

Complex Analysis, Week 1, Part 1

complex_number_algebra_1

Algebra of complex numbers

Complex number: $z = x + iy$

$$|z| = \sqrt{x^2 + y^2} \quad zz^* = |z|^2$$

$$x = \operatorname{Re} z, \quad y = \operatorname{Im} z; \quad z^* z_1 z = z_2 z^*$$

$$\frac{z^* z_1 z}{x_1^2 + y_1^2} = \frac{z_2}{z_1}, \quad \rightarrow z_1^{-1}$$

$$z_1 \cdot z_2 = (x_1 + iy_1)(x_2 + iy_2) = x_1 x_2 - y_1 y_2 + i(x_1 y_2 + x_2 y_1)$$

$$\underline{z_1 z = z_2}, \quad z = \frac{z_2}{z_1} = z_2 \cdot z_1^{-1}.$$

$$z_1 = x_1 \in \mathbb{R}$$

$$z = \frac{x_2 + iy_2}{x_1} = \frac{x_2}{x_1} + i \frac{y_2}{x_1}$$

$$z^* = x - iy$$

$$zz^* = x^2 + y^2$$

Z1 to the power of -1 is defined as you see as Z1 star divided by the square of its modulus.

Example I

$$z(1+2i) = 3+i \quad z = \frac{3+i}{1+2i} \cdot \frac{1-2i}{1-2i} = \frac{5-5i}{5} = 1-i$$

MISIS

complex_number_algebra_2

Algebra of complex numbers

$$|z| = \sqrt{x^2 + y^2} \quad zz^* = |z|^2$$

$$z^* z_1 z = z_2 z^*$$

$$\frac{z^* z_1 z}{x_1^2 + y_1^2} = \frac{z_2}{z_1}, \quad \rightarrow z_1^{-1} = \frac{z_1^*}{|z_1|^2}$$

identity. So what was discussed so far was called in fact an algebraic representation of a complex

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Complex Analysis, Week 1, Part 1

complex_number_geometry_1

Algebra of complex numbers

Geometric representation

$$z_1 + z_2$$

has the components X1 + X2 and Y1 and Y2.
But to understand better how a geometrical

Complex Analysis, Week 1, Part 1

complex_number_geometry_2

Algebra of complex numbers

Geometric representation

$$|z + 1| = 2$$

$$z + 1 = z - (-1)$$

$$z = x + iy$$

$$|(x + 1) + iy| = 2$$

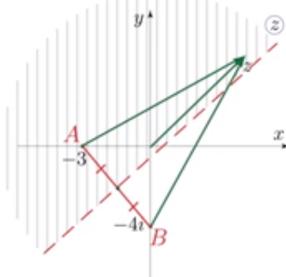
Then you square both parts of this equation and expand the left-hand side like X plus 1 squared

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Algebra of complex numbers

$$|z + 3| < |z + 4i| \quad |z + 3| = |z + 4i| \text{ (Boundary)}$$



$$z + 3 = z - (-3)$$

$$z + 4i = z - (-4i)$$

$$(x_0, y_0) = \left(-\frac{3}{2}, -2\right)$$

$$\overline{AB} = (3, -4)$$

$$n = (4, 3)$$

$$\frac{x - x_0}{n_x} = \frac{y - y_0}{n_y}$$

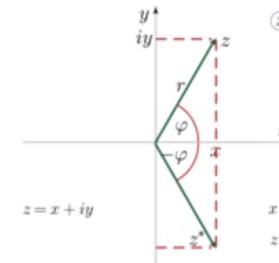
$$\frac{x - \left(-\frac{3}{2}\right)}{4} = \frac{y + 2}{3}$$

$$y = \frac{3}{4}x - \frac{7}{8}$$

minus 2 so it's minus 7/8 and that's it that's our equation for a boundary. And of course you



Algebra of complex numbers



$$z = x + iy$$

$$x = r \cos \varphi, \quad y = r \sin \varphi$$

$$z = r(\cos \varphi + i \sin \varphi)$$

$$|z| = r, \quad \varphi = \arg z$$

$$z^* = x - iy = r(\cos \varphi - i \sin \varphi) = r[\cos(-\varphi) + i \sin(-\varphi)]$$

$$z_1 = r_1(\cos \varphi_1 + i \sin \varphi_1)$$

$$z_2 = r_2(\cos \varphi_2 + i \sin \varphi_2)$$

$$z_1 \cdot z_2 = r_1 r_2 [(\cos \varphi_1 \cos \varphi_2 - \sin \varphi_1 \sin \varphi_2) + i(\cos \varphi_1 \sin \varphi_2 + \cos \varphi_2 \sin \varphi_1)]$$

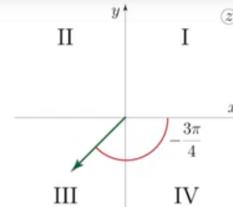
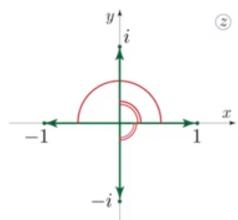
Now we conclude that when we divide two complex numbers the modulus are divided while their

$$\frac{z_1}{z_2} = \frac{z_1 z_2^*}{z_2 z_2^*} = \frac{r_1 r_2^* [\cos(\varphi_1 - \varphi_2) + i \sin(\varphi_1 - \varphi_2)]}{r_2^2}$$

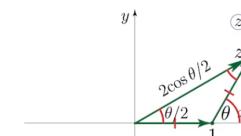
$$= \frac{r_1}{r_2} [\cos(\varphi_1 - \varphi_2) + i \sin(\varphi_1 - \varphi_2)]$$



Algebra of complex numbers

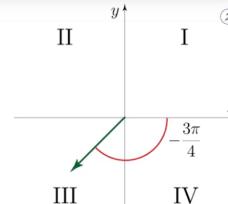


-pi/2 or 3*pi/2, depending on your choice.
So this is how trigonometric representation



$$2\cos^2 \frac{\theta}{2} + 2\sin \frac{\theta}{2} \cos \frac{\theta}{2}$$

$$z = 2\cos \frac{\theta}{2} \left(\cos \frac{\theta}{2} + i \sin \frac{\theta}{2} \right)$$



2 cosine of 30 by 2, we obtain the same trigonometric representation.

Eulers_identity_1

Algebra of complex numbers

 $e^{i\theta}$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \quad e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

$$e^{i\theta} = \cos \theta + i \sin \theta \quad \text{Euler's identity}$$

$$z = |z|(\cos \theta + i \sin \theta) = |z|e^{i\theta} \quad \text{exponential form of a complex number}$$

$$z_1 z_2 = |z_1||z_2|e^{i(\theta_1 + \theta_2)}$$

$$\left| \frac{a_{n+1}}{a_n} \right|_{n \rightarrow \infty} < 1 \quad (\text{Convergence})$$

$$\left| \frac{z}{n} \right|_{n \rightarrow \infty} = 0 \quad \text{for any finite } z$$

$$e^{i\theta} = \sum_{n=0}^{\infty} \frac{(i\theta)^n}{n!} = 1 + i\theta - \frac{\theta^2}{2!} - i\frac{\theta^3}{3!} + \frac{\theta^4}{4!} + i\frac{\theta^5}{5!} + \dots$$

$$= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} + \dots \right) + i \left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} \dots \right)$$

 $\cos \theta$ $\sin \theta$

The same goes for the division. Even more, if you recall that cos and sin function are



Algebra of complex numbers

$$e^{i\theta + 2\pi n} = \cos(\theta + 2\pi n) + i \sin(\theta + 2\pi n) = e^{i\theta}, \quad n \in \mathbb{Z}$$

$$e^{-i\theta} = \cos \theta - i \sin \theta$$

$$\cos \theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$$

$$\sin \theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

$$\cosh i\theta = \cos \theta, \quad \sinh i\theta = i \sin \theta$$



Eulers_identity_2

$$e^{i\theta} = \cos \theta + i \sin \theta \quad \text{Euler's identity}$$

$$z = |z|(\cos \theta + i \sin \theta) = |z|e^{i\theta} \quad \text{exponential form of a complex number}$$

$$z_1 z_2 = |z_1||z_2|e^{i(\theta_1 + \theta_2)}$$



coincides with cosine theta while sin hyperbolic of i*theta is equal to sine theta over i.

exp_repr_1

Algebra of complex numbers

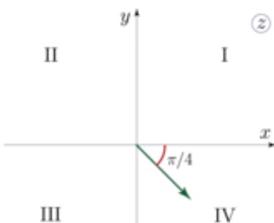
$$z = 1 + i^{123}$$

$$z = \sqrt{2}e^{-}$$

$$i = e^{i\pi/2} \quad i^{123} = e^{i\pi/2(120+3)} = e^{i\pi/2 + 3\pi/2} = e^{3\pi/2} = -i$$

$$z = 1 - i$$

$$|z| = \sqrt{2}$$



$$\arg z = -\pi/4$$

and this way our complex number is represented as square root of 2 times $e^{-\pi i/4}$.



Algebra of complex numbers

$$z = \frac{(1-i)^6}{(1+i\sqrt{3})^5}$$

$$1-i = \sqrt{2}e^{-i\pi/4}$$

$$z_2 = 1 + i\sqrt{3} = 2\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)$$

$$\cos \varphi = \frac{1}{2}, \quad \sin \varphi = \frac{\sqrt{3}}{2} \quad \arg z_2 = \varphi = \frac{\pi}{3}$$

$$z_2 = 2e^{i\pi/3}$$

$$z = \frac{[\sqrt{2}]^6 e^{-i\pi/2}}{2^5 e^{5\pi i/3}} = \frac{8}{32} \frac{e^{i\pi/2}}{e^{-i\pi/3}} = \frac{1}{4} e^{5\pi i/6}$$



exp_repr_2



it's a - pi i / 3 which gives us 1/4 times $e^{5\pi i/6}$ and this is our final answer.

exp_repr_3

Algebra of complex numbers

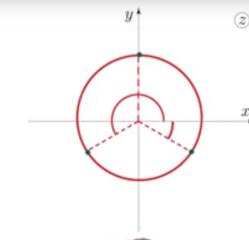
$$\begin{aligned} S &= \sin \theta + \sin 2\theta + \dots + \sin n\theta = \operatorname{Im}(e^{i\theta} + e^{2i\theta} + \dots + e^{ni\theta}) \\ &= q + q^2 + \dots + q^n \\ &= q(1 + q + \dots + q^{n-1}) = q \frac{q^n - 1}{q - 1} \\ &= \operatorname{Im} \frac{e^{in\theta} - 1}{e^{i\theta} - 1} = \operatorname{Im} \frac{e^{i\theta} e^{in\theta/2} \cancel{\sin \frac{n\theta}{2}}}{e^{i\theta/2} \cancel{\sin \frac{\theta}{2}}} = \frac{\sin \frac{n\theta}{2}}{\sin \frac{\theta}{2}} \operatorname{Im} e^{i\theta + in\theta/2 - i\theta/2} \\ &= e^{i\alpha} \pm e^{i\beta} \\ e^{i\alpha} + e^{i\beta} &= e^{i(\alpha + \beta)/2} (e^{i(\alpha - \beta)/2} + e^{-i(\alpha - \beta)/2}) = e^{i(\alpha + \beta)/2} 2 \cos \frac{\alpha - \beta}{2} \\ e^{i\theta} - 1 &= e^{i\theta/2} (e^{i\theta/2} - e^{-i\theta/2}) = e^{i\theta/2} 2i \sin \frac{\theta}{2} \\ e^{in\theta} - 1 &= e^{in\theta/2} 2i \sin \frac{n\theta}{2} \\ &= \frac{\sin \frac{n\theta}{2}}{\sin \frac{\theta}{2}} \sin \frac{(n+1)\theta}{2} \end{aligned}$$

exponential representation of a complex number,
namely the solution of the simplest power type



exp_repr_4

$$\begin{aligned} z^3 &= -i \\ z = |z|e^{i\varphi} &\quad z^3 = |z|^3 e^{3i\varphi} \quad -i = e^{-i\pi/2 + 2\pi n} \\ |z|^3 = 1 &\rightarrow |z| = 1 \\ 3\varphi = -\frac{\pi}{2} + 2\pi n &\quad \varphi = -\frac{\pi}{6} + \frac{2\pi n}{3} \\ z_n = e^{-i\pi/6 + 2\pi n/3} & \\ n = 0: z_0 &= e^{-i\pi/6} \\ n = 1: z_1 &= e^{i\pi/2} = i \\ n = 2: z_2 &= e^{7\pi i/6} \\ n = 3: z_3 &= e^{-i\pi/6 + 2\pi i} = z_0 \end{aligned}$$



so we see that these roots split the unit circle into three equal parts,

Analytic functions

$$\begin{aligned} f'(z_0) &= \left. \frac{f(z) - f(z_0)}{z - z_0} \right|_{z \rightarrow z_0} \\ \text{I} \downarrow & \quad \text{IV} \downarrow \\ \Delta z \sim \Delta f &= \Delta u + i\Delta v = \frac{\partial u}{\partial x} \Delta x + \frac{\partial u}{\partial y} \Delta y + i \left(\frac{\partial v}{\partial x} \Delta x + \frac{\partial v}{\partial y} \Delta y \right) \\ \text{---} & \quad \text{---} \quad \text{---} \quad \text{---} \\ \frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y} \end{aligned}$$

$$\begin{aligned} \text{I} + \text{IV} &= \frac{\partial u}{\partial x} (\Delta x + i\Delta y) \\ \downarrow & \quad \downarrow \\ \text{II} + \text{III} &= i \left(\frac{\partial v}{\partial x} \Delta x - \frac{\partial u}{\partial y} i\Delta y \right) = \frac{\partial v}{\partial x} (\Delta x + i\Delta y) \\ \frac{\partial v}{\partial x} &= -\frac{\partial u}{\partial y} \end{aligned}$$

see that due to our conditions both derivatives are equal to each other so due to their importance

Cauchy-Riemann_theory_1



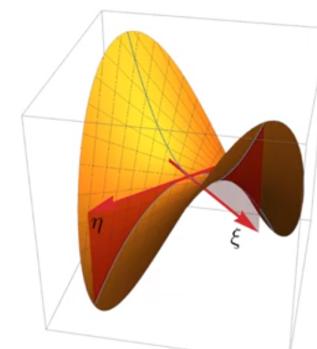
Analytic functions

$$\begin{aligned} \Delta f &= \left(\frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} \right) \Delta z \quad \frac{\Delta f}{\Delta z} \text{ is independent of } \Delta z \\ df &= \left(\frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} \right) dz = \frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} - \frac{\partial u}{\partial x} \\ &\quad i\frac{\partial v}{\partial y} + \frac{\partial u}{\partial y} = i\frac{\partial f}{\partial y} \end{aligned}$$



Analytic functions

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \quad \text{Cauchy-Riemann conditions}$$



Cauchy-Riemann_theory_2



Consequence II

$$u(x + \Delta x, y + \Delta y) = u(x, y) + \frac{1}{2} u_{xx} (\Delta x)^2 + u_{xy} \Delta x \Delta y + \frac{1}{2} u_{yy} (\Delta y)^2$$

$$v_{xy} - v_{yx} - u_{xy}^2 = -u_{xy}^2 < 0 \quad \frac{1}{2} (\Delta x \Delta y) \begin{pmatrix} u_{xx} & u_{xy} \\ u_{xy} & u_{yy} \end{pmatrix} \begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix}$$



of a function of a complex variable can't have minima or maximum but just saddles, has tremendous

Analytic functions



$$f(z) = z^2$$

$$\operatorname{Re} f = u(x,y) = x^2 - y^2$$

$$\operatorname{Im} f = v(x,y) = 2xy$$

$$\frac{\partial u}{\partial x} = 2x, \quad \frac{\partial v}{\partial y} = 2x$$

$$\frac{\partial u}{\partial y} = -2y, \quad \frac{\partial v}{\partial x} = 2y$$



and indeed $\frac{\partial u}{\partial y}$ is equal to $-\frac{\partial v}{\partial x}$. So everything seems fine now let us consider somewhat maybe less

Analytic functions

$$v = 2xy + C$$

$$w = \ln f = \ln |f| e^{i \arg f} = \ln |f| + i \arg f$$

$$\operatorname{Re} w = \ln |f|, \quad \operatorname{Im} w = \arg f$$



$$u = r^2 \cos 2\varphi = r^2 (\cos^2 \varphi - \sin^2 \varphi) = x^2 - y^2$$

$$\frac{\partial u}{\partial x} = 2x = \frac{\partial v}{\partial y}$$

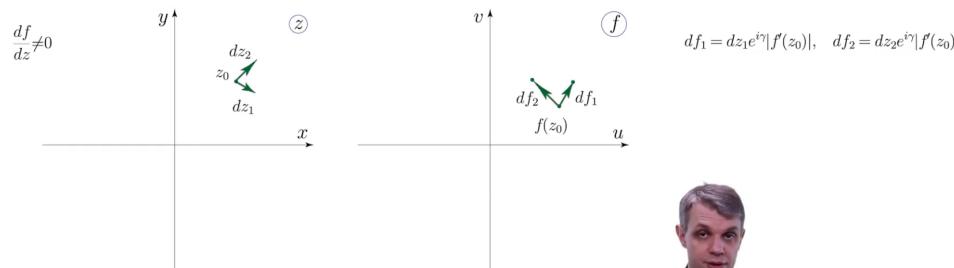
$$v = 2xy + \psi(x)$$

$$\frac{\partial v}{\partial x} = 2y + \psi'(x) = -\frac{\partial u}{\partial y} = 2y$$

$$\psi'(x) = 0 \rightarrow \psi(x) = C$$

$f(z)$ is equal to $e^{z^2} e^{ic}$ and this way you see that indeed Cauchy-Riemann conditions

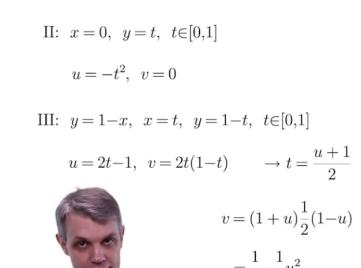
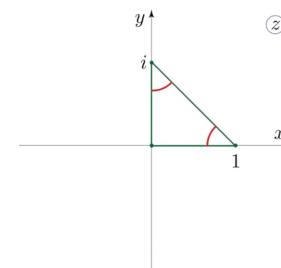
Analytic functions



in complex analysis, because whenever you deal with integral and you perform the change of

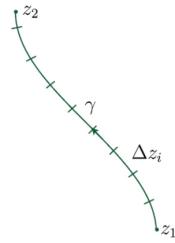


Analytic functions



one and one at this edge points meaning that the corresponding tangent angles are 45 degrees.

Analytic functions

 $f(z)$

$$S = \sum_i f(z_i) \Delta z_i \xrightarrow{\Delta z_i \rightarrow 0} \int_{\gamma} f(z) dz = \int_{\gamma} (u + iv)(dx + idy)$$

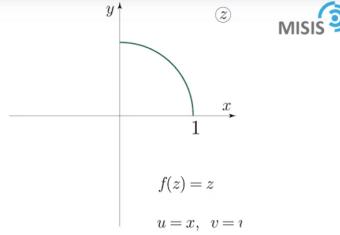
$$= \int_{\gamma} (udx - vdy) + i \int_{\gamma} (udy + vdx)$$

function $f(z) = z$. So u is equal to x while v equals y . And here we go: the integral now



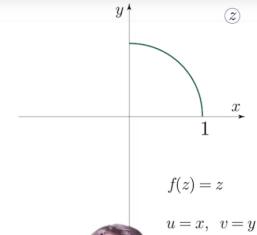
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Analytic functions



$$\int (xdx - ydy) + i \int (xdy + ydx) = \int \frac{1}{2} d(x^2 - y^2) + i \int d(xy)$$

$$= \int \frac{1}{2} d(x^2 - y^2 + 2xyi) = \int_1^i \frac{1}{2} dz^2 = -1$$



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rather on the position of its initial and end points. There is a fundamental reason for this in