Tree & Binary Trees (4)

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Outline

- Introduction
- Weighted Path Length
- Huffman tree
- Huffman Codes

Introduction

- We usually encode strings by assigning **fixed-length codes** to all characters in the alphabet (for example, 8-bit coding in ASCII).
- However, if different characters occur with different frequencies, we can save memory and reduce transmittal time by using **variable-length encoding**.
- The idea is to assign shorter codes to characters that occur more often.

Introduction

relative frequencies of the letters of the alphabet:

Letter	Frequency	Letter	Frequency
A	77	N	67
B	17	O	67
C	32	P	20
D	42	Q	5
E	120	R	59
F	24	S	67
G	17	T	85
H	50	U	37
I	76	V	12
J	4	W	22
K	7	X	4
L	42	Y	22
M	24	Z	2

The letter 'E' appears about 60 times more often than the letter 'Z.'

Introduction

- We must be careful when assigning variable-length codes.
- For example, let us encode **e** with 0, **a** with 1, and **t** with 01. How can we then encode the word **tea**?
- The encoding is **0101**.
- Unfortunately, this encoding is ambiguous. It could also stand for **eat**, **eaea**, or **tt**.
- Of course this coding is unacceptable, because it results in loss of information.

Introduction

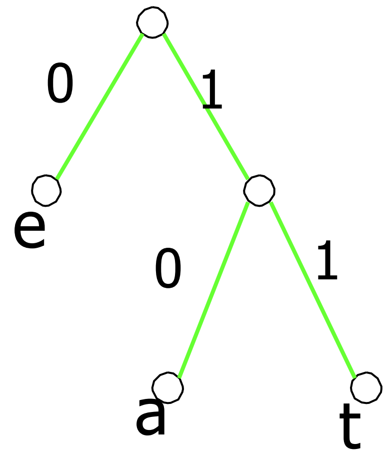
- To avoid such ambiguities, we can use **prefix codes**. In a prefix code, the bit string for a character never occurs as the **prefix** (first part) of the bit string for another character.
- For example, the encoding of **e** with 0, **a** with 10, and **t** with 11 is a prefix code. How can we now encode the word **tea**?
- The encoding is **11010**.
- This bit string is unique, it can only encode the word **tea**.

Introduction

- We can represent prefix codes using binary tree, where the characters are the labels of the leaves in the tree.
- The edges of the tree are labeled so that an edge leading to a left child is assigned a 0 and an edge leading to a right child is assigned a 1.
- The bit string used to encode a character is the sequence of labels of the edges in the unique path from the root to the leaf labeled with this character.

Introduction

- The tree corresponding to our example:



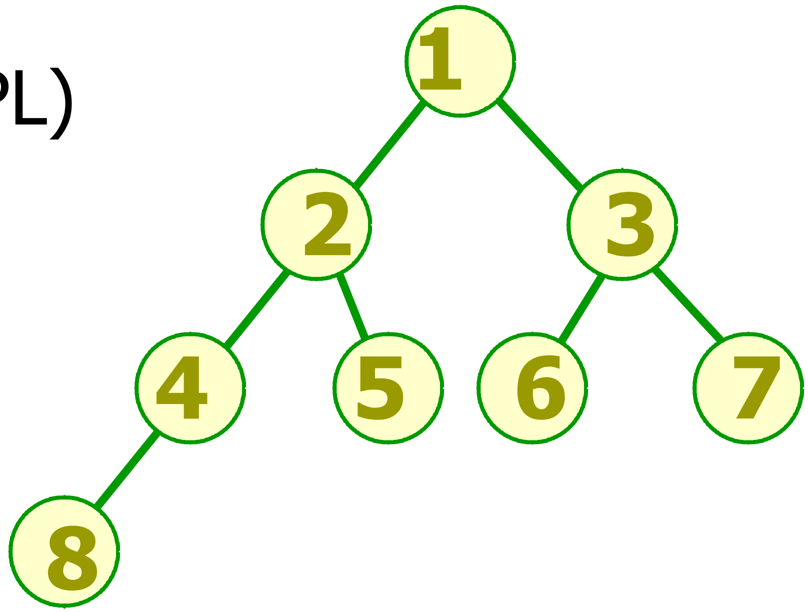
In a tree, no leaf can be the ancestor of another leaf. Therefore, no encoding of a character can be a prefix of an encoding of another character (prefix code).

Introduction

- To determine the optimal encoding for a given string, we first have to find the frequencies of characters in that string. Let us consider the following string:
  - eeadfeejjeggebeeggddehhhececddeciedee
  - It contains 1×a, 1×b, 3×c, 6×d, 15×e, 1×f, 4×g, 3×h,1×i, and 2×j.
- We can use **Huffman's** algorithm to build the optimal coding tree.

Path Length (PL)

- If $n_1, n_2, \ldots, n_k$ is a sequence of nodes in the tree such that n_i is the parent of n_{i+1} for $1 \leqslant i < k$, then this sequence is called a path from n_1 to n_k. The length of the path is $k - 1$.
- Path Length of tree (PL)



PL = 3*1+2*3 = 9

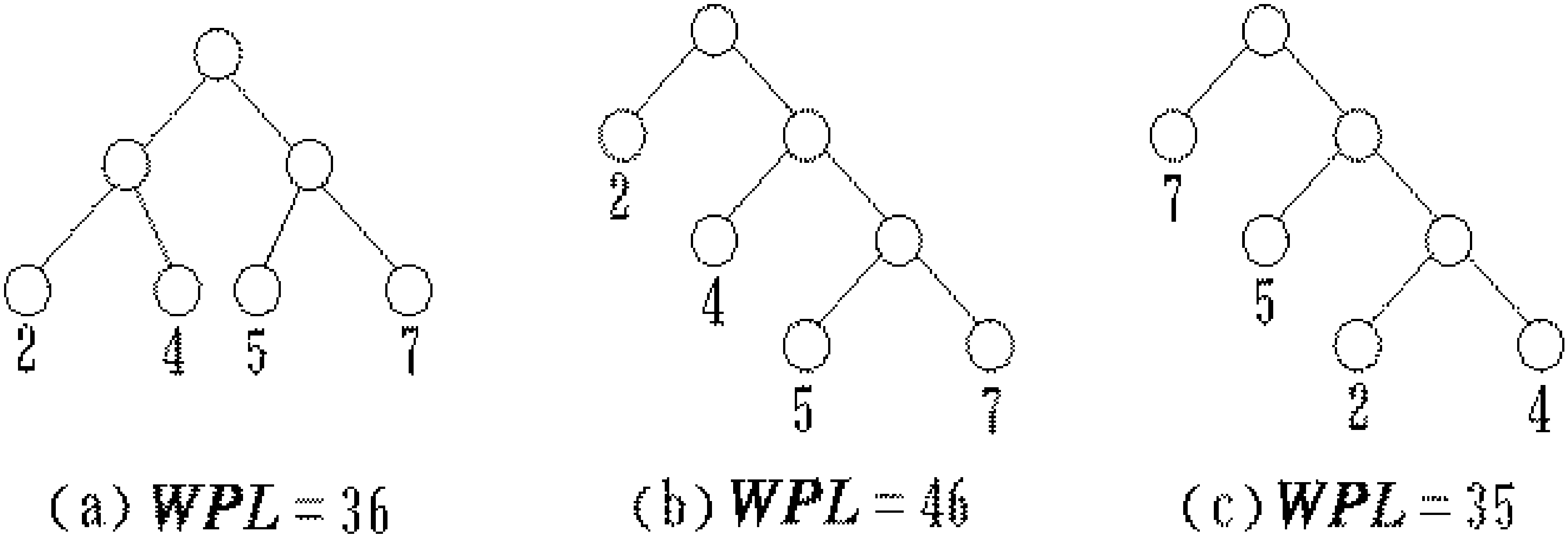
Weighted Path Length

- weighted path length of a leaf is its weight times its depth.
- weighted path length of a tree is the sum of weighted path lengths of every leaf.

$$WPL = \sum_{k=1}^{n} w_k \cdot PL_k$$

- **Huffman tree has the minimum WPL**

Huffman tree



- **C is Huffman tree.**

Building Human Trees

- Create a collection of n initial Huffman trees, each of which is a single leaf node containing one of the letters. Put the n partial trees onto a list in ascending order by weight (frequency).
- Next, remove the first two trees (the ones with lowest weight) from the list. Join these two trees together to create a new tree whose root has the two trees as children, and whose weight is the sum of the weights of the two trees. Put this new tree back on the list in the correct place necessary to preserve the order of the list.
- This process is repeated until all of the partial Huffman trees have been combined into one.

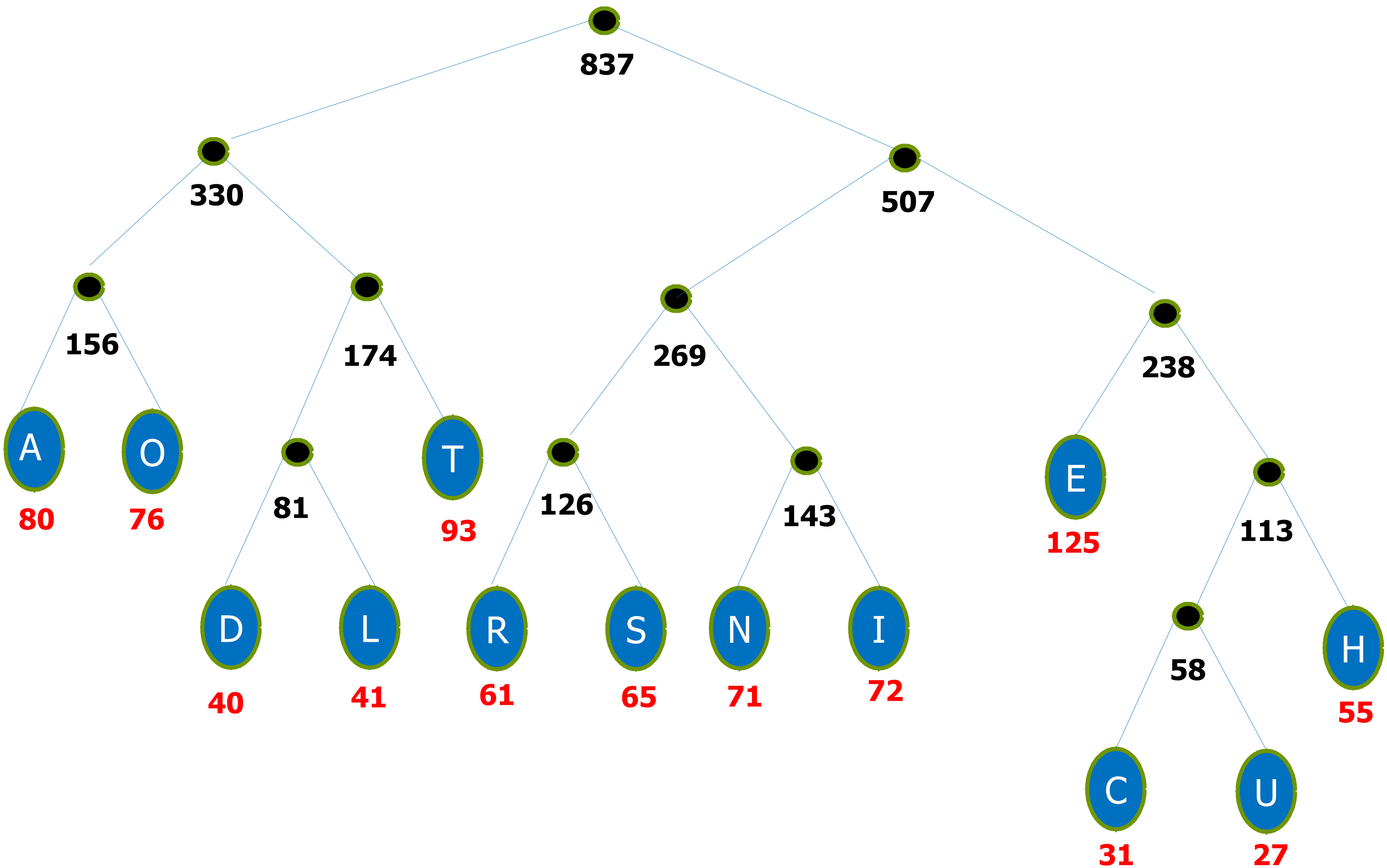
Example

- Character count in text:

Char	E	T	A	O	I	N	S	R	H	L	D	C	U
Freq	125	93	80	76	72	71	65	61	55	41	40	31	27

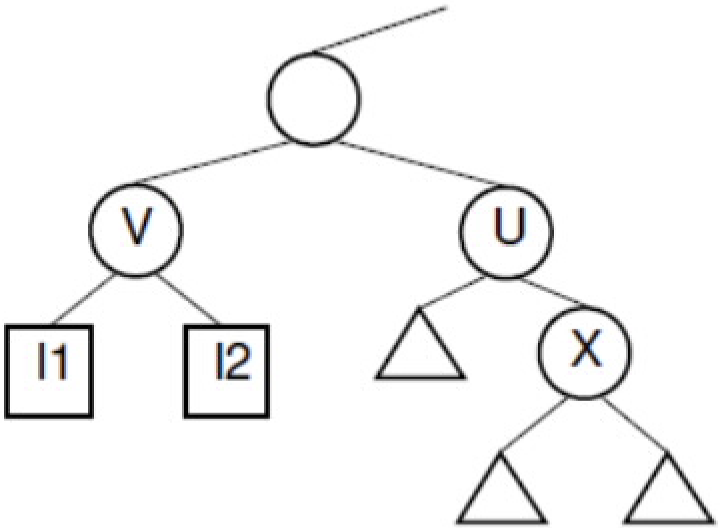
- At first, there are 13 partial trees.

Huffman Tree Construction



Lemma

- **Lemma 5.1** For any Huffman tree built by function **buildHuff** containing at least two letters, the two letters with least frequency are stored in siblings nodes whose depth is at least as deep as any other leaf nodes in the tree.
- **Proof:**



Theorem

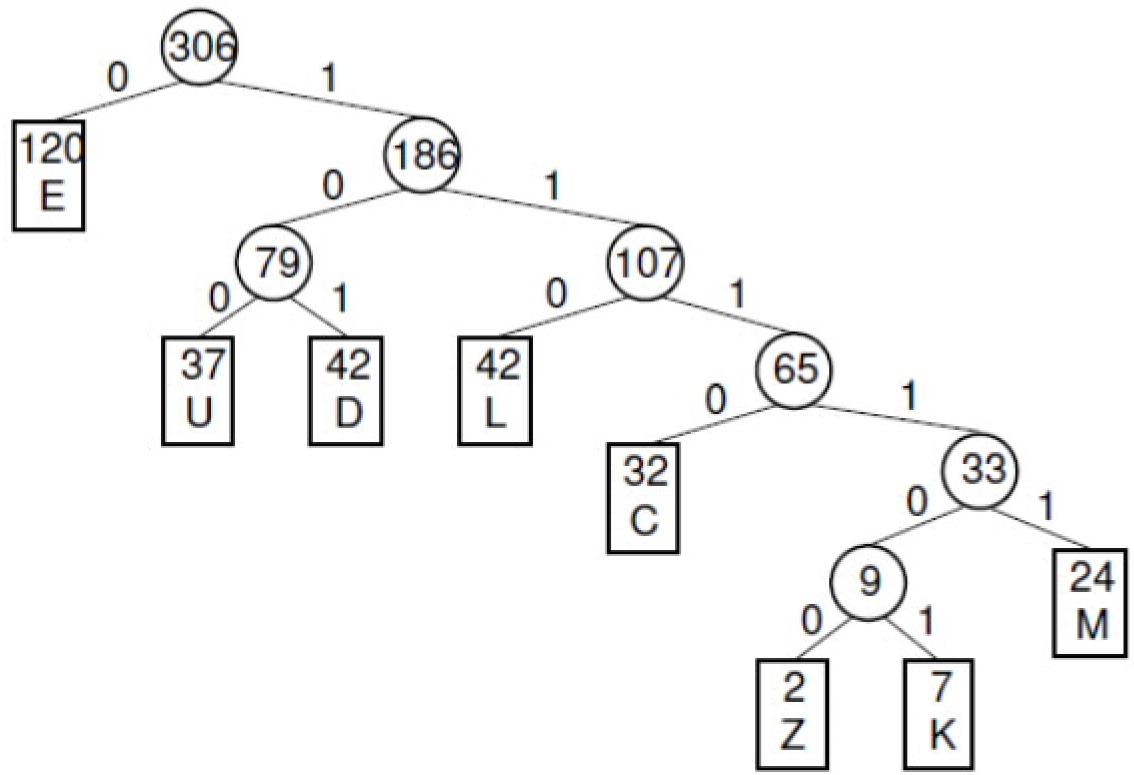
- Theorem 5.3 Function **buildHuff** builds the Huffman tree with the minimum external path weight for the given set of letters.

- **Proof:** The proof is by induction on n, the number of letters
  - Base Case: For n = 2, there are only two possible trees
  - Induction Hypothesis: Assume that any tree created by **buildHuff** that contains n - 1 leaves has minimum external path length
  - Induction Step: Given a Huffman tree T **with n** leaves, $n \geqslant 2$, suppose that $w_1 \leqslant w_2 \leqslant \ldots \leqslant w_n$ where w_1 to w_n are the weights of the letters. Call V the parent of the letters with frequencies w_1 and w_2.

Assigning Huffman Codes

- Example:

Letter	C	D	E	K	L	M	U	Z
Frequency	32	42	120	7	42	24	37	2



Using Huffman Codes

From the Huffman tree, we can get the codes for all eight letters.

Letter	Freq	Code	Bits
C	32	1110	4
D	42	101	3
E	120	0	1
K	7	111101	6
L	42	110	3
M	24	11111	5
U	37	100	3
Z	2	111100	6

Encoding

- replace each letter in the string with its binary code. A lookup table can be used for this purpose.
- Using the code generated by example Huffman tree
  - "DEED" is represented by the bit string "10100101"
  - "MUCK" is represented by the bit string "111111001110111101."

Decoding

- Decoding a bit string begins at the root of the tree. To take branches depending on the bit value — left for ‘0’ and right for ‘1’ — until reaching a leaf node. This leaf contains the first character in the message. Then to process the next bit in the code restarting at the root to begin the next character.
- To decode the bit string “1011001110111101”
  - “DUCK”

Prefix Property

- **Huffman codes certainly have the prefix property because any prefix for a code would correspond to an internal node, while all codes correspond to leaf nodes.**

Implementation: Huffman Tree Nodes(1)

```
template <class Elem>
class HuffNode {    //Node abstract base class
public:
   virtual int weight() = 0;
   virtual bool isLeaf() = 0;
   virtual HuffNode* left() const = 0;
   virtual void setLeft(HuffNode*) = 0;
   virtual HuffNode* right() const = 0;
   virtual void setRight(HuffNode*) = 0;
};
```

Implementation: Huffman Tree Nodes(2)

```
template <class Elem>      //leaf node subclass
class LeafNode: public HuffNode<Elem> {
private:
  Freqpair<Elem>* it;        //Frequency pair
public:
  LeafNode(const Elem& val, int freq)     //constructor
    { it = new Freqpair<Elem>(val,freq);  }
  int weight()   { return it->weight(); }   //Return frequency
  Freqpair<Elem>* val()   {  return it;  }
  bool isLeaf()   { return true; }
  virtual HuffNode* left() const  { return NULL; }
  virtual void setLeft (HuffNode*)  {  }
  virtual HuffNode* right() const  { return NULL; }
  virtual void setRight(HuffNode*)  {  }
};
```

Implementation: Huffman Tree Nodes(3)

```
template <class Elem>     //Internal node subclass
class IntlNode: public HuffNode<Elem> {
private:
   HuffNode<Elem>*  lc;           //left child
   HuffNode<Elem>*  rc;           //right child
   int wgt;                       //Subtree weight
public:
   IntlNode(HuffNode<Elem> *  l; HuffNode<Elem> * r)
     { wgt = l->weight() + r->weight();   lc = l;  rc = r; }
   int weight()    { return  wgt; }     //Return frequency
   bool isLeaf()   { return  false; }
   HuffNode<Elem>* left() const  { return  lc; }
   void setLeft(HuffNode<Elem>* b)
     { lc = (HuffNode*)b;  }
   HuffNode<Elem>* right() const  { return  rc; }
   void setRight(HuffNode<Elem>* b)
     { rc = (HuffNode*)b;  }
};
```

Class Declaration: Frequency Pair Object

```
template <class Elem>
class FreqPair  {     //An element / frequency pair
private:
   Elem it;                //An element of some sort
   int freq;
public:
   FreqPair(const Elem& e, int f)    //Constructor
     {  it = e;     freq = f;   }
   ~FreqPair() {  }                        //Destructor
   int weight()  {  return  freq; }  //Return the weight
   Elem& val()  {  return  it;   }    //Return the element
};
```

Class Declaration: Huffman Tree

```
template <class Elem>
class  HuffTree  {
private:
   HuffNode<Elem>*  theRoot;
public:
   HuffTree(Elem& val,  int freq)
      { theRoot = new LeafNode<Elem>(val,freq);  }
   HuffTree(HuffTree<Elem>*  l, HuffTree<Elem>*  r)
      { theRoot = new IntlNode<Elem>(l->root(), r->root()); }
   ~ HuffTree() {  }
   HuffNode<Elem>* root()  {  return  theRoot;  }
   int weight()   {  return  theRoot->weight();   }
};
```

Class Declaration: Huffman Tree(2)

```
template <class Elem>  class HHCompare  {
public:
   static bool lt (HuffTree<Elem>* x, HuffTree<Elem>*  y)
      { return x->weight() < y->weight();  }
   static bool eq(HuffTree<Elem>* x, HuffTree<Elem>*  y)
      { return x->weight() = = y->weight();  }
   static bool gt(HuffTree<Elem>* x, HuffTree<Elem>*  y)
      { return x->weight() > y->weight();  }
};
```

Huffman Tree Construction 1

```
template <class Elem>  HuffTree<Elem>*
buildHuff(SLList<HuffTree<Elem>*,HHCompare<Elem> >* fl) {
  HuffTree<Elem>* temp1, *temp2, *temp3;
  for (fl->setStart(); fl->leftLength()+fl->rightLength()>1;
       fl->setStart())  {          //While at least two items left
     fl->remove(temp1);           //Pull first two trees off the list
     fl->remove(temp2);
     temp3 = new HuffTree<Elem>(temp1, temp2);
     fl->insert(temp3);            //Put the new tree back on list
     delete temp1;                 //Must delete the remnants
     delete temp2;                 //   of the trees we created
  }
  return temp3;
}
```

Huffman Tree Construction 2

```
// Build a Huffman tree from a collection of frequencies
template <typename E> HuffTree<E>*
buildHuff(HuffTree<E>** TreeArray, int count) {
  heap<HuffTree<E>*,minTreeComp>* forest =
    new heap<HuffTree<E>*, minTreeComp>(TreeArray,
                                         count, count);
  HuffTree<char> *temp1, *temp2, *temp3 = NULL;
  while (forest->size() > 1) {
    temp1 = forest->removefirst();   // Pull first two trees
    temp2 = forest->removefirst();   //   off the list
    temp3 = new HuffTree<E>(temp1, temp2);
    forest->insert(temp3);  // Put the new tree back on list
    delete temp1;          // Must delete the remnants
    delete temp2;          //   of the trees we created
  }
  return temp3;
}
```

-End-