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1 HMM

1.1 Algorithm

In [1]:
```
import numpy as np
from scipy.special import gammaln

# Global Parameters
y=np.loadtxt("hw5-2.data",delimiter=", ",dtype= 'int')
ytest=np.loadtxt("hw5-2.test",delimiter=", ",dtype= 'int')
m=3
K=2
t=len(y)
niter=10

# Initialize parameters to optimize
Pi=np.zeros([m,1],dtype=np.float128)+1/m
a=np.zeros([m,m],dtype=np.float128)+1/m
Lambda=np.array([[1, 5],[50,100],[200,300]],dtype=np.float128)+0.
mix=np.zeros([m,K],dtype=np.float128)+1/K
```

In [2]:
```
def EM_HMM(Pi,a,Lambda,mix,m,K,t,y,ytest,niter):
    Pi=Pi
    a=a
    Lambda=Lambda
    mix=mix
    l_train=np.zeros([niter],dtype=np.float128)
    l_test=np.zeros([niter])

    for i0 in range(niter):

        ######## Probabilities

        # Compute P(yt|qti=1,ptk=1)
        def YgivenQP_f(y):
            res=np.zeros([t,m,K],dtype=np.float128)
            for i in range(m):
                for k in range(K):
                    res[:,i,k]= -Lambda[i,k]+y*np.log(Lambda[i,k])-\
                    gammaln(y+1)
            return(np.exp(res))
        YgivenQP=YgivenQP_f(y)
        YgivenQP_test=YgivenQP_f(ytest)
```

```python
# Compute P(yt|qti=1)
def YgivenQ_f(YgivenQP):
    res=np.zeros([t,m],dtype=np.float128)
    for i in range(m):
        for k in range(K):
              res[:,i]=res[:,i]+YgivenQP[:,i,k]*mix[i,k]

    return(res)
YgivenQ=YgivenQ_f(YgivenQP)
YgivenQ_test=YgivenQ_f(YgivenQP_test)

# Compute alpha=P(y1..yt-1,qti=1)
def alpha_f(YgivenQ):
    res=np.zeros([t,m],dtype=np.float128)
    res[0,:]=YgivenQ[0,:]*np.transpose(Pi)
    for t1 in range(1,t):
        s=np.zeros([1,m])
        for j in range(m):
            s=s+res[t1-1,j]*YgivenQ[t1,:]*a[j,:]
        res[t1,:]=s
    return(res)
alpha=alpha_f(YgivenQ)
alpha_test=alpha_f(YgivenQ_test)

# Compute beta=P(yt+1..yT,qti=1)
def beta_f(YgivenQ):
    res=np.ones([t,m],dtype=np.float128)
    for t1 in range(t-2,-1,-1):
        for i in range(m):
            s=0
            for j in range(m):
                s=s+res[t1+1,j]*a[i,j]*YgivenQ[t1+1,j]
            res[t1,i]=s
    return(res)
beta=beta_f(YgivenQ)
beta_test=beta_f(YgivenQ_test)

# Compute p(y)
def py_f(alpha,beta):
    return(np.sum(alpha*beta,axis=1))
py=py_f(alpha,beta)
py_test=py_f(alpha_test,beta_test)

# Compute gamma=P(qti=1|y)
def gamma():
    res=alpha*beta
    for t1 in range(t):
        res[t1,]=res[t1,]/py[t1]
    return(res)
gamma=gamma()

# Compute dzeta
def dzeta():
```

```
    res=np.zeros([t-1,m,m],dtype=np.float128)
    for i in range(m):
        for j in range(m):
            for t1 in range(t-1):
                res[t1,i,j]=alpha[t1,i]*YgivenQ[t1+1,j]*\
                beta[t1+1,j]*a[i,j]/py[t1]
    return(res)
dzeta=dzeta()


# Compute P(qti=1,ptj=1|y)
def QPgivenY():
    res=np.zeros([t,m,K],dtype=np.float128)
    for i in range(m):
        for k in range(K):
            for t1 in range(t):
                res[t1,i,k]=YgivenQP[t1,i,k]*mix[i,k]*\
                gamma[t1,i]/YgivenQ[t1,i]
    return(res)
QPgivenY=QPgivenY()


######## Expectation

# Compute gamma0
gamma0 = gamma[0,:]

# Compute Aij
def A():
    res=np.zeros([m,m],dtype=np.float128)
    for i in range(m):
        for j in range(m):
            res[i,j]=np.sum(dzeta[:,i,j])
    return(res)
A=A()

# Compute Bik
def B():
    res=np.zeros([m,K],dtype=np.float128)
    for i in range(m):
        for k in range(K):
            res[i,k]=np.sum(QPgivenY[:,i,k]*y)
    return(res)
B=B()


# Compute Cik
def C():
    res=np.zeros([m,K],dtype=np.float128)
    for i in range(m):
        for k in range(K):
            res[i,k]=np.sum(QPgivenY[:,i,k])
    return(res)
C=C()
```

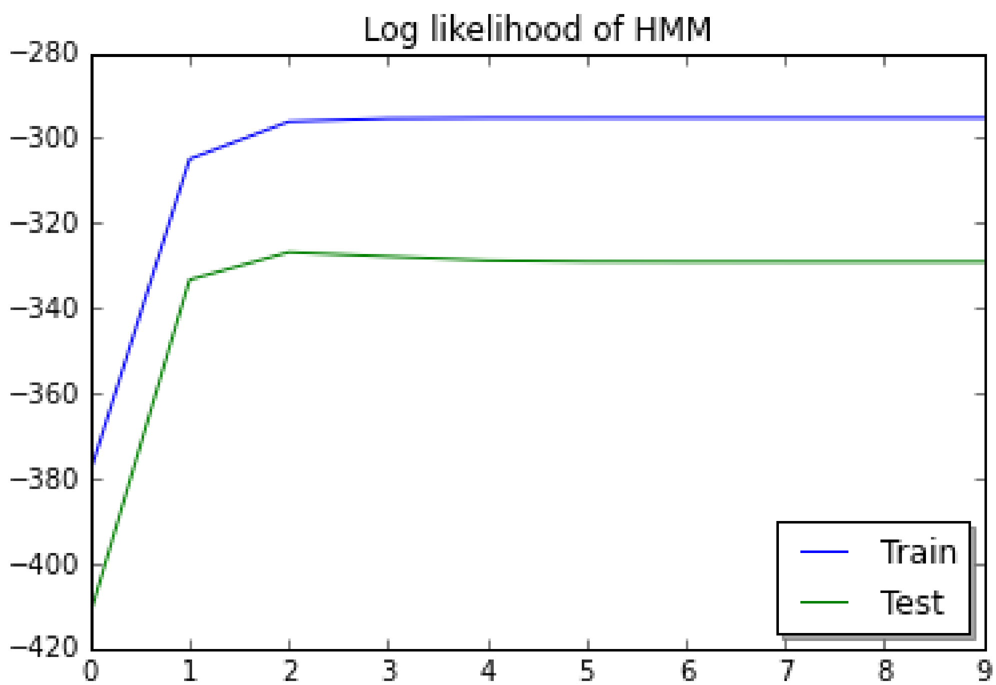
```
        ######## Maximization
        # Update a
        def a():
            a=A
            for i in range(m):
                a[i,:]=a[i,:]/np.sum(a[i,:])
            return(a)
        a=a()

        # Update lambda
        Lambda=B/C

        # Update mix
        def mix():
            mix=C
            for i in range(m):
                mix[i,:]=mix[i,:]/np.sum(mix[i,:])
            return(mix)
        mix=mix()

        # Compute the expected likelihood
        def likelihood_train():
            res=np.sum(np.log(Pi)*gamma0)+np.sum(A*np.log(a))+\
            np.sum(B*(np.log(Lambda/np.exp(1))))-np.sum(gammaln(y+1))
            return(res)

        # Compute the train log-likelihood
        l_train[i0]=np.log(py[t-1])

        # Compute the test log-likelihood
        l_test[i0]=np.log(py_test[t-1])
    return(A,B,C,Lambda,mix,a,l_train,l_test)

EM_HMM=EM_HMM(Pi,a,Lambda,mix,m,K,t,y,ytest,niter)
```

1.2 Results

```
In [3]: # Lambda
        EM_HMM[3]

Out[3]: array([[ 1.0357935,  15.862358],
               [ 40.502892,  106.05553],
               [ 188.80634,  272.43183]], dtype=float128)

In [4]: # mix
        EM_HMM[4]

Out[4]: array([[ 0.84402282,  0.15597718],
               [ 0.24993662,  0.75006338],
               [ 0.45000642,  0.54999358]], dtype=float128)

In [5]: # a
        EM_HMM[5]
```

```
Out[5]: array([[ 0.19245884,  0.73062408,  0.076917078],
               [ 0.17383237,  0.08696938,  0.73919825],
               [ 0.79999533,  0.15000475,  0.049999924]], dtype=float128)
```

```
In [6]: %matplotlib inline
        import matplotlib.pyplot as plt
        fig, ax = plt.subplots()
        ax.plot(range(10), EM_HMM[6], '-', label='Train')
        ax.plot(range(10), EM_HMM[7], '-', label='Test')
        plt.title('Log likelihood of HMM')
        ax.legend(loc='lower right', shadow=True)
        plt.show()
```



2 Mixture model

2.1 Algorithm

```
In [7]: # Global parameters
        K=6
        p=np.zeros([K],dtype=np.float128)+1/K
        lambdas=np.array([1, 5,50,100,200,300],dtype=np.float128)+0.
        niter=10

        # Density
        def dpoisson(x,lmd):
            logd= -lmd+x*np.log(lmd)-gammaln(x+1)
            return(np.exp(logd))
```

```
In [8]: def EM_Mixtuure(lambdas,p,K,t,y,ytest,niter):
            l_train=np.zeros([niter],dtype=np.float128)
            l_test=np.zeros([niter],dtype=np.float128)

            for i0 in range(niter):
                ###### Expectation

                # Compute the joint probability of yt and kth component
                def q(k,t1):
                    return(p[k]*dpoisson(y[t1],lambdas[k]))
                def q_test(k,t1):
                    return(p[k]*dpoisson(ytest[t1],lambdas[k]))

                # Compute the conditional probability of kth component given yt
                def P(k,t1):
                    norm=0
                    for ktilde in range(K):
                        norm = norm+q(ktilde,t1)
                    return(q(k,t1)/norm)


                ###### Maximization

                #Z
                def Z(k):
                    res=0
                    for t1 in range(t):
                        res=res+P(k,t1)
                    return(res)

                # Update lambdas
                def update_lambdas():
                    for k in range(K):
                        res=0
                        for t1 in range(t):
                            res=res+P(k,t1)*y[t1]
                        lambdas[k]=res/Z(k)
                update_lambdas()

                # Update p
                def update_k():
                    for k in range(K):
                        p[k]=Z(k)/t
                    return(p)
                update_k()


                ###### Log Likelihood

                def loglikelihood(p,q):
                    res=0
                    for t1 in range(t):
                        norm=0
                        for ktilde in range(K):
```

```
                    norm = norm+q(ktilde,t1)
                res=res+np.log(norm)
            return(res)

        l_train[i0]=loglikelihood(p,q)
        l_test[i0]=loglikelihood(p,q_test)

    return(lambdas,p,l_train,l_test)

EM_Mixtuure=EM_Mixtuure(lambdas,p,K,t,y,ytest,niter)
```

2.2 Results

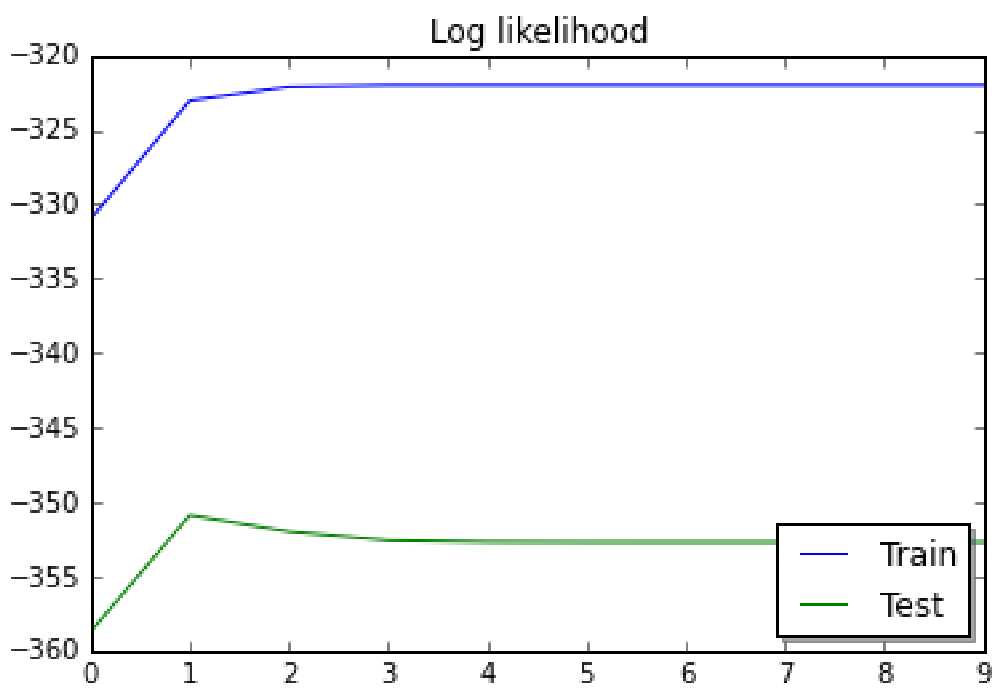
In [9]:
```
# lambdas
EM_Mixtuure[0]
```

Out[9]:
```
array([ 1.035924,  15.882874,  40.497836,  106.05555,  188.80653,
        272.43183], dtype=float128)
```

In [10]:
```
# mixture components p
EM_Mixtuure[1]
```

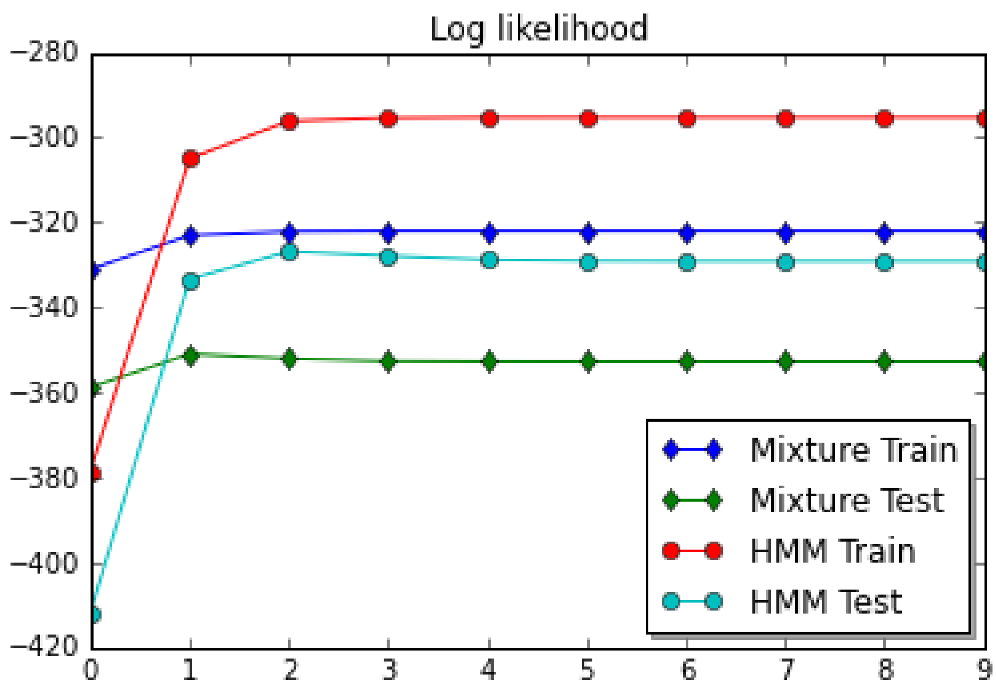
Out[10]:
```
array([ 0.31352911,  0.057954703,  0.085659048,  0.25714275,  0.12857314,
        0.15714125], dtype=float128)
```

In [11]:
```
%matplotlib inline
import matplotlib.pyplot as plt
fig, ax = plt.subplots()
ax.plot(range(10), EM_Mixtuure[2], '-', label='Train')
ax.plot(range(10), EM_Mixtuure[3], '-', label='Test')
plt.title('Log likelihood')
ax.legend(loc='lower right', shadow=True)
plt.show()
```



3 Comparison between HMM and Mixture

In [12]:
```
%matplotlib inline
import matplotlib.pyplot as plt
fig, ax = plt.subplots()
ax.plot(range(10), EM_Mixtuure[2], '-d', label='Mixture Train')
ax.plot(range(10), EM_Mixtuure[3], '-d', label='Mixture Test')
ax.plot(range(10), EM_HMM[6], '-o', label='HMM Train')
ax.plot(range(10), EM_HMM[7], '-o', label='HMM Test')
plt.title('Log likelihood')
ax.legend(loc='lower right', shadow=True)
plt.show()
```



4 Trees

4.1 Algoritm

In [13]:
```
import numpy as np

# Global Parameters
y=np.loadtxt("hw5-4.data",delimiter=", ")
t=len(y)
mu0=0
mu1=2
```

```
v0=1
v1=1
a=0.5 # alpha in the problem set
alphaX1=np.array([0.5,0.5],dtype=np.float128)
niter=10
```

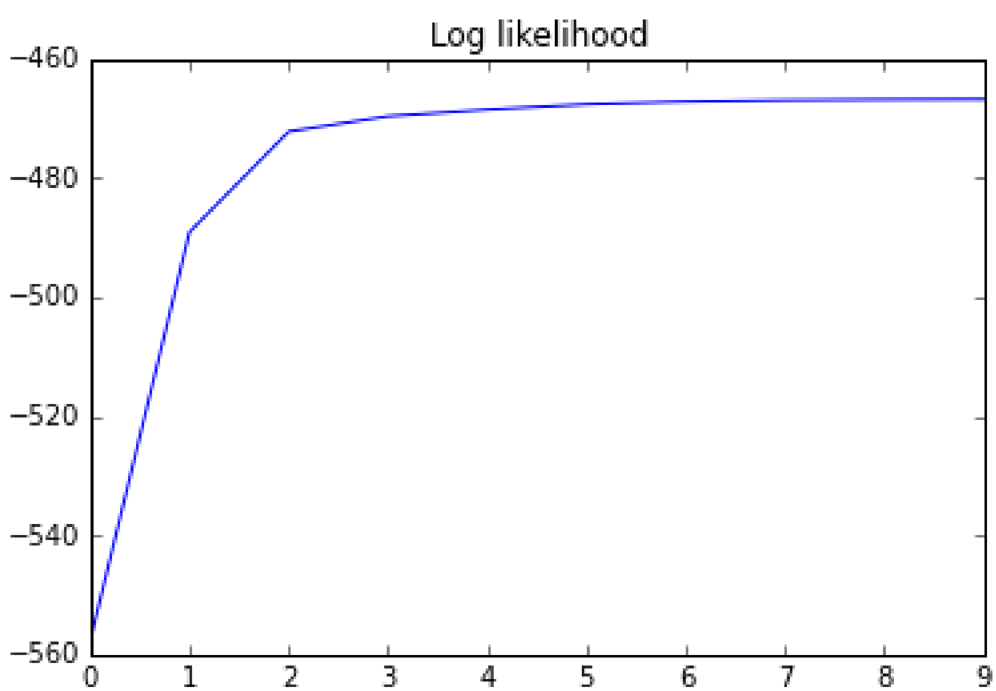
```
In [14]: def Tree_EM(mu0,mu1,v0,v1,a,alphaX1,y,niter):
             l_train=np.zeros([niter],dtype=np.float128)

             for i0 in range(niter):

                 ###### Expectation

                 # Compute P(Yt|X=i)
                 def YgivenX():
                     res=np.zeros([t,2],dtype=np.float128)
                     for t1 in range(t):
                         res[t1,0]=np.exp(-(y[t1]-mu0)**2/(2*v0))/np.sqrt(2*np.pi*v0)
                         res[t1,1]=np.exp(-(y[t1]-mu1)**2/(2*v1))/np.sqrt(2*np.pi*v1)
                     return(res)
                 YgivenX=YgivenX()

                 # Compute P(X2i|Xi) or P(X2i+1|Xi)
                 def SgivenX():
                     return(np.array([[1-a,a],[a,1-a]],dtype=np.float128))
                 SgivenX=SgivenX()

                 # Compute beta(xi)
                 def beta_f():
                     res=np.zeros([t,2],dtype=np.float128)
                     res2=np.zeros([t,2],dtype=np.float128)
                     maxchild=np.zeros([np.power(2,7)-1,2,2],dtype=np.float128)
                     # Fill leaves with normal distributions
                     for t1 in range(np.power(2,7),np.power(2,8)):
                         res[t1-1,:]=YgivenX[t1-1,:]
                         res2[t1-1,:]=YgivenX[t1-1,:]
                     # i begins at one like real nodes indices
                     for i in range(np.power(2,7)-1,0,-1):
                         for j1 in range(2):
                             for j2 in range(2):
                                 # Maxchild
                                 up0=res2[2*i-1,j1]*res2[(2*i+1)-1,j2]*YgivenX[i-1,0]*\
                                 SgivenX[j1,0]*SgivenX[j2,0]
                                 up1=res2[2*i-1,j1]*res2[(2*i+1)-1,j2]*YgivenX[i-1,1]*\
                                 SgivenX[j1,1]*SgivenX[j2,1]
                                 if res2[i-1,0] < up0:
                                     res2[i-1,0]=up0
                                     maxchild[i-1,0,:]=[j1,j2]
                                 if res2[i-1,1]<up1:
                                     res2[i-1,1]=up1
                                     maxchild[i-1,1,:]=[j1,j2]
                                 # Beta
                                 res[i-1,0]=res[i-1,0]+res[2*i-1,j1]*res[(2*i+1)-1,j2]*\
                                 YgivenX[i-1,0]*SgivenX[j1,0]*SgivenX[j2,0]
```

```
                res[i-1,1]=res[i-1,1]+res[2*i-1,j1]*res[(2*i+1)-1,j2]*\
                YgivenX[i-1,1]*SgivenX[j1,1]*SgivenX[j2,1]
    return(res,maxchild)

beta_f=beta_f()
beta=beta_f[0]
maxchild=beta_f[1]

# Compute P(yD2i+1|xi)
def DgivenX():
    res=np.zeros([np.power(2,7)-1,2],dtype=np.float128)
    for i in range(1,np.power(2,7)):
        for j in range(2):
            res[i-1,0]=res[i-1,0]+beta[(2*i+1)-1,j]*SgivenX[j,0]
            res[i-1,1]=res[i-1,1]+beta[(2*i+1)-1,j]*SgivenX[j,1]
    return(res)
DgivenX=DgivenX()

# Compute P(yD2i|xi)
def GgivenX():
    res=np.zeros([np.power(2,7)-1,2],dtype=np.float128)
    for i in range(1,np.power(2,7)):
        for j in range(2):
            res[i-1,0]=res[i-1,0]+beta[2*i-1,j]*SgivenX[j,0]
            res[i-1,1]=res[i-1,1]+beta[2*i-1,j]*SgivenX[j,1]
    return(res)
GgivenX=GgivenX()

# Compute alpha(xi)
def alpha():
    res=np.zeros([t,2],dtype=np.float128)
    # Initialze alpha(x1)
    res[0,0]=alphaX1[0]
    res[0,1]=alphaX1[1]
    # i begins at one like real nodes indices
    for i in range(1,np.power(2,7)):
        for j in range(2):
            res[2*i-1,0]=res[2*i-1,0]+res[i-1,j]*\
            YgivenX[i-1,j]*DgivenX[i-1,j]*SgivenX[0,j]
            res[2*i-1,1]=res[2*i-1,1]+res[i-1,j]*\
            YgivenX[i-1,j]*DgivenX[i-1,j]*SgivenX[1,j]
            res[(2*i+1)-1,0]=res[(2*i+1)-1,0]+res[i-1,j]*\
            YgivenX[i-1,j]*GgivenX[i-1,j]*SgivenX[0,j]
            res[(2*i+1)-1,1]=res[(2*i+1)-1,1]+res[i-1,j]*\
            YgivenX[i-1,j]*GgivenX[i-1,j]*SgivenX[1,j]
    return(res)
alpha=alpha()

# Compute p(y)
def py():
    return(np.sum(alpha*beta,axis=1))
py=py()

# Compute p(xi|y)
```

```python
def XgivenY():
    res=alpha*beta
    for i in range(t):
        res[i,:]=res[i,:]/py[i]
    return(res)
XgivenY=XgivenY()

# Compute p(xi,x2i|y) or p(xi,x2i+1|y)
def XSgivenY():
    res=np.zeros([np.power(2,7)-1,2,2],dtype=np.float128)
    for t1 in range(1,np.power(2,7)):
        for j in range(2):
            for xi in range(2):
                for xj in range(2):
                    res[t1-1,xi,xj]=res[t1-1,xi,xj]+alpha[t1-1,xi]*\
                    beta[2*t1-1,xj]*beta[(2*t1+1)-1,j]*\
                    YgivenX[t1-1,xi]*SgivenX[xj,xi]*SgivenX[j,xi]
    return(res/py[0])
XSgivenY=XSgivenY()

###### Maximization

# Update a
def newa():
    res_diff=0
    for i in range(1,np.power(2,7)):
        res_diff=res_diff+2*(XSgivenY[i-1,0,1]+XSgivenY[i-1,1,0])
    return(res_diff/(t-1))
a=newa()

# Update sigma0^2
def newv0():
    num=0
    den=0
    for i in range(t):
        den=den+XgivenY[i,0]
        num=num+XgivenY[i,0]*np.power(y[i]-mu0,2)
    return(num/den)
v0=newv0()

# Update sigma1^2
def newv1():
    num=0
    den=0
    for i in range(t):
        den=den+XgivenY[i,1]
        num=num+XgivenY[i,1]*np.power(y[i]-mu1,2)
    return(num/den)
v1=newv1()

# Update mu0
def newmu0():
    num=0
    den=0
```

```
            for i in range(t):
                num=num+y[i]*XgivenY[i,0]
                den=den+XgivenY[i,0]
            return(num/den)
        mu0=newmu0()

        # Update mu1
        def newmu1():
            num=0
            den=0
            for i in range(t):
                num=num+y[i]*XgivenY[i,1]
                den=den+XgivenY[i,1]
            return(num/den)
        mu1=newmu1()


        ###### Log-likelihood
        l_train[i0]=np.log(py[0])

    return(mu0,mu1,v0,v1,a,l_train,maxchild)

Tree_EM=Tree_EM(mu0,mu1,v0,v1,a,alphaX1,y,niter)
```

In [15]:
```
%matplotlib inline
import matplotlib.pyplot as plt
plt.plot(Tree_EM[5])
plt.title('Log likelihood')
plt.show()
```



In [16]:
```
#mu0
Tree_EM[0]
```

Out[16]: -0.99501356955284888094

In [17]:
```
#mu1
Tree_EM[1]
```

Out[17]: 2.1592947568198361604

In [18]:
```
#v0
Tree_EM[2]
```

Out[18]: 0.62216359423016470031

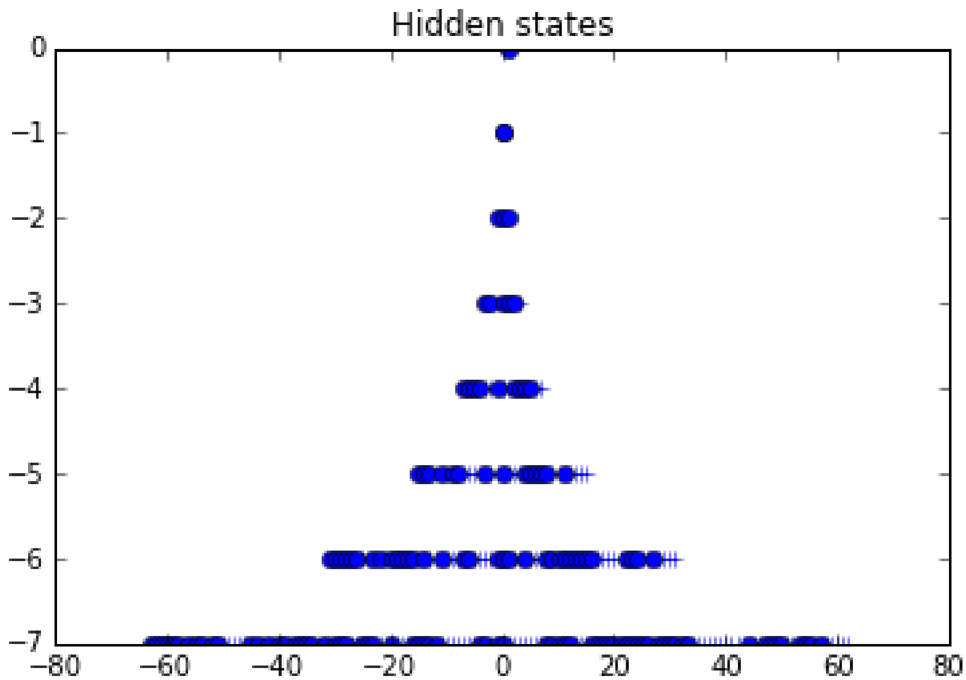
In [19]:
```
#v1
Tree_EM[3]
```

Out[19]: 1.5632186874985169451

In [20]:
```
#a
Tree_EM[4]
```

Out[20]: 0.13863891645807134905

In [21]:
```
###### Most likely hidden states

%matplotlib inline
# Maxchild
mchild=Tree_EM[6]
# init plot
xaxis=np.zeros([t],dtype=np.int32)
yaxis=np.zeros([t],dtype=np.int32)
for i in range(1,t):
    xaxis[i-1]=(i-3*np.power(2,np.int(np.log2(i))-1)+.5)
    yaxis[i-1]=-np.int(np.log2(i))

#labels
def labels():
    res=np.ones([t],dtype=np.int32)
    for i in range(1,np.power(2,7)):
        res[2*i-1]=mchild[i-1,res[i-1],0]
        res[(2*i+1)-1]=mchild[i-1,res[i-1],1]
    return(res)
labels=(labels()==1)

# plot
plt.plot(xaxis,yaxis,'+')
plt.plot(xaxis[labels],yaxis[labels],'bo')
plt.title('Hidden states')
plt.show()
```


Hidden states
0
−1
−2
−3
−4
−5
−6
−7
−80
−60
−40
−20
0
20
40
60
80