Discrete Optimization

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Comparing two formulations

To compare two formulations P^1 and P^2 with the same integer feasible points, we consider their respective linear relaxations P^1_{LP}, P^2_{LP}.

Comparing two formulations

P^1 is better than P^2 if

$$P^1_{LP} \subset P^2_{LP}$$

Ideal formulation

If $\mathcal{F} = \{x_1, \ldots, x_k\}$ is the set of feasible solutions, an ideal formulation is

$$\mathrm{conv}(\mathcal{F})$$

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Comparing two formulations : the facility location problem

In the uncapacitated facility location problem (UFL), we are given a set of potential facilities (with a fixed cost f_i when open) to open to serve a list of clients (with cost c_{ij}). What is the set of facilities to open, and which clients should they serve in order to minimize the cost ?

Variables

$y_i = 1$ if facility i is open (0 otherwise)
$x_{ij} = 1$ if facility i serves client j (0 otherwise)

Formulation 1

$$\begin{aligned} \min \ & \sum_i f_i y_i + \sum_{i,j} c_{ij} x_{ij} \\ \text{subject to } & \sum_i x_{ij} = 1 \quad \text{for all } j \\ & x_{ij} \leq y_i \\ & x_{ij} \in \{0,1\}, y_i \in \{0,1\}. \end{aligned}$$

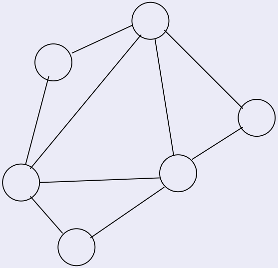
Aggregated formulation

$$\begin{aligned} \min \ & \sum_i f_i y_i + \sum_{i,j} c_{ij} x_{ij} \\ \text{subject to } & \sum_i x_{ij} = 1 \quad \text{for all } j \\ & \sum_{j=1}^{n} x_{ij} \leq n y_i \\ & x_{ij} \in \{0,1\}, y_i \in \{0,1\}. \end{aligned}$$

Comparing two formulations for graph problems

The minimum spanning tree

Let $G = (V, E)$ be an undirected graph. Every edge has a cost c_e. We look for the tree with the minimum total cost.



Constraints to encode :

- A tree should have $n - 1$ edges where n is the number of nodes
- A tree cannot have a cycle or equivalently
  A tree must be connected

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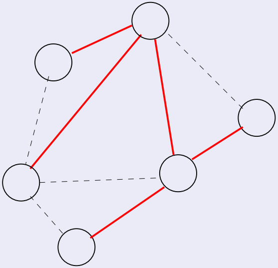

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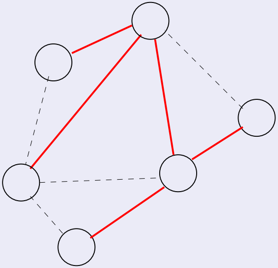

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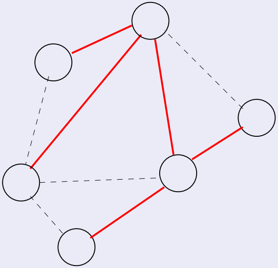

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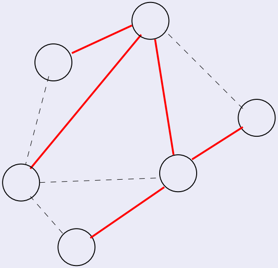

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Subtour elimination formulation

Integer formulation

$$P^I_{sub} = \{x_e \in \{0,1\} \mid \sum_{e \in E} x_e = n-1$$
$$\sum_{e \in E(S)} x_e \leq |S| - 1, \quad S \subset V, S \neq \emptyset, V \}$$

Linear programming relaxation

$$P_{sub} = \{x_e \in [0,1] \mid \sum_{e \in E} x_e = n-1$$
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Cutset formulation

Integer formulation

$$P^I_{cut} = \{x_e \in \{0,1\} \mid \sum_{e \in E} x_e = n-1$$
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Comparing the two formulations

Theorem

- $P_{sub} \subset P_{cut}$ and the inclusion is sometimes strict
- P_{cut} can have fractional extreme points

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The traveling salesman problem

Subtour elimination formulation

$$P^I_{tspsub} = \{x_e \in \{0,1\} \mid \sum_{e \in \delta(\{i\})} x_e = 2 \quad \text{for all } i \in V$$
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If P_{tspsub} and P_{tspcut} are the respective linear relaxations,

$$P_{tspsub} = P_{tspcut}$$

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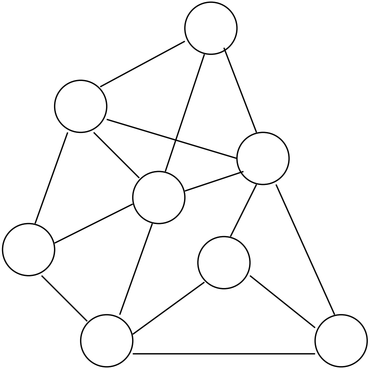
The matching problem

Consider a set of N pilots that must be matched by teams of two.
Each pilot has certain skills (languages that he speaks, operations that he is able to perform, ...)
For each pair of pilots, we define a reward c_{ij} that corresponds to the fact that these two pilots are matched together.

What is the best way to divide the n pilots into $\frac{N}{2}$ teams of 2 pilots in order to maximize the reward.



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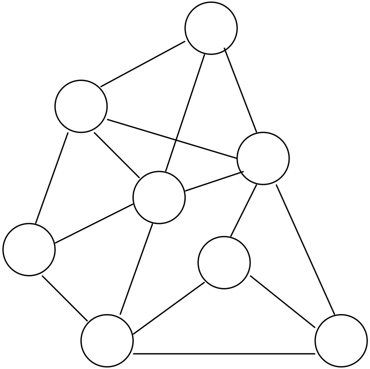

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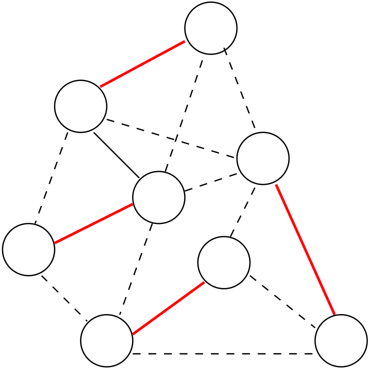

The matching problem

Simple formulation

$$
\begin{aligned}
\text{minimize} \quad & \sum_{e \in E} c_e x_e \\
\text{subject to} \quad & \sum_{e \in \delta(\{i\})} x_e = 1 \\
& x_e \in \{0, 1\}
\end{aligned}
$$

The linear relaxation is not the convex hull of all feasible solutions.

Enhanced formulation

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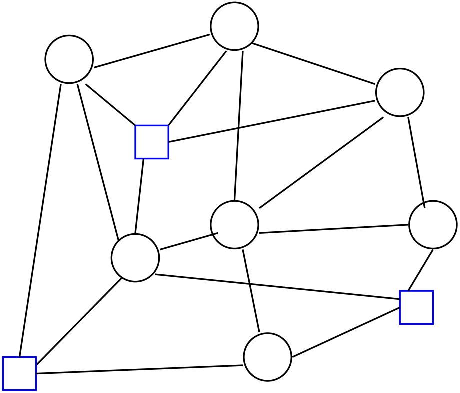
The Steiner tree problem

Given a graph and some terminal nodes, find the shortest tree that connects all terminal nodes.
Very similar to the minimum spanning tree but this version is NP-hard !



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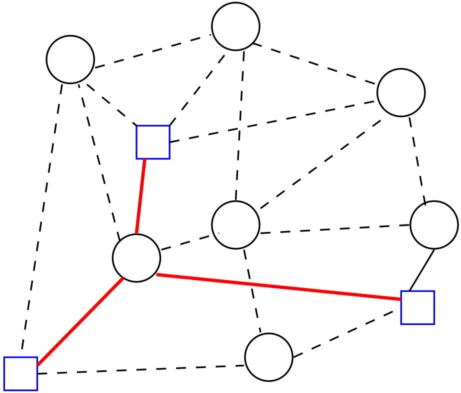
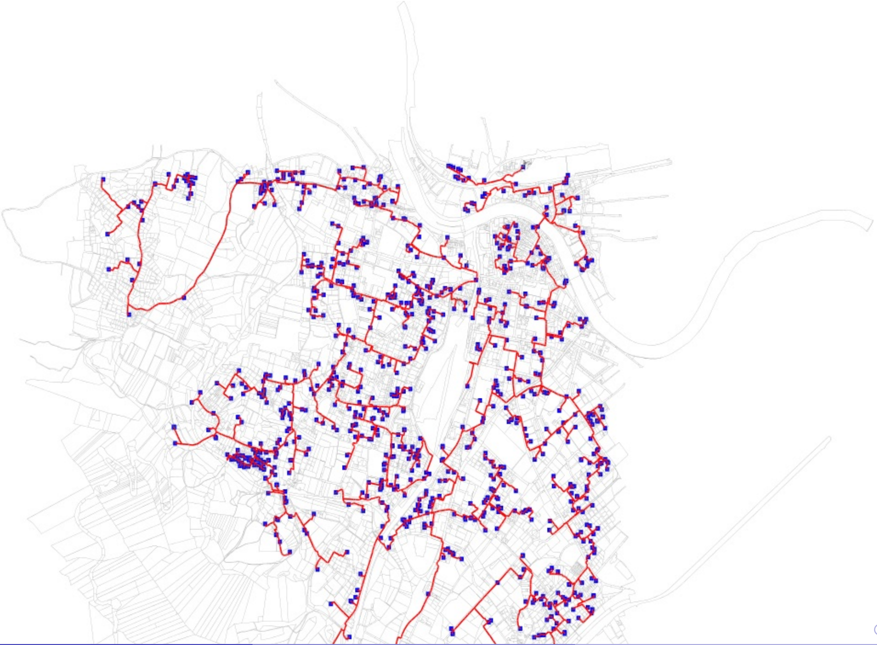


Steiner tree real-world instance



3 formulations of the Steiner tree problem

Simple formulation

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\text{subject to} \quad & \sum_{e \in \delta(S)} x_e \geq 1 \qquad S \subset V,\ S \cap T \neq \emptyset, T \\
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\end{aligned}
$$

Enhanced formulation

- $V_i \cap T \neq \emptyset, i = 1, \ldots, p$
- $V_i \cap V_j = \emptyset, i \neq j$
- $V_1 \cup \cdots \cup V_p = V$

$$
\begin{aligned}
\text{minimize} \quad & \sum_{e \in E} c_e x_e \\
\text{subject to} \quad & \sum_{e \in \delta(V_1, \ldots, V_p)} x_e \geq p - 1 \\
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Directed formulation

$$
\begin{aligned}
\text{minimize} \quad & \sum_{(i,j)\in A} c_{ij} y_{ij} \\
\text{subject to} \quad & \sum_{(i,j): i\in S, j\in V\setminus S} y_{ij} \geq 1, \quad S \subset V, 1 \in S, S \cap T \neq T \\
& y_{ij} + y_{ji} \leq 1 \\
& y_{ij} \in \{0, 1\}
\end{aligned}
$$

Comparing the formulations

$$Z_{Steiner} \leq Z_{partition} \leq ZD_{steiner}$$

where Z correspond to linear relaxations.