I. INTEGRATION WITH RESIDUES I

In this lecture, we will practice with evaluation of real integrals. This is a very beautiful application of complex analysis: we will find out that many of the real integrals which may seem rather non-trivial from a real point of view, can be easily evaluated by promoting the integrands from real to complex functions. Before turning to general considerations, let us consider some simple example.

$$I = \int_{-\infty}^{\infty} f(x)dx, \quad f(x) = \frac{1}{1+x^2}. \tag{1}$$

Let us start with computing this integral via purely real methods, a very straightforward approach:

$$I = \int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \int_{-\infty}^{\infty} d \arctan x = \frac{\pi}{2} - (-\frac{\pi}{2}) = \pi. \tag{2}$$

Now let me show you how the real integrals are taken using complex analysis.

As is always with powerful methods, when applied to simple situations, they may seem even more involved. The advantage is however, that powerful methods allows to solve cases previously untractable.

First of all, complex analysis likes to operate with closed contours. Indeed, all Cauchy relations are formulated for such contours.

Now we will treat our real integral as an integral taken along a straight line contour positioned in a complex plane. To employ powerful Cauchy theorems we need to close the contour. Well, how do we do that?

In complex analysis we will do this in almost every example. The general prescription is that we should close the contour with some simple curve. The hope is that the integral along this additional curve either disappears or is easily computed and is given by some simple value.

In our case how do we connect two infinite ends of the straight line? Well, for example, by an lower or upper infinite semicircle. Therefore, instead of initial integral we will consider a close contour integral (see Fig).

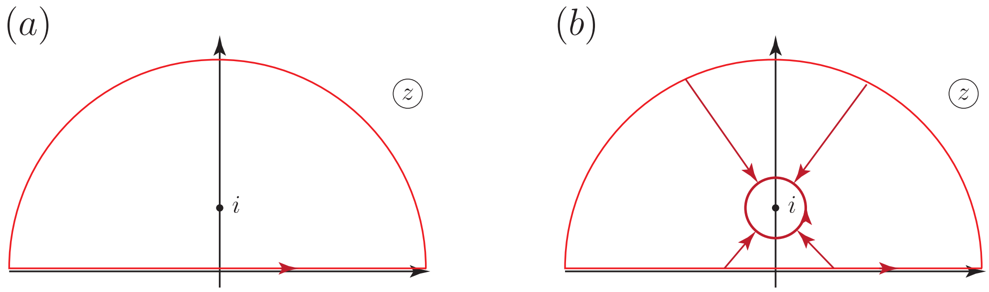

FIG. 1: Integration contour for evaluation of I.

And remember the main consequence of Cauchy integral theorem. The closed contour integral retains its value under any deformation of the contour as long as this deformation doesn't touch the singularities of the integrand. Here our integrand has singularities at points $z = \pm i$. One of them is positioned inside the contour. So we shrink the contour to such extent that it becomes an infinitesimal circle round the pole $z = i$. And, as it turns out, this integral is extremely easily computable with the help of a suitable parametrization.

But of course, this is not our original integral. We modified it with the additional ark. However, the elegance of this ark-trick is that the integrals along the infinite semicircle are usually quite straightforward to evaluate. The principal observation is that as we move along the circle, the complex number z remains large at all points of the circle. Therefore, instead of original complicated integrand we may use its asymptotics for large z. And the asymptotics is always much simpler than the original function. Like in our case. It is simply equal to $1/z^2$.

Now, we integrate $1/z^2 dz$ along some contour. The result is the difference of antiderivatives $1/z$ at the endpoints of the arc. But those are zeroes! So the integral along the big arc simple vanishes. So, despite the fact that we modified our original integral, turning it into the closed contour integral, in reality we didn't change its value. The closed contour integral is equal to our original integral.

With this trick we reduced the task to the computation of an infinitesimal circular integral round the pole $z = i$. We introduce a parametrization $z = i + \varepsilon e^{i\varphi}$. Next we expand our integrand as $1/(z-i)(z+i)$. As we did many times before, $dz/(z-i) = i d\varphi$. As a result, we have the integral:

$$\int_0^{2\pi} \frac{id\varphi}{2i + \varepsilon e^{i\varphi}} \tag{3}$$

We discard ε in the denominator and obtain $1/2 \int d\varphi = \pi$. The same answer.

Although in this example you may find that a purely real approach is simpler than the complex one, you can't but notice a certain geometrical elegance of the complex analysis approach.

And in our next video I'll give you powerful theorems which will essentially automate the whole procedure of tackling such integrals.

A. Residue theorem

In the previous video I outlined a certain technique of tackling the integrals. Before we continue with practicing this technique let us pinpoint the crucial step in what we did. We managed to reduce the computation of original integral to evaluating an integral over a small loop surrounding a singularity at $z = i$. Moreover, everything that mattered at this stage was the behaviour of the integrand in the vicinity of this pole and was entirely defined by the behavior of the integrand near this pole. If you think a little what we actually did, we Laurent expanded the integrand near the pole.

That is the main reason we studied attentively Laurent expansions of functions during our entire previous week.

Now, time has come to introduce the central definition of our course. The residue of a function $f(z)$ at the point $z = z_0$ which is an isolated singularity of $f(z)$ is defined as coefficient c_{-1} of an associated Laurent series:

$$f(z) = ... + \frac{c_{-1}}{(z - z_0)} + c_0 + ... \tag{1}$$

Symbolically, this is denoted as follows:

$$\mathrm{Res}_{z=z_0} f(z) = c_{-1}. \tag{2}$$

Before we state the residue theorem, let us also introduce one more important concept, the one of meromorphic function. A function that is analytic on all of domain D except for a set of isolated points, which are poles of the function, is called meromorphic function on D.

Residue theorem

Residue theorem concerns the integral I of a meromorphic function over a positively oriented non-self-intersecting closed curve γ. It states that

$$\oint_\gamma f(z)\, dz = 2\pi i \sum \mathrm{Res}_{z=z_k} f(z), \qquad \text{Residue theorem} \tag{3}$$

with the sum running over singular points of $f(z)$ inside γ, a_k. (see Fig. 1(a))

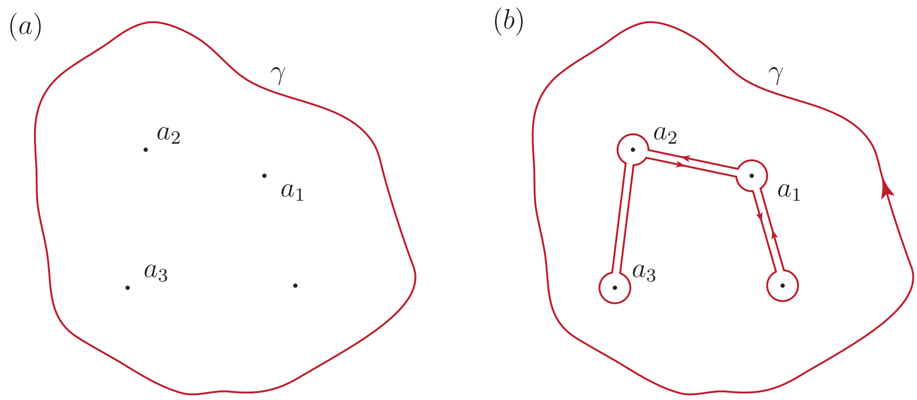

FIG. 1: Towards the residue theorem

Now let me remind you what is the positive orientation of the contour with respect to some domain. We say that the contour circles around some domain in a positive direction if this domain stays to the left as we move along the contour.

For example, as we move along the circle in the counterclockwise direction, we may say that it is a positive orientation of the contour with respect to the inner disk. But if we move in the clockwise direction along the same contour we may say that the contour is positively oriented with respect to the exterior of the disk.

The proof of the residue theorem is relatively straightforward.

But before addressing it let us prove a relatively simple preliminary identity. Let us consider an integral along a circle centered at some point a passed in the counterclockwise direction (see Fig 2) Then, we have the following identity:

$$\oint_\gamma (z - a)^n\, dz = \begin{cases} 0, & n \neq -1, \\ 2\pi i, & n = -1. \end{cases} \tag{4}$$

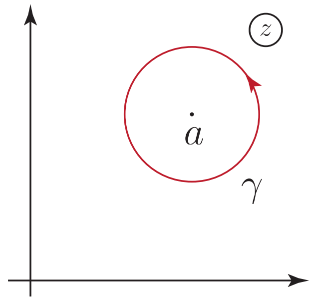

FIG. 2: Towards the residue theorem

To prove it we introduce a parametrization $z = a + re^{i\varphi}$ where r is the radius of a circle. Then we obtain:

$$\oint_\gamma (z-a)^n \, dz = r^{n+1} \int_0^{2\pi} e^{i(n+1)\varphi} i d\varphi \tag{5}$$

And, of course, the last integral vanishes due to periodicity of anti derivative with the only exception of the case $n = -1$ when it is equal to $2\pi i$.

Now, back to the proof of the residue theorem. As the first step, we can deform the contour to a combination of infinitesimal circles surrounding all singularities a_k inside the integration contour and straight lines. This deformation formes a dumbbell-like shape and doesn't change the value of the integral, since no singularities are crossed (see Fig. 1(b)).

Next, the combination of the integrals along the pairs of straight segments disappears (since they are infinitely close to each other and passed in opposite directions).

Therefore, in the end we are left with integrals along the infinitesimal circles, or rather the sum of such integrals.

$$\oint_\gamma f(z) \, dz = \sum_k \oint_{\gamma_k} f(z) \, dz, \tag{6}$$

where γ_k is an infinitesimal circle surrounding a_k. Let us compute one of these integrals, assuming that Laurent series in the vicinity of a_k has the following form:

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z - a_k)^n. \tag{7}$$

Using identity (4) we immediately see, that only c_{-1} term survives. Therefore, each integral along the respective infinitesimal circle is given by $2\pi i$ times the corresponding coefficient c_{-1} of the Laurent expansion, that is, the residue.

Thus, the residue theorem (3) is proven.

Finally, let us formulate a general approach, generalizing our exercise with computation of $\int_{-\infty}^{\infty} \frac{dx}{1+x^2}$ above. In order to evaluate real integrals, the residue theorem is used in the following manner: the integrand is extended to the complex plane and its residues are computed (which is usually easy), and a part of the real axis is extended to a closed curve by attaching a half-circle in the upper or lower half-plane, forming a semicircle (sometimes we close the contour with rectangular shape, it depends on the integrand). The integral over this curve can then be computed using the residue theorem. Often, the half-circle part of the integral will tend towards zero as the radius of the half-circle grows, leaving only the real-axis part of the integral, the one we were originally interested in.

Before proceeding with practice in evaluation of integrals along the real line, let us start with a simple exercise in contour integration.

Example 1

Let us consider the function

$$f(z) = \frac{1}{z(z^2+1)}$$

and compute $\int_{C_i} f(z)dz$ along the contours, shown on the Fig. 3. Note that all singularities of $f(z)$ are simple poles and the residues can be read off the following expansion:

$$f(z) = \frac{1}{z} + \frac{-1/2}{z+i} + \frac{-1/2}{z-i}.$$

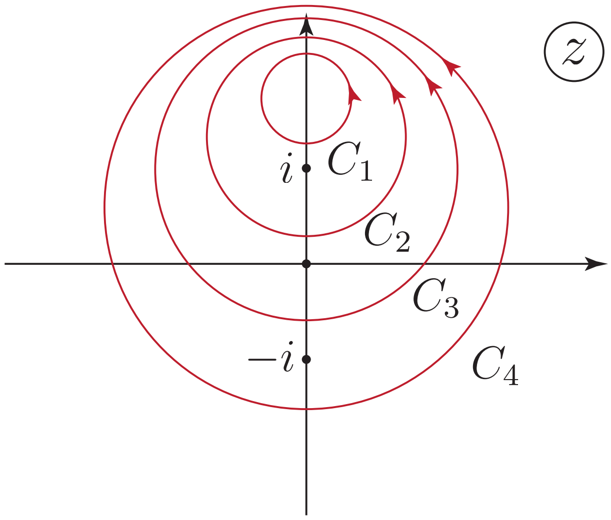

FIG. 3: Integration contours C_i.

- $\int_{C_1} f(z)dz = 0$, as there are no residues inside C_1.
- $\int_{C_2} f(z)dz = -\pi i$, as the only residue inside C_2 is $-1/2$.
- $\int_{C_3} f(z)dz = \pi i$, as there are two residues inside C_3: $1, -1/2$.
- $\int_{C_4} f(z)dz = 0$, as there are three residues inside C_3: $1, -1/2, -1/2$ and their sum vanishes.

I. RESIDUE AT INFINITY

In the previous video we proved the residue theorem by shrinking the contour so, nothing left except the infinitesimal circles round the poles. And this way we achieved a great simplification culminating in the formulation of the residue theorem.

But there is one more way to simplify an integral. And it is as important as the previous one.

Suppose we have an integral of the meromorphic function in the complex plane along some closed contour γ. Now, instead of shrinking, let us try to expand the contour. Naturally as we do so we encounter singularities which are positioned outside the contour. Our final goals is to turn the integral into an infinite circle.

But as a result of contours catching the singularities we obtain a more interesting curve. It consists of the infinite circle, a number of infinitesimal circles passed in the clockwise direction round the poles and straight linear pairs of segments stretching from poles to infinity.

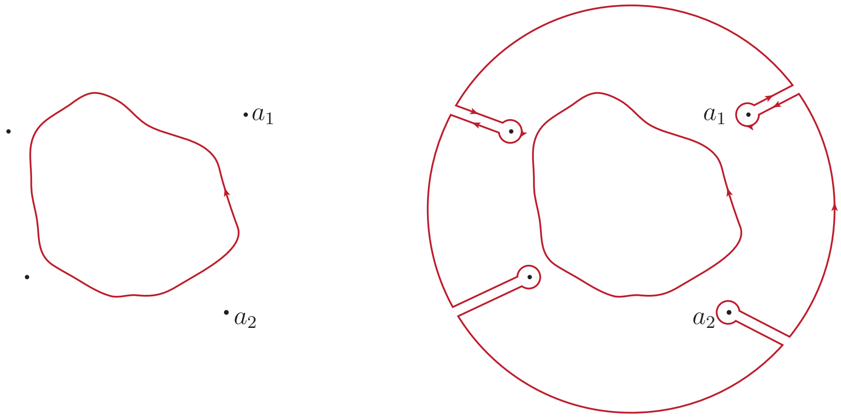

FIG. 1: Towards residue at infinity.

As before, we argue that the integrals along straight infinitely close segments passed in the opposite directions cancel each other. Now the infinitesimal integral round each pole is equal to $2\pi i$ times the residue of the integrand at this pole but with an opposite sign! Indeed, as you see, the orientation is clockwise. As a result, the integral is equal to $-2\pi i$ times the sum of the residues outside the original contour plus the integral over an infinite circle.

And now let us discuss this infinite circular integral. Let us perform the Laurent expansion of our function at $z \to \infty$.

In general,

$$f(z) = \sum_{n=-\infty}^{\infty} c_n z^n. \tag{1}$$

If the expansion has only finite amount of terms with positive powers of z, meaning the expansion starts like this:

$$f(z) = c_n z^n + c_{n-1} z^{n-1} + c_0 + \frac{c_{-1}}{z} + ... \tag{2}$$

then they say that the function has a pole of order n at infinity.

Such an expansion is called the Laurent expansion of the function at infinity.

Now, as we pointed out in the previous video, the integral of each term in the expansion is equal to zero with the exception of the term c_{-1}. This way, the integral along the infinite circle is equal to

$$\int_C f(z)\, dz = 2\pi i c_{-1}. \tag{3}$$

Therefore, we obtained a rather interesting result for an arbitrary closed contour integral. It is equal to the $-2\pi i$ times the sum of the residues outside the contour + $2\pi i$ times coefficient c_{-1} of the Laurent expansion of the function at infinity.

$$\int_\gamma f(z)\, dz = -2\pi i \sum_{a_i \in \text{out}} \operatorname*{res}_{z=a_i} f(z) + 2\pi i c_{-1} \tag{4}$$

For aethetic considerations, this coefficient c_{-1} with minus sign is called the residue of the function at infinity:

$$\operatorname*{res}_{z=\infty} f(z) = -c_{-1}. \tag{5}$$

And this way we obtain a beautiful complementary of the residue theorem. Now it is formulated as follows: the integral of the meromorphic in a complex plane function along a closed contour is found either as a $2\pi i$ times the sum of the residues inside the contour or $2\pi i$ times the sum of the residues outside the contour including the residue at infinity.

Also remember that the residue at infinity is just a clever expression of the integral over an infinite circle.

Now, the theorem proved has a very beautiful consequence:

Since for an arbitrary integral we may write down:

$$\oint = 2\pi i \sum \operatorname*{res}_{inside} f(z) = -2\pi i \sum \operatorname*{res}_{outside} f(z) \tag{6}$$

the sum of the residues of the function in an entire complex plane (including the residue at infinity is zero).

For a specific example, let us reconsider the integral we discussed in the previous video, $\int_{C_4} f(z)dz = 0$ with

$$f(z) = \frac{1}{z(z^2+1)}$$

and contour C_4 being a circle with radius 2, centered at $z = 0$. At a previous approach to this integral, we made a straightforward use of the residue theorem. This required to sum up contribution of the three residues, corresponding to simple poles of $f(z)$ inside C_4, that is $z = 0, \pm i$. Equipped with the notion of residue at infinity, we may observe that the integral can be evaluated in a simpler manner:

$$\int_{C_4} f(z)dz = -2\pi i \mathrm{Res}_{z=\infty} f(z)$$

and residue at infinity can be read off Laurent expansion of $f(z)$ at infinity. What we do, we simply expand our function in terms of inverse powers of z

$$f(z) = \frac{1}{z^3(1+1/z^2)} = \frac{1}{z^3} - \frac{1}{z^5} + ... \tag{7}$$

and we see that there is no term with $1/z$. The function simply decays too fast. Therefore,

$$\mathrm{res}_{z=\infty} f(z) = 0.$$

Hence, we obtain the same answer.

I. RIEMANN SPHERE

As you probably start to feel, the infinity plays a special role in complex analysis, quite unlike the role in real calculus. Though the variety of infinitely distant points in a plane form a circle (Fig. 1(a)), sometimes we expand our function in the vicinity of infinity, or introduce a pole and residue at infinity.

In other words, the infinity sometimes behaves like a regular point in complex analysis. Riemann was the first who suggested a nice geometrical interpretation of this observation. Let us put a sphere (now known as a Riemann sphere) on a complex plane. Then every point of the complex plane can be projected on a sphere with the help of so-called stereographic projection. We connect the point on a plane with the north pole of a sphere. The intersection with sphere gives us the image of the point on a plane on a sphere. It is obvious that every point on a complex plane has a unique projection on a sphere (see Fig. 1(b)).

The reverse is almost true with only one exception: the north pole of a sphere. It doesn't have a distinct counterpart in a complex plane. On the contrary, it seems that every infinitely distant point in a plane is projected onto a north pole of a Riemann sphere. If you use your imagination, you will easily understand that the circle of a large radius on a complex plane is projected on to infinitesimal circle near the north pole on a Riemann sphere (Fig. 1(c)).

To make the correspondence completely 1 to 1, we introduce a formally infinitely distant point on a complex plane $z = \infty$ which corresponds to the north pole of a Riemann sphere. Number $z = \infty$ doesn't take part in arithmetic operations as ordinary complex numbers. However, they say that the sequence z_n convergence to infinity if for any positive M we can find number n_0 starting from which $|z_n| > M$. This terminology is justified because the stereographic projection of our sequence indeed converges to the north pole.

A complex plane with addition of an infinitely distant point is called the extended complex plane. It is equivalent to the sphere, or like topologiest say the two objects are diffeomorphic manifolds. The extended complex plane is therefore compact.

As a function on the complex plane is understood as a mapping between two complex planes, we now understand the function in extended complex plane as the mapping between two Riemann spheres (Fig. 1(d)). In particular, the notion of an infinite limit of the function is no longer unusual. It simply means that the corresponding value of a function is on a north pole on a sphere of its values.

It is not hard to prove that circle on a sphere are projected as circles on a Riemann sphere (Fig. 1(e)). While it is obvious that the line in complex plane becomes a circle on a sphere passing through the north pole. Indeed, to build a projection of a line simply draw a plane via the north pole of a sphere and the line in a plane. The intersection of the obtained this way plane with a sphere is the projection of a line. But on the other hand, the intersection of a plane and a sphere is always a circle (Fig. 1(f).

The neighborhood of infinity is understood as the exterior of a circle of radius $|z| > R$. Geometrically, this is easy to understand in terms of stereographic projection. If we have the circle of radius R then the region $|z| > R$ is projected on to an interior of a circle surrounding the north pole of a sphere.

All the definition of the limits, connectedness are carried over a Riemann surface without any change.

The Riemann sphere is a very useful geometrical concept, which is suitable to work with when we deal with infinities in a complex plane. Though we won't use it much in our course, in differential geometry and algebraic topology it has many beautiful applications.

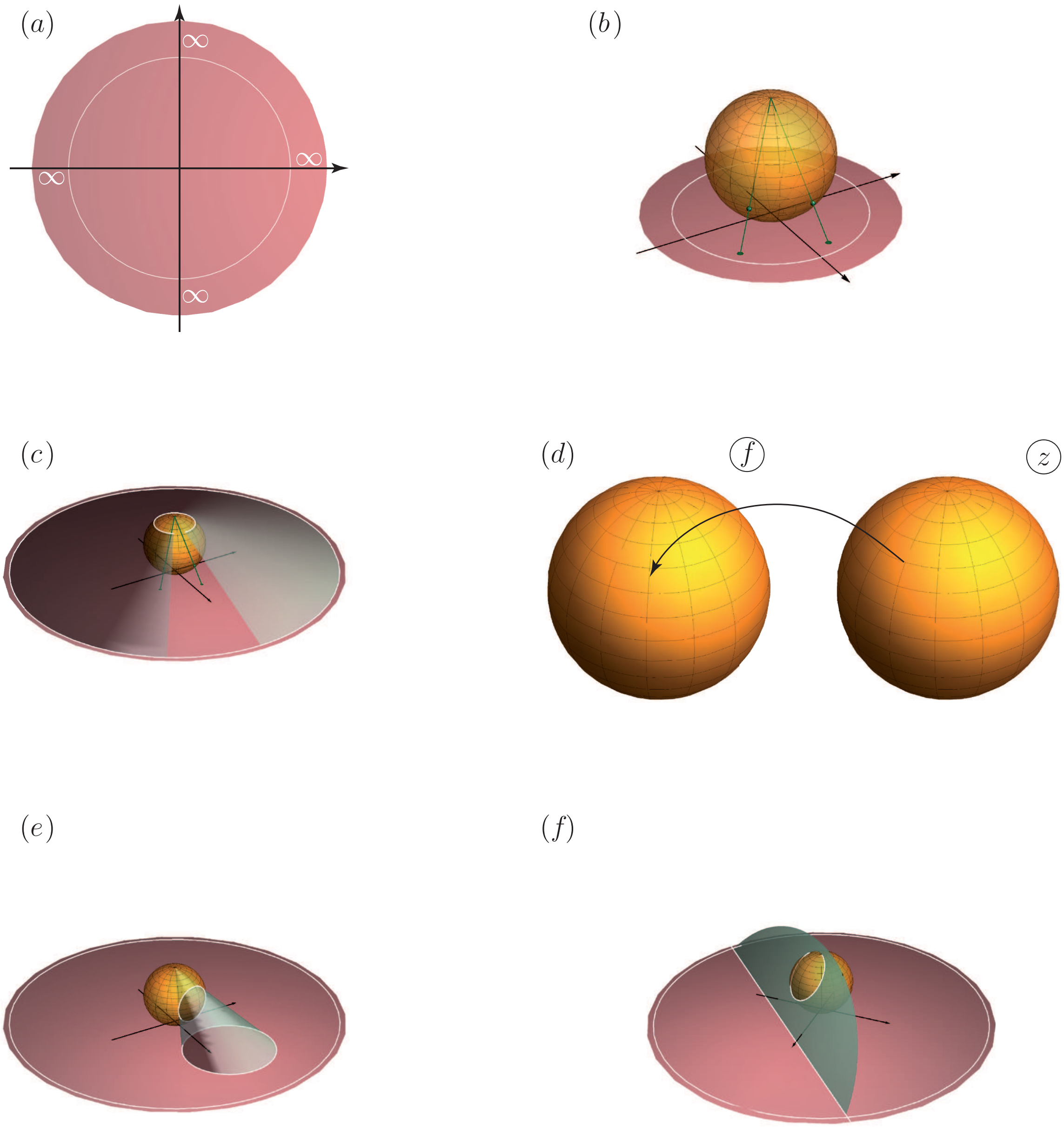

FIG. 1: Riemann sphere.

I. INTEGRATION (AND SUMMATION) WITH RESIDUES II

Evaluation of residues

So far we realized that for the computation of closed-contour integrals in a complex plane we need to compute the residues at poles.

Recall that basic definition of residues implies that they can be extracted from the respective Laurent series expansions, and one defines the residue as the coefficient c_{-1} of the power expansion.

So this seminar will be dedicated to the polishing of our technique with residues.

And we start with simple examples.

Example 1

Consider the computation of $\mathrm{Res}(\frac{e^z}{z^3}, 0)$. Here $z = 0$ is a third order pole.

Using the power series for e^z, we find the Laurent series:

$$\frac{e^z}{z^3} = \frac{1 + z + \frac{z^2}{2} + \frac{z^3}{6} + ...}{z^3} = \frac{1}{z^3} + \frac{1}{z^2} + \frac{1}{2}\frac{1}{z} + \frac{1}{6} + ...$$

and the residue equals $\mathrm{Res}(\frac{e^z}{z^3}, 0) = \frac{1}{2}$.

Example 2

Compute the residue:

$$\operatorname*{res}_{z=\infty} e^{a/z} z^n$$

where n is arbitrary positive integer.

This time we need to expand in $1/z$ powers. We start with the exponential:

$$e^{a/z} = \sum_{k=0}^{\infty} \frac{a^k}{z^k k!}$$

To extract $1/z$ term in $z^n e^{a/z}$ we need z^{n+1} term in the expansion of the exponential. It is $a^{n+1}/(n+1)!$

As a result the residue in question is

$$z = \infty \mathrm{res} e^{a/z} z^n = \frac{a^{n+1}}{(n+1)!}$$

Example 3 Find the residues of the function

$$f(z) = \frac{1}{(z-1)^2(z^2+1)}$$

at all final points.

As we see, the denominator has 3 roots $z = 1$ (second order root) and $z = \pm i$ (first order roots).

Hence we have the respective poles of $f(z)$ of the same order.

I. $z = 1$.

We perform the Laurent expansion of $f(z)$ near the point $z = 1$. As usual, we make a change $z = 1 + \varepsilon$ to obtain:

$$f(z) = \frac{1}{\varepsilon^2(2 + 2\varepsilon + \varepsilon^2)} \equiv \frac{1}{\varepsilon^2}\frac{1}{2 + 2\varepsilon + \varepsilon^2} \tag{1}$$

To obtain the residue we need to extract the coefficient in $1/ve$ term. We see that the second fraction in (1) is regular at $\varepsilon = 0$ and it can be easily Taylor expanded in ε: $A + B\varepsilon + ...$. We don't need any higher order terms. The accuracy upt to linear term in ε is enough. Combined with $1/\varepsilon^2$ in front it gives what we need.

Therefore we discard ε^2 term in the denominator:

$$f(z) = \frac{1}{\varepsilon^2}\frac{1}{2 + 2\varepsilon} = \frac{1}{2\varepsilon^2}\frac{1}{1 + \varepsilon} \tag{2}$$

and perform the geometric expansion of the second fraction: $(1+\varepsilon)^{-1} = 1-\varepsilon+....$

As a result we obtain the following Laurent expansion:

$$f(z) = \frac{1}{\varepsilon^2}\frac{1}{2+2\varepsilon} = \frac{1}{2\varepsilon^2} - \frac{1}{2\varepsilon} + ... \tag{3}$$

And extract the residue:

$$\operatorname*{res}_{z=1} f(z) = -\frac{1}{2}. \tag{4}$$

II. $z=i$.

This time the necessary change is $z = i+\varepsilon$. For transparency we expand the denominator into simple factors:

$$f(z) = \frac{1}{(z-1)^2(z-i)(z+i)} = \frac{1}{\varepsilon}\frac{1}{(i+\varepsilon-1)^2(2i+\varepsilon)} \tag{5}$$

And again, we have $1/\varepsilon$ term in front of the regular in ε function: $1/[(i-1+\varepsilon)^2(2i+\varepsilon)] = A+B\varepsilon+....$ That means we need to retain only zero order term in this function, i.e. totally drop ε:

$$f(z) = \frac{1}{\varepsilon}\frac{1}{(i-1)^2 2i} + ... = \frac{1}{4\varepsilon} \tag{6}$$

As a result we read out the residue:

$$\operatorname*{res}_{z=i} f(z) = \frac{1}{4}. \tag{7}$$

III. $z=-i$.

In this case we could in principle do the same expansion but we opt for the different (short) path. As we remember, the sum of all the residues (including the residue at infinity) of a complex function vanishes. But, the reading out the residue is often a very easy task. What is the leading asymptotic behavior of our function at infinity? Obviously, it is:

$$f(z) = \frac{1}{z^4}, \quad z\to\infty. \tag{8}$$

That means $1/z$ term is absent in the expansion. The function simply decays too fast. Therefore the residue at infinity is 0. But from this we immediately conclude that:

$$\operatorname*{res}_{z=-i} f(z) + \operatorname*{res}_{z=i} f(z) + \operatorname*{res}_{z=1} f(z) = 0 \qquad \Rightarrow \operatorname*{res}_{z=-i} f(z) = \frac{1}{4}. \tag{9}$$

Example 4 Find the residues of the function:

$$f(z) = \frac{\cos z}{(z^2+1)^2}. \tag{10}$$

Here we have two second order poles at points $z=\pm i$.

Let us study the case $z=i$. We make an expansion $z=i+\varepsilon$:

$$f(z) = \frac{\cos z}{(z+i)^2(z-i)^2} = \frac{1}{\varepsilon^2}\frac{\cos(i+\varepsilon)}{(2i+\varepsilon)^2} = \frac{1}{\varepsilon^2}\frac{\cosh 1\cos\varepsilon - i\sinh 1\sin\varepsilon}{(2i+\varepsilon)^2} \tag{11}$$

As before we have $1/\varepsilon^2$ term in front of the regular in ε function. Hence, we Taylor-expand it in ε up to first order in ε. That implies $\cos\varepsilon = 1$ while $\sin\varepsilon = \varepsilon$:

$$f(i+\varepsilon) = -\frac{1}{4\varepsilon^2}\frac{\cosh 1 - i\varepsilon\sinh 1}{(1-i\varepsilon/2)^2}. \tag{12}$$

Next, we perform the geometric-type expansion of the denominator:

$$\frac{1}{(1-i\varepsilon/2)^2} = 1 + i\varepsilon + ... \tag{13}$$

And as a result we have:

$$f(i+\varepsilon) = -\frac{1}{4\varepsilon^2}(\cosh 1 - i\varepsilon \sinh 1)(1+i\varepsilon) = -\frac{1}{4\varepsilon^2}(\cosh 1 + i\varepsilon \cosh 1 - i\varepsilon \sinh 1) \tag{14}$$

Finally, we read out the residue:

$$\operatorname*{res}_{z=i} f(z) = -\frac{i}{4}(\cosh 1 - \sinh 1) = -\frac{i}{4e}. \tag{15}$$

In a similar manner, we expand near $z = -i$ to extract the residue. But again, a different, smarter path is possible. Let us try to figure out what the residue at infinity of the function might be. On the one hand, it seems we need to expand function $\cos z$ at large z and collect infinite amount of terms which is not a particular pleasant task.

On the other hand, if we look at function $f(z)$ we immediately notice that it is an even function. It can't have $1/z$ terms in its expansion. Therefore,

$$\operatorname*{res}_{z=\infty} f(z) = 0. \tag{16}$$

But that means, of course, that

$$\operatorname*{res}_{z=-i} f(z) = -\operatorname*{res}_{z=i} f(z) = \frac{i}{4e}. \tag{17}$$

And this complete our first practice with residues. In the next section we will discover a general identity which allows for the computation of the residue of any function without constructing explicit Laurent expansion.

I. GENERAL FORMULA FOR THE RESIDUE

As you practised with computation of residues, you probably noticed that sometimes, the Laurent expansion can become a bit tedious.

That is why I'll derive for you now the general formula for the residue which avoids the necessity of building a Laurent expansion.

It is extremely helpful in many cases I'll show how it works right after the derivation. We'll also learn an, I'd say, lighter version of it, which will be encountered quite often.

As a start let us write down the expression for the Laurent expansion of the function in the vicinity of nth order pole:

$$f(z) = \frac{c_{-n}}{(z-a)^n} + \frac{c_{-n+1}}{(z-a)^{n-1}} + \frac{c_{-n+2}}{(z-a)^{n-2}} + \frac{c_{-n+3}}{(z-a)^{n-3}} ... + \frac{c_{-1}}{z-a} + c_0 + c_1(z-a) + ...$$

Next we multiply both parts by $(z-a)^n$ to obtain

$$f(z)(z-n)^n = c_{-n} + c_{-n+1}(z-a) + c_{-n+2}(z-a)^2 + c_{-n+3}(z-a)^3 ... + c_{-1}(z-a)^n + c_0(z-a)^n + c_1(z-a)^{n+1} + ...$$

And now we start taking derivatives from both parts, one after another.

After the first differentiation the term c_{-n} which is a constant part in the r.h.s. disappears. The rest of the terms lower the power of (z-a) factor by one and acquire prefactors:

$$[f(z)(z-n)^n]' = c_{-n+1} + 2c_{-n+2}(z-a) + 3c_{-n+2}(z-a)^2 ... + (n-1)c_{-1}(z-a)^{n-2} + nc_0(z-a)^{n-1} + ...$$

Now terms c_{-n+1} is constant. And after the second differentiation it will disappear:

$$[f(z)(z-n)^n]'' = 2c_{-n+2} + 3 \cdot 2c_{-n+2}(z-a) ... + (n-1)(n-2)c_{-1}(z-a)^{n-3} + n(n-1)c_0(z-a)^{n-2} + ...$$

and all other powers are diminished by one more unity. Look at term c_{-1}. It acquired the factor $n(n-1)$ after second differentiation. Now make a guess, what would be this factor after $n-1$ differentiations?

Of course $(n-1)!$.

Generally, after the second differentiation I hope the pattern becomes more or less clear. Each differentiation removes frontal terms $c_{-n},\ c_{-n+1},\ c_{n-2}$ and so on. After $n-1$ differentiations all the terms from c_{-n} to c_{-2} will be gone. Just check it yourself. And the power of factor $z-a$ at the coefficient c_{-1} weill be diminished by (n-1) units. That means, this power will simply disappear.

So, after such a differentiation our r.h.s. will look as

$$[f(z)(z-n)^n]^{(n-1)} = (n-1)!c_{-1} + n!c_0(z-a) + ...$$

On the r.h.s. under the dots I concealed senior powers of $(z-a)$. Then we simply take the limit $z=a$ on both sides. Then all the powers of $z-a$ will disappear in the r.h.s. and we obtain:

$$[f(z)(z-n)^n]^{(n-1)}\Big|_{z=a} = (n-1)!c_{-1}.$$

And this is the general formula for the residue of the function with the pole $z=a$ of nth order:

$$\operatorname*{res}_{z=a} f(z) = \frac{1}{(n-1)!}\frac{d^{n-1}}{dz^{n-1}} f(z)(z-a)^n\Big|_{z=a} \tag{1}$$

Now let us see how it is applied to particular cases.

Example 1.

$$f(z) = \frac{\cos z}{(z-1)^2}$$

Here, we have a second order pole $z=1$. According to our formula for the residue we obtain:

$$\operatorname*{res}_{z=1} f(z) = \frac{d}{dz} f(z)(z-1)^2\Big|_{z=1} = \cos' z\Big|_{z=1} = -\sin 1. \tag{2}$$

Example 2. Let us study our previous example and see if general formula (1) is more effective.

$$f(z) = \frac{1}{(z-1)^2(z^2+1)}$$

$z = 1$ is the second order pole, hence:

$$\operatorname*{res}_{z=1} f(z) = \frac{d}{dz} f(z)(z-1)^2\Big|_{z=1} = \left(\frac{1}{z^2+1}\right)'\Big|_{z=1} = -\frac{2z}{(z^2+1)^2}\Big|_{z=1} = -\frac{1}{2}; \tag{3}$$

Now, let us address the residue at point $z = i$. It is a first order pole (which is also called *simple pole*). For this situation we write down the simplified formula:

$$\operatorname*{res}_{z=a} f(z) = f(z)(z-a)\Big|_{z=a} \qquad \textit{simple pole.} \tag{4}$$

We expand the denominator $f(z)$ into factors and perform the cancellation:

$$\operatorname*{res}_{z=i} f(z) = \frac{(z-i)}{(z-i)(z+i)(z-1)^2}\Big|_{z=i} = \frac{1}{2i(i-1)^2} = \frac{1}{4}. \tag{5}$$

And similarly, we obtain the residue at point $z = -i$.

We see that sometimes general formula (1) can be much faster and effective than the Laurent expansion.

Example 3.

$$f(z) = \frac{1}{(z^2+1)^3}$$

Here, $z = \pm i$ are 3rd order poles. From the general residue formula we obtain:

$$\operatorname*{res}_{z=i} f(z) = \frac{1}{2!}\frac{d^2}{dz^2}\frac{1}{(z+i)^3}\Big|_{z=i} = \frac{1}{2}\frac{3\cdot 4}{(z+i)^5}\Big|_{z=i} = \frac{3}{16i}. \tag{6}$$

And similarly, for the point $z = -i$.

And let us discuss an additional formula which is very helpful once we deal with a first order pole. If a function has a first order pole in some point $z = a$ then it can always be represented as a ratio of two functions where the function in denominator has the first order root, while the function in nominator doesn't vanish at this point:

$$f(z) = \frac{h(z)}{g(z)}, \quad h(a) \neq 0.$$

Then we make a Taylor expansion of the function in the denominator restricting it to only first nonzero term: $g(z) = g'(a)(z-a)$. Then in the vicinity of $z = a$ the first term of the Laurent expansion of our original function looks as follows:

$$f(z) = \frac{h(a)}{g'(a)(z-a)}, \quad h(a) \neq 0.$$

Here, we restricted the expansion of the function in nominator by its first term as well, setting $h(z) = h(a)$. But then, we have precisely the expression of the type $\sim 1/(z-a)$. And the coefficient in front of it is our residue. Therefore, for simple poles we have the following nice formula

$$\mathrm{res} f = \frac{h(a)}{g'(a)}. \qquad \textit{first order pole.} \tag{7}$$

Let us see how suitable the formula can be in a typical situation.

Example 4.

$$f(z) = \frac{1}{z^3+1}. \tag{8}$$

Find the residues at all finite points.

Solving the equation $z^3 + 1 = 0$ we find that the function has three first order poles:

$$z_i = \begin{cases} e^{i\pi/3} \\ -1 \\ e^{-i\pi/3} \end{cases} \tag{9}$$

According to formula (7) the residue of the function at each of these points read:

$$\operatorname*{res}_{z=z_i} f(z) = \frac{1}{3z_i^2} \tag{10}$$

This way, with a single computation we obtained three residues simultaneously:

$$\operatorname*{res}_{z=z_i} f(z) = \begin{cases} \frac{1}{3}e^{-2\pi i/3}, & z = e^{i\pi/3} \\ \frac{1}{3}, & z = -1, \\ \frac{1}{3}e^{2\pi i/3}, & z = e^{-i\pi/3} \end{cases} \tag{11}$$

Evaluation of integrals

Example 1
Let us now practice with evaluation of integrals via residue theorem.
Compute the following trigonometric integral:

$$\int_0^{2\pi} \frac{d\theta}{5-3\sin\theta}$$

The trigonometric integrals with the integration domain spanning the period of the integrand are always taken in a similar manner. Let us first convert this integral into a contour one, over C – a circle centered at 0 of radius 1, oriented positively. This is achieved by substitution $z = e^{i\theta}$. Note that $dz = ie^{i\theta}d\theta = izd\theta$, so that $d\theta = dz/(iz)$. Also, $sin(\theta) = (z - z^{-1})/(2i)$. We obtain:

$$\int_0^{2\pi} \frac{d\theta}{5-3\sin\theta} = \oint_C \frac{dz}{iz(5-\frac{3z-3/z}{2i})} = \oint_C \frac{-2dz}{3z^2-10iz-3}.$$

The singularities of the integrand are simple poles at $z_1 = 3i$ and $z_2 = i/3$. Only one of them, z_2 is inside the integration contour C and has the residue

$$\mathrm{Res}(\frac{-2dz}{3z^2-10iz-3}, i/3) = -i/4.$$

As a result,

$$\int_0^{2\pi} \frac{d\theta}{5-3\sin\theta} = 2\pi i(-i/4) = \pi/2.$$

Example 2
Compute the following integral

$$I = \int_{-\infty}^{\infty} \frac{dx}{(x^4+1)}. \tag{1}$$

Of course, this integral can be easily computed with the help of ordinary analysis, but our goal here is to practise residue theorem.

Note that the integral at first sight is not equivalent to any closed contour integral. However, the standard trick is to close the contour by a semiarc. The arc is used quite often as the closure of the infinite contour. That is because the modulus of the argument of the function, z, is always large as we move along the arc and we may use the asymptotics of the function instead of the function itself. Therefore we close the contour with upper semicircle (see Fig. 1). And,

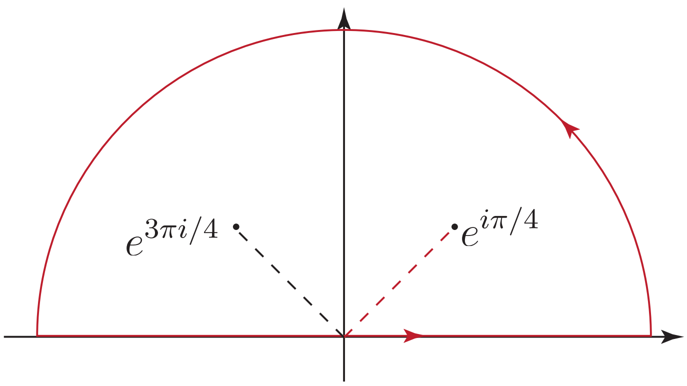

FIG. 1: Towards the computation of integral (1)

naturally, the hope is that the arc integral either vanishes or is reduced to some trivial number. Here, the asymptotics of the integrand at large z is $1/z^4$. It is indeed, quite simple. As we introduce the arc parametrization $z = Re^{i\theta}$ where θ changes from 0 to π, differential becomes large though: $dz = Rie^{i\theta}d\theta$. In the end, the integral behaves as

$$\frac{1}{R^3}\int_0^\pi ...d\theta \to 0, \quad R \to \infty.$$

Therefore, instead of original integral we now consider a complex closed contour integral:

$$\oint f(z)\,dz, \quad f(z) = \frac{1}{z^4+1}. \tag{2}$$

We just proved that this integral is, in fact, just our original integral. On the other hand we may compute it using residue theorem:

$$I = \oint f(z)\,dz = 2\pi i\Big(\operatorname*{res}_{z=e^{i\pi/4}} f(z) + \operatorname*{res}_{z=e^{3i\pi/4}} f(z)\Big) \tag{3}$$

Here we included only those residues which corresponds to the pole inside the close contour. Those are the first order poles. And the residues are easily computed by a simplified formula with derivative from the previous section.

$$\operatorname*{res}_{z=z_0} f(z) = \frac{1}{4z_0^3}$$

As a result we obtain the residues:

$$\operatorname*{res}_{z=e^{i\pi/4}} f(z) = \frac{1}{4e^{3\pi i/4}} = -\frac{1}{4}e^{i\pi/4}.$$

$$\operatorname*{res}_{z=e^{3i\pi/4}} f(z) = \frac{1}{4e^{9\pi i/4}} = \frac{1}{4}e^{-i\pi/4}.$$

The sum of two residues can be combined into a sine function $-(i/2)\sin\frac{\pi}{4} = -i/(2\sqrt{2})$. And finally, we the value of the integral:

$$I = \frac{\pi}{\sqrt{2}}$$

A. Integral III

In this quick example I'd like to discuss with you one more neat example of an integral. It's quite simple but the reason I'd like to address it is because it has some peculiarity which will make you much more cautious in all your future examples of integrals. The integral is as follows:

$$I = \int_{-\infty}^{\infty} \frac{dx}{x+ia} \tag{1}$$

Formally, this integral is divergent. It can be made convergent if we understand it as the symmetric limit:

$$I = \lim_{R\to\infty} \int_{-R}^{R} \frac{dx}{x+ia}$$

and we will treat it as this from now on.

Let us employ our closure technique. We complement our contour with an upper semicircle (see Fig. 1).

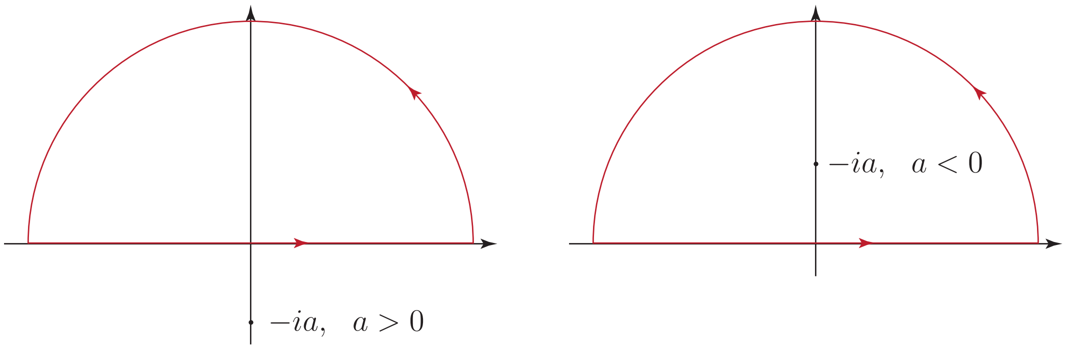

FIG. 1: Towards the computation of integral (1)

Let us prove quickly that this arc integral doesn't change anything, as usual. The parametrization is $z = Re^{i\theta}$. The function's asympotic is $1/R$. Be very careful, when estimating this arc integrals. The integrand may decay, but don't forget to keep in mind additional large factor R coming from differential. This way we have:

$$\int_{\text{arc}} \frac{dz}{z} = \int_0^{\pi} \frac{Rie^{i\theta}d\theta}{Re^{i\theta}} = \int_0^{\pi} d\theta = i\pi.$$

So this time this arc integral doesn't vanish, but is reduced to a simple number.

Therefore, our new closed contour integral reads:

$$\oint = I + i\pi$$

The closed contour integral, as you probably already see by now depends on the sign of a. Let us consider 2 cases.

Suppose $a > 0$, then there are no residues inside the contour, the function is regular. And the closed contour integral is equal to zero. Therefore, the initial integral is equal to:

$$I = -i\pi$$

Suppose now, $a < 0$. Then, there is a residue inside the contour. It is a first order pole and the residue is equal to 1. Then our closed contour integral, according to Cauchy residue theorem is equal to $2\pi i$. As a result, the initial integral is equal to

$$I = 2\pi i - \pi i = \pi i.$$

As a result our Integral is, in fact, nothing but a sign function, up to some constant term.

Therefore, we obtained the integral representation of the sign function:

$$\mathrm{sign} a = -\frac{1}{\pi i} \int\limits_{-\infty}^{\infty} \frac{dx}{z + ia}.$$

Why this result may be interesting? Because sign is a super non-analytic function of a variable a and it is impossible to operate with this function in a complex plane as is.

What we have just obtained is an integral representation of this function in terms of much more regular functions, namely meromorphic functions in a complex plane.

I. INTEGRATION WITH JORDAN'S LEMMA

We will now consider a very important class of integrals of the following form:

$$\int_{-\infty}^{\infty} e^{i\lambda z} f(z)dz$$

for $f(z)$ holomorphic on a half-plane. The importance of such integrals is in the fact, that any Fourier transform is given by this type of an integral.

We already understand that the crucial ingredient in evaluation of this kind of integral via residue theorem is the completion of the contour by an arc (in the upper or lower half–plane of z). For integration via residues to be practical, it is desirable for the integral over this arc to vanish. From what we have learn't from our previous example, this would require for the function $f(z)$ to satisfy

$$\lim_{|z|\to\infty} f(z)z = 0$$

uniformly in $0 \le \arg z \le \pi$. Amazingly, it turns out that this requirement can be relaxed to

$$\lim_{|z|\to\infty} f(z) = 0$$

uniformly in $0 \le \arg z \le \pi$. This relaxation is possible thanks to oscillatory nature of the integrand, namely, the exponential function which helps the convergence. The precise statement is known as Jordan's lemma.

Jordan's lemma

Consider positive a contour C_R – semicircle of radius R in the upper half–plane, centered at $z = 0$. Let the function $f(z)$ satisfy $\lim_{|z|\to\infty} f(z) = 0$ uniformly in $0 \le \arg z \le \pi$, then for $\lambda > 0$:

$$\lim_{R\to\infty} \int_{C_R} e^{i\lambda z} f(z)dz = 0.$$

The proof is simple and demonstrates an important technique of estimation of oscillatory integrals, so let us follow it.

First of all, let us grasp what the uniform convergence of f means. It means that for any radius R, $\max\{|f(z)|\}\Big|_{z\in C_R} < \varepsilon_R$ and $\varepsilon_R \to 0$ as $R \to \infty$

For arbitrary positive ϵ let us choose R_0 such that $|f(z)| < \epsilon/\pi$ for all $|z| > R_0$. Then for $\rho > \rho_0$ we have:

$$\left|\int_{C_\rho} e^{imz} f(z)dz\right| = \left|\int_0^\pi e^{im\rho(cos\phi+i\sin\phi)} f(\rho e^{i\phi})\rho e^{i\phi} i d\phi\right| < \frac{\epsilon}{\pi}\rho\int_0^\pi e^{-m\rho\sin\phi}d\phi = \frac{2\epsilon}{\pi}\rho\int_0^{\pi/2} e^{-m\rho\sin\phi}d\phi.$$

Next, using for $0 \le \phi \le \pi/2$ the inequality $\frac{\sin\phi}{\phi} < \frac{\sin\pi/2}{\pi/2} = 2/\pi$ we arrive to

$$\left|\int_{C_\rho} e^{imz} f(z)dz\right| < \frac{2\epsilon}{\pi}\rho\int_0^{\pi/2} e^{-2m\rho\phi/\pi}d\phi = \frac{\epsilon}{m}\left(1 - e^{-m\rho}\right) < \frac{\epsilon}{m}.$$

Hence, at appropriately large ρ the integral can be made arbitrary small and the Jordan's lemma is proven.

Obviously, if you compute the integral along the lower semicircle, the statement stays the same, but this time the integral vanishes for negative λ.

Let us now consider examples of oscillatory integrals.

Let us compute, for real a the following integral:

$$I(a) = \int_{-\infty}^{\infty} \frac{e^{iax}}{x+i}dx.$$

We need to complete the integration line to the closed contour. The proper way to do it depends on the sign of a.

For $a > 0$ we should close the contour in the upper half–plane (see Fig. 1(a)).

Note that the integrand is analytic inside the closed contour and hence the contour integral vanishes. The integral over the arc tends zero by Jordan's lemma and hence the original integral also vanishes.

$$I(a) = 0, \quad a > 0$$

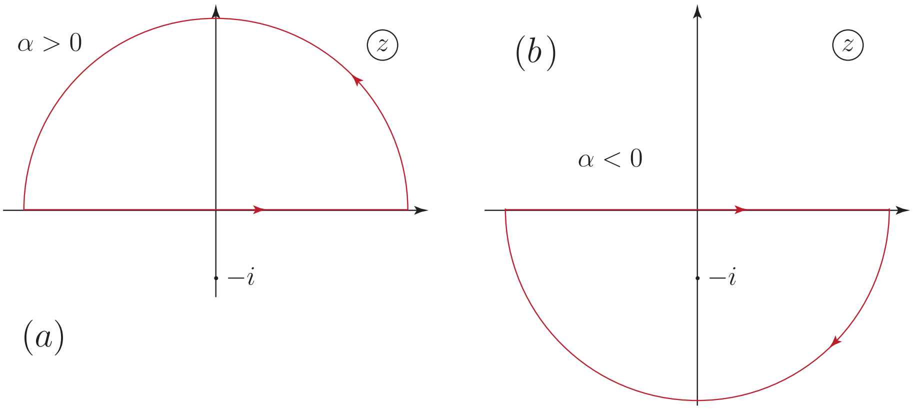

FIG. 1: Towards the application of Jordan's lemma.

For $a < 0$ the contour should be closed in the lower half–plane. (see Fig. 1(b)) In this case, the integral over the arc vanishes again and the original integral equals

$$2\pi i \times (\text{residues of } \frac{e^{iax}}{x+i}) \text{ inside the integration contour}$$

which equals $I(a) = 2\pi i e^{a}, \quad a < 0.$

Consider the integral

$$I = \int_0^\infty \frac{\cos x}{x^2+a^2} dx \tag{1}$$

There are two issues here. First, the contour doesn't go from $-\infty$ to ∞. Second, there is cos instead of exponential in the integrand meaning we have a combination of exponentials instead of a single exponential. As a result we won't be able to employ Jordan's lemma. As we will see, we will resolve these two issues simultaneously.

First we need to extend the contour to minus infinity. To do this let us split the cos term into a sum of two exponentials and change the variable form $-x$ to x in the second integral. The last step is done to make exponentials to have the increment of the same sign. We obtain:

$$I = \frac{1}{2}\int\limits_0^\infty \frac{e^{ix}}{x^2+a^2} dx + \frac{1}{2}\int\limits_0^{-\infty} \frac{e^{ix}}{x^2+a^2}(-dx) \tag{2}$$

Interchanging the limits of integration in the last integral we absorb the minus sign and obtain:

$$I = \frac{1}{2}\int\limits_{-\infty}^\infty \frac{e^{ix}}{x^2+a^2} dx. \tag{3}$$

Therefore, we achieved our goal. The integral now spans the entire real axis and contains a single exponential with positive increment $\lambda = 1$. This way, keeping in mind future application of Jordan's lemma, we close the contour with an upper semi circle (see Fig. 1). We promote our integrand in to a complex plane $e^{iz} f(z) = e^{iz}/[2(z^2+a^2)]$ and

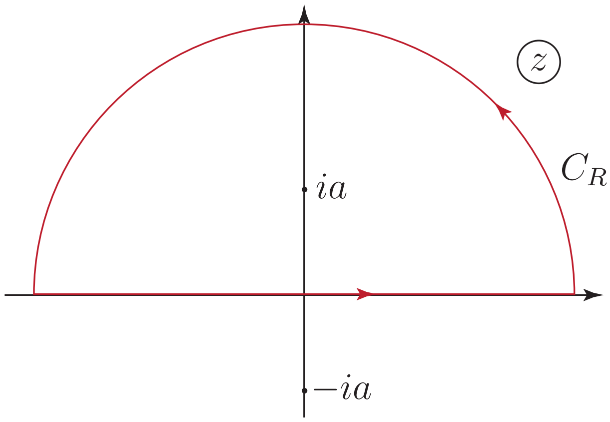

FIG. 1: Towards the application of Jordan's lemma.

consider a closed contour integral:

$$\oint e^{iz} f(z)\, dz = \int\limits_{-\infty}^{\infty} + \int_{C_R}. \tag{4}$$

The integral over upper semicircle C_R vanishes due to Jordan's lemma, since $f(z)$ decays uniformly with respect to $\arg z$ as $z \to \infty$.

Therefore, the closed contour integral is simply equal to the original integral I. We compute the former using residue theorem. Only one first order pole is positioned inside the integration contour $z = ia$. Hence:

$$\oint e^{iz} f(z)\, dz \equiv I = 2\pi i \operatorname*{res}_{z=ia} \frac{e^{iz}}{2(z^2+a^2)} = 2\pi i \frac{e^{-a}}{4ia} = \frac{\pi}{2a} e^{-a} \tag{5}$$

Here, we assume, of course, that $a > 0$.

Let us compute, for $a > 0$ and $b > 0$ the following integral:

$$I = \int_0^\infty \frac{\cos 2ax - \cos 2bx}{x^2} dx. \tag{1}$$

First of all, let us make sure that the integral is well defined and study the behavior of the integrand at $x \to 0$. It is a good practice since it reveals valuable information about the structure of the integral as a whole.

Taylor expanding the nominator to second order in x we obtain:

$$\frac{\cos 2ax - \cos 2bx}{x^2} \to 2(b^2 - a^2)\Big|_{x \to 0} \tag{2}$$

Therefore, the integrand is well - defined. Now let us proceed with computation as in previous examples. We expand cosines into sum of exponentials and change the variables in the exponentials with *negative* increments. This way we obtain the integral along the entire real axis:

$$I = \frac{1}{2} \int_{-\infty}^\infty \frac{e^{2iax} - e^{2ibx}}{x^2} dx. \tag{3}$$

However (3) is already ill defined. Indeed, its integrand has a first order pole at $x = 0$:

$$f(z) = \frac{1}{2} \frac{e^{2iaz} - e^{2ibz}}{z^2} \to \frac{i(a-b)}{z}\Big|_{z \to 0} \tag{4}$$

This presents a problem. Apparently, we made a mistake when transforming the integral. We encourage the reader to figure out this mistake independently. The question is, how to tackle the integral.

As we understand, point $x = 0$ is potentially problematic. Therefore, we consider the original integral as a limit of a slightly modified integral:

$$I = \lim_{\varepsilon \to 0} I_\varepsilon \equiv \lim_{\varepsilon \to 0} \int_\varepsilon^\infty \frac{\cos 2ax - \cos 2bx}{x^2} dx. \tag{5}$$

Here ε is an infinitesimal positive parameter. This way we exclude the potentially dangerous point $x = 0$ from our analysis.

Now we use the same algebra and turn our integral into a form suitable for application for Jordan's lemma, but this time however we will not be able to obtain the integral from minus infinity to plus infinity of difference of two exponentials. Rather we'll obtain two separate integrals from - infinity to - epsilon and from + epsilon to + infinity of the same integrand.

$$I_\varepsilon = \int_{-\infty}^{-\varepsilon} \frac{e^{2iax} - e^{2ibx}}{x^2} dx + \int_\varepsilon^\infty \frac{e^{2iax} - e^{2ibx}}{x^2} dx \tag{6}$$

So this way there is no problem with singularity but there is a trade-off now: we have two separate integrals instead of one and in mathematics there is a special name for this combination of integrals. It is called a principal value integration and is denoted as integral with cross sign:

$$⨍ \equiv \Big[\int_{-\infty}^{-\varepsilon} + \int_\varepsilon^\infty \Big]_{\varepsilon \to 0} \tag{7}$$

What it means is that we take the integral of some function along some contour and whenever the contour meets a singularity of the integrand it is split and offset by infinitesimal distance to the left and to the right of the singularity (see Fig. 1).

This procedure is called the integration in the sense of a principal value. Therefore, we reduced our original integral to a more suitable form but over a split contour. However, it is the closed contour that we were aiming at, not the split one.

What shall we do now? Well, it is simple. We close our contour. We'll connect our separate pieces at the origin with infinitesimal upper or lower semicircle - it's up to us to choose, but I choose upper semicircle for the reason which will hopefully be clear to you in a couple of minutes. Then I'll connect to infinite edges of this contour, but this time infinite upper semicircle, keeping in mind the application of Jordan's lemma. Now we'll promote our function into a complex plane and consider a closed contour integral (see Fig. 2).

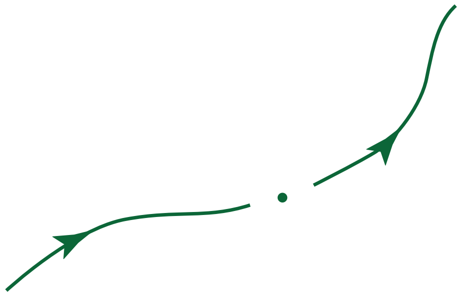

FIG. 1: Towards the principal value integration.

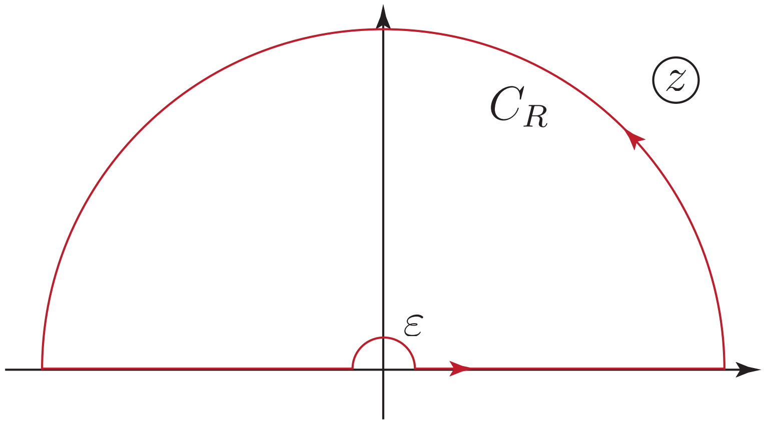

FIG. 2: Towards the evaluation of the integral.

Of course, this integral consists of our principal value integral which is nothing but our initial integral plus the integral along infinitesimal upper semicircle and plus the integral along the upper infinite semicircle C_R:

$$\oint = \fint + \int_{\varepsilon} + \int_{C_R} \tag{8}$$

Due to Jordan's lemma the infinite semicircular integral tends to zero as the radius of the circle tends to infinity, because it's a combination of two exponentials with positive increments a and b. And the preexponential function g(z) which is equal to $1/z^2$ tends to zero uniformly with respect to the argument of the complex number z. Now the closed contour integral itself. from residue theorem,

it should be equal to $2\pi i$ times the sum of the residues inside the contour. But there are no poles inside this contour: our function is analytic in it, so this closed contour integral is in fact, equal to 0.

$$\oint = 0 \tag{9}$$

Miraculously, our principal value integral is now equal to the integral along the infinitesimal semi-circle at the origin.

$$\fint = -\int_{\varepsilon} \tag{10}$$

And this is from my point of view is a charm of complex analysis: we started with a complicated integral from zero to plus infinity and we reduced it to an integral along some infinitesimal circle. And of course, this integral is way easier to compute because we don't need the full function to do this. But just it's Taylor series near the origin. So let us perform the Taylor expansion of our f(z) function.

The first term in the numerator was already obtained by us in (4): $i(a-b)/z$ plus some regular terms. Now, why don't we need these regular terms?

That's because when we integrate along this infinitesimal circle, they will vanish as the radius of the circle tends to zero. This way we only need singular terms, and there is only one of them. And it's what is written in (4).

Now let's plug in this expansion into our integral, and we'll obtain

$$\fint = -i(a-b)\int\limits_{\varepsilon} \frac{dz}{z} \tag{11}$$

Next we introduce the standard parametrization $z = \epsilon e^{i\varphi}$, and therefore dz/z is simply $id\varphi$ and since φ changes from π to 0, this integral is equal to $-i\pi$. Therefore, we obtain:

$$I \equiv \fint = -i(a-b)(-i\pi) = \pi(b-a). \tag{12}$$