Quaternion Quantum Field Theory Demystified: The Method Applied

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Quaternion quantum field theory is introduced. The goal is for every equation that plays a role in quantum field theory gets rewritten using real-valued quaternions. Like the correspondence principle before it, the method is simple and systematic: keep 4-vectors together, drop factors of i, keep the constants, but make the expression dimensionless. The differences between classical, relativistic and quantum mechanics equations are based on their constants and form. The uncertainty principles for position/momentum, and energy/time appear in the same expression, a result of the product rule of calculus. The method will shun the most famous equation in physics, $E = m c^2$, because momentum is omitted. The square of energy-momentum will be used in its place. Substitution in that equation leads directly to the Klein-Gordon equation.

The path to the Dirac equation is more complicated. One needs to know that the 16 gamma matrices can be represented by quaternion triple products. Pre- and post-multiply a quaternion by each combination of the four basis vectors accomplishes the feat. Particular sets of quaternion gamma operators and choices of inertial reference frames can lead to wave functions whose scalar is either positive or negative definite.

A way to visualize quantum field theory using analytical animations is begun. The simplest animation is for an inertial observer. The same number gets added iteratively. The classical view is what one would expect: a ball moving in a straight line at a steady pace. The quantum view show the ball following the same path on average, but the next step cannot be known. Applying the quaternion gamma matrices to inertial path creates 16 possible histories. There is a huge amount of work ahead for the visualization project, but it will look interesting and can be shared with a far greater audience than quaternion quantum field theory equations ever will.

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1. The Method

The book "Quantum Field Theory Demystified" by David McMahon provides a survey of the core ideas and techniques of the subject. This work will present a collection of quaternion variations on chapters from that book. As much as possible, every equation will be written as a quaternion function. This will assure expressions are always consistent with special relativity. When time is always placed next to space as it should be, equations take on a different look. There is an on-going tension because physicists deal with energy separately from momentum, even though both live in the same 4-vector. The hallmark of Nature is her consistency, so we will use that as a guiding light.

To be more precise, the following four rules will be used to transform from standard QFT to QQFT, quaternion quantum field theory:

1 - Keep 4-vectors together. Examples include (t, $\vec{R}$/c), (E, $\vec{P}$ c) and $(\frac{1}{c}\frac{\partial}{\partial t}, \vec{\nabla})$.

2 - Drop all factors of i. Quaternions have 3 imaginary numbers already.

3 - Keep all constants, such as c and $\hbar$.

4 - Make equations dimensionless via Planck units: $t/\sqrt{G\hbar/c^5}$, $\vec{R}/\sqrt{G\hbar/c^3}$, $E/\sqrt{\hbar c^5/G}$, $\vec{P}\Big/\sqrt{\hbar c^3/G}$, $m\Big/\sqrt{\hbar c/G}$, $\sqrt{G\hbar/c^5}\ \frac{\partial}{\partial t}$, and $\sqrt{G\hbar/c^3}\ \vec{\nabla}$.

In an effort to be as consistent as Nature, "the method" will always be followed.

2. Introduction to QQFT, Quaternion Quantum Field Theory

"*Quantum field theory* is a theoretical framework that combines *quantum mechanics* with *special relativity*."

So begins David McMahon's book right on target.

"Generally speaking, quantum mechanics is a theory that describes the behavior of small systems, such as atoms and individual electrons. Special relativity is the study of high energy physics, that is, the motion of particles and systems at velocities near the speed of light (but without gravity)."

The description is both true and useless to readers of any book filled with only words and equations. Which equation applies to the small? Which to the quick? One would prefer a description that applies to equations.

The first and fastest method is to look for constants. If the speed of light c is in the expression, special relativity is relevant. If there is Planck's constant $\hbar$, then quantum mechanics is in play. If Newton's gravitational constant G is on the page, then gravity is involved. The Schrödinger equation has just the Planck's constant so cannot be the subject of quantum field theory. Both the Dirac and Klein-Gordon equations have $\hbar$ and c and thus are part of quantum field theory.

When written with quaternion operators, the structure of the equations also reveal their subject matter. Consider Newton's second law:

Newton's second law written with quaternion operators

$$\left(0, \vec{F}\right) = \left(\sqrt{\frac{G\hbar}{c^5}}\, \frac{\partial}{\partial t}, \vec{0}\right) \left(\sqrt{\frac{G}{\hbar c}}\, m, \vec{0}\right) \left(0, \frac{1}{c} \frac{\partial \vec{R}}{\partial t}\right) = \left(0, \frac{G}{c^4} \left(m \frac{\partial^2 \vec{R}}{\partial t^2} + \frac{\partial \vec{R}}{\partial t} \frac{\partial m}{\partial t}\right)\right) \tag{1}$$

The presence of the constants in the operators (the zeros) means the equation cannot adhere to the rules of special relativity. The factors of c make the force dimensionless, but are not involved with mixing a scalar and 3-vector, the sign of special relativity at work. Newton's law is exceptionally useful. The claim here is that it is easy to spot an equation that is not consistent with special relativity.

Using the Plank units make the everyday experience quite tiny. For example, the gravitational acceleration on the Earth is:

$$g \sqrt{\frac{G\hbar}{c^7}} = 9.8 \frac{m}{s^2} \sqrt{\frac{6.67 \text{x} 10^{-11} \frac{m^3}{kg\, s^2}\, 6.62 \text{x} 10^{-34} \frac{kg\, m^2}{s}}{(299\,792\,458\, m/s)^7}} = 4.42 * 10^{-51} \tag{2}$$

The trivial size of gravity on Earth is consistent with our trivial role in the entire Universe.

Newton's second law produces three imaginary numbers. Many other equations in physics deliver a scalar or real number. This indicates a classical equation. When one has an equation with both the real and imaginary numbers, the equation is in the domain of quantum mechanics.

Newton's second law is classical by all three criteria: the units, the constants in the operators, and the pure imaginary result.

It was special relativity that put the 4-vectors together. It was quantum mechanics that made use of the complex values. It is quantum field theory that uses both: complete 4-vectors without constants like zero or one, and the results are complex values.

The uncertainty principle for position/momentum and energy/time - common as they are - hide two accounting issues. One should not think of the variation in position separately from the variation in time, nor the variation in energy separately from that of momentum. Someone in a rocket ship will say your change in energy is his change in momentum. If one went ahead and put these in the proper 4-vector, there would not be a way to form a product. With quaternions, one must put time next to space, energy next to momentum, and the product is well-defined:

Quaternion uncertainty

$$\left(\Delta t \Big/ \sqrt{\frac{G\hbar}{c^5}}, \Delta \vec{R} \Big/ \sqrt{\frac{G\hbar}{c^3}}\right) \left(\Delta E \Big/ \sqrt{\hbar c^5 / G}, \Delta \vec{P} \Big/ \sqrt{\hbar c^3 / G}\right) \tag{3}$$

Quaternion uncertainty

$$= \left(\Delta E\, \Delta t - \Delta \vec{P} \cdot \Delta \vec{R},\ c\, \Delta t \Delta \vec{P} + \Delta \vec{R}\, \Delta E / c + \Delta \vec{R} \times \Delta \vec{P}\right) / \hbar$$

Only the hbar appears in the scalar term seen in the normalization of plane waves, Newton's gravitational constant politely bowing out. The exponentials that appear so often in field theory do not make sense unless the quantity in question is dimensionless.

Quaternions contain an i, j, and k, so we do not need to add these to our expressions. This will be quite odd, since the factor of i is repeated almost as much as hbar. Pruning in good.

Commutator relations that are vital to quantum mechanics can be easily justified. Consider the position and momentum operators, symbolized by putting a hat on X or P. Operators must act on a wave function to have any meaning, but this accounting detail is often skipped. We will not do that to keep the rules consistent. Another problem with such notation is that it hides the relationship between the two operators. In the position representation, the position operator X will just be the position x, but momentum will be the operator hbar times the derivative with respect to x, $P = \hbar \frac{\partial}{\partial x}$. Calculate the commutator of the momentum and position operators action on the wave function:

Uncertainty from product rule

$$\sqrt{\frac{c^3 \hbar}{G}} \sqrt{\frac{G \hbar}{c^3}} \left(x P_x \psi - P_x x \psi\right) = \hbar \left(x \frac{\partial \psi}{\partial x} - \frac{\partial x \psi}{\partial x}\right) = -\hbar \psi \tag{4}$$

This may be the only expression in this work that has units in order to line up with the definition of the uncertainty principle.

If one uses a different combination of operators, say the position y with momentum in the x direction, that can and should always disappear:

No uncertainty

$$\sqrt{\frac{c^3 \hbar}{G}} \sqrt{\frac{G \hbar}{c^3}} \left(P_x y \psi - y P_x \psi\right) = \hbar \left(\frac{\partial y \psi}{\partial x} - y \frac{\partial \psi}{\partial x}\right) = 0 \tag{5}$$

The uncertainty principle is a consequence of the product rule of calculus, nothing more or less. When one says two variables are conjugates, it means there is some kind of link via calculus between the two: position and momentum in the x direction are linked whether one is using a position or momentum representation, but not position along y and momentum along x operators.

The most famous equation in all of physics as far as the popular culture is concerned - $E = m c^2$ - is not valid (true for all). Energy transforms like the first component of a 4-vector, while mass is invariant under a Lorentz transformation and the speed of light is a constant. The equation is true for one and only one observer, where the relativistic velocity is zero. And as is our practice, we will not write an equation with the energy without also including the momentum. Here is a dimensionless expression written with energy, momentum and mass that is true for all inertial observers:

Energy and mass definition

$$\left(E \Big/ \sqrt{\hbar c^5 / G}\, ,\ \vec{P} \Big/ \sqrt{\hbar c^3 / G}\right) \tag{6}$$

Energy and mass definition

$$= \sqrt{\frac{G}{\hbar c}}\, m \left(\cosh\left(\operatorname{arctanh}\left(\frac{|V|}{c}\right)\right), \sinh\left(\operatorname{arctanh}\left(\frac{|V|}{c}\right)\right) \frac{\vec{V}}{c}\right);$$

Write γ and $\gamma\vec{\beta}$ as hyperbolic functions because when squared, the link between the Lorentz covariant energy and momentum with the invariant mass is due to this hyperbolic trig identity:

Hyperbolic trig identity yields Lorentz invariance

$$\cosh^2(\alpha) - \sinh^2(\alpha) = 1 \tag{7}$$

As a sanity check, do a few simple number games to confirm the relationship between hyperbolic functions and gamma/-beta notation:

Check hyperbolic functions versus gamma and gamma beta

$$\begin{aligned} \cosh(\operatorname{arctanh}(.5)) &= \frac{1}{\sqrt{1 - .5^2}} = 1.1547 \\ \sinh(\operatorname{arctanh}(.5)) &= \frac{.5}{\sqrt{1 - .5^2}} = 0.57735 \end{aligned} \tag{8}$$

There is an advantage to the gammas and betas when you think about this inequality:

gamma is greater than gamma beta

$$\gamma > \left|\gamma \vec{\beta}\right| \tag{9}$$

This means that energy is always greater than the magnitude of the momentum.

What is more important than the definition of the 4-momentum is the square of the 4-momentum:

Valid relationship between energy momentum and mass

$$\frac{G}{\hbar c^5}\left(E, \vec{P} c\right)^2 = \frac{G}{\hbar c^5}\left(E^2 - P^2 c^2, 2 c E \vec{P}\right) = \frac{G}{\hbar c} m^2 \left(1, 2 \gamma^2 \vec{\beta}\right) \tag{10}$$

This equation is both true in any reference frame and *complete*. It is a perfectly fine operation to multiply energy times momentum, yet no graduate-level books I have paged through use this information. It is this equation that will be the basis for formulating quaternion quantum field theory.

The Schrödinger equation is written like so:

Schrodinger wave equation

$$\frac{-\hbar^2}{2 m} \frac{\partial^2 \psi}{\partial x^2} + V\psi = i\hbar \frac{\partial \psi}{\partial t} \tag{11}$$

According to the rules of this method, the changes in time and changes in space should be kept together, the factor of *i* should be dropped, and the equation should be dimensionless. Here is the rewrite:

Schrodinger rearranged

$$\sqrt{\frac{G}{\hbar c^5}} \left(\hbar \frac{\partial}{\partial t} + \frac{\hbar^2}{2 m} \nabla^2\right) \psi = \sqrt{\frac{G}{\hbar c^5}} V \psi \tag{12}$$

Note this is a scalar operator acting on a quaternion-valued wave function. Consider what is needed to generate the Laplacian:

Laplacian and curl of a curl

$$\left(0, \vec{\nabla}\right) \left(0, \vec{\nabla}\right)^* \psi = \left(\nabla^2, -\vec{\nabla} \times \vec{\nabla}\right) \psi \tag{13}$$

This has the Laplacian, and a curl of a Del. One recurring theme is that physics often ignores what goes on in the vector part of expressions. Another way to say it is that vector parts often find their utility via the scalar. Since the operator is a scalar, the time derivative must end up there too. While it is easy enough to add a time derivative to one of the terms, the other gets the constant 1 which has different units from the 1/length provided by the spacial derivative. To make those units work out requires an hbar/mc because the units for Planck's constant are mass length^2/time:

Quaternion Schrodinger equation

$$\sqrt{\frac{G}{\hbar c^5}} \left(\left(\hbar \frac{\partial}{\partial t}, \hbar c \vec{\nabla}\right) \left(1, \frac{\hbar}{2 m c} \vec{\nabla}\right)^* + \left(\left(\hbar \frac{\partial}{\partial t}, \hbar c \vec{\nabla}\right) \left(1, \frac{\hbar}{2 m c} \vec{\nabla}\right)^*\right)^*\right) \psi \tag{14}$$

Quaternion Schrodinger equation

$$= \sqrt{\frac{G}{\hbar c^5}} \left(\hbar \frac{\partial}{\partial t} + \frac{\hbar^2}{2 m} \nabla^2, \vec{0}\right) \psi = \sqrt{\frac{G}{\hbar c^5}} \left(V, \vec{0}\right) \psi$$

The product of a 4-derivative with a dimensionless spacial derivative generates the Schrödinger equation. What is interesting about the units is that only hbar is needed next to the operators, not the speed of light or the gravitational constant.

Substitution in the square of energy - momentum leads to the form of the Klein - Gordon equation :

Quaternion Klein-Gordon

$$\frac{G}{\hbar c^5}\left(E, \vec{P} c\right)^2 : E \to \hbar \frac{\partial}{\partial t}, \vec{P} \to \hbar c \vec{\nabla} \Longrightarrow \frac{G \hbar}{c^5} \left(\frac{\partial^2}{\partial t^2} - c^2 \nabla^2, 2 c \frac{\partial}{\partial t} \vec{\nabla}\right) \psi = \frac{G}{\hbar c} m^2 \left(1, 2 \beta \gamma^2\right) \psi \tag{15}$$

Bring all the constants over to the left hand side of the equation:

Quaternion Klein-Gordon

$$\left(\frac{\hbar}{m c^2}\right)^2 \left(\frac{\partial^2}{\partial t^2} - c^2 \nabla^2, 2 c \frac{\partial}{\partial t} \vec{\nabla}\right) \psi = \left(1, 2 \beta \gamma^2\right) \psi \tag{16}$$

As we have come to expect, the phase is not used at this time, but it is well-formed.

The Dirac equation is sometimes described by the square root of the Klein-Gordon equation because squaring the Dirac equation results in the Klein-Gordon equation. This feat is accomplished by using the 16 gamma matrices (more information can be found here: http://mathworld.wolfram.com/DiracMatrices.html). Work by others (J. Löpez-Bonilla, L. Rosales-Roldàn) has show that these 16 gamma matrices can be written as quaternion *triple* products with a simple form: pre- and post-multiply a quaternion by one of the four basis vectors. Write the Dirac equation with the time and space operators on the same side, dropping the factor of i:

Dirac, rearranged

$$\hbar \left(\frac{1}{c} \frac{\partial}{\partial t} - \vec{\alpha} \cdot \vec{\nabla}, \vec{0}\right) \psi = \beta \, m \, c^2 \, \psi \tag{17}$$

If we hope to treat the time derivative as we do the space derivative, then it only makes sense to have a gamma matrix act on the time derivative, not the invariant mass. The goal now looks something more like this:

Dirac, rearranged beta

$$\frac{\hbar}{m \, c^2} \left(\beta \frac{\partial}{\partial t} - c \, \vec{\alpha} \cdot \vec{\nabla}, \vec{0}\right) \psi = \psi \tag{18}$$

If a gamma matrix is implemented as a quaternion triple product, how can we write this as quaternion expression? The Del belongs in the vector. Forget about the alpha and beta matrices for a moment:

Dirac, rearranged without alpha or beta

$$\frac{\hbar}{m \, c^2} \left(\frac{\partial}{\partial t}, c \, \vec{\nabla}\right) \psi = \psi \tag{19}$$

The terms in this equation can be hit on both sides with a basis vector to do the work of gamma matrices. The test is to see if the square of the triple product gives exactly the same scalar operator as generated by the Klein-Gordon equation. It should not be a surprise if hitting both sides of the 4-derivative with the identity element, one gets exactly the Klein-Gordon equation, even the phase:

Quaternion Dirac equation squared

$$\left(\frac{\hbar}{m \, c^2} \left(1 \left(\frac{\partial}{\partial t}, c \, \vec{\nabla}\right) 1\right)\right)^2 \psi = \left(\frac{\hbar}{m \, c^2}\right)^2 \left(\frac{\partial^2}{\partial t^2} - c^2 \, \nabla^2, 2 \, c \, \frac{\partial}{\partial t} \vec{\nabla}\right) \psi = \left(1, 2 \, \beta \, \gamma^2\right) \psi \tag{20}$$

There are three other triples that generate the same scalar operator, but have distinct the 3-vectors:

Quaternion Dirac equations squared with different phases

$$\left(\frac{\hbar}{m \, c^2} \left(i \left(\frac{\partial}{\partial t}, c \, \vec{\nabla}\right) i\right)\right)^2 \psi = \left(\frac{\hbar}{m \, c^2}\right)^2 \left(\frac{\partial^2}{\partial t^2} - c^2 \, \nabla^2, 2 \, c \, \frac{\partial}{\partial t} \frac{\partial}{\partial x}, -2 \, c \, \frac{\partial}{\partial t} \frac{\partial}{\partial y}, -2 \, c \, \frac{\partial}{\partial t} \frac{\partial}{\partial z}\right) \tag{21}$$

Quaternion Dirac equations squared with different phases

$$\left(\frac{\hbar}{m \, c^2} \left(j \left(\frac{\partial}{\partial t}, c \, \vec{\nabla}\right) j\right)\right)^2 \psi = \left(\frac{\hbar}{m \, c^2}\right)^2 \left(\frac{\partial^2}{\partial t^2} - c^2 \, \nabla^2, -2 \, c \, \frac{\partial}{\partial t} \frac{\partial}{\partial x}, 2 \, c \, \frac{\partial}{\partial t} \frac{\partial}{\partial y}, -2 \, c \, \frac{\partial}{\partial t} \frac{\partial}{\partial z}\right)$$

Quaternion Dirac equations squared with different phases

$$\left(\frac{\hbar}{m \, c^2} \left(k \left(\frac{\partial}{\partial t}, c \, \vec{\nabla}\right) k\right)\right)^2 \psi = \left(\frac{\hbar}{m \, c^2}\right)^2 \left(\frac{\partial^2}{\partial t^2} - c^2 \, \nabla^2, -2 \, c^2 \, \frac{\partial}{\partial t} \frac{\partial}{\partial x}, -2 \, c \, \frac{\partial}{\partial t} \frac{\partial}{\partial y}, 2 \, c \, \frac{\partial}{\partial t} \frac{\partial}{\partial z}\right)$$

Notice that if you were to sum all four of these gamma actions together, the phase would add up to zero.

Imagine one is in a reference frame where the 3-vector part of a static wave function is zero. This kind of thing happens if one is in the rest frame of a static charge. All the changes are due to spatial derivatives.

only spacial derivatives

$$\frac{\hbar}{m \, c^2} \left(\frac{\partial}{\partial t}, c \, \vec{\nabla}\right) \left(\psi[R], \vec{0}\right) = \frac{\hbar}{m \, c^2} \left(0, c \, \vec{\nabla} \, \psi[R]\right) \tag{22}$$

Square this to get a scalar invariant, noting that we have chosen a reference frame where the relativistic velocity is zero:

only spacial derivatives squared is negative definite

$$\left(\frac{\hbar}{m \, c^2} \left(0, c \, \vec{\nabla} \, \psi[R]\right)\right)^2 = \left(\frac{\hbar}{m \, c^2}\right)^2 \left(-(c \, \nabla \psi[R])^2, \vec{0}\right) = \left(1, \vec{0}\right) \left(\psi[R], \vec{0}\right) \tag{23}$$

This creates an accounting issue. The left hand side is necessarily negative no matter what the derivative of the wave function happens to be. The right hand will be negative if and only if the scalar wave function is negative. A positive definite operator must be found to complement this negative definite.

One of the guiding principles of this effort is that Nature uses every math tool available to cover every situation possible. So far, only four of sixteen quaternion gamma operators have been used. Use one of the neglected quaternion gamma operator on the differential operator:

gamma operator jumble

$$\frac{\hbar}{m \, c^2} \, i \, \nabla \, j \, \psi = \frac{\hbar}{m \, c^2} \left(c \, \frac{\partial}{\partial z}, c \, \frac{\partial}{\partial y}, -c \, \frac{\partial}{\partial x}, \frac{\partial}{\partial t}\right) \psi \tag{24}$$

This time chose the reference frame such that the derivative of the wave function with respect to z is the only non-zero term. Why this odd choice? It is a compliment to the previous selection that effectively set the scalar derivative to zero while the 3-vector derivative was filled out. This is the way to make the scalar derivative non-zero, but the 3-derivative is zero.

Only a scalar derivative

$$\frac{\hbar}{m\,c^2}\left(\left(i\left(\frac{\partial}{\partial t}, c\,\vec{\nabla}\right) j + \left(i\left(\frac{\partial}{\partial t}, c\,\vec{\nabla}\right) j\right)^*\right)\right)\psi/2 = \frac{\hbar}{m\,c^2}\left(c\,\frac{\partial\,\psi}{\partial z}, \vec{0}\right) \tag{25}$$

Square this:

Only a scalar derivative squared is positive definite

$$\left(\frac{\hbar}{m\,c^2}\left(\left(i\left(\frac{\partial}{\partial t}, c\,\vec{\nabla}\right) j + \left(i\left(\frac{\partial}{\partial t}, c\,\vec{\nabla}\right) j\right)^*\right)\right)\psi/2\right)^2 = \left(\frac{\hbar}{m\,c^2}\right)^2\left(\left(c\,\frac{\partial\psi}{\partial z}\right)^2, \vec{0}\right) = \left(1, \vec{0}\right)\left(\psi[R], \vec{0}\right) \tag{26}$$

The left hand side this time is necessarily positive. The right hand side will be positive if and only if the scalar wave function is positive.

This is not the only way to get a necessarily positive square spatial derivative. One could do a gamma operation on the wave function:

Gamma wave function jumble

$$i\,\psi\,j = (\psi z, -\psi y, -\psi x, \psi E) \tag{27}$$

Take the derivative of this mixed up wave function. As before, pick a reference frame such that the 3-derivative is zero:

The negative scalar derivative

$$\frac{\hbar}{m\,c^2}\left(\left(\frac{\partial}{\partial t}, c\,\vec{\nabla}\right) i\,\psi\,j + \left(\left(\frac{\partial}{\partial t}, c\,\vec{\nabla}\right) i\,\psi\,j\right)^*\right)\Big/\,2 = \frac{\hbar}{m\,c^2}\left(-c\,\frac{\partial\,\psi E}{\partial z}, \vec{0}\right) \tag{28}$$

Although the sign of the derivative is now negative, that is a result of the order of the *i* and *j* (reversing the order reverses the sign). Square this relation:

The negative scalar derivative squared is postive definite

$$\left(\frac{\hbar}{m\,c^2}\left(\left(\frac{\partial}{\partial t}, c\,\vec{\nabla}\right) i\,\psi\,j + \left(\left(\frac{\partial}{\partial t}, c\,\vec{\nabla}\right) i\,\psi\,j\right)^*\right)\Big/\,2\right)^2 = \left(\frac{\hbar}{m\,c^2}\right)^2\left(\left(c\,\frac{\partial\psi}{\partial z}\right)^2, \vec{0}\right) = \left(1, \vec{0}\right)\left(\psi[R], \vec{0}\right) \tag{29}$$

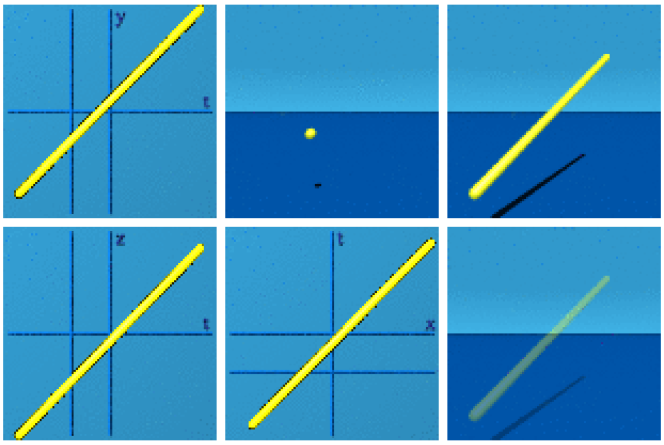
This will be true if and only if the wave function is positive.

To recap the Dirac equation results, the negative definite come from choosing a reference frame so the scalar wave function is zero. Take the spatial 3-derivative, square that, and the result is negative definite. The positive definite comes from choosing a reference frame where the 3-vector derivatives are zero. Square a scalar and the result is positive definite.

These observations lead to speculations about particles and their antiparticles. A Feynman diagram can be interpreted as a particle traveling forward in time, or its antiparticle traveling backward in spacetime. It is required that we sum over all possible histories. To get complete coverage of the wave function from minus values to positive ones, we need both the necessarily positive and negative squares of spatial derivatives of the wave function. If the necessarily negative square expression is multiplied by -1, then one gets the same expression as necessarily positive one for the scalar. Note that the phases of the two will be distinct. While there remains a huge amount of work to solidify this speculation, let it be a motivator for the work to come.

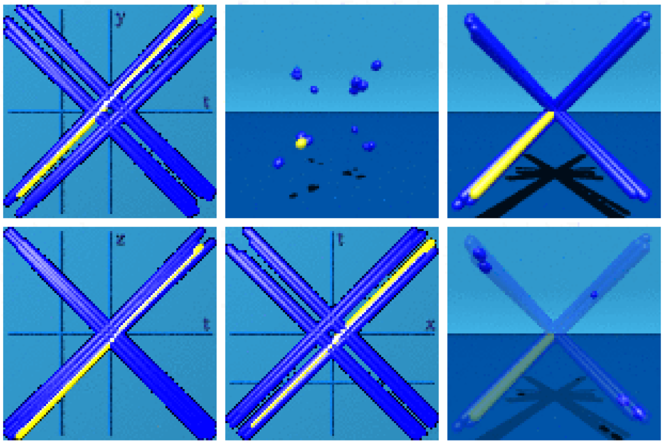
What is the root of the difference between classical and relativistic quantum field theory? Both are second-order differential equations. It is the Schrödinger equation that pins down time with its dimensionless spacial derivative. A single particle can obey the odd rules of quantum mechanics.

A consequence of freeing up changes in time is that we are obligated to think of many particles at once. I have embraced this idea because I have a visualization of what the gamma matrices do to a classical particle. Software was written by the author to create animations of an arbitrary collection of quaternions (http://visualphysics.org/content/downloads). Consider the following animation of an inertial particle moving along a path through spacetime:



The animation is created by starting at a point in spacetime, then adding the same amount of change in spacetime a thousand times. The path is inertial because the amount of change is constant. At first glance, you might think that this animation is completely classical. The upper middle panel is the animation of the classical particle. It is the lower right panel that is the one for classical quantum mechanics. The upper right is the superposition of all states achieved by both the classical and quantum mechanical systems. We know for certain that the particle will be somewhere on that line, but we cannot know until we look, where the particle will be. On average, the energy and momentum of the classical versus quantum systems (the top-center versus lower-right) will be the same.

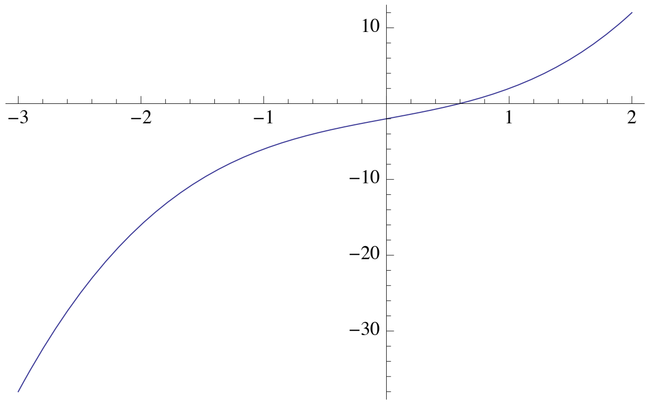
To see the effects of the gamma matrix, take each of the events in the linear animation and multiply them by the 16 possible basis vector triple products. The result is sixteen lines:



If the gamma matrices are doing something like this algebraically, then relativistic quantum mechanics is about multiple particles in different places in spacetime, all trying to tell the story of how something happened. By including all these stories, correct average behavior is predicted.

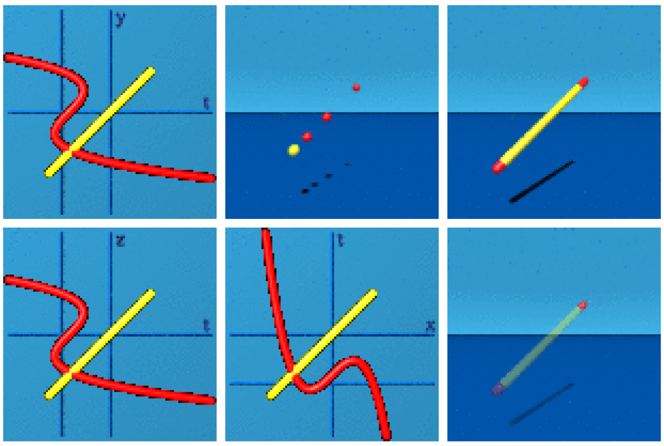
With this animation in mind, the need to use space and time as parameters becomes clear. In classical physics, a particle exists at a place and time within the limits of the uncertainty principle. In relativistic quantum mechanics, not only is there the uncertainty principle, but there are multiple particles to describe. The wave function becomes a field whose job involves the creation and annihilation of particles throughout a volume of spacetime.

Creation and annihilation? Surely physics must be joking Mr. Feynman! Fortunately, this visual interpretation of relativistic quantum mechanics provides an answer. Consider a simple polynomial, $q^3 + 3\,q - 2$. *Mathematica* knows how to plot this:



The lesson learned from the Schrödinger equation was that time was fixed at second order. An analogous can be said about this graph: come back in a second or a hundred years, and it will look the same.

What happens if we animate this polynomial, using a collection of events that move from (-3, -3, -3, -3) in spacetime to (2, 2, 2, 2)?



The inertial input for the polynomial is in yellow. The output of the polynomial is in red. There is initially one red dot on the screen. Then two red dots are created for a total of three dots. They move along their separate ways until two collide and annihilate each other. The animation ends with one particle. You are probably focusing on the classical animation, the top middle, but on average, the same thing is happening in the lower right. What you see in the animation depends on when you look. The number of red particles does change.

How can we deal with this situation? We will want to know, on average, how much energy and momentum a particle has, and on average, how many particles there are in a system. If those two values can be calculated, then we will have a useful description of a system that is dynamic to second order in spacetime.