{-\infty}^{\infty}=\frac{\pi}{2}-\left(-\frac{\pi}{2}\right)=\pi$$

$$\oint = I \qquad z=i+\varepsilon e^{i\varphi} \qquad dz=\varepsilon e^{i\varphi} i\,d\varphi$$

$$\text{Integrand}=\frac{dz}{(z+i)(z-i)} \qquad \frac{dz}{z-i}=i\,d\varphi$$

$$\oint=\int_0^{2\pi}\frac{i\,d\varphi}{2i+\cancel{\varepsilon e^{i\varphi}}}=\frac{1}{2}\int_0^{2\pi}d\varphi=\pi$$



$$f(z)=\frac{1}{1+z^2}$$

$$z=\pm i$$

$$\frac{1}{1+z^2}\to\frac{1}{z^2}, \quad |z|\to\infty$$

Well, despite that complex analysis treatment seems to be more involved, here still you

4:58 / 5:25

Speed 1.0x HD

at plus infinity minus infinity and that means that this arc integral is actually equal to

zero. So despite the fact that we changed our original integral, our closed contour

integral is equal to the initial integral, and eventually it's reduced to this infinitesimal

circle integral round point z equals I. That is the amazing consequence of application

of Cauchy's integral theorem. And now let's compute this circular integral. As usual,

we introduce the parameterization z equals I plus epsilon times e^{i phi}, where dz is

equal to epsilon e^{i phi}I dphi. And decompose our integrand as dz over z plus i times z

minus i. Well, dz over z minus i is simply reduced to i d phi so in the end we are left

with the integral from zero to two pi I dphi over two I plus epsilon times e^{i phi}. But

epsilon is tending to zero so we discard this epsilon term in the denominator and obtain

1/2 of the integral of d phi which is again equal to pi.

Well, despite that complex analysis treatment seems to be more involved, here still you

can't deny its geometrical butane. So in our next video I will give you powerful theorems,

which will provide you with formidable tools of computing that kind of integrals, and it

will eventually automate your procedure of tackling these integrals.

Complex analysis, Week 3, Part 2

Watch later

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Integration with residues

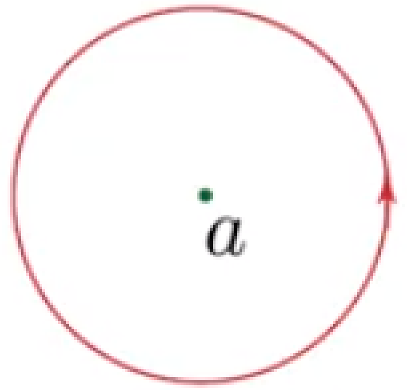
MISIS

$$f(z) = \ldots + \frac{c_{-1}}{(z-z_0)} + \ldots$$

$$\operatorname*{res}_{z=z_0} f(z) = c_{-1}$$

$$\oint_\gamma f(z)dz = 2\pi i \sum_{z_i \in in} \operatorname*{res}_{z=z_i} f(z)$$

$$\oint \underline{(z-a)^n} dz = \begin{cases} 0, & n \neq -1 \\ 2\pi i, & n = -1 \end{cases}$$



$$z = a + re^{i\varphi}, \quad dz = re^{i\varphi} i d\varphi$$

$$\int_0^{2\pi} r^{n+1} e^{i(n+1)\varphi} i d\varphi = \begin{cases} \left. \dfrac{r^{n+1} e^{i(n+1)\varphi}}{i(n+1)} \right|_0^{2\pi} = 0, & n \neq -1 \\ \displaystyle\int_0^{2\pi} i d\varphi = 2\pi i, & n = -1 \end{cases}$$

I dphi and obtain 2 pi I. So that completes our proof and now back to the Residual theorem.

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Well, to prove this elementary property, let us introduce a familiar to us already parametrization

z = a + r e^{i phi}, where r is the radius of the circle. Then dz = r e^{i phi} I dphi

and let's plug in this parametrization in our integrand. So what we obtain is

the integral over phi from say 0 to 2 pi r to the power of n+1 e^{I(n + 1)phi} I dphi.

And we see that the integrand is exponential so the antiderivative would be also an exponential

function which is periodic. So when you integrate on a segment of 2 pi the difference of

antiderivatives will vanish due to the periodicity of the function with the only exception of a

single situation, when n is equal to negative one. Well, in this situation the exponential in

the integral disappears: it is turned into unity and in this case we simply integrate

I dphi and obtain 2 pi I. So that completes our proof and now back to the Residual theorem.

Well, suppose we have an arbitrary closed counter gamma,

but say in the counterclockwise direction. And let's position some singularities

of our function inside this contour. The main consequence of Cauchy's integral theorem tells

us that we may deform the contour in an arbitrary manner without the integral changing its value,

as long as the deformation doesn't cross the singularities. So what we do: we deform this

contour into a combination of infinitesimal circles around each pole, connected by straight

infinitely close lines, forming a dumbbell-like shape. Well, first of all, let's address this

linear infinite closed segments of our contour. Well, each pair is passed in opposite directions

and due to the fact that they are infinitely close to each other, these integrals eventually cancel

each other, because the function is essentially the same on both parts of these linear segments,

and the directions are opposite. So this way our original closed contour integral is reduced to the

Complex analysis, Week 3, Part 2

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Integration with residues

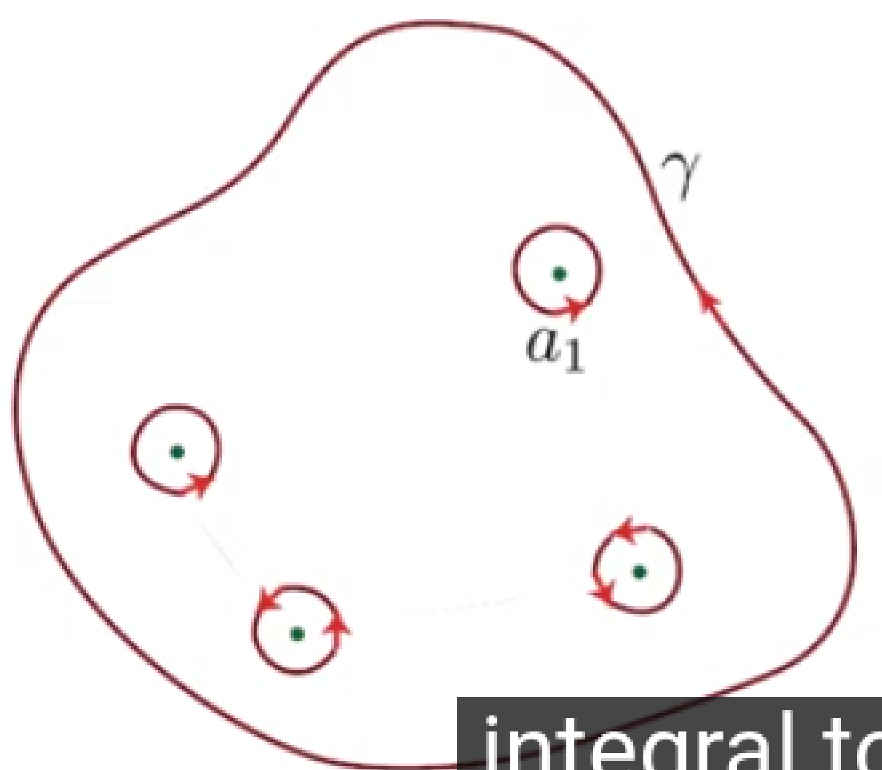
MISIS

$$f(z) = \ldots + \frac{c_{-1}}{(z-z_0)} + \ldots$$

$$\oint f(z)dz = \sum_{n=-\infty}^{\infty} c_n \oint (z-a_1)^n dz = 2\pi i c_{-1} = 2\pi i \operatorname*{res}_{z=a_1} f(z)$$

$$\operatorname*{res}_{z=z_0} f(z) = c_{-1}$$

$$\oint_\gamma f(z)dz = 2\pi i \sum_{z_i \in in} \operatorname*{res}_{z=z_i} f(z)$$



integral to the computation of the residuals of the function inside a particular contour.

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the neighborhood of the pole,

our integrand can be represented by the Laurent expansion near this pole. So let's write it down.

It's a sum from minus infinity to infinity c_n times z-a_1 to the power of n. Now, once we

start integrating each term of this sum, we'll immediately realize that all of these integrals

cancel. We just proved it with only one exception of the term containing power negative 1 where

this integral will give us 2 pi I. So the integral around the first infinitesimal circle

will simply be equal to 2 pi I times the corresponding coefficient in front of this

power c_{-1}. But this is nothing but the residue of this function at point a_1.

And of course, we may repeat the same procedure for all the other poles. And necessary,

we'll recover the statement of the theorem. Now let's think a little what this theorem

tells us. The theorem effectively reduces the computation of an arbitrary complicated

integral to the computation of the residuals of the function inside a particular contour.

But the computation of the residuals requires only the skill of Laurent expansion, and as we

saw earlier, the skill requires only the knowledge of algebra and some differential analysis,

like Taylor expansion. And this is the marvel of complex analysis: it turned the integration

into essentially differential procedure, which is an exponentially easier task.

So to see how this works in practice, let's study some simple example.

Let's consider a function f(z) equals one over z times z squared plus one. And we'd

like to compute the closed contour integral of this function along the following contours:

contour c1, contour c2, contour c3 and contour c4. Now let's have a closer look at

the analytic structure of our function. Well, it has simple poles at point z=0 and z=+- I.

Now let's simplify this function just a little bit. So first of all we decompose the fraction

1 over z squared plus 1 as 1 over z minus I minus 1 over z plus I

6:58 / 10:45 Speed 1.0x HD

integration_with_residues__3

Complex analysis, Week 3, Part 2

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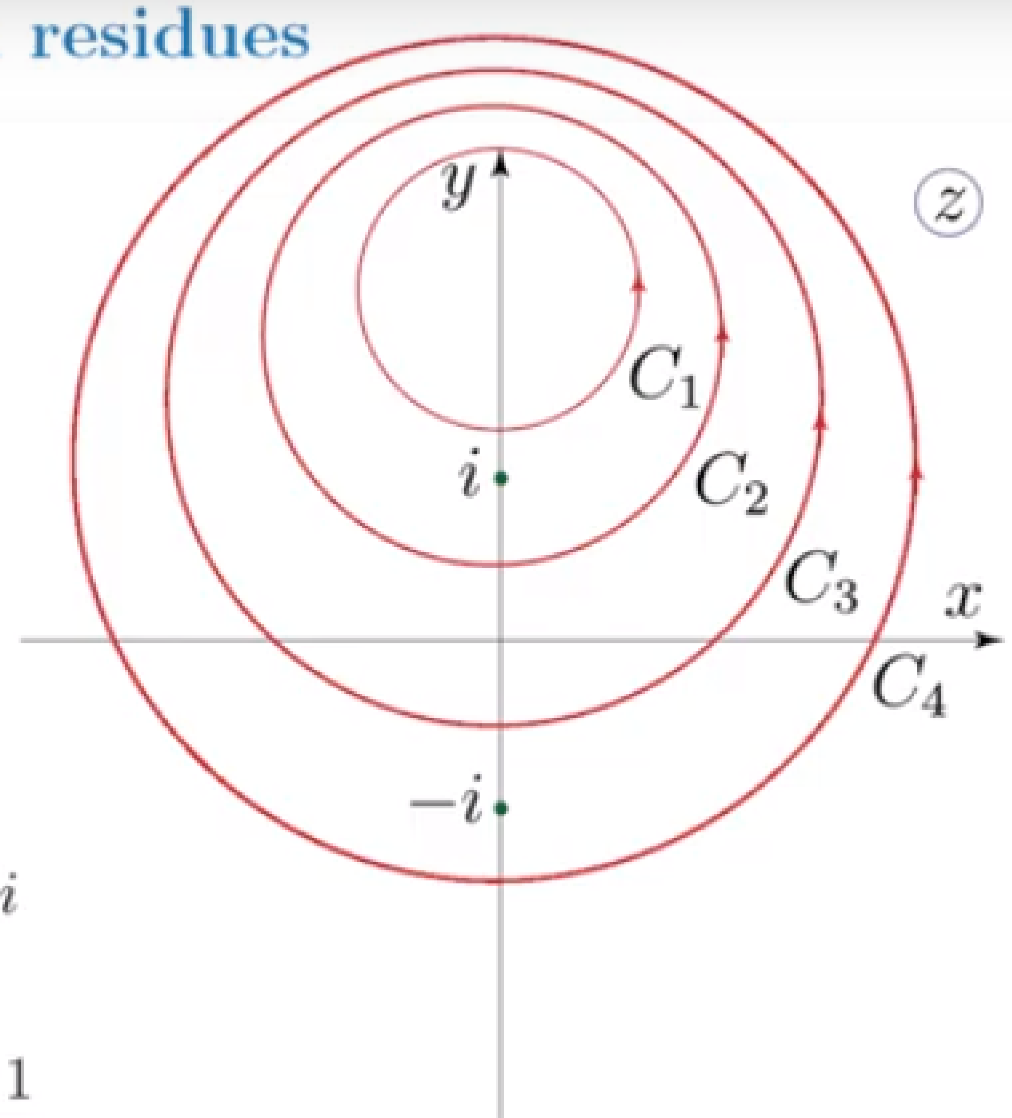
MISIS

Integration with residues

$$f(z) = \frac{1}{z(z^2+1)}$$



$$\oint_{C_1} = 0 \qquad \oint_{C_2} = 2\pi i\left(-\frac{1}{2}\right) = -\pi i$$

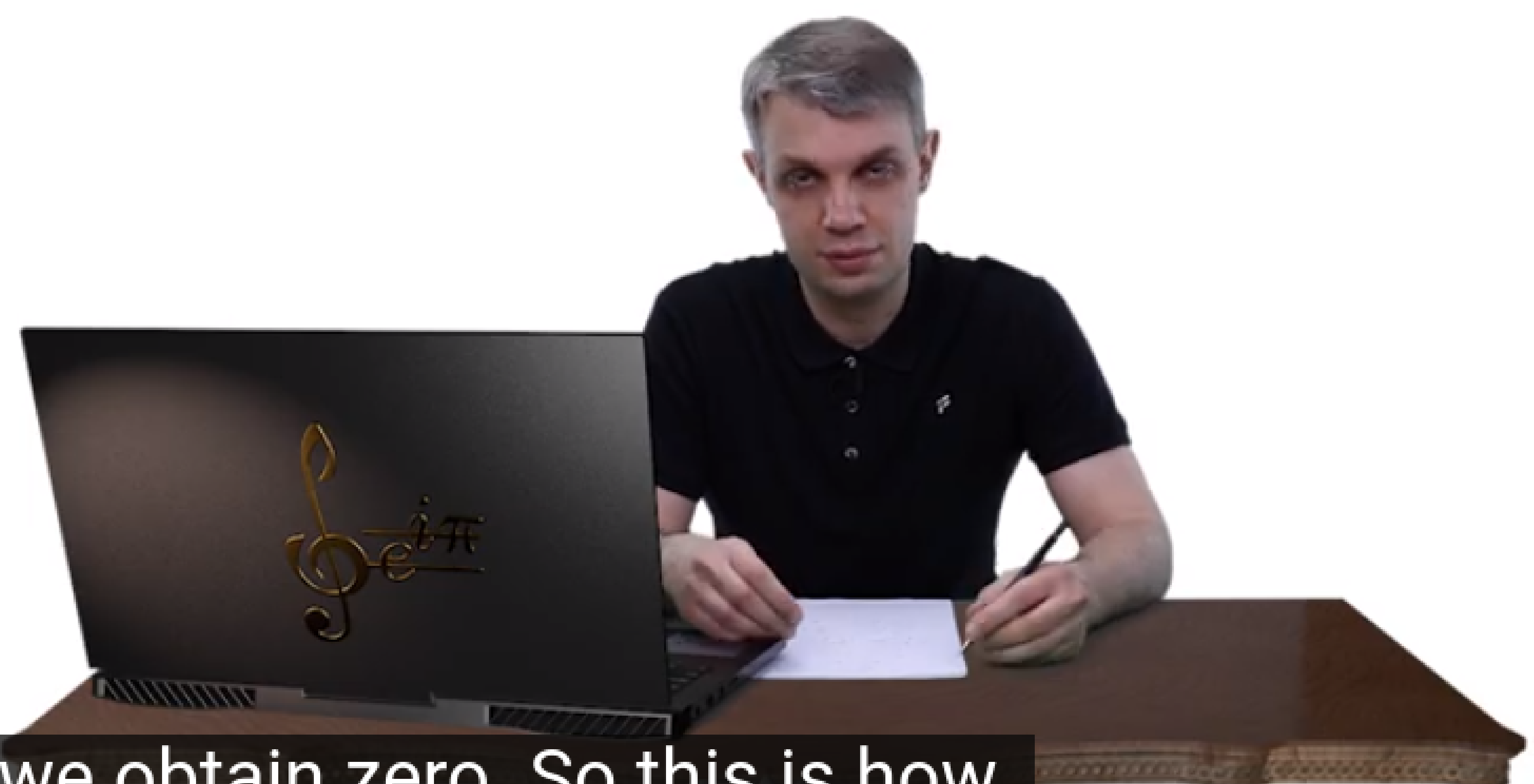
$$\oint_{C_3} = 2\pi i\left(-\frac{1}{2}+1\right) = \pi i \qquad \oint_{C_4} = 2\pi i\left(1-\frac{1}{2}-\frac{1}{2}\right) = 0$$

Simple poles: $z = 0, \pm i$

$$f(z) = \frac{1}{z}\left(\frac{1}{z-i} - \frac{1}{z+i}\right)\frac{1}{2i}$$

$$= \left(\left(\frac{1}{z-i} - \frac{1}{z}\right)\frac{1}{i} - \left(\frac{1}{z} - \frac{1}{z+i}\right)\frac{1}{i}\right)\frac{1}{2i}$$

$$= \frac{1}{z} - \frac{1}{2}\frac{1}{z-i} - \frac{1}{2}\frac{1}{z+i} \qquad \operatorname*{res}_{z=0} f(z) = 1, \quad \operatorname*{res}_{z=\pm i} f(z) = -\frac{1}{2}$$



minus one half, and we obtain zero. So this is how residue theory works in most elementary examples.

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the residues of our function at point zero and plus minus I. The residue at point zero

is equal to 1 while the residuals at points plus minus I are equal to minus one half.

Now let's use the Residual theorem to compute the respective integrals the integral over contour c1.

Well, there are no poles inside this contour, so that no residues, and the integral is equal

to zero. Now the integral over contour c2 is equal to 2 pi I times the residual of function:

there is only one pole inside this contour (point I) and the residue is equal to minus one half,

and the integral is equal to minus pi I.

Now, the integral over contour c3: now there are two residues inside: at point z=0 and at point z=I

and the integral is equal to 2 pi I times the sum of these two residues,

which is minus one half plus one, and we obtain pi I. And finally, the integral over contour c4,

which is equal to 2 pi I times the sum of all three residues, which is one minus one half and

minus one half, and we obtain zero. So this is how residue theory works in most elementary examples.

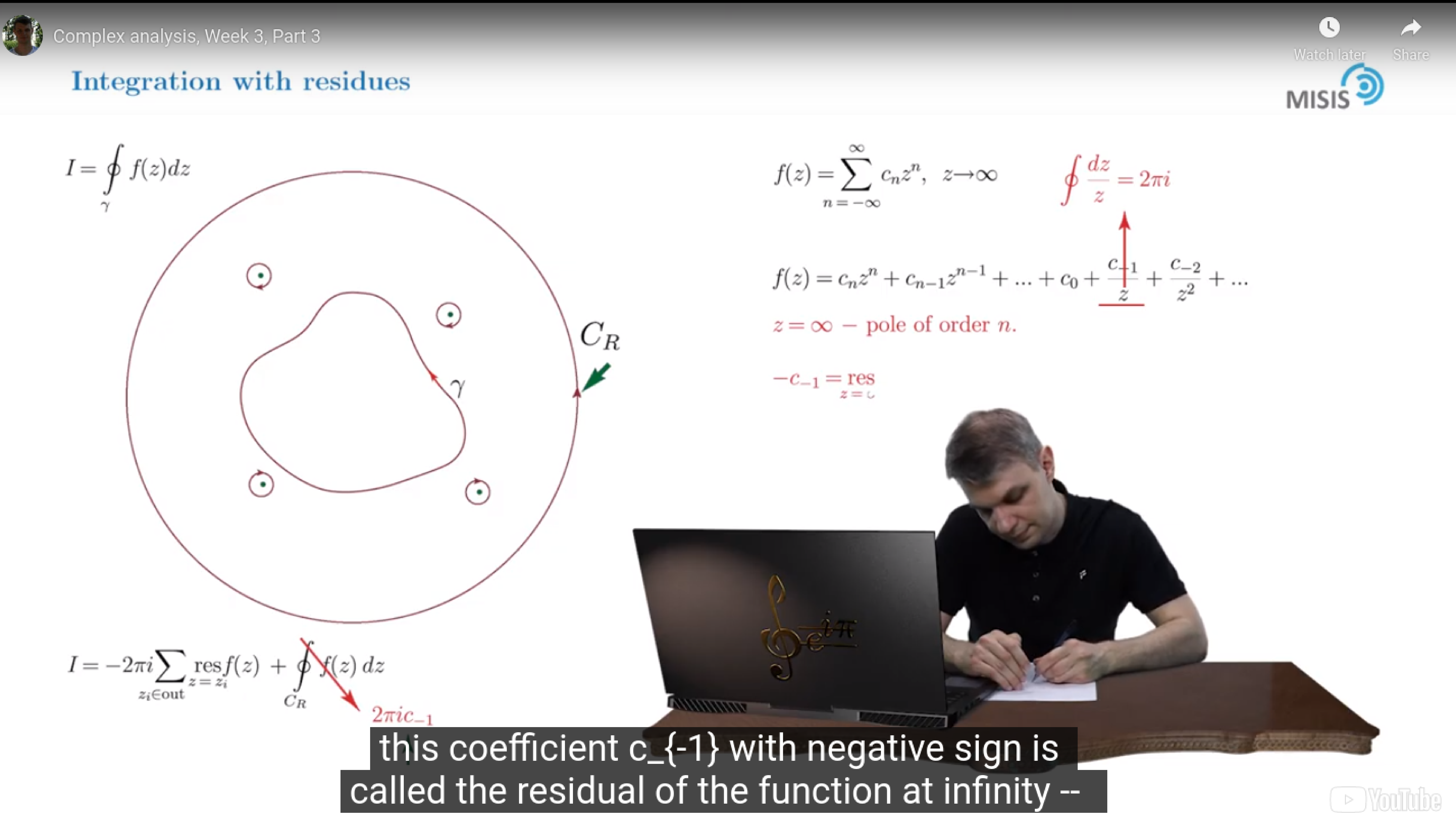
Well, in our next video we'll introduce a slightly different technique of computing the close contour

integrals -- not by the shrinking of the contour, but on the contrary, while expanding them,

and will introduce an important concept of residue of the function at infinity.

10:24 / 10:45 Speed 1.0x HD

residue_at_infinity__1



corresponding expansion,

and in general it will have the following structure:

the sum over n from minus infinity to infinity c_n times z^n. So now if this expression has

only finite amount of terms so it looks like: c_n times z_n plus c_{n-1} times z^{n-1} and so on.

Then they say that the function has a pole of order n at infinity

and such an expansion for large values of z is called the Laurent expansion of the function

at infinity. Now we take our integral but as we proved earlier all those integrals around this

circle will disappear with the only exception of the term with power n equals negative and in this

case this integral will be equal to 2 pi I. So in the end our integral along this large circle

will be equal to 2 pi I times c_{-1}. And so we have quite an interesting result: our integral is

equal to -2 pi I times the sum of the residues of the function outside this contour plus 2 pi

I times this expansion coefficient c_{-1} at infinity. And from aesthetic considerations,

this coefficient c_{-1} with negative sign is called the residual of the function at infinity --

and this is the formal definition. And remember that the residual infinity is just a clever

way of expressing the integral over an infinite circle. And finally, we can formulate a beautiful

complementary to our initial residue theorem. Now the full residual theorem goes as follows:

the integral of a meromorphic function along a closed positively oriented contour is equal to 2

pi I times the sum of the residues of the function inside this contour or minus 2 pi i times the

sum of the residues of the function outside this contour, including the residual at infinity.

Now the theorem as I stated has an amazing consequence. From what we see, we immediately

conclude that the sum of the residues of a meromorphic function in the entire complex plane

is equal to zero, if we take into account the residue at infinity.

As a specific example, let's consider the integral from our previous video.

So we integrated function f(z), which was one over z times z squared plus one over contour c4.

3:45 / 7:10 Speed 1.0x HD

residue_at_infinity__2

Complex analysis, Week 3, Part 3

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Integration with residues

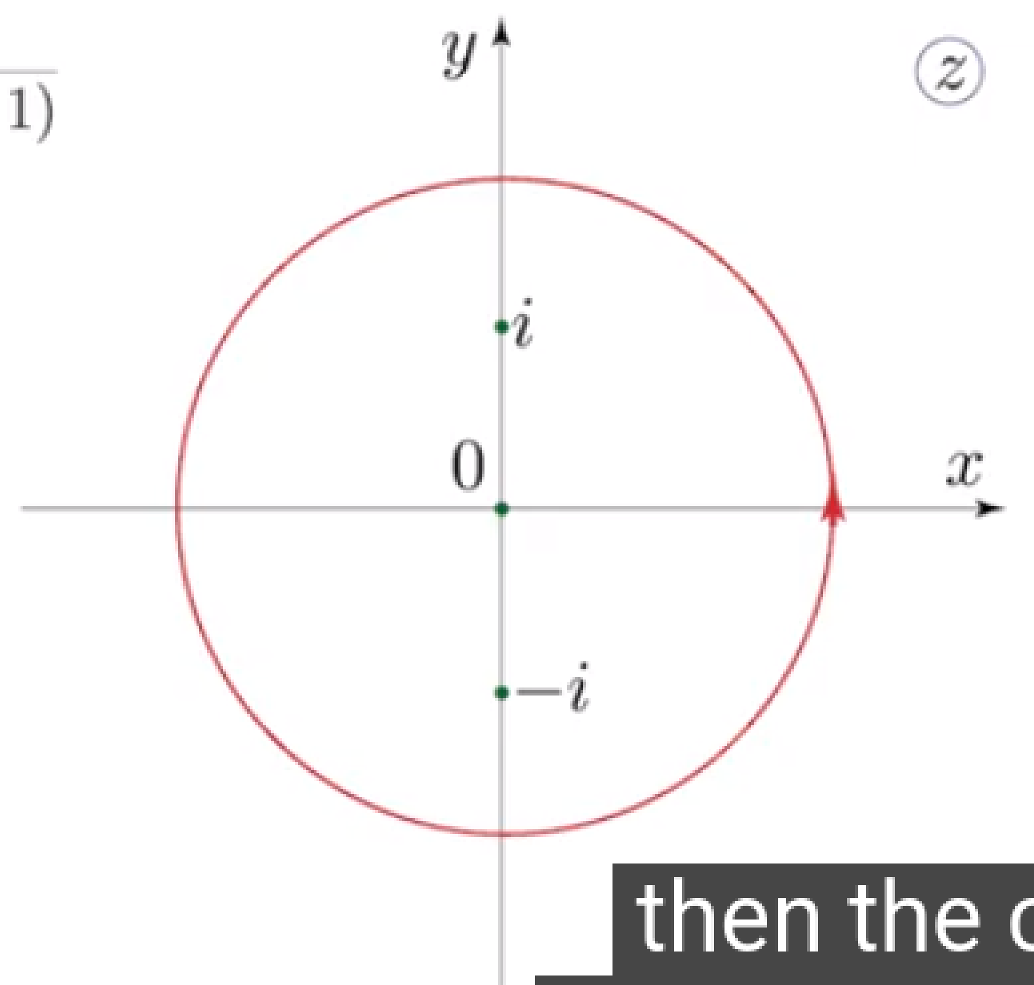
MISIS

$$I = \oint_\gamma f(z)dz = 2\pi i \sum_{z_i \in in} \operatorname*{res}_{z=z_i} f(z) = -2\pi i \sum_{z_i \in out} \operatorname*{res}_{z=z_i} f(z)$$

$f(z)$ – meromorphic

$$\sum_{z_i \in C} \operatorname*{res}_{z=z_i} f(z) = 0$$

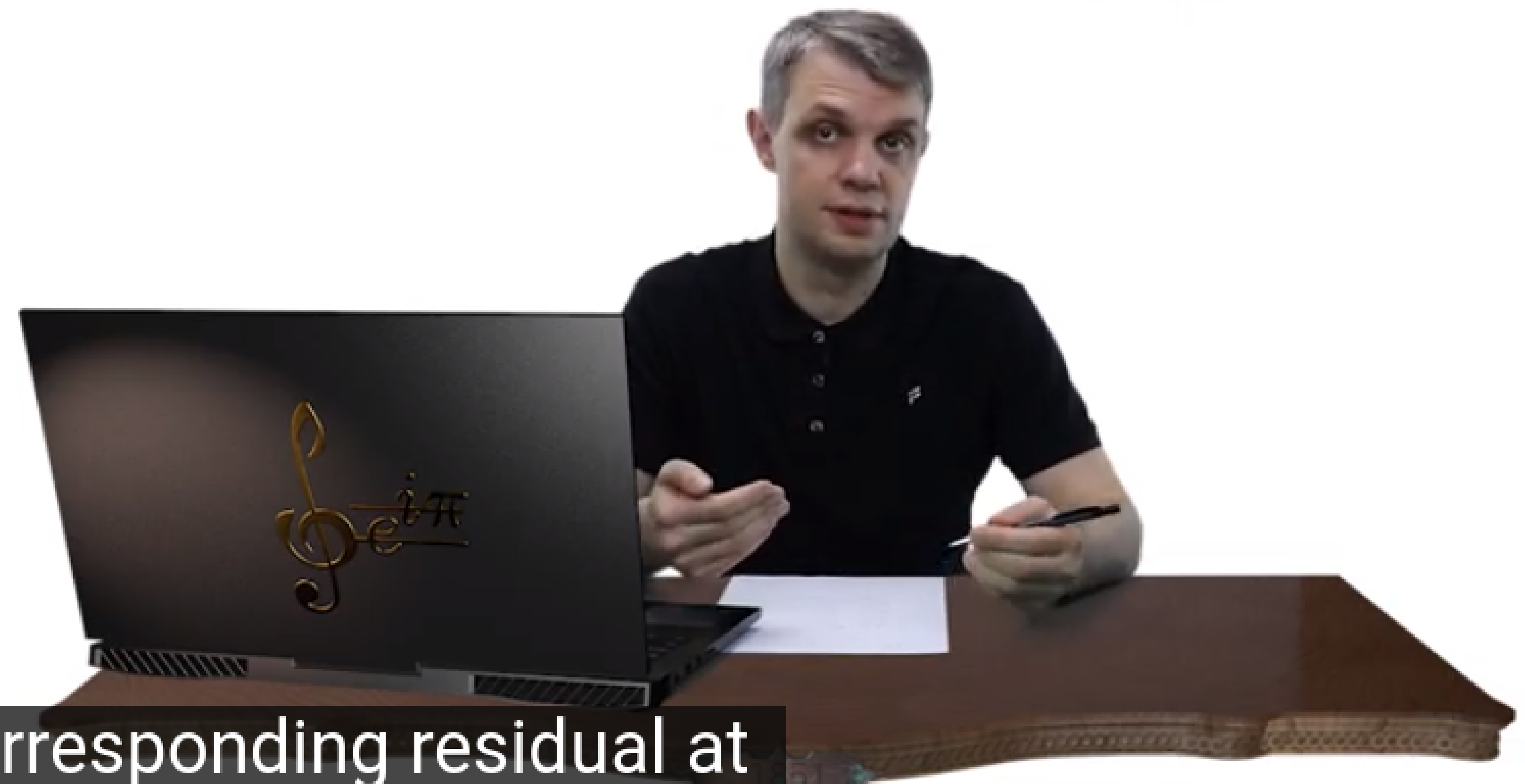
$$f(z) = \frac{1}{z(z^2+1)}$$



$$\oint f(z)\,dz = -2\pi i \operatorname*{res}_{z=\infty} f(z) = 0$$

$$f(z) = \frac{1}{z^3(1+1/z^2)} = \frac{1}{z^3} - \frac{1}{z^5} + \ldots$$

$$\operatorname*{res}_{z=\infty} f(z) = 0. \qquad f(z) \underset{z\to\infty}{\sim} \frac{1}{z^3}$$



then the corresponding residual at
infinity is zero. And the whole integral

6:55 / 7:10 Speed 1.0x HD

a geometric expansion

to obtain one over z cubed minus one over z to the power of five

and plus so on. And we see that there is no one over z term because the function just decays too

fast. And this way the residual at infinity is equal to zero. And so the integral itself is zero

and of course one could see that the residue of the function at infinity is zero from the

very start: you just need to figure out the asymptotic behavior of our function at infinity.

And you see that it decays as one always a cubed so no one over z term.

So in the end if you have some experience with residuals at infinity sometimes you just don't

need to write anything to compute integrals: you just figure out that there are no poles outside

the contour, then you're thinking a little bit about the asymptotic of your function -- usually

it's clearly seen from the very start and if your function decays faster than one over z,

then the corresponding residual at infinity is zero. And the whole integral

is zero. And this completes our discussion of the residual theorem. I hope you enjoyed it.

Riemann_sphere__1

Complex analysis, Week 3, Part 4

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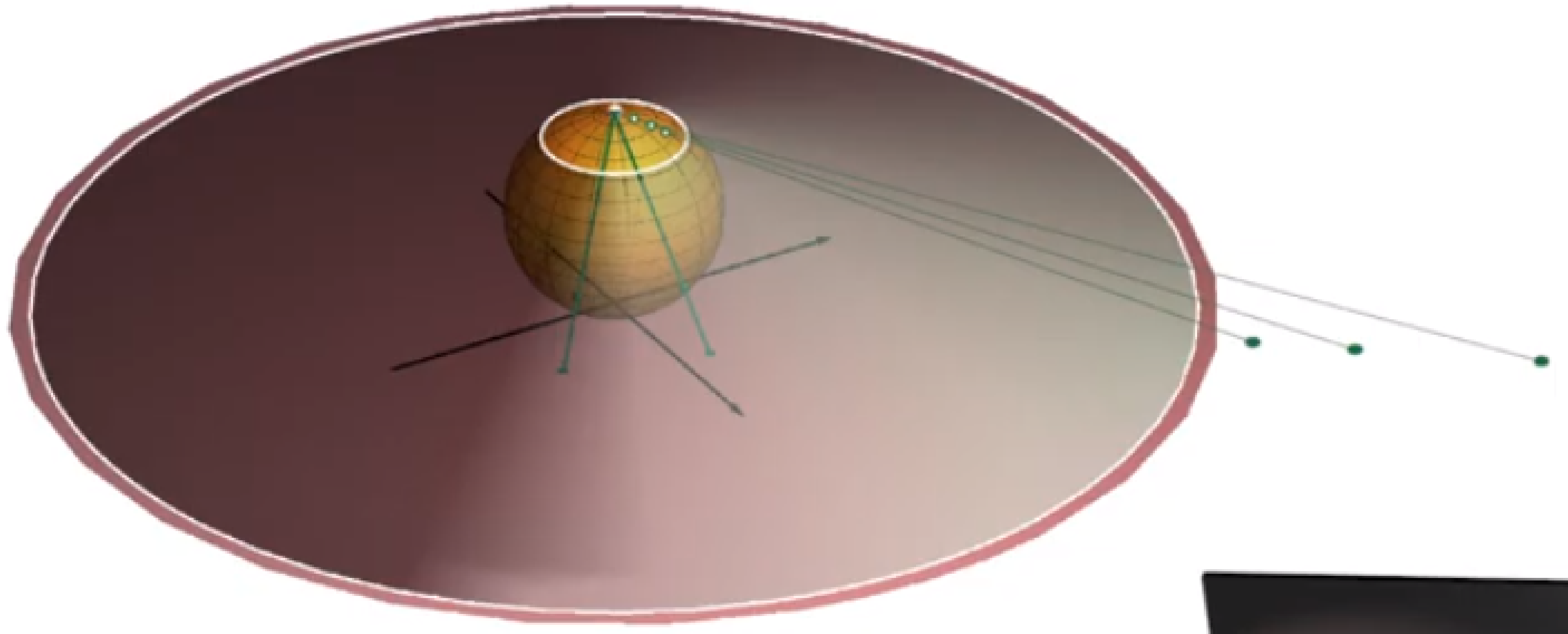
Riemann sphere

MISIS

4. $\mathbb{C}\cup\infty$ = extended complex plane



1. Riemann sphere $= S_2$

2. Stereographic projection

3. $z = \infty \to$ north pole of S_2

$\lim_{n\to\infty} z_n = \infty$ if for any $M>0$

there is n_0: $|z_n|>M$, for any $n>n_0$

two objects are differmorphic manifolds. The extended complex plane is therefore compact.

2:12 / 4:16 Speed 1.0x HD

because if you think a little

bit then you understand that the infinitesimal circle around the north pole on the riemann sphere

is a projection of an infinite circle in a complex plane. So to make these correspondence

true one-to-one, we add an additional point to a complex plane, which we call infinity.

And the image of this point on a sphere is a north pole of a sphere. Number z = infinity

doesn't take part in arithmetic calculations like an ordinary complex number, but they say that

the sequence z_n converges to infinity if for any positive number m there is number n0 such,

that the modulus of z_{n0} is greater than m. This terminology is justified because the

stereographic projection of the sequence onto Riemann sphere does converge to the north pole.

A complex plane with an addition of infinitely distant point is called an extended complex plane.

It is equivalent to the sphere, or the topologist would say that the

two objects are differmorphic manifolds. The extended complex plane is therefore compact.

As a function on a complex plane is understood as a mapping between two complex planes, the

function on extended complex plane is understood at the mapping between two Riemann spheres.

And this way such a concept as an infinite limit of a function doesn't look so unusual

anymore. And indeed it simply means that the corresponding value of our function

is positioned on the north pole of a sphere of its values. It's not hard to prove that

the circle on a plane becomes a circle on the Riemann sphere. Also, any line on a complex

plane is projected onto a circle on a Riemann sphere, and the letter is always obvious. Indeed,

to see this, let's draw a plane through a line on a complex plane and a north pole of a sphere

then the intersection of this plane with the Riemann sphere forms a projection of a line

on the Riemann sphere. But on the other hand the intersection of a plane with a sphere

is always a circle. In fact, there is a beautiful demonstration of these observations. Suppose, we

Riemann_sphere__2

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is always a circle. In fact, there is a beautiful demonstration of these observations. Suppose, we

have a collection of rings made of wire that are positioned on a sphere into second and its north

pole as projection lines. You may use an ordinary point-like source of light and here it is.

Now let's talk a little bit more about the similarity of topology of a Riemann sphere

and the complex plane. The neighborhood of infinity in the complex plane is understood as

an exterior of a circle of radius r. So the equation is: the modulus of z is greater than r.

But if you make a projection of this region onto a Riemann sphere,

you will immediately see that it is an interior of a circle surrounding the north pole.

All other definitions of limits connectedness are translated onto Riemann sphere without any change.

This way, the Riemann sphere is very useful geometrical object, which is nice to work with,

when you deal with infinities in a complex plane. Now we won't use it much in our

course but in differential geometry or algebraic topology it has many beautiful applications.

3:27 / 4:16

Speed 1.0x

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Riemann_sphere__3

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3:57 / 4:16 Speed 1.0x HD

integration_with_residues__1

Complrx analysis, Week 3, Part 5

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Integration with residues

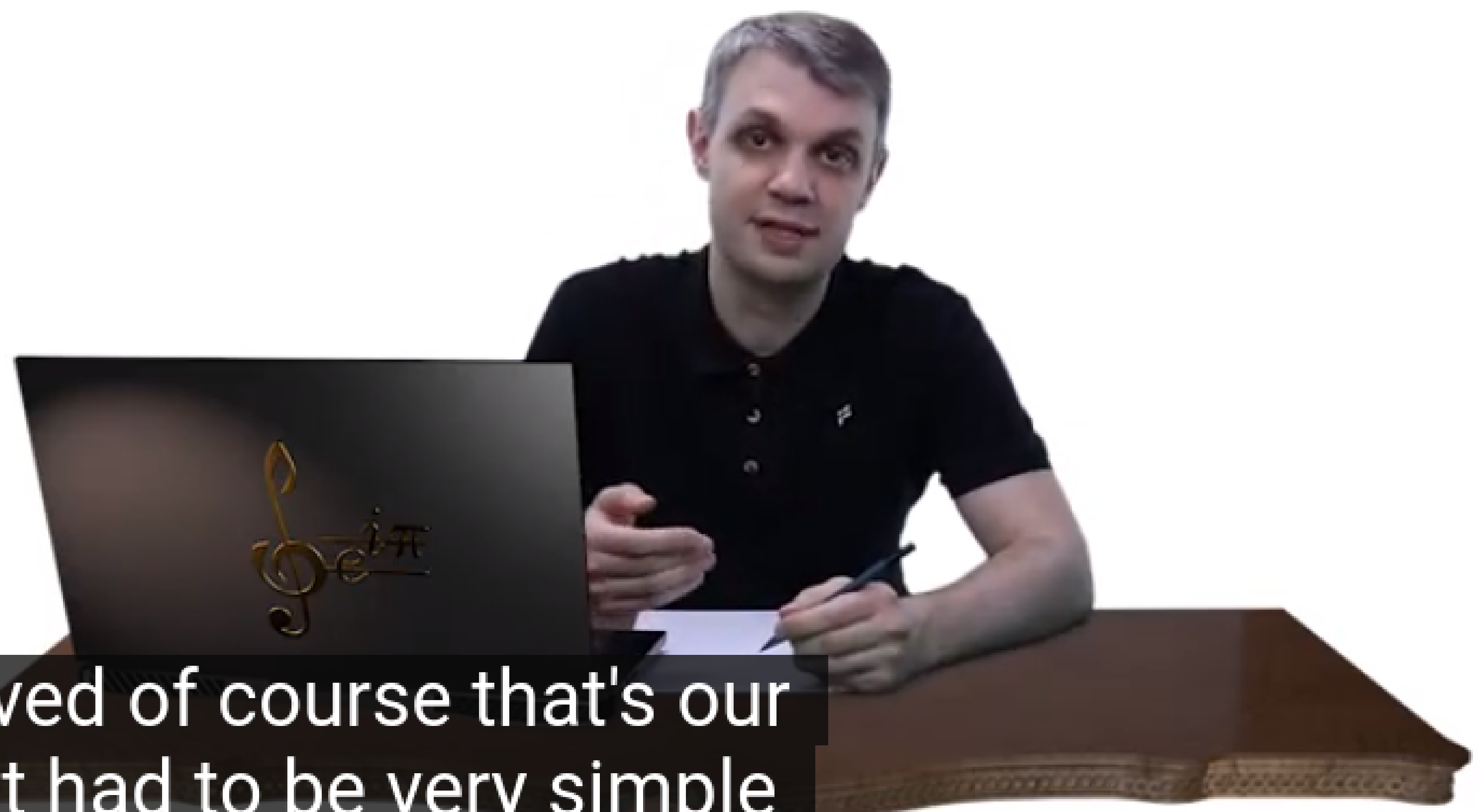
MISIS

$$f(z) = \frac{e^z}{z^3} \qquad \operatorname*{res}_{z=0} f(z) = \frac{1}{2}$$

$$z = 0\text{: third order pole} \qquad f(z) \sim \frac{1}{z^3}, \quad z \to 0$$

$$f(z) \underset{z\to 0}{=} \frac{1 + z + \frac{z^2}{2!} + \dots}{z^3} = \dots + \frac{1}{2!z} + \dots$$

$$c_{-1} = \frac{1}{2!} = \frac{1}{2}$$



but don't get deceived of course that's our first example and it had to be very simple

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of 1/z^3.

And the task is to find the residual at the origin. So obviously point z=0

is a third order pole. Well, how to see this? The exponential at the origin behaves as one

and we obtain: one over z^3 cubed behavior and the vicinity of zero, so indeed it's a third

order pole. And to extract the residual we simply perform a Laurent expansion of our exponential

in the vicinity of zero. as a remember we are hiding from one over the term in our expansion

so we have one plus z plus z squared over 2 factorial and plus so on always e cubed

and we don't need any high order terms because everything we need is already here

we need the term z squared in denominator because combined with z cubed in the denominator

it will give us one over the term in our expansion and we see that the corresponding

coefficient c negative one is simply one over two factorial so one half and that's our residual

but don't get deceived of course that's our first example and it had to be very simple

So the next example our function f(z) is equal to exponential to the power of a/z,

where a is some parameter, times z to the power of n, where n is some positive integer. And the

asignment is to find the residue of this function at infinity. Well, to find the residue at infinity

we need to expand this function for large values of z. So basically we perform 1/z expansion

and this is essentially a Taylor series for our exponential. So we write down the full series,

and in this expansion we need only 1/z to the power of (n+1) term,

because combined with z to the power of n. It will give us one over z term in the expansion,

so our coefficient c_{-1} in the Laurent expansion will look like a to the power of n+1 divided by

(n+1)! and the residual at infinity is minus c_{-1}. And so we obtain the answer.

The next example: function f(z) equals 1/((z-1)^2(z^2+1)). And the assignment

is finding the residues of this function at all finite points. The pole candidates are the

1:45 / 11:42 Speed 1.0x HD

Integration with residues

$$f(z) = e^{a/z} z^n, \; n - \text{natural} \qquad \operatorname*{res}_{z=\infty} f(z) = -c_{-1} = -\frac{a^{n+1}}{(n+1)!}$$

$$e^{a/z} = \sum_{n=0}^{\infty} \frac{a^n}{n! z^n}$$

$$c_{-1} = \frac{a^{n+1}}{(n+1)!}$$

that the corresponding

coefficient c negative one is simply one over two factorial so one half and that's our residual

but don't get deceived of course that's our first example and it had to be very simple

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The next example: function f(z) equals 1/((z-1)^2(z^2+1)). And the assignment

is finding the residues of this function at all finite points. The pole candidates are the

zeros of the denominator, and we have the second order zero: z=1 and two first order zeros z=+-I.

And since our nominator is constant, we indeed conclude that z=1 is the second order

pole while z=+- I are first order poles. First let's find the residue at point z=1.

So we introduce a change of variables z-1=epsilon and we expand our function in epsilon. So f(z)

equals 1 over epsilon squared times 2 plus 2 epsilon plus epsilon squared. And let's rewrite

this fraction as 1 over epsilon squared times 1 over two plus two epsilon plus epsilon squared.

As usual, we are cutting for one over epsilon term, but here we already have

a prefactor of one epsilon squared, it is multiplied by some expression

which can be Taylor expanded in epsilon. And what I would argue is that we only need to keep

integration_with_residues__3

Complrx analysis, Week 3, Part 5

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Integration with residues

MISIS

$$f(z)=\frac{1}{(z-1)^2(z^2+1)}$$

$z=1$ (II order), $z=\pm i$ (I order)

1. $z=1.$

$$z-1=\varepsilon$$

$$f(z)=\frac{1}{\varepsilon^2(2+2\varepsilon+\varepsilon^2)}=\frac{1}{\varepsilon^2}\underline{\underline{\frac{1}{2\cancel{+2\varepsilon+\varepsilon^2}}}} \quad A+B\varepsilon+\ldots$$

$$=\frac{1}{2\varepsilon^2}\frac{1}{1+\varepsilon}=\frac{1}{2\varepsilon^2}(1-\varepsilon+\ldots)=\frac{1}{\varepsilon^2}-\frac{1}{2}\frac{1}{\varepsilon}+\ldots$$

$$c_{-1}=-\frac{1}{2} \qquad \operatorname*{res}_{z=1} f(z)=-\frac{1}{2}$$

2. $z=i.$

$$z-i=\varepsilon$$

$$f(i+\varepsilon)=\frac{1}{(z+i)(z-i)(z-1)^2}=\boxed{\frac{1}{\varepsilon(2i\cancel{+\varepsilon})(i-1\cancel{+\varepsilon})^2}} \quad A+B\varepsilon$$

$$c_{-1}=\frac{1}{2i(i-1)^2}$$

times I minus one squared. now I minus one squared is equal to minus two I, and we obtain 1/4.

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6:30 / 11:42 Speed 1.0x HD

point.

Now let's find the residue at point z equals I. And again as before, we introduce a new variable

z - I equals epsilon and expand in epsilon. So f(I + epsilon) equals 1 over z plus I

times z minus I times z minus 1 squared. And then we plug in this change and obtain 1 over epsilon

times 2I plus epsilon times I minus 1 plus epsilon squared. And again we have a prefactored

term 1 over epsilon times some expression machine with Taylor expanded and epsilon.

And using the same reasoning as before I want to argue that it's enough to keep only zero order

terms in epsilon in this expansion. Why? because if you repetitive again is: a plus b epsilon.

Then again we see that this b epsilon term is redundant: it produces regular and epsilon term.

So we don't need it and that means that we can cross out epsilon in the rest of the terms in

the denominator. And we already got our c_{-1} coefficient, here it is. It's one over two I

times I minus one squared. now I minus one squared is equal to minus two I, and we obtain 1/4.

Now the third residue. i could of course repeat the same procedure for point z = -I

but here I'd like to show you some workaround. We remember the theorem that the sum of all the

residuals of the function including the residual at infinity is equal to zero.

Now look at this function: what do you think could be residual of this function at infinity?

As you remember, the residual at infinity is given by the asymptotic behavior of our function

at large values of z. But here the asymptotic behavior is pretty obvious, it's 1 over z to

the power of 4. It decays pretty quickly, it doesn't have one over z term in its expansion

near infinity. So the residual at infinity is simply equal to zero. It's clear and that means

that the sum of all three remaining residuals at point 1, I and -I is equal to 0. We already

computed two of these residues, we obtained minus 1/2 and 1/4. And so the third residue is 1/4.

Integration with residues

$$f(z) = \frac{1}{(z-1)^2(z^2+1)}$$

$z=1$ (II order), $z=\pm i$ (I order)

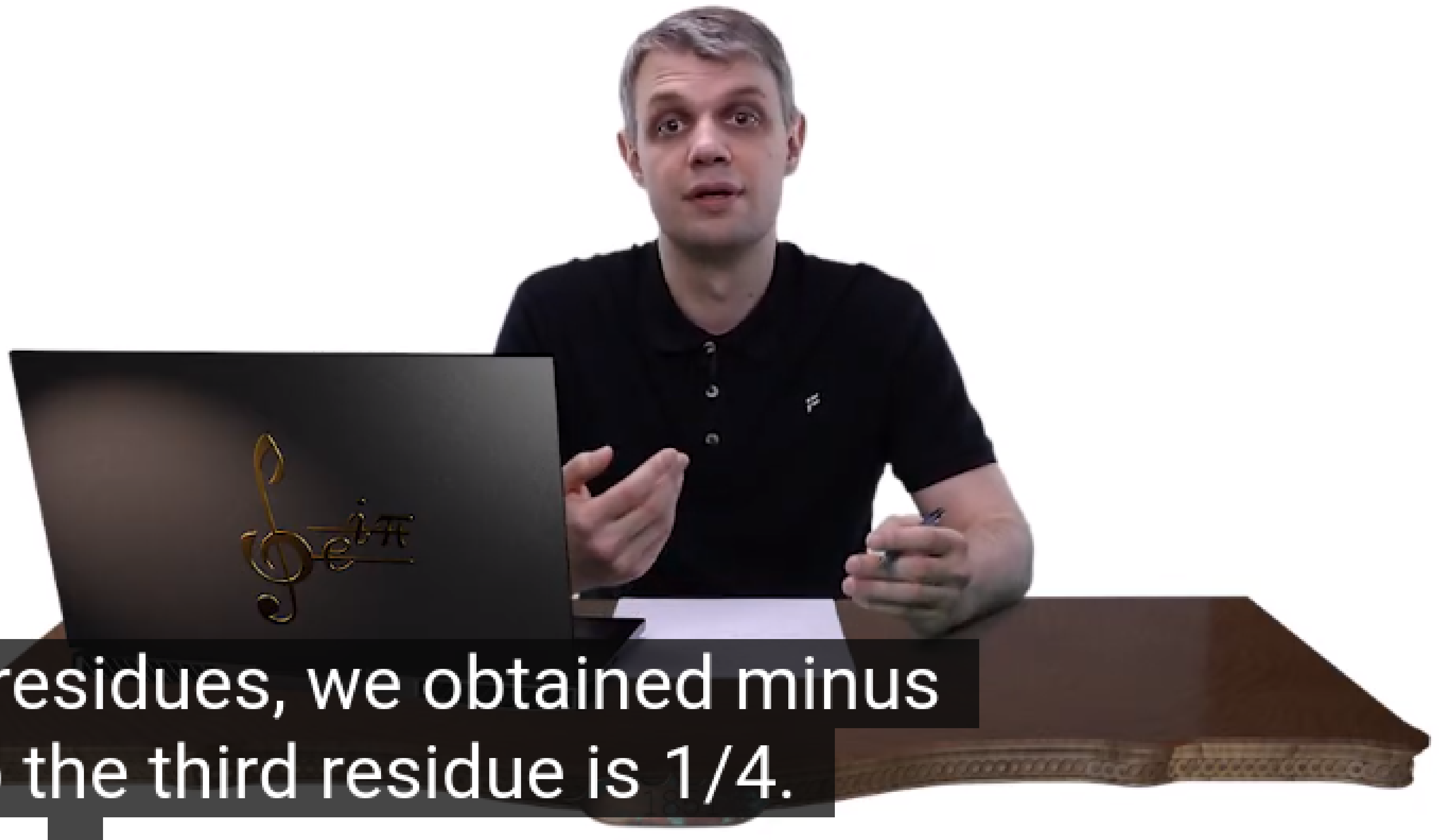
1. $z=1$.

$$\operatorname*{res}_{z=1} f(z) = -\frac{1}{2}$$

$$f(z) \underset{z\to\infty}{\sim} \frac{1}{z^4} \qquad \operatorname*{res}_{z=\infty} f(z) = 0$$

2. $z=i$.

$$\operatorname*{res}_{z=i} f(z) = \frac{1}{4}$$



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that the sum of all three remaining residuals at point 1, I and -I is equal to 0. We already

computed two of these residues, we obtained minus 1/2 and 1/4. And so the third residue is 1/4.

And our final example for this lecture: function f(z) equals cos(z)/(z^2+1)^2.

As before, the poles are the zeroes of the denominator, there are two of them: z = +- I

and they are the second order zeros. And since the nominator doesn't vanish at this point,

then these points are second order poles of this function.

So let us compute the residual at point z=I. Again we change the variable, z - I equals epsilon.

And here we go, f(I + epsilon) equals to cos(I plus epsilon) over z plus I squared

times z minus I squared. Well, z minus I squared is converted to epsilon squared, while z plus I

squared becomes 2i plus epsilon squared. Well, we have one epsilon squared term as a pre-factor

and the remaining expression is cosine of I plus epsilon divided by 2i plus epsilon squared.

And we need to Tailor expand this second term and keep only first of the terms in epsilon,

as you remember from your previous exercise. So let's do this.

integration_with_residues__5

Complrx analysis, Week 3, Part 5

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MISIS

Integration with residues

$$f(z)=\frac{\cos z}{(z^2+1)^2}$$

$$c_{-1}=-\frac{1}{4}i(\overbrace{\cosh 1-\sinh 1}^{\frac{1}{e}})=-\frac{i}{4e}=\operatorname*{res}_{z=i}f(z)$$

$$z=\pm i$$

1. $z=i$.

$$z-i=\varepsilon$$

$$f(i+\varepsilon)=\frac{\cos(i+\varepsilon)}{(z+i)^2(z-i)^2}=\frac{\cos(i+\varepsilon)}{\varepsilon^2(2i+\varepsilon)^2}=\frac{1}{\varepsilon^2}\frac{\cos(i+\varepsilon)}{(2i+\varepsilon)^2}$$

$$=\frac{1}{\varepsilon^2}\frac{\overset{\cosh 1}{\cancel{\cos i}}\,\overset{1}{\cancel{\cos\varepsilon}}-\overset{i\sinh 1}{\cancel{\sin i}}\,\overset{\varepsilon}{\cancel{\sin\varepsilon}}}{-4(1-\frac{1}{2}i\varepsilon)^2}=-\frac{1}{4\varepsilon^2}\frac{\cosh 1-i\sinh 1\,\varepsilon}{(1-\frac{1}{2}i\varepsilon)^2}$$

$$=-\frac{1}{4\varepsilon^2}(1+i\varepsilon)(\cosh 1-i\sinh 1\,\varepsilon)$$

$$()()=\ldots i\varepsilon\cosh 1-i\varepsilon\sinh 1\ldots$$

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epsilon squared. Now retaining only first of the terms in denominator we substitute cosine

of epsilon with one and sine epsilon with epsilon, and cosine of I is turned into cosine hyperbolic

of one, while sine of y is turned into I sine hyperbolic of 1. And the only remaining thing

is the expansion of our function in denominator. It's a binomial expansion with power negative 2,

and we obtain: 1 plus I epsilon times cosine of 1 minus I sine of 1 times epsilon. And now we need

to multiply braces and collect only those terms which contain first powers of epsilon. And here

we go: we have I epsilon times cosine hyperbolic of 1 minus I epsilon times sine hyperbolic of one.

Then combining this with one over epsilon squared prefactor we obtain our c_{-1} coefficient:

it's equal to minus 1/4 times i times cosine hyperbolic of one - sine hyperbolic of one.

The difference of these two hyperbolic functions will produce one over e. As the result we obtain

minus I over four e and this is the residue of our function and point z=I.

Now the residue at point z equals negative I. Well, in principle we could repeat all

these computations for this point and I actually strongly advise you to do so just for practice.

But to save time again I'll use a workaround as before. Let us use e theorem about the sum

of the residuals: it should be equal to zero. Well, there are two residues of this function

at finite points, z equals plus minus I, and the residual at infinity. The residual at infinity

is the coefficient at one with the power of the expansion at infinity Bbut here is our function,

look at it: it's an even function of z. It can't have one over z terms in its expansion,

it only has even powers of z. So the residual at infinity is automatically zero. And that

means that the second residue of our function is simply minus the residue we just obtained,

so it's I over four e. And that completes our practice, so enjoy your homework.

10:42 / 11:42 Speed 1.0x HD

integration_with_residues__6

Complrx analysis, Week 3, Part 5

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Integration with residues

MISIS

$$f(z) = \frac{\cos z}{(z^2+1)^2}$$

$$c_{-1} = -\frac{1}{4} i \left(\overset{\frac{1}{e}}{\cosh 1 - \sinh 1}\right) = -\frac{i}{4e} = \operatorname*{res}_{z=i} f(z)$$

$z = \pm i$

1. $z = i.$

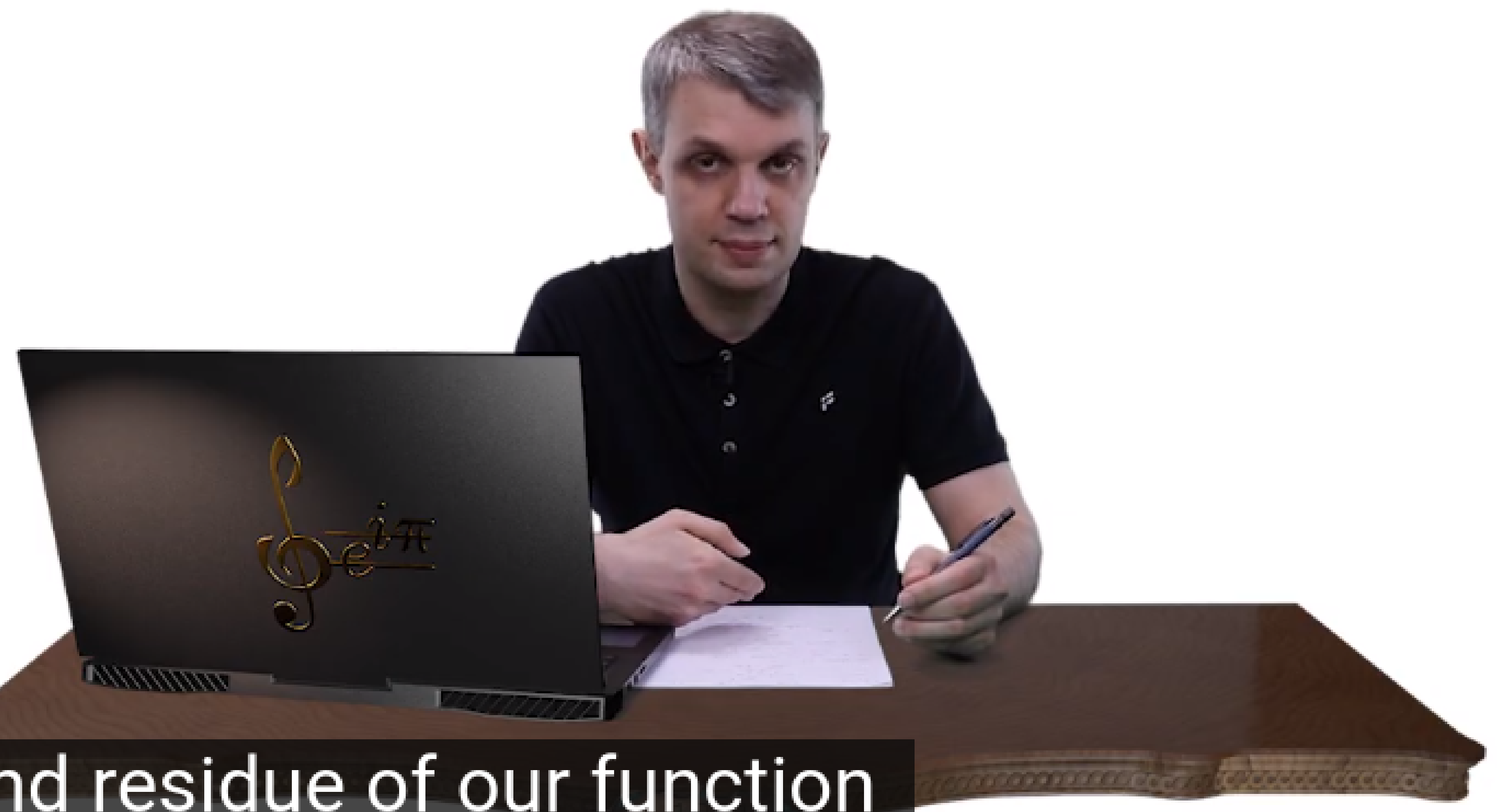
$$z - i = \varepsilon$$

$$f(i+\varepsilon) = \frac{\cos(i+\varepsilon)}{(z+i)^2(z-i)^2} = \frac{\cos(i+\varepsilon)}{\varepsilon^2(2i+\varepsilon)^2} = \frac{1}{\varepsilon^2}\frac{\cos(i+\varepsilon)}{(2i+\varepsilon)^2}$$

$$= \frac{1}{\varepsilon^2}\frac{\overset{\cosh 1}{\cancel{\cos i}}\,\overset{1}{\cancel{\cos\varepsilon}} - \overset{i\sinh 1}{\cancel{\sin i}}\,\overset{\varepsilon}{\cancel{\sin\varepsilon}}}{-4(1-\frac{1}{2}i\varepsilon)^2} = -\frac{1}{4\varepsilon^2}\frac{\cosh 1 - i\sinh 1\,\varepsilon}{(1-\frac{1}{2}i\varepsilon)^2}$$

$$= -\frac{1}{4\varepsilon^2}(1+i\varepsilon)(\cosh 1 - i\sinh 1\,\varepsilon)$$

$$()() = \ldots i\varepsilon\cosh 1 - i\varepsilon\sinh 1 \ldots$$



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so it's I over four e. And that completes our practice, so enjoy your homework.

11:29 / 11:42 Speed 1.0x HD

general_formula_for_the_residue__1

Complex analysis, Week 3, Part 6

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Integration with residues

MISIS

$$f(z) = \frac{c_{-n}}{(z-a)^n} + \frac{c_{-n+1}}{(z-a)^{n-1}} + \dots + \frac{c_{-1}}{z-a} + c_0 + \dots$$

$$\operatorname*{res}_{z=a} f(z) = \frac{1}{(n-1)!}\frac{d^{n-1}}{dz^{n-1}} f(z)(z-a)^n$$

$$(z-a)^n f(z) = c_{-n} + c_{-n+1}(z-a) + c_{-n+2}(z-a)^2 + c_{-n+3}(z-a)^3 + \dots$$

$$+ c_{-1}(z-a)^{n-1} + c_0(z-a)^n + \dots$$

$$\frac{d}{dz}(z-a)^n f(z) = c_{-n+1} + c_{-n+2}2(z-a) + c_{-n+3}3(z-a)^2 + \dots$$

$$+ c_{-1}(n-1)(z-a)^{n-2} + c_0 n(z-a)^{n-1} + \dots$$

$$\frac{d^2}{dz^2}(z-a)^n f(z) = 2c_{-n+2} + 2\cdot 3c_{-n+3}(z-a) + \dots$$

$$+ c_{-1}(n-1)(n-2)(z-a)^{n-3} + c_0 n(n-1)(z-a)^{n-2} + \dots$$

$$\frac{d^{n-1}}{dz^{n-1}}(z-a)^n f(z) = (n-1)!c_{-1} + \cancel{c_0 n!(z-a) + \dots}$$

d n minus 1 over dz n minus 1 f of z times z minus a to the power of n. And then we take the limit

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c minus 1 term. And its power: initially it was n minus one but after n minus one

differentiations will become zero. So naturally the first term again will have no z minus a power

so let's write it down on the left hand side. We have n minus 1's derivative.

But on the right hand side we'll have n minus 1 factorial time

c minus 1 plus c_0 times obviously n factorial times z minus a to the first power,

right, and then all the rest of the terms with the raising powers of z minus a. So as you probably

observe, we are getting closer to our goal. We need to isolate term c_{-1} and we almost

achieved it. And our final step is setting z=a in both parts of this equation. This way

we'll eliminate all the terms, which stand to the right of the c_{-1} expression and here

it is. This way we obtained the desired formula for our residual: the residual of function f(z)

at point z equals a contains n minus 1 derivatives so it's 1 over n minus 1 factorial

d n minus 1 over dz n minus 1 f of z times z minus a to the power of n. And then we take the limit

z equals a at the end of the calculation. And let's see how this formula works in practice.

Our first example would be a function f(z) = cosine of z over z minus one to the second power.

Well, first of all this function has a second order pole at point z equals one. So let's find

the residual at this pole. So we use our formula 1 over 1 factorial (which is 1) the first derivative

of f(z) times z minus 1 squared. And then we set z equals 1 at the end of the calculation.

So once we plug in the function f(z), we immediately note that z minus 1 squared in

denominator and denominator cancel each other, and we are left with the first derivative of

cosine of z at point z equals one. And so we obtain the result: minus sine of one.

Quite fast and effective! Our second example is in fact the example

from our previous video. Function f(z) equals 1 over z minus 1 squared times z squared plus 1.

And let's find the residue at point z equals 1 and z equals plus minus i

5:19 / 12:46 Speed 1.0x HD

general_formula_for_the_residue__2

Complex analysis, Week 3, Part 6

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Integration with residues

MISIS

$$\operatorname*{res}_{z=a} f(z) = \frac{1}{(n-1)!}\frac{d^{n-1}}{dz^{n-1}} f(z)(z-a)^n \Big|_{z\to a}$$

$z = i$

Simple pole: $\operatorname*{res}_{z=a} f(z) = f(z)(z-a)\Big|_{z\to a}$

$$\operatorname*{res}_{z=i} f(z) = \frac{1}{(z+i)\cancel{(z-i)}(z-1)^2}\cancel{(z-i)}\Big|_{z=i} = \frac{1}{(i-1)^2 2i} = \frac{1}{4}$$

1. $f(z) = \dfrac{\cos z}{(z-1)^2}$

$$\operatorname*{res}_{z=1} f(z) = \frac{1}{1}\frac{d}{dz} f(z)(z-1)^2\Big|_{z\to 1} = \frac{d}{dz}\cos z\Big|_{z=1} = -\sin 1$$

2. $f(z) = \dfrac{1}{(z-1)^2(z^2+1)}$

$z = 1,\ z = \pm i$

$$\operatorname*{res}_{z=1} f(z) = \frac{d}{dz}\frac{1}{(z^2+1)\cancel{(z-1)^2}}\cancel{(z-1)^2}\Big|_{z=1}$$

$$= -\frac{2z}{(z^2+1)^2}\Big|_{z=1} = -\frac{1}{2}$$

this is precisely our answer which we obtained using the Laurent expansion.

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8:35 / 12:46 Speed 1.0x HD

minus 2 over 4 = -1/2.

The next pole: z=I. Well, actually that's a first order pole. And now I think it's a

good time to write down a simplified version of our general formula for a simple pole. Indeed,

we see that since n=1, this formula doesn't require any differentiation at

all. So let's write this down. The residual of function f(z) at its arbitrary simple pole z=a

is simply given by the expression: f(z) times z-a when z is tending to a. And now let's apply

this formula here. So we take our function f(z) and multiply it by z-I. But before we do this

let's expand the denominator: z squared plus 1 = z plus I times z minus I.

And again, as before we have this cancellation: z - I in the denominator and nominator. And setting

z equals I we obtain the final expression for the residual: it's 1 over I minus 1 squared

times 2I. I minus 1 squared is simply minus 2I and we obtain 1/4. And as you remember

this is precisely our answer which we obtained using the Laurent expansion.

And finally the third example: f(z) is equal to 1/(z^2 + 1)^3. We see that this function has third

order poles at points z = +-I. So let's find the residue of this function say at point z equals I.

Again, we employ our formula and this time it will require the second order differentiation.

So we have one over two factorial, which is one half, the second derivative of 1/(z^2 + 1)^

multiplied by (z-I)^3. And as usual it's desirable to expand the function in the denominator

and let's do this. So we obtain (z-I)^3 / (z-I)^3 times (z + I)^3. So z-I cubed

cancelled and we are left with the second derivative of 1 over (z + I)^3.

And this derivative is simply 12 over (z+I)^5 and then we set z=I

and obtain the result: 6 over (2I)^5 which is minus three sixteenths of I. So now you

probably noticed that the main advantage of this formula is that it works so quick.

And as a final remark let's obtain an alternative formula for first order pole. It's also very

general_formula_for_the_residue__3

Complex analysis, Week 3, Part 6

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Integration with residues

$$\operatorname*{res}_{z=a} f(z) = \frac{1}{(n-1)!}\frac{d^{n-1}}{dz^{n-1}} f(z)(z-a)^n\Big|_{z\to a}$$

$$= \frac{1}{2}\frac{12}{(z+i)^5}\Big|_{z=i} = \frac{6}{}$$

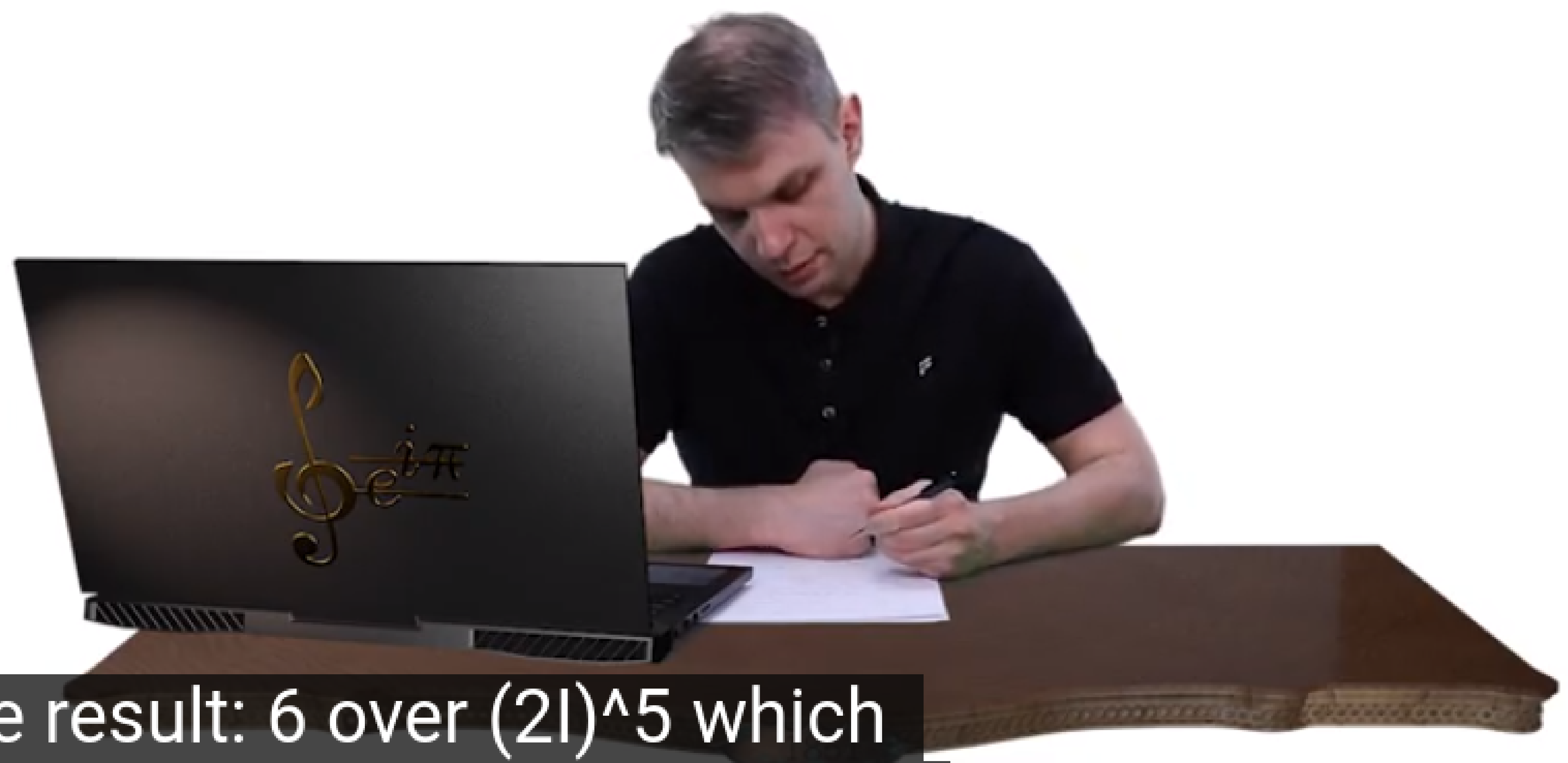
Simple pole: $\operatorname*{res}_{z=a} f(z) = f(z)(z-a)\Big|_{z\to a}$

3. $f(z) = \dfrac{1}{(z^2+1)^3}$

$z = \pm i$ (III order)

$$\operatorname*{res}_{z=i} f(z) = \frac{1}{2!}\frac{d^2}{dz^2}\frac{1}{(z^2+1)^3}(z-i)^3\Big|_{z=i}$$

$$= \frac{1}{2}\frac{d^2}{dz^2}\frac{\cancel{(z-i)^3}}{\cancel{(z-i)^3}(z+i)^3}\Big|_{z=i}$$



and obtain the result: 6 over (2I)^5 which
is minus three sixteenths of I. So now you

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z equals I we obtain the final expression for the residual: it's 1 over I minus 1 squared

times 2I. I minus 1 squared is simply minus 2I and we obtain 1/4. And as you remember

this is precisely our answer which we obtained using the Laurent expansion.

And finally the third example: f(z) is equal to 1/(z^2 + 1)^3. We see that this function has third

order poles at points z = +-I. So let's find the residue of this function say at point z equals I.

Again, we employ our formula and this time it will require the second order differentiation.

So we have one over two factorial, which is one half, the second derivative of 1/(z^2 + 1)^

multiplied by (z-I)^3. And as usual it's desirable to expand the function in the denominator

and let's do this. So we obtain (z-I)^3 / (z-I)^3 times (z + I)^3. So z-I cubed

cancelled and we are left with the second derivative of 1 over (z + I)^3.

And this derivative is simply 12 over (z+I)^5 and then we set z=I

and obtain the result: 6 over (2I)^5 which is minus three sixteenths of I. So now you

probably noticed that the main advantage of this formula is that it works so quick.

And as a final remark let's obtain an alternative formula for first order pole. It's also very

suitable and is used quite often. So suppose our function has a first order pole that means that

this function can always be represented as a ratio of two functions. The function in the nominator

doesn't have the root at this pole while the function in the denominator

has a first order root, like this: f(z) is represented as h(z) over g(z)

where h(a) is non-zero while g(a) is zero and the zero is of the first order. Now let's write

down the leading taylor expansions for both of these functions in the vicinity of point

z=a. For h(z) function the leading term will be simply h(a) while for g(z) function it will be

g'(a)(z-a). And now look at this formula. It's just the leading term of its Laurent expansion

near the first order pole z=a, so this way this prefactor h(a) over g'(a) is nothing

general_formula_for_the_residue__4

Complex analysis, Week 3, Part 6

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Integration with residues

MISIS

$$\operatorname*{res}_{z=a} f(z) = \frac{1}{(n-1)!}\frac{d^{n-1}}{dz^{n-1}} f(z)(z-a)^n\Big|_{z\to a}$$

Simple pole: $\operatorname*{res}_{z=a} f(z) = f(z)(z-a)\Big|_{z\to a}$

$$\operatorname*{res}_{z=a} f(z) = \frac{h(a)}{g'(a)}$$

$$f(z) = \frac{1}{z^3+1}$$

$z=-1,\ \ z=e^{\pm i\pi/3}.$

$h(z)=1,\ \ g(z)=z^3+1$

$$\operatorname*{res}_{z} f(z) = \frac{1}{3z^2} = \begin{cases} \frac{1}{3}, & z=-1, \\ \frac{1}{3}e^{\mp 2i\pi/3}, & z=e^{\pm i\pi/3} \end{cases}$$

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the residue of the function at a simple pole is simply equal to h(a) over g'(a).

And let's address a quick example to see just how it works. Consider function f(z) equals 1/(z^3+1).

And the assignment is to find the residues of this function at all finite points. So the function

in the denominator z^3+1 has three distinct roots of the first order. So the function has

three simple poles at point z=-1, and then points e^{+- I pi /3}. And now let's use our formula.

h(z) in this case is equal to 1, while g(z) is equal to z^3 plus 1. Ao according to our formula,

the residual of f(z) at each of these points is given by a simple expression: it's 1(3z^2)

where z is equal to either to negative 1 or e^{+- I pi /3}. Well that's why this

formula is so suitable in many cases: it gives a general formula for the residual

at any pole of the function. And now we simply plug in our z's and obtain the residue: 1/3

at point z equals minus one and 1/3 times e^{-+ 2I pi /3} for the rest of the points.

And that's it for residues so now you have a good start for integration techniques.

12:34 / 12:46 Speed 1.0x HD

integrating_with_residuals_1__1

Complex Analysis, week 3, Part 7

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Integration with residues

MISIS

$$I = \int_0^{2\pi} \frac{d\varphi}{5-3\sin\varphi}$$

$$z = e^{i\varphi}$$

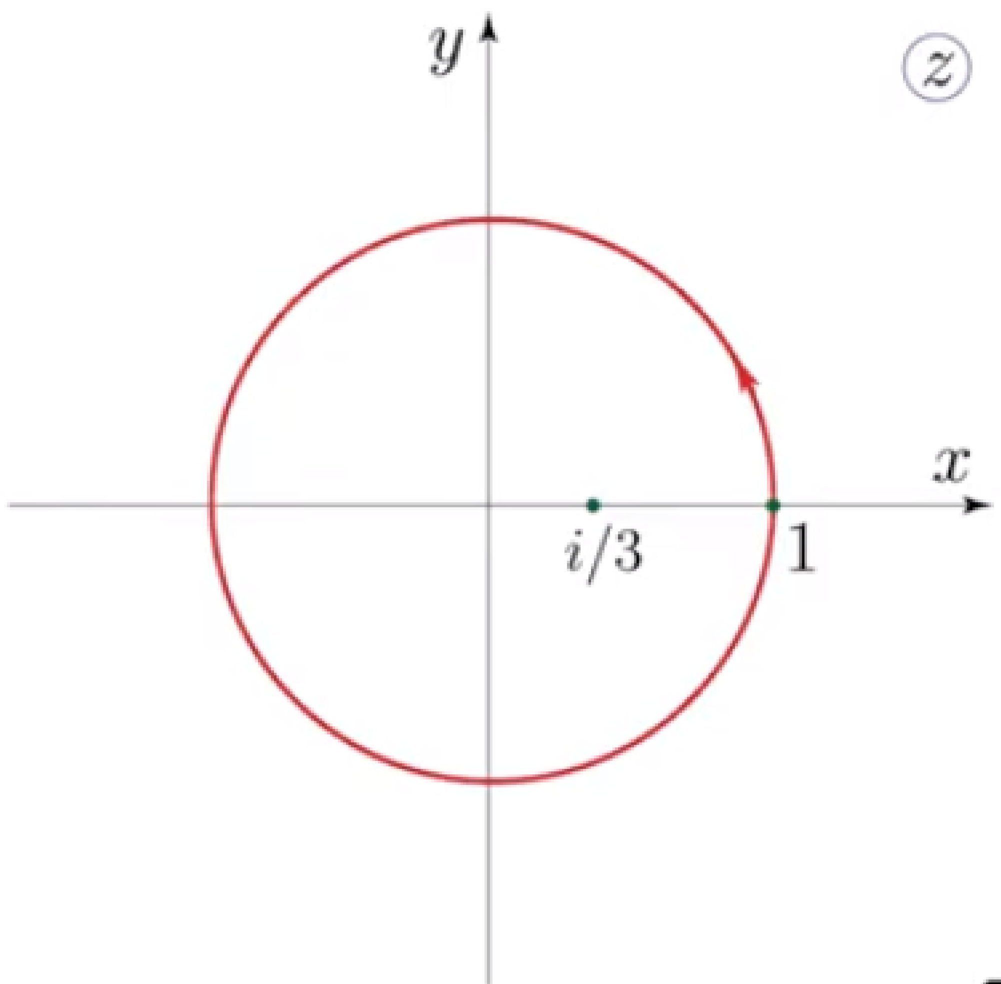
$$dz = e^{i\varphi} i\, d\varphi, \quad d\varphi = \frac{dz}{iz}$$

$$\sin\varphi = \frac{1}{2i}\left(z - \frac{1}{z}\right)$$

$$I = \oint f(z)\,dz, \quad f(z) = -\frac{2}{3z^2-10iz-3}$$

$$z = 3i, \quad z = \frac{i}{3}$$

$$f(z) = \frac{h(z)}{g(z)}, \quad \operatorname*{res}_{z=z_0} f(z) = \frac{h(z_0)}{g'(z_0)}$$



$$h(z) = -2, \quad g(z) = 3z^2 - 10iz - 3$$

$$\operatorname*{res}_{z=i/3} f(z) = \left.\frac{-2}{6z-10i}\right|_{z=i/3} = \frac{1}{4i}$$

$$I = 2\pi i \frac{1}{4i} = \frac{\pi}{2}$$

equal to 2 pi i times 1 over 4 i which gives pi/2. And that completes our calculation.

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3:16 / 8:19 Speed 1.0x HD

region inside is simply a

unit disk. And now let's find the poles of our function which are simply the zeros of

our denominator. In denominator we have a quadratic polynomial, and its roots are z=3I

and z=I/3. Those are first order roots and that means they are simple poles of our function.

And that means that to compute the residuals of the function it's enough to use a shortcut

formula, which we discussed in one of our previous lectures. Namely, if a function can

be represented as a ratio of two functions h and g then the residue of our function at

a simple pole is equal to h(z_0) / g'(z_0). here our h function is minus 2, while our

g function is this second order polynomial. This way the residual of our function and

point z=I/3 is equal to -2 divided by the derivative of our polynomial 6 z minus 10I

taken at point z = i/3. And we obtain 1 over 4 i and as a result our integral is simply

equal to 2 pi i times 1 over 4 i which gives pi/2. And that completes our calculation.

Now let's start with the next example. And this time our integration domain is spanning

from minus infinity to plus infinity of function dx over x to the power of 4 plus 1. In principle,

you can solve this integral using elementary tools of real calculus. But the computation

is tedious and a little bit cumbersome. So let's see how things works in the realm of

complex analysis. And in complex analysis things work only for closed contour integrals,

so we need to devise some closure of this contour. And whenever we deal with infinite

domain of integration the most often are used closure either upper or lower semi-arc. In

this case let's opt for upper semicircle. Now let's promote our integrand function into

a complex plane f(z) equals 1 over z to the power of 4 plus 1, and study this closed contour

integral f(z) dz, which naturally consists of our initial integral plus the integral

along their upper semicircle. And the reason we introduced this

integrating_with_residuals_1__2

Complex Analysis, week 3, Part 7

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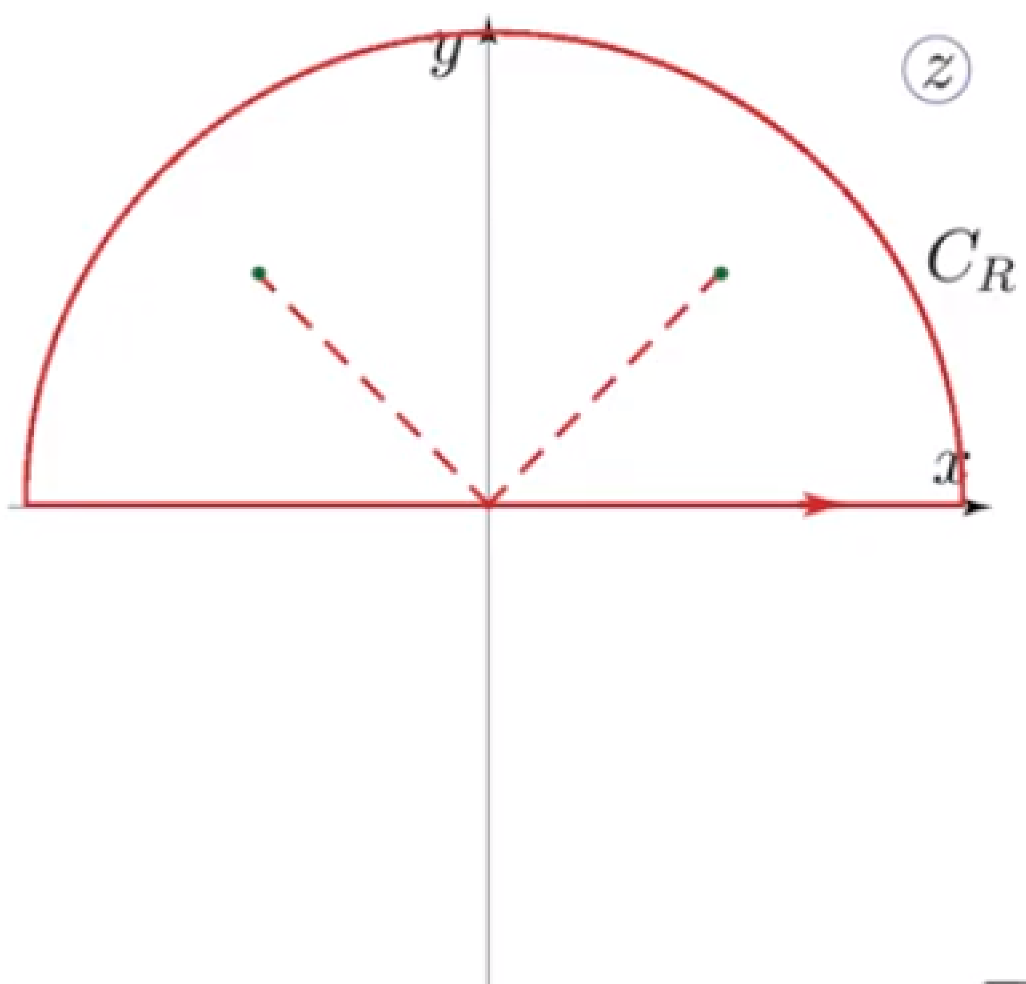
MISIS

Integration with residues

$$I = \int_{-\infty}^{\infty} \frac{dx}{x^4+1}$$



$$f(z) = \frac{1}{z^4+1}$$

$$\oint = I$$

$$z^4+1=0 \qquad z_n = e^{i\pi/4 + i\pi n/2}, \quad n \in Z$$

$$z_1 = e^{i\pi/4}, \quad z_2 = e^{3\pi i/4}$$

$$h(z) = 1, \quad g(z) = z^4+1$$

$$\operatorname*{res}_{z=z_0} f(z) = \frac{1}{4z_0^3}$$

$$\operatorname*{res}_{\exp(i\pi/4)} f(z) = \frac{1}{4e^{3\pi i/4}} \quad \operatorname*{res}_{\exp(3i\pi/4)} f(z) = \frac{1}{4e^{9\pi i/4}}$$

$$-\frac{1}{4}e^{i\pi/4} \qquad \frac{1}{4}e^{-i\pi/4}$$

$$I = 2\pi i\left(-\frac{1}{4}e^{i\pi/4} + \frac{1}{4}e^{-i\pi/4}\right) = 2\pi i\left(\frac{-2i\sin\frac{\pi}{4}}{4}\right) = \pi\sin\frac{\pi}{4} = \frac{\pi}{\sqrt{2}}$$

which is pi times sin(pi/4) which yields pi/sqrt(2).
And that completes our first practice. In

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8:04 / 8:19 Speed 1.0x HD

and g(z) which is equal

to z raised to the power of four plus one. As a result, the general formula for the residual

is one over four z_0 cubed where z_0 is the position of the corresponding pole. And this

way we obtained for the residues 1/4 times e^{3 pi I/ by 4} for the reside at point z_0

= e^{I pi / 4} and 1 over 4 times e^{9 pi I/4} for the residue at point e^{3 pi I / 4}.

And let us simplify the expression for these residuals just to make them more suitable

for future calculations. The first residue can be transformed into -1/4 times e^{I pi/4},

while the second residue can be transformed into 1/4 e^{-I pi/4}. So finally we have for

our integral of the following expression: 2 pi I times minus 1/4 e^{I pi/4} plus 1/4

times e^{- I pi/4}. And this sum can be organized into a sine function: in the braces we obviously

have -2I sin(pi/4) divided by 4. In this way we obtain our final answer for the integral

which is pi times sin(pi/4) which yields pi/sqrt(2). And that completes our first practice. In

our next videos we'll study more examples and we'll even learn some new theorems which

simplify the calculations.

integrating_with_residuals_2__1

Complex Analysis, week 3, Part 8

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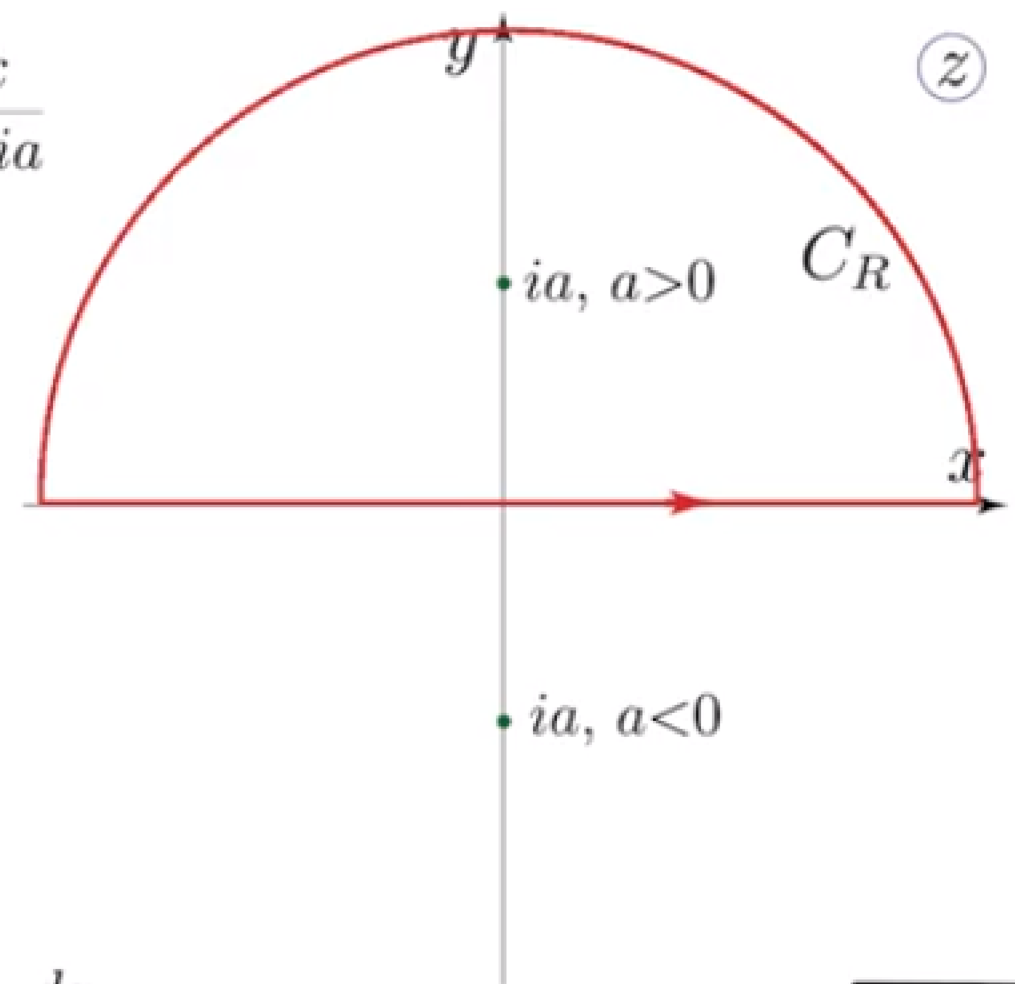
Integration with residues

MISIS

$$I=\int_{-\infty}^{\infty}\frac{dx}{x-ia} \quad \to \lim_{R\to\infty}\int_{-R}^{R}\frac{dx}{x-ia}$$



$$\operatorname*{res}_{z=ia} f(z)=1 \qquad \oint = 2\pi i \ \to\ I=\pi i,\ \ a>0.$$

$$\oint = 0, \ \to\ I=-\pi i,\ \ a<0$$

$$I=I(a)=\pi i \operatorname{sign} a$$

$$f(z)=\frac{1}{z-ia}$$

$$f(z)\to\frac{1}{z},\quad z\to\infty$$

$$z=Re^{i\varphi},\quad dz=\underline{R}e^{i\varphi}id\varphi, \qquad \frac{dz}{z}=id\varphi$$

$$\int_{C_R}=\int_0^{\pi} id\varphi = i\pi \qquad \oint = I+i\pi$$

containing only regular functions you can unleash the whole power of complex analysis in your work.

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a month. So my message to you is:

stay alert in considering this integral. Well the continuation is all material right because if a

is positive then we have precisely one pole inside our contour and the residual at this

point is equal to one. So our closed contour integral in this case is equal to two pi I

and our original integral for positive a is simply pi I. If a is negative

then there are no poles inside the contour and the closed contour integral is equal to zero,

and our original integral is reduced to - pi I for negative a. Therefore, our answer for this

integral can be expressed by a so-called sign function and we obtain: pi I times sign(a).

So what we got here is a nice integral representation of a sign function. and

it's very useful in applications because a sign function is aggressively non-analytic. So when you

encounter it you can't use complex analysis, but once you substitute it with an integral

containing only regular functions you can unleash the whole power of complex analysis in your work.

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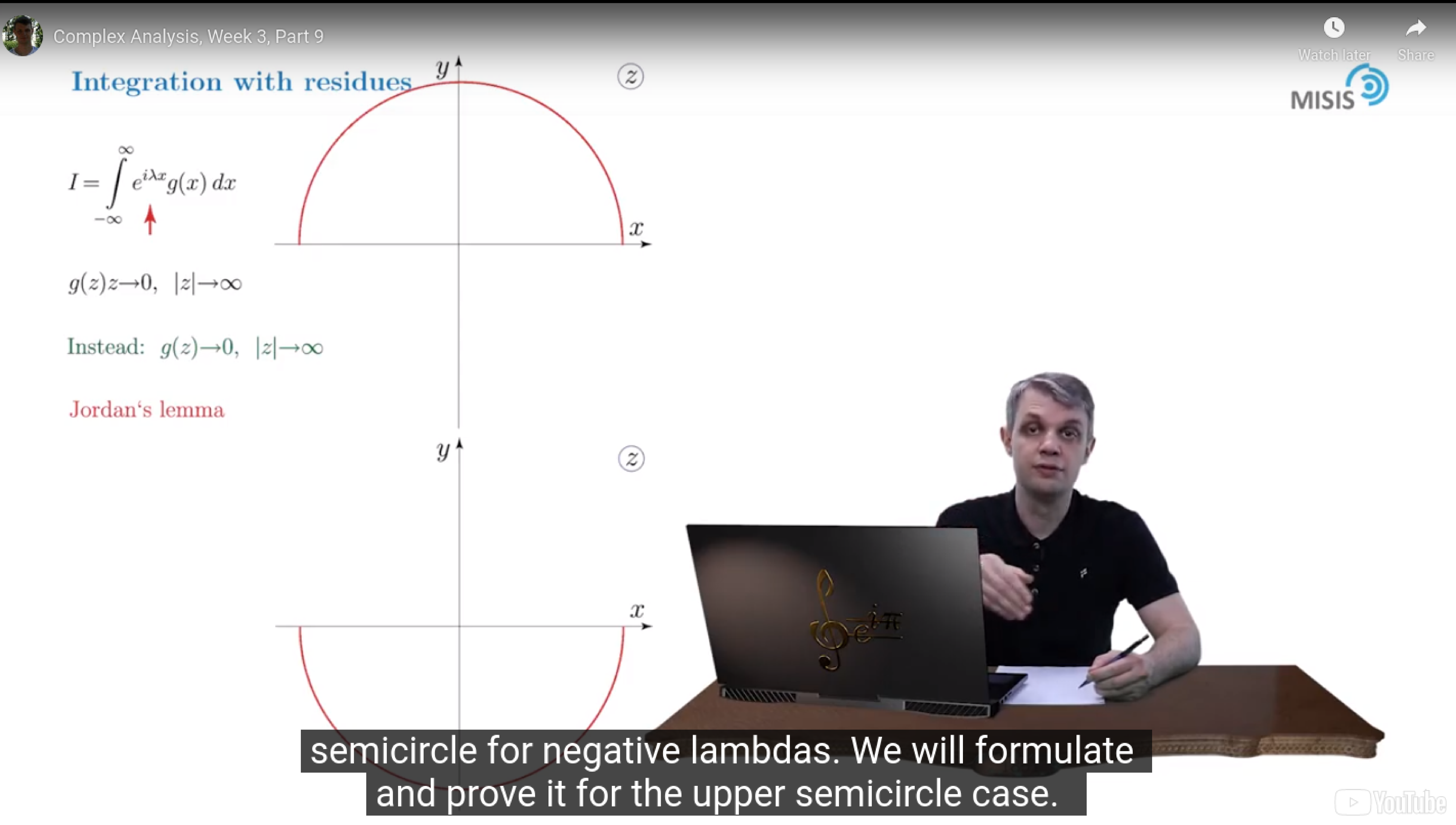
Jordans_lemma__1



theorem to be practical,

the integrals along these arcs need to vanish. Naively as we would expect from our previous

video this would require the function g(z) to decay at z tending to infinity faster than 1/z.

But in reality, this condition can be relaxed and substituted with condition

of g(z) simply tending to zero as the modulus of z tends to infinity.

In fact, the condition is slightly more subtle, but we'll return to this in a minute.

This relaxation is possible due to the presence of the exponential function in our integrand.

Indeed the exponential function is suppressed for positive numbers if we go upward in the

complex semi-plane and for negative lambdas if we move downwards in the complex semi-plane. So the

precise statement is known as Jordan's lemma. It is formulated for two types of integrals:

the integral along the upper semi-circle with positive lambdas, and the integral along lower

semicircle for negative lambdas. We will formulate and prove it for the upper semicircle case.

The statement for a lower semicircle is completely symmetric and the proof will be

your homework exercise. And the formulation is as follows: suppose we have an integral

along the upper semi-circle of radius r tending to plus infinity and the integrand is of the form

e^{i lambda z} g(z) dz. Now if lambda is positive and the function g(z) tends to zero uniformly with

respect to its argument as r tends to infinity, then the whole integral tends to zero.

First of all, I need to clarify what the uniform convergence of function g(z) really means. And in

this context it's equivalent to the following statement. We say that the function tends to

zero uniformly with respect to its argument as the radius of the arc tends to infinity, if the

maximum value of its modulus on the arc tends to zero as the radius of the arc tends to infinity.

And now let us prove the theorem. We need to build an estimate for our integral.

And our first step is the usage of triangle inequality: the modulus of the integral is always

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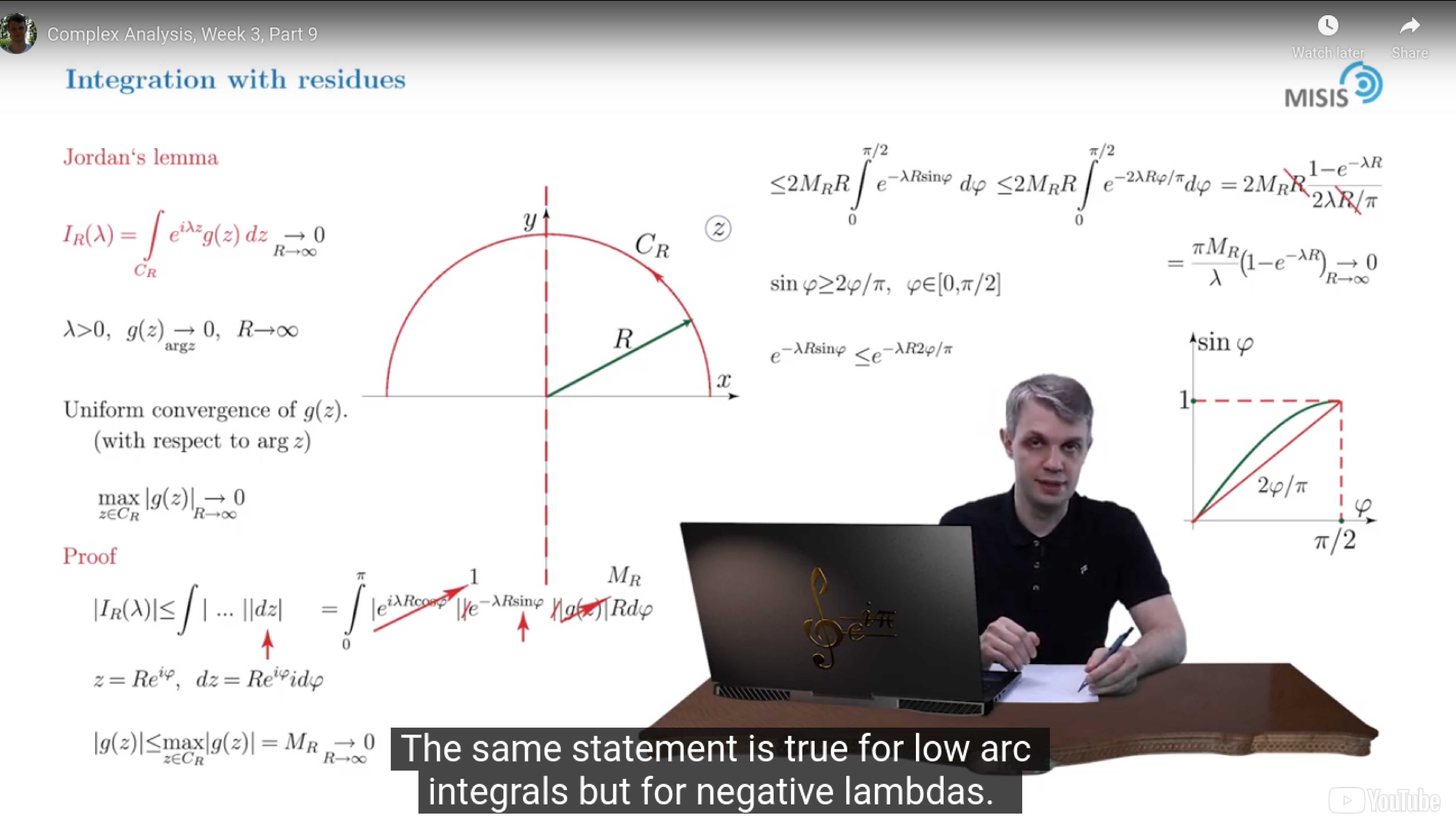
Jordans_lemma__2



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higher than the corresponding line which connects the origin and the point 1 and pi/2.

The equation of this line is 2 phi / pi and this way you obtain a very useful inequality: sin(phi)

is always greater than 2 phi / pi for phi belonging to the segment from 0 to pi/2.

Now flipping the sign and exponentiating this inequality, we obtain the crucial inequality

for our problem: e^{ - lambda r sin phi} is always smaller than e ^{ - lambda r 2 phi /

pi} and therefore we obtain the next estimate for our integral. It's less than M_r times r

times the integral over a simple e^{ - lambda r 2 phi/ pi} d phi. And this integral is

easily taken with antiderivatives, and the answer is 1 minus e^{ - lambda r}

divided by {2 lambda r / pi} and we see that large prefector r is

compensated by the same large factor in the denominator and obtain our final estimate.

And as M_r tends to zero as r tends to infinity the whole integral tends to zero.

The same statement is true for low arc integrals but for negative lambdas.

But now let's address some simple example just to see how the theorem works in practice.

We'll take the following integral from minus infinity to plus infinity: e^{I alpha x} / (x + I)

dx for positive and negative alphas. Let us first consider the case of positive

alphas. Our first step is to complete the integration contour and keeping in mind the

possible usage of Jordan's lemma let's complete the integration counter with an upper semicircle.

Now let's promote our integrand into a complex plane and denote it as f(z)

and separately let's denote our g(z) function as 1/(z+I). Obviously, our new closed contour

integral is equal to our original integral plus this semi-circular arc integral.

And the integrand is precisely of the form which enters the Jordan's lemma:

our g(z) function decays is 1/z for large values of z and obviously

tends to zero independently of its argument. So this part of Jordan's lemma is satisfied.

Jordans_lemma__3

Complex Analysis, Week 3, Part 9

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Integration with residues

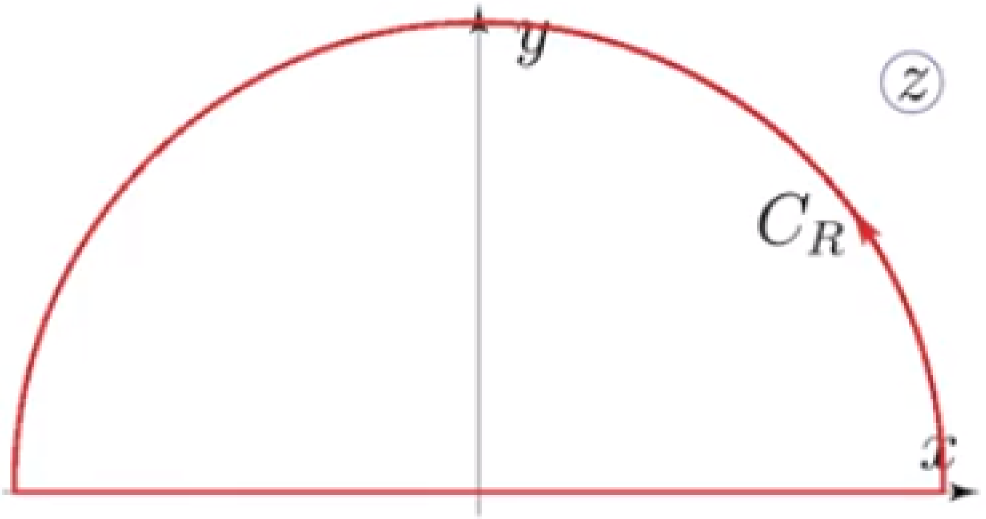
MISIS

$$I=\int_{-\infty}^{\infty}\frac{e^{i\alpha x}}{x+i}dx \quad \text{for } \alpha>0 \text{ and } \alpha<0.$$

$\alpha>0.$

$$f(z)=\frac{e^{i\alpha z}}{z+i}$$

$$g(z)=\frac{1}{z+i}$$



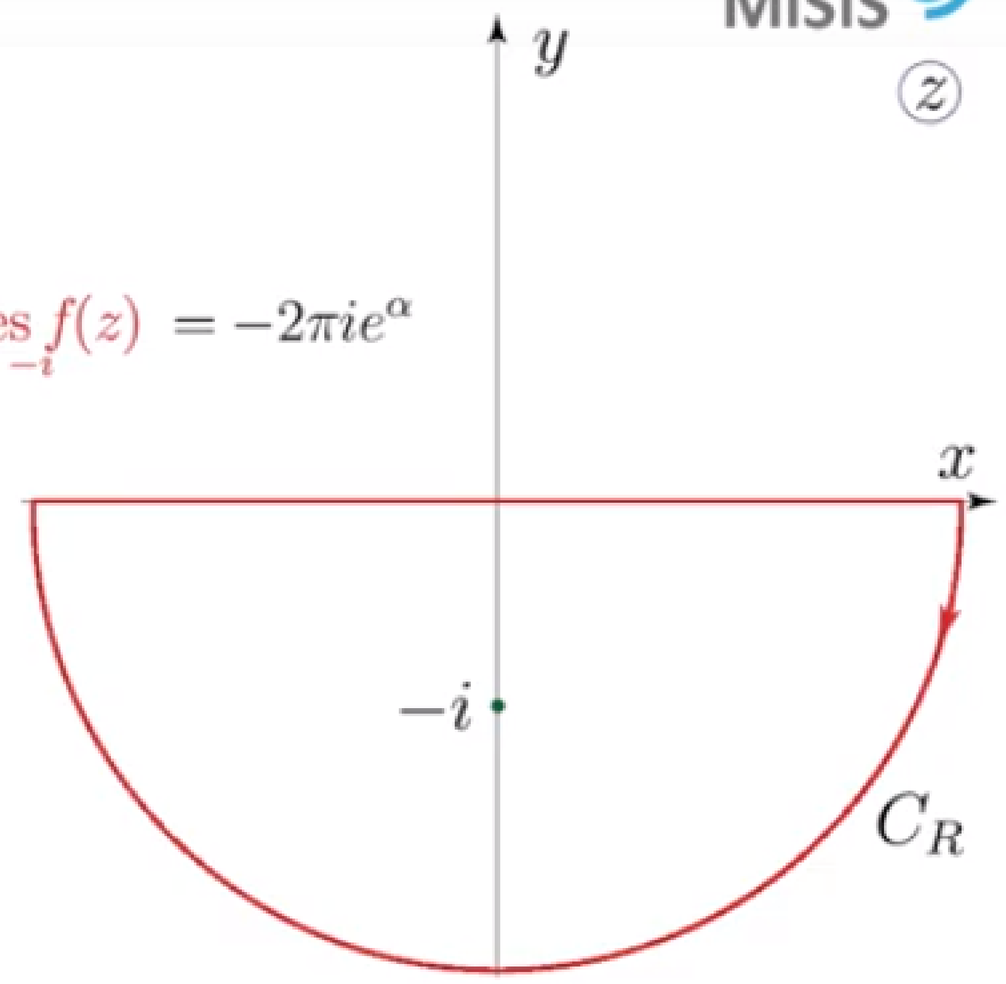
$$\oint = I + \underbrace{\int_{C_R}}_{\to 0} = 0$$

$$g(z)\to\frac{1}{z}\to 0,\ \ |z|\to\infty,\ \ \text{for any arg } z$$

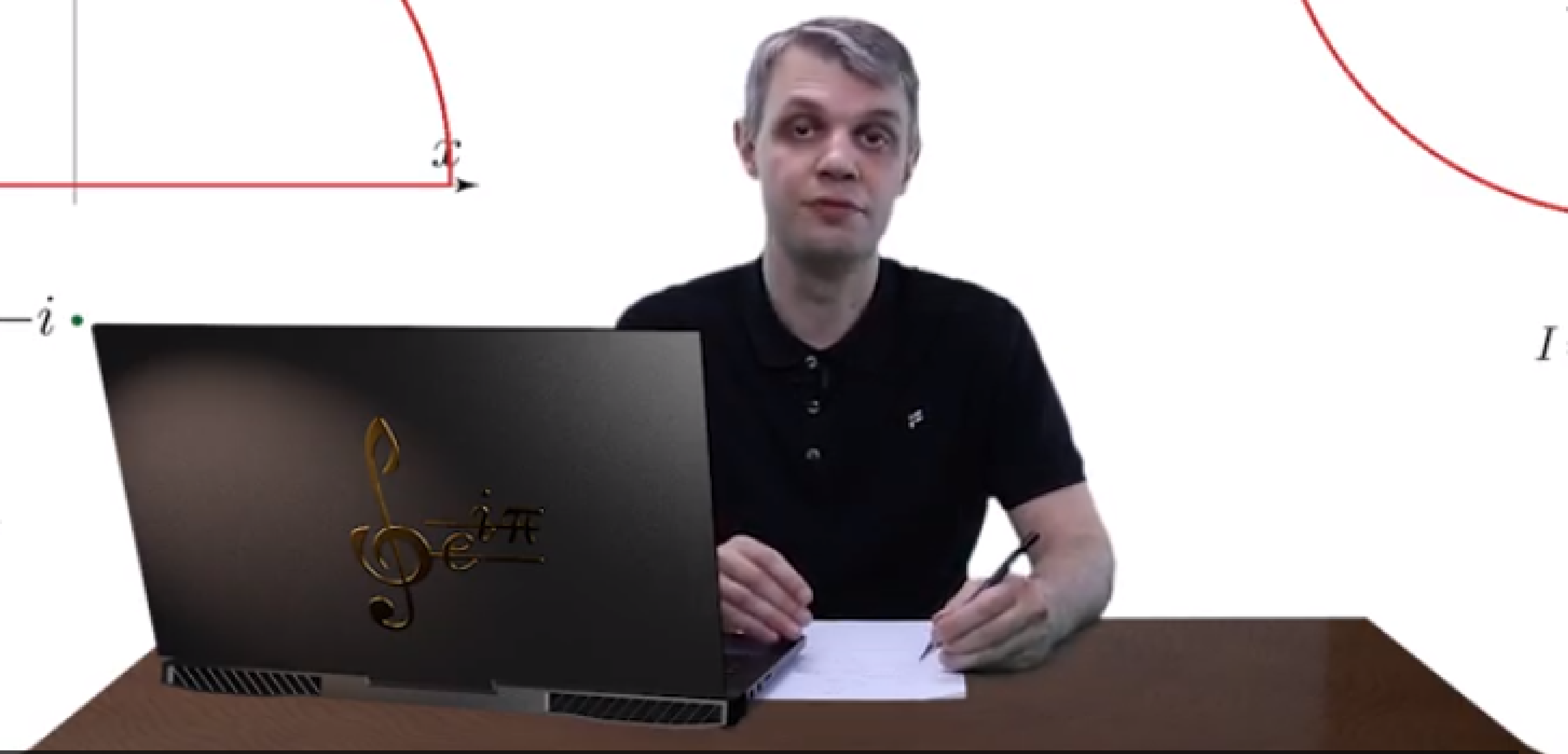
$I=0,\ \ \alpha>0$

$\alpha<0$

$$\oint = I + \underbrace{\int_{C_R}}_{\to 0} = -2\pi i \operatorname*{res}_{z=-i} f(z) = -2\pi i e^{\alpha}$$



$$I=I(\alpha)=-2\pi i e^{\alpha}\theta(-\alpha)$$



of Jordan's lemma. Now next we just will practice more with it and study more interesting examples.

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integral.

And now we may use Residue theorem: namely, the closed contour integral is equal to 2 pi I times

the sum of the residues of our function inside this contour. But in this particular case our

contour is passed in negative direction, because as we move along it the region inside stays to our

right. And that is why the closed contour integral is equal to actually minus 2 pi I times the sum

of the residues of our function inside. So always pay attention to the orientation of your contour.

So we obtain - 2 pi I times the residue of our function f(z)

at point z =- I and the residue of the function is trivially evaluated and we obtain - 2 pi I times

e^{alpha}. And this way we completed the computation of our integral.

The answer can be expressed by unit step function, namely: -2 pi I times e^{alpha}Theta(-alpha),

where theta is a unit step function. So we are done with our first example of the usage

of Jordan's lemma. Now next we just will practice more with it and study more interesting examples.

9:49 / 10:01 Speed 1.0x HD