Generalized Local-to-Unity Models*

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Abstract

We introduce a generalization of the popular local-to-unity model of time series persistence by allowing for p autoregressive roots and $p-1$ moving average roots close to unity. This generalized local-to-unity model, GLTU(p), induces convergence of the suitably scaled time series to a continuous time Gaussian ARMA($p, p-1$) process on the unit interval. Our main theoretical result establishes the richness of this form of limiting processes, in the sense that they can well approximate a large class of stationary Gaussian processes in the total variation norm. We show that Campbell and Yogo's (2006) popular inference method for predictive regressions fails to control size in the GLTU(2) model with empirically plausible parameter values, and we propose a limited-information Bayesian framework for inference in the GLTU(p) model and apply it to quantify the uncertainty about the half-life of deviations from Purchasing Power Parity.

Keywords: Continuous time ARMA process; Convergence; Approximability

JEL Codes: C22; C51

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1 Introduction

This paper proposes a flexible asymptotic framework for the modelling of persistent time series. Our starting point is an empirical observation: For many macroeconomic time series, such as the unemployment rate, interest rates, labor's share of national income, real exchange rates, price earnings ratios, etc., tests for an autoregressive unit root are often inconclusive, or rejections are not exceedingly significant. As such, the unit root model is a natural benchmark for empirically plausible persistence modelling. At the same time, most economic models assume that these time series are stationary. What is more, econometric techniques based on an assumption of an exact unit root can yield highly misleading inference under moderate deviations of the unit root model, as demonstrated by Elliott (1998).

These concerns have generated a large literature on econometric modelling and inference with the local-to-unity model.[1] Specifically, a stationary local-to-unity (LTU) model of the scalar time series $x_{T,t}$ is of the form

$$(1-\rho_T L)(x_{T,t}-\mu)=u_t, \quad t=1,\ldots,T \tag{1}$$

where L is the lag operator, $\rho_T = 1-c/T$ for some fixed $c>0$ and u_t is a mean-zero I(0) disturbance satisfying a functional central limit theorem $T^{-1/2}\sum_{t=1}^{\lfloor \cdot T\rfloor} u_t \Rightarrow W(\cdot)$ with W a Wiener process with variance equal to the long-run variance of u_t. In this model

$$T^{-1/2}(x_{T,\lfloor \cdot T\rfloor}-x_{T,1}) \Rightarrow J_1(\cdot)-J_1(0) \tag{2}$$

where J_1 is a stationary Ornstein-Uhlenbeck process with parameter c, the continuous time analogue of an AR(1) process. The process (1) is the local asymptotic alternative of an autoregressive unit root (cf. Elliott, Rothenberg, and Stock (1996), Elliott (1999)). As such, it is impossible to perfectly discriminate between a LTU process and a unit root process, even as $T\to\infty$. Correspondingly, for any finite c, the measure of $J_1(\cdot)-J_1(0)$ is mutually absolutely continuous with respect to the measure of W, the continuous time analogue of a unit root process. LTU asymptotics thus properly reflect the empirical ambivalence of unit root tests noted above.

[1]See Bobkoski (1983), Chan and Wei (1987), Phillips (1987), Stock (1991), Elliott and Stock (1994), Cavanagh, Elliott, and Stock (1995), Wright (2000), Moon and Phillips (2000), Elliott and Stock (2001), Gospodinov (2004), Valkanov (2003), Torous, Valkanov, and Yan (2004), Rossi (2005), Campbell and Yogo (2006), Jansson and Moreira (2006) and Mikusheva (2007, 2012), among many others.

But the LTU model is clearly not the only persistence model with this feature, even with attention restricted to stationary models. After all, the properties of the limiting process J_1 are governed by a single parameter c, and the long-range dependence of J_1 are those of a continuous time AR(1). For instance, in the LTU model, the correlation between $x_{T,\lfloor sT \rfloor}$ and $x_{T,\lfloor rT \rfloor}$ converges to $e^{-c|s-r|}$. It is not clear why this very particular form of long-range dependence should adequately model the persistence properties of macroeconomic time series.

This paper proposes a more flexible asymptotic framework by allowing for p autoregressive roots and $p-1$ moving-average roots local-to-unity for $p \geq 1$, so that

$$(1-\rho_{T,1}L)(1-\rho_{T,2}L)\cdots(1-\rho_{T,p}L)(x_{T,t}-\mu) = (1-\gamma_{T,1}L)\cdots(1-\gamma_{T,p-1}L)u_t,$$

where $\rho_{T,j} = 1-c_j/T$ and $\gamma_{T,j} = 1-g_j/T$ for fixed $\{c_j\}_{j=1}^p$ and $\{g_j\}_{j=1}^{p-1}$ (with some conditions on these parameters as specified in Section 2 below). With $p = 1$, this "generalized local-to-unity" model GLTU(p) nests the familiar LTU model (1). A first result of this paper is the convergence of the GLTU(p) model, that is, in analogy to (2),

$$T^{-1/2}(x_{T,\lfloor \cdot T \rfloor} - x_{T,1}) \Rightarrow J_p(\cdot) - J_p(0) \tag{3}$$

where J_p is a stationary continuous time Gaussian ARMA($p, p-1$) process with parameters $\{c_j\}_{j=1}^p$ and $\{g_j\}_{j=1}^{p-1}$.

The GLTU(p) model sets the difference between the number of local-to-unity autoregressive and moving-average parameters to exactly one. This ensures that the limit process J_p still resembles a Wiener process: For instance, if instead $(1-\rho_{T,1}L)(1-\rho_{T,2}L)(x_{T,t}-\mu) = u_t$, the large sample properties of $x_{T,t}$ would be more akin to an I(2) process, with the suitably scaled limit of $x_{T,\lfloor \cdot T \rfloor} - x_{T,1}$ converging to a limit process that is absolutely continuous with respect to the measure of an *integrated* Wiener process $\int_0^{\cdot} W(r)dr$. In contrast, the measure of $J_p(\cdot) - J_p(0)$ is mutually absolutely continuous with respect to the measure of W, so just as for the LTU model, the GLTU(p) model cannot be perfectly discriminated from the unit root model, even asymptotically.

While clearly more general than the standard local-to-unity model (1), one might still worry about the appropriateness of the GLTU(p) model for generic persistence modelling of macroeconomic time series. Our main theoretical result addresses this concern by establishing the richness of the GLTU(p) model class. Recall that the total variation distance between two probability measures is the difference in the probability they assign to an event, maximized over all events. We show in Section 3 below that for any given stationary Gaussian limiting process G whose measure of $G(\cdot) - G(0)$ is mutually absolutely continuous with respect to the measure of W, and a mild regularity constraint on the spectral density of G, for any $\varepsilon > 0$ there exists a finite p_ε and GLTU(p_ε) model such that the measure of the induced limiting process J_{p_ε} is within ε of the measure of G in total variation norm. In other words, for small ε, the stochastic properties of J_{p_ε} and G are nearly identical. Thus, positing a GLTU model is nearly without loss of generality for the large sample modelling of persistent stationary processes that cannot be distinguished from a unit root process with certainty even in the limit.

In practice, applications of the GLTU model involve the choice of a finite p and the determination of the corresponding $2p-1$ GLTU(p) model parameters $\{c_j\}_{j=1}^{p}$ and $\{g_j\}_{j=1}^{p-1}$. This is perfectly analogous to the modelling of generic covariance stationary processes as finite order AR, MA or ARMA process. The implementation is relatively harder for the GLTU mode, however: As noted above, since the LTU model cannot be perfectly discriminated from the unit root model, the parameter c in (1) cannot be consistently estimated. By the same logic, neither the value of p, nor the parameters $\{c_j\}_{j=1}^{p}$ and $\{g_j\}_{j=1}^{p-1}$ for a given p can be consistently estimated. This impossibility is simply the flip-side of the arguably attractive property of GLTU asymptotics to appropriately capture the empirical ambivalence of unit root tests.

With that in mind, we suggest a limited-information framework for likelihood based inference with the GLTU(p) model. Note that (3) implies, for any fixed integer N

$$\{T^{-1/2}(x_{T,\lfloor jT/N \rfloor} - x_{T,1})\}_{j=1}^{N} \Rightarrow \{J_p(j/N) - J_p(0)\}_{j=1}^{N}. \tag{4}$$

Thus, with attention restricted to the N observations on the left-hand side of (4), large-sample inference about the GLTU parameters is equivalent to inference given N discretely sampled observations from a continuous time Gaussian ARMA($p, p-1$) process. But this latter problem is well-studied (cf. Phillips (1959), Jones (1981), Bergstrom (1985), Jones and Ackerson (1990), for example), and we show how to obtain a numerically accurate approximation to the likelihood by a straightforward Kalman filter.

The computational convenience of this approach opens the door to the numerical determination of asymptotically valid frequentist inference in the GLTU(p) model, at least for a given moderately large value of p. For instance, one could in principle construct a confidence set for the $2p-1$ GLTU parameters, potentially followed by a Bonferroni or projection-based method if the parameter of interest is a lower-dimensional function of the GLTU parameters. Alternatively, the generic approaches to valid confidence set construction in the presence of nuisance parameters in Dufour (2006) or Elliott, Müller, and Watson (2015) could be employed. This includes as the special case the construction of asymptotically valid specification tests of $H_0 : p = p_0$ again $H_1 : p > p_0$, which could potentially guide the choice of p in practice.

We leave the important task of deriving such procedures to future work. Here we merely highlight the empirical relevance of the GLTU model and the feasibility of limited-information likelihood based inference in two conceptually distinct exercises.

First, we show that inference methods derived to be valid in the LTU model can be highly misleading under an empirically plausible GLTU(2) model. In particular, we consider Campbell and Yogo's (2006) popular test for stock return predictability. By construction, this test controls size in the LTU model. But we find that in the GLTU(2) model, it exhibits severe size distortions, even if the GLTU(2) parameters are restricted to be within a two log-points neighborhood of the peak of the limited-information likelihood for the price-dividends ratio. In other words, unless one has good reasons to impose that the long-range persistence patterns of potential stock price predictors are of the AR(1) type, the Campbell and Yogo (2006) test is not a reliable test of the absence of predictability.

Second, and more constructively, we conduct limited-information Bayesian inference about the half-life of the US/UK real exchange rate deviations, using the long-span data from Lothian and Taylor (1996). While most of the frequentist complications of inference with persistent time series do not arise in a Bayesian setting (Sims and Uhlig (1991)), the appeal of the asymptotic limited information analysis (4) relative to a full information approach is that it avoids having to model (and hence potentially misspecify) the short run dynamics and marginal distribution of the exchange rate. We find that the GLTU model with $p \geq 3$ has considerably better limited-information fit than those with $p = 1$ or 2, while at the same time leading to much larger half-life estimates. This illustrates that allowing the generality of the GLTU model can substantially alter conclusions about economic quantities of interest.

This paper contributes to a large literature on alternative models of persistence, such as the fractional model (see Robinson (2003) for an overview), or more recently, the three parameter generalization of Müller and Watson (2016). A relatively close analogue to the

GLTU model is the VAR(1) LTU model considered by Phillips (1988), Stock and Watson (1996), Stock (1996) or Phillips (1998): The marginal process for a scalar time series of a VAR(1) LTU model is in the GLTU class, since sums of finite order AR processes are finite order ARMA processes with particular coefficient restrictions.

Our main theoretical result on the approximability of continuous time Gaussian processes is related in spirit to the approximability of the second order properties of discrete time stationary processes by the finite order ARMA class—see, for instance, Theorem 4.4.3 of Brockwell and Davis (1991) for a textbook exposition. The continuous time case is subtly different, though, since spectral densities are then functions on the entire real line (and not confined to the interval $[-\pi, \pi]$). What is more, we obtain approximability in total variation distance, and not just for a metric on second order properties. We are not aware of any closely related results in the literature.

The remainder of the paper is organized as follows. Section 2 introduces the GLTU(p) model in detail and formally establishes its limiting properties. Section 3 studies the richness of the GLTU(p) model class and contains the main theoretical result. Section 4 develops a straightforward Kalman filter to evaluate the limited-information likelihood. Section 5 contains the two empirical illustrations, and is followed by a concluding Section 6. Proofs are collected in an appendix.

2 The GLTU(p) Model

2.1 Set-up

We make the following assumptions about the building blocks of the GLTU(p) model

$$(1-\rho_{T,1}L)(1-\rho_{T,2}L)\cdots(1-\rho_{T,p}L)(x_{T,t}-\mu) = (1-\gamma_{T,1}L)\cdots(1-\gamma_{T,p-1}L)u_t \tag{5}$$

where $\rho_{T,j} = 1 - c_j/T$ and $\gamma_{T,j} = 1 - g_j/T$.

Condition 1 *(i) The innovations $\{u_t\}_{t=-\infty}^{\infty}$ are mean-zero covariance stationary with absolutely summable autocovariances and satisfy $T^{-1/2}\sum_{t=1}^{\lfloor \cdot T \rfloor} u_t \Rightarrow W(\cdot)$, where $W(\cdot)$ is a Wiener process of variance ω^2.*

(ii) The parameters $\{c_j\}_{j=1}^{p}$ and $\{g_j\}_{j=1}^{p-1}$ do not depend on T and have positive real parts. They can be complex valued, but if they are, then they appear in conjugate pairs, so that

the polynomials $a(z) = \prod_{j=1}^{p}(c_j + z) = z^p + \sum_{j=1}^{p} a_j z^{p-j}$ *and* $b(z) = \prod_{j=1}^{p-1}(g_j + z) = z^{p-1} + \sum_{j=0}^{p-2} b_j z^j$ *have real coefficients.*

(iii) For all T*, the process* $\{x_{T,t}\}_{t=-\infty}^{\infty}$ *is covariance stationary.*

The high-level Condition 1 (i) allows for flexible weak dependence in the innovations u_t. Part (ii) ensures that a covariance stationary distribution of $x_{T,t}$ exists, and that the limiting continuous time Gaussian ARMA process J_p is stationary and minimum phase. Part (iii) ensures that $\{x_{T,t}\}_{t=1}^{T}$ is covariance stationary, implicitly restricting the initial condition $(x_{T,0}, \ldots, x_{T,-p+1})$.

As noted in the introduction, the GLTU(p) model obviously nests the familiar LTU model in (1) as a special case with $p = 1$. Maybe more interestingly, the sum of two independent LTU processes $x_{T,t}^{A}$ and $x_{T,t}^{B}$ with parameters c_A and c_B and long-run variances $\omega^2/2$ and $\omega^2/2$ is recognized as a GLTU(2) model with parameters $c_1 = c_A$, $c_2 = c_B$ and $g_1 = (c_A^2 + c_B^2)/(2(c_A + c_B))$. The GLTU($p$) model thus also encompasses aggregations of p independent LTU models. It is more general than that, though, since any aggregation of independent LTU models yields a monotone spectral density function for the limiting process, while the spectral density function of J_p has no such constraint.

2.2 Limit Theory

Following Brockwell (2001), a mean-zero stationary continuous time Gaussian ARMA($p, p-1$) process J_p with parameters $\{c_j\}_{j=1}^{p}$, $\{g_j\}_{j=1}^{p-1}$ and ω^2 of Condition 1 (ii), denoted CARMA($p, p-1$) process in the following, can be written as a scalar *observation*

$$J_p(s) = \mathbf{b}'\mathbf{X}(s) \tag{6}$$

of the $p \times 1$ *state* process $\mathbf{X}$ with

$$\mathbf{X}(s) = e^{\mathbf{A}s}\mathbf{X}(0) + \int_0^s e^{\mathbf{A}(s-r)}\mathbf{e}dW(r) \tag{7}$$

where $\mathbf{X}(0) \sim \mathcal{N}(0, \mathbf{\Sigma})$ is independent of the scalar Wiener process W of variance ω^2,

$$\mathbf{A} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_p & -a_{p-1} & -a_{p-2} & \cdots & -a_1 \end{pmatrix}, \mathbf{e} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} b_0 \\ b_1 \\ \vdots \\ b_{p-2} \\ 1 \end{pmatrix}$$

and the coefficients a_j and b_j are defined in Condition 1 (ii). The covariance matrix of $\mathbf{X}(0)$, and hence $\mathbf{X}(s)$, is given by

$$\mathbf{\Sigma} = E[\mathbf{X}(0)\mathbf{X}(0)'] = \omega^2 \int_{-\infty}^{0} e^{-\mathbf{A}r}\mathbf{e}\mathbf{e}'e^{-\mathbf{A}'r}dr, \tag{8}$$

the autocovariance function of J_p is $\gamma_p(r) = E[J_p(s)J_p(s+r)] = \mathbf{b}'e^{\mathbf{A}|r|}\mathbf{\Sigma}\mathbf{b}$, and, with $i = \sqrt{-1}$, the spectral density $f_p : \mathbb{R} \mapsto \mathbb{R}$ of J_p satisfies

$$f_{J_p}(\lambda) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\lambda r}\gamma_p(r)dr = \frac{\omega^2}{2\pi}\frac{|b(i\lambda)|^2}{|a(i\lambda)|^2}. \tag{9}$$

It is not hard to see that under Condition 1, $f_{J_p}(\lambda)$ is a ratio of polynomials in λ^2 with real coefficients of order p and $p-1$, respectively. Also, Theorem III.17 of Ibragimov and Rozanov (1978) implies that the measure of J_p is mutually absolutely continuous with the measure of J_1 for any fixed $c = c_1 > 0$. But the measure of $J_1(\cdot) - J(0)$ is mutually absolutely continuous with the measure of W, so the same holds true for $J_p(\cdot) - J_p(0)$.

The state space representation (6) and (7) is not obviously related to the usual state space representation of the discrete time ARMA process (5). But it turns out that one can rewrite the latter in the form

$$x_{T,t} = \mathbf{b}'\mathbf{Z}_{T,t} + \mu \tag{10}$$

$$\mathbf{Z}_{T,t} = (\mathbf{I} + \mathbf{A}/T)\mathbf{Z}_{T,t-1} + \mathbf{e}u_t \tag{11}$$

where $\mathbf{Z}_{T,t} \in \mathbb{R}^p$, mimicking (6) and (7). This is the key step in the proof of the following theorem, which establishes the large sample relationship between the GLTU(p) model and the corresponding CARMA($p, p-1$) model J_p.

Theorem 1 *Under Condition 1, the GLTU(p) model satisfies* $T^{-1/2}(x_{T,\lfloor \cdot T\rfloor} - \mu) \Rightarrow J_p(\cdot)$.

3 Richness of the GLTU(p) Model Class

In this section we explore the range of large sample persistence patterns that GLTU(p) models can induce. By Theorem 1, this amounts to studying the richness of the CARMA($p, p-1$) processes on the unit interval. As discussed in the introduction, we focus on processes that are stationary, and that cannot be distinguished from a unit root process with certainty, even asymptotically. Thus the question becomes: Does there exist a stationary Gaussian process G on the unit interval such that the measure of $G(\cdot) - G(0)$ is absolutely continuous with respect to the measure of W, yet G has a substantially different distribution than any CARMA$(p, p-1)$, for all finite p?

The following theorem shows that the answer is no, at least under an additional technical assumption about the spectral density of G.

Theorem 2 *Let G be a mean-zero continuous time stationary Gaussian process on the unit interval satisfying*

(i) $G(\cdot) - G(0)$ is absolutely continuous with respect to the measure of W, and

(ii) G has a spectral density $f_G : \mathbb{R} \to [0, \infty)$ satisfying $\sup_\lambda (1 + \lambda^2) f_G(\lambda) < \infty$ and $\inf_\lambda (1 + \lambda^2) f_G(\lambda) > 0$.

Then for any $\varepsilon > 0$, there exists a CARMA$(p_\varepsilon, p_\varepsilon - 1)$ process J_{p_ε} such that the total variation distance between the measures of G and J_{p_ε} is smaller than ε.

Theorem 2 asserts that a very large class of stationary Gaussian processes on the unit interval can be arbitrarily well approximated by a CARMA$(p, p-1)$ process, at least with sufficiently large p and suitably chosen parameters. In conjunction with Theorem 1, this implies that the GLTU(p) class is a nearly unrestricted starting point for approximating stationary forms of persistence that are not entirely distinct from the unit root model in large samples. To be precise, suppose the large sample properties of $x_{T,t}$ are characterized by the convergence $\hat{G}_T(s) = T^{-1/2}(x_{T,[sT]} - \mu) \Rightarrow G(s)$. Then there exists a GLTU(p_ε) model with large sample properties characterized by J_{p_ε}, and for any function $\psi(\hat{G}_T)$ that is sufficiently continuous for the continuous mapping theorem to hold, the large sample stochastic properties of $\psi(\hat{G}_T)$ are nearly indistinguishable from those of $\psi(J_{p_\varepsilon})$, for all such ψ.

Note that the spectral density of an Ornstein-Uhlenbeck process J_1 with mean reverting parameter $c = 1$ is given by $f_{J_1}(\lambda) = (2\pi)^{-1}\omega^2/(\lambda^2 + 1)$, so that $\lim_{\lambda\to\infty}(1 + \lambda^2) f_{J_1}(\lambda) = \omega^2/(2\pi)$. In fact, it follows from (9) that $\lim_{\lambda\to\infty}(1 + \lambda^2) f_{J_p}(\lambda) = \omega^2/(2\pi)$ for any CARMA$(p, p-1)$ process. Yet the regularity assumption in part (ii) of Theorem 2 only requires that $(1 + \lambda^2) f_G(\lambda)$ is bounded away from zero and infinity uniformly in λ, but not that it converges as $\lambda \to \infty$ (so Theorem 2 covers cases where $\sup_\lambda (1 + \lambda^2)|f_G(\lambda) - f_{J_{p_\varepsilon}}(\lambda)|$ is large, even for small ε). It also immediately follows from Theorem III.17 of Ibragimov and Rozanov (1978) that if $(1 + \lambda^q) f_G(\lambda)$ is bounded away from zero and infinity uniformly in

λ for some $q > 1$, then for any $q \neq 2$, the measure of G is orthogonal to the measure of J_1, and hence the first assumption in Theorem 2 is violated. We thus consider assumption (ii) a fairly mild regularity condition.

The proof of Theorem 2 is involved. We leverage classic results on the mutual absolute continuity (but not approximability) of Gaussian measures by Ibragimov and Rozanov (1978) to obtain a bound on the entropy norm between the measures of a countable set of characterizing random variables $\{\psi_j(G)\}_{j=1}^{\infty}$ and those of potential approximating process in terms of their spectral densities, and then apply a locally compact version of the Stone-Weierstrass theorem to uniformly approximate f_G by some $f_{J_{p_\varepsilon}}$. See the appendix for details.

4 A Limited-Information Likelihood Framework

In this section we suggest a framework for conducting large sample inference with the GLTU model. A natural place to start would be the likelihood of J_p. Pham-Dinh (1977) derives the likelihood but notes that it is "too complicated for practical use" (page 390). What is more, it wouldn't be appropriate to treat $T^{-1/2}(x_{T,[\cdot T]} - \mu)$ as a realization of $J_p(\cdot)$ directly, since Theorem 1 only establishes weak convergence. To make further progress, we restrict attention to inference that is a function only of the N random variables $\{x_{T,\lfloor jT/N \rfloor}\}_{j=1}^{N}$, for some given finite integer N.

The following result is immediate from Theorem 1 and the continuous mapping theorem.

Corollary 1 *Under Condition 1, for any fixed integer $N \geq 1$,*

$$\{T^{-1/2}(x_{T,\lfloor jT/N \rfloor} - \mu)\}_{j=1}^{N} \Rightarrow \{J_p(j/N)\}_{j=1}^{N}. \tag{12}$$

An asymptotically justified limited-information likelihood of the GLTU(p) model is thus given by the likelihood of a discretely sampled CARMA($p, p-1$) process. The number N determines the resolution of the limited-information "lens" through which we view the original data $\{x_{T,t}\}_{t=1}^{T}$. The convergence in Theorem 1, and thus in (12), are approximations that show that under a wide range of weak dependence of u_t, Central Limit Theorem type effects yield large sample Gaussianity and a dependence structure that is completely dominated by the long-run dependence properties of the GLTU(p) model. In finite samples, a large N takes these approximations seriously even on a relatively fine grid, so in general, a large N reduces the robustness of the resulting inference. At the same time, a small N leads to a fairly uninformative limited-information likelihood.[2] The choice of N thus amounts to a classic efficiency vs. robustness trade-off. In our applications, we set $N = 50$.

As noted in the introduction, there are a number of suggestions in the literature on how to obtain the likelihood of a discretely sampled CARMA$(p, p-1)$ process. One potential difficulty is the computation of covariance matrices involving matrix exponentials (cf. (8)). If the local-to-unity AR roots are distinct, then the companion matrix $\mathbf{A}$ is diagonalizable, so one can rotate the system by the matrix of eigenvectors to avoid this difficulty. But in general, this yield a complex valued system, which requires additional care. What is more, one might not want to rule out a pair of identical local-to-unity AR roots a priori.

We now develop an alternative approach for the computation of the likelihood of $\{J_p(j/N)\}_{j=1}^N$ that avoids these difficulties. To this end, consider the discrete time Gaussian ARMA$(p, p-1)$ process

$$(1-\rho_{T_0,1}L)\cdots(1-\rho_{T_0,p}L)(x^0_{T_0,t}-\mu) = (1-\gamma_{T_0,1}L)\cdots(1-\gamma_{T_0,p-1}L)u^0_t \tag{13}$$

for $t = 1, \dots, T_0$, where T_0 is large, $u^0_t \sim iid\mathcal{N}(0, \omega^2)$ and $\rho_{T_0,j}$, $\gamma_{T_0,j}$ are defined below (5). As in (10) and (11), $x^0_{T_0,t}$ has the state space representation

$$\begin{aligned} x^0_{T_0,t} &= \mathbf{b}'\mathbf{Z}^0_{T_0,t} + \mu && (14)\\ \mathbf{Z}^0_{T_0,t} &= (\mathbf{I}_p + \mathbf{A}/T_0)\mathbf{Z}^0_{T_0,t-1} + \mathbf{e}u^0_t && (15) \end{aligned}$$

with $\mathbf{\Omega}^0_{T_0} = E[\mathbf{Z}^0_{T_0,0}(\mathbf{Z}^0_{T_0,0})']$ satisfying $\operatorname{vec}\mathbf{\Omega}^0_{T_0} = \omega^2(\mathbf{I}_{p^2} - (\mathbf{I}_p + \mathbf{A}/T_0)\otimes(\mathbf{I}_p + \mathbf{A}/T_0))^{-1}\operatorname{vec}(\mathbf{ee}')$.

With $T = T_0$ and $u_t = u^0_t$, the model (13) clearly satisfies Condition 1, so by Corollary 1, $\{T_0^{-1/2}(x^0_{T_0,\lfloor jT_0/N\rfloor} - \mu)\}_{j=1}^N \Rightarrow \{J_p(j/N)\}_{j=1}^N$ as $T_0 \to \infty$. Furthermore, since $\{x^0_{T_0,t}\}_{t=1}^{T_0}$ is a Gaussian process, this further implies convergence of the corresponding first two moments. Thus, the Gaussian likelihood of $\{T_0^{-1/2}(x^0_{T_0,\lfloor jT_0/N\rfloor} - \mu)\}_{j=1}^N$ approximates the likelihood of $\{J_p(j/N)\}_{j=1}^N$ arbitrarily well as $T_0 \to \infty$. An accurate approximation to the asymptotically justified limited-information likelihood for the $2p+1$ parameters μ, ω^2, $\{c_j\}_{j=1}^p$ and $\{g_j\}_{j=1}^{p-1}$ of the GLTU(p) model can therefore be obtained from a straightforward application of the Kalman filter with state (15) and observations $x^0_{T_0,\lfloor jT_0/N\rfloor} = x_{T,\lfloor jT/N\rfloor}$, $j = 1, \dots, N$, with all other observations of $x^0_{T_0,t}$ treated as missing. In our applications, we found that setting $T_0 = 1000$ leads to results that remain numerically stable also for larger values of T_0.

[2] The number of observations N thus plays a similar role to the number of cosine regression coefficients q in the low-frequency extraction approach of Müller and Watson (2017); in combination with the Kalman filter described below, the approach based on (12) is computationally much more advantageous, however.

A remaining difficulty is the restriction on the parameters $\{c_j\}_{j=1}^{p}$ and $\{g_j\}_{j=1}^{p-1}$ of Condition 1 (ii). Here we follow Jones (1981), who noted that under Condition 1 (ii), one can rewrite $a(z)$ and $b(z)$ as a product of quadratic factors (and a linear factor if p is odd), where each quadratic factor collapses a potentially conjugate pair of roots into a quadratic polynomial with positive coefficients. For instance, if $c_1 = c_1^r + c_1^i i$ and $c_2 = c_1^r - c_1^i i$ with $c_1^r > 0$ and $c_1^i \in \mathbb{R}$, then $(z+c_1)(z+c_2) = (c_1^i)^2 + (c_1^r)^2 + 2c_1^r z + z^2$, and if c_1 and c_2 are real and positive, $(z+c_1)(z+c_2) = c_1c_2 + (c_1+c_2)z + z^2$. Either way, the resulting quadratic polynomial is of the form $h_1^2 + 2h_2 z + z^2$, with $h_1, h_2 > 0$, and in this parameterization $c_{1,2} = h_2 \pm \sqrt{h_2^2 - h_1^2}$. The same argument applies to the MA polynomial, so in such a reparameterization, the limited-information likelihood evaluation as described above involves no complex numbers at all.

5 Applications

This section describes two applications of the GLTU(p) model and the limited-information framework of the last section. The first application considers frequentist tests of the null hypothesis of no predictability in a regression framework where the predictor is highly persistent. A large literature has considered this problem in a framework where the predictor is assumed to follow the LTU model: see, for instance, Elliott and Stock (1994), Cavanagh, Elliott, and Stock (1995), Campbell and Yogo (2006), Jansson and Moreira (2006) and Elliott, Müller, and Watson (2015). We consider the size properties of the popular test by Campbell and Yogo (2006) when the predictor is in fact generated by the GLTU(2) model. Using the CRSP data set for the price dividend ratio, we find that empirically plausible values of the GLTU(2) parameters induce large size distortions.

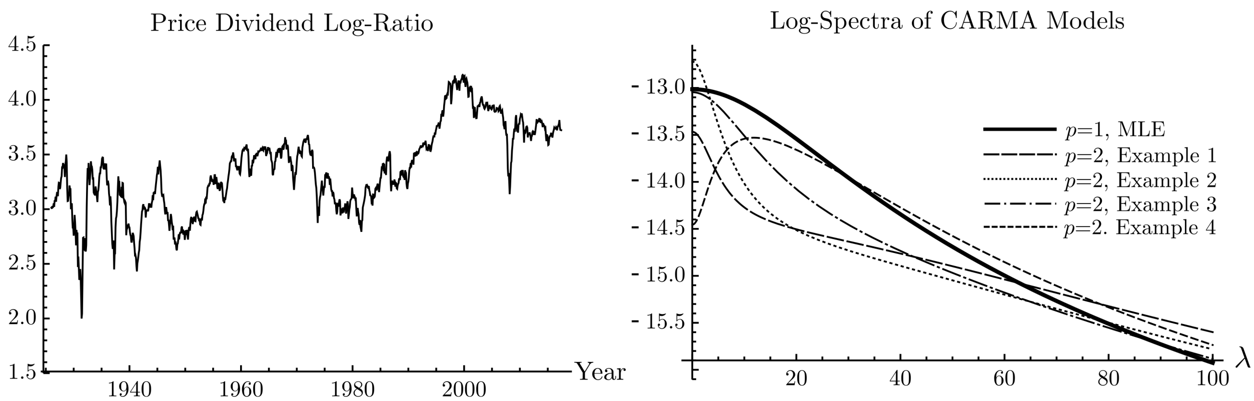
Our second application concerns the quantification of mean reversion in real exchange rates predicted by the theory of purchasing power parity, applied to the long-span data assembled by Lothian and Taylor (1996).

5.1 Predictive Regression with a Persistent Predictor

Let $y_{T,t}$ denote the excess stock return in period t, and let $x_{T,t-1}$ denote a potential predictor variable observed at $t-1$. Campbell and Yogo (2006) consider the regression

$$y_{T,t} \quad = \quad \mu_y + \beta x_{T,t-1} + e_t, \tag{16}$$

Figure 1: CRSP Price Dividend Ratio and Empirically Plausible Limiting Log-Spectra



$$(1 - \rho_T L)\,(x_{T,t} - \mu) \;\; = \;\; u_t \tag{17}$$

where $\rho_T = 1 - c/T$ for fixed $c > 0$, and the mean-zero disturbances (e_t, u_t) are weakly dependent in the sense that $T^{-1/2} \sum_{t=1}^{\lfloor \cdot T \rfloor} (e_t, u_t)' \Rightarrow (W_e, W)' = \mathbf{W}(\cdot)'$ with $\mathbf{W}$ a bivariate Wiener process with correlation r_{eu} (see Appendix A of Campbell and Yogo (2006) for a precise statement of the conditions). By construction, Campbell and Yogo's (2006) test of the null hypothesis of no predictability $H_0 : \beta = 0$ against $H_1 : \beta \neq 0$ is asymptotically valid under this LTU assumption for the predictor, that is it rejects a true null hypothesis at most 10% of the times in repeated samples.[3]

As an empirical illustration, we follow Campbell and Yogo (2006) and consider the monthly excess return on the NYSE/AMSE value-weighted monthly index for $y_{T.t}$ and the corresponding price-dividend log-ratio, averaged over the preceding 12 months, for $x_{T.t}$, constructed from the database of the Center for Research in Security Prices (CRSP). We update the Campbell and Yogo (2006) data set to 1098 monthly observations from 1926:1-2018:6. The left panel in Figure 1 plots $x_{T,t}$.

Now suppose that in contrast to Campbell and Yogo's assumption, $x_{T,t}$ follows a GLTU(2)

[3] Technically, Campbell and Yogo (2006) assume a non-stationary LTU model with zero initial condition. We find in unreported results that in analogy to the results below, an empirically plausible GLTU(2) with zero initial condition still induces large size distortions.

Table 1: Four GLTU(2) Parameters and Resulting Null Rejection Probability of Campbell-Yogo (2006) Test

Example No.	1	2	3	4
Value of c_1	70.9	70.0	60.6	29.8
Value of c_2	4.4	4.4	11.1	6.0
Value of g_1	7.3	11.7	24.3	3.3
Null rejection probability	49.8%	46.6%	41.2%	40.3%

model

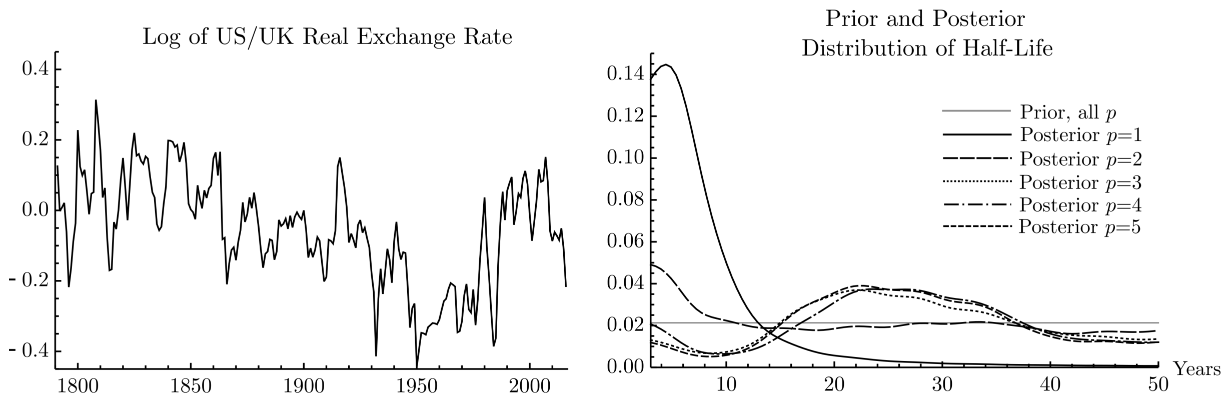
$$(1 - \rho_{T,1}L)(1 - \rho_{T,2}L)(x_{T,t} - \mu) = \left(1 - \gamma_{T,1}L\right) u_t$$

and Condition 1 holds. To obtain plausible parameters of the GLTU(2) model, we first maximize the limited-information likelihood as described in Section 4 in the $p = 1$ LTU model, yielding a value of c equal to 24.0. Call values of $\{c_1, c_2, g_1\}$ "empirically plausible" for the GLTU(2) model if the profiled value over μ and ω^2 of the limited-information likelihood is within two log-points of the LTU maximum likelihood. For such values, consider a data generating process with $T = 1098$, $\beta = 0$, $(e_t, u_t)'$ i.i.d. mean-zero normal and correlation equal to $r_{eu} = -0.951$, which is the value of r_{eu} estimated by Campbell and Yogo's procedure under the LTU model assumption.

Under this GLTU(2) data generating process, we compute the rejection probability of Campbell and Yogo's (2006) nominal 10% level two-sided test of no predictability (the test is invariant to translation shifts and scale transformations of $y_{T,t}$ and $x_{T,t}$, so the variances of e_t and u_t, as well as the means μ and μ_y are immaterial). Since $\beta = 0$ in the data generating process, numbers larger than 10% indicate size distortions. In Table 1, we report the parameter values for four fairly distinct empirically plausible GLTU(2) parameters that induce severe size distortions. The right panel in Figure 1 plots the corresponding log spectral densities of the limiting CARMA(2,1) model, along with the limiting Ornstein-Uhlenbeck process with $c = 24.0$, that is at the limited-information MLE.

We conclude from this exercise that the LTU model is not well suited for generic persistence modelling: Empirically plausible deviations from the LTU model grossly invalidate inference.

Figure 2: Bayesian Limited-Information Analysis of US/UK Real Exchange Rates



5.2 Persistence of Deviations from Purchasing Power Parity

Lothian and Taylor (1996) assembled long-term data on the log US/UK real exchange rate from 1791 to 1990 and estimated half-life deviations of approximately 6 years based on an AR(1) specification. We consider the same data extended[4] through 2016, $x_{T,t}$, and plotted in the left panel of Figure 2. We are interested in quantifying for how long deviations from purchasing power parity persist assuming that the exchange rate $x_{T,t}$ follows a GLTU(p) model.

The traditional definition of the half-life is based on the impulse response of the Wold innovation to $x_{T,t}$, which in general depends not only on the GLTU(p) parameters $\{c_j\}$ and $\{g_j\}$, but also on the short-run dynamics of u_t. See, for instance, Andrews and Chen (1994), Murray and Papell (2002) or Rossi (2005). At the same time, as discussed in Taylor's (2003) survey, the literature on real exchange rates emphasizes mean reversion *in the long run*, and often applies corresponding augmented Dickey-Fuller regressions, which in the context of the LTU model amount to inference about c (also see Murray and Papell (2005) and Stock (1991)).

Impulse responses are most meaningful in the context of a structural model, where innovations are given an explicit interpretation. But the structural interpretation of Wold innovations to the real exchange rates is not obvious. We therefore define the half-life in

[4]The extension is based on the FRED series DEXUSUK, SWPPPI and WPSFD49207 for recent values of the exchange rate, and UK and US producer price indeces.

terms of the following thought experiment: Given the model parameters, suppose we learn that the value of the stationary process $x_{T,t}$ at the time $t = 0$ is one unconditional standard deviation above its mean, but we don't observe any other values of $x_{T,t}$. What is the smallest horizon τ such that the best linear predictor of $x_{T,t}$ given $x_{T,0}$ is within 1/2 unconditional standard deviations of its mean, for all $t \geq \tau$?

The best linear predictor of $x_{T,t}$ given $x_{T,0}$ is proportional to the correlation between $x_{T,0}$ and $x_{T,t}$. Assuming that u_t has more than two moments, Theorem 1 implies that

$$T^{-1}E[(x_{T,0} - \mu)(x_{T,\lfloor sT \rfloor} - \mu)] \to E[J_p(0)J_p(s)] = \mathbf{b}'e^{\mathbf{A}|s|}\mathbf{\Sigma}\mathbf{b}$$

so that we obtain the large sample approximation

$$\tau \approx T \inf_r \{r : \frac{\mathbf{b}'e^{\mathbf{A}|s|}\mathbf{\Sigma}\mathbf{b}}{\mathbf{b}'\mathbf{\Sigma}\mathbf{b}} \leq 1/2 \text{ for all } s \geq r\}. \tag{18}$$

For $p = 1$, that is in the LTU model, this definition of a half-life is equivalent to the half-life of the impulse response relative to the "long-run" shock u_t, which in large samples becomes the impulse response function of J_1. But for $p > 2$, this equivalence breaks down, since the impulse response function of J_p is equal to $\mathbf{1}[s \geq 0]\mathbf{b}'e^{\mathbf{A}s}\mathbf{e}$ (cf. (6) and (7)), while the autocovariance function is $\mathbf{b}'e^{\mathbf{A}|s|}\mathbf{\Sigma}\mathbf{b}$. The definition (18) also differs from measuring the decay of deviations by the the sum of autoregressive coefficients in the AR(∞) representation, which in a GLTU model equals to $1 - d_T/T$ with $d_T \to d = a(0)/b(0) = \prod_{j=1}^{p} c_j / \prod_{j=1}^{p-1} g_j$ (as implied by (9)).

In order to avoid evaluating the matrix exponential in (18), note that $\mathbf{b}'e^{\mathbf{A}|s|}\mathbf{\Sigma}\mathbf{b}$ can be arbitrarily well approximated by the autocovariance function of the discrete stationary state space system (14) and (15) as $T_0 \to \infty$, so that

$$\tau \approx T \inf_r \{r : \frac{\mathbf{b}'(\mathbf{I}_p + \mathbf{A}/T_0)^{\lfloor sT_0 \rfloor}\mathbf{\Omega}_{T_0}^0\mathbf{b}}{\mathbf{b}'\mathbf{\Omega}_{T_0}^0\mathbf{b}} \leq 1/2 \text{ for all } s \geq r\}. \tag{19}$$

We again find that choosing $T_0 = 1000$ generates numerically stable results.

We consider the GLTU model with $p = 1, 2, \cdots, 5$, and conduct inference based on the limited-information likelihood for $N = 50$ as introduced in Section 4. We employ the h parameterization of $\{c_j\}_{j=1}^p$ and $\{g_j\}_{j=1}^{p-1}$ as discussed there, collected in the vector $\mathbf{h} = (h_1^c, \ldots, h_p^c, h_1^g, \ldots, h_{p-1}^g)' \in \mathbb{R}^p \times \mathbb{R}^{p-1}$, where for p odd, $c_p = 2h_p^c$, and for p even, $g_{p-1} = 2h_{p-1}^g$. We restrict each element in $\mathbf{h}$ to be in the interval $(0, 40]$, so that the real components of all c_j and g_j are smaller than 80 (a root with $c = 80$ implies very fast mean reversion, with

Table 2: Bayesian Limited-Information Analysis of US/UK Real Exchange Rates

$p =$	1	2	3	4	5
maximal log-likelihood	28.4	30.7	33.1	36.2	37.5
posterior median half-life	6.3	23.7	26.4	27.1	26.4
90% posterior interval	(3.6; 21.2)	(4.1; 47.0)	(8.1; 46.3)	(5.4; 45.9)	(9.5; 45.7)

a half-life of $(\ln 2)T/80 \approx 1.96$ years). We choose the usual improper uninformative priors for the location and scale parameters μ and ω^2. Let $\tau(\mathbf{h})$ be the half-life in (18) implied by a given value of $\mathbf{h}$. Then we let the prior on $\mathbf{h}$ be proportional to $g(\tau(\mathbf{h}))$, where the function $g : \mathbb{R} \mapsto [0, \infty)$ is such that the implied prior distribution of $\tau(\mathbf{h})$ is uniform on the interval $[3, 50]$. In this way, the prior on the object of interest $\tau(\mathbf{h})$ is controlled and independent of p. (In practice, g is computed by generating many draws from a prior with $g = g_0$ for some initial guess g_0, and g is then given by $\mathbf{1}[3 \leq \tau(\mathbf{h}) \leq 50]g_0(\tau(\mathbf{h}))$ divided by a kernel estimate of the resulting density of $\tau(\mathbf{h})$.)

The posterior is obtained by a standard Gibbs sampler with a random walk Metropolis-Hastings step for $\mathbf{h}$. With the Kalman filter approximation to the limited-information likelihood of Section 4 and the corresponding half-life approximation (19), evaluation is very fast and it only takes minutes to generate 100,000 draws, even for $p = 5$.

The last two rows of Table 2 provide summary statistics for the posterior half-life for $1 \leq p \leq 5$, and the right panel of Figure 2 plots the posterior densities. For $p = 1$, the posterior for the half-life is unimodal with a mode of around 4.5 years and a median of 6.3, more or less in line with the original results of Lothian and Taylor (1996). But letting $p > 1$ leads to posteriors with much more mass at substantially longer half-lives. This accords qualitatively with Murray and Papell's (2005) finding of longer half-life point estimates when allowing for many lags in the autoregression, although their half-lives are computed from impulse responses or sums of autoregressive coefficients, and are thus not directly comparable.

Remarkably, the posterior densities in Figure 2 for $p = 3, 4, 5$ are very similar to each other. It seems that once the model is flexible enough, the implications settle, with a posterior mode of the half-life at around 25 years. The maximized values of the log-likelihood is more than 4.7 log-points larger for $p \geq 3$ compared to $p = 1$, suggesting that the additional flexibility provided by the GLTU model is preferred by the data.[5] Overall, these results suggest that deviations from purchasing power parity are considerably more persistent than an analysis based on the LTU model suggests.

6 Conclusion

This paper introduces the GLTU(p) model as a natural generalization of the popular local-to-unity approach to modelling stationary time series persistence. The main theoretical result concerns the richness of this model class: The asymptotic properties of a very large class of persistent processes can be well approximated by some GLTU(p) model. What is more, we suggest a straightforward approximation to the limited-information asymptotic likelihood of the GLTU(p) model. The GLTU(p) model thus seems a convenient starting point for the modelling of persistent time series in macroeconomics and finance.

[5] More formally, the low-frequency stationarity test LFST of Müller and Watson (2008) rejects the null hypothesis of stationarity of the LTU innovations $y_t - (1 - \hat{c}_{\mathrm{MLE}}/T)y_{t-1}$ at the 5% level for $10 \leq q \leq 50$ ($q = 10$ corresponds to testing stationarity for frequencies below 45 year cycles, and $q = 50$ corresponds to frequencies below 9 year cycles).

7 Appendix

7.1 Proof of Theorem 1

We first show that $x_{T,t}$ has representation (20) and (21). Set $\prod_{j=1}^{p}(z-\rho_{T,j}) = z^p + \sum_{j=1}^{p}\phi_{T,j}z^{p-j}$ and $\prod_{j=1}^{p-1}(z-\gamma_{T,j}) = z^{p-1} + \sum_{j=0}^{p-2}\theta_{T,j}z^j$. The usual state-space representation of the ARMA$(p,p-1)$ process $x_{T,t}$ with innovations u_t is

$$
\begin{aligned}
x_{T,t} &= \boldsymbol{\theta}_T'\mathbf{V}_t + \mu & (20)\\
\mathbf{V}_{T,t} &= \mathbf{\Phi}_T\mathbf{V}_{T,t-1} + \mathbf{e}u_t & (21)
\end{aligned}
$$

where

$$
\mathbf{V}_{T,t} = \begin{pmatrix} v_{T,t-p+1} \\ v_{T,t-p+2} \\ \vdots \\ v_{T,t-1} \\ v_{T,t} \end{pmatrix},\ \mathbf{\Phi}_T = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -\phi_{T,p} & -\phi_{T,p-1} & -\phi_{T,p-2} & \cdots & -\phi_{T,1} \end{pmatrix},\ \boldsymbol{\theta}_T = \begin{pmatrix} \theta_{T,0} \\ \theta_{T,1} \\ \vdots \\ \theta_{T,p-2} \\ 1 \end{pmatrix}.
$$

Let $\mathbf{c} = (c_1,\ldots,c_p)'$ and $\mathbf{g} = (g_1,\ldots,g_{p-1})'$ with elements ordered ascendingly by the real parts, and define the corresponding vectors $\boldsymbol{\rho}_T = (\rho_{T,1},\ldots,\rho_{T,p})'$ and $\boldsymbol{\gamma}_T = (\gamma_{T,1},\ldots,\gamma_{T,p-1})'$.

Since $\mathbf{\Phi}_T$ and $\mathbf{A}$ are companion matrices, and the roots of $z^p + \sum_{j=1}^{p}\phi_{T,j}z^{p-j}$ and $a(z)$ are $\boldsymbol{\rho}_T$ and $-\mathbf{c}$, respectively, they allow the Jordan decomposition (cf. Brand (1964))

$$
\begin{aligned}
\mathbf{\Phi}_T &= \mathbf{Q}(\boldsymbol{\rho}_T)\mathbf{J}(\boldsymbol{\rho}_T)\mathbf{Q}(\boldsymbol{\rho}_T)^{-1} & (22)\\
\mathbf{A} &= \mathbf{Q}(-\mathbf{c})\mathbf{J}(-\mathbf{c})\mathbf{Q}(-\mathbf{c})^{-1} & (23)
\end{aligned}
$$

where for any $\mathbf{a} = (a_1,\ldots,a_k)' \in \mathbb{C}^k$, $k \in \{p-1,p\}$ with $\mathrm{Re}(a_j) \le \mathrm{Re}(a_{j+1})$, $\mathbf{J}(\mathbf{a})$ is a Jordan matrix with Jordan blocks corresponding to common values of a_i, and $\mathbf{Q}(\mathbf{a})$ are the corresponding (generalized) eigenvectors. More specifically, the m columns of $\mathbf{Q}(\mathbf{a})$ corresponding to the value a of multiplicity m contain the values $\frac{d^l}{dz^l}z^{j-1}|_{z=a}/l!$, $j=1,\ldots,k$, $l=0,\ldots,m-1$.

From (23), we also have

$$
\mathbf{I} + \mathbf{A}/T = \mathbf{Q}(-\mathbf{c})(\mathbf{I} + \mathbf{J}(-\mathbf{c})/T)\mathbf{Q}(-\mathbf{c})^{-1}. \tag{24}
$$

Let $\mathbf{F}$ be the $p \times p$ lower triangular Pascal matrix, that is, the first j entries in row j of $\mathbf{F}$ contain the jth binomial coefficients, and let $\mathbf{D}_T = \operatorname{diag}(1, T^{-1}, \ldots, T^{1-p})$. Further, let $\mathbf{D}_T^c$ be a diagonal matrix where the diagonal elements corresponding to a Jordan block of $\mathbf{A}$ of size m are equal to $1, T, \ldots, T^{m-1}$. Then, with $\mathbf{P}_T = \mathbf{F}\mathbf{D}_T$ we have from a straightforward calculation

$$\begin{aligned} \mathbf{Q}(\boldsymbol{\rho}_T) &= \mathbf{P}_T \mathbf{Q}(-\mathbf{c}) \mathbf{D}_T^c \\ \mathbf{D}_T^c \mathbf{J}(\boldsymbol{\rho}_T)(\mathbf{D}_T^c)^{-1} &= \mathbf{I} + \mathbf{J}(-\mathbf{c})/T \end{aligned} \tag{25}$$

so that from (22) and (24)

$$\boldsymbol{\Phi}_T = \mathbf{P}_T(\mathbf{I} + \mathbf{A}/T)\mathbf{P}_T^{-1}. \tag{26}$$

Furthermore, since $z^{p-1} + \sum_{j=0}^{p-2} \theta_{T,j} z^j = \prod_{j=1}^{p-1}(z - \gamma_{T,j})$, we have $\boldsymbol{\theta}_T' \mathbf{Q}(\boldsymbol{\gamma}_T) = 0$, and similarly, $\mathbf{b}'\mathbf{Q}(-\mathbf{g}) = 0$. Now as in (25), $\mathbf{Q}(\boldsymbol{\gamma}_T) = \mathbf{P}_T \mathbf{Q}(-\mathbf{g}) \mathbf{D}_T^g$ for some diagonal matrix $\mathbf{D}_T^g$ with nonzero diagonal elements, so that also $\boldsymbol{\theta}_T' \mathbf{P}_T \mathbf{Q}(-\mathbf{g}) = 0$. Since $\mathbf{Q}(-\mathbf{g})$ is of full column rank (cf. Theorem 2 of Brand (1964)), we conclude that $\boldsymbol{\theta}_T' \mathbf{P}_T$ is a scalar multiple of $\mathbf{b}'$. The last element of $\mathbf{b}'$ is equal to one, and the last element of $\boldsymbol{\theta}_T' \mathbf{P}_T$ is equal to T^{1-p}, so that

$$\boldsymbol{\theta}_T' \mathbf{P}_T = T^{1-p} \mathbf{b}'. \tag{27}$$

Finally, from $\mathbf{P}_T \mathbf{e} = T^{1-p} \mathbf{e}$,

$$\mathbf{P}_T^{-1} \mathbf{e} = T^{p-1} \mathbf{e}. \tag{28}$$

From (26), (27) and (28) it follows that the system (20) and (21) can equivalently be written as (10) and (11).

For an arbitrary valued matrix $\mathbf{B}$, let $||\mathbf{B}||$ its largest singular value. In the following, let T be large enough so that $|\rho_{T,j}|^2 = 1 - 2\operatorname{Re}(c_j)/T + |c_j|^2/T^2 \leq (1 - \frac{1}{2}\operatorname{Re}(c_1)/T)^2$ for all $j = 1, \ldots, p$, so that from (26), also

$$||\mathbf{I} + \mathbf{A}/T|| \leq 1 - \tfrac{1}{2}\operatorname{Re}(c_1)/T. \tag{29}$$

Now from (10) and (11), for any fixed integer $K > 0$,

$$T^{-1/2}(x_{T,\lfloor sT \rfloor} - \mu) = R_T(s) + T^{-1/2}\mathbf{b}' \sum_{t=-KT+1}^{\lfloor sT \rfloor} (\mathbf{I} + \mathbf{A}/T)^{\lfloor sT \rfloor - t} \mathbf{e} u_t$$

where $R_T(s) = T^{-1/2}\mathbf{b}'(\mathbf{I}+\mathbf{A}/T)^{\lfloor sT \rfloor + KT}\mathbf{Z}_{-KT}$ and $\mathbf{Z}_{-KT} = \sum_{t=0}^{\infty}(\mathbf{I}+\mathbf{A}/T)^t \mathbf{e} u_{-KT-t}$, and we write $\mathbf{Z}_t$ for $\mathbf{Z}_{T,t}$ to ease notation. Since the autocovariances of u_t are absolutely summable, the spectral density of u_t exists and is bounded on $[-\pi, \pi]$. Let a bound be $\tilde{\sigma}_u^2/(2\pi)$. For any given T and $\mathbf{w} \in \mathbb{R}^p$, the variance of the time invariant linear filter $\mathbf{w}'\mathbf{Z}_{-KT}$ is thus weakly smaller than the variance of $\mathbf{w}'\tilde{\mathbf{Z}}_{-KT}$, where $\tilde{\mathbf{Z}}_{-KT} = \sum_{t=0}^{\infty}(\mathbf{I} + \mathbf{A}/T)^t \mathbf{e}\tilde{u}_{-KT-t}$ with $\tilde{u}_t \sim iid(0, \tilde{\sigma}_u^2)$. Furthermore

$$\mathrm{Var}[T^{-1/2}\tilde{\mathbf{Z}}_{-KT}] = \tilde{\sigma}_u^2 T^{-1} \sum_{t=0}^{\infty} (\mathbf{I} + \mathbf{A}/T)^t \mathbf{e}\mathbf{e}'(\mathbf{I} + \mathbf{A}'/T)^t$$

so that from (29)

$$|| \mathrm{Var}[T^{-1/2}\tilde{\mathbf{Z}}_{-KT}]|| \leq \tilde{\sigma}_u^2 ||\mathbf{e}\mathbf{e}'|| T^{-1} \sum_{t=0}^{\infty} (1 - \tfrac{1}{2}\mathrm{Re}(c_1)/T)^{2t} = O(1).$$

Thus, $||T^{-1/2}\mathbf{Z}_{-KT}|| = O_p(1)$. Using again (29), we obtain

$$\begin{aligned}
\sup_{0 \leq s \leq 1} |R_T(s)| &\leq ||\mathbf{b}|| \cdot ||T^{-1/2}\mathbf{Z}_{-KT}|| \cdot \sup_s (1 - \tfrac{1}{2}\mathrm{Re}(c_1)/T)^{\lfloor sT \rfloor + KT} \\
&\leq |\mathbf{b}|| \cdot ||T^{-1/2}\mathbf{Z}_{-KT}|| \cdot \exp[-\tfrac{1}{2}K\,\mathrm{Re}(c_1)]
\end{aligned} \tag{30}$$

so that $R_T(\cdot)$ converges in probability to zero as $K \to \infty$.

Furthermore, under Condition 1,

$$W_T(\cdot) = T^{-1/2} \sum_{t=-\lfloor KT \rfloor}^{\lfloor \cdot T \rfloor} u_t \Rightarrow W(\cdot) - W(-K)$$

where W is a Wiener process on the interval $[-K, 1]$ of variance ω^2 normalized to $W(0) = 0$. By summation by parts,

$$\begin{aligned}
& T^{-1/2}\mathbf{b}' \sum_{t=-\lfloor KT \rfloor + 1}^{\lfloor sT \rfloor} (\mathbf{I} + \mathbf{A}/T)^{\lfloor sT \rfloor - t} \mathbf{e} u_t \\
= \quad & \mathbf{b}'\mathbf{e}W_T(s) + \mathbf{b}'\mathbf{A}T^{-1} \sum_{t=-\lfloor KT \rfloor + 1}^{\lfloor sT \rfloor} (\mathbf{I} + \mathbf{A}/T)^{\lfloor sT \rfloor - t} \mathbf{e} W_T(\frac{t-1}{T}) \\
\Rightarrow \quad & \mathbf{b}'\mathbf{e}(W(s) - W(-K)) + \mathbf{b}'\mathbf{A} \int_{-K}^{s} e^{\mathbf{A}(s-r)} \mathbf{e}(W(r) - W(-K)) dr \\
= \quad & \mathbf{b}' \int_{-K}^{s} e^{\mathbf{A}(s-r)} \mathbf{e} dW(r) \\
= \quad & R_0(s) + J_p(s)
\end{aligned}$$

where $R_0(s) = -\mathbf{b}' e^{\mathbf{A}(s+K)}\mathbf{X}(-K)$ with $\mathbf{X}(-K) \sim \mathcal{N}(0, \mathbf{\Sigma})$ independent of W as in (7), the convergence relies on the well-known identity $e^{s\mathbf{A}} = \lim_{T\to\infty}(\mathbf{I} + \mathbf{A}/T)^{\lfloor sT \rfloor}$ for all s, and the second equality follows from the stochastic calculus version of integration by parts. Since $\sup_{0\leq s\leq 1} ||e^{\mathbf{A}(s+K)}|| \to 0$ as $K \to \infty$, $\sup_{0\leq s\leq 1} |R_0(s)|$ converges in probability to zero as $K \to \infty$. As noted below (30), the same holds for $R_T(\cdot)$. But convergence in probability implies convergence in distribution, and K was arbitrary, so the result follows.

7.2 Proof of Theorem 2

Overview

The proof of Theorem 2 relies heavily on the framework developed by Ibragimov and Rozanov (1978), denoted IR78 in the following. As discussed there, a continuous time Gaussian process on the unit interval can be described in terms of a countably infinite sequence of random variables (cf. (39) and the discussion in the proof of Lemma 4 below), whose distribution can be expressed in terms of the spectral density of the underlying process (cf. (38) and the discussion below (39)). The challenge in the proof of Theorem 2 is to establish that the "infinite tail" of this sequence contributes negligibly to the total variation distance. Intuitively, this must hold for some appropriate definition of tail if the two measures are equivalent, and appropriate equivalence results are obtained by IR78. But the construction of this tail must be such that its contribution is negligible *uniformly* over a sufficiently rich class of potential approximating processes. To this end, the sequence of random variables (and hence its tail) is constructed as a function of the properties of two Gaussian processes whose spectral densities form an upper and lower bound on the class of potential approximating spectral density functions (cf. (34), (35) and (36)), which turns out to be suitable to obtain such a uniform bound (cf. (41), (42) and (45)). With the contribution from the tail controlled, the approximability of the distribution of the finite dimensional non-tail part of the sequence of random variables follows with some additional work from Lemmas 1 and 2 below.

We first state Lemmas 1 and 2. We write z^* for the conjugate of the complex number z, and $\mathbf{v}^*$ for the conjugate transpose of a complex vector $\mathbf{v}$.

Lemma 1 *Let $\mathcal{C}_0$ be the space of continuous real valued functions on $[0, \infty)$ which vanish at infinity. For any $\vartheta_0 \in \mathcal{C}_0$ and $\varepsilon > 0$, there exists an integer $q \geq 1$ such that* $\sup_{\lambda\geq 0} |\vartheta_0(\lambda) -$

$\vartheta(\lambda)| < \varepsilon$, *where* ϑ *is a rational function of the form*

$$\vartheta(\lambda) = \frac{\sum_{j=0}^{q-1} e_j^n \lambda^{2j}}{\prod_{j=1}^{q}(\lambda^2 + e_j^d)} \tag{31}$$

with $e_j^d > 0$ *and* $e_j^n \in \mathbb{R}$, $j = 0, \ldots, q$.

Proof. Note that functions of the form ϑ form a vector subspace of $\mathcal{C}_0$ which is closed under multiplication of functions, that is, they form a sub-algebra on $\mathcal{C}_0$. It is easily seen that this sub-algebra separate points and vanishes nowhere. The locally compact version of the Stone-Weierstrass Theorem thus implies the result. ■

Lemma 2 *Let* $\xi_p(\lambda^2)$ *be a polynomial with real coefficients of order* $p-1$ *in* λ^2 *such that* $\xi_p(\lambda^2) > 0$ *for all* $\lambda \in \mathbb{R}$, *and with unit coefficient on* $(\lambda^2)^{p-1}$. *Then there exists polynomial* b *of order* $p-1$ *of the form* $b(z) = \prod_{i=1}^{p-1}(z + g_j)$ *with* g_j *as described in Condition 1 (ii) such that* $\xi_p(\lambda^2) = |b(i\lambda)|^2$ *for all* $\lambda \in \mathbb{R}$.

Proof. By the fundamental theorem of algebra, and since $\xi_p(\lambda^2) > 0$ for all $\lambda \in \mathbb{R}$, $\xi_p(\lambda^2) = \prod_{j=1}^{p-1}(\lambda^2 + \eta_j)$, where the η_j's are of two types: real and positive, or complex with positive real part, and in conjugate pairs. Now for $0 < \eta_j \in \mathbb{R}$, $\lambda^2 + \eta_j = |i\lambda + g_j|^2$ with $g_j = \sqrt{\eta_j}$. For $\eta_j = \eta_{j'}^* \in \mathbb{C}$ for $j \neq j'$,

$$(\lambda^2 + \eta_j)(\lambda^2 + \eta_{j'}) = \lambda^4 + 2\,\mathrm{Re}(\eta_j)\lambda^2 + |\eta_j|^2 = |i\lambda + g_j|^2 |i\lambda + g_j^*|^2$$

where $g_j = (\sqrt{|\eta_j| + \mathrm{Re}(\eta_j)} + \sqrt{|\eta_j| - \mathrm{Re}(\eta_j)}i)/\sqrt{2}$. ■

Without loss of generality, assume $\omega^2 = 2\pi$. In the following, we write G_1 for G, and f_1 for its spectral density. Let $f_0(\lambda) = (1+\lambda^2)^{-1}$ be the spectral density of the Ornstein-Uhlenbeck process with mean reversion parameter equal to unity, denoted G_0, and let

$$\delta_0 = \min(\tfrac{1}{2}\inf_{\lambda}(1+\lambda^2)f_1(\lambda), 1). \tag{32}$$

Let P_0 and P_1 be the measures of G_1 and G_0, respectively. By Theorem III.17 of IR78, equivalence of P_0 and P_1 implies

$$\int (\frac{f_1(\lambda)}{f_0(\lambda)} - 1)^2 d\lambda < \infty \tag{33}$$

and here and below, integrals are over the entire real line unless indicated otherwise.

Define

$$\overline{f}(\lambda) = \max(f_1(\lambda), f_0(\lambda)) + \frac{\delta_0}{(1+\lambda^2)^2} \tag{34}$$

$$\underline{f}(\lambda) = \min(f_1(\lambda), f_0(\lambda)) - \frac{\delta_0}{(1+\lambda^2)^2} \tag{35}$$

and let f_2 be some function satisfying

$$\underline{f}(\lambda) \leq f_2(\lambda) \leq \overline{f}(\lambda) \text{ for all } \lambda. \tag{36}$$

From (33),

$$\int (\frac{\underline{f}(\lambda)}{f_0(\lambda)} - 1)^2 d\lambda < \infty, \ \int (\frac{\overline{f}(\lambda)}{f_0(\lambda)} - 1)^2 d\lambda < \infty \tag{37}$$

so that also for all f_2 satisfying (36), $\int (f_2(\lambda)/f_0(\lambda) - 1)^2 d\lambda < \infty$. Since $\overline{f}$, $\underline{f}$ and f_2 are nonnegative integrable real functions, there exist corresponding correlation functions that are positive definite. By the development in Section I.2 of IR78, there hence exist corresponding stationary Gaussian processes $\overline{G}$, $\underline{G}$ and G_2 with spectral densities $\overline{f}$, $\underline{f}$ and f_2 and measures $\overline{P}$, $\underline{P}$ and P_2, respectively. Theorem III.17 of IR78 and (37) implies that $\overline{P}$, $\underline{P}$ and P_2 are equivalent to P_0, and hence also to P_1.

For $\psi, \varphi : \mathbb{R} \mapsto \mathbb{C}$ functions of the type $\psi(\lambda) = \sum_{k=1}^{n} c_k e^{i\lambda t_k}$ for $t_k \in [0,1]$ and $c_k \in \mathbb{R}$, define the inner product

$$\langle \psi, \varphi \rangle_{F_1} = \int \psi(\lambda)\varphi(\lambda)^* f_1(\lambda) d\lambda. \tag{38}$$

Let $L(F_1)$ be the corresponding Hilbert space. Analogously, define the inner products $\langle \psi, \varphi \rangle_{F_2}$, $\langle \psi, \varphi \rangle_{\overline{F}}$ and $\langle \psi, \varphi \rangle_{\underline{F}}$, and corresponding Hilbert spaces $L(F_2)$, $L(\overline{F})$ and $L(\underline{F})$. Since the measures P_1, P_2, $\overline{P}$ and $\underline{P}$ are equivalent, so are the Hilbert spaces, as noted on page 71 of IR78. Define the linear operator $A : L(F_1) \mapsto L(F_2)$ via $A\psi = \psi$, let A^* be its adjoint, and define the self-adjoint operator $\Delta : L(F_1) \mapsto L(F_1)$ via $\Delta\psi = \psi - A^* A\psi$, so that

$$\langle \Delta\psi, \varphi \rangle_{F_1} = \langle \psi, \varphi \rangle_{F_1} - \langle \psi, \varphi \rangle_{F_2}$$

and analogously for $\overline{\Delta}$ and $\underline{\Delta}$ (that is, $\langle \overline{\Delta}\psi, \varphi \rangle_{F_1} = \langle \psi, \varphi \rangle_{F_1} - \langle \psi, \varphi \rangle_{\overline{F}}$ and $\langle \underline{\Delta}\psi, \varphi \rangle_{F_1} = \langle \psi, \varphi \rangle_{F_1} - \langle \psi, \varphi \rangle_{\underline{F}}$). By Theorem III.4 of IR78, equivalence of the measures P_1, P_2, $\overline{P}$ and $\underline{P}$ implies that the operators Δ, $\overline{\Delta}$ and $\underline{\Delta}$ are Hilbert-Schmidt.

Let ψ_k be an arbitrary orthonormal sequence in $L(F_1)$, and define the $n \times 1$ vector $\boldsymbol{\eta}_n$ of Gaussian complex valued random variables

$$\eta(\psi_k) = \int \psi_k(\lambda) d\Phi_i(\lambda) \text{ for } k = 1, \ldots n \tag{39}$$

where Φ_i is the stochastic spectral measure such that $G_i(s) = \int e^{i\lambda s} d\Phi_i(\lambda)$, $i = 1, 2$ (cf. chapter I.6 of IR78). Then $E[\eta(\psi_k)] = 0$ under both P_1 and P_2, $E[\eta(\psi_j)\eta(\psi_k)] = \langle\psi_j, \psi_k\rangle_{F_1} = \mathbf{1}[i = k]$ under P_1, and $E[\eta(\psi_j)\eta(\psi_k)] = \langle\psi_j, \psi_k\rangle_{F_2}$ under P_2. Thus, under P_1, $\boldsymbol{\eta}_n \sim \mathcal{N}(\mathbf{0}, \mathbf{I}_n)$, and under P_2, $\boldsymbol{\eta}_n \sim \mathcal{N}(\mathbf{0}, \boldsymbol{\Sigma}_n)$, where $\boldsymbol{\Sigma}_n$ has elements $\langle\psi_j, \psi_k\rangle_{F_2}$. Since P_1 and P_2 are equivalent, $\boldsymbol{\Sigma}_n$ is positive definite for any n (cf. page 76 of IR78). Let $\mathbf{v}_{kn} \in \mathbb{C}^n$, $k = 1, \ldots, n$ be a set of eigenvectors of $\boldsymbol{\Sigma}_n$ with associated eigenvalues $\sigma_{kn}^2 \leq \sigma_{k-1,n}^2$ for all $k = 2, \ldots, n$, so that $\mathbf{v}_{kn}^* \boldsymbol{\eta}_n \sim iid\mathcal{N}(0, 1)$ under P_1, and $\mathbf{v}_{kn}^* \boldsymbol{\eta}_n$ are independent $\mathcal{N}(0, \sigma_{kn}^2)$ under P_2. Let d_n be the entropy distance between the distribution of $\boldsymbol{\eta}_n$ under P_1 and P_2, that is the sum of the two corresponding Kullback-Leibler divergences. By a straightforward calculation (cf. equation (III.2.4) of IR78),

$$d_n = \tfrac{1}{2} \sum_{k=1}^{n} [(\frac{1}{\sigma_{kn}^2} - 1) + (\sigma_{kn}^2 - 1)].$$

Define

$$D_n = \sum_{k=1}^{n} (1 - \sigma_{kn}^2)^2$$

and with $\lambda_k(\mathbf{B})$ denoting the kth largest eigenvalue of the Hermitian matrix $\mathbf{B}$, we have

$$\begin{aligned} D_n &= \sum_{k=1}^{n} (1 - \lambda_k(\boldsymbol{\Sigma}_n))^2 \\ &= \sum_{k=1}^{n} \lambda_k((\mathbf{I}_n - \boldsymbol{\Sigma}_n)^2) \\ &= \operatorname{tr}((\mathbf{I}_n - \boldsymbol{\Sigma}_n)^2) \end{aligned}$$

so that

$$D_n = \sum_{j,k=1}^{n} |\langle\psi_j, \psi_k\rangle_{F_1} - \langle\psi_j, \psi_k\rangle_{F_2}|^2 = \sum_{j,k=1}^{n} |\langle\Delta\psi_j, \psi_k\rangle_{F_1}|^2. \tag{40}$$

The following straightforward Lemma establishes a useful relationship between d_n and D_n.

Lemma 3 *For any* $0 < \delta < 1/4$, $D_n < \delta$ *implies* $d_n < \delta$.

Proof. Note that $\sum_{k=1}^{n} (\sigma_k^2 - 1)^2 < 1/4$ implies $1/2 < \sigma_k^2 < 3/2$ for all $k = 1, \ldots, n$, but for such σ_k^2, $\frac{1}{2}[(\frac{1}{\sigma_k^2} - 1) + (\sigma_k^2 - 1)] \leq (\sigma_k^2 - 1)^2$, which implies the result. ■

Let $\overline{\mathbf{\Sigma}}_n$ be the $n \times n$ Hermitian matrix with elements $\langle \psi_j, \psi_k \rangle_{\overline{F}}$. Then for any $\mathbf{v} = (v_1, \ldots, v_n)' \in \mathbb{C}^n$, $\mathbf{v}^*(\overline{\mathbf{\Sigma}}_n - \mathbf{\Sigma}_n)\mathbf{v} = \sum_{j,k=1}^n v_k^* v_j \langle \varphi_{jn}, \varphi_{kn} \rangle_{(\overline{F}-F_2)} = \langle \sum_{j=1}^n v_j \varphi_{jn}, \sum_{k=1}^n v_k \varphi_{kn} \rangle_{(\overline{F}-F_2)} \geq 0$ from (36). Therefore, by Weyl's inequality, $\overline{\sigma}_{kn}^2 = \lambda_k(\overline{\mathbf{\Sigma}}_n) \geq \lambda_k(\mathbf{\Sigma}_n) + \lambda_n(\overline{\mathbf{\Sigma}}_n - \mathbf{\Sigma}_n) \geq \lambda_k(\mathbf{\Sigma}_n) = \sigma_{kn}^2$ for all k, so that

$$\overline{D}_n = \sum_{j,k=1}^n |\langle \overline{\Delta} \psi_j, \psi_k \rangle_{F_1}|^2 = \sum_{k=1}^n (1 - \overline{\sigma}_{kn}^2)^2 \geq \sum_{k=1}^n \mathbf{1}[\sigma_{kn}^2 > 1](1 - \sigma_{kn}^2)^2.$$

By an analogous argument, also

$$\underline{D}_n = \sum_{j,k=1}^n |\langle \underline{\Delta} \psi_j, \psi_k \rangle_{F_1}|^2 \geq \sum_{k=1}^n \mathbf{1}[\sigma_{kn}^2 < 1](1 - \sigma_{kn}^2)^2$$

so that

$$D_n \leq \overline{D}_n + \underline{D}_n. \tag{41}$$

Now let $\overline{\varphi}_k$ be a complete set of eigenvectors of the operator $\overline{\Delta}$, with associated eigenvalues $1 - \overline{\sigma}_k^2$, that is $\overline{\Delta}\overline{\varphi}_k = (1 - \overline{\sigma}_k^2)\overline{\varphi}_k$, and $\overline{\varphi}_k$ form an orthonormal basis in $L(\overline{F})$. Define $\underline{\varphi}_k$ and $\underline{\sigma}_k^2$ analogously relative to the operator $\underline{\Delta}$. Since $\overline{\Delta}$ and $\underline{\Delta}$ are Hilbert-Schmidt, $\sum_{k=1}^\infty (1 - \overline{\sigma}_k^2)^2 < \infty$ and $\sum_{k=1}^\infty (1 - \underline{\sigma}_k^2)^2 < \infty$, so that for any $\epsilon > 0$, there exists n_ϵ such that $\sum_{k=n_\epsilon}^\infty (1 - \overline{\sigma}_k^2)^2 < \epsilon/2$ and $\sum_{k=n_\epsilon}^\infty (1 - \underline{\sigma}_k^2)^2 < \epsilon/2$. Let $\overline{L}_\epsilon^0 \subset L(F_1)$ and $\underline{L}_\epsilon^0 \subset L(F_1)$ be the finite spaces spanned by $\overline{\varphi}_k$ and $\underline{\varphi}_k$, $k = 1, \ldots, n_\epsilon$, respectively, and let L_ϵ^1 be the orthogonal complement of $L_\epsilon^0 = \overline{L}_\epsilon^0 \cup \underline{L}_\epsilon^0$ relative to $\langle \cdot, \cdot \rangle_{F_1}$, so that $L(F_1) = L_\epsilon^0 \cup L_\epsilon^1$. Note that L_ϵ^0 and L_ϵ^1 do not depend on f_2. Let $\overline{L}_\epsilon^1$ be the space spanned by $\overline{\varphi}_k$, $k = n_\epsilon + 1, n_\epsilon + 2, \ldots$ For any orthonormal sequence ψ_k in L_ϵ^1, since $L_\epsilon^1 \subset \overline{L}_\epsilon^1$

$$\begin{aligned}
\overline{D}_n &= \sum_{j,k=1}^n |\langle \overline{\Delta} \psi_j, \psi_k \rangle_{F_1}|^2 \\
&\leq \sum_{j=1}^n ||\overline{\Delta} \psi_j||_{\overline{L}_\epsilon^1}^2 \\
&= \sum_{j=1}^n \sum_{k=n_\epsilon+1}^\infty |\langle \overline{\Delta} \psi_j, \overline{\varphi}_k \rangle_{F_1}|^2 \\
&= \sum_{k=n_\epsilon+1}^\infty \sum_{j=1}^n |\langle \overline{\Delta} \overline{\varphi}_k, \psi_j \rangle_{F_1}|^2 \\
&\leq \sum_{k=n_\epsilon+1}^\infty ||\overline{\Delta} \overline{\varphi}_k||_{\overline{L}_\epsilon^1}^2
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{j,k=n_\epsilon+1}^{\infty} |\langle \overline{\Delta}\overline{\varphi}_j, \overline{\varphi}_k\rangle_{F_1}|^2 \\
&= \sum_{j,k=n_\epsilon+1}^{\infty} (1-\overline{\sigma}_k^2)^2 |\langle \overline{\varphi}_j, \overline{\varphi}_k\rangle_{F_1}|^2 \\
&= \sum_{k=n_\epsilon+1}^{\infty} (1-\overline{\sigma}_k^2)^2 < \epsilon/2
\end{aligned}
$$

where the inequalities follow from Bessel's inequality. By the analogous argument, also $\underline{D}_n \leq \epsilon/2$. Thus, from (41), for any orthonormal sequence ψ_k in L_ϵ^1,

$$
D_n \leq \epsilon. \tag{42}
$$

Now let ψ_k^ϵ, $k = 1, \ldots, m_\epsilon \leq 2n_\epsilon$ be an orthonormal basis of L_ϵ^0, and let ψ_k^ϵ, $k = m_\epsilon + 1, m_\epsilon + 2, \ldots$ be an orthonormal basis of L_ϵ^1, so that ψ_k^ϵ, $k = 1, 2, \ldots$ is an orthonormal basis of $L(F_1)$. Note that the sequence ψ_k^ϵ does not depend on f_2. Let $\mathfrak{U}_m^\epsilon$ be the σ-field generated by the Gaussian random variables $\eta(\psi_k^\epsilon)$ as defined in (39) for $k = 1, \ldots, m$, $i = 1, 2$, and let $\mathfrak{U}^\epsilon$ be the σ-field generated by $\eta(\psi_k^\epsilon)$, $k = 1, 2, \ldots$. Define

$$
D_m^\epsilon = \sum_{j,k=1}^{m} |\langle \Delta\psi_j^\epsilon, \psi_k^\epsilon\rangle_{F_1}|^2.
$$

We have the following Lemma.

Lemma 4 *For all $0 < \epsilon_0 < 1/2$, $\sup_m D_m^\epsilon < \epsilon_0^2$ implies that the total variation distance between P_1 and P_2 is smaller than ϵ_0.*

Proof. As discussed on page 65 of IR78, the distribution on the σ-field $\mathfrak{U}^\epsilon$ equivalently characterizes the distribution of G_i relative to the σ-fields generated by the cylindric sets of the paths $G_i(\cdot)$ under P_i, $i = 1, 2$, so it suffices to show that

$$
\sup_{\mathcal{A}\in\mathfrak{U}^\epsilon} |P_2(\mathcal{A}) - P_1(\mathcal{A})| \leq \epsilon_0. \tag{43}
$$

Let d_m^ϵ be the entropy distance between the distribution of $\eta(\psi_k^\epsilon)$, $k = 1, \ldots, m$ under P_1 and P_2. By Lemma 3, $d_m^\epsilon \leq \epsilon_0^2$. Thus, by Pinsker's inequality

$$
\sup_{\mathcal{A}_m\in\mathfrak{U}_m^\epsilon} |P_2(\mathcal{A}_m) - P_1(\mathcal{A}_m)| \leq \epsilon_0 \text{ for all } m. \tag{44}
$$

Now suppose (43) does not hold. Then there exists $\mathcal{A} \in \mathfrak{U}^\epsilon$ such that $P_2(\mathcal{A}) - P_1(\mathcal{A}) > \epsilon_0$. Construct a sequence of events $\mathcal{A}_m \in \mathfrak{U}_m^\epsilon$ such that $P_i(\mathcal{A}_m \ominus \mathcal{A}) \to 0$ for $i = 1, 2$ as on page 77 of IR78, where $\mathcal{A}_m \ominus \mathcal{A}$ is the symmetric difference $\mathcal{A}_m \ominus \mathcal{A} = (\mathcal{A}_m \cup \mathcal{A}) \backslash (\mathcal{A}_m \cap \mathcal{A})$. Then from $\mathcal{A} \subseteq \mathcal{A}_m \cup (\mathcal{A}_m \ominus \mathcal{A})$ and $\mathcal{A}_m \subseteq \mathcal{A} \cup (\mathcal{A}_m \ominus \mathcal{A})$, we have $|P_i(\mathcal{A}) - P_i(\mathcal{A}_m)| \leq P_i(\mathcal{A}_m \ominus \mathcal{A})$ for $i = 1, 2$. We thus obtain $P_2(\mathcal{A}_m) - P_1(\mathcal{A}_m) \to P_2(\mathcal{A}) - P_1(\mathcal{A}) > \epsilon_0$, contradicting (44), and the lemma is proved. ■

Given that the choice of $0 < \epsilon$ was arbitrary, in light of Lemma 4 it suffices to show that for some CARMA implied f_2 satisfying (36), $\sup_m D_m^\epsilon < 2\epsilon$, say. Now for all $m > m_\epsilon$, from (40)

$$
\begin{aligned}
D_m^\epsilon &\leq \sum_{j,k=m_\epsilon+1}^{m} |\langle \Delta\psi_j^\epsilon, \psi_k^\epsilon \rangle_{F_1}|^2 + 2 \sum_{j=1}^{m_\epsilon} \sum_{k=1}^{m} |\langle \Delta\psi_j^\epsilon, \psi_k^\epsilon \rangle_{F_1}|^2 \qquad (45) \\
&\leq \epsilon + 2 \sum_{j=1}^{m_\epsilon} \sum_{k=1}^{m} |\langle \Delta\psi_j^\epsilon, \psi_k^\epsilon \rangle_{F_1}|^2
\end{aligned}
$$

where the second inequality follows from (42). Further

$$
\begin{aligned}
\sum_{k=1}^{m} |\langle \Delta\psi_j^\epsilon, \psi_k^\epsilon \rangle_{F_1}|^2 &= \sum_{k=1}^{m} |\langle (\frac{f_2}{f_1} - 1)\psi_j^\epsilon, \psi_k^\epsilon \rangle_{F_1}|^2 \\
&\leq \langle (\frac{f_2}{f_1} - 1)\psi_j^\epsilon, (\frac{f_2}{f_1} - 1)\psi_j^\epsilon \rangle_{F_1}^2 \\
&= \int (\frac{f_2(\lambda)}{f_1(\lambda)} - 1)^2 |\psi_j^\epsilon(\lambda)|^2 f_1(\lambda) d\lambda
\end{aligned}
$$

where the inequality follows from Bessel's inequality by viewing $L(F_1)$ as a subspace of the Hilbert space of measurable square integrable functions with inner-product $\langle \cdot, \cdot \rangle_{F_1}$. Thus

$$
\sup_m D_m^\epsilon \leq \epsilon + 2 \sum_{j=1}^{m_\epsilon} \int (\frac{f_2(\lambda)}{f_1(\lambda)} - 1)^2 |\psi_j^\epsilon(\lambda)|^2 f_1(\lambda) d\lambda. \qquad (46)
$$

From equation (II.1.3) of IR78, every $\psi \in L(F_1)$ can be represented in the form

$$
\psi(\lambda) = c_0 + (1 + i\lambda) \int_0^1 e^{i\lambda t} c(t) dt
$$

for some real c_0 and some square integrable function $c : [0, 1] \mapsto \mathbb{R}$. Thus

$$
|\psi(\lambda)| \leq c_0 + \sqrt{1 + \lambda^2} \int_0^1 c(t) dt
$$

$$\leq \ c_0 + \sqrt{1+\lambda^2}\sqrt{\int_0^1 c^2(t)dt}$$

where the second inequality follows from Jensen's inequality, so that

$$\sup_\lambda |\psi(\lambda)|^2 f_1(\lambda) < \infty.$$

Thus, (46) implies that for some $M_\epsilon < \infty$ that does not depend on f_2, for all f_2 satisfying (36),

$$\sup_m D_m^\epsilon \leq \epsilon + M_\epsilon \int (\frac{f_2(\lambda)}{f_1(\lambda)} - 1)^2 d\lambda. \tag{47}$$

It thus suffices to show that there exists a CARMA implied f_2 satisfying (36) that makes $\int_0^\infty (f_2(\lambda)/f_1(\lambda) - 1)^2 d\lambda$ arbitrarily small.

Let $h_1(\lambda) = f_1(\lambda)/f_0(\lambda) - 1$ and $h_2(\lambda) = f_2(\lambda)/f_0(\lambda) - 1$. Recalling the definition of δ_0 in (32), we have

$$\int_0^\infty (\frac{f_2(\lambda)}{f_1(\lambda)} - 1)^2 d\lambda \leq \tfrac{1}{4}\delta_0^{-2} \int_0^\infty \left(h_1(\lambda) - h_2(\lambda)\right)^2 d\lambda$$

and it suffices to show that for any $\epsilon_1 > 0$, there exists a CARMA implied h_2 such that $\int_0^\infty \left(h_1(\lambda) - h_2(\lambda)\right)^2 d\lambda < 2\epsilon_1$.

For any $\tilde{h}_1(\lambda)$, from $(a-b)^2 \leq 2(a^2+b^2)$,

$$\begin{aligned}\int_0^\infty \left(h_1(\lambda) - h_2(\lambda)\right)^2 d\lambda \ &= \ \int_0^\infty \left(h_1(\lambda) - \tilde{h}_1(\lambda) - h_2(\lambda) + \tilde{h}_1(\lambda)\right)^2 d\lambda \\ &\leq \ 2\int_0^\infty (h_1(\lambda) - \tilde{h}_1(\lambda))^2 d\lambda + 2\int_0^\infty (h_2(\lambda) - \tilde{h}_1(\lambda))^2 d\lambda.\end{aligned}$$

By (33), $\int_0^\infty h_1(\lambda)^2 d\lambda < \infty$. Thus, there exists $K < \infty$ such that $\int_K^\infty h_1(\lambda)^2 d\lambda < \epsilon_1/2$. Let $\chi_K(\lambda) = 1$ for $\lambda \leq K$, $\chi_K(\lambda) = 0$ for $\lambda \geq K+1$ and $\chi_K(\lambda) = K+1-\lambda$ otherwise, and define $\tilde{h}_1(\lambda) = \chi_K(\lambda)h_1(\lambda)$. Then $\int_0^\infty (h_1(\lambda) - \tilde{h}_1(\lambda))^2 d\lambda \leq \epsilon_1/2$, and since f_1 is continuous by standard Fourier arguments (see, for instance, Proposition 4.1 in Stein and Shakarchi (2005)), so is $\tilde{h}_1$. It thus suffices to show that there exists a CARMA implied h_2 that makes $\int_0^\infty (h_2(\lambda) - \tilde{h}_1(\lambda))^2 d\lambda$ smaller than $\epsilon_1/2$.

Now

$$\int_0^\infty (h_2(\lambda) - \tilde{h}_1(\lambda))^2 d\lambda = \int_0^\infty (1+\lambda^2)^{-2}(\vartheta_2(\lambda) - \tilde{\vartheta}_1(\lambda))^2 d\lambda$$

with $\vartheta_2(\lambda) = (1+\lambda^2)h_2(\lambda)$ and $\tilde{\vartheta}_1(\lambda) = (1+\lambda^2)\tilde{h}_1(\lambda)$. Note that $\tilde{\vartheta}_1$ is continuous, and $\lim_{\lambda\to\infty}\tilde{\vartheta}_1(\lambda) = 0$. Thus, by Lemma 1, for any $\delta > 0$, there exists an integer q and a rational function ϑ_2 of the form (31) such that

$$\sup_{\lambda} |\vartheta_2(\lambda) - \tilde{\vartheta}_1(\lambda)| < \delta. \tag{48}$$

We have $\int_0^\infty (1+\lambda^2)^{-2}(\vartheta_2(\lambda)-\tilde{\vartheta}_1(\lambda))^2 d\lambda \le \delta^2 \int_0^\infty (1+\lambda^2)^{-2} d\lambda$, which can be made arbitrarily small by choosing δ small. From the definitions of ϑ_2 and h_2, we have

$$f_2(\lambda) = \frac{1}{1+\lambda^2} + \frac{\vartheta_2(\lambda)}{(1+\lambda^2)^2}$$

so the implied $f_2(\lambda)$ is a rational function in λ^2 of degree $p = q+2$ in the denominator $p-1$ in the numerator.

Furthermore, for all $\delta < \delta_0$, we have uniformly in λ,

$$\begin{aligned} f_2(\lambda) &\le \frac{1}{1+\lambda^2} + \frac{\tilde{\vartheta}_1(\lambda)+\delta_0}{(1+\lambda^2)^2} \\ &= \frac{1}{1+\lambda^2} + \frac{\chi_K(\lambda)((1+\lambda^2)^2 f_1(\lambda) - (1+\lambda^2)) + \delta_0}{(1+\lambda^2)^2} \\ &= \chi_K(\lambda)f_1(\lambda) + (1-\chi_K(\lambda))f_0(\lambda) + \frac{\delta_0}{(1+\lambda^2)^2} \le \overline{f}(\lambda) \end{aligned}$$

and similarly,

$$\begin{aligned} f_2(\lambda) &\ge \frac{1}{1+\lambda^2} + \frac{\tilde{\vartheta}_1(\lambda)-\delta_0}{(1+\lambda^2)^2} \\ &= \chi_K(\lambda)f_1(\lambda) + (1-\chi_K(\lambda))f_0(\lambda) - \frac{\delta_0}{(1+\lambda^2)^2} \ge \underline{f}(\lambda) \end{aligned}$$

so that f_2 satisfies (36). In particular, since $\underline{f}(\lambda) > 0$ for all λ, the numerator of $f_2(\lambda)$ is a positive rational function, so by Lemma 2, f_2 has the form of the spectral density of a CARMA$(p, p-1)$ process.

References

ANDREWS, D. W. K., AND H. CHEN (1994): "Approximately Median-Unbiased Estimation of Autoregressive Models," *Journal of Business and Economic Statistics*, 12, 187–204.

BERGSTROM, A. R. (1985): "The estimation of parameters in non-stationary higher-order continuous-time dynamic models," *Econometric Theory*, 1, 369–385.

BOBKOSKI, M. J. (1983): "Hypothesis Testing in Nonstationary Time Series," *unpublished Ph.D. thesis, Department of Statistics, University of Wisconsin.*

BRAND, L. (1964): "The Companion Matrix and Its Properties," *The American Mathematical Monthly*, 71, 629–634.

BROCKWELL, P. J. (2001): "Continuous-time ARMA processes," in *Handbook of Statistics 19; Stochastic Processes: Theory and Methods*, ed. by D. N. Shanbhag, and C. R. Rao, vol. 19, pp. 249–276. Elsevier.

BROCKWELL, P. J., AND R. A. DAVIS (1991): *Time Series: Theory and Methods*. Springer, New York, second edn.

CAMPBELL, J. Y., AND M. YOGO (2006): "Efficient Tests of Stock Return Predictability," *Journal of Financial Economics*, 81, 27–60.

CAVANAGH, C. L., G. ELLIOTT, AND J. H. STOCK (1995): "Inference in Models with Nearly Integrated Regressors," *Econometric Theory*, 11, 1131–1147.

CHAN, N. H., AND C. Z. WEI (1987): "Asymptotic Inference for Nearly Nonstationary AR(1) Processes," *The Annals of Statistics*, 15, 1050–1063.

DUFOUR, J.-M. (2006): "Monte Carlo Tests with Nuisance Parameters: A General Approach to Finite-Sample Inference and Nonstandard Asymptotics," *Journal of Econometrics*, 133, 443–477.

ELLIOTT, G. (1998): "The Robustness of Cointegration Methods When Regressors Almost Have Unit Roots," *Econometrica*, 66, 149–158.

——— (1999): "Efficient Tests for a Unit Root When the Initial Observation is Drawn From its Unconditional Distribution," *International Economic Review*, 40, 767–783.

ELLIOTT, G., U. K. MÜLLER, AND M. W. WATSON (2015): "Nearly Optimal Tests When a Nuisance Parameter is Present Under the Null Hypothesis," *Econometrica*, 83, 771–811.

Elliott, G., T. J. Rothenberg, and J. H. Stock (1996): “Efficient Tests for an Autoregressive Unit Root,” *Econometrica*, 64, 813–836.

Elliott, G., and J. H. Stock (1994): “Inference in Time Series Regression When the Order of Integration of a Regressor is Unknown,” *Econometric Theory*, 10, 672–700.

——— (2001): “Confidence Intervals for Autoregressive Coefficients Near One,” *Journal of Econometrics*, 103, 155–181.

Gospodinov, N. (2004): “Asymptotic Confidence Intervals for Impulse Responses of Near-Integrated Processes,” *Econometrics Journal*, 7, 505–527.

Ibragimov, I. A., and Y. A. Rozanov (1978): *Gaussian random processes*. Springer Verlag, Berlin.

Jansson, M., and M. J. Moreira (2006): “Optimal Inference in Regression Models with Nearly Integrated Regressors,” *Econometrica*, 74, 681–714.

Jones, R. H. (1981): “Fitting a continuous time autoregression to discrete data,” in *Applied time series analysis II*, ed. by D. F. Findley, pp. 651–682. Academic Press, New York.

Jones, R. H., and K. M. Ackerson (1990): “Serial correlation in unequally spaced longitudinal data,” *Biometrika*, 77, 721–732.

Lothian, J. R., and M. P. Taylor (1996): “Real Exchange Rate Behavior: The Recent Float from the Perspective of the Past Two Centuries,” *Journal of Political Economy*, 104, 488–509.

Mikusheva, A. (2007): “Uniform Inference in Autoregressive Models,” *Econometrica*, 75, 1411–1452.

——— (2012): “One-dimensional inference in autoregressive models with the potential presence of a unit root,” *Econometrica*, 80(1), 173–212.

Moon, H. R., and P. C. Phillips (2000): “Estimation of autoregressive roots near unity using panel data,” *Econometric Theory*, 16(06), 927–997.

Müller, U. K., and M. W. Watson (2008): “Testing Models of Low-Frequency Variability,” *Econometrica*, 76, 979–1016.

——— (2016): “Measuring Uncertainty about Long-Run Predictions,” *Review of Economic Studies*, 83.

——— (2017): "Low-Frequency Econometrics," in *Advances in Economics: Eleventh World Congress of the Econometric Society*, ed. by B. Honoré, and L. Samuelson, vol. II, pp. 63–94. Cambridge University Press.

Murray, C. J., and D. H. Papell (2002): "The purchasing power parity persistence paradigm," *Journal of International Economics*, 56, 1–19.

——— (2005): "The purchasing power parity puzzle is worse than you think," *Empirical Economics*, 30(3), 783–790.

Pham-Dinh, T. (1977): "Estimation of parameters of a continuous time Gaussian stationary process with rational spectral density," *Biometrika*, 64, 385–399.

Phillips, A. W. (1959): "The estimation of parameters in systems of stochastic differential equations," *Biometrika*, 46, 67–76.

Phillips, P. C. B. (1987): "Towards a Unified Asymptotic Theory for Autoregression," *Biometrika*, 74, 535–547.

——— (1988): "Regression Theory for Near-Integrated Time Series," *Econometrica*, 56, 1021–1043.

——— (1998): "Impulse Response and Forecast Error Variance Asymptotics in Nonstationary VARs," *Journal of Econometrics*, 83, 21–56.

Robinson, P. M. (2003): "Long-Memory Time Series," in *Time Series with Long Memory*, ed. by P. M. Robinson, pp. 4–32. Oxford University Press, Oxford.

Rossi, B. (2005): "Confidence Intervals for Half-Life Deviations from Purchasing Power Parity," *Journal of Business and Economic Statistics*, 23, 432–442.

Sims, C. A., and H. Uhlig (1991): "Understanding Unit Rooters: A Helicopter Tour," *Econometrica*, 59, 1591–1599.

Stein, E., and R. Shakarchi (2005): *Real Analysis: Measure Theory, Integration, and Hilbert Spaces*, Princeton Lectures in Analysis. Princeton University Press, Princeton.

Stock, J. H. (1991): "Confidence Intervals for the Largest Autoregressive Root in U.S. Macroeconomic Time Series," *Journal of Monetary Economics*, 28, 435–459.

——— (1996): "VAR, Error Correction and Pretest Forecasts at Long Horizons," *Oxford Bulletin of Economics and Statistics*, 58, 685–701.

Stock, J. H., and M. W. Watson (1996): "Confidence Sets in Regressions with Highly Serially Correlated Regressors," *Working Paper, Harvard University.*

Taylor, M. P. (2003): "Purchasing power parity," *Review of International Economics*, 11(3), 436–452.

Torous, W., R. Valkanov, and S. Yan (2004): "On predicting stock returns with nearly integrated explanatory variables," *The Journal of Business*, 77(4), 937–966.

Valkanov, R. (2003): "Long-Horizon Regressions: Theoretical Results and Applications," *Journal of Financial Economics*, 68, 201–232.

Wright, J. H. (2000): "Confidence Intervals for Univariate Impulse Responses with a Near Unit Root," *Journal of Business and Economics Statistics*, 18, 368–373.