The Predicted Precession of the Perihelion of a Planet Based on a Solution to GEM's Field Equations

In a Yahoo discussion group, a question was raised as to whether the GEM unified field proposal was consistent with the precession of the perihelion of Mercury results. My stock reply is that the coefficients of the GEM metric are identical to those of the Schwarzschild metric only to first order PPN accuracy, the level used in the calculations, ergo the results must be identical. I like a short, solid reply.

I also like a long-winded one, because it shows all the nuts and bolts. I have read in many places about the precession of the perihelion of Mercury, yet didn't get how they actually did the darn calculation. There were always a few steps that I did not follow. While reading through the Sean Carroll's Lecture notes on GR, I decided to try and figure out the details. Here I write it all out. This is not easy or short, but for those willing to work at it, might be a unique information source.

Start with the GEM field equations with the gauge choice of a constant 4-potential, then for a spherically symmetric, non-rotating, uncharged mass, the metric which solves the field equations is:

$$d\tau^2 = \exp(-2\tfrac{GM}{c^2R})dt^2 - \exp(2\tfrac{GM}{c^2R})dR^2/c^2 - (R/c)^2(d\theta^2 + \sin^2\theta d\phi^2)$$

A couple of procedural things. I like to baby step through derivations: if there is one division to make, I write it out, just like I would do with a pencil, but this time with TeX. Because I was initially trained to get the units right, I still keep that tradition going as an internal consistency check, instead of adopting natural units.

Let's make the metric simpler by making some reasonable assumptions about the angle θ, namely that it is always in a plane at angle $\pi/2$ so that $d\theta = 0$ and $\sin^2\theta = 1$. Write out the metric with these substitutions:

$$d\tau^2 = \exp(-2\tfrac{GM}{c^2R})dt^2 - \exp(2\tfrac{GM}{c^2R})dR^2/c^2 - (R/c)^2 d\phi^2$$

What one is suppose to do is work with the equation of motion, looking for a constant of the equation of motion. Sometime though, it is quicker to cheat a little, and just divide the above expression by $d\tau^2$ to get here:

$$1 = \exp(-2\tfrac{GM}{c^2R})(\tfrac{dt}{d\tau})^2 - 1/c^2\exp(2\tfrac{GM}{c^2R})(\tfrac{dR}{d\tau})^2 - (R/c)^2(\tfrac{d\phi}{d\tau})^2$$

Let's get rid of the exponential in front of the $dR/d\tau$.

$$\exp(-2\tfrac{GM}{c^2R}) = \exp(-4\tfrac{GM}{c^2R})(\tfrac{dt}{d\tau})^2 - 1/c^2(\tfrac{dR}{d\tau})^2 - \exp(-2\tfrac{GM}{c^2R})(R/c)^2(\tfrac{d\phi}{d\tau})^2$$

For our solar system, the exponent is really tiny, so we can use the Taylor series expansion to one term of the exponent. One slight trick is that the -4 one is like a -2 exponent squared. Approximate away:

$(1-2\frac{GM}{c^2R})=(1-2\frac{GM}{c^2R})^2(\frac{dt}{d\tau})^2-1/c^2(\frac{dR}{d\tau})^2-(1-2\frac{GM}{c^2R})(R/c)^2(\frac{d\phi}{d\tau})^2$

Notice that this expression is not a function of either time t or angle ϕ. This means there is a conserved quantity associated with change in the time (energy E) and a change in angle (angular momentum L). Come back at a later time, the expression stays the same. Spin around a few degrees, the metric stays the same. Something like this happens for Lagrange densities. The same logic applies to metrics. This time the idea goes by a different name: Killing vectors. I do not understand enough of the details, sorry, but it turns out that a Killing vector works like this:

$(\text{conserved thingie})=(\text{Killing vector foo})(\text{velocity foo})$

Do this for energy and angular momentum, normalizing to a test mass:

$E/mc^2=K_tV_t=(1-2\frac{GM}{c^2R},0,0,0)(\frac{dt}{d\tau},0,0,0)=(1-2\frac{GM}{c^2R})\frac{dt}{d\tau}$

$L/mc=K_\phi V_\phi=(0,0,0,R/c)(0,0,0,R\,\frac{d\phi}{d\tau})=R^2/c\,\frac{d\phi}{d\tau}$

Calculate the squares of these:

$(E/mc^2)^2=(1-2\frac{GM}{c^2R})^2(\frac{dt}{d\tau})^2$

$(L/mc)^2=R^4(\frac{d\phi}{d\tau})^2/c^2$

Plug these back into the metric expression:

$(1-2\frac{GM}{c^2R})=(E/\mathrm{mc}^2)^2-1/c^2(\frac{dR}{d\tau})^2-(1-2\frac{GM}{c^2R})(L/\mathrm{mc})^2/R^2$

Let's pause and catch our breath, there still is a long way to go. All that has happened is to introduce two constant quantities, the energy E and angular momentum L, into the metric expression. The equation still displays the "bending factor", the $(1-2\frac{GM}{c^2R})$, which is equal to one for Newtonain gravity. We could stop here because this expression is identical to the one for GR, but let's continue.

The next task is to do classical celestial mechanics. Talk about an area of weakness for me! The way this is done is to work with the two invariants: energy because Newtonain gravity is conservative, and angular momentum because the system is isolated. The cause of the planets whirling around is a 1/R potential, gravity. For that reason, we need to change the variable.

$U=1/R$

$\frac{dR}{d\tau}=\frac{dR}{d\phi}\frac{d\phi}{d\tau}=(-\frac{1}{U^2}\,\frac{dU}{d\phi})(LU^2/m)=-\,L/m\,\frac{dU}{d\phi}$

Plug these into the equation-that-used-to-be-the-metric:

$(1-2\frac{GM}{c^2}\,U)=(E/\mathrm{mc}^2)^2-L^2/(m^2\,c^2)(\frac{dU}{d\phi})^2-(1-2\frac{GM}{c^2}\,U)(UL/\mathrm{mc})^2$

Multiply this out for U, and bring all the terms to one side of the equation:

$0 = (E/\mathrm{mc}^2)^2 - L^2/(m^2 c^2)(\frac{dU}{d\phi})^2 - 1 + 2\,\frac{GM}{c^2}\,U - U^2 L^2/(m^2 c^2) + 2\,\frac{GM}{c^2} U^3\,L^2/(m^2 c^2)$

Take the derivative with respect to ϕ which will drop a few constant terms.

$0 = -\,2L^2/(m^2 c^2)\,\frac{dU}{d\phi}\,\frac{d^2U}{d\phi^2} + 2\,\frac{GM}{c^2}\,\frac{dU}{d\phi} - 2\,U\frac{dU}{d\phi}\,L^2/(m^2 c^2) + 6\,\frac{GM}{c^2}\,U^2\,\frac{dU}{d\phi}\,L^2/(m^2 c^2)$

Divide all the terms by $-\,2L^2\frac{dU}{d\phi}/(m^2 c^2)$:

$0 = \frac{d^2U}{d\phi^2} - \frac{GM}{c^2 L^2}\,m^2 c^2 + U - 3\,\frac{GM}{c^2}\,U^2$

The first three terms are classical Newtonian gravitational physics. The fourth term is the correction required by GR and GEM. First we need to understand the Newtonian gravity solution, ignoring the correction. Here is the Newtonian expression to solve:

$0 = \frac{d^2U}{d\phi^2} - \frac{GM}{c^2 L^2}\,m^2 c^2 + U$

Guess a solution. Fortunately, this differential equation is a variation on a common equation with a cosine solution. Thing is, the circle will be a tad eccentric, so include the eccentricity ϵ:

Guess: $U = \frac{GM}{c^2 L^2}\,m^2 c^2 (1 + \epsilon\cos(\phi - \phi_0))$

$\frac{dU}{d\phi} = -\,\frac{GM}{c^2 L^2}\,m^2 c^2\,\epsilon\,\sin(\phi - \phi_0)$

$\frac{d^2U}{d\phi^2} = -\,\frac{GM}{c^2 L^2}\,m^2 c^2\,\epsilon\cos(\phi - \phi_0)$

Put it all together:

$-\,\frac{GM}{c^2 L^2}\,m^2 c^2\,\epsilon\cos(\phi - \phi_0) - \frac{GM}{c^2 L^2}\,m^2 c^2 + \frac{GM}{c^2 L^2}\,m^2 c^2 + \frac{GM}{c^2 L^2}\,m^2 c^2\,\epsilon\cos(\phi - \phi_0) = 0$
QED

At the perihelion, $\cos(\phi - \phi_0) = 1$. Now will can get an expression for R at the perihelion with no corrections:

$U = \frac{1}{R} = \frac{GM}{c^2 L^2}\,m^2 c^2 (1 + \epsilon)$

the perihelion distance is $R = a(1 - \epsilon)$ for an ellipse with a semi-major axis of a. Plug this in, and divide by $(1 + \epsilon)$:

$\frac{1}{a(1 - \epsilon^2)} = \frac{GM}{c^2 L^2}\,m^2 c^2$

This will be useful, because in a short while we will see a $\frac{GM}{c^2 L^2} m^2 c^2$, and we can drop the $\frac{1}{a(1 - \epsilon^2)}$ right in.

How do we now deal with the correction? Well the Newtonian solution is pretty darn good, so we can use that as a place to start. The correction term has U^2, so calculate that:

$U^2 = (\frac{GM}{c^2 L^2}\,m^2 c^2)^2 (1 + 2\epsilon\cos(\phi - \phi_0) + \epsilon^2\cos^2(\phi - \phi_0))$

Put this back into the differential equation:

$$0 = \frac{d^2 U}{d\phi^2} - \frac{GM}{c^2 L^2} m^2 c^2 + U - 3\,\frac{GM}{c^2}\left(\frac{GM}{c^2 L^2} m^2 c^2\right)^2 (1 + 2\epsilon\cos(\phi - \phi_0) + \epsilon^2 \cos^2(\phi - \phi_0))$$

Which of these three terms matters? The first one is a constant term, much smaller than the constant term we already have since it has a factor of $(GM/c^2)^3$. A small number cubed can be safely ignored. The second term oscillates just like the Newtonian solution, so it will always be adding into the results. This "on resonance" terms will continually add up so the term matters. The squared cosine term will not be in tune with the dominant Newtonian solution, so it will add a periodic perturbation to the path, a form of noise. Keep only the middle term of U^2:

$$0 = \frac{d^2 U}{d\phi^2} - \frac{GM}{c^2 L^2} m^2 c^2 + U - 6\,\frac{G^3 M^3}{c^6 L^4} m^4 c^4 \epsilon \cos(\phi - \phi_0)$$

What is the solution for this differential equation? If we can find an expression that drops the cosine term, then that expression with the previous solution would do the trick. We will need a sine or cosine, and something to make terms drop.

Guess: $U_{\text{correction}} = \phi \sin(\phi - \phi_0)$

$$\frac{dU}{d\phi} = \phi\cos(\phi - \phi_0) + \sin(\phi - \phi_0)$$

$$\frac{d^2 U}{d\phi^2} = -\,\phi\sin(\phi - \phi_0) + \cos(\phi - \phi_0) + \cos(\phi - \phi_0)$$

$$\frac{d^2 U}{d\phi^2} + U = -\,\phi\sin(\phi - \phi_0) + 2\cos(\phi - \phi_0) + \phi\sin(\phi - \phi_0) = 2\cos(\phi - \phi_0)$$

The right coefficients in front of this correction term will cancel the $-6\cos$ term. Combine the correction term with the Newtonian solution:

$$U = \frac{GM}{c^2 L^2} m^2 c^2 (1 + \epsilon\cos(\phi - \phi_0)) + 3\,\frac{G^3 M^3}{c^6 L^4} m^4 c^4 \epsilon\phi\sin(\phi - \phi_0)$$

Cosine is an even function, sine is an odd function, and $\phi\sin\phi$ is an even function. Because $\frac{G^3 M^3}{c^6 L^4}$ is so small, only the first term of its Taylor series expansion is going to matter, and it will behave like it subtracts a bit off of a cosine. Bring the correction into the cosine, noticing a GM/L^2 is already shared:

$$U \simeq \frac{GM}{c^2 L^2} m^2 c^2 \left(1 + \epsilon\cos\left(\phi - \phi_0 - 3\,\frac{G^2 M^2}{c^4 L^2} m^2 c^2 \phi\right)\right)$$

How much does ϕ change? In one revolution it would go 2π. Calculate the ratio of no correction to correction:

$$\Delta\phi_{1\,\text{revolution}} = 2\pi\,\frac{1}{1 - 3\,\frac{G^2 M^2}{c^4 L^2} m^2 c^2} \simeq 2\pi\left(1 + 3\,\frac{G^2 M^2}{c^4 L^2} m^2 c^2\right)$$

The advance is $6\pi\,\frac{G^2 M^2}{c^4 L^2} m^2 c^2$. Now we can use the result found earlier for the major axis and the eccentricity$[\frac{1}{a(1-\epsilon^2)} = \frac{GM}{c^2 L^2} m^2 c^2]$:

$$\text{Advance/revolution} = 6\pi\,\frac{GM}{a(1-\epsilon^2)c^2}$$

This is what appears in all the books. I think this is tough to get all the details clear and correct. I hope I acheived that goal.