Discrete Optimization

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Modeling techniques

We consider several concepts that can be well modeled by integer programs

Binary choice

A choice between 2 alternatives is modeled through a $0,1$-variable.

Example

The knapsack problem

$$\begin{aligned} \text{maximize} \quad & \sum_{i=1}^{n} c_i x_i \\ \text{subject to} \quad & \sum_{i=1}^{n} a_i x_i \leq b \\ & x_i \in \{0,1\} \text{ for all } i = 1, \ldots, n. \end{aligned}$$

Modeling techniques

Forcing constraints

If decision B is taken then decision A must be taken.

$$\begin{aligned} x &= 1 && \text{if decision } A \text{ is taken} \\ x &= 0 && \text{otherwise} \end{aligned}$$

$$\begin{aligned} y &= 1 && \text{if decision } B \text{ is taken} \\ y &= 0 && \text{otherwise} \end{aligned}$$

The constraint reads

$$y \leq x$$

Example Facility location problem

Modeling techniques

Disjunctive constraints

Consider $x \geq 0, a \geq 0, c \geq 0$. We want to model an OR constraint :

$$a^T x \geq b \quad \text{or} \quad c^T x \geq d$$

We introduce a variable $y \in \{0, 1\}$ that represents whether constraint 1 or constraint 2 is satisfied.

$$a^T x \geq yb \quad \text{and} \quad c^T x \geq (1 - y)d.$$

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Modeling techniques

Restricted range of values

Suppose we want to formulate $x \in \{a_1, a_2, \ldots, a_m\}$.

We introduce m binary variables y_j.

$$x = \sum_{j=1}^{m} a_j y_j, \quad \sum_{j=1}^{m} y_j = 1, \quad y_j \in \{0,1\}$$

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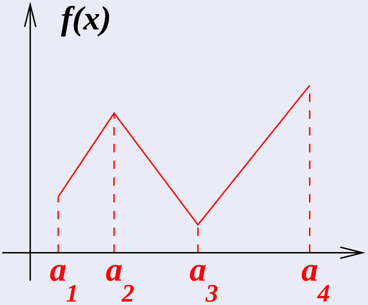
Modeling techniques

Arbitrary piecewise linear cost functions



Introduce $y_i \in \{0, 1\}$ such that

$$y_i = 1 \quad \text{if } x \in [a_i, a_{i+1}]$$
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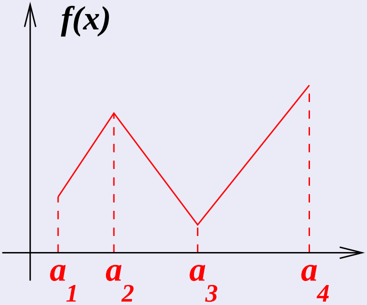

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Guidelines for strong formulation

The linear relaxation

Given

$$\begin{aligned} \min\ & c^T x + d^T y \\ \text{s.t.}\ & Ax + By = b \\ & x, y \geq 0 \\ & x \in \mathbb{Z}^n. \end{aligned}$$

Its linear relaxation is defined as

$$\begin{aligned} \min\ & c^T x + d^T y \\ \text{s.t.}\ & Ax + By = b \\ & x, y \geq 0 \\ & x \in \mathbb{R}^n. \end{aligned}$$

The linear relaxation gives important information about the optimal value of an integer program.

Reminder : linear programming

If the objective is linear and the constraints are linear, we talk about linear programming (LP) or linear optimization.

LP in standard form

$$\begin{aligned} &\min c^T x \\ &\text{s.t. } Ax = b \\ &\quad x \in \mathbb{R}^n_+ \end{aligned}$$

Definition

A polyhedron is a set $\{x \in \mathbb{R}^n | Ax \geq b\}$

A set of the form $Ax \leq b$ is also a polyhedron.
A set $\{x \in \mathbb{R}^n | Ax = b, x \geq 0\}$ is a polyhedron in standard form.

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Graphic representation

We can represent a problem in two dimensions graphically.

Example :

$$
\begin{aligned}
\max\ & x_1 + 2x_2 && (1)\\
& -x_1 + 2x_2 \leq 1 && (2)\\
& -x_1 + x_2 \leq 0 && (3)\\
& 4x_1 + 3x_2 \leq 12 && (4)\\
& x_1,\ x_2 \geq 0 && (5)
\end{aligned}
$$

Graphic representation

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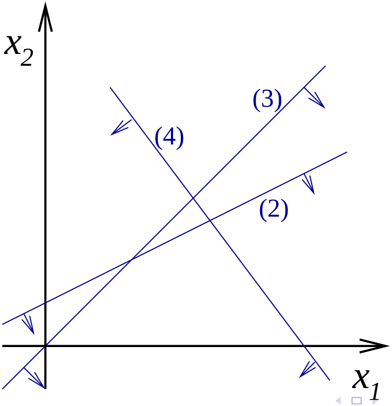

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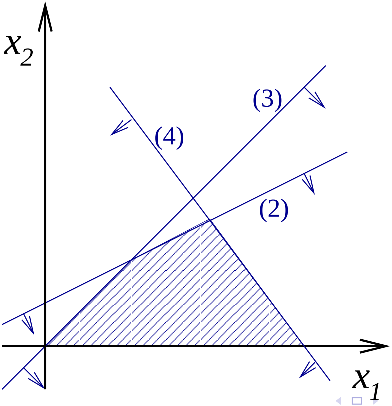

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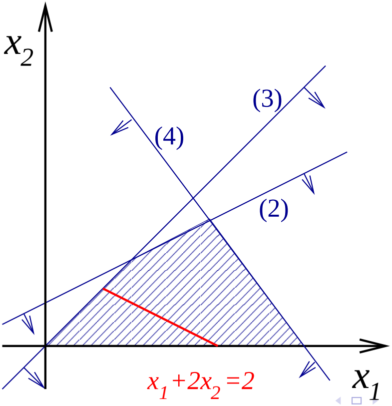

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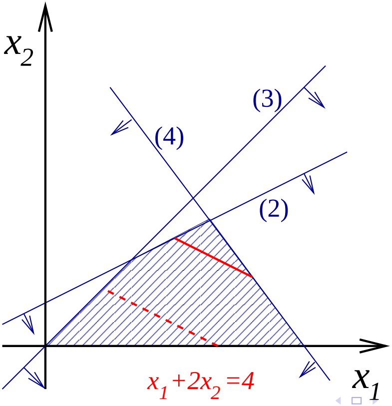

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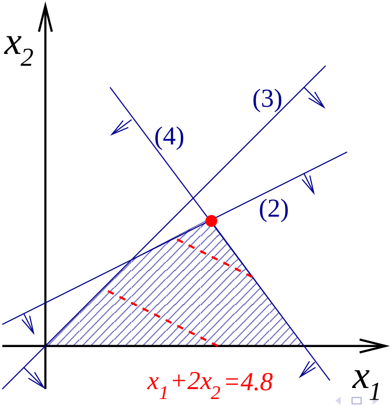

Extreme points and vertices

Definition

Let P be a polyhedron. A point $x \in P$ is an extreme point of P if there do not exist two points $y, z \in P$ such that x is a convex combination of y and z.

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In the example we had a unique solution at a vertex of the polyhedron.
Some degenerate cases can lead to different solutions.

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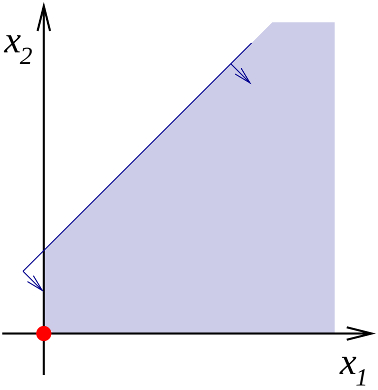

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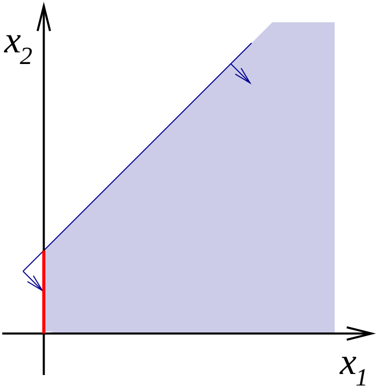

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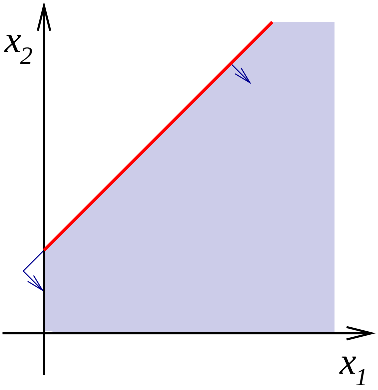

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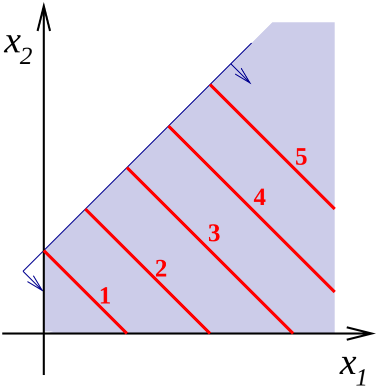

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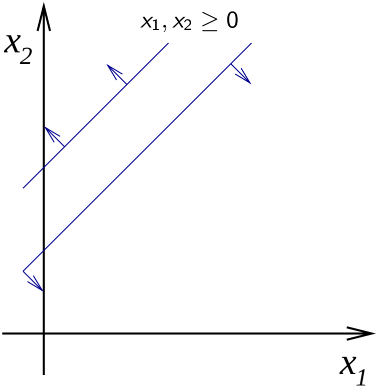

Bases of a polyhedron

We subdivide the equalities and inequalities into three categories :

$$\begin{aligned} a_i^T x &\geq b_i \qquad & i \in M_{\geq} \\ a_i^T x &\leq b_i \qquad & i \in M_{\leq} \\ a_i^T x &= b_i \qquad & i \in M_{=} \end{aligned}$$

Definition

Let $\bar{x}$ be a point satisfying $a_i^T \bar{x} = b_i$ for some $i \in M_{\geq}, M_{\leq}$ or $M_{=}$. The constraint i is said to be active or tight.

Bases of a polyhedron

Definition

Let P be a polyhedron and let $\bar{x} \in \mathbb{R}^n$.

(a) $\bar{x}$ is a basic solution if

- all equalities ($i \in M_=$) are active
- among the active constraints, there are n linearly independent

(b) if $\bar{x}$ is a basic solution that satisfies all constraints, then $\bar{x}$ is a feasible basic solution.

Theorem

Let P be a polyhedron and let $\bar{x} \in P$. The three following statements are equivalent.

(i) $\bar{x}$ is a vertex

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Main messages

- The problem

$$\begin{aligned} \min\ & c^T x \\ \text{subject to } & a_{(i)}^T x \leq b_{(i)} \qquad i = 1, \ldots, m \\ & x \in \mathbb{R}_+^{\ltimes} \end{aligned}$$

can be solved efficiently both in theory and in practice.
Problems with thousands of variables and constraints can be solved in seconds.

- An optimal solution can always be found among vertices
- The simplex algorithm always outputs a vertex as an optimal solution.
- If you add a new constraint to the problem, you can reoptimize very quickly using the simplex algorithm.

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