complex_number_algebra__1

Algebra of complex numbers

Complex number: $z = x + iy$

$x = \mathrm{Re}\, z, \quad y = \mathrm{Im}\, z;$

$i^2 = -1.$

$z_1 \cdot z_2 = (x_1 + iy_1)(x_2 + iy_2) = x_1x_2 - y_1y_2 + i(x_1y_2 + x_2y_1)$

$\underline{z_1 z = z_2}, \quad z = \frac{z_2}{z_1} = z_2 \cdot z_1^{-1}\ .$

$z_1 = x_1 \in \mathrm{R}$

$z = \frac{x_2 + iy_2}{x_1} = \frac{x_2}{x_1} + i\frac{y_2}{x_1}$

$z^* = x - iy$

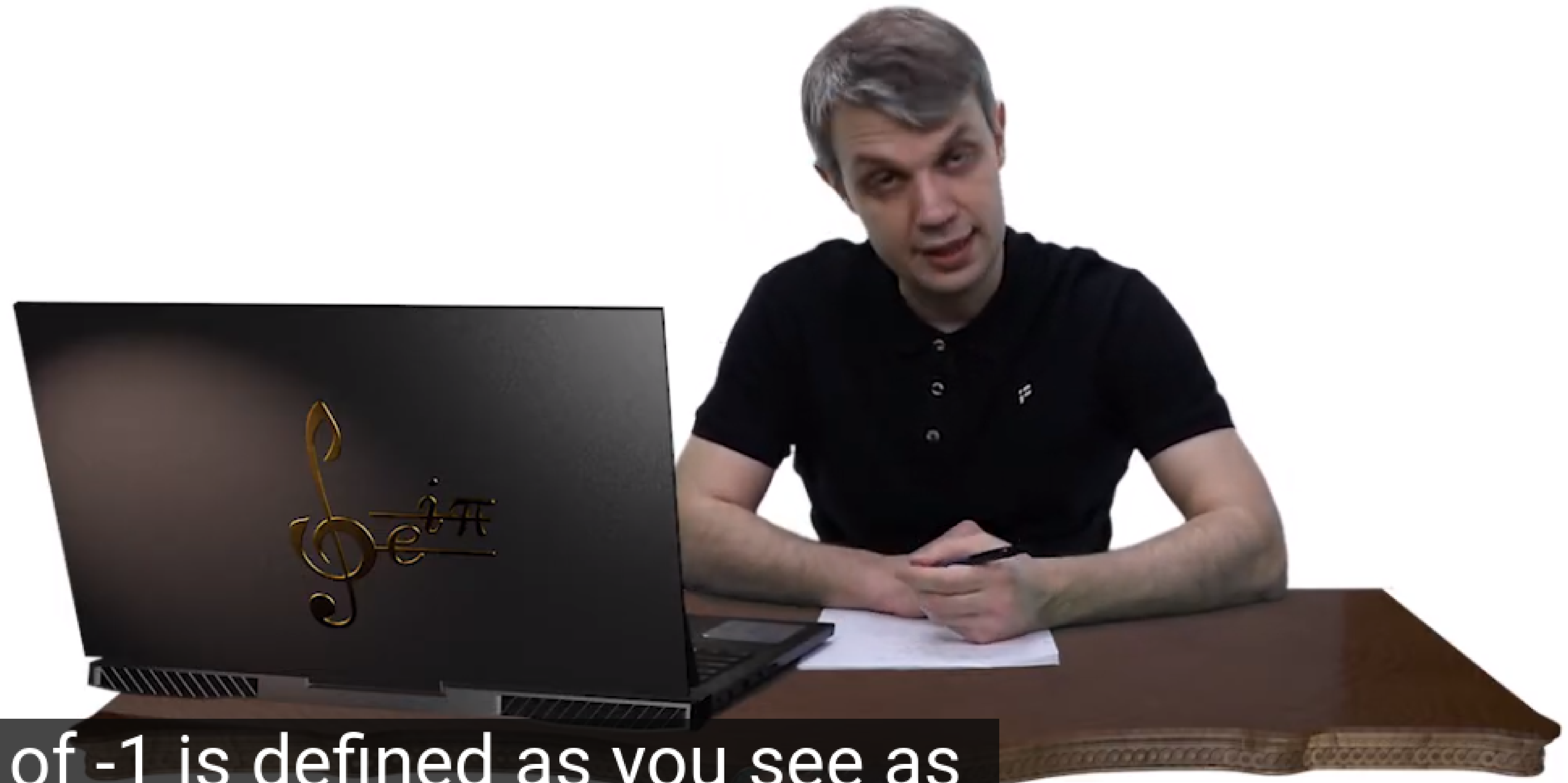
$zz^* = x^2 + y^2$

$|z| = \sqrt{x^2 + y^2}$

$zz^* = |z|^2$

$\underbrace{z_1^* z_1}_{x_1^2 + y_1^2} z = z_2 z_1^*$

$z = \frac{z_2 z_1^*}{|z_1|^2} = \frac{z_2}{z_1}, \ \rightarrow z_1^{-1}$



Z1 to the power of -1 is defined as you see as
Z1 star divided by the square of its modulus.

Algebra of complex numbers

MISIS

Example I

$$|z| = \sqrt{x^2 + y^2} \qquad zz^* = |z|^2$$

$$z(1+2i) = 3+i \qquad z = \underbrace{\frac{3+i}{1+2i} \cdot \frac{1-2i}{1-2i}}_{5} = \frac{5-5i}{5} = 1-i$$

$$\underbrace{z_1^* z_1}_{x_1^2 + y_1^2} z = z_2 z_1^* \qquad z = \frac{z_2 z_1^*}{|z_1|^2} = \frac{z_2}{z_1}, \rightarrow z_1^{-1} = \frac{z_1^*}{|z_1|^2}$$

identity. So what was discussed so far was called
in fact an algebraic representation of a complex

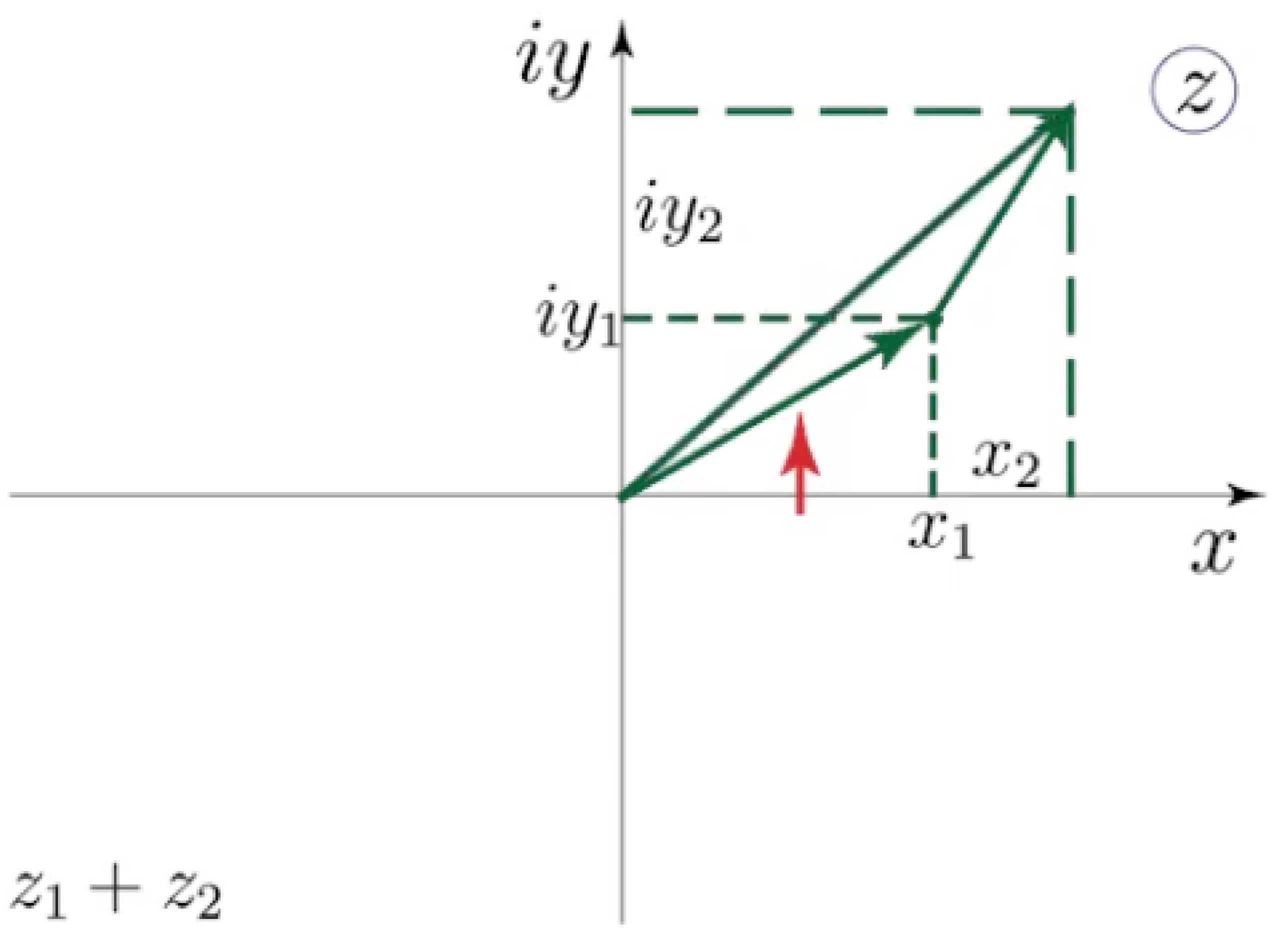
Complex Analysis, Week 1, Part 1.

complex_number_geometry__1

Watch later Share

Algebra of complex numbers

Geometric representation



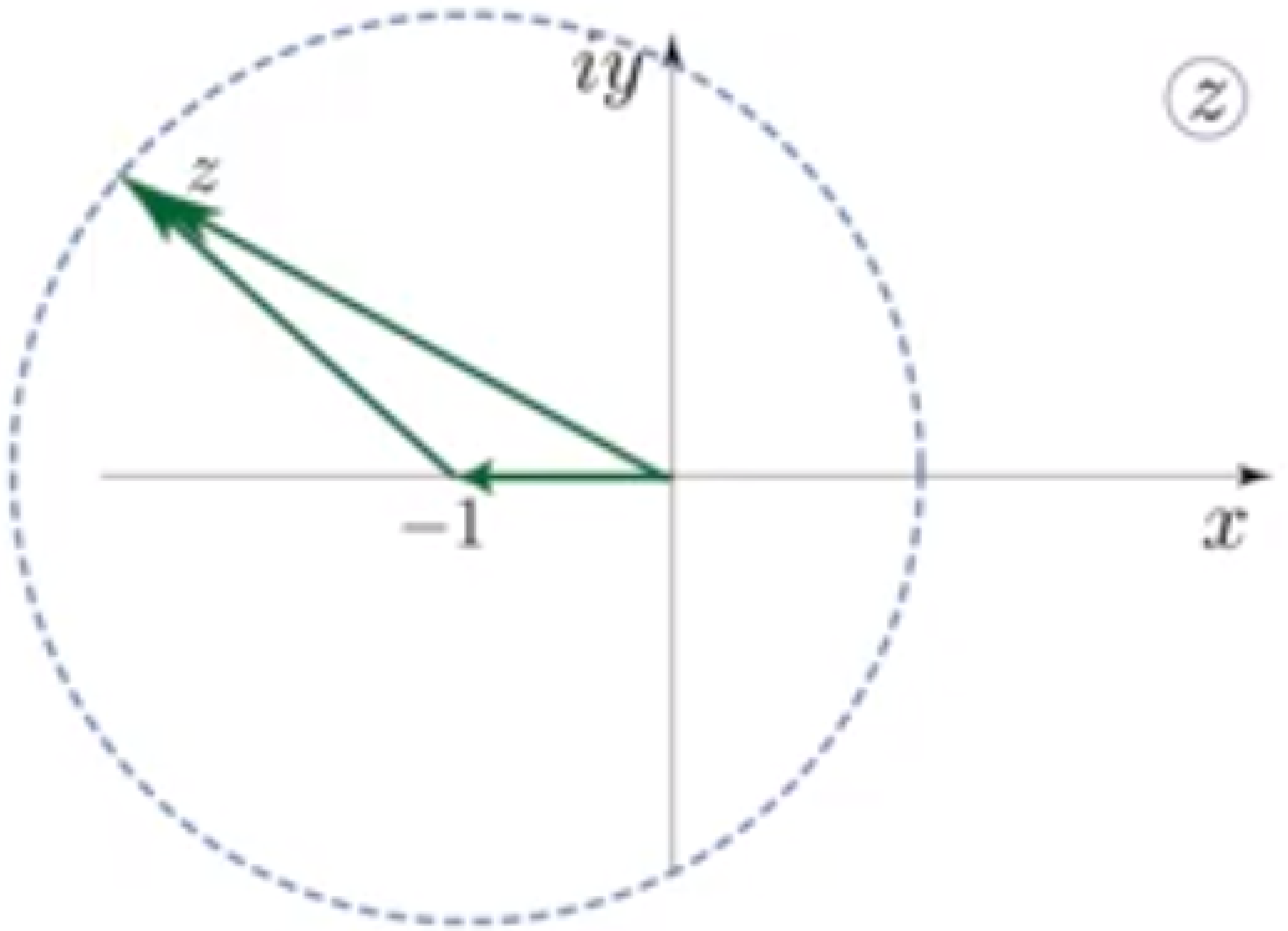
$z_1 + z_2$

has the components X1 + X2 and Y1 and Y2.
But to understand better how a geometrical

6:45 / 9:00 Speed 1.50x HD

Algebra of complex numbers

Geometric representation



$|z + 1| = 2$

$z + 1 = z - (-1)$

$z = x + iy$

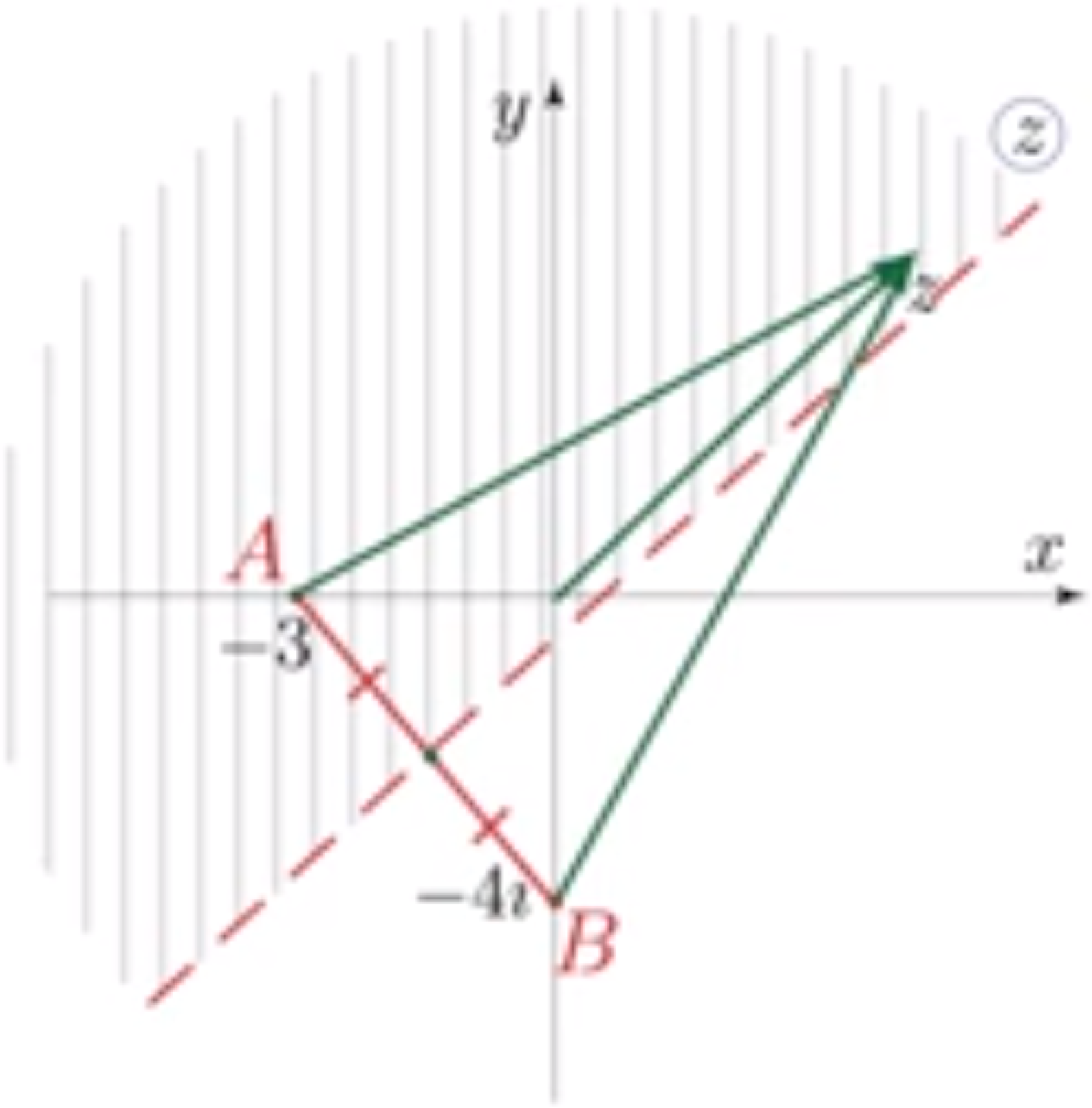
$|(x + 1) + iy| = 2$

Then you square both parts of this equation and
expand the left-hand side like X plus 1 squared

Algebra of complex numbers

$|z+3|<|z+4i|$

$|z+3|=|z+4i|$ (Boundary)



$$z+3=z-(-3)$$

$$z+4i=z-(-4i)$$

$$(x_0,y_0)=\left(-\frac{3}{2},-2\right)$$

$$\overline{AB}=(3,-4)$$

$$n=(4,3)$$

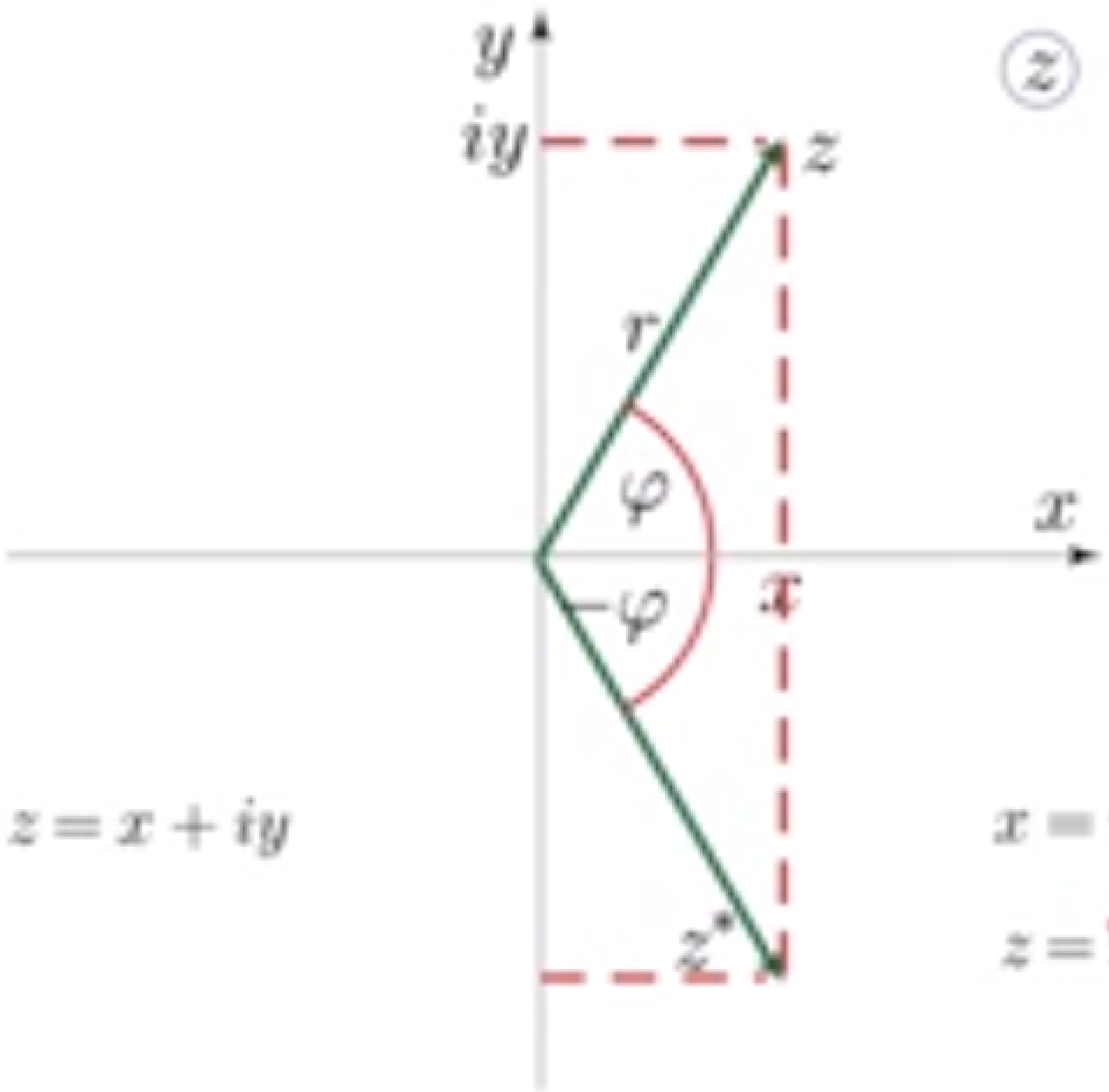
$$\frac{x-x_0}{n_x}=\frac{y-y_0}{n_y}$$

$$\frac{x-(-\frac{3}{2})}{4}=\frac{y+2}{3}$$

$$y=\frac{3}{4}x-\frac{7}{8}$$

minus 2 so it's minus 7/8 and that's it that's
our equation for a boundary. And of course you

Algebra of complex numbers



$$\frac{z_1}{z_2} = \frac{z_1 z_2^*}{\underbrace{\cancel{z_2 z_2^*}}_{r_2^2}} = \frac{r_1 \cancel{r_2}[\cos(\varphi_1-\varphi_2) + i\sin(\varphi_1-\varphi_2)]}{\cancel{r_2^2}}$$

$$= \frac{r_1}{r_2}[\cos(\varphi_1-\varphi_2) + i\sin(\varphi_1-\varphi_2)]$$

$z = x + iy$

$x = r\cos\varphi, \quad y = r\sin\varphi$

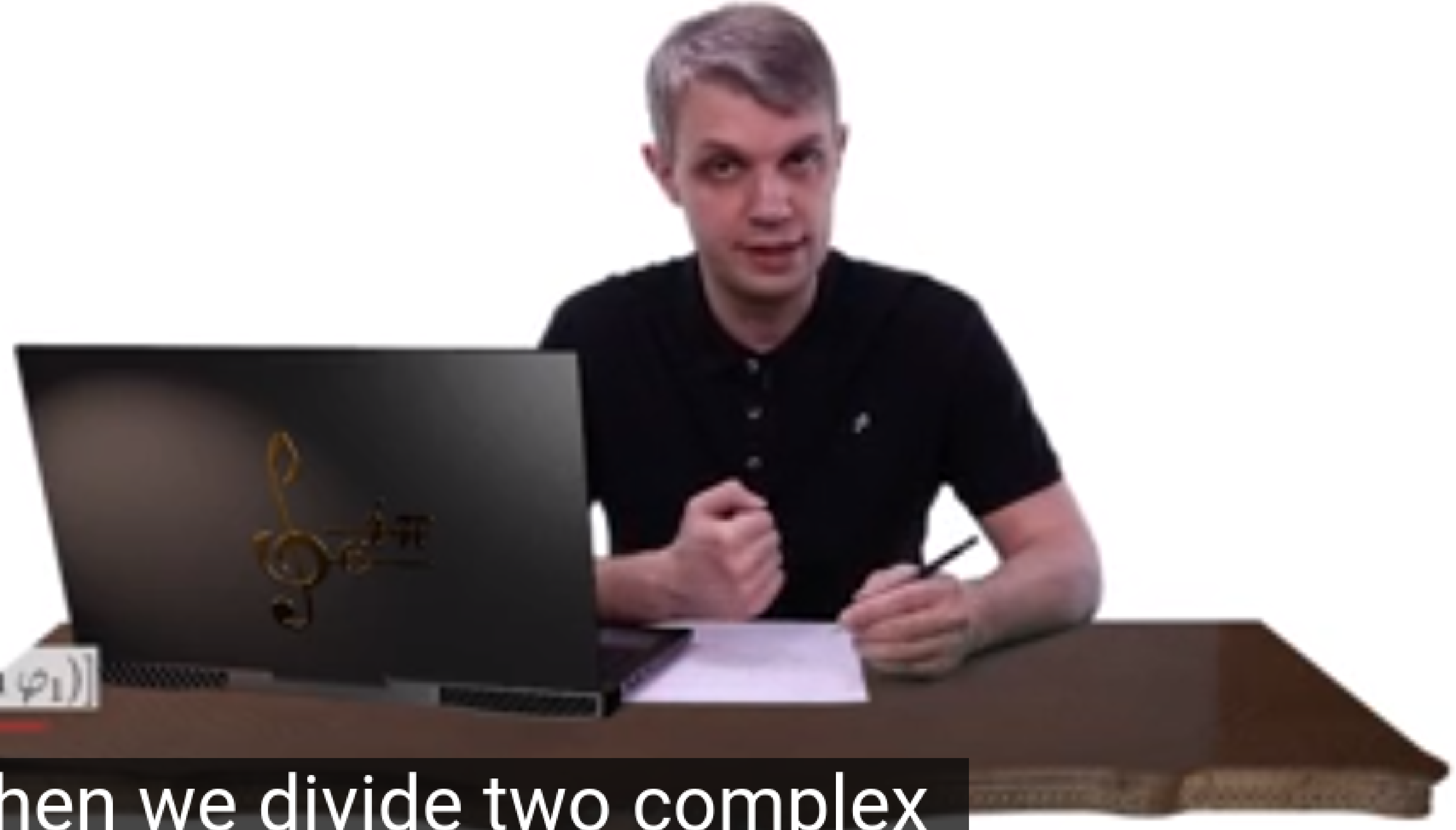
$z = \underbrace{r}_{|z|}(\cos\varphi + i\sin\varphi) \qquad \varphi = \arg z$

$z^* = x - iy = r(\cos\varphi - i\sin\varphi) = r[\cos(-\varphi) + i\sin(-\varphi)]$

$z_1 = r_1(\cos\varphi_1 + i\sin\varphi_1)$

$z_2 = r_2(\cos\varphi_2 + i\sin\varphi_2)$

$$z_1 \cdot z_2 = r_1 r_2[(\underbrace{\cos\varphi_1\cos\varphi_2 - \sin\varphi_1\sin\varphi_2}_{\cos(\varphi_1+\varphi_2)}) + i(\underbrace{\cos\varphi_1\sin\varphi_2 + \cos\varphi_2\sin\varphi_1}_{\sin(\varphi_1+\varphi_2)})]$$



Now we conclude that when we divide two complex numbers the modulus are divided while their

Complex analysis, week 1, Part 3.

trig_representation__2

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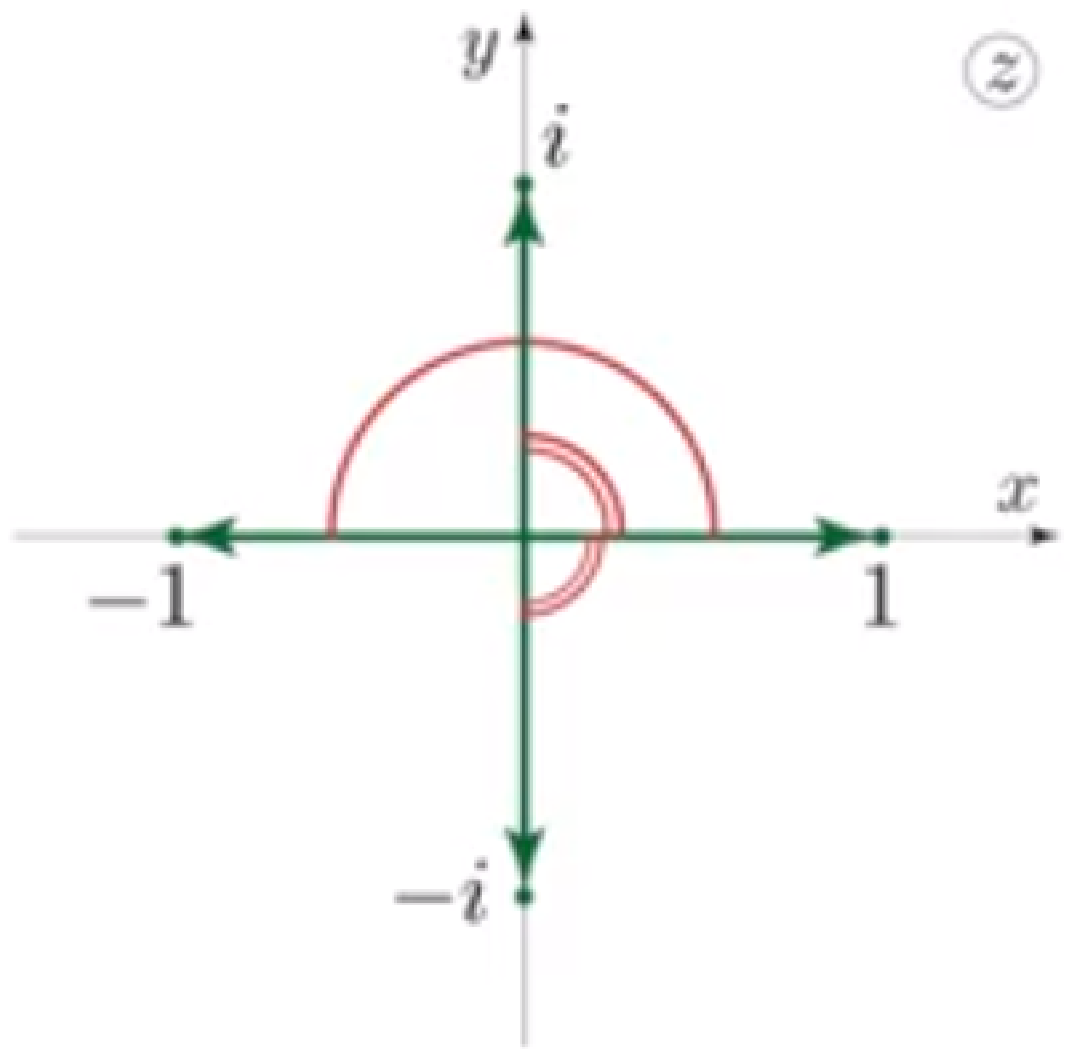
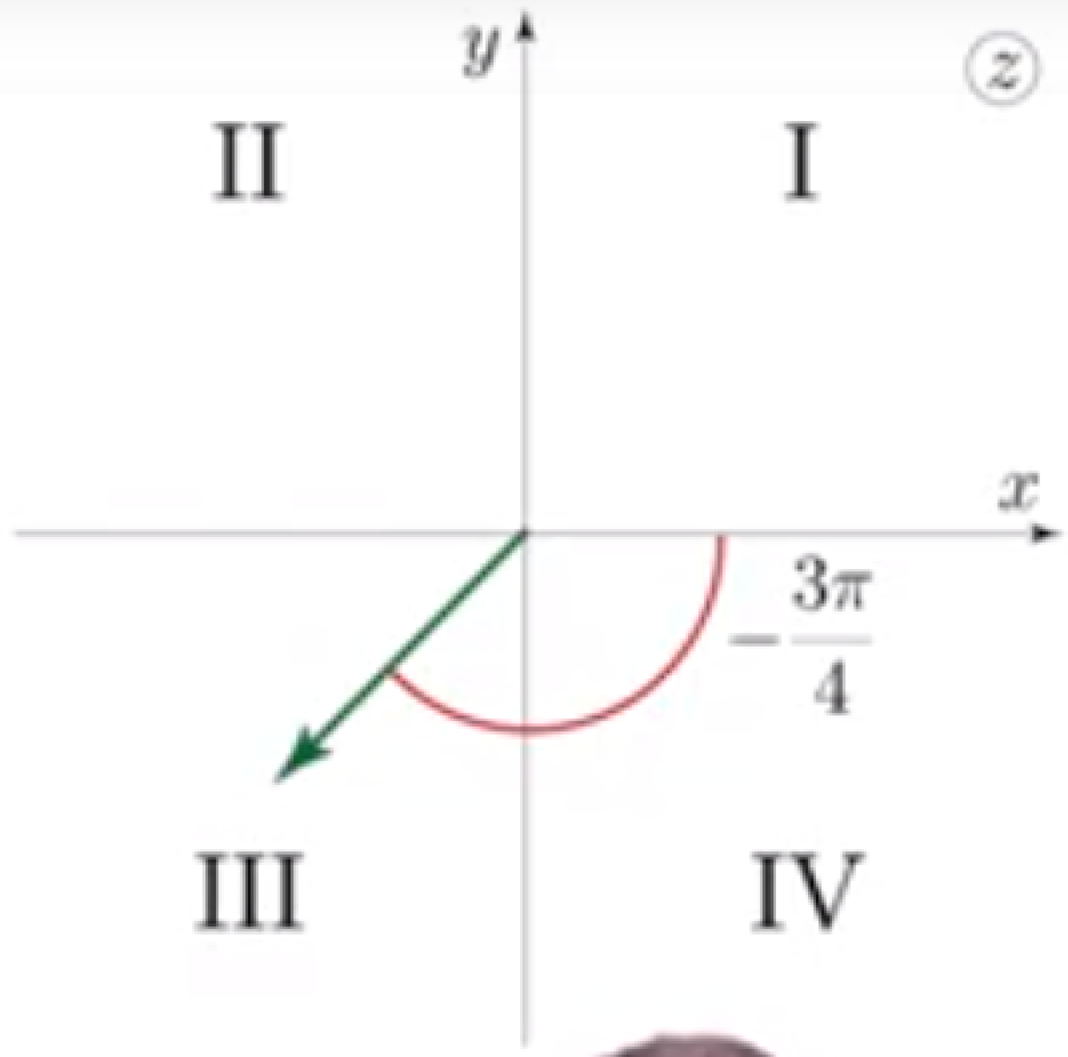
Algebra of complex numbers

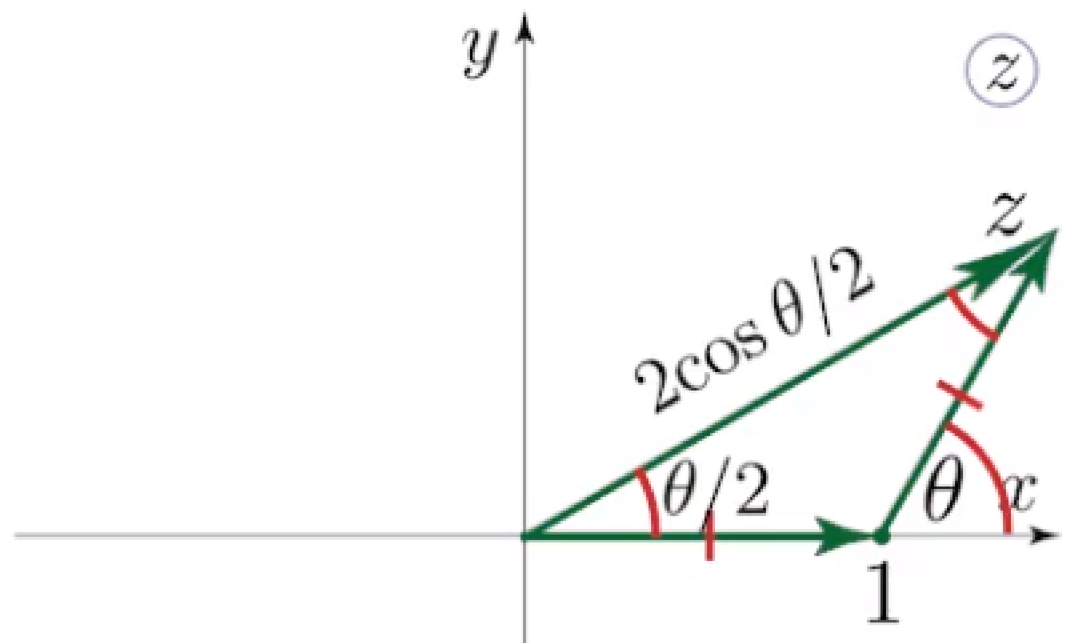


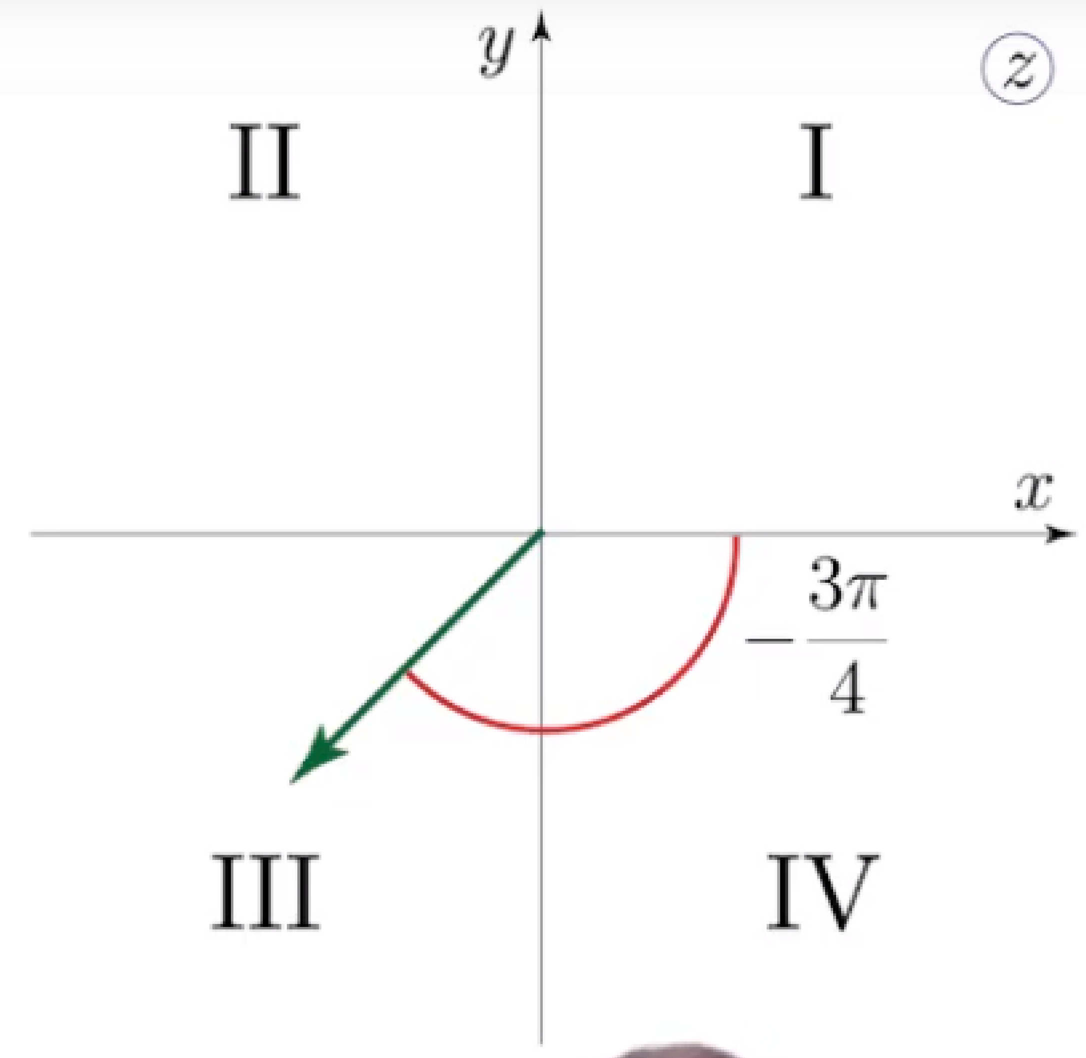


-pi/2 or 3*pi/2, depending on your choice.
So this is how trigonometric representation

9:25 / 11:26 Speed 1.0x HD

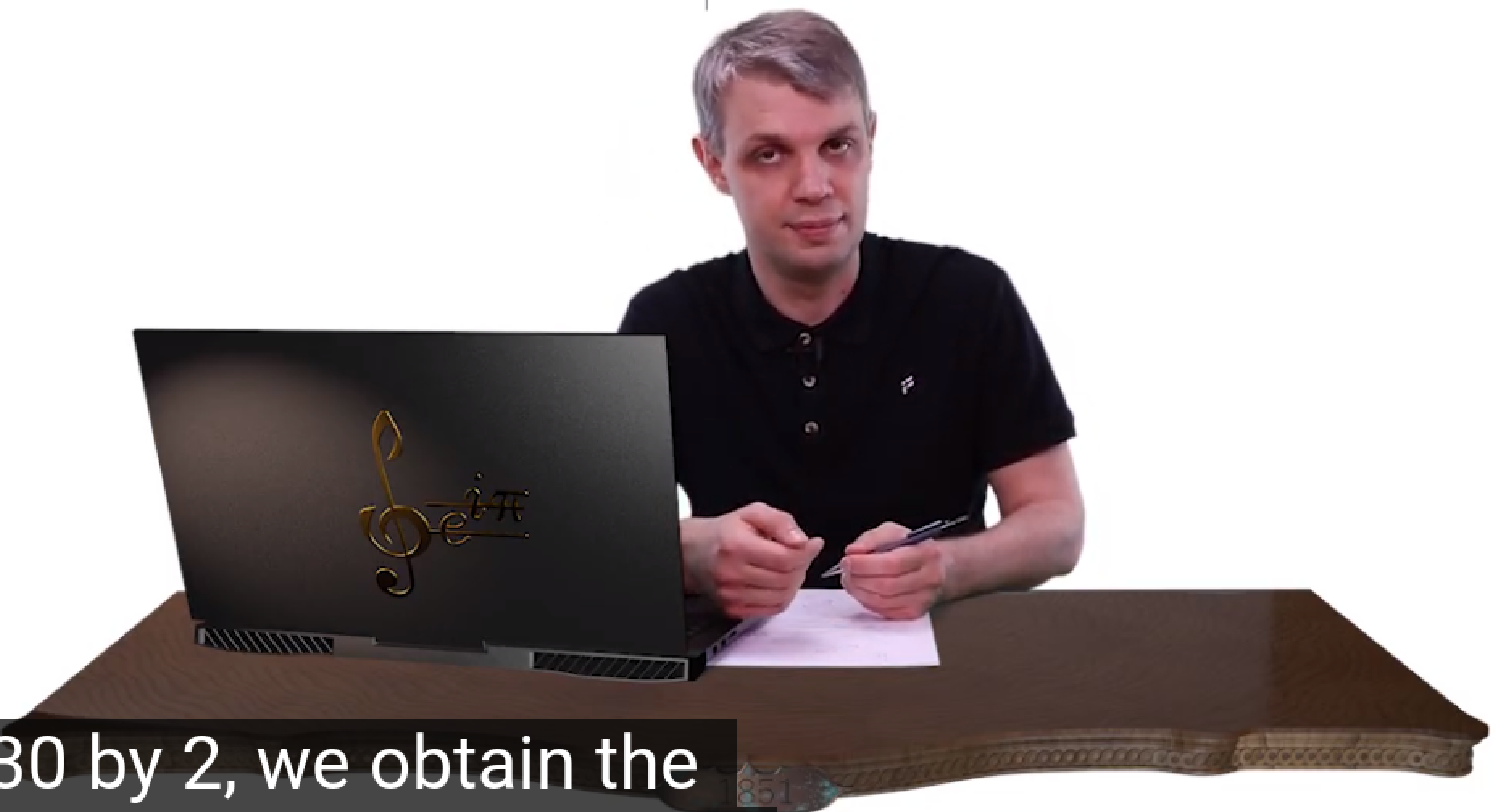
Algebra of complex numbers





$$z = \overbrace{1+\cos\theta}^{2\cos^2\frac{\theta}{2}} + i\overbrace{\sin\theta}^{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}$$

$$z = 2\cos\frac{\theta}{2}\left(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right)$$



2 cosine of 30 by 2, we obtain the
same trigonometric representation.

Algebra of complex numbers

$e^{i\theta}$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \qquad e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

$$\left|\frac{a_{n+1}}{a_n}\right|_{n\to\infty} < 1 \quad \text{(Convergence)}$$

$$\left|\frac{z}{n}\right|_{n\to\infty} = 0 \text{ for any finite } z$$

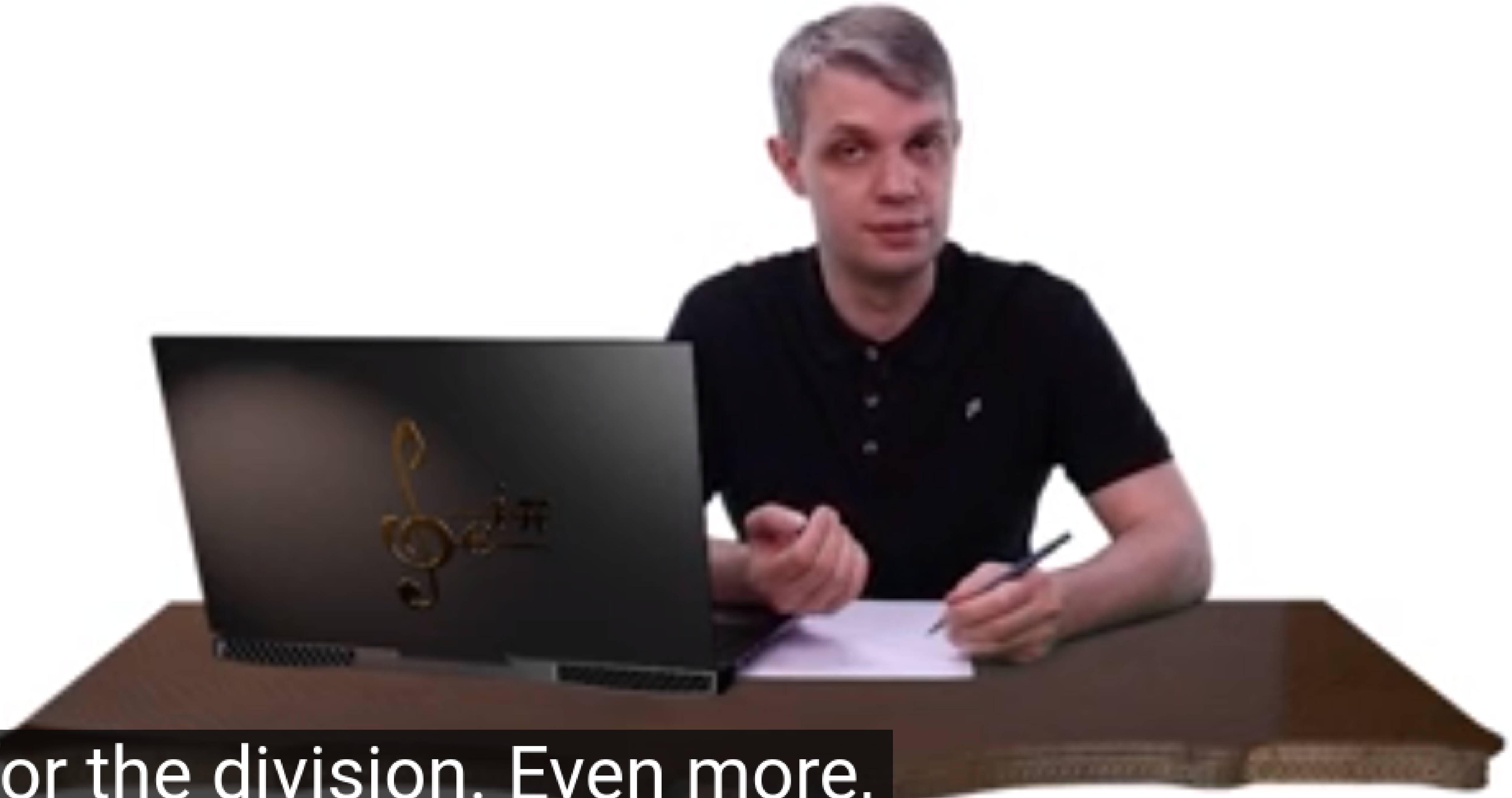
$$e^{i\theta} = \sum_{n=0}^{\infty} \frac{(i\theta)^n}{n!} = 1 + i\theta - \frac{\theta^2}{2!} - i\frac{\theta^3}{3!} + \frac{\theta^4}{4!} + i\frac{\theta^5}{5!} + \ldots$$

$$= \underbrace{\left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} + \ldots\right)}_{\cos\theta} + i\underbrace{\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} \ldots\right)}_{\sin\theta}$$

$e^{i\theta} = \cos\theta + i\sin\theta$ Euler's identity

$z = |z|(\cos\theta + i\sin\theta) = |z|e^{i\theta}$ exponential form of a complex number

$z_1 z_2 = |z_1||z_2|e^{i(\theta_1 + \theta_2)}$



The same goes for the division. Even more, if you recall that cos and sin function are

Eulers_identity__2

Algebra of complex numbers

$$e^{i\theta+2\pi i n}=\cos(\theta+2\pi n)+i\sin(\theta+2\pi n)=e^{i\theta},\quad n\in Z$$

$$e^{i\theta}=\cos\theta+i\sin\theta \quad \text{Euler's identity}$$

$$e^{-i\theta}=\cos\theta-i\sin\theta$$

$$+$$

$$e^{i\theta}=\cos\theta+i\sin\theta$$

$$z=|z|(\cos\theta+i\sin\theta)=|z|e^{i\theta}$$ exponential form of a complex number

$$z_1z_2=|z_1||z_2|e^{i(\theta_1+\theta_2)}$$

$$\cos\theta=\frac{e^{i\theta}+e^{-i\theta}}{2}$$

$$\sin\theta=\frac{e^{i\theta}-e^{-i\theta}}{2i}$$

$$\cosh i\theta=\cos\theta,\qquad \sinh i\theta=i\sin$$

coincides with cosine theta while sin hyperbolic
of i*theta is equal to sine theta over i.

Algebra of complex numbers

MISIS

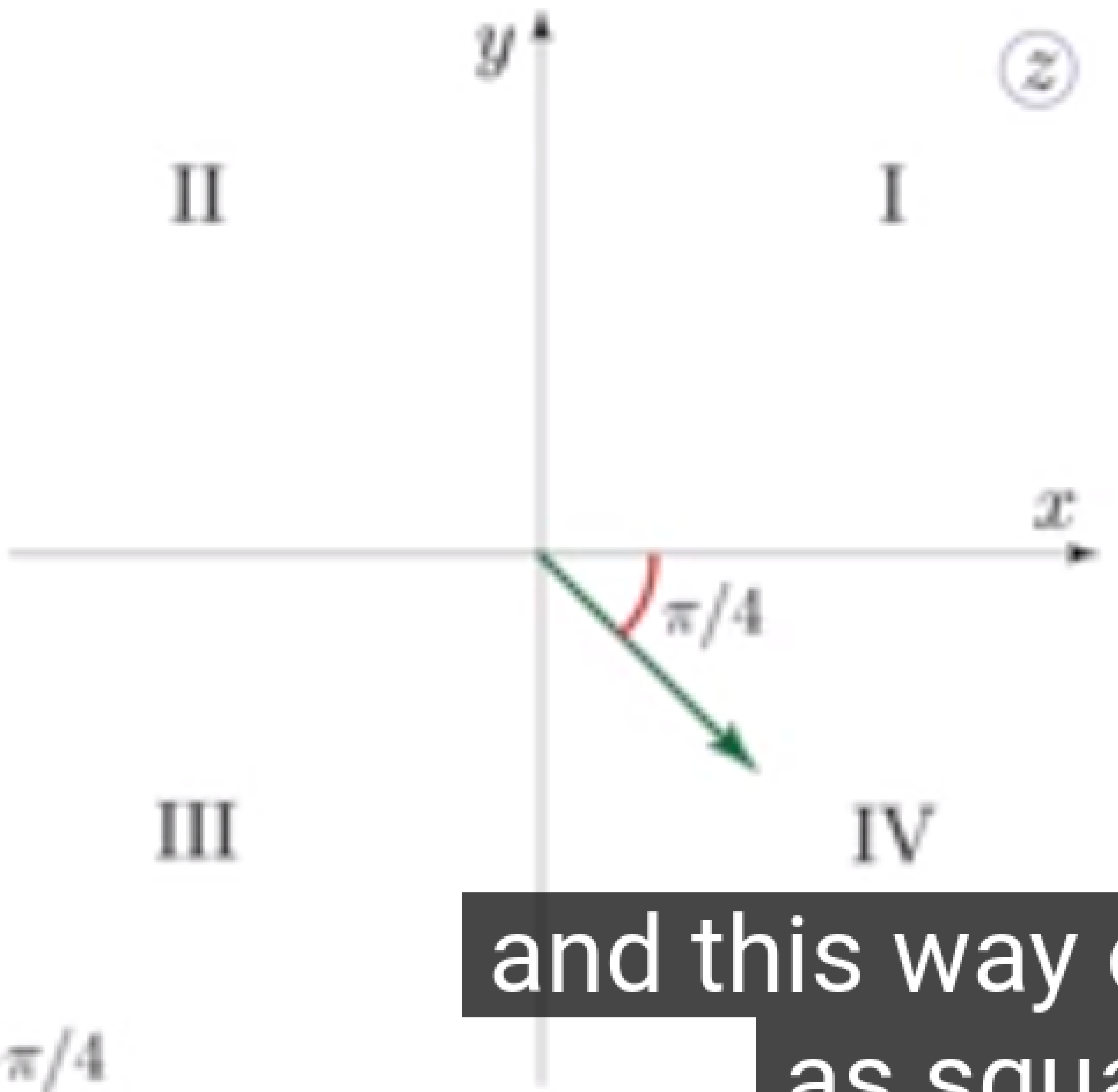
$z = 1 + i^{123}$

$z = \sqrt{2}e^{-}$

$i = e^{i\pi/2}$ $i^{123} = e^{i\pi/2(120+3)} = e^{\cancel{60\pi i}+3\pi i/2} = e^{3\pi i/2} = -i$

$z = 1 - i$

$|z| = \sqrt{2}$



$\arg z = -\pi/4$

and this way our complex number is represented
as square root of 2 times e^{-pi*i/4}.

Algebra of complex numbers

MISIS

$$z = \frac{(1-i)^6}{(1+i\sqrt{3})^5}$$

$$1-i = \sqrt{2}e^{-i\pi/4}$$

$$z_2 = 1 + i\sqrt{3} = 2\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)$$

$$\cos\varphi = \frac{1}{2}, \quad \sin\varphi = \frac{\sqrt{3}}{2} \qquad \arg z_2 = \varphi = \frac{\pi}{3}$$

$$z_2 = 2e^{i\pi/3}$$

$$e^{-\cancel{2\pi i} + \pi i/2}$$

$$z = \frac{[\sqrt{2}]^6 e^{\cancel{-3\pi i/2}}}{2^5 e^{\cancel{5\pi i/3}}} = \frac{8}{32}\cdot\frac{e^{\pi i/2}}{e^{-i\pi/3}} = \frac{1}{4}e^{5\pi i/6}$$

$$e^{\cancel{2\pi i} - \pi i/3}$$

it's a - pi i / 3 which gives us 1/4 times
e^{5 pi i/6} and this is our final answer.

Algebra of complex numbers

MISIS

$$S = \sin\theta + \sin 2\theta + \ldots + \sin n\theta = \mathrm{Im}(e^{i\theta} + e^{2i\theta} + \ldots e^{ni\theta})$$

$$q + q^2 + \ldots + q^n$$

$$= q(1 + q + \ldots + q^{n-1}) = q\frac{q^n - 1}{q - 1}$$

$$= \mathrm{Im}\, e^{i\theta}\frac{e^{in\theta} - 1}{e^{i\theta} - 1} = \mathrm{Im}\, e^{i\theta}\frac{e^{in\theta/2}\cancel{2}i\sin\frac{n\theta}{2}}{e^{i\theta/2}\cancel{2}i\sin\frac{\theta}{2}} = \frac{\sin\frac{n\theta}{2}}{\sin\frac{\theta}{2}}\mathrm{Im}\, e^{i\theta + in\theta/2 - i\theta/2}$$

$$e^{i\alpha} \pm e^{i\beta}$$

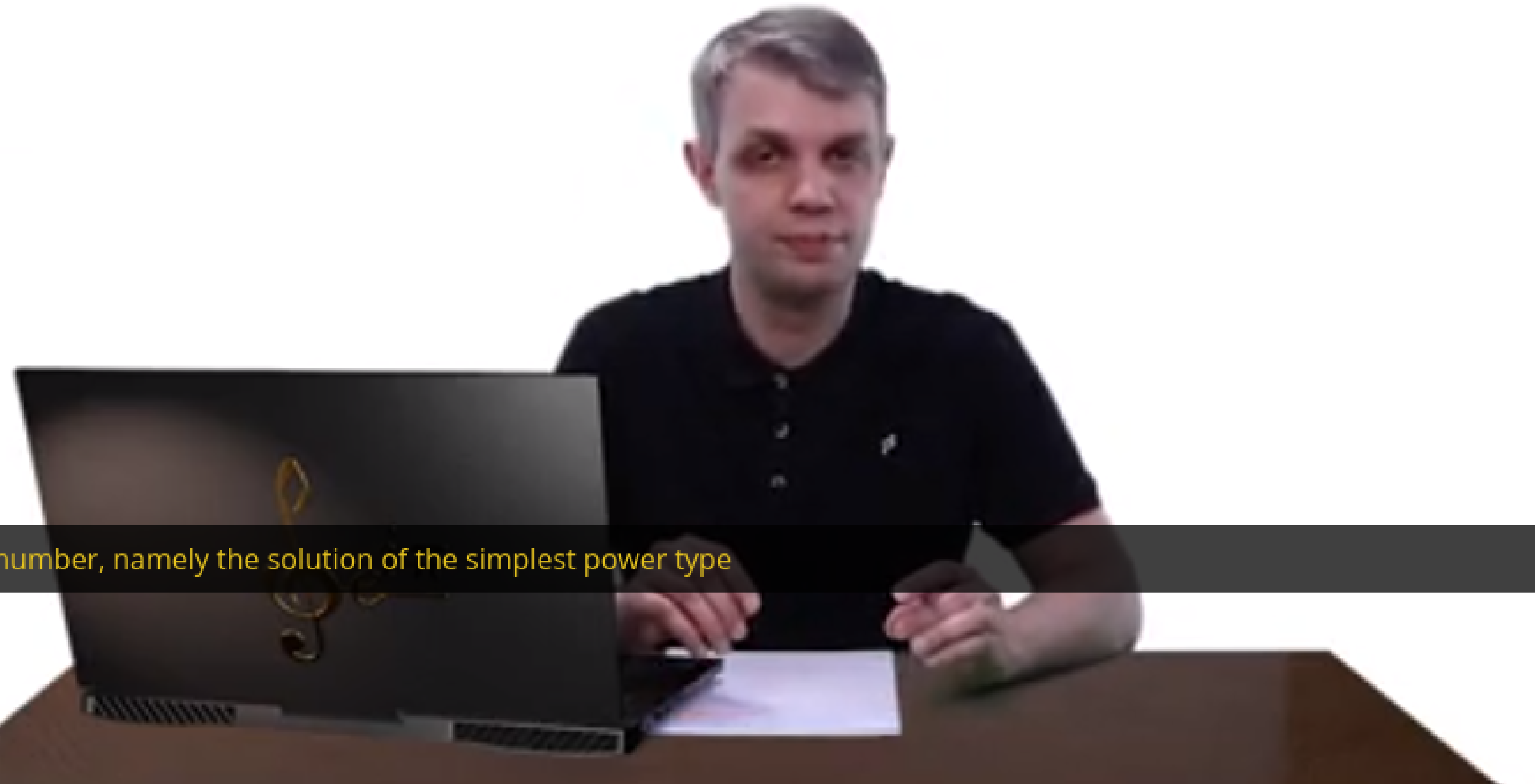
$$e^{i\alpha} + e^{i\beta} = e^{i(\alpha+\beta)/2}\left(e^{i(\alpha-\beta)/2} + e^{-i(\alpha-\beta)/2}\right) = e^{i(\alpha+\beta)/2} 2\cos\frac{\alpha-\beta}{2}$$

$$e^{i\theta} - 1 = e^{i\theta/2}\left(e^{i\theta/2} - e^{-i\theta/2}\right) = e^{i\theta/2} 2i\sin\frac{\theta}{2}$$

$$e^{in\theta} - 1 = e^{in\theta/2} 2i\sin\frac{n\theta}{2}$$

exponential representation of a complex number, namely the solution of the simplest power type

$$= \frac{\sin\frac{n\theta}{2}}{\sin\frac{\theta}{2}}\sin\frac{(n+1)\theta}{2}$$



exponential representation of a complex number,
namely the solution of the simplest power type

MISIS

Algebra of complex numbers

$z^3 = -i$

$z = |z|e^{i\varphi} \quad z^3 = |z|^3 e^{3i\varphi} \qquad -i = e^{-i\pi/2 + 2\pi i n}$

$|z|^3 = 1 \;\rightarrow\; |z| = 1$

$3\varphi = -\dfrac{\pi}{2} + 2\pi n \qquad \varphi = -\dfrac{\pi}{6} + \dfrac{2\pi n}{3}$

$z_n = e^{-i\pi/6 + 2\pi i n/3}$

$n = 0:\;\; z_0 = e^{-i\pi/6}$

$n = 1:\;\; z_1 = e^{i\pi/2} = i$

$n = 2:\;\; z_2 = e^{7\pi i/6}$

$n = 3:\;\; z_3 = e^{-i\pi/6 + \cancel{2\pi i}} = z_0$

so we see that these roots split the
unit circle into three equal parts,

Complex analysis, Week 1, Part 6

Cauchy-Riemann_theory__1

Watch later

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Analytic functions

$$f'(z_0) = \left.\frac{f(z)-f(z_0)}{z-z_0}\right|_{z\to z_0}$$

$$\Delta z \sim \Delta f = \Delta u + i\Delta v = \overset{\mathrm{I}}{\frac{\partial u}{\partial x}\Delta x} + \frac{\partial u}{\partial y}\Delta y + i\left(\frac{\partial v}{\partial x}\Delta x + \overset{\mathrm{IV}}{\frac{\partial v}{\partial y}\Delta y}\right)$$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

$$\mathrm{I} + \mathrm{IV} = \frac{\partial u}{\partial x}\underbrace{(\Delta x + i\Delta y)}_{\Delta z}$$

$$\mathrm{II} + \mathrm{III} = i\left(\frac{\partial v}{\partial x}\Delta x - \frac{\partial u}{\partial y} i\Delta y\right) = \frac{\partial v}{\partial x}\underbrace{(\Delta x + i\Delta y)}_{\Delta z}$$

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

$$\Delta f = \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)\Delta z \qquad \frac{\Delta f}{\Delta z} \text{ is independent of } \Delta z$$

$$\frac{df}{dz} = \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right) = \frac{\partial f}{\partial x} = \frac{\partial v}{\partial y} - \frac{\partial u}{\partial y} = \frac{\partial f}{i\partial y}$$

$$\frac{i\partial v}{i\partial y} + \frac{\partial u}{i\partial y}$$

see that due to our conditions both derivatives
are equal to each other so due to their importance

6:12 / 12:09

Speed 1.0x

HD

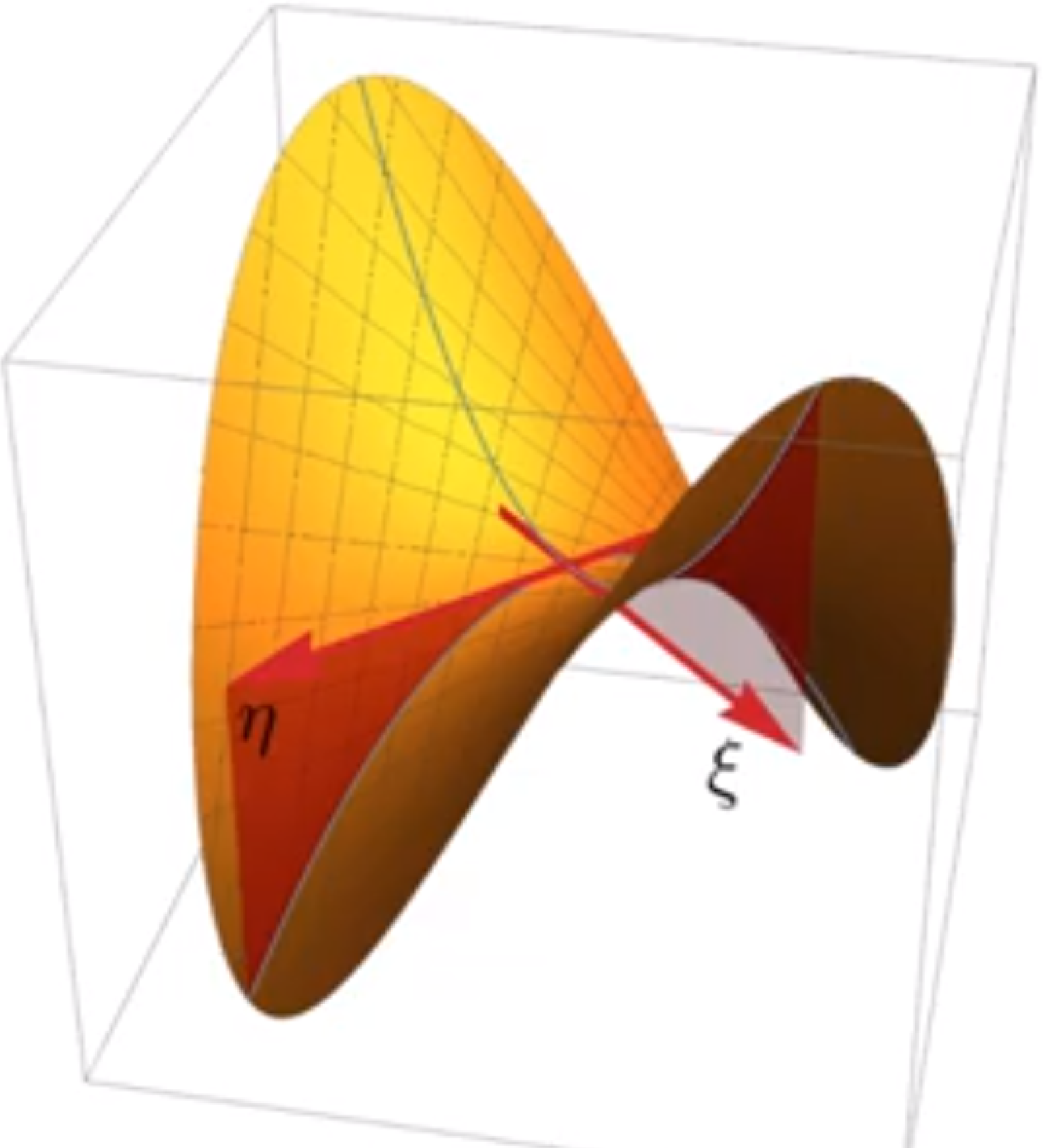
Analytic functions

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \qquad \text{Cauchy–Riemann conditions}$$

Consequence II

$$u(x+\Delta x, y+\Delta y) = u(x,y) + \underline{\frac{1}{2}u_{xx}(\Delta x)^2 + u_{xy}\Delta x\Delta y + \frac{1}{2}u_{yy}(\Delta y)^2}$$

$$\overset{v_{xy}}{u_{xx}}\overset{-v_{xy}}{u_{yy}} - u_{xy}^2 = -v_{xy}^2 - u_{xy}^2 < 0 \qquad \frac{1}{2}(\Delta x \ \ \Delta y)\underline{\begin{pmatrix} u_{xx} & u_{xy} \\ u_{xy} & u_{yy} \end{pmatrix}}\begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix}$$



$$u \approx u(x_0, y_0) + \lambda_1\xi^2 + \lambda_2\eta^2 + \ldots$$

of a function of a complex variable can't have
minima or maximum but just saddles, has tremendous

Analytic functions

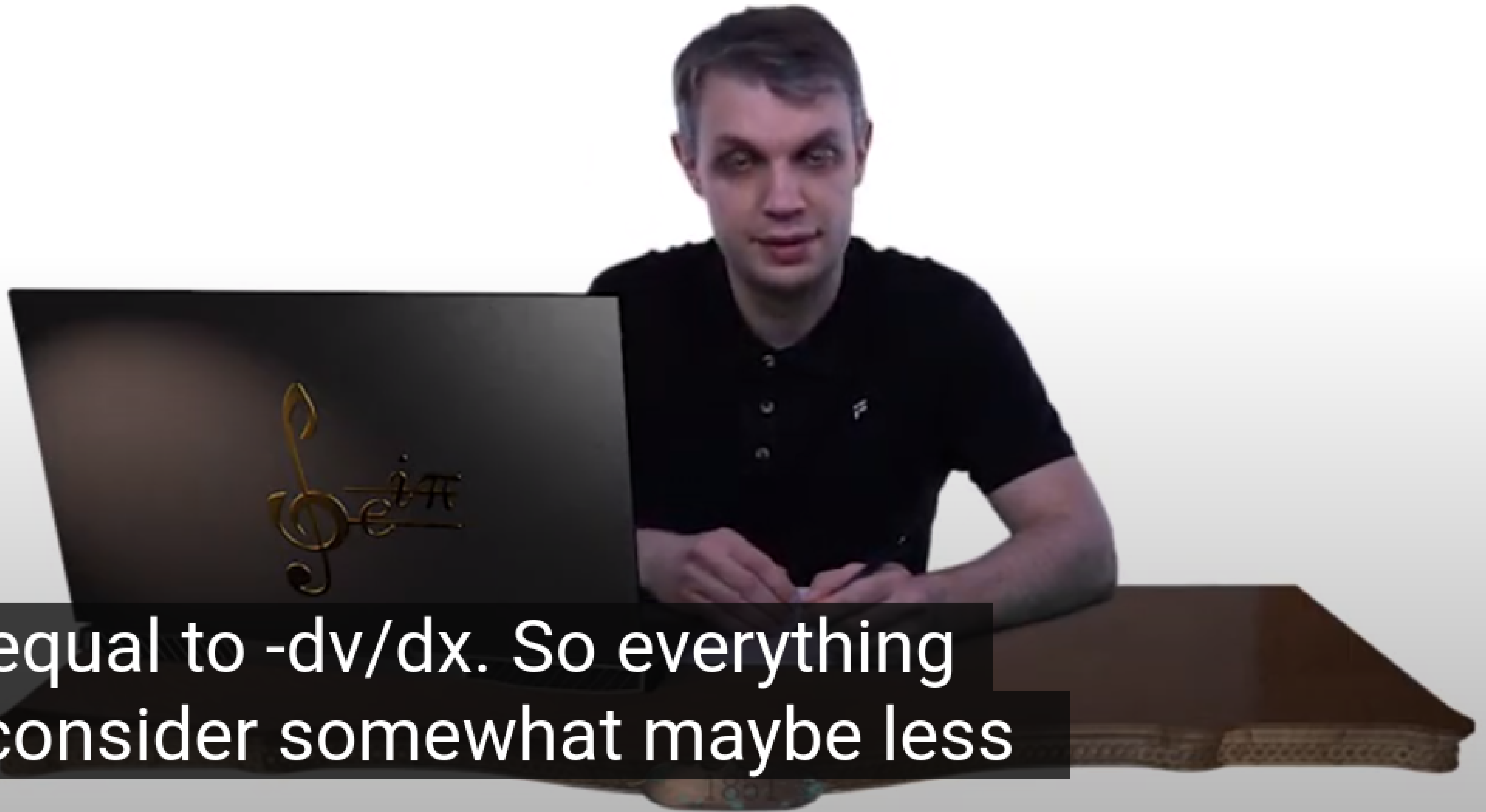
$$f(z) = z^2$$

$$\mathrm{Re}\, f = u(x,y) = x^2 - y^2$$

$$\mathrm{Im}\, f = v(x,y) = 2xy$$

$$\frac{\partial u}{\partial x} = 2x, \quad \frac{\partial v}{\partial y} = 2x$$

$$\frac{\partial u}{\partial y} = -2y, \quad \frac{\partial v}{\partial x} = 2y$$



and indeed du/dy is equal to -dv/dx. So everything
seems fine now let us consider somewhat maybe less

Analytic functions

$$|f| = \exp\left(r^2 \cos 2\varphi\right)$$

$$w = \ln f = \ln|f|e^{i\arg f} = \ln|f| + i\arg f$$

$$\operatorname{Re} w = \ln|f|, \quad \operatorname{Im} w = \arg f$$

$$u = r^2 \cos\, 2\varphi = r^2(\cos^2\varphi - \sin^2\varphi) = x^2 - y^2$$

$$\frac{\partial u}{\partial x} = 2x = \frac{\partial v}{\partial y}$$

$$v = 2xy + \psi(x)$$

$$\frac{\partial v}{\partial x} = 2y + \psi'(x) = -\frac{\partial u}{\partial y} = 2y$$

$$\psi'(x) = 0 \rightarrow \psi(x) = C$$

$$v = 2xy + C$$

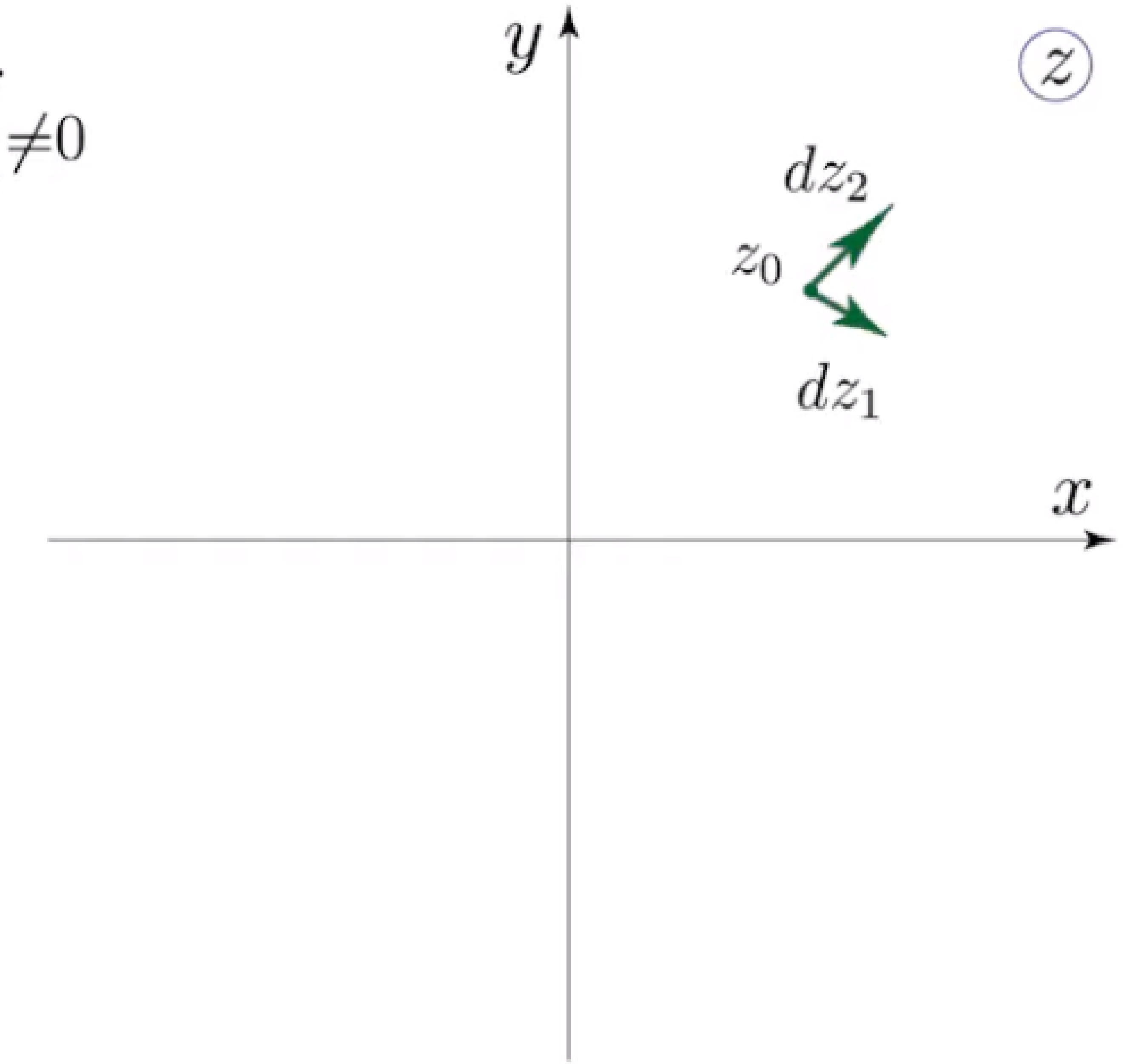
$$w = x^2 - y^2 + 2xyi + iC = (x+iy)^2 + iC$$

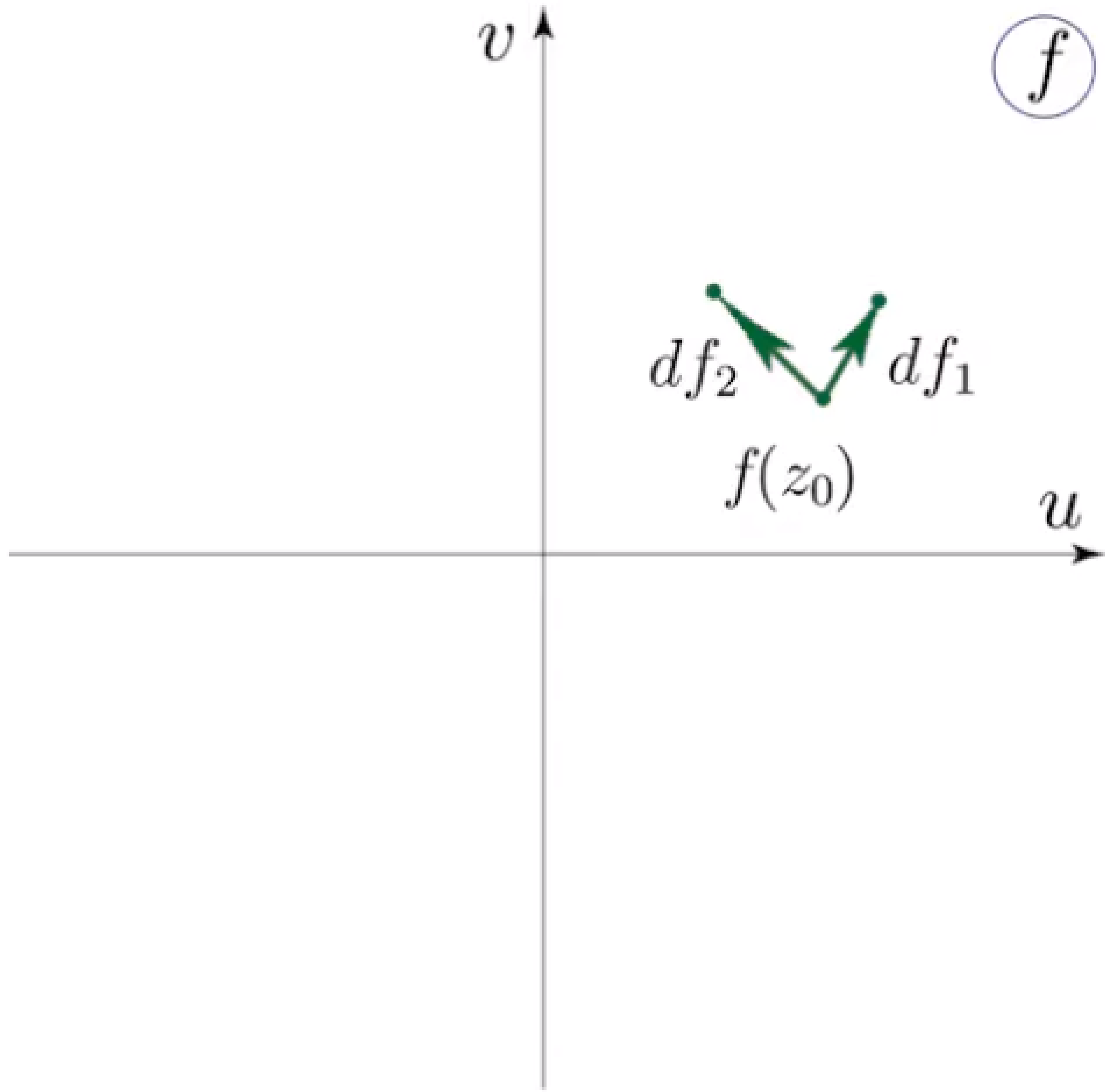
$$f(z) = \exp(z^2)\exp(iC)$$

f(z) is equal to e^{z^2}e^{ic} and this way
you see that indeed Cauchy-Riemann conditions

Analytic functions

$$\frac{df}{dz} \neq 0$$





$$df_1 = dz_1 e^{i\gamma}|f'(z_0)|, \quad df_2 = dz_2 e^{i\gamma}|f'(z_0)|$$

$$f(z_0 + dz_1) = f(z_0) + f'(z_0)dz_1 = f(z_0) + df_1$$

$$f(z_0 + dz_2) = f(z_0) + f'(z_0)dz_2 = f(z_0) + df_2$$

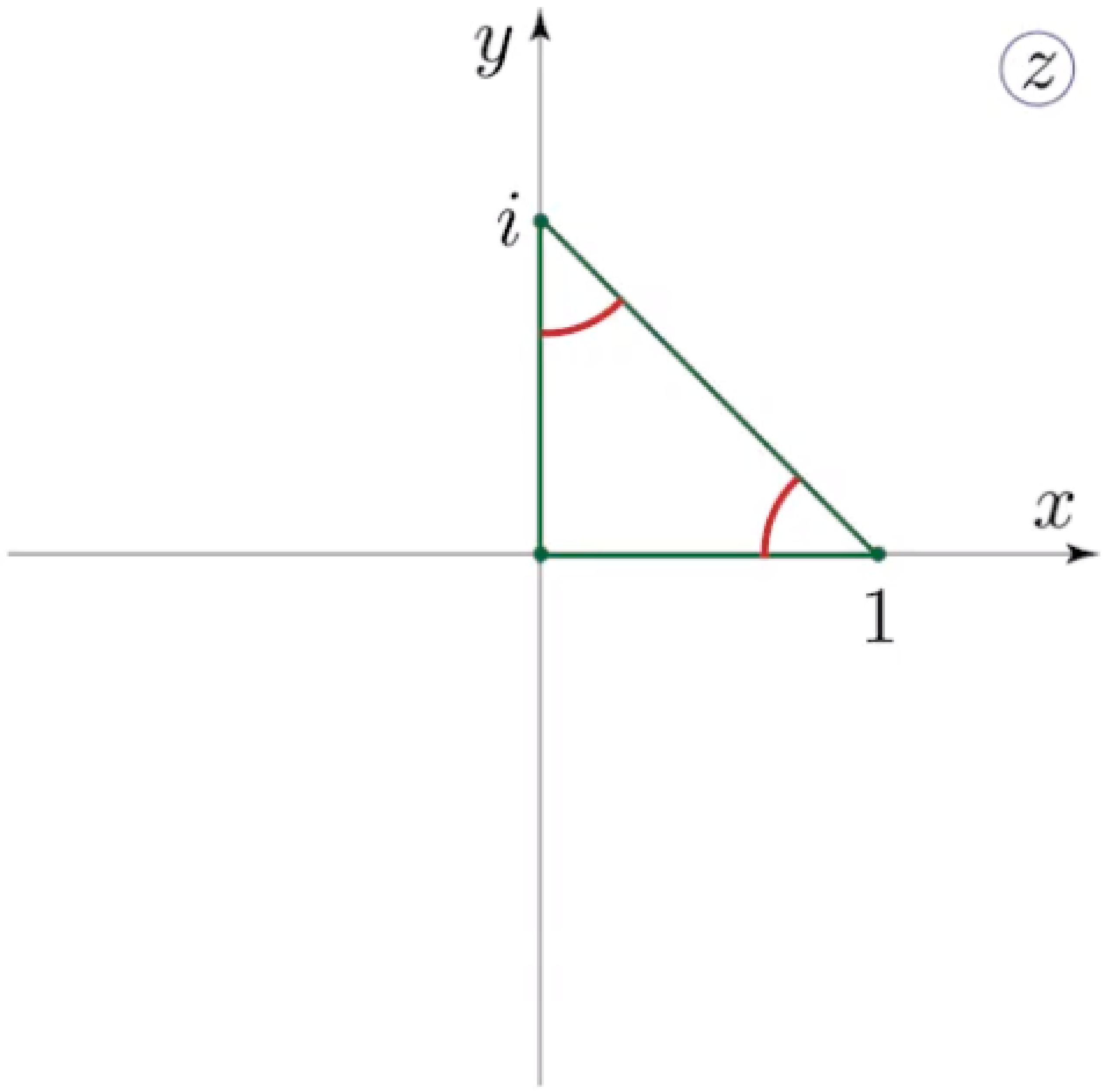
$$dz_1 = |dz_1|e^{i\alpha_1} \qquad f'(z_0) = |f'(z_0)|e^{i\gamma}$$

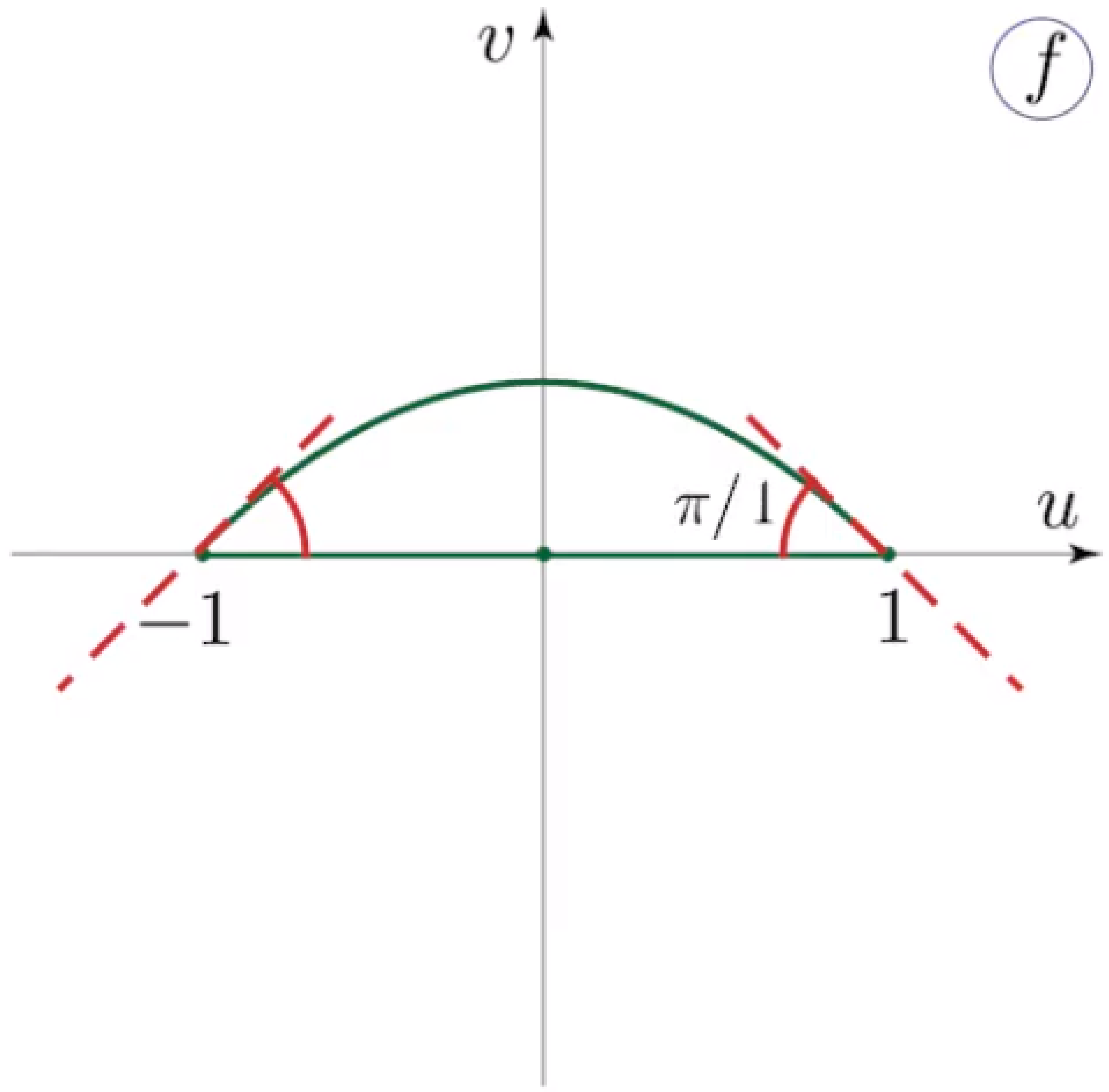
$$dz_2 = |dz_2|e^{i\alpha_2}$$

in complex analysis, because whenever you deal
with integral and you perform the change of

Analytic functions

MISIS





II: $x=0, \quad y=t, \quad t\in[0,1]$

$$u=-t^2, \quad v=0$$

III: $y=1-x, \quad x=t, \quad y=1-t, \quad t\in[0,1]$

$$u=2t-1, \quad v=2t(1-t) \qquad \to t=\frac{u+1}{2}$$

$$v=(1+u)\frac{1}{2}(1-u)$$

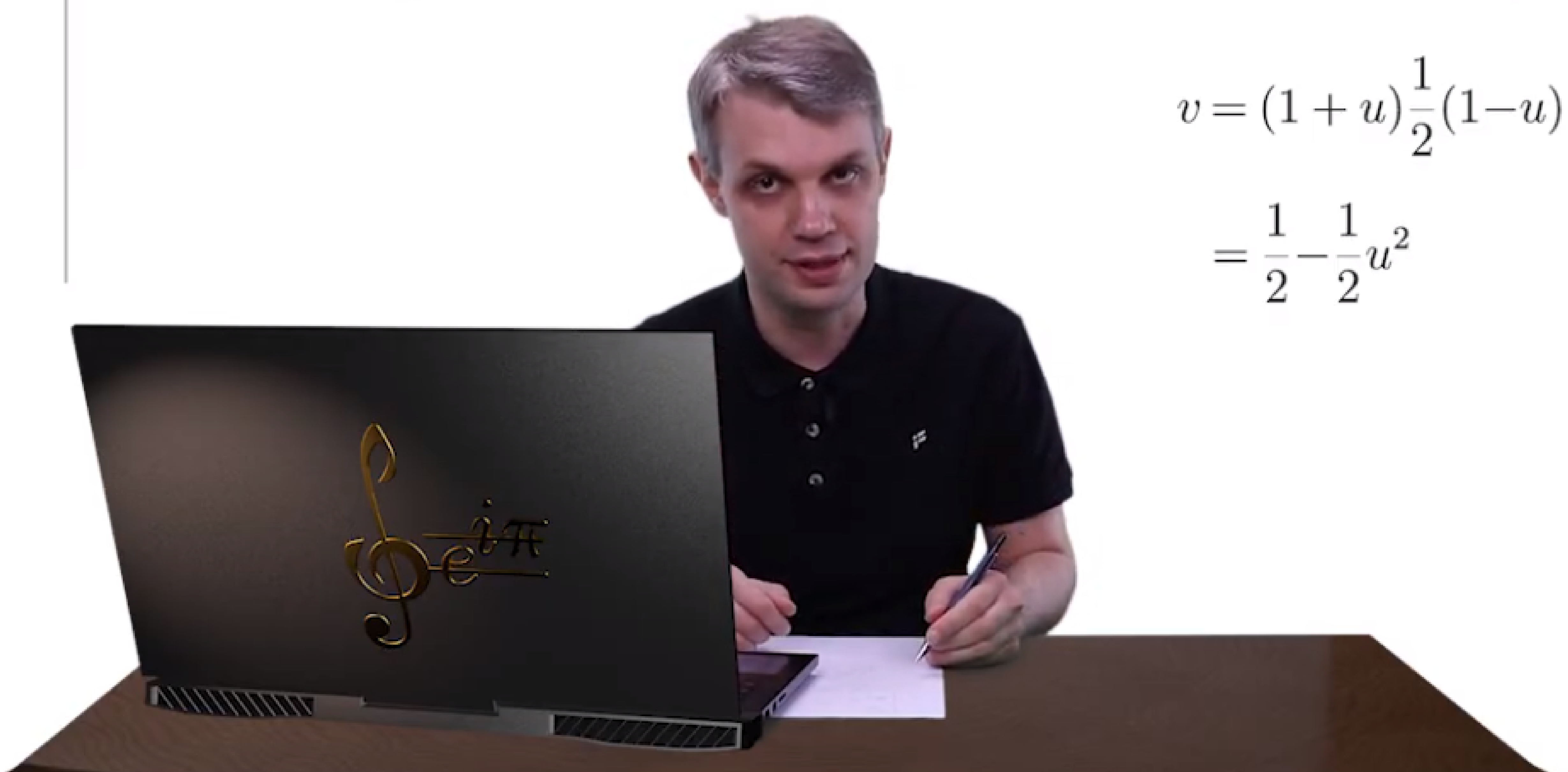
$$=\frac{1}{2}-\frac{1}{2}u^2$$

$$f(z)=z^2 = u+iv = x^2-y^2+2ixy$$

$$u=x^2-y^2, \quad v=2xy$$

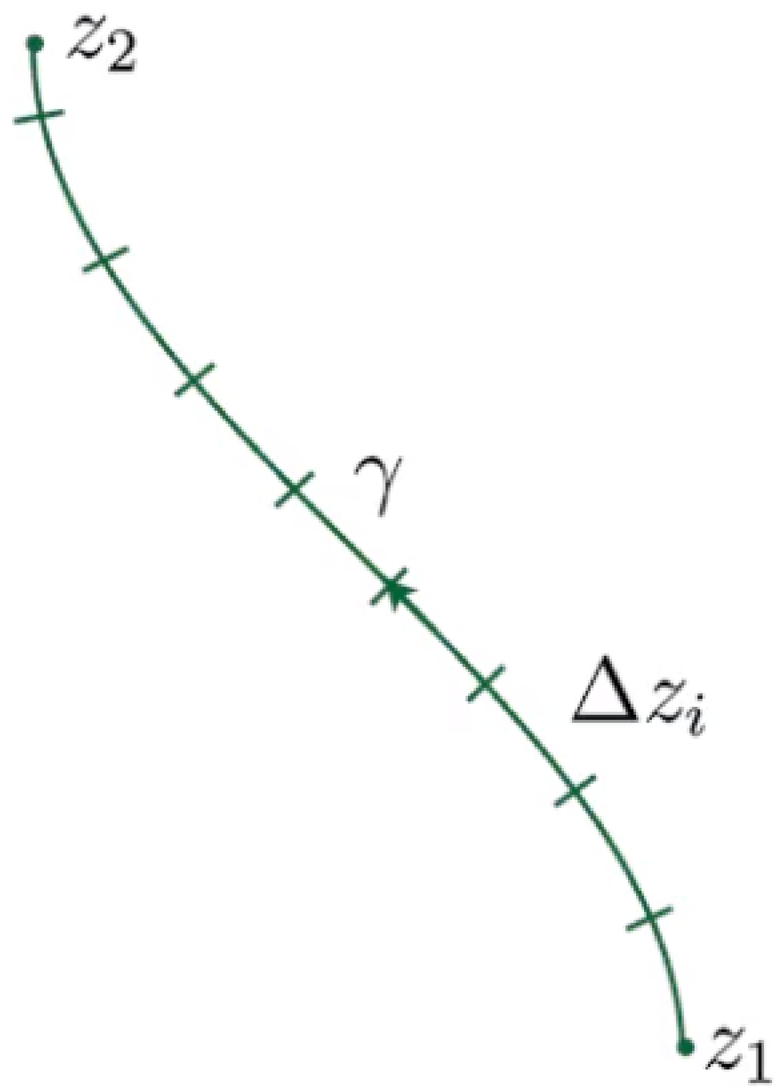
I: $x=t, \quad y=0, \quad t\in[0,1]$

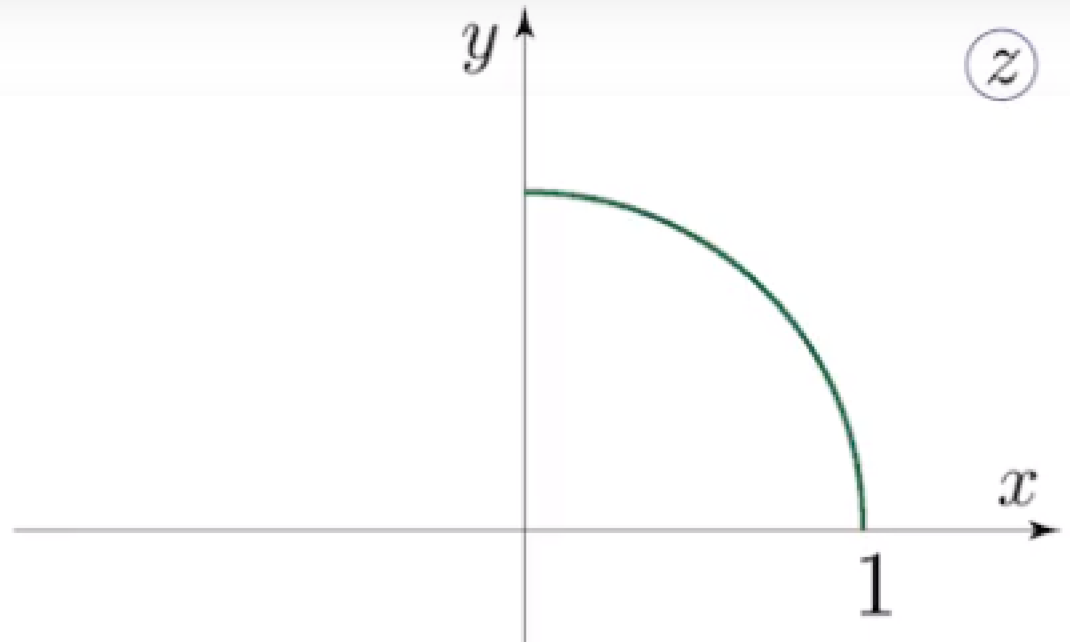
$$u=t^2, \quad v=0$$



one and one at this edge points meaning that
the corresponding tangent angles are 45 degrees.

Analytic functions





$f(z) = z$

$u = x, \quad v = y$

$f(z)$

$$S = \sum_i f(z_i)\Delta z_i \underset{\Delta z_i \to 0}{\to} \int_\gamma f(z)\,dz = \int_\gamma (u+iv)(dx+idy)$$

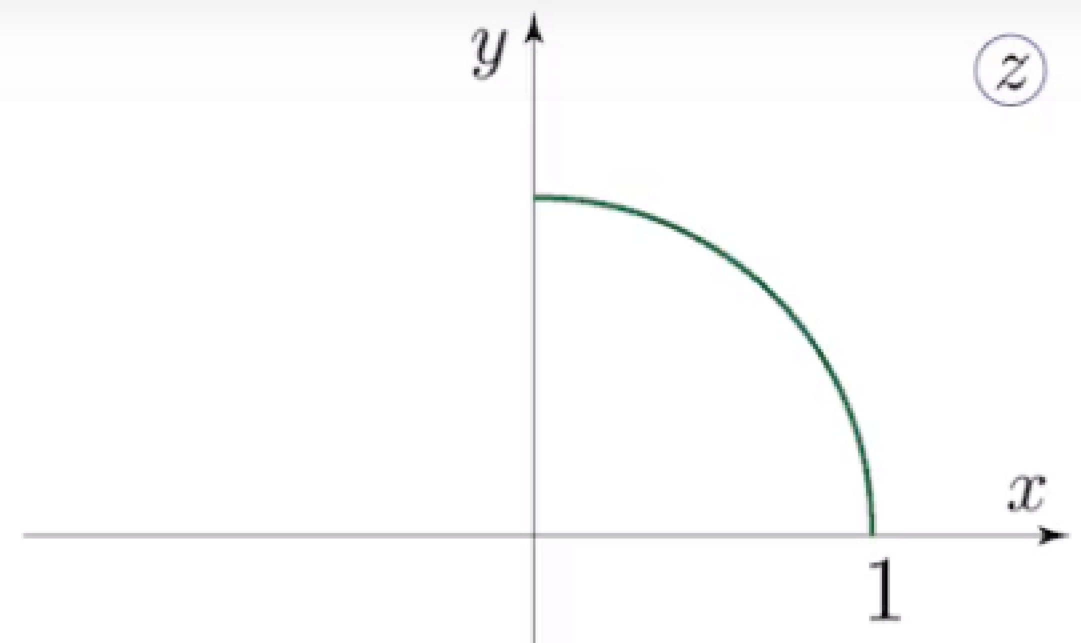
$$= \int_\gamma (udx - vdy) + i\int_\gamma (udy + vdx)$$

function f(z) = z. So u is equal to x while
v equals y. And here we go: the integral now

Analytic functions

$$\int (xdx - ydy) + i\int (xdy + ydx) \quad = \int \frac{1}{2} d(x^2 - y^2) + i\int d(xy)$$

$$= \int \frac{1}{2} d(x^2 - y^2 + 2xyi) \quad = \int_1^i \frac{1}{2} dz^2 = -1$$



$f(z) = z$

$u = x, \quad v = y$

rather on the position of its initial and end points. There is a fundamental reason for this in