Discrete Optimization

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Valid inequalities

We have seen that having a good formulation is crucial to obtain a (fast)-solving problem.
Main issue : how to automatically improve a formulation.

Definition

Let $P \subseteq \mathbb{R}^n$. An inequality $\sum_{j=1}^{n} a_j x_j \leq b$ is valid if it is satisfied by all points $x \in P$.

Typically, we want to derive valid inequalities for the set of integral solutions and at the same time cut off some part of the linear programming relaxation.

The rounding principle

Let $P = \{x \in \mathbb{Z}^n \mid Ax \leq b\}$ and $P_{LP} = \{x \in \mathbb{R}^n \mid Ax \leq b\}$ be the corresponding linear programming relaxation.

If $x \leq c$ is valid for P_{LP} then $x \leq \lfloor c \rfloor$ is valid for P.

The Chvatal-Gomory procedure

- Compute a nonnegative combination of the rows of the LP formulation

$$(u^T A)x \leq u^T b, \qquad (u \geq 0)$$

- The inequality

$$(\lfloor u^T A \rfloor)x \leq \lfloor u^T b \rfloor$$

  is valid for P.

Gomory's fractional cutting plane algorithm

- Based on the simplex algorithm applied to the linear relaxation of the MIP
- automatically generate and apply cuts until solution is integer
  - if optimal solution is fractional, use the information provided by the optimal basis to generate cuts (apply the Chvatal-Gomory procedure)
- terminates in a finite number of iterations if combined with the right simplex pivoting rule
- not very successful in practice, hence **branch-and-cut**.

The Basic Mixed Integer inequality

2D case

Let $X = \{(x, y) \in \mathbb{R}_+ \times \mathbb{Z}_+ \mid x + y \geq b\}$ and $f = b - \lfloor b \rfloor > 0$.
Then

$$\frac{x}{f} + y \geq \lceil b \rceil$$

is valid for X

Corollary

Let $X = \{(x, y) \in \mathbb{R}_+ \times \mathbb{Z}_+ \mid y \leq b + x\}$ and $f = b - \lfloor b \rfloor > 0$.
Then

$$y \leq \lfloor b \rfloor + \frac{x}{1 - f}$$

is valid for X

Mixed Integer Rounding (MIR) cut

Let

$$X = \{(x, y) \in \mathbb{R}_+ \times \mathbb{Z}^2_+ \mid a_1 y_1 + a_2 y_2 \leq b + x\},$$
$$f = b - \lfloor b \rfloor > 0,$$

and

$$f_i = a_1 - \lfloor a_i \rfloor, \ i = 1, 2$$

with

$$f_1 \leq f \leq f_2.$$

Then

$$\lfloor a_1 \rfloor y_1 + \left(\lfloor a_2 \rfloor + \frac{f_2 - f}{1 - f} \right) y_2 \leq \lfloor b \rfloor + \frac{x}{1 - f}$$

is valid for X.

Superadditivity : preliminary definitions

Superadditive function

The function $F : D \subseteq \mathbb{R}^m \mapsto \mathbb{R}$ is superadditive if

$$F(a_1) + F(a_2) \leq F(a_1 + a_2)$$

for all $a_1, a_2 \in D$ such that $a_1 + a_2 \in D$.

Remark : F superadditive $\Rightarrow F(0) \leq 0$.

Non-decreasing function

The function $F : D \subseteq \mathbb{R}^m \mapsto \mathbb{R}$ is non-decreasing if

$$F(a_1) \leq F(a_2)$$

for all $a_1, a_2 \in D$ such that $a_1 \leq a_2$.

Superadditivity

If $F : \mathbb{R}^m \mapsto \mathbb{R}$ is superadditive, non-decreasing and satisfies $F(0) = 0$, then the inequality

$$\sum_{j=1}^{n} F(A_j)x_j \leq F(b)$$

is valid for conv(P) with $P = \{x \in \mathbb{Z}_+^n | Ax \leq b\}$.

Proof, comparison to MIR

Strong inequalities

- Inequalities $\pi x \leq \pi_0$ and $\lambda \pi x \leq \lambda \pi_0$ are identical if $\lambda > 0$.
- An inequality $\pi x \leq \pi_0$ dominates $\mu x \leq \mu_0$ if there exists $u > 0$ with

$$\pi \geq u\mu \quad \text{and} \quad \pi_0 \leq u\mu_0$$

if we work in a polyhedron $P \subset \mathbb{R}^n_+$.

Polyhedra, faces and facets

- n points $x^{(1)}, \ldots, x^{(k)}$ are affinely independent if $x^{(2)} - x^{(1)}, \ldots, x^{(k)} - x^{(1)}$ are linearly independent or equivalently if $(x^{(1)}, 1), \ldots, (x^{(k)}, 1)$ are linearly independent.
- The dimension d of a polyhedron P is the maximum number of affinely independent points in P minus 1.
- F is a face of P if $F = \{x \in P : \pi x = \pi_0\}$ for some valid inequality $\pi x \leq \pi_0$.
- F is a facet if F is a face of P of dimension $\dim(P) - 1$.

Facets of $\text{conv}(P)$ are the valid inequalities that we are looking for !

Knapsack covers

We consider the knapsack set

$$X = \{x \in \{0,1\}^n \mid \sum_{j=1}^{n} a_j x_j \leq b\}.$$

Definition

A set C is a cover if $\sum_{j \in C} a_j > b$.

A cover inequality

If C is a cover, the cover inequality

$$\sum_{j \in C} x_j \leq |C| - 1$$

is valid for X.

Lifting a cover inequality

Consider an inequality $\sum_{j\in C} x_j \leq |C| - 1$.
Consider a variable $i \notin C$ that we would like to lift, namely we want to give it a coefficient in the cover inequality.

$$\begin{aligned} \alpha_i = |C| - 1 - \max & \sum_{j\in C} x_j \\ \text{s. t.} & \sum_{j\in C} \leq b - a_i \\ & x_j \in \{0,1\}. \end{aligned}$$

Branch-and-cut : used in all MIP solvers nowadays

- Branch-and-bound combined with cutting plane algorithm
- uses several families of cuts, depending on the problem (MIR, covers, ...)
- typically starts as a cutting plane algorithm, then branches
- at each node, decide to branch or to generate and add cuts
- cuts are often node specific, i.e. cannot be imported in other parts of the tree without care.