Derive the Einstein field eqs. from the Hilbert action

1. Start with the Hilbert action

$$S = \int dx^4 \left(\kappa \sqrt{-g}\, R + \sqrt{-g}\, \mathcal{L}_m \right) \qquad \kappa = \frac{c^4}{16\pi G}\left(\frac{kg\, m}{s^2}\right) \quad R\left(\frac{1}{m^2}\right) \quad \text{Energy Density}\left(\frac{kg}{m s^2}\right)$$

2. Vary w/respect to the metric tensor $g_{\mu\nu}$

$$\delta S = \int dx^4 \left(\kappa \frac{\delta(\sqrt{-g}\, R)}{\delta g^{\mu\nu}} + \frac{\delta(\sqrt{-g}\, \mathcal{L}_m)}{\delta g^{\mu\nu}} \right) \delta g^{\mu\nu}$$

3. Write the product rule for 1^{st} term, pulling back a $\sqrt{-g}$

$$\delta S = \int \sqrt{-g}\, dx^4 \left(\kappa \left(\frac{\delta R}{\delta g^{\mu\nu}} + \frac{R}{\sqrt{-g}} \frac{\delta \sqrt{-g}}{\delta g^{\mu\nu}} \right) + \frac{1}{\sqrt{-g}} \frac{\delta(\sqrt{-g}\, \mathcal{L}_m)}{\delta g^{\mu\nu}} \right)$$

4. Examine the 1^{st} term in 3:

$$\frac{\delta R}{\delta g^{\mu\nu}} = \frac{\delta g^{\mu\nu} R_{\mu\nu}}{\delta g^{\mu\nu}} = R_{\mu\nu} \overset{1}{\cancel{\frac{\delta g^{\mu\nu}}{\delta g^{\mu\nu}}}} + \cancel{g^{\mu\nu} \frac{\delta R_{\mu\nu}}{\delta g^{\mu\nu}}}$$

makes no contribution
A total derivative
Non-trivial to show

5. Examine the 2^{nd} term in 3:

$$\frac{R}{\sqrt{-g}} \frac{\delta \sqrt{-g}}{\delta g^{\mu\nu}} = -\frac{1}{2} \frac{R}{\cancel{\sqrt{-g}}} \frac{1}{\cancel{\sqrt{-g}}} (-1) \cancel{g}\, g^{\mu\nu} \frac{\delta g_{\mu\nu}}{\delta g^{\mu\nu}} = -\frac{1}{2} g_{\mu\nu} R$$

6. Examine the 3^{rd} term in 3:

$$\frac{1}{\sqrt{-g}} \frac{\delta(\sqrt{-g}\, \mathcal{L}_m)}{\delta g^{\mu\nu}} \equiv -\frac{1}{2} T_{\mu\nu} \quad \text{the stress-Energy tensor}$$

7. Variation is an extremum if integrand $= \emptyset$

$$\frac{c^4}{16\pi G} \left(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R - \frac{1}{2} T_{\mu\nu} \right) = 0$$

$$\boxed{R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{8\pi G}{c^4} T_{\mu\nu}}$$