CS217 Data Structures and Algorithm Analysis

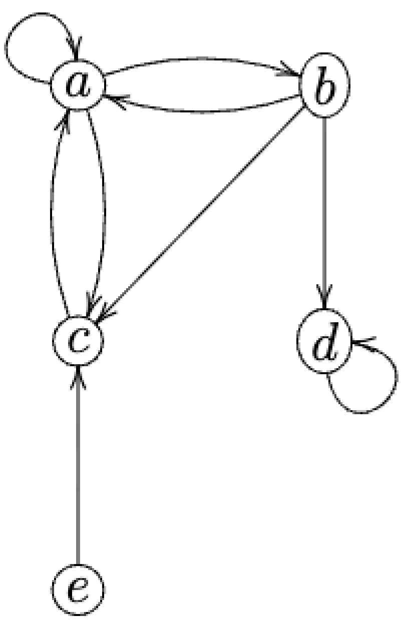
Fall term 2025

Exercise Sheet 14

Handout: December 9th **Deadline:** December 16th, 4pm

Question 14.1 (0.25 marks)

Perform a depth-first search on the following graph visiting nodes in alphabetical order. Assume that all adjacency lists are sorted alphabetically. Write down the timestamps and the π-value of each node.



Answer:

adjacency list:

a: a, b, c

b: a, c, d

c: a

d: d

e: c

Starting DFS from vertex a:

- Vertex a: $time = 1/, \pi = \text{NULL}, color = \text{gray}$
- Vertex b: $time = \infty, \pi = \text{NULL}, color = \text{white}$
- Vertex c: $time = \infty, \pi = \text{NULL}, color = \text{white}$
- Vertex d: $timed = \infty, \pi = \text{NULL}, color = \text{white}$
- Vertex e: $time = \infty, \pi = \text{NULL}, color = \text{white}$

We explore the vertices in the following order:

Visit vertex a.

- Explore neighbors of a: a, b, c
  - Skip neighbor a since it has already been visited.
  - Mark b as visited with timestamp 2 and set its parent to a.
  - Vertex b: $time = 2/, \pi = \text{a}, color = \text{grey}$
  - Explore neighbors of b: a, c, d
    - Skip neighbor a since it has already been visited.
    - Mark c as visited with timestamp 3 and set its parent to b.
    - Vertex c: $time = 3/, \pi = \text{b}, color = \text{grey}$
      - Explore neighbors of c: a
      - Skip neighbor a since it has already been visited.
      - Mark c as black and finish at timestamp 4.
      - Vertex c: $time = 3/4, \pi = \text{b}, color = \text{black}$
      - Backtrack to vertex b.
    - Mark d as visited with timestamp 5 and set its parent to b.
    - Vertex d: $time = 5/, \pi = \text{b}, color = \text{grey}$
      - Explore neighbors of d: d
      - Skip neighbor d since it has already been visited.
      - Mark d as black and finish at timestamp 6.
      - Vertex d: $time = 5/6, \pi = \text{b}, color = \text{black}$
      - Backtrack to vertex b.
    - Mark b as black and finish at timestamp 7.
    - Vertex b: $time = 2/7, \pi = \text{a}, color = \text{black}$
  - Backtrack to vertex a.
  - Skip neighbor c since it has already been visited.
  - Mark a as black and finish at timestamp 8.
  - Vertex a: $time = 1/8, \pi = \text{NULL}, color = \text{black}$

visit vertex e.

- Mark e as visited with timestamp 9 and set its parent to NULL.
- Vertex e: $time = 9/, \pi = \text{NULL}, color = \text{grey}$
  - Explore neighbors of e: c
  - Skip neighbor c since it has already been visited.
  - Mark e as black and finish at timestamp 10.
  - Vertex e: $time = 9/10, \pi = \text{NULL}, color = \text{black}$

After DFS traversal completes, we have the following timestamps and π values for each vertex:

- Vertex a: $time = 1/8, \pi = \text{NULL}, color = \text{black}$
- Vertex b: $time = 2/7, \pi = \text{a}, color = \text{black}$
- Vertex c: $time = 3/4, \pi = \text{b}, color = \text{black}$
- Vertex d: $time = 5/6, \pi = \text{b}, color = \text{black}$
- Vertex e: $time = 9/10, \pi = \text{NULL}, color = \text{black}$

Question 14.2 (0.5 marks)

Prove or refute the following claim: if some depth-first search on a directed graph yields precisely one back edge, then all depth-first searches on this graph yield precisely one back edge.

Answer:

The statement is false. Consider the following counterexample:

adjacency list:
A: B
B: C
C: A, B

Performing a depth-first search (DFS) starting from vertex A:

- Start at A: mark A as gray.
  - Explore neighbor B: mark B as gray.
    - Explore neighbor C: mark C as gray.
      - Explore neighbor A: A is gray, so we have found a back edge (C -> A).
      - Explore neighbor B: B is gray, so we have found another back edge (C -> B).
      - Mark C as black.
    - Mark B as black.

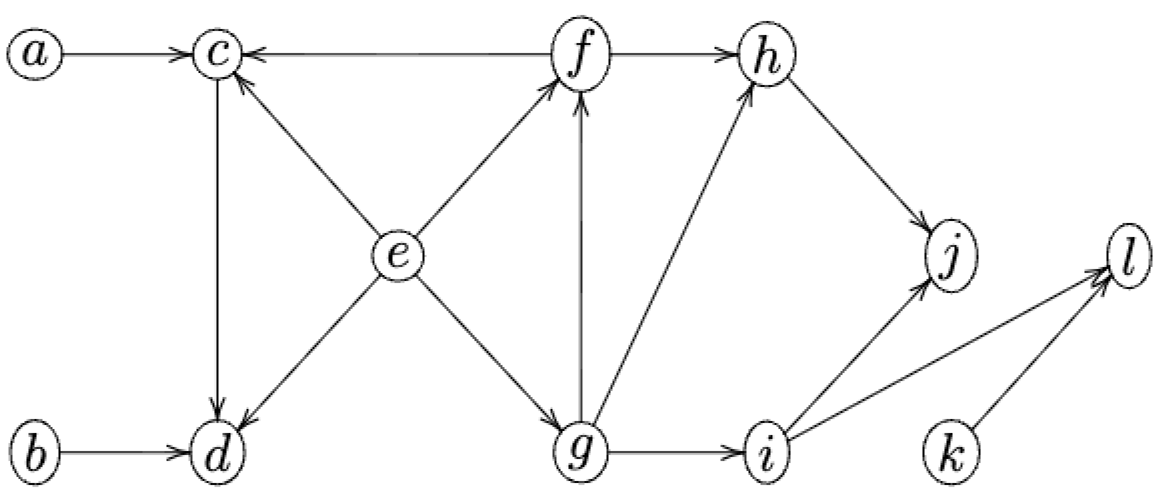
- Mark A as black.
  In this DFS traversal, we found two back edges: (C -> A) and (C -> B).

Now, let's perform a DFS starting from vertex C:

- Start at C: mark C as gray.
  - Explore neighbor A: mark A as gray.
    - Explore neighbor B: mark B as gray.
      - Explore neighbor C: C is gray, so we have found a back edge (B -> C).
      - Mark B as black.
    - Mark A as black.
  - Explore neighbor B: B is black, (C -> B) is a forward edge, not a back edge.
  - Mark C as black.
    In this DFS traversal, we found only one back edge: (B -> C).

Question 14.3 (0.25 marks)

Run TOPOLOGICAL-SORT on the following directed acyclic graph. Assume that depth-first search visits nodes in alphabetical order and that adjacency lists are sorted alphabetically.



Answer:

We start by initializing the vertices with their respective properties:

adjacency list:
a: c

b: d

c: d

d:

e: c, d, f, g

f: c, h

g: f, h, i

h: j

i: j, l

j:

k: l

l:

- Vertex a: $time = \infty, \pi = \text{NULL}, color = \text{gray}$
- Vertex b: $time = \infty, \pi = \text{NULL}, color = \text{white}$
- Vertex c: $time = \infty, \pi = \text{NULL}, color = \text{white}$
- Vertex d: $timed = \infty, \pi = \text{NULL}, color = \text{white}$
- Vertex e: $time = \infty, \pi = \text{NULL}, color = \text{white}$
- Vertex f: $time = \infty, \pi = \text{NULL}, color = \text{white}$
- Vertex g: $time = \infty, \pi = \text{NULL}, color = \text{white}$
- Vertex h: $time = \infty, \pi = \text{NULL}, color = \text{white}$
- Vertex i: $time = \infty, \pi = \text{NULL}, color = \text{white}$
- Vertex j: $time = \infty, \pi = \text{NULL}, color = \text{white}$
- Vertex k: $time = \infty, \pi = \text{NULL}, color = \text{white}$
- Vertex l: $time = \infty, \pi = \text{NULL}, color = \text{white}$

Initialize an empty list to hold the topological order.
List L = []
Call DFS on the graph:
Starting DFS from vertex a:

- Explore neighbors of a: c
  - Mark c as visited with timestamp 1 and set its parent to a.
  - Vertex c: $time = 1/, \pi = \text{a}, color = \text{grey}$
    - Explore neighbors of c: d
      - Mark d as visited with timestamp 2 and set its parent to c.
      - Vertex d: $time = 2/, \pi = \text{c}, color = \text{grey}$
        - d has no neighbors.
        - Mark d as black and finish at timestamp 3.
        - Vertex d: $time = 2/3, \pi = \text{c}, color = \text{black}$

            - Insert d at the front of L. List L = [d].
          - Backtrack to vertex c.
        - Mark c as black and finish at timestamp 4.
        - Vertex c: $time = 1/4, \pi = \mathrm{a}, color = \mathrm{black}$
        - Insert c at the front of L. List L = [c, d].
    - Backtrack to vertex a.
- Mark a as black and finish at timestamp 5.
- Vertex a: $time = 0/5, \pi = \mathrm{NULL}, color = \mathrm{black}$
- Insert a at the front of L. List L = [a, c, d].

Continuing DFS from vertex b:

- mark b as visited with timestamp 6 and set its parent to NULL.
- Vertex b: $time = 6/, \pi = \mathrm{NULL}, color = \mathrm{grey}$
    - Explore neighbors of b: d
    - d is already black, so no update.
- Mark b as black and finish at timestamp 7.
- Vertex b: $time = 6/7, \pi = \mathrm{NULL}, color = \mathrm{black}$
- Insert b at the front of L. List L = [b, a, c, d].

Continuing DFS from vertex e:

- mark e as visited with timestamp 8 and set its parent to NULL.
- Vertex e: $time = 8/, \pi = \mathrm{NULL}, color = \mathrm{grey}$
    - Explore neighbors of e: c, d, f, g
    - c is already black, so no update.
    - d is already black, so no update.
    - Mark f as visited with timestamp 9 and set its parent to e.
    - Vertex f: $time = 9/, \pi = \mathrm{e}, color = \mathrm{grey}$
        - Explore neighbors of f: c, h
        - c is already black, so no update.
        - Mark h as visited with timestamp 10 and set its parent to f.
        - Vertex h: $time = 10/, \pi = \mathrm{f}, color = \mathrm{grey}$
            - Explore neighbors of h: j
            - Mark j as visited with timestamp 11 and set its parent to h.
            - Vertex j: $time = 11/, \pi = \mathrm{h}, color = \mathrm{grey}$
                - j has no neighbors.
                - Mark j as black and finish at timestamp 12.
                - Vertex j: $time = 11/12, \pi = \mathrm{h}, color = \mathrm{black}$

            - Insert j at the front of L. List L = [j, b, a, c, d].
        - Backtrack to vertex h.
    - Mark h as black and finish at timestamp 13.
    - Vertex h: $time = 10/13, \pi = \text{f}, color = \text{black}$
    - Insert h at the front of L. List L = [h, j, b, a, c, d].
    - Backtrack to vertex f.
  - Mark f as black and finish at timestamp 14.
  - Vertex f: $time = 9/14, \pi = \text{e}, color = \text{black}$
  - Insert f at the front of L. List L = [f, h, j, b, a, c, d].
  - Backtrack to vertex e.
  - Mark g as visited with timestamp 15 and set its parent to e.
  - Vertex g: $time = 15/, \pi = \text{e}, color = \text{grey}$
    - Explore neighbors of g: f, h, i
    - f is already black, so no update.
    - h is already black, so no update.
    - Mark i as visited with timestamp 16 and set its parent to g.
    - Vertex i: $time = 16/, \pi = \text{g}, color = \text{grey}$
        - Explore neighbors of i: j, l
        - j is already black, so no update.
        - Mark l as visited with timestamp 17 and set its parent to i.
        - Vertex l: $time = 17/, \pi = \text{i}, color = \text{grey}$
            - l has no neighbors.
            - Mark l as black and finish at timestamp 18.
            - Vertex l: $time = 17/18, \pi = \text{i}, color = \text{black}$
            - Insert l at the front of L. List L = [l, f, h, j, b, a, c, d].
        - Backtrack to vertex i.
    - Mark i as black and finish at timestamp 19.
    - Vertex i: $time = 16/19, \pi = \text{g}, color = \text{black}$
    - Insert i at the front of L. List L = [i, l, f, h, j, b, a, c, d].
    - Backtrack to vertex g.
  - Mark g as black and finish at timestamp 20.
  - Vertex g: $time = 15/20, \pi = \text{e}, color = \text{black}$
  - Insert g at the front of L. List L = [g, i, l, f, h, j, b, a, c, d].
  - Backtrack to vertex e.
- Mark e as black and finish at timestamp 21.
- Vertex e: $time = 8/21, \pi = \text{NULL}, color = \text{black}$
- Insert e at the front of L. List L = [e, g, i, l, f, h, j, b, a, c, d].

Continuing DFS from vertex k:

- mark k as visited with timestamp 22 and set its parent to NULL.
- Vertex k: $time = 22/, \pi = \mathrm{NULL}, color = \mathrm{grey}$
  - Explore neighbors of k: l
  - l is already black, so no update.
- Mark k as black and finish at timestamp 23.
- Vertex k: $time = 22/23, \pi = \mathrm{NULL}, color = \mathrm{black}$
- Insert k at the front of L. List L = [k, e, g, i, l, f, h, j, b, a, c, d].

Finally we have completed the DFS traversal. The topological order is given by the list L:
Topological Sort Order: [k, e, g, i, l, f, h, j, b, a, c, d]

Question 14.4 (0.5 marks)

Recall from the lecture that DFS can be used to check whether a directed graph $G = (V, E)$ is acyclic or not, and that DFS runs in time $\Theta(|V| + |E|)$.

Give an algorithm that checks whether or not an undirected graph $G = (V, E)$ is acyclic and that runs in time only $O(|V|)$.

Answer:

Intuitively, for an undirected graph, if $|E| \geq |V|$, then the graph must contain at least one cycle.

```
For edge in adjacency_list:
    count += 1
    if count == |V|:
        return "Cyclic"
```

During the counting, if the count reaches |V|, stop immediately and return "Cyclic". If the count finishes and |E|<|V|, run the standard DFS cycle detection algorithm. If DFS finds a back edge, return "Cyclic". Otherwise, return "Acyclic".

Time Complexity Analysis:

Step 1: We process at most |V| edges. If the graph has more edges, we terminate early. Thus, this step takes O(|V|).

Step 2: This step runs standard DFS, which takes $O(|V|+|E|)$. However, this step is only executed when $|E|<|V|$. Substituting $|E|$, the runtime becomes $O(|V|+|V|)=O(|V|)$.

Total Time: $O(|V|)$.