STAT230 HW 5
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1 Theory

1.1

I note E the elite tolerance, R the repression and M the mass tolerance.

$$\begin{aligned} R &= \beta_1 M + \beta_2 E + \delta \qquad (1) \\ &= X\beta + \delta \qquad (2) \end{aligned}$$

Where $X = \begin{bmatrix} M & E \end{bmatrix}$ and $\beta = \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix}$.
Using the OLS estimate:

$$\hat{\beta} = (X^T X)^{-1} X^T R \qquad (3)$$

Here, we have:

$$X^T X = n \begin{bmatrix} 1 & 0.52 \\ 0.52 & 1 \end{bmatrix} \qquad (4)$$

$$X^T R = n \begin{bmatrix} -0.26 \\ -0.42 \end{bmatrix} \qquad (5)$$

Therefore:

$$\hat{\beta} = \begin{bmatrix} 1 & 0.52 \\ 0.52 & 1 \end{bmatrix}^{-1} \begin{bmatrix} -0.26 \\ -0.42 \end{bmatrix} = \begin{bmatrix} -0.057 \\ -0.390 \end{bmatrix} \qquad (6)$$

1.2

Using the fact that:

$$1 = \hat{\beta_1}^2 + \hat{\beta_2}^2 + 2\hat{\beta}_1\hat{\beta}_2 * 0.52 + \hat{\sigma}^2 \tag{7}$$

We come up with:

$$\hat{\sigma}^2 = 0.821 \tag{8}$$

However, with the multiplicative correction term $\frac{n}{n-p}$ in front of $\hat{\sigma}^2$, we have:

$$\hat{\sigma}^2 = 0.896 \tag{9}$$

We will use this value for the following questions.

1.3

Since $n = 36$ understate SE. Here the intercept variable is masked and I choose $p = 3$.

$$\begin{aligned} SE &= \left(\hat{\sigma}^2 (X^T X)^{-1}_{(1,1)}\right)^{1/2} && (10) \\ &= 0.185 && (11) \end{aligned}$$

Note that, here we have: $(X^T X)^{-1}_{(1,1)} = (X^T X)^{-1}_{(2,2)}$

1.4

$$\begin{aligned} Var(\hat{\beta}_1 - \hat{\beta}_2|X) &= Var(\hat{\beta}_1) + Var(\hat{\beta}_2|X) - 2Cov(\hat{\beta}_1, \hat{\beta}_2|X) && (12) \\ &= V_{11} + V_{22} - 2V_{12} && (13) \end{aligned}$$

Where:

$$\begin{aligned} V &= \hat{\sigma}^2 (X^T X)^{-1} && (14) \\ &= \frac{\hat{\sigma}^2}{n} \begin{bmatrix} 1 & 0.52 \\ 0.52 & 1 \end{bmatrix}^{-1} && (15) \\ &= \begin{bmatrix} 0.034 & -0.018 \\ -0.018 & 0.034 \end{bmatrix} && (16) \end{aligned}$$

Therefore, using the formula above:

$$\begin{aligned} Var(\hat{\beta}_1 - \hat{\beta}_2|X) &= 0.104 && (17) \\ SE(\hat{\beta}_1 - \hat{\beta}_2) &= 0.322. && (18) \end{aligned}$$

Since $\hat{\beta}_1 - \hat{\beta}_2 = 0.33$, the difference is not significant; the standard error is too high. If we take β_1 alone, it seems to be not significant, and β_2 seems to be significant. However, we cannot say that the difference is not significant because the null hypothesis of $\beta_1 = \beta_0$ is viable. We will see that next when performing the p-values of the t-tests.

2 Code

2.1

Here I compute the estimate of β, and its adjusted variance.

```
## # Paramaters
n = 36
p = 3

## # Compute A = X'X
A = matrix(c(1,0.52,0.52,1),ncol=2)*n
beta_hat = solve(A,c(-0.26,-0.42)*n)
beta_hat

## [1] -0.05701754 -0.39035088

## # Compute adjusted variance
var_hat = n/(n-p)*(1-beta_hat[1]^2-beta_hat[2]^2-
                     2*beta_hat[1]*beta_hat[2]*0.52)
var_hat

## [1] 0.8958852
```

Then, noticing that the standard error (SE) is the same for $\hat{\beta}_1$ or $\hat{\beta}_2$, we have:

```
## # Compute adjusted SE
# SE for both beta coefficients
SE = sqrt(var_hat*solve(A)[1,1])
SE

## [1] 0.1846854

## # Compute adjusted SE
```

```
V = var_hat*solve(A)
var_diff = V[1,1]+V[2,2]-2*V[1,2]
SE_diff = sqrt(var_diff)
diff = beta_hat[1]-beta_hat[2]
SE_diff

## [1] 0.32201
```

2.2

We know that, under the null, $\hat{\beta}/SE$ is following a student distribution with $n-p$ degrees of freedom. I then compute the p-values of the coefficients. It is useful to note that since the t distribution is two sided, I can calculate the p-value with $2P(X > |t|)$ where X is distributed as a student-t distribution.

```
## # Student test for beta 1.
SE1 = sqrt(var_hat*solve(A)[1,1])
t1 = beta_hat[1]/SE1
2*(1-pt(abs(t1),df=n-p)) #pvalue

## [1] 0.7594691

## # Student test for beta 2.
SE2 = sqrt(var_hat*solve(A)[2,2])
t2 = beta_hat[2]/SE2
2*(1-pt(abs(t2),df=n-p)) #pvalue

## [1] 0.04219431
```

Since the p-value for $\hat{\beta}_1$ is greater than 0.1, we cannot say anything about rejecting the null hypothesis. However, the p-value for $\hat{\beta}_2$ is less than 0.05. Therefore, it makes sense to reject the null hypothesis and say that $\hat{\beta}_2$ is significant.

2.3

Now, when doing the t-test to the difference $\hat{\beta}_1 - \hat{\beta}_2$, we observe the following:

```r
## # Student test for the difference of beta 1 and beta 2
t = (beta_hat[1]-beta_hat[2])/SE_diff
2*(1-pt(abs(t),df=n-p)) #pvalue

## [1] 0.3081186
```

Here, the p-value is big enough not to reject the null hypothesis $\hat{\beta}_1 = \hat{\beta}_2$. This is due to the correlation betwen β_1 and β_2. As stating in the Theory part, β_1 alone is not significant. However, we cannot say that the difference $\beta_1 - \beta_0$ is not significant by looking at the p-values because the null hypothesis of $\beta_1 = \beta_0$ cannot be rejected that easily.

As a side note, I would add that we have to be careful with the definition of p-values. Having a high p-value means that the data is likely with true null, and a low p-value means that the data is unlikely with a true null, no more.