A. Introduction (Video 1.1)

We will start with some very simple definitions and operations. Of course, we will not spend too much time of them and those of you who are already familiar with this material please stay with us for a while. Apart from algebraic aspects of complex numbers, we will exercise in geometrical interpretations which will be later very useful for construction of regular branches of multi-valued functions and computation of complicated integrals in the following lectures.

1. Definition of a complex number and algebraic operations

$$z = x + iy, \quad x, y \in R, \quad i^2 = -1$$

where $x = \operatorname{Re} z$, and $y = \operatorname{Im} z$.

It is easy to construct a multiplication rule:

$$z_3 = z_1 z_2 = (x_1 + iy_1)(x_2 + iy_2) = \underbrace{x_1 x_2 - y_1 y_2}_{\operatorname{Re} z_3} + i \underbrace{(x_1 y_2 + x_2 y_1)}_{\operatorname{Im} z_3} \tag{1}$$

It immediately follows that complex numbers inherit commutativity and distributivity from real numbers. A bit less trivial fact which makes the definition above very useful is that complex number can also be divided! For division of complex number z by a real number x_0 it is of course trivial, as division can be substituted by multiplication by an inverse number:

$$z/x_0 = \frac{1}{x_0}(x + iy) = \frac{x}{x_0} + i\frac{y}{x_0}. \tag{2}$$

Before turning to division of complex number by complex number, let us define the conjugation operation:

$$z = x + iy, \quad z^* = x - iy. \tag{3}$$

Note that the product zz^* is a real number:

$$zz^* = (x + iy)(x - iy) = x^2 + y^2 \tag{4}$$

The quantity $|z|$ defined as

$$|z| = \sqrt{x^2 + y^2} \tag{5}$$

is called modulus of a complex number z. As a result, we see $zz^* = |z|^2$.

Now, let us come back to division of complex numbers. To this end, we have to be able to find solutions to the following equation:

$$zz_1 = z_2, \tag{6}$$

for complex number $z = x + iy$, considering $z_1 = x_1 + iy_1$, $z_2 = x_1 + iy_1$ as given complex number. This equation has a formal solution

$$z = z_2/z_1 = z_2 z_1^{-1} \tag{7}$$

But what is actually z_2/z_1? To understand it, let us multiply our equation by z_1^*:

$$z|z_1|^2 = z_2 z_1^* \tag{8}$$

As a result

$$z = \frac{z_2 z_1^*}{|z_1|^2} \tag{9}$$

We find that inversion should be defined as follows:

$$z_1^{-1} = \frac{z_1^*}{|z_1|^2} \tag{10}$$

Let us consider a little exercise and compute the fraction $\frac{z_1}{z_2}$ where

$$z_1 = 2 - 5i, \quad z_2 = -3 + 4i.$$

We find:

$$\frac{z_1}{z_2} = \frac{(2-5i)(-3-4i)}{25} = -\frac{26}{25} + \frac{7}{25}i$$

2. Geometric interpretation

It is often convenient to think of a complex number as a 2D vector, see Fig. 1. Here $z = a + ib$ and two coordinates are equal to a and b. Then complex number -1 is represented by a unit vector oriented to the left and complex number i – by a unit vector forming $\pi/2$ angle with the x-axis. Naturally, sum or difference of complex numbers can

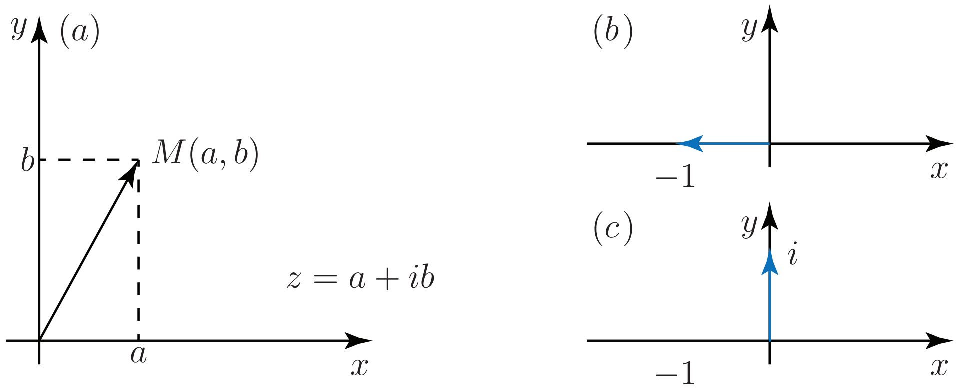

FIG. 1: Geometric interpretation of complex numbers.

be interpreted as sum or difference of associated vectors in the complex plane.

3. Example of geometric interpretation

Let us now practice with determination of loci of certain complex equations. Find the loci of the following equation: $|z+1| = 1$. Let us start with drawing a vector to a given point z, and will think of $z+1$ as $z-(-1)$. That is, let us think of $z+1$ as a vector pointing from -1 to z. According to the definition of our set, the length of this vector equals one. Thus, the logic of our equation is a unit circle centered at -1. This problem of course allows for an analytic solution which is less elegant though:

$$|z+1| = |x+iy+1| = \sqrt{(x+1)^2+y^2} = 1 \;\Rightarrow (x+1)^2 + y^2 = 1$$

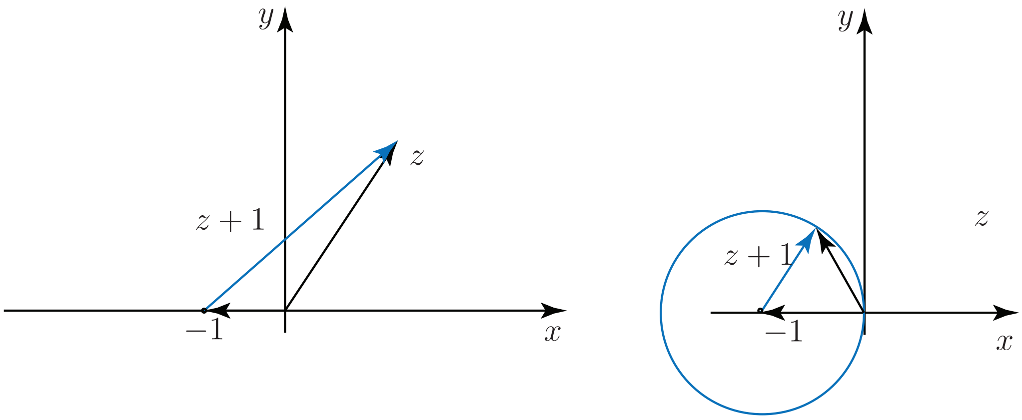

FIG. 2: The loci of equation $|z+1|=1$.

A. (Video 1.2)

B. Loci of the complex equations (Video 1.2)

Let us consider another example. Here we have to describe the complex plane set defined by the inequality:

$$|z+3| < |z+4i|. \tag{1}$$

The simplest way to approach this kind of problem is to establish the loci of the equation, defining the boundary of the desired set:

$$|z+3| = |z+4i|. \tag{2}$$

As in the previous example, let us interpret the complex numbers $z+3$ and $z+4i$ as vectors pointing from $A=-3$ and $B=-4i$ to z, correspondingly. The lengths of the vectors $z+3$ and $z+4i$ coincide if and only if z belongs to a line l, perpendicular to AB and crossing this segment in a middle. The vector AB is described by components $(3,-4)$ and the normal to this vector has components $(1,3/4)$. The line l intersects AB at the point $(x_0,y_0)=(-3/2,-2)$. Hence, l has the following equation:

$$y = \frac{3}{4}(x-x_0)+y_0 = \frac{3}{4}\left(x+\frac{3}{2}\right)-2 = \frac{3}{4}x-\frac{7}{8}. \tag{3}$$

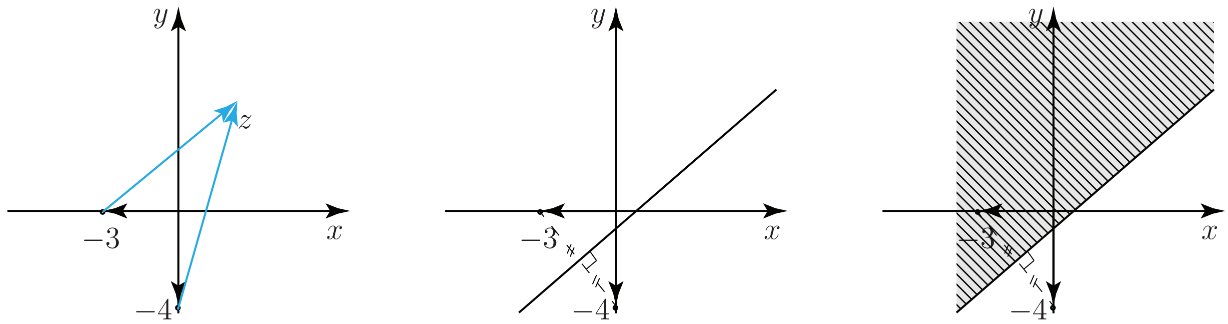

FIG. 1: Graphical solution to the equation $|z+3| = |z+4i|$

Again, this problem admits an explicit analytic solution:

$$|z+3| = \sqrt{(x+3)^2+y^2} = |z+4i| = \sqrt{x^2+(y+4)^2} \;\Rightarrow (x+1)^2+y^2 = 1$$

Squaring both parts of the equation, we establish the same result Eq. (3).

After determining the domain, we can easily establish that the region we are looking for lies to the right of the boundary.

A. Video 1.3

1. Trigonometric interpretation

Trigonometric interpretation is naturally connected to the geometric one. Its construction is achieved by switching the coordinate system to the polar one and represent z by $|z|$ and φ. The polar angle φ counted from the positive direction of the real axis is called argument of a complex number z.

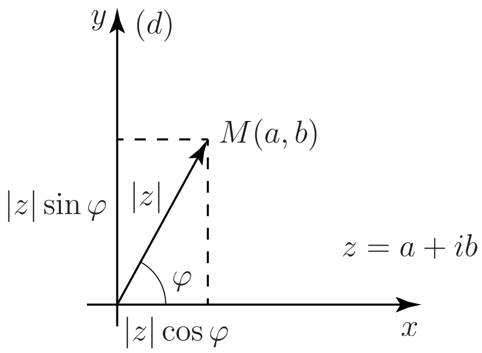

FIG. 1: Trigonometric form of a complex number

$$\varphi = \arg z. \tag{1}$$

A sort of ambiguity is present here, as this angle is defined up to addition of integer factors of 2π: this fact will be very important in what follows later. Now, to summarize our discussion of trigonometric representation, we wrap it up by the following equation:

$$z = a + ib = |z|\cos\varphi + i|z|\sin\varphi = |z|(\cos\varphi + i\sin\varphi). \tag{2}$$

Thus, a complex number is uniquely recovered from its modulus and argument.

Complex conjugation keeps the modulus intact and an argument can be considered as changing sign:

$$\begin{aligned} z^* = a - ib = |z|\cos\varphi - i|z|\sin\varphi &= |z|(\cos\varphi - i\sin\varphi) \\ &= |z|(\cos(-\varphi) + i\sin(-\varphi)). \end{aligned}$$

Let us inspect the effect of multiplication and division on the modulus and argument.

$$\begin{gathered} z_1 = |z_1|(\cos\varphi_1 + i\sin\varphi_1), \quad z_2 = |z_2|(\cos\varphi_2 + i\sin\varphi_2), \\ z_3 = z_1 z_2 = |z_1||z_2|(\cos\varphi_1 + i\sin\varphi_1)(\cos\varphi_2 + i\sin\varphi_2) = \\ |z_1||z_2|(\cos\varphi_1\cos\varphi_2 - \sin\varphi_1\sin\varphi_2 + i(\cos\varphi_1\sin\varphi_2 + \cos\varphi_2\sin\varphi_1)) \\ = |z_1||z_2|(\cos(\varphi_1+\varphi_2) + i\sin(\varphi_1+\varphi_2)) = |z|(\cos\varphi + i\sin\varphi). \end{gathered}$$

Quite nicely, the modulus multiply under the product and arguments – add up:

$$|z_1 z_2| = |z_1||z_2|, \quad \arg(z_1 z_2) = \arg z_1 + \arg z_2 \tag{3}$$

For division operation, we find:

$$\begin{gathered} z_1 = |z_1|(\cos\varphi_1 + i\sin\varphi_1), \quad z_2 = |z_2|(\cos\varphi_2 + i\sin\varphi_2), \\ z_3 = z_1/z_2 = \frac{z_1 z_2^*}{|z_2|^2} = \frac{|z_1||z_2|}{|z_2|^2}(\cos\varphi_1 + i\sin\varphi_1)(\cos\varphi_2 - i\sin\varphi_2) = \\ \frac{|z_1|}{|z_2|}(\cos\varphi_1\cos\varphi_2 + \sin\varphi_1\sin\varphi_2 + i(\cos\varphi_1\sin\varphi_2 - \cos\varphi_2\sin\varphi_1)) \\ = \frac{|z_1|}{|z_2|}(\cos(\varphi_1-\varphi_2) + i\sin(\varphi_1-\varphi_2)) = |z|(\cos\varphi + i\sin\varphi). \end{gathered}$$

Thus, the modulus of the result of division is just a result of division of modulus and the arguments should be subtracted:

$$\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}, \quad \arg\left(z_1 - z_2\right) = \arg z_1 - \arg z_2 \tag{4}$$

This allows for a very simple interpretation of the power function. Indeed, raising the complex number z to the power n multiplies the argument by a factor of n times and the modulus (a real number) is simply raised to the same n-th power.

Consider a number of unit modulus:

$$z_0 = \cos\theta + i\sin\theta, \quad |z_0| = 1 \tag{5}$$

Then:

$$z_0^n = (\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta \tag{6}$$

Let us consider an example. Compute argument and modulus of the complex number $z = -1 - i$ and write down the trigonometric representation. Reduce the argument to the interval

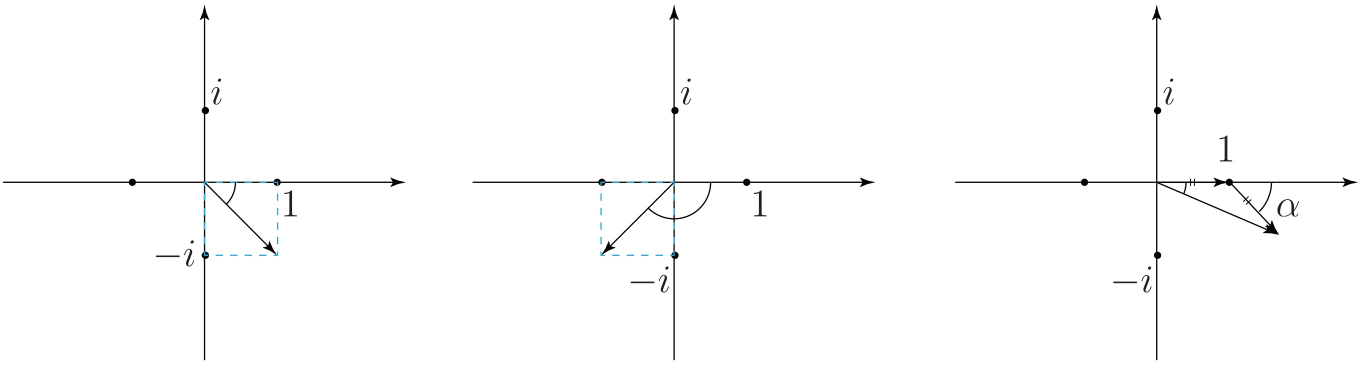

FIG. 2: Computation of arguments of complex numbers

We find $|z| = \sqrt{2}$

$$z = |z|(\cos\theta + i\sin\theta) = \sqrt{2}\left(-\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}\right) \tag{7}$$

and hence:

$$\cos\theta = -\frac{1}{\sqrt{2}}, \rightarrow \theta = \pm\frac{3\pi}{4}, \quad \sin\theta = -\frac{1}{\sqrt{2}} \Rightarrow \theta = -\frac{3\pi}{4} \tag{8}$$

So, $\arg z = -3\pi/4$, which can be also easily read off the Fig. 2(b)

$$z = \sqrt{2}\left(\cos\left(-\frac{3\pi}{4}\right) + i\sin\left(-\frac{3\pi}{4}\right)\right) \quad \text{trignometric form} \tag{9}$$

I. (VIDEO 1.6)

A. 1.6 Differentiation of a complex function. Cauchy-Riemann conditions

So far we dealt with complex algebra only. Now we will advance our knowledge and start discussing the infinitesimal properties of complex numbers.

First if all let us define a complex function. The function of complex variable is called a mapping f which for any point z of some domain assign some complex value. We can say that we define the function as a mapping between regions of two complex planes. The complex plane of z and the complex plane of f, values of our complex function.

Now we'd like to build differential analysis in the complex plane. Before we start though we need to define the notion of limit and convergence in the complex plane.

Fortunately, all these notions are carried over from real analysis with minimal modification.

For example,

The limit of sequence

$\{z_n\}$ is defined as the complex number a such that for any $\varepsilon > 0$ there exist number n_0 such that for any point inside a punctured circle of radius ε centered at a and for any $n > n_0$, $|z_n - a| < \varepsilon$.

In the same manner, the limit of the function is defined.

B. Cauchy-Riemann conditions

However we will face some difficulty when defining the derivative of the complex function.

Let us split the function into its real and imaginary parts.

$$f = u + iv$$

The question now is how to differentiate the function with respect to the change of complex variable z. Here is the issue. We have to define the change of the function as we shift the coordinate (x, y) in 2D. But, obviously, the change of the function will depend on a particular direction in which we move from point (x, y) to point $(x + \Delta x, y + \Delta y)$. Just imagine, say the, relief of the function $u(x, y)$. Then you get the geometric idea.

Let us write down the differential of the function f:

$$\Delta f = \Delta u + i\Delta v = \underbrace{\frac{\partial u}{\partial x}\Delta x}_{I} + \underbrace{\frac{\partial u}{\partial y}\Delta y}_{II} + i\Big(\underbrace{\frac{\partial v}{\partial x}\Delta x}_{III} + \underbrace{\frac{\partial v}{\partial y}\Delta y}_{IV}\Big)$$

What we'd like to introduce is a not a directional derivative but a different type of derivative, defined as the ratio of complex numbers.

Definition The function is called differentiable at point $z = z_0$ if exists a limit:

$$f'(z_0) = \frac{f(z) - f(z_0)}{z - z_0}\Big|_{z\to z_0} = \frac{\Delta f}{\Delta z}\Big|_{z\to 0} = A \tag{1}$$

which is called a derivative of a function at point z_0.

And we want this limit to be independent of the direction Δx, Δy. That means we need to require that Δf to be proportional $\Delta x + i\Delta y$.

Let us combine term I and IV and II and III. Let us look at the first combination. We have Δx and $i\Delta y$ with different factors. But if we require that

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \tag{2}$$

then these terms will be combined into the combination we seek for:

$$\Delta f_{I,iV} = \frac{\partial u}{\partial x}(\Delta x + i\Delta y). \tag{3}$$

The same goes for terms II and III if we require:

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}, \tag{4}$$

And we have:

$$\Delta f_{II,III} = i\frac{\partial v}{\partial x}(\Delta x + i\Delta y). \tag{5}$$

This way we obtain the following expression for the differential of the function:

$$\Delta f = \Big(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\Big)(\Delta x + i\Delta y). \tag{6}$$

And we get the desired limit which is independent of the direction Δz:

$$\frac{\Delta f}{\Delta z} = \Big(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\Big). \tag{7}$$

The conditions (2) and (4) are called Cauchy-Riemann's conditions. Cauchy-Riemann equations are necessary and sufficient conditions of the existance of the limit defining the derivative of the function at point z_0.

We have just proved the sufficiency. The necessity is even easier to prove. We just differentiated the function f in x and iy directions:

$$\frac{df}{dx} = \frac{\partial u}{\partial x} + i\frac{\partial v}{dy}, \quad \frac{df}{idy} = -i\frac{\partial u}{\partial y} + \frac{\partial v}{dy}. \tag{8}$$

This way, they establish a criteria of differentiability of the function at point $z = z_0$.

These are the first relations Riemann introduces in his doctorial thesis creating the field of Complex Analysis.

Definition

The function which is differential at every point of some region D is called analytic, or holomorphic (or regular) in this region.

C. The consequences of Cauchy-Riemann conditions.

1. The real and imaginary parts of a complex function are harmonic functions in a complex plane.

$$\Delta u = 0, \quad \Delta v = 0 \tag{9}$$

This is easily proven as follows:

$$\partial^2_{xx}u = \partial^2 v_{xy}, \quad \partial^2_{yy}u = -\partial^2 v_{yx}. \tag{10}$$

summing up these identities and recalling that for any smooth function $\partial^2_{xy}v = \partial^2_{yx}v$ we obtain the necessary equation. The same goes for v function.

2. The only critical points of the real and imaginary parts of a complex function are saddle points.

First of all, the point where $f'(z_0) = 0$ is called a critical point:

$$f'(z_0) = 0 \quad \textit{critical points}. \tag{11}$$

In this point $\partial_x u = 0, \;\; \partial_y u = 0$. (Indeed, since $\partial_x u = 0$ and $\partial_x v = 0$ then according to Caucy-Riemann conditions we have $\partial_x v = -\partial_y u = 0$)

Therefore, the critical point of the function of complex variable is simultaneously the critical point of u -function as a function of (x,y).

Now, this critical point may be of three possible types, defined by the main curvatures of the relief of the function u: maximum, minimum or saddle.

So you are always able to find orthogonal directions such that in rotated coordinate system: $x = x_0 + \alpha_{11}\xi + \alpha_{12}\eta, \;\; y = y_0 + \alpha_{21}\xi + \alpha_{22}\eta$ the behavior of the function in the vicinity of the critical point will be:

$$u(x + \Delta x, y + \Delta y) = u(x, y) + \lambda_1\xi^2 + \lambda_2\eta^2 \tag{12}$$

If both lambdas are positive we deal with minimum, if negative, we deal with maximum, and if they are of different sign, we have a saddle (maximum in one direction and minimum in the other).

The curvatures of the function is obtained from its second differential:

$$u(x + \Delta x, y + \Delta y) = u(x, y) + \frac{1}{2}u_{xx}\Delta x^2 + \frac{1}{2}u_{xy}\Delta x\Delta y + \frac{1}{2}u_{yy}\Delta y^2 \tag{13}$$

the second term is better to recast in the matrix form:

$$u(x+\Delta x, y+\Delta y) = u(x,y) + \frac{1}{2}(\Delta x, \Delta y)\begin{pmatrix} u_{xx} & u_{xy} \\ u_{xy} & u_{yy} \end{pmatrix}\begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix} \tag{14}$$

To see the type of the criticality we rotate the coordinate system so the matrix in new coordinates becomes diagonal. But we don't even need to do it. We know that the determinant of the matrix stays unchanged under the orthogonal transformation of the matrix.

We want to prove that the eigenvalues of this matrix have different signs. That means its determinant (which is the product of its eigenvalues) ought to be negative.

So let us compute the determinant:

$$u_{xx}u_{yy} - u_{xy}^2 = -v_{xy}^2 - u_{xy}^2 < 0 \tag{15}$$

In the second half we used Cauchy-Riemann conditions.

This simple fact, that a complex function has no maximums or minimums but only saddles has tremendous consequences for the whole field of complex analysis. It gives the name to significant part of asymptotic analysis called saddle-point approximations.

And it is all the consequence of differentiability of the function.

I. 1.7 APPLICATIONS OF CAUCHY-RIEMANN CONDITIONS

No let us consider some neat examples of application of Cauchy - Riemann conditions to get used to them and realize their importance.

Let us see if they do work on example of simplest functions.

Example 1 Let us check the function $f(z) = z^2$. We have:

$$f = u + iv = x^2 - y^2 + 2ixy \quad \Rightarrow u = x^2 - y^2, \quad v = 2xy; \tag{1}$$

As a result we obtain:

$$\frac{\partial u}{\partial x} = 2x, \quad \frac{\partial v}{\partial y} = 2x$$
$$\frac{\partial u}{\partial y} = -2y, \quad \frac{\partial v}{\partial x} = 2y$$

And we see, that Cauchy-Riemann conditions are satisfied.

In fact, Cauchy-Riemann conditions are so powerful that they allow to restore the full function from its real or imaginary parts (up to some additive constant).

Example 2

Given the function:

$$u(x, y) = x^3 + 6x^2y - 3xy^2 - 2y^3$$

The additional condition $f(0) = 0$. Find the function $f = u + iv$.

We start from the first pair of Cauchy-Riemannn conditions:

$$\frac{\partial u}{\partial x} = 3x^2 + 12xy - 3y^2 = \frac{\partial v}{\partial y}$$

Integrating we restore v up to an arbitrary function of x:

$$v = \int \frac{\partial u}{\partial x} dy + \psi(x) = \int (3x^2 + 12xy - 3y^2) dy + \psi(x) = 3x^2y + 6xy^2 - y^3 + \psi(x)$$

The second pair of Cauchy-Riemannn conditions yields:

$$-\frac{\partial v}{\partial x} = -6xy - 6y^2 - \psi'(x) = \frac{\partial u}{dy} = 6x^2 - 6xy - 6y^2 \;\Rightarrow\; \psi'(x) = -6x^2 \Rightarrow\; \psi(x) = -2x^3 + \text{const}.$$

Taking into account initial condition $f(0) = 0$ we conclude that const $= 0$ and obtain:

$$v = 3x^2y + 6xy^2 - y^3 - 2x^3 \tag{2}$$

Now we may restore f as the function of z using relations $x = (z + z^*)/2$ and $y = (z - z^*)/(2i)$:

$$v = \left(-1 - \frac{i}{2}\right) z^3 - \left(1 - \frac{i}{2}\right) (z^*)^3$$
$$u = \left(\frac{1}{2} - i\right) z^3 + \left(\frac{1}{2} + i\right) (z^*)^3$$

As a result:

$$f = u + iv = (1 - 2i)z^3. \tag{3}$$

Example 3

Given the modulus of the analytic function:

$$|f| = e^{r^2 \cos 2\varphi} \tag{4}$$

find the full function f.

First, we notice that the function is given in polar coordinates.

Second, we recall that Cauchy-Riemann conditions are not written for the modulus of the function, so we need to resolve this obstacle. The key consideration comes from the observation that if f is analytic then $w = \ln f$ is also analytic. Hence, if we decompose the function:

$$f = |f|e^{\mathrm{arg}} \ \Rightarrow w = \ln f = \ln |f| + i\mathrm{arg}\, f \tag{5}$$

Hence:

$$w = u + iv, \qquad u = \ln |f|, \quad v = \arg f. \tag{6}$$

And we may write down Cauchy-Riemann conditions for u and v. We switch to cartesian coordinates:

$$u = r^2 \cos 2\varphi = r^2(\cos^2 \varphi - \sin^2 \varphi) = x^2 - y^2. \tag{7}$$

And then we proceed along similar to Example 2 lines:

$$\frac{\partial u}{\partial x} = 2x = \frac{\partial v}{\partial y} \ \Rightarrow \ v = 2xy + \psi(x)$$

Next:

$$-\frac{\partial v}{\partial x} = -2y - \psi'(x) = \frac{\partial u}{dy} = -2y \ \Rightarrow \ \psi'(x) = 0 \ \Rightarrow \ \psi(x) = \mathrm{const}$$

Hence,

$$w = u + iv = x^2 - y^2 + 2xyi + i\mathrm{const} = z^2 + i\mathrm{const} \tag{8}$$

As a result our initial function:

$$f = e^w = e^{i\mathrm{const}} e^{z^2} \tag{9}$$

is restored up to some multiplicative constant.

I. VIDEO 1.8

II. 1.8 CONFORMAL MAPPINGS

The last chapter of this lecture is dedicated to one more application of analytic functions. Namely, the conformal mappings.

Let us consider some complex function $f(z)$ in some domain such that its derivative df/dz never vanishes in this domain. As we pointed out earlier complex function f can be considered as a mapping between complex domain of its argument to complex domain of its values. Therefore, let us draw two complex planes z and f.

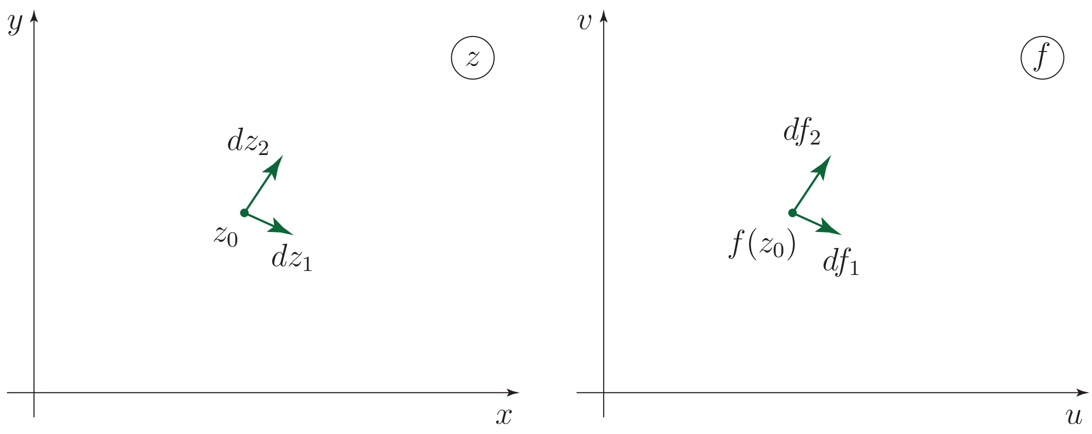

FIG. 1:

Then let us take some point z_0 and draw two infinitesimal vectors dz_1 and dz_2 from this point in complex plane z.

Obviously point z_0 is mapped onto $f(z_0)$ in the complex plane f. Analogously, points z_0+dz_1 and z_0+dz_2 are also mapped onto corresponding points in complex plane f: $f(z_0+dz_1)$ and $f(z_0+dz_2)$. These can be simplified using Taylor expansion of function f:

$$f(z_0+dz_1) = f(z_0) + \frac{df}{dz}\Big|_{z_0} dz_1,$$
$$f(z_0+dz_2) = f(z_0) + \frac{df}{dz}\Big|_{z_0} dz_2.$$

This way we obtain the image of vectors dz_1 and dz_2. Those are vectors

$$df_1 = \frac{df}{dz}\Big|_{z_0} dz_1,$$
$$df_2 = \frac{df}{dz}\Big|_{z_0} dz_2.$$

Now let us characterize the initial vectors by their moduli and its arguments:

$$dz_1 = |dz_1|e^{i\alpha_1}, \quad dz_2 = |dz_2|e^{i\alpha_2}$$

Then we introduce the same representation for the derivative:

$$\frac{df}{dz}\Big|_{z=z_0} = \left|\frac{df}{dz}\right|e^{i\gamma}$$

As a result the mapped vectors df_1 and df_2 read:

$$df_1 = \left|\frac{df}{dz}\right|e^{i(\gamma+\alpha_1)}|dz_1|, \quad df_2 = \left|\frac{df}{dz}\right|e^{i(\gamma+\alpha_2)}|dz_2|$$

This way we see, that the mapping by a holomorphic function turns the infinitesimal vectors at a particular point by the same angle γ given by the argument of the derivative of the function at this point. Therefore, the angle between the vectors is retained by the mapping, while the lengths of the vectors is multiplied by the same number given by the modulus of the derivative of the function.

Also, I leave it up to you to prove that an infinitesimal circle in complex plane z is turned into infinitesimal circle in complex plane f. The mapping which satisfies these two conditions is called conformal.

We may say that any holomorphic function with non-zero derivative in some domain implements a conformal mapping of this domain. The condition is, in fact, a little bit more strict, but it is irrelevant for our introductory discussion. What I'd like you to learn, however, is how to build the images of contours under conformal mapping. It is an essential skill in complex analysis because whenever you deal with an integral in a complex plane, and you make a change of a complex variable, you need to understand how your integration contour is transformed.

So, let us study a simple example.

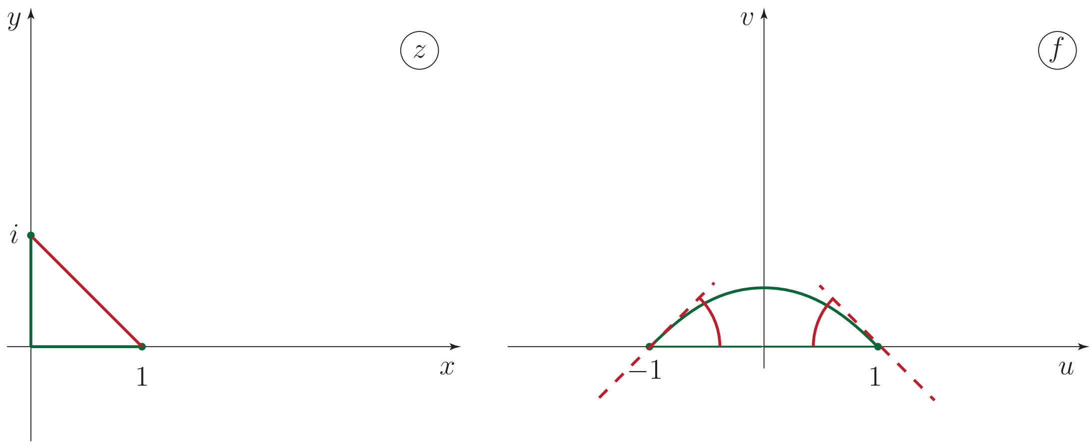

FIG. 2:

We consider the right triangle with apexes at the origin and points 1 and i. Let us see, how it is transformed by a conformal mapping $f = z^2$.

First of all, let us again redraw the complex plane z and the complex lane $w = u + iv$. Our function realises the mapping $u = x^2 - y^2$ and $v = 2xy$.

Then, let us parameterize the first side of our triangle $x = t,\ y = 0,\ t \in [0, 1]$ then, its image becomes $u = t^2, \ \ v = 0$, that is, a unit segment starting at the origin going to the right along u-axis.

The parametrization for the next side, $x = 0,\ y = t,\ $. We obtain $u = -t^2,\ v = 0$, that is the unit segment starting at the origin but going to the left along u-axis.

Finally, the segment connecting 1 and i. It is the part of a line with equation $y = 1 - x$. We parameterize $x = t$ and y as $1 - t$ to obtain:

$$u = x^2 - y^2 = (x - y)(x + y) = (2t - 1), \quad v = 2xy = 2t(1 - t) \tag{1}$$

To obtain the curve $v(u)$, we express t via $t = (1/2)(u + 1)$ and, as a result we get:

$$v = \frac{1}{2}(u + 1)(1 - u) = \frac{1}{2} - \frac{1}{2}u^2. \tag{2}$$

And we see, that we obtained a parabolic curve connecting the edge points -1 and 1. This is how the conforming map is done. Now if you compute the tangents of the parabola at $u = \pm 1$ you will see that they are equal to ± 1. That means the respective angles are $\pm\pi/4$ which precisely corresponds to the acute angles of our initial right triangle in z plane.

And indeed, our mapping preserves the angles between lines as it should. However, the $\pi/2$ angle is destroyed, it is turned into π. Try to understand why this happened.

The last issue I'd like to touch in this week is the integration along the curve in the complex plane. Certainly if we can define the sum of complex numbers, the limit and convergence, then, obviously, we can define the notion of the integral.

Suppose we have some contour starting at point z_1 and ending at z_2 in a complex plane and suppose we have some function $f(z)$.

Then we can split the contour into small linear segments Δz_i and compose the sum:

$$I = \sum_i f(z_i)\Delta z_i \tag{3}$$

Then shrinking the step of our partition we obtain a well defined limit which is called an integral of a complex function along the contour:

$$I = \int\limits_{z_0}^{z_1} f(z)dz. \tag{4}$$

The integral can be split into two real 2D curve integrals in a natural way. If $f = u + iv$ and $dz = dx + idy$ than

$$I = \int\limits_{\gamma} (udx - vdy) + i \int\limits_{\gamma} (udy + vdx). \tag{5}$$

For example, we may consider an integral along the upper right quarter circle of the function $f(z) = z$. The integral can be rewritten as

$$I = \int\limits_{\gamma} (xdx - ydy) + i \int\limits_{\gamma} (xdy + ydx) = \frac{1}{2} \int\limits_{\gamma} d(x^2 - y^2) + i \int\limits_{\gamma} d(xy) = \frac{1}{2} \int\limits_{\gamma} d(x^2 - y^2 + 2ixy) \tag{6}$$

$$= \frac{1}{2} \int\limits_{\gamma} dz^2 = \frac{1}{2} z_1^2 - \frac{1}{2} z_0^2 = -\frac{1}{2} - \frac{1}{2} = -1 \tag{7}$$

As we see, the two dimensional integral can be reduced to a simple antiderivative but taken in a complex plane. We understand that the value of the integral doesn't depend here on the integration path, but on the position of the beginning and end points of the contour.

There is a fundamental reason for this in complex analysis, it is hidden in the analyticity of the omplex function and we will essentially dedicate the rest of course to the exploration of this unique property. But upto now I think that is it for this week.