Discrete Optimization

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ULg - Institut Montefiore

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Contents of the lecture

- Formulations
  Guidelines for strong formulations
- Branch-and-bound algorithm
  The best known technique to solve integer programs in general
- Flow problems
- Matching and assignment problems
- Dynamic programming
- Cutting planes for integer programs
- Totally unimodular problems
- Lagrangian relaxations

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Modeling techniques

We consider several concepts that can be well modeled by integer programs

Binary choice

A choice between 2 alternatives is modeled through a $0,1$-variable.

Example

The knapsack problem

$$\begin{aligned} \text{maximize} \quad & \sum_{i=1}^{n} c_i x_i \\ \text{subject to} \quad & \sum_{i=1}^{n} a_i x_i \leq b \\ & x_i \in \{0,1\} \text{ for all } i = 1, \ldots, n. \end{aligned}$$

Modeling techniques

Forcing constraints

If decision B is taken then decision A must be taken.

$$\begin{aligned} x &= 1 && \text{if decision } A \text{ is taken} \\ x &= 0 && \text{otherwise} \end{aligned}$$

$$\begin{aligned} y &= 1 && \text{if decision } B \text{ is taken} \\ y &= 0 && \text{otherwise} \end{aligned}$$

The constraint reads

$$x \leq y$$

Example Facility location problem

Modeling techniques

Disjunctive constraints

Consider $x \geq 0, a \geq 0, c \geq 0$. We want to model an OR constraint :

$$a^T x \geq b \quad \text{or} \quad c^T x \geq d$$

We introduce a variable $y \in \{0, 1\}$ that represents whether constraint 1 or constraint 2 is satisfied.

$$a^T x \geq yb \quad \text{and} \quad c^T x \geq (1 - y)d.$$

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Restricted range of values

Suppose we want to formulate $x \in \{a_1, a_2, \ldots, a_m\}$.

We introduce m binary variables y_j.

$$x = \sum_{j=1}^{m} a_j y_j, \quad \sum_{j=1}^{m} y_j = 1, \quad y_j \in \{0,1\}$$

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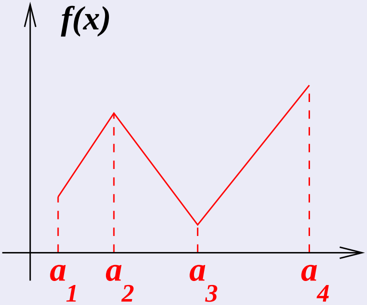
Modeling techniques

Arbitrary piecewise linear cost functions



Introduce $y_i \in \{0, 1\}$ such that

$$\begin{aligned} y_i &= 1 \quad \text{if } x \in [a_i, a_{i+1}] \\ y_i &= 0 \quad \text{if } x \notin [a_i, a_{i+1}] \end{aligned}$$

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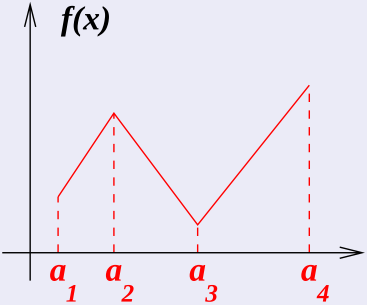

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Guidelines for strong formulation

The linear relaxation

Given

$$\begin{aligned} \min\ & c^T x + d^T y \\ \text{s.t.}\ & Ax + By = b \\ & x, y \geq 0 \\ & x \in \mathbb{Z}^n. \end{aligned}$$

Its linear relaxation is defined as

$$\begin{aligned} \min\ & c^T x + d^T y \\ \text{s.t.}\ & Ax + By = b \\ & x, y \geq 0 \\ & x \in \mathbb{R}^n. \end{aligned}$$

The linear relaxation gives important information about the optimal value of an integer program.

Reminder : linear programming

If the objective is linear and the constraints are linear, we talk about linear programming (LP) or linear optimization.

LP in standard form

$$
\begin{aligned}
\min\ & c^T x \\
\text{s.t. } & Ax = b \\
& x \in \mathbb{R}^n_+
\end{aligned}
$$

Definition

A polyhedron is a set $\{x \in \mathbb{R}^n | Ax \geq b\}$

A set of the form $Ax \leq b$ is also a polyhedron.
A set $\{x \in \mathbb{R}^n | Ax = b, x \geq 0\}$ is a polyhedron in standard form.

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Graphic representation

We can represent a problem in two dimensions graphically.

Example :

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\begin{aligned}
\max\ x_1 + 2x_2 & & (1)\\
-x_1 + 2x_2 &\leq 1 & (2)\\
-x_1 + x_2 &\leq 0 & (3)\\
4x_1 + 3x_2 &\leq 12 & (4)\\
x_1,\quad x_2 &\geq 0 & (5)
\end{aligned}
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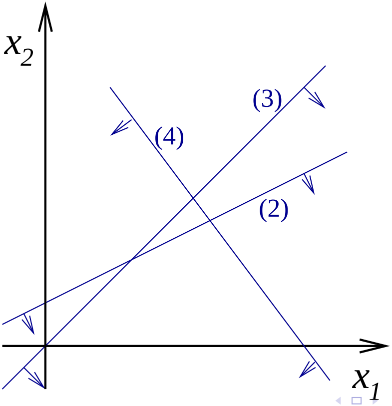

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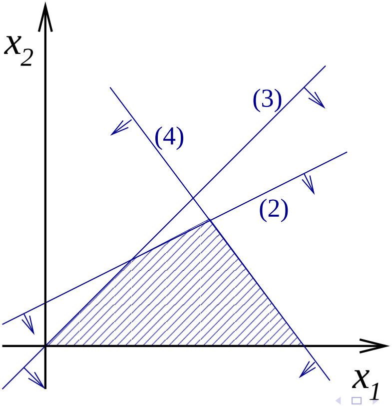

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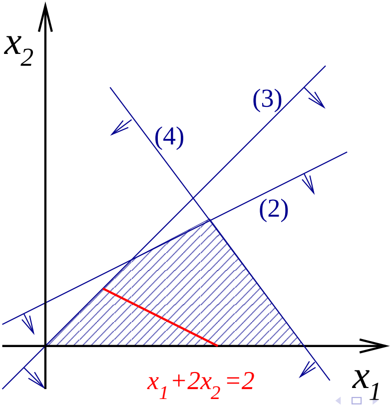

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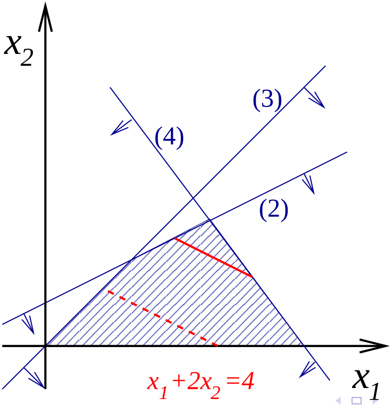

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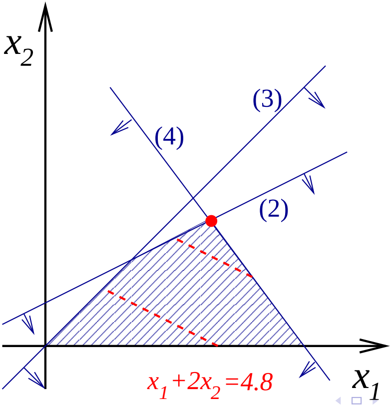

Extreme points and vertices

Definition

Let P be a polyhedron. A point $x \in P$ is an extreme point of P if there do not exist two points $y, z \in P$ such that x is a convex combination of y and z.

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In the example we had a unique solution at a vertex of the polyhedron.
Some degenerate cases can lead to different solutions.

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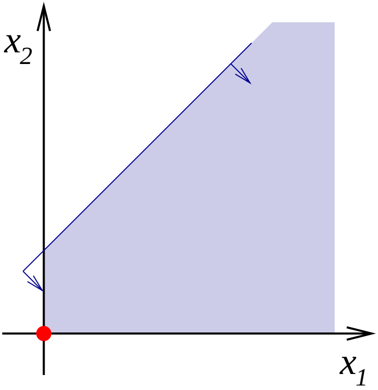

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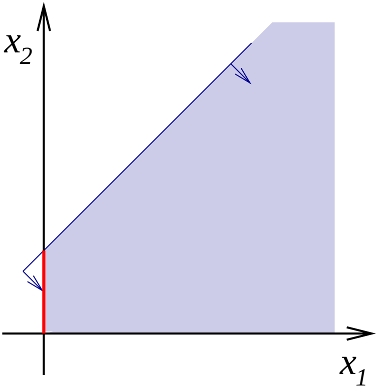

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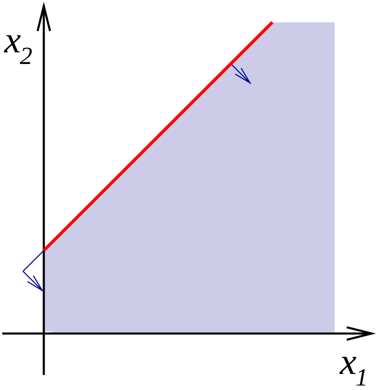

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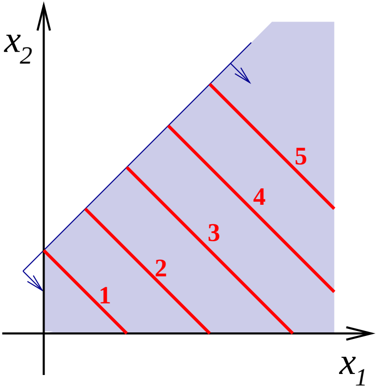

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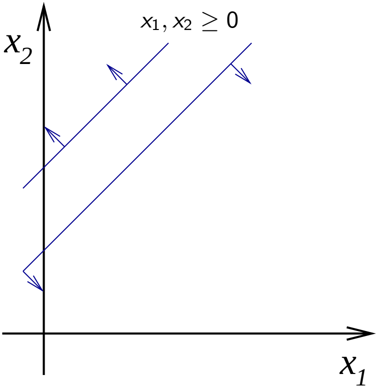

Bases of a polyhedron

We subdivide the equalities and inequalities into three categories :

$$\begin{aligned} a_i^T x \geq b_i \qquad & i \in M_\geq \\ a_i^T x \leq b_i \qquad & i \in M_\leq \\ a_i^T x = b_i \qquad & i \in M_= \end{aligned}$$

Definition

Let $\bar{x}$ be a point satisfying $a_i^T \bar{x} = b_i$ for some $i \in M_\geq, M_\leq$ or $M_=$. The constraint i is said to be active or tight.

Bases of a polyhedron

Definition

Let P be a polyhedron and let $\bar{x} \in \mathbb{R}^n$.

(a) $\bar{x}$ is a basic solution if
- all equalities ($i \in M_=$) are active
- among the active constraints, there are n linearly independent

(b) if $\bar{x}$ is a basic solution that satisfies all constraints, then $\bar{x}$ is a feasible basic solution.

Theorem

Let P be a polyhedron and let $\bar{x} \in P$. The three following statements are equivalent.

(i) $\bar{x}$ is a vertex

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Comparing two formulations

To compare two formulations P^1 and P^2 with the same integer feasible points, we consider their respective linear relaxations P^1_{LP}, P^2_{LP}.

Comparing two formulations

P^1 is better than P^2 if

$$P^1_{LP} \subset P^2_{LP}$$

Ideal formulation

If $\mathcal{F} = \{x_1, \ldots, x_k\}$ is the set of feasible solutions, an ideal formulation is

$$\mathrm{conv}(\mathcal{F})$$

Example : The facility location problem

Example : The pigeonhole principle

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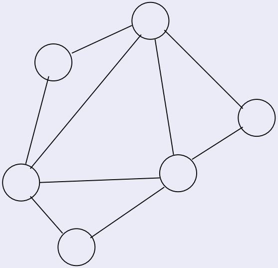
Comparing two formulations for graph problems

The minimum spanning tree

Let $G = (V, E)$ be an undirected graph. Every edge has a cost c_e. We look for the tree with the minimum total cost.



Constraints to encode :

- A tree should have $n - 1$ edges where n is the number of nodes
- A tree cannot have a cycle or equivalently
  A tree must be connected

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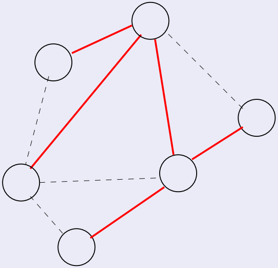

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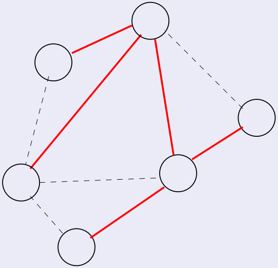

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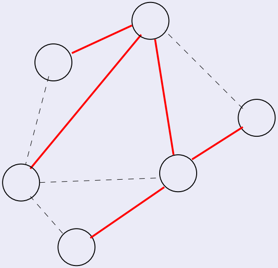

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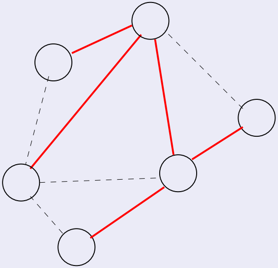

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Subtour elimination formulation

Integer formulation

$$P^I_{sub} = \{x_e \in \{0,1\} \mid \sum_{e \in E} x_e = n-1$$
$$\sum_{e \in E(S)} x_e \leq |S|-1, \quad S \subset V, S \neq \emptyset, V\ \}$$

Linear programming relaxation

$$P_{sub} = \{x_e \in [0,1] \mid \sum_{e \in E} x_e = n-1$$
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Cutset formulation

Integer formulation

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The traveling salesman problem

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$$P^I_{tspsub} = \{x_e \in \{0,1\} \mid \sum_{e \in \delta(\{i\})} x_e = 2 \quad \text{for all } i \in V$$
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