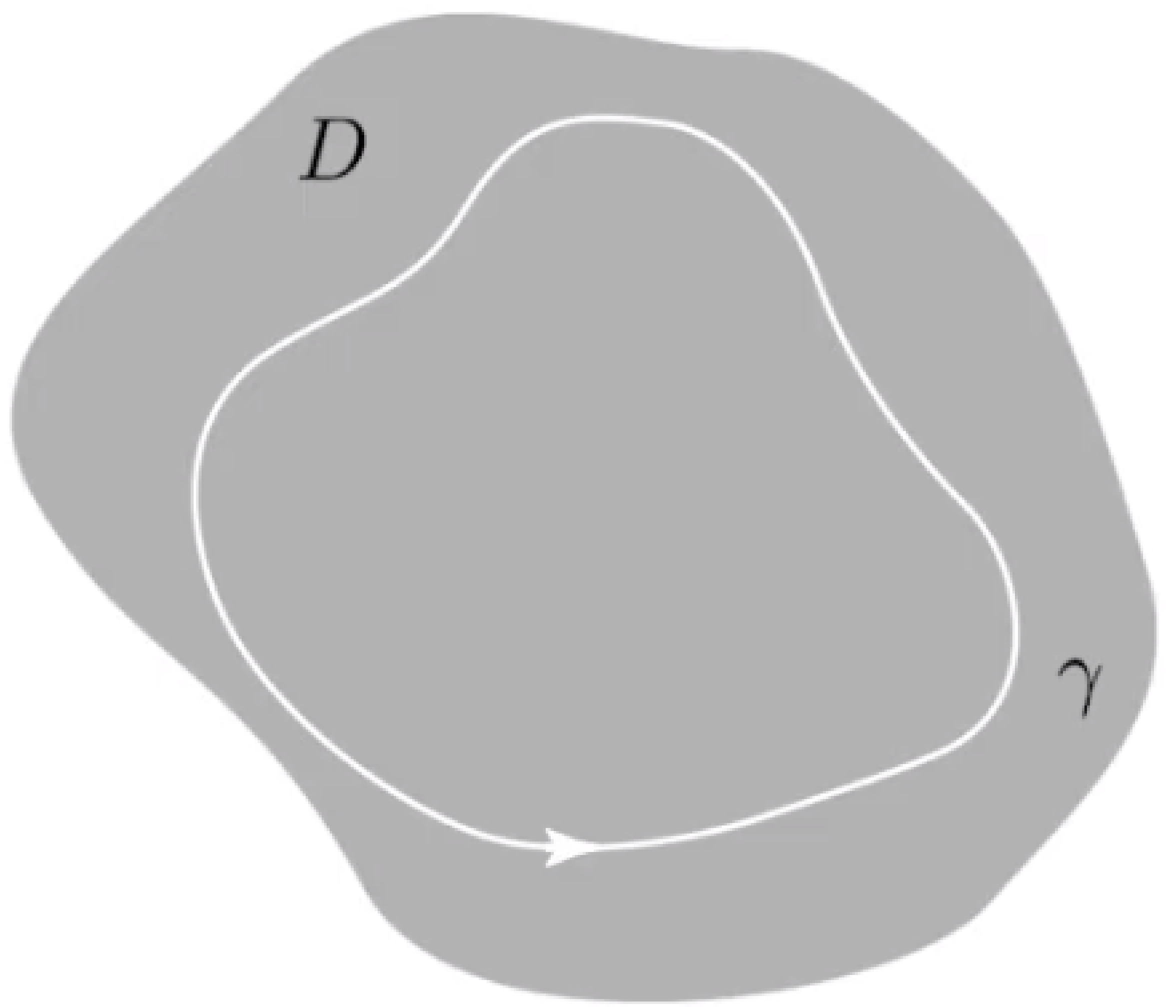
Cauchy theorem

Cauchy's integral theorem



$f(z)$ analytic in D

$$\oint_\gamma f(z)dz = 0$$

$$f = u + iv, \quad dz = dx + idy$$

$$\oint_\gamma = \oint_\gamma udx - vdy + i\oint_\gamma udy + vdx$$

Green's formula:

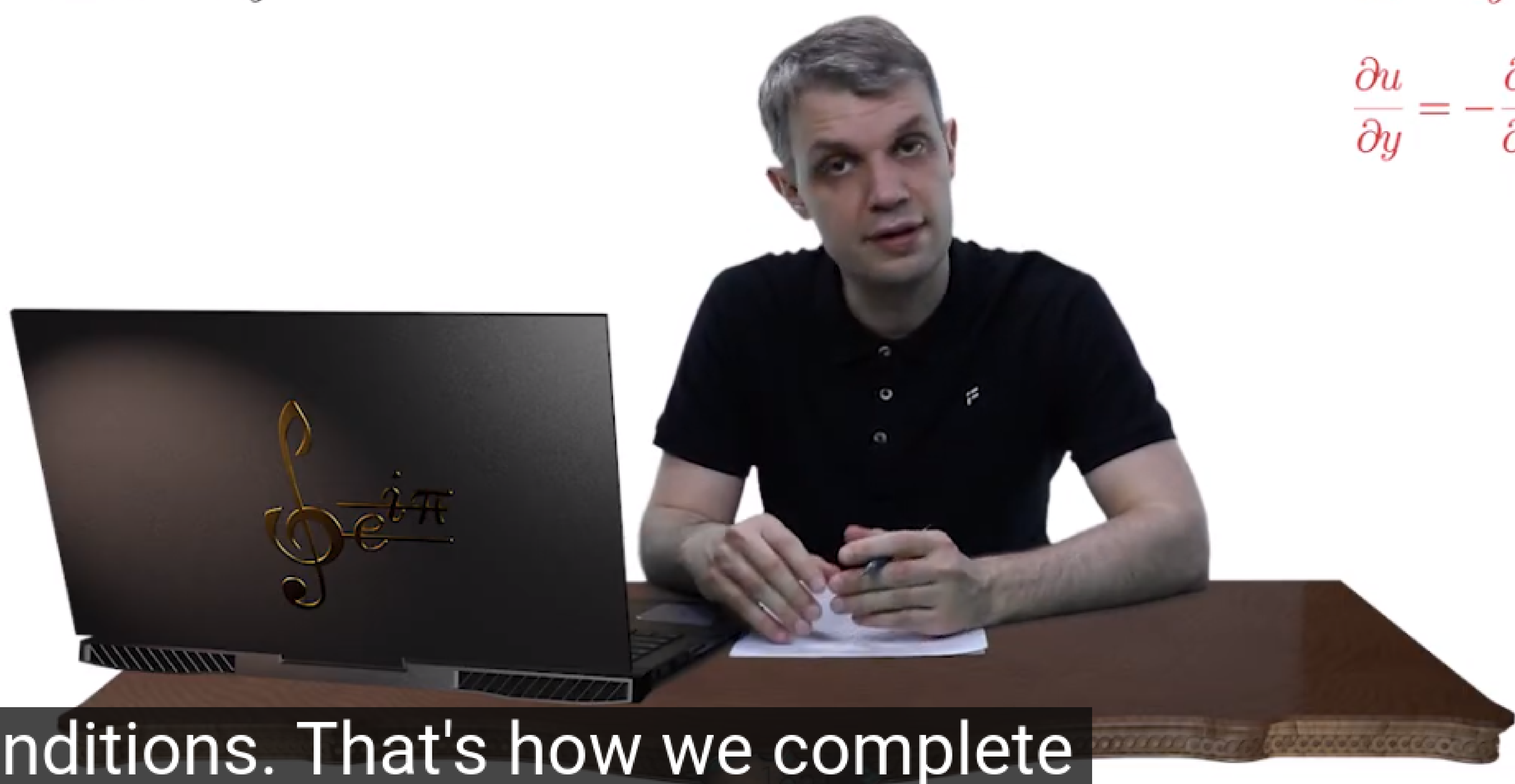
$$\oint Pdx + Qdy = \int_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy$$

$$u = P, \quad -v = Q \qquad \oint_\gamma udx - vdy = \int_D \cancelto{0}{\left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)} dxdy$$

$$v = P, \quad u = Q \qquad \oint_\gamma udy + vdx = \int_D \cancelto{0}{\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right)} dxdy$$

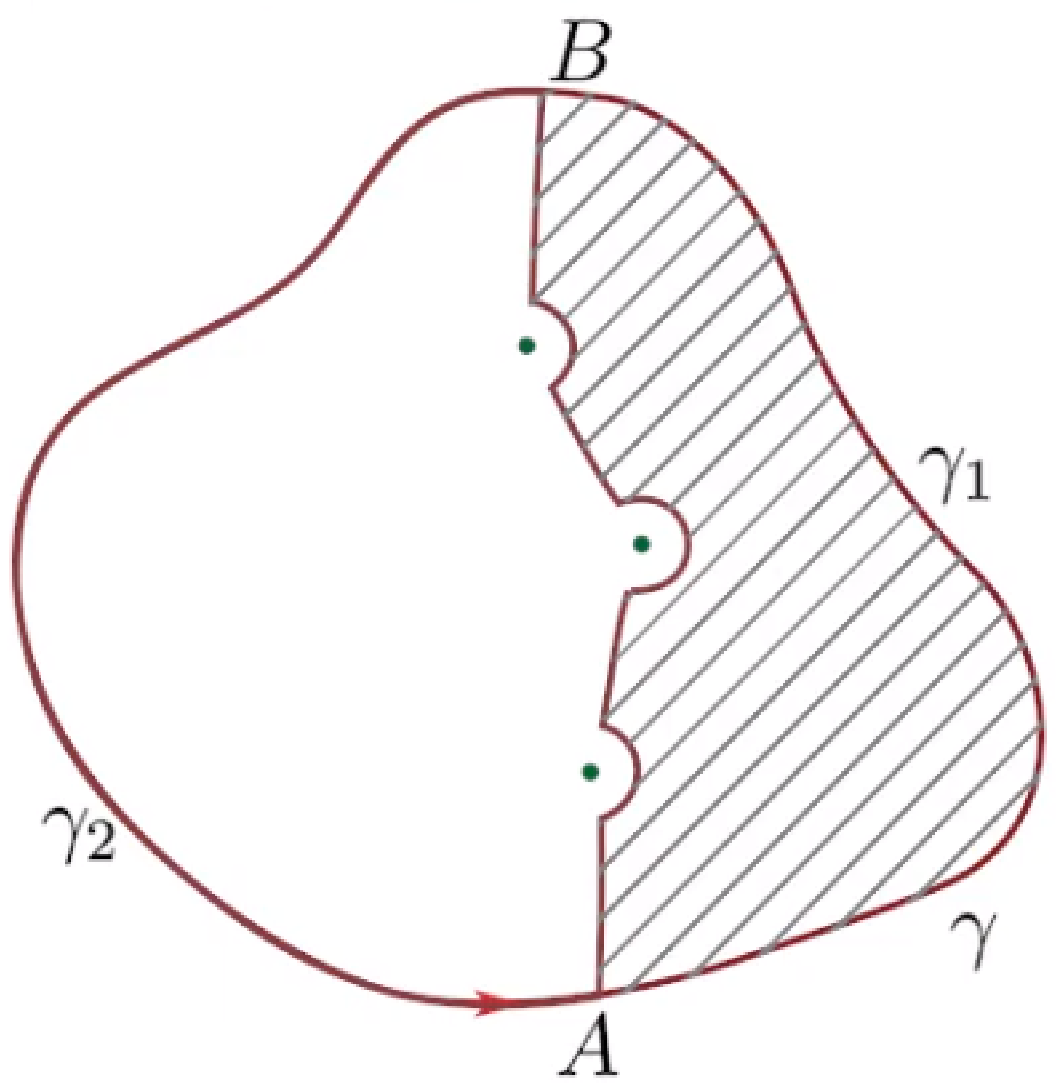
$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$



Cauchy-Riemann conditions. That's how we complete the proof of Cauchy integral formula. Now

Cauchy theorem

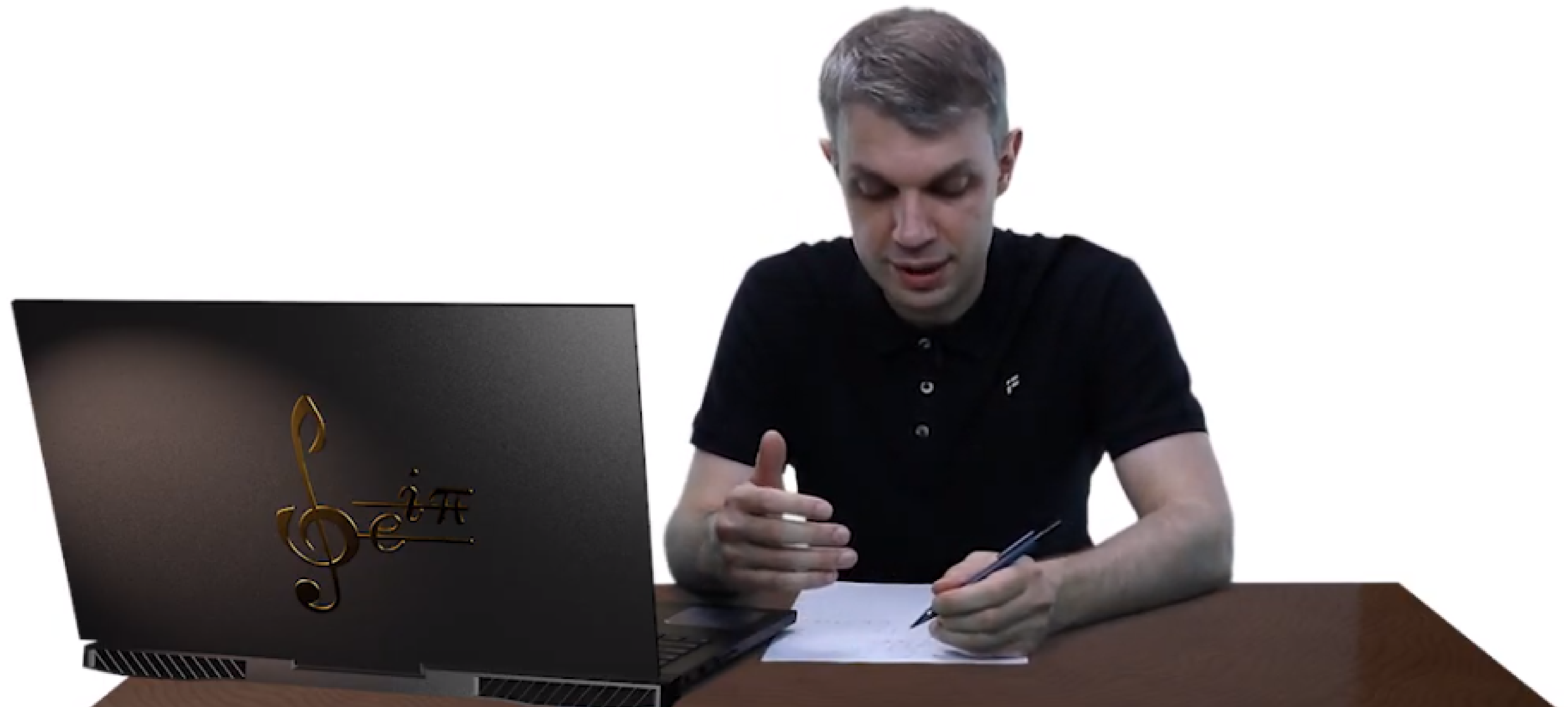


$$I = \oint_\gamma f(z)dz$$

$$\oint_{AB+\gamma_2} = \oint_\gamma$$

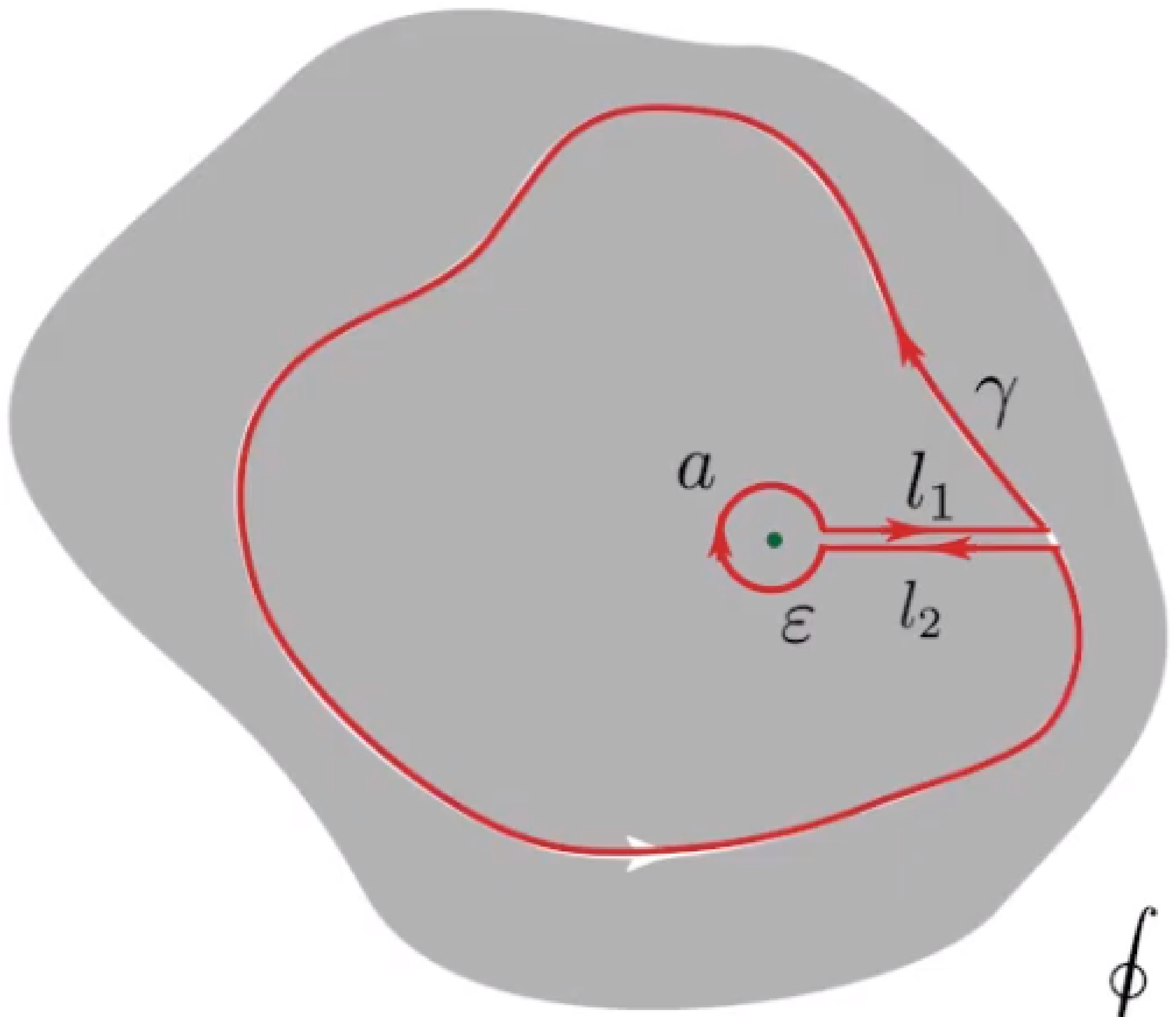
$$\oint_{\gamma_1+BA} = 0$$

$$\oint_{\gamma_1+BA} + \oint_{\gamma_2+AB} = \oint_{\gamma_1} + \cancel{\oint_{BA}} + \oint_{\gamma_2} + \cancel{\oint_{AB}}$$



to the integral along the contour gamma_1
plus gamma_2 which is nothing but our initial

Cauchy theorem



$$f(a) = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z-a}dz$$

$$\gamma_1 = \gamma + l_2 + \varepsilon + l_1$$

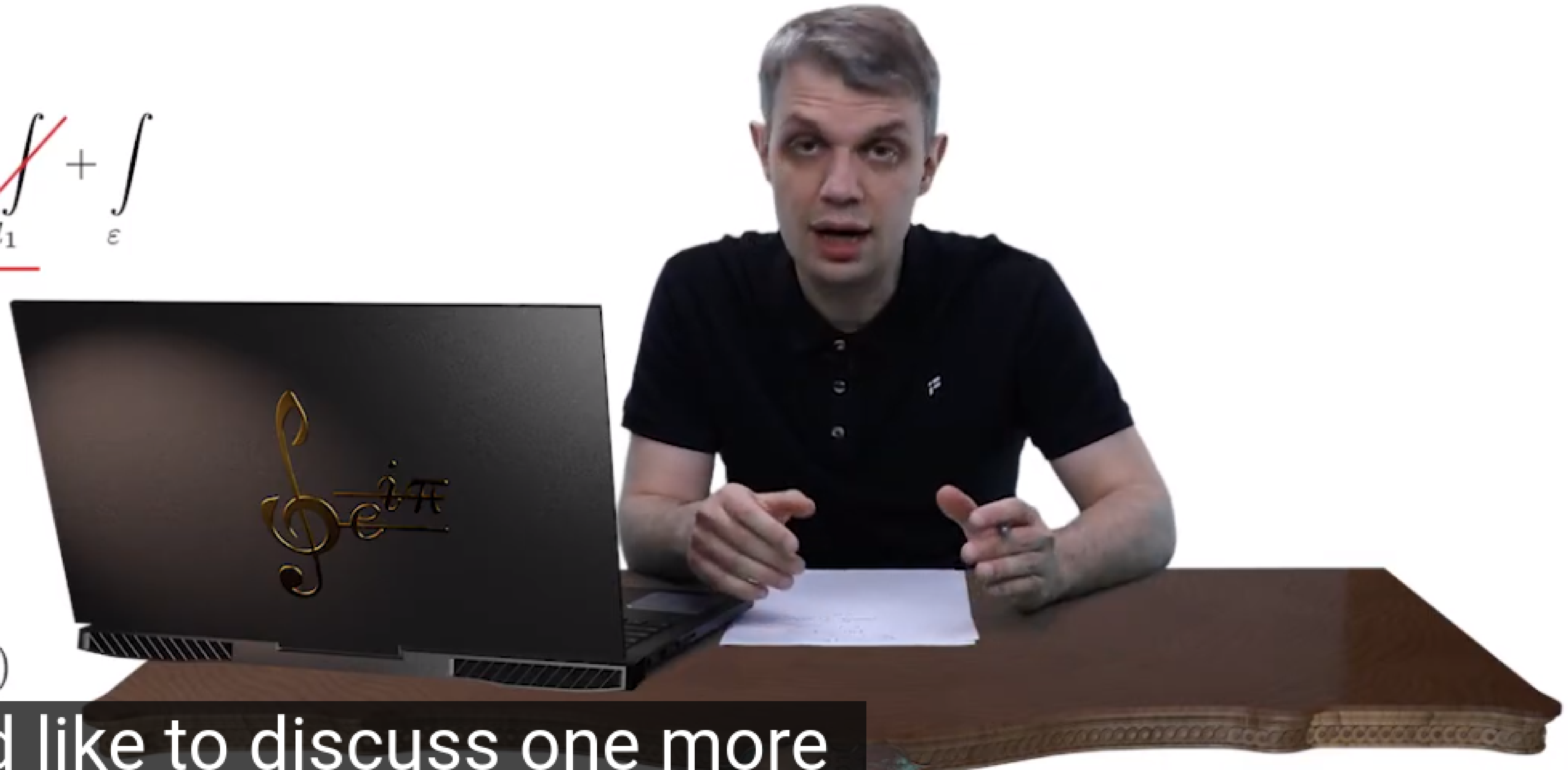
$$\oint_{\gamma_1} \frac{f(z)}{z-a}dz = 0$$

$$\oint_{\gamma_1} = \oint_\gamma + \cancel{\int_{l_2}} + \cancel{\int_{l_1}} + \int_\varepsilon$$

$$\oint_\gamma = -\int_\varepsilon \qquad z = a + \varepsilon e^{i\varphi}, \quad \varphi \in [0, -2\pi]$$

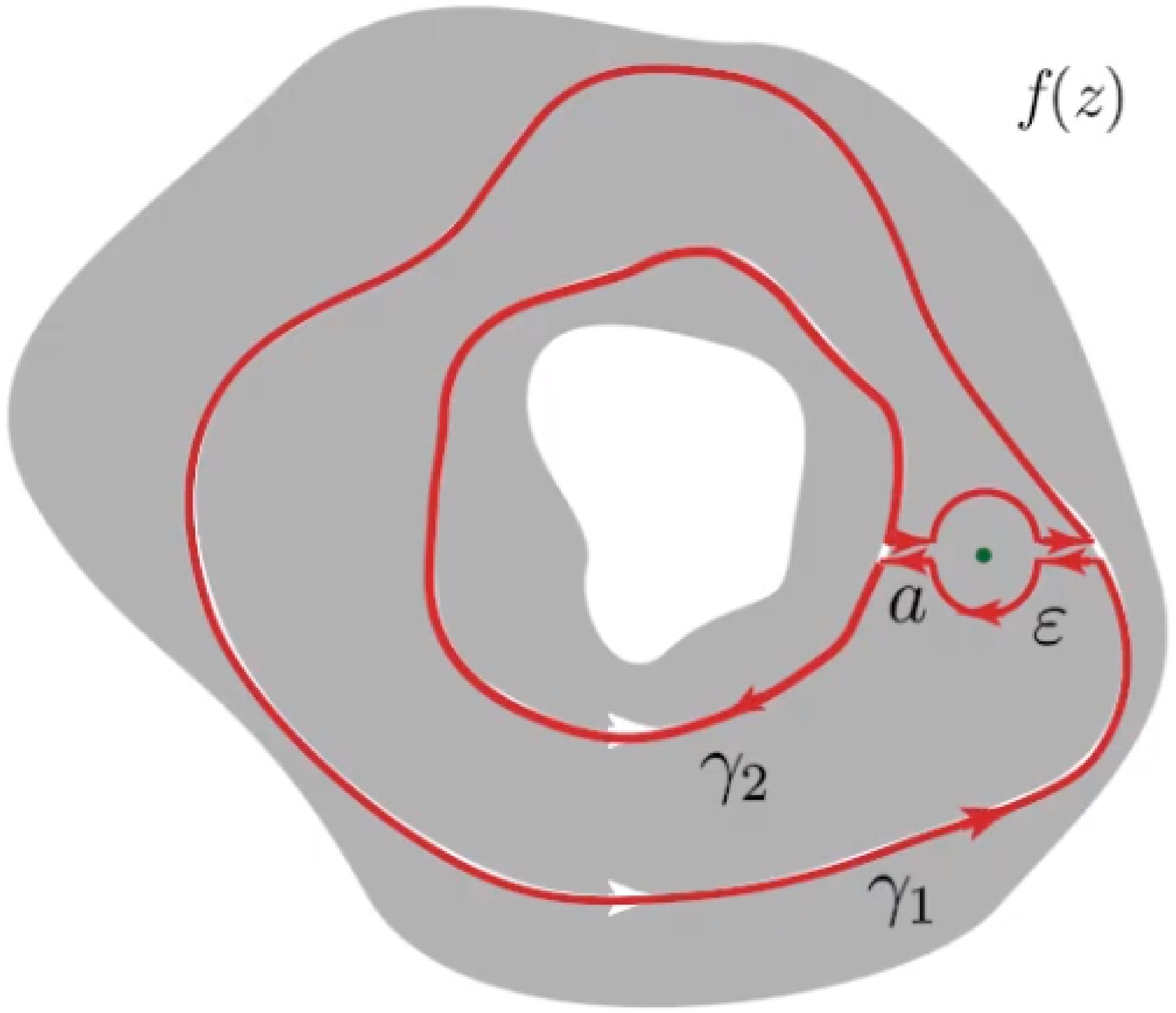
$$dz = \varepsilon e^{i\varphi} i d\varphi \qquad \frac{dz}{z-a} = i d\varphi$$

$$\int_\varepsilon = \int_0^{-2\pi} i d\varphi f(a + \cancel{\varepsilon e^{i\varphi}})\Big|_{\varepsilon\to 0} = \int_0^{-2\pi} i d\varphi f(a) = -2\pi i f(a)$$



And now I'd like to discuss one more
generalization of this result, because up to now

Cauchy theorem



$$f(a) = \frac{1}{2\pi i}\oint_{\gamma_1} \frac{f(z)}{z-a}dz - \frac{1}{2\pi i}\oint_{\gamma_2} \frac{f(z)}{z-a}dz$$

$$\gamma_1 + l_1 + l_2 + l_3 + l_4 - \gamma_2 + \varepsilon : \quad \oint \frac{f(z)}{z-a}dz = 0$$

$$\oint = \int_{\gamma_1} - \int_{\gamma_2} + \cancelto{-2\pi i f(a)}{\int_{\varepsilon}} = 0$$

f(a), and this basically completes our proof even for this more complicated case.

Taylor expansion

$$f(z)$$

$$f(a) = \frac{1}{2\pi i}\oint\limits_R \frac{f(z)}{z-a}dz$$

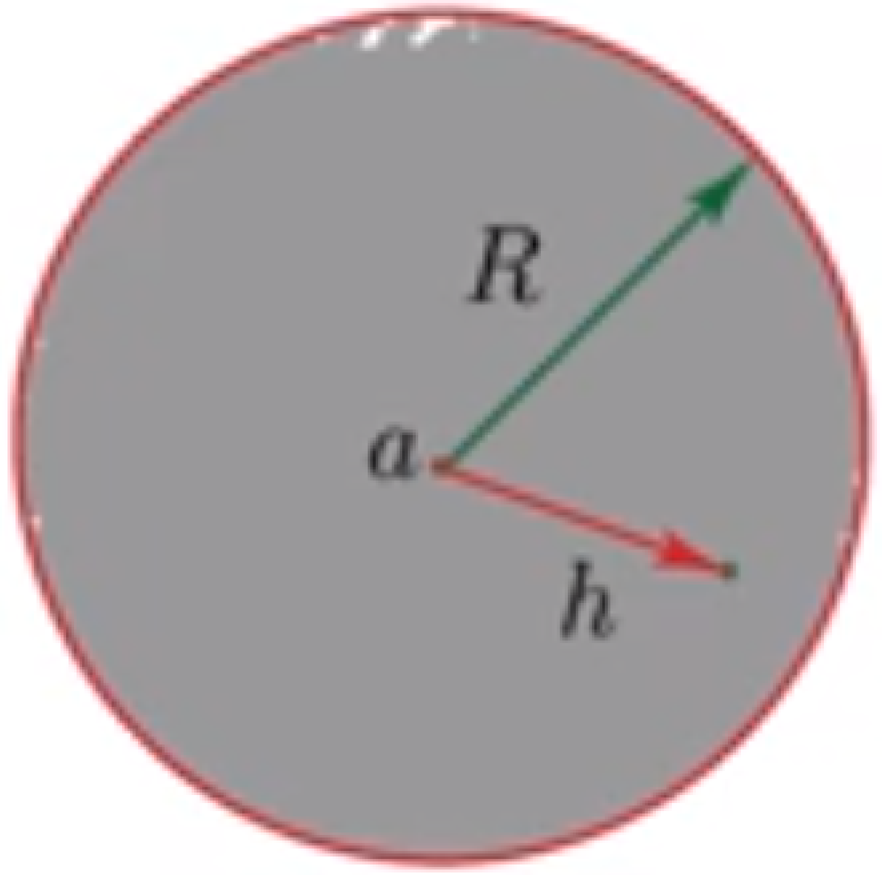
$$f'(a) = \frac{1}{2\pi i}\oint\limits_R \frac{f(z)}{(z-a)^2}dz$$

$$f^{(n)}(a) = \frac{n!}{2\pi i}\oint\limits_R \frac{f(z)}{(z-a)^{n+1}}dz$$

$$z = a + h, \quad |h| < R$$

$$f(a+h) = \frac{1}{2\pi i}\oint\limits_R \frac{f(z)}{z-a-h}dz \qquad \frac{|h|}{R} < 1 \qquad |z-a| = R$$

$$\frac{1}{z-a-h} = \frac{1}{z-a}\frac{1}{1-\frac{h}{z-a}} = \frac{1}{z-a}\sum_{n=0}^{\infty}\left(\frac{h}{z-a}\right)^n$$



$$f(a+h) = \sum_{n=0}^{\infty} h^n \underline{\frac{1}{2\pi i}\oint\limits_R \frac{f(z)}{(z-a)^{n+1}}dz} = \sum_{n=0}^{\infty} \underbrace{\frac{f^{(n)}(a)}{n!}}_{c_n} h^n$$

$$c_n = \frac{1}{2\pi i}\oint\limits_R \frac{f(z)}{(z-a)^{n+1}}dz$$

Taylor expansion

MISIS

$f(z)$

$$f(a)=\frac{1}{2\pi i}\oint_R \frac{f(z)}{z-a}dz$$

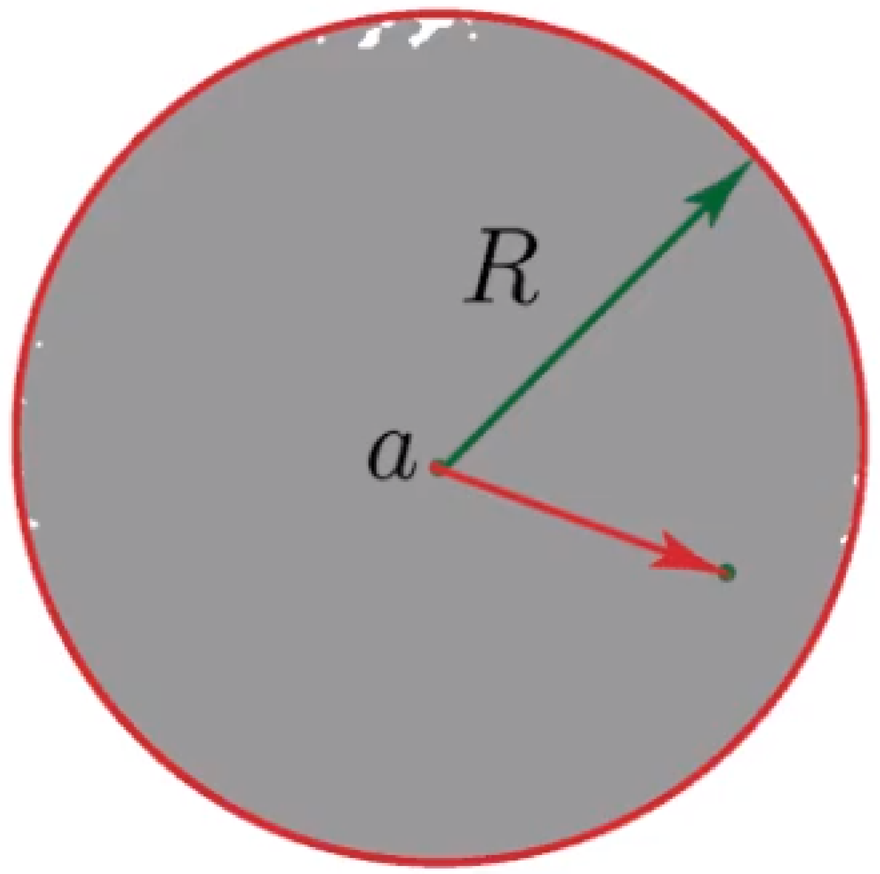
$$f'(a)=\frac{1}{2\pi i}\oint_R \frac{f(z)}{(z-a)^2}dz$$

$$f^{(n)}(a)=\frac{n!}{2\pi i}\oint_R \frac{f(z)}{(z-a)^{n+1}}dz$$

$z=a+h$

$$\frac{1}{1-q}=1+q+q^2+\dots+q^n+\frac{q^{n+1}}{1-q}$$

$$f(a+h)=\sum_{l=0}^{n} h^l\frac{1}{2\pi i}\oint_R \frac{f(z)}{(z-a)^{l+1}}dz+r_n(h)$$



$$=\sum_{l=0}^{n}\frac{f^{(n)}(a)}{n!}h^n+r_n(h)$$

$$r_n(h)=\frac{1}{2\pi i}\oint_R f(z)\frac{h^{n+1}}{(z-a)^{n+1}(z-a-h)}dz$$

$$\left|\int \dots dz\right|\leq\int|\dots||dz|$$

$$|z-a-h|\geq\overset{R}{|z-a|}-\overset{kR}{|h|},\quad 0<k<1$$

$$|f(z)|\leq M$$

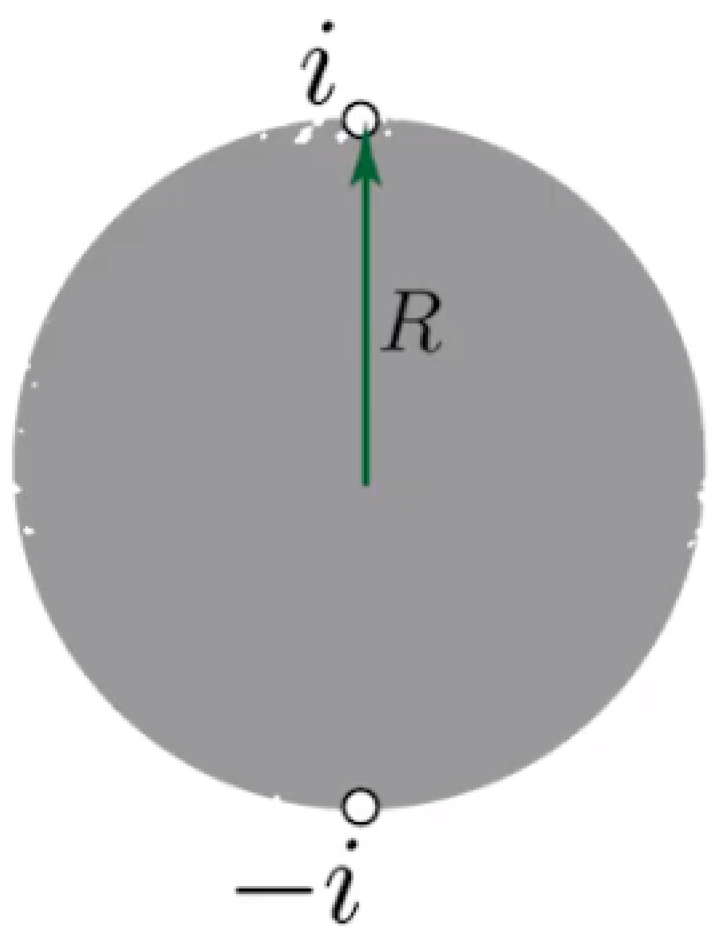
$$|r_n(h)|\leq\frac{1}{2\pi}|h|^{n+1}\int_0^{2\pi}\frac{R\,d\varphi}{R(1-k)R^{n+1}}M=M\frac{k^{n+1}}{1-k}\xrightarrow[n\to\infty]{}0$$

Taylor expansion

$$f(z) = \frac{1}{1+z^2} = 1 - z^2 + z^4 - z^6 + \ldots$$

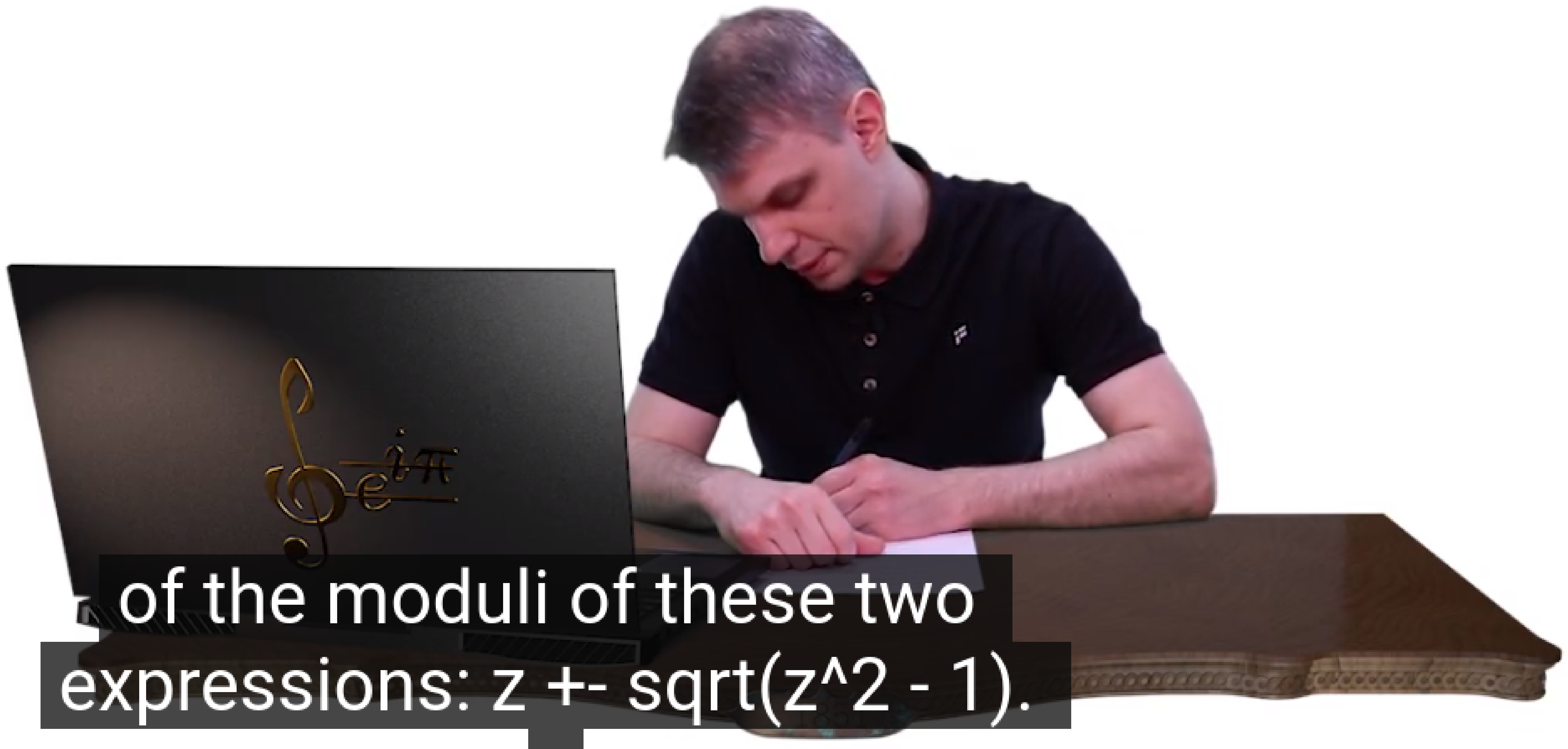
$R = 1$

Singularities: $z^2 = -1, \rightarrow \quad z = \pm i$



$$(1 - 2hz + h^2)^{-1/2} = 1 + P_1(z)h + P_2(z)h^2 + \ldots$$

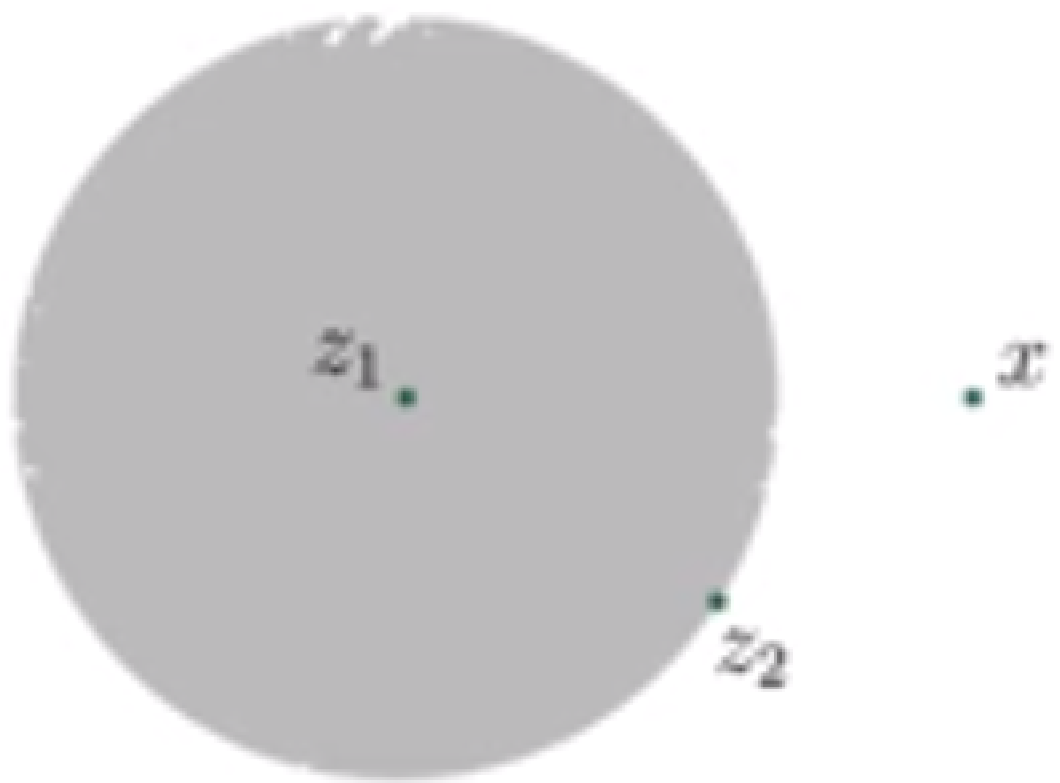
$$h = z \pm \sqrt{z^2-1} \rightarrow R = \min\{|z \pm \sqrt{z^2-1}$$

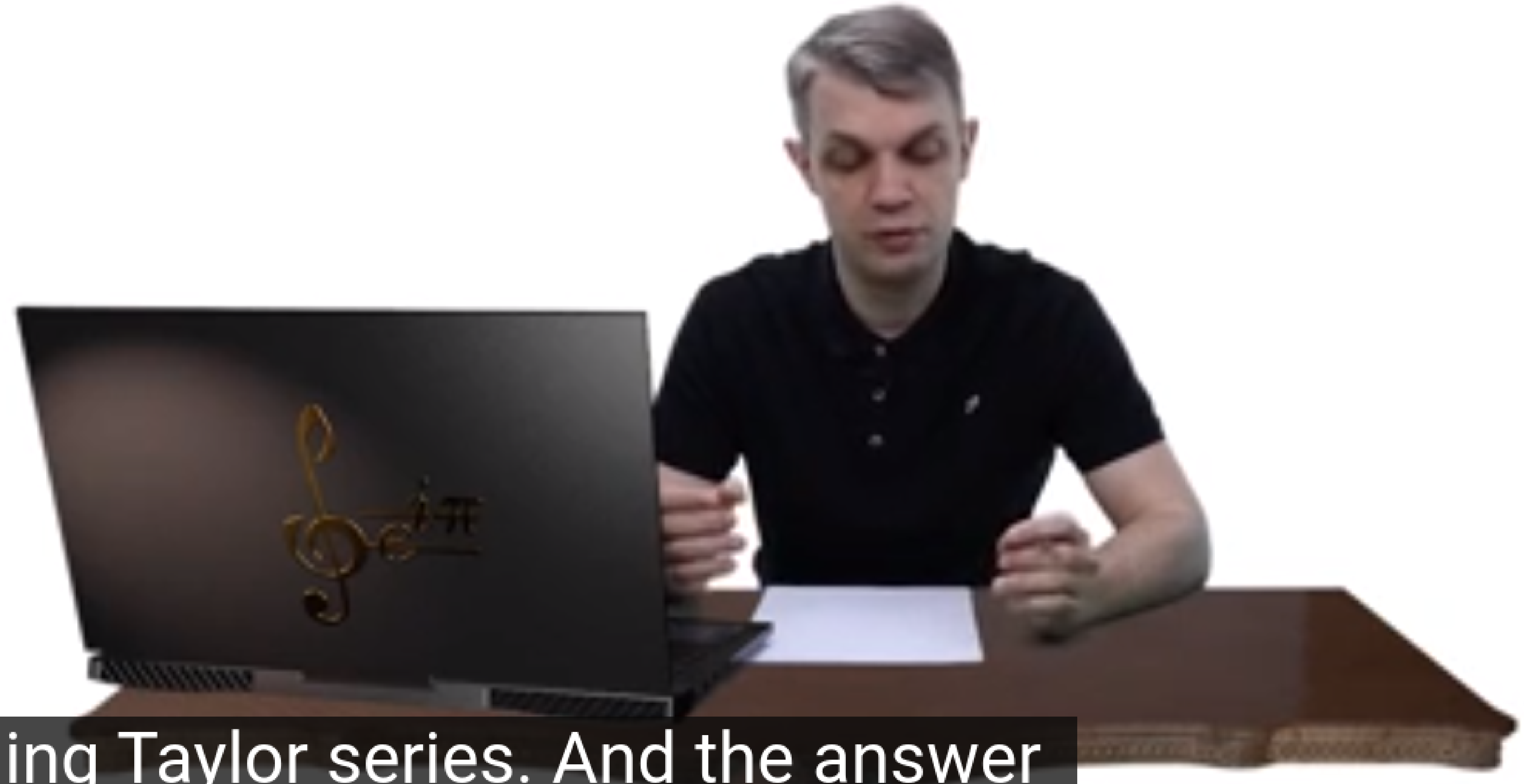


of the moduli of these two expressions: z +- sqrt(z^2 - 1).

Laurent expansion

$R = |z_2 - z_1|$



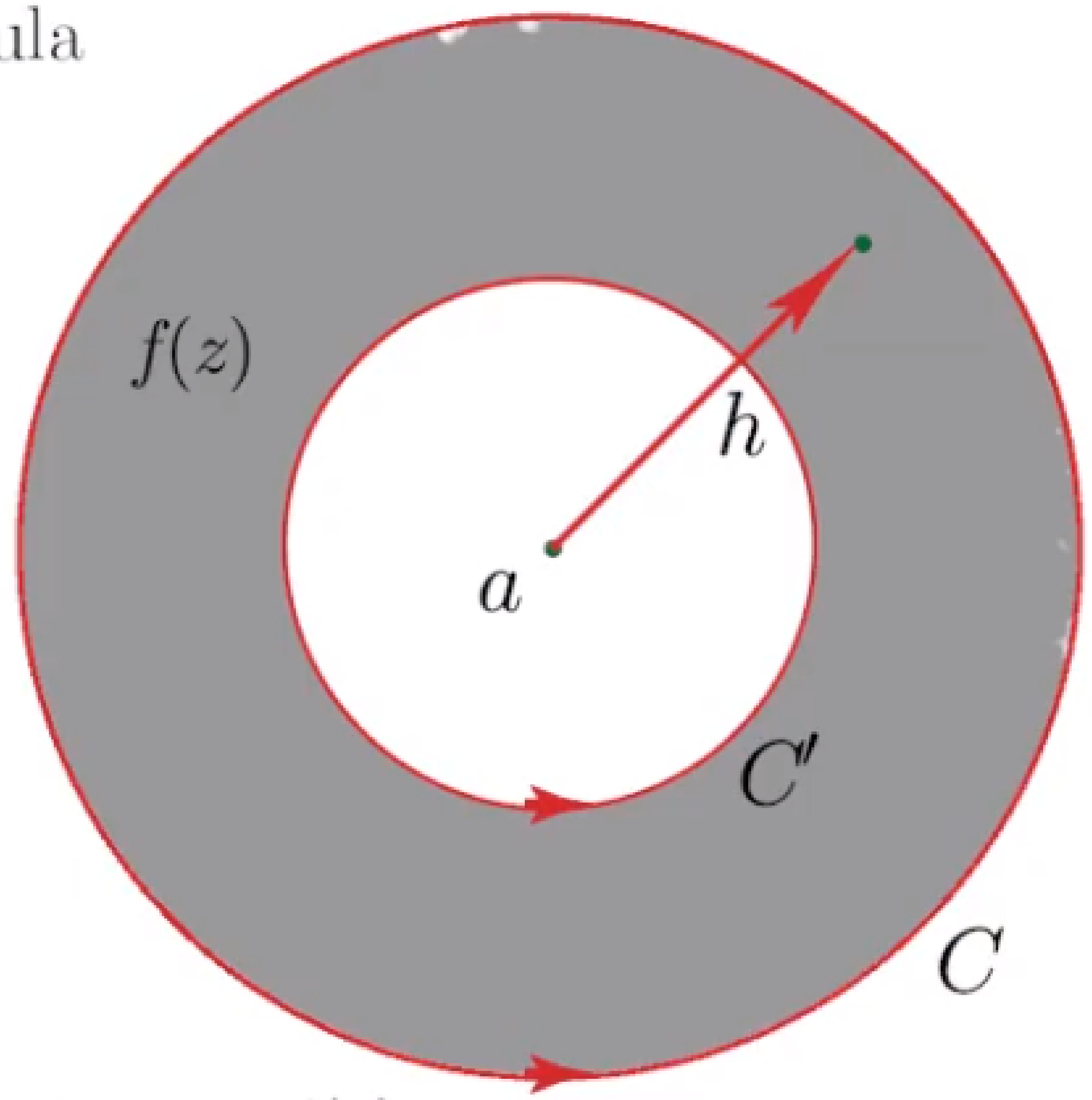


of the corresponding Taylor series. And the answer is: yes, we can, but at the price of incorporating

Laurent expansion

MISIS

We write down Cauchy formula for some arbitrary point $z=a+h$ inside the ring.



$$= \underline{\sum_{n=0}^{\infty} a_n h^n} + \underline{\sum_{n=1}^{\infty} \frac{b_n}{h^n}}$$ Laurent expansion

$$b_n = \frac{1}{2\pi i}\oint_{C'} f(z)(z-a)^{n-1}dz$$

$$f(a+h) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z-a-h}dz - \underline{\frac{1}{2\pi i}\oint_{C'} \frac{f(z)}{z-a-h}dz}$$

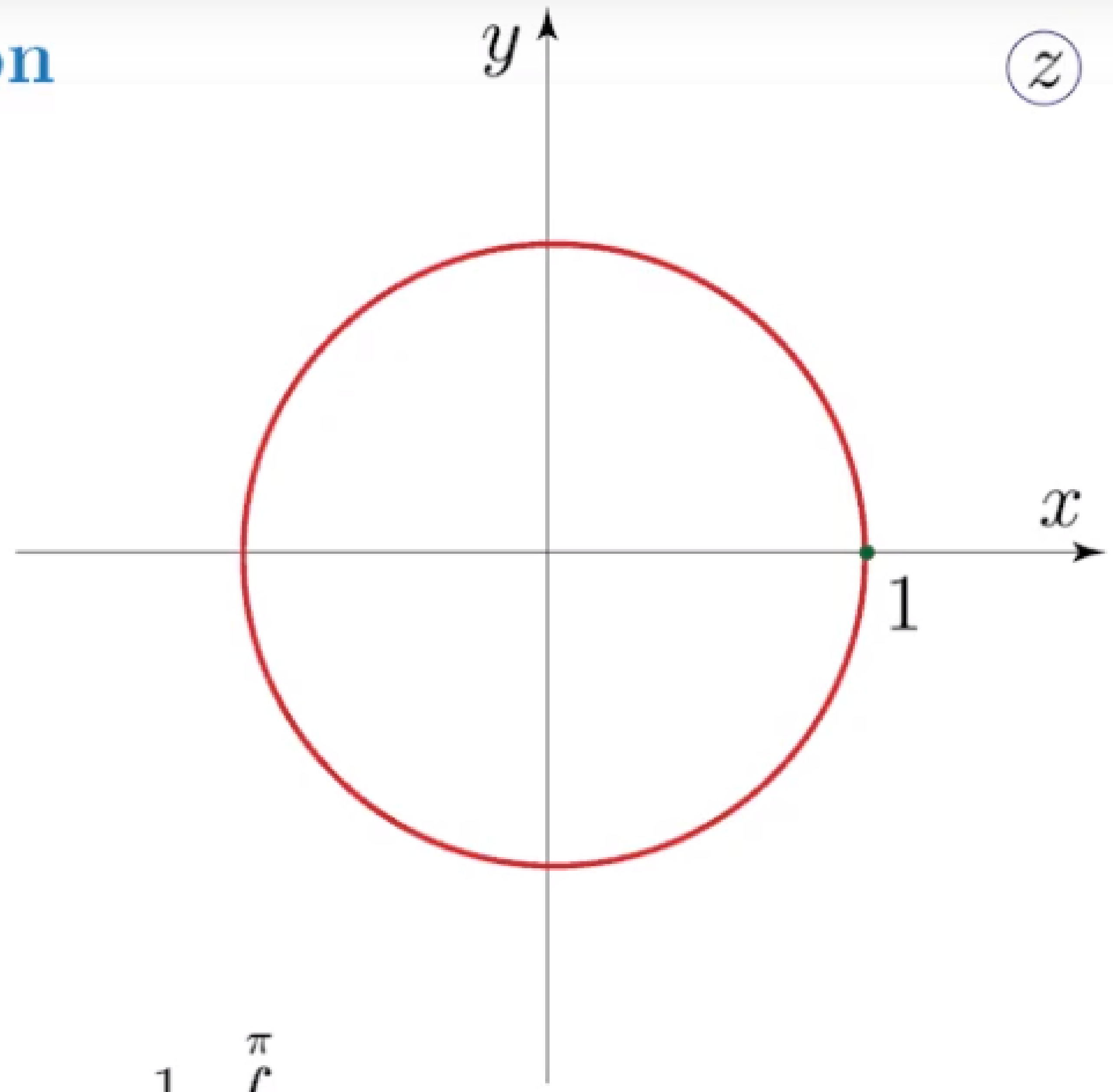
$$\downarrow$$

$$\sum_{n=0}^{\infty} a_n h^n \quad + \frac{1}{2\pi i h}\oint_{C'} \sum_{n=0}^{\infty} f(z)\left(\frac{z-a}{h}\right)^n dz$$

$$\underline{a_n = \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-a)^{n+1}}dz}$$

$$|z-a|<|h| \qquad \frac{1}{z-a-h} = -\frac{1}{h}\frac{1}{1-\frac{z-a}{h}} \qquad \left|\frac{z-a}{h}\right|<1$$

Laurent expansion



$$f(z)=e^{\frac{x}{2}(z-1/z)}$$

$$z=0$$

$$z=e^{i\varphi},\quad \varphi\in[-\pi,\ \pi]$$

$$\frac{dz}{z}=id\varphi$$

$$a_n=\frac{1}{2\pi i}\int_{-\pi}^{\pi}e^{ix\sin\varphi-in\varphi}\ id\varphi=\frac{1}{2\pi}\int_{-\pi}^{\pi}\cos\left(x\sin\varphi-n\varphi\right)d\varphi=\mathrm{J}_n(x)$$

$$a_n=\frac{1}{2\pi i}\oint\frac{f(z)}{(z-a)^{n+1}}dz$$

$$b_n=\frac{1}{2\pi i}\oint f(z)(z-a)^{n-1}dz$$

$$b_n=\frac{1}{2\pi i}\int_{-\pi}^{\pi}e^{ix\sin\varphi+in\varphi}\ id\varphi\ =(-1)^n a_n$$

$$\varphi\to\pi-\varphi$$

$$f(z)=\sum_{n=0}^{\infty}\mathrm{J}_n(x)z^n+\sum_{n=1}^{\infty}(-1)^n\mathrm{J}_n(x)z^{-n}$$

times z to the power of negative n. And that's it: that completes our first discussion of the

Laurent expansion

$$f(z)=\frac{1}{z^2-z-2} \qquad \text{Center: } z=0,\ \frac{3i}{2}\in D$$

$$z=2,\ \ z=-1 \rightarrow f(z)=\frac{1}{(z-2)(z+1)}=\left(\frac{1}{z-2}-\frac{1}{z+1}\right)\frac{1}{3}$$

$$\frac{1}{z+1}=1-z+z^2+\ldots \qquad R=1$$

$$\frac{1}{z+1}=\frac{1}{z(1+\frac{1}{z})}=\frac{1}{z}\sum_{n=0}^{\infty}\frac{(-1)^n}{z^n}$$

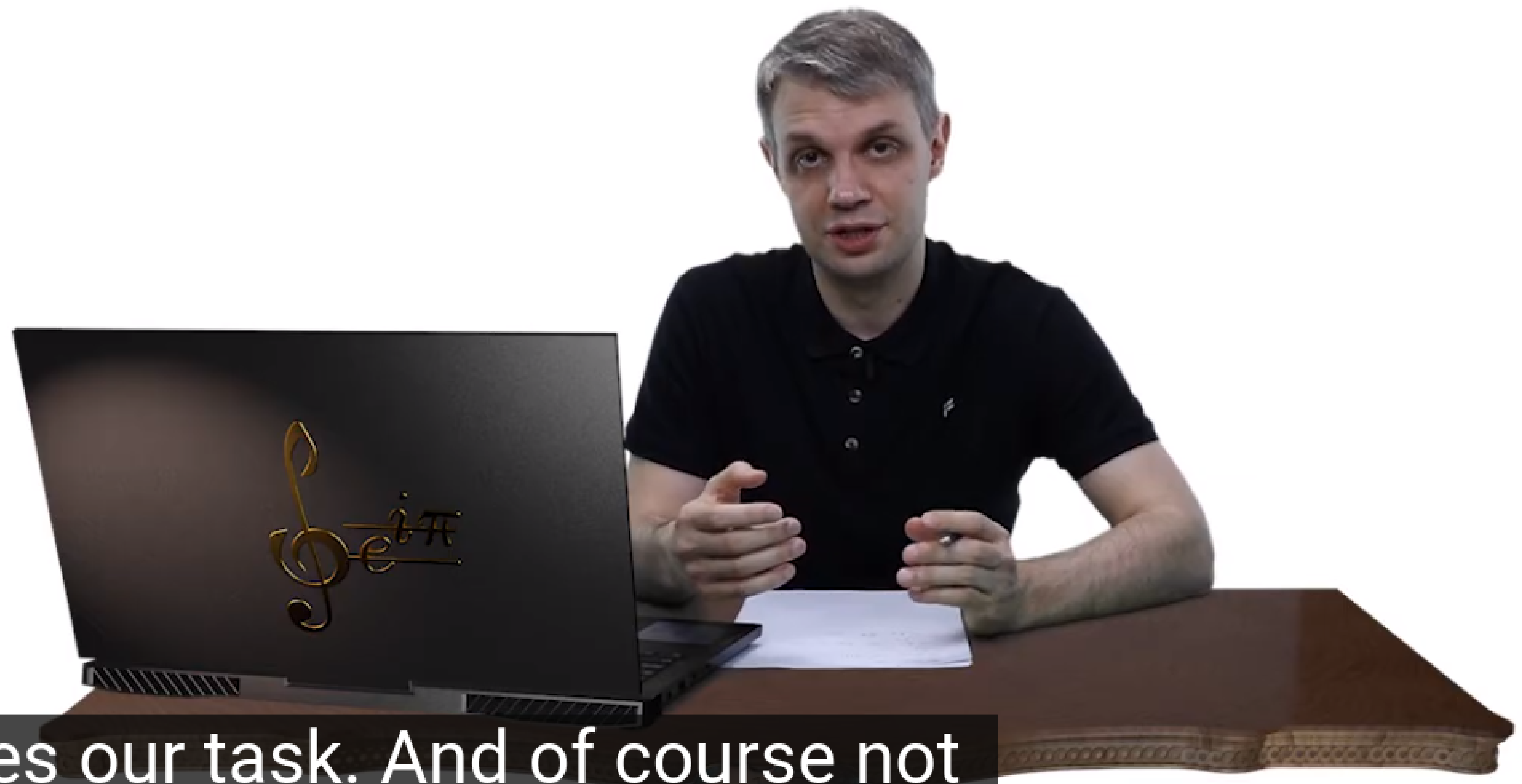
$$\left|\frac{1}{z}\right|<1 \rightarrow |z|>1$$

$$\frac{1}{z-2}=-\frac{1}{2(1-\frac{z}{2})}=-\frac{1}{2}\sum_{n=0}^{\infty}\frac{z^n}{2^n}$$

$$\left|\frac{z}{2}\right|<1 \rightarrow |z|<2$$

$$f(z)=-\frac{1}{6}\sum_{n=0}^{\infty}\frac{z^n}{2^n}-\frac{1}{3}\sum_{n=0}^{\infty}\frac{(-1)^n}{z^{n+1}}$$

$$1<|z|<2$$



And this completes our task. And of course not many functions have this suitable algebraic

Laurent expansion

$$f(z)=\frac{1}{(\sin z)^3} \qquad z=\pi n,\ \ n\in Z \qquad\qquad R=\pi$$

$$z=\pi n+\varepsilon$$

$$f(z)=\frac{1}{[\sin(\pi n+\varepsilon)]^3}=\frac{(-1)^n}{\underline{(\sin\varepsilon)}^3}=\frac{(-1)^n}{(\varepsilon-\varepsilon^3/6)^3}=\frac{(-1)^n}{\varepsilon^3(1-\varepsilon^2/6)^3}$$

$$\sin\varepsilon=\underline{\varepsilon-\frac{\varepsilon^3}{3!}}+\frac{\varepsilon^5}{5!}+\dots$$

$$(1+x)^\alpha=1+\alpha x+\dots \qquad \alpha=-3,\ \ x=-\frac{\varepsilon^2}{6}$$

$$\frac{1}{(1-\varepsilon^2/6)^3}=1-3\left(-\frac{\varepsilon^2}{6}\right)+\dots$$

$$f(\pi n+\varepsilon)=\underline{(-1)^n\left(\frac{1}{\varepsilon^3}+\frac{1}{2\varepsilon}\right)}+\dots$$

$$f(z)=\frac{(-1)^n}{(z-\pi n)^3}+\frac{(-1)^n}{2(z-\pi n)}+\dots$$

And this completes our discussion of the third method of the Laurent expansions.

Types of singularities

MISIS

1. Isolated singularity

$f(z)$. $\quad z_0, \quad U\colon 0<|z-z_0|<r, \quad f(z)$ is analytic on U

Laurent series

I. $f(z) = c_0 + c_1(z-z_0) + c_2(z-z_0)^2 + \dots$

$$f(z) = \frac{\sin z}{z} \qquad z=0, \quad f(0)=1$$

II. $f(z) = \dfrac{c_{-n}}{(z-z_0)^n} + \dfrac{c_{-n+1}}{(z-z_0)^{n-1}} + \dots + c_0 + c_1(z-z_0) + \dots$

z_0 – pole of order n.

$$f(z) = \frac{1}{(\sin z)^3}$$

III. $f(z) = \dots + \dfrac{c_{-1}}{z-z_0} + c_0 + c_1(z-z_0) + \dots$

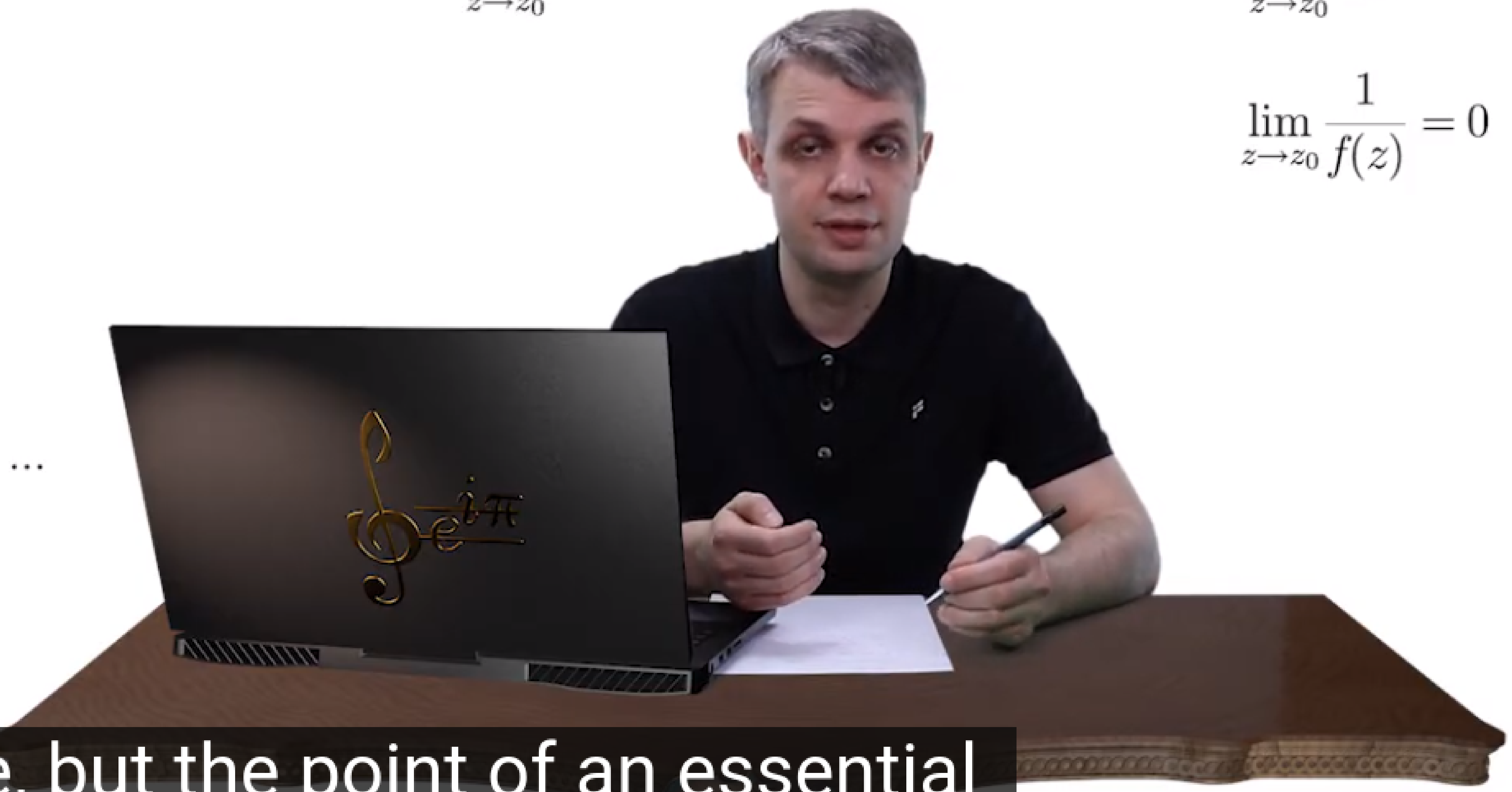
z_0 – essential singularity

$$f(z) = e^{1/z} = \sum_{n=0}^{\infty} \frac{1}{n!z^n}$$

Es. sing.: $\lim_{z \to z_0} f($

Pole: $\lim_{z \to z_0} f(z) = \infty$

$$\lim_{z \to z_0} \frac{1}{f(z)} = 0$$



of a Riemann sphere, but the point of an essential singularity is such that the limit of the function

Types of singularities

1. Isolated singularity

Es. sing.: $\lim\limits_{z\to z_0} f(z)$ does not exist

Definition of the limit by Heine:

$$\lim_{z\to z_0} f(z) = M$$

for any $\{z_n\} \underset{n\to\infty}{\to} z_0$

$$\{f(z_n)\} \underset{n\to\infty}{\to} M$$

$$f(z) = e^{1/z} \qquad z_0 = 0$$

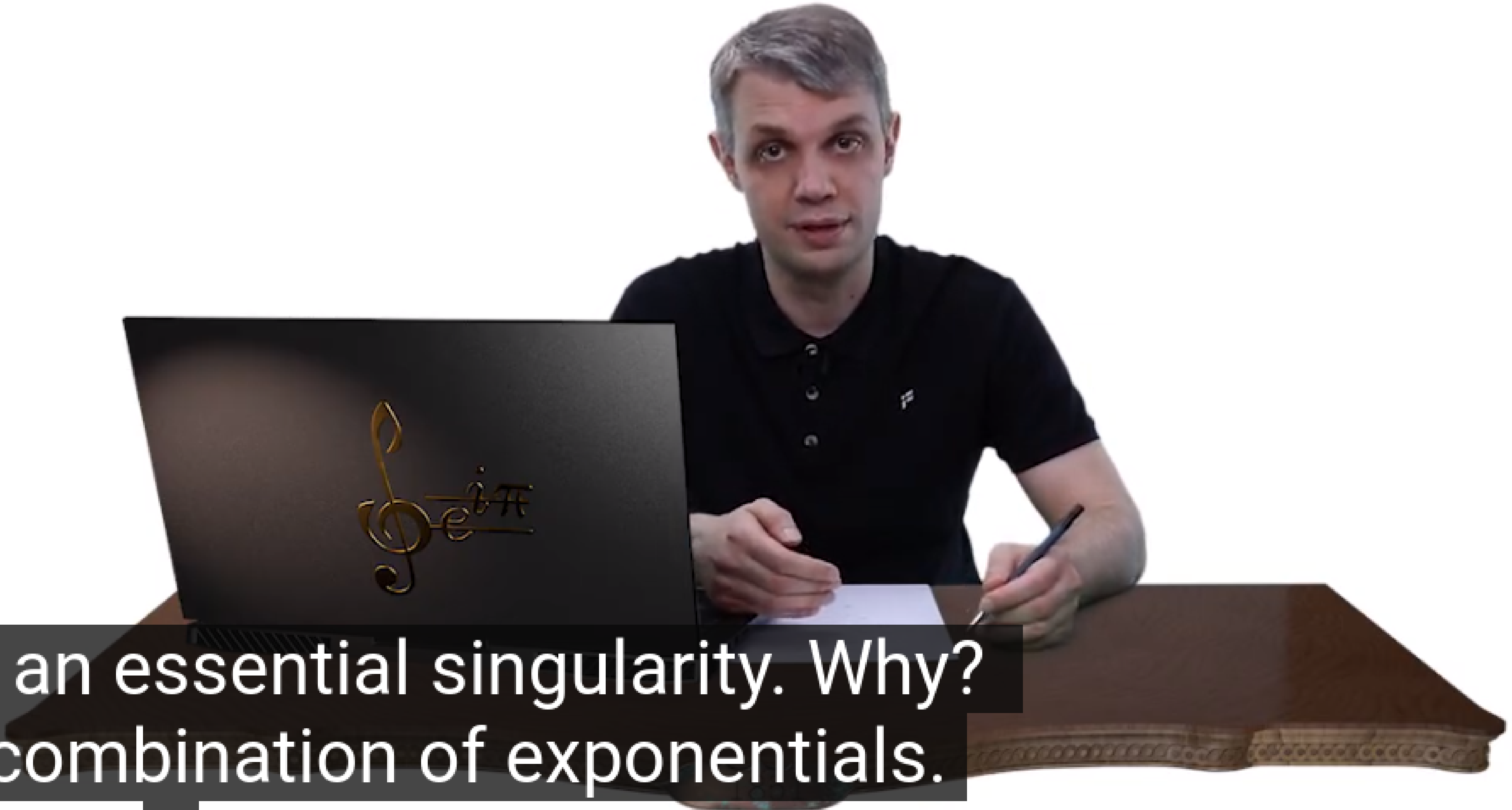
$$z_n = \frac{1}{2\pi i n}, \ \lim_{n\to\infty} z_n = 0$$

$$f(z_n) = e^{2\pi i n} = 1, \ \lim_{n\to\infty} f(z_n) = 1$$

$$z_n = -\frac{1}{n}, \ \lim_{n\to\infty} z_n = 0$$

$$f(z_n) = e^{-n}, \ \lim_{n\to\infty} f(z_n) = 0$$

$$f(z) = \sin\frac{1}{z}, \quad z = 0 \ - \text{essential singularity}$$



z equals 0 would be an essential singularity. Why?
because sine is a combination of exponentials.

Types of singularities

MISIS

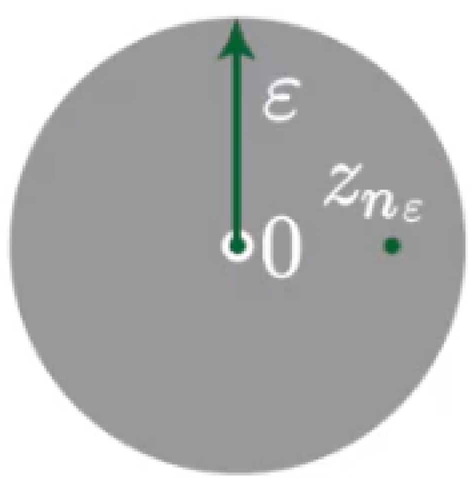
1. Isolated singularity

2. Non−isolated singularity

$$f(z) = \tan\frac{1}{z}$$

$$\frac{1}{z} = \frac{\pi}{2} + \pi n, \to z_n = \frac{1}{\frac{\pi}{2} + \pi n}$$

$z = 0$ − non−isolated singularity

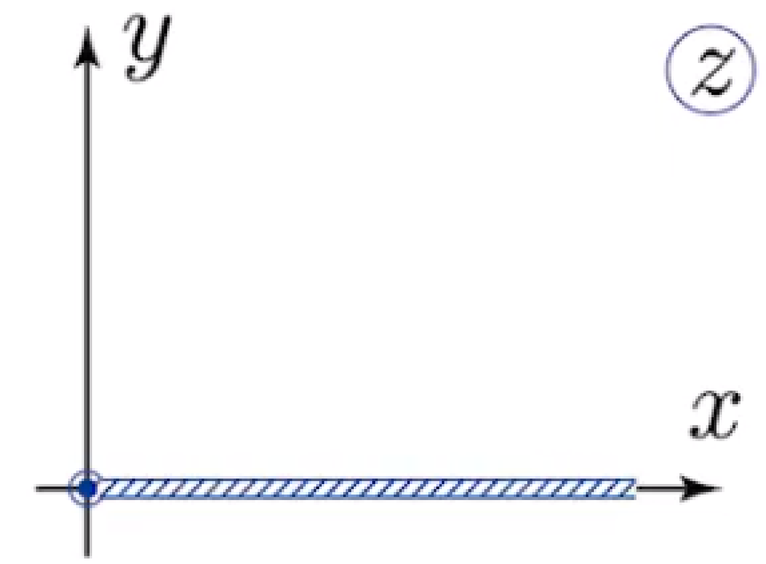


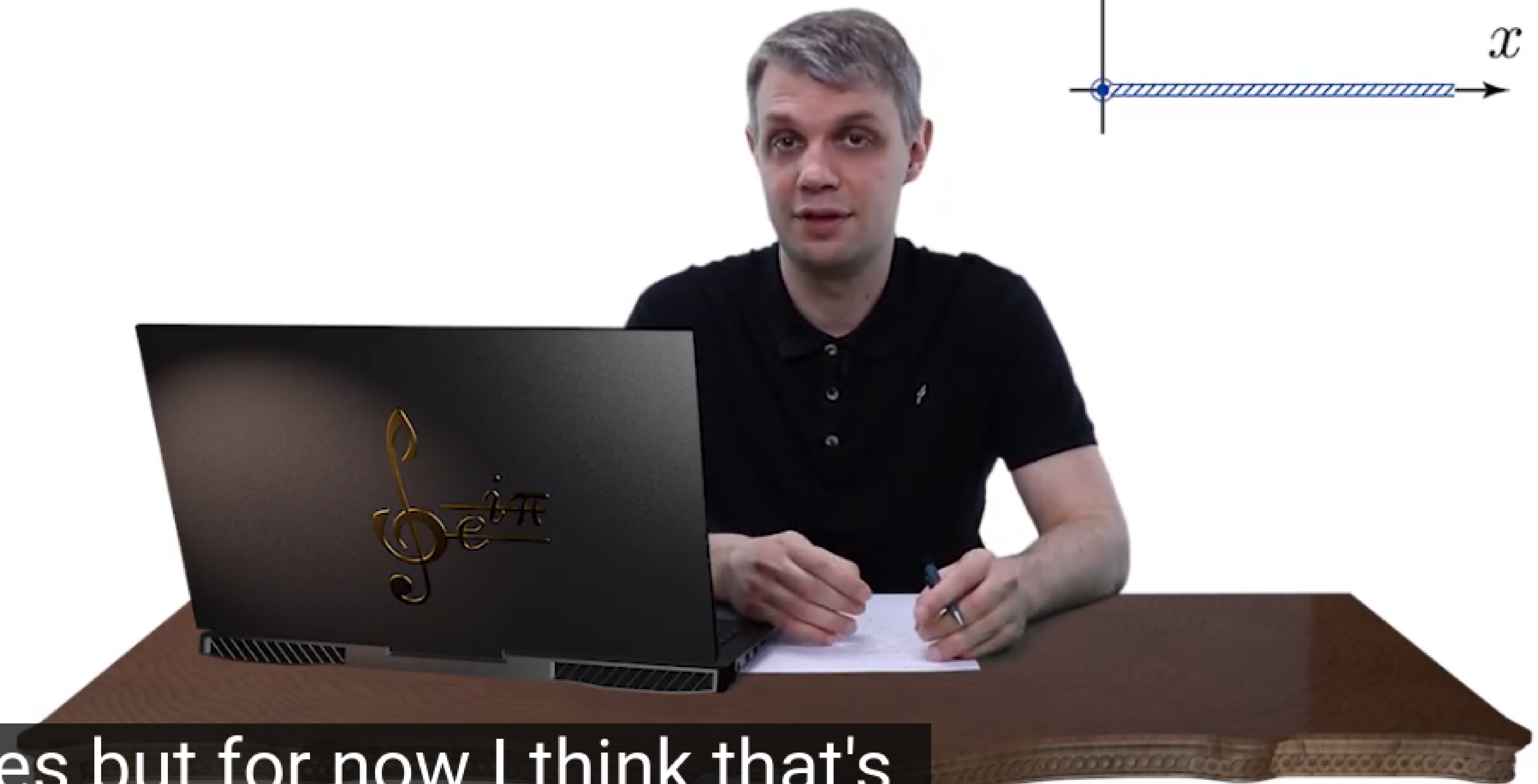
$$n_\varepsilon > \frac{1}{\pi\varepsilon} - \frac{1}{2}$$

3. Branch point

$f(z) = \sqrt{z}$, $\ln z$

$z = 0$ − branch point





our future lectures but for now I think that's
it and good luck with your homework exercises!