

# Discrete Optimization

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# The maximum flow problem

Let  $G = (V, E)$  be a **directed graph** with a **source node**  $s$  and a **sink node**  $t$  and with **capacities**  $u_j$  on each edge  $e_j$ .

## The maximum flow problem

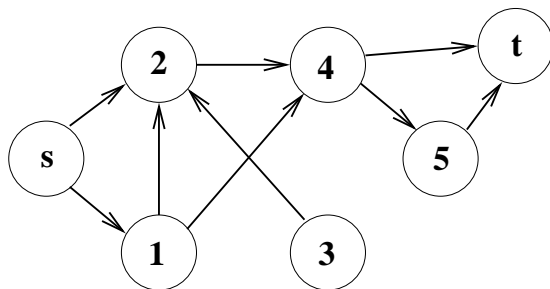
Find the maximum flow sent from  $s$  to  $t$  where the flow **entering** each node (except  $s$  and  $t$ ) is equal to the flow **leaving** each node and where the flow on **each edge** satisfies the capacity.

# The max flow problem

Send the maximum flow from a node  $s$  (**source**) to a node  $t$  (**sink**).

Each **edge** has a **capacity**

**Example** of a feasible flow

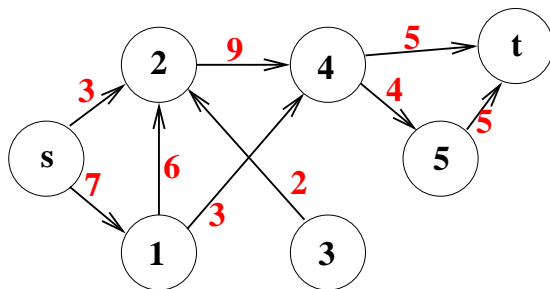


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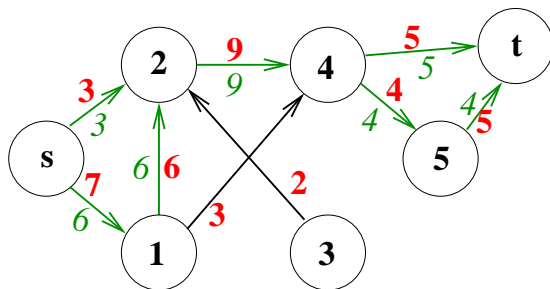


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# The minimum cut problem

## The minimum $s - t$ -cut problem

An  $s - t$ -cut is a partition of the nodes of  $G$  into  $V_1$  and  $V_2$  where  $s \in V_1$  and  $t \in V_2$ . The **value** of an  $s - t$ -cut is equal to the **sum of the capacities** of all edges joining one node of  $V_1$  to one node of  $V_2$ .

The **min-cut problem** is the problem of finding the  $s - t$ -cut that has the minimal value.

## Important result

Max flow = min cut

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## Examples of applications

### The problem of representatives

A town has  $r$  residents  $R_1, \dots, R_r$ ,  $q$  clubs  $C_1, \dots, C_q$  and  $p$  political parties  $P_1, \dots, P_p$ . Each resident is member of **at least** one club and can belong to **exactly one** political party.

Each club must nominate one of its members to represent it in the town's council in such a way that the number of council members belonging to the political party  $P_k$  is **at most**  $u_k$ .

Problem with applications in **resource assignment settings**.



# Examples of applications

## Matrix rounding problem

We are given a **matrix**  $D \in \mathbb{R}^{p \times q}$  with row sums  $\alpha_i$  and column sums  $\beta_j$ . We can decide to round any element in the matrix to the **lower** or the **upper** integer.

How can we round each element such that, for each row and each column, the sum of the **rounded elements** is equal to a **rounding of the sums**.

This is called a **consistent rounding**.

# Examples of applications

## Scheduling on uniform parallel machines

We want to schedule  $J$  jobs on  $M$  parallel machines.

Each job  $j$  has a **processing time**  $p_j$ , a **release date**  $r_j$  and a **due date**  $d_j$ . Furthermore we allow **preemption**.

# Examples of applications for the min-cut problem

## Distributed computing on a 2-processor computer

Consider a **large computer program** that has to be run on **two processors**. The code is subdivided in **tasks** that have a different processing time on both processors. Furthermore there is a **communication cost** if some tasks are performed on **different processors**.

## Example of application of the min-cut problem

### Segmentation of an image

Consider a picture taken of an object : for example, an MRI image of a heart. We may want to detect, for each pixel, whether it actually belongs to the heart or whether it belongs to the background.

It is possible to create a **similarity function** for each pair of pixel that is **large** if the 2 pixels are likely to belong to the same part of the picture.

## The residual network

Given a flow  $x$ , the **residual capacity**  $r_{ij}$  is the maximum additional flow that can be **added** to arc  $(i, j)$  or **removed** from arc  $(j, i)$ .

$$r_{ij} = (u_{ij} - x_{ij}) + x_{ji}$$

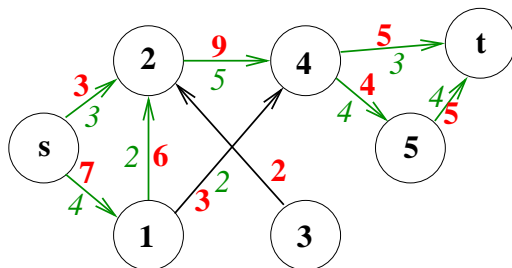
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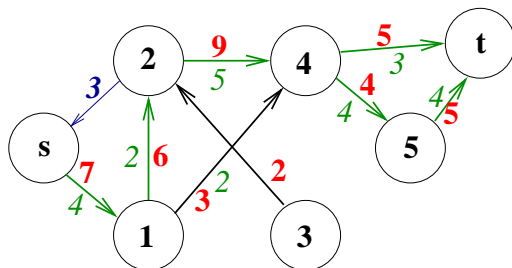


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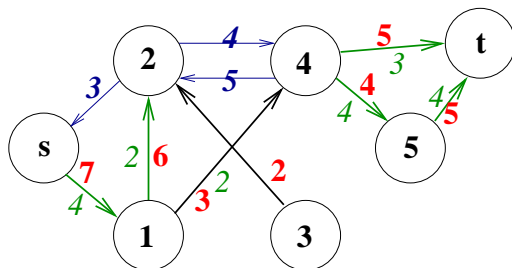


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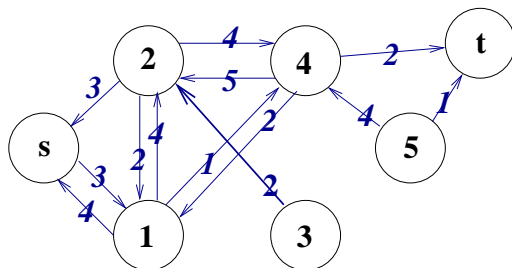
# Flows and cuts

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## $(s - t)$ -cuts

### Definition of an $(s - t)$ -cut

A cut is a partition of the **node set** into two subsets  $S, \bar{S}$  such that

- $S \cup \bar{S} = V$
- $S \cap \bar{S} = \emptyset$
- It is an  **$(s - t)$ -cut** if  $s \in S$  and  $t \in \bar{S}$ .

The **capacity** of an  $(s - t)$ -cut is the sum of the capacities of the **forward arcs** of the cut :

$$u(S, \bar{S}) = \sum_{(i,j) \in (S, \bar{S})} u_{ij}.$$

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**Property** : The value of any flow is less or equal to the capacity of **any cut** in the network. This is a **weak duality** property.

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# Generic augmenting path algorithm

## Definition

A **directed path** from the source  $s$  to the sink  $t$  in the residual network is an **augmenting path**.

The residual capacity of the augmenting path is the **minimum** residual capacity over all arcs in the path.

Idea : augment step by step the value of the current flow using paths in the residual network.

The value by which we can augment the flow is less or equal to the **residual capacity** of any cut.

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# Generic augmenting path algorithm

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x:=0;  
while  $G(x)$  contains a directed path from  $s$  to  $t$  do  
    Find an augmenting path  $P$   
     $\delta := \min\{r_{ij} \mid (i,j) \in P\}$   
    Augment  $\delta$  units of flow along  $P$  and update  $G(x)$ 
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Important questions :

Does the algorithm terminate ?

Does it provide an optimal solution ?

What is its running time ?



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# Labeling algorithm

This is a first implementation of the augmenting path algorithm that **does not run in polynomial time**.

**Idea** : Perform a breadth-first-search or a depth-first-search on the graph to **label** the nodes. If the sink node is labeled, we have found an **augmenting path**.

The algorithm is correct

If, at an iteration, the sink node **cannot be labeled**, then the **flow is maximal**.

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# The max flow-min cut theorem

## Theorem

The maximum value of the flow from  $s$  to  $t$  equals the minimum capacity among all  $(s - t)$ -cuts.

## Augmenting path theorem

A flow  $x^*$  is a **maximum flow** if and only if the residual network  $G(x^*)$  **contains no augmenting path**.

## Integrality theorem

If all arc capacities are **integer**, the maximum flow problem has an **integer** maximum flow.

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# Complexity of the labeling algorithm

## Theorem

The labeling algorithm solves the maximum flow problem in  $\mathcal{O}(mnU)$  operations.

This implies that in a such an implementation, the algorithm **does not run** in polynomial time in the **worst case**.



# Combinatorial implications

- The maximum number of **arc-disjoint paths** from  $s$  to  $t$  is equal to the minimum number of arcs whose **removal disconnects**  $s$  from  $t$ .
- The maximum number of **node-disjoint** paths from  $s$  to  $t$  is equal to the minimum number of **nodes whose removal** disconnects  $s$  from  $t$ .
- Matchings and covers are strong dual for **bipartite graphs**.

## Flows with lower bounds

### Optimizing from a feasible flow

Just change the computation of the **residual capacity**

$$r_{ij} = (u_{ij} - x_{ij}) + (x_{ji} - l_{ji})$$

### Establishing a feasible flow

- Add an arc  $(t, s)$  of infinite capacity and transform the problem into finding a feasible circulation.
- At each node, compute

$$b(i) = \sum_{j|(i,j) \in A} l_{ji} - \sum_{j|(j,i) \in A} l_{ij}$$

- Put only upper bounds on the arcs to find  $x'$
- Create a dummy source node  $s'$  and a dummy sink node  $t'$ .
- Create an edge from  $s'$  to  $i$  if  $b(i) > 0$  with capacity  $b(i)$ .
- Create an edge from  $i$  to  $t'$  if  $b(i) < 0$  with capacity  $|b(i)|$ .
- The problem is feasible if there exists a flow with total value  $\sum_i b(i)$
- The potential feasible flow is obtained as  $x_{ij} := x'_{ij} + l_{ij}$

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