Discrete Optimization

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Polynomially solvable problems

Can we study some types of problems where the linear relaxation always solves the discrete problem.

Cramer rule

A solution of $Ax = b$ is given by x with

$$x_i = \frac{\det(A^i)}{\det(A)},$$

with A^i is obtained by taking A and replacing its i^{th} column by b.

A polyhedron has all integer vertices if all determinant of all square submatrices is equal to 1, 0, or -1.

Total unimodularity

Definition

- A matrix $A \in \mathbb{Z}^{m \times n}$ is unimodular if the determinant of each basis is 1 or -1.
- A matrix $A \in \mathbb{Z}^{m \times n}$ is totally unimodular (TU) if the determinant of each square submatrix of A is $0, 1, -1$.

Proposition

- A matrix A is TU iff $[A, I]$ is unimodular.
- A matrix A is TU iff $\begin{pmatrix} A \\ -A \\ I \\ -I \end{pmatrix}$ is TU
- A matrix A is TU iff A^T is TU

Total unimodularity in integer problems

Theorem

- A is unimodular iff $P(b) = \{x \in \mathbb{R}^n_+ \mid Ax = b\}$ is integral for all $b \in \mathbb{Z}^m$.
- A is TU iff $P(b) = \{x \in \mathbb{R}^n_+ \mid Ax \leq b\}$ is integral for all $b \in \mathbb{Z}^m$.

Corollary

- A is TU iff $\{x \mid Ax = b, 0 \leq x \leq u\}$ is integral for all integral vectors b and u.
- A is TU iff $\{x \mid a \leq Ax \leq b, l \leq x \leq u\}$ is integral for all integral vectors a, b, l, u.

Recognizing total unimodularity

Proposition

A matrix A is TU iff each collection Q of rows of A can be partitioned into two parts so that the sum of the rows in one part minus the sum of the rows in the other part is a vector with entries $0, +1, -1$.

Theorem

The following matrices are TU

- The node-arc indicence matrix of a directed graph
- The node-edge incidence matrix of an undirected bipartite graph
- A $\{0, 1\}$-matrix in which each column has its ones consecutively (also known as interval matrix)

Example of discrete problems solvable with the LP

- The maximum flow problem
- The shortest path problem
- The minimum cost flow problem
- The matching problem in a bipartite graph