Cow Synchronization: Modeling Reference Sheet

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1 State Variables and Observable Modes

We model the biological status of a single cow by

$$w = (x, y; \theta) \in [0,1] \times [0,1] \times \Theta. \tag{1}$$

The real variables x and y represent, respectively, the extent of desire to eat and lie down of the cow, and

$$\theta \in \Theta = \{\mathcal{E}, \mathcal{R}, \mathcal{S}\}. \tag{2}$$

Here:

- $\mathcal{E}$ = Eating,
- $\mathcal{R}$ = Ruminating (lying down),
- $\mathcal{S}$ = Standing.

Description: Each cow is modeled with two hidden internal states, a desire to eat and a desire to lie down, both evolving over time. The observable behavior θ depends on these internal needs and switches according to threshold rules. The full cow state w includes both the hidden dynamics and the visible action.

2 Uncoupled Dynamics of a Single Cow

We model the dynamics of a single cow in different states using

$$(\mathcal{E}) \text{ Eating state:} \quad \begin{cases} \dot{x} = -\alpha_2 x, \\ \dot{y} = \beta_1 y. \end{cases} \tag{3}$$

$$(\mathcal{R}) \text{ Resting state:} \quad \begin{cases} \dot{x} = \alpha_1 x, \\ \dot{y} = -\beta_2 y. \end{cases} \tag{4}$$

$$(\mathcal{S}) \text{ Standing state:} \quad \begin{cases} \dot{x} = \alpha_1 x, \\ \dot{y} = \beta_1 y. \end{cases} \tag{5}$$

where the calligraphic letters inside parentheses indicate the corresponding values of θ. For biological reasons, the parameters α_1, α_2, β_1, and β_2 must all be positive real numbers. They can be interpreted as follows:

$$\begin{cases} \alpha_1 : \text{rate of increase of hunger}, \\ \alpha_2 : \text{decay rate of hunger}, \\ \beta_1 : \text{rate of increase of desire to lie down}, \\ \beta_2 : \text{decay rate of desire to lie down}. \end{cases}$$

3 Observable Behavior Switching Rules

The cow's observable behavior θ evolves according to the internal states (x, y) by

$$\theta \to \begin{cases} \mathcal{E} & \text{if} \quad \theta \in \{\mathcal{R}, \mathcal{S}\} \quad \text{and} \quad x = 1, \\ \mathcal{R} & \text{if} \quad \theta \in \{\mathcal{E}, \mathcal{S}\} \quad \text{and} \quad x < 1,\ y = 1, \\ \mathcal{S} & \text{if} \quad \theta \in \{\mathcal{E}, \mathcal{R}\} \quad \text{and} \quad x < 1,\ y = \delta \ (\text{or } x = \delta,\ y < 1). \end{cases} \tag{6}$$

where δ is a small positive threshold (e.g., $\delta = 0.01$) used to prevent the cow from becoming stuck at the $(x, y) = (0, 0)$ point.

Description:

- When hunger reaches its maximum, cows switch to eating.
- When lying desire reaches its maximum (and hunger is under control), cows switch to ruminating.
- When either need falls very low, cows switch to standing.

4 Discrete Dynamics and the Poincaré Map

Rather than continuously tracking cows at every moment, we focus on key events when their internal states (x, y) reach important thresholds.

We define a special set Σ where transitions occur:

$$\Sigma \equiv \{(x, y; \theta) \,|\, x = 1,\ \delta \leq y \leq 1,\ \theta = \mathcal{E}\} \cup \{(x, y; \theta) \,|\, \delta \leq x < 1,\ y = 1,\ \theta = \mathcal{R}\} = \partial\mathcal{E} \cup \partial\mathcal{R}, \tag{3}$$

where $\partial\mathcal{E}$ and $\partial\mathcal{R}$ denote the relevant boundaries.

We extend Σ to a larger set Σ' by adding low-threshold crossings:

$$\Sigma' \equiv \Sigma \cup \{(x, y; \theta) \,|\, x = \delta,\ \delta \leq y < 1\} \cup \{(x, y; \theta) \,|\, \delta \leq x < 1,\ y = \delta\} = \partial\mathcal{E} \cup \partial\mathcal{R} \cup \partial\mathcal{S}_y \cup \partial\mathcal{S}_x, \tag{4}$$

where $\partial\mathcal{S}_x$ and $\partial\mathcal{S}_y$ represent additional boundaries involving $x = \delta$ and $y = \delta$.

Description:

- We focus on key events when a cow's internal state (x, y) hits a critical threshold.
- The set Σ includes natural switching points: $x = 1$ or $y = 1$.
- The extended set Σ' adds low-hunger and low-lying desire thresholds ($x = \delta$, $y = \delta$).
- This simplifies the model by focusing only on important transitions instead of continuous tracking.

The discrete dynamics are governed by a **map** g, which:

- Takes a cow's current (x, y, θ) at the boundary Σ',
- Predicts where the cow will land next,
- Determines the cow's new observable behavior.

The detailed cases for the discrete map g are given in the next section.

5 Discrete Mapping Rules

The discrete map g determines how a cow's state (x, y, θ) evolves after hitting a boundary in Σ'.

Each case corresponds to a cow hitting a boundary ($x = 1$, $y = 1$, $x = \delta$, or $y = \delta$) and switching behavior accordingly.

Mapping Rules by Cases

Case (a) : Starting from $\theta = \mathcal{E}$, hitting $x = 1$.

$$\text{If} \quad y \geq \frac{\beta_1}{\alpha_2}, \quad g(x = 1, \delta \leq y \leq 1; \mathcal{E}) = \left(\frac{\alpha_2}{\beta_1} y,\, 1;\, \mathcal{R}\right) \tag{5}$$

Case (b) : Starting from $\theta = \mathcal{E}$, hitting $x = 1$.

$$\text{If} \quad y < \frac{\beta_1}{\alpha_2}, \quad g(x = 1, \delta \leq y \leq 1; \mathcal{E}) = \left(\delta,\, \delta^{-\frac{\beta_1}{\alpha_2}} y;\, \mathcal{S}\right) \tag{6}$$

Case (c) : Starting from $\theta = \mathcal{R}$, hitting $y = 1$.

$$\text{If} \quad x \geq \frac{\alpha_1}{\beta_2}, \quad g(\delta \leq x < 1, y = 1; \mathcal{R}) = \left(1,\, \frac{\beta_2}{\alpha_1} x;\, \mathcal{E}\right) \tag{7}$$

Case (d) : Starting from $\theta = \mathcal{R}$, hitting $y = 1$.

$$\text{If} \quad x < \frac{\alpha_1}{\beta_2}, \quad g(\delta \leq x < 1, y = 1; \mathcal{R}) = \left(\delta^{-\frac{\alpha_1}{\beta_2}} x,\, \delta;\, \mathcal{S}\right) \tag{8}$$

Case (e) : Starting from $\theta = \mathcal{S}$, hitting $x = \delta$.

$$\text{If} \quad y \leq \frac{\beta_1}{\alpha_1}, \quad g(x = \delta, \delta \leq y < 1; \mathcal{S}) = \left(1,\, \delta^{-\frac{\beta_1}{\alpha_1}} y;\, \mathcal{E}\right) \tag{9}$$

Case (f) : Starting from $\theta = \mathcal{S}$, hitting $x = \delta$.

$$\text{If} \quad y > \frac{\beta_1}{\alpha_1}, \quad g(x = \delta, \delta \leq y < 1; \mathcal{S}) = \left(\frac{\alpha_1}{\beta_1} y,\, 1;\, \mathcal{R}\right) \tag{10}$$

Case (g) : Starting from $\theta = \mathcal{S}$, hitting $y = \delta$.

$$\text{If} \quad x \geq \frac{\alpha_1}{\beta_1}, \quad g(\delta < x < 1, y = \delta; \mathcal{S}) = \left(1,\, \delta^{-\frac{\alpha_1}{\beta_1}} x;\, \mathcal{E}\right) \tag{11}$$

Case (h) : Starting from $\theta = \mathcal{S}$, hitting $y = \delta$.

$$\text{If} \quad x < \frac{\alpha_1}{\beta_1}, \quad g(\delta < x < 1, y = \delta; \mathcal{S}) = \left(\delta^{-\frac{\alpha_1}{\beta_1}} x,\, 1;\, \mathcal{R}\right) \tag{12}$$

Summary of Mapping Rules

The discrete mapping rules (cases a–h) govern how a cow's internal state (x, y) and observable behavior θ update when crossing a boundary in Σ'.

Each rule describes a transition event depending on:

- Which variable (x or y) triggered the event,
- The cow's current behavior θ at the time of the boundary crossing.

Together, these rules allow modeling a cow's behavior as a sequence of continuous flows interrupted by discrete jumps at boundary crossings.

6 Coupled Dynamics for Synchronization

To model interactions between cows, we introduce a coupling mechanism based on the idea that cows become hungrier when they see others eating and have a greater desire to lie down when they see others lying down.

We define indicator functions on the set $\Theta = \{\mathcal{E}, \mathcal{R}, \mathcal{S}\}$:

$$\chi_\psi(\theta) = \begin{cases} 1, & \text{if } \theta = \psi, \\ 0, & \text{otherwise.} \end{cases} \tag{13}$$

Using these indicators, the single-cow dynamics can be rewritten compactly as:

$$\begin{aligned} \dot{x} &= \alpha(\theta)x, \\ \dot{y} &= \beta(\theta)y, \end{aligned} \tag{14}$$

where

$$\begin{aligned} \alpha(\theta) &= -\alpha_2\chi_{\mathcal{E}}(\theta) + \alpha_1\chi_{\mathcal{R}}(\theta) + \alpha_1\chi_{\mathcal{S}}(\theta), \\ \beta(\theta) &= \beta_1\chi_{\mathcal{E}}(\theta) - \beta_2\chi_{\mathcal{R}}(\theta) + \beta_1\chi_{\mathcal{S}}(\theta). \end{aligned} \tag{15}$$

Now, for a herd of n cows, indexed by i, the **coupled dynamics** are:

$$\begin{aligned} \dot{x}_i &= \left[\alpha^{(i)}(\theta_i) + \frac{\sigma_x}{k_i}\sum_{j=1}^{n} a_{ij}\chi_{\mathcal{E}}(\theta_j)\right] x_i, \\ \dot{y}_i &= \left[\beta^{(i)}(\theta_i) + \frac{\sigma_y}{k_i}\sum_{j=1}^{n} a_{ij}\chi_{\mathcal{R}}(\theta_j)\right] y_i, \end{aligned} \tag{16}$$

where:

- $a_{ij}(t) = 1$ if cow i perceives cow j at time t, and 0 otherwise,
- $k_i = \sum_{j=1}^{n} a_{ij}$ is the number of cows visible to cow i,
- σ_x and σ_y are non-negative coupling strengths.

Description:

- Each cow evolves according to its own internal dynamics, modified by interactions with neighboring cows.
- Cows feel hungrier when they observe others eating and feel more desire to lie down when they observe others ruminating.
- The adjacency matrix A defines the social network between cows at any given time.

7 Measuring Synchronization

To quantify the level of synchronization between cows, we track the times at which each cow switches observable behaviors.

For each cow i:

- Let $\tau^{(i)}$ be the sequence of times at which cow i switches to the eating state $\mathcal{E}$,
- Let $\kappa^{(i)}$ be the sequence of times at which cow i switches to the ruminating state $\mathcal{R}$.

Pairwise Synchronization Measures

Given two cows i and j, assuming $\tau^{(i)}$ and $\tau^{(j)}$ are vectors of the same length K, the **eating synchronization error** between cows i and j is defined as

$$\Delta_{ij}^{\mathcal{E}} \equiv \left\langle \left| \tau_k^{(i)} - \tau_k^{(j)} \right| \right\rangle = \frac{1}{K} \sum_{k=1}^{K} \left| \tau_k^{(i)} - \tau_k^{(j)} \right|, \tag{17}$$

where $\langle \cdot \rangle$ denotes time-averaging.

Similarly, the **ruminating synchronization error** is defined as

$$\Delta_{ij}^{\mathcal{R}} \equiv \left\langle \left| \kappa_k^{(i)} - \kappa_k^{(j)} \right| \right\rangle. \tag{18}$$

Smaller values of $\Delta_{ij}^{\mathcal{E}}$ and $\Delta_{ij}^{\mathcal{R}}$ indicate stronger synchronization between cows.

Group Synchronization Measures

For a herd of n cows, the overall group synchronization is obtained by averaging over all cow pairs:

$$\Delta^{\mathcal{E}} \equiv \frac{1}{n^2} \sum_{i,j} \Delta_{ij}^{\mathcal{E}}, \tag{19}$$

$$\Delta^{\mathcal{R}} \equiv \frac{1}{n^2} \sum_{i,j} \Delta_{ij}^{\mathcal{R}}. \tag{20}$$

The **aggregate synchronization** is then defined by

$$\Delta \equiv \Delta^{\mathcal{E}} + \Delta^{\mathcal{R}}. \tag{21}$$

Description:

- Synchronization is measured by comparing the switching times between cows.
- Lower synchronization errors imply that cows switch between behaviors (eating and ruminating) at more similar times.
- The aggregate synchronization Δ captures the total mismatch across the herd.

8 Numerical Exploration of Herd Synchrony

We perform numerical simulations to investigate synchronization behavior in small herds.

Simulation Setup for Two Coupled Cows

We consider a herd consisting of two cows with nearly identical but slightly mismatched parameters:

$$\alpha_1^{(1,2)} = 0.05 \pm \epsilon, \quad \alpha_2^{(1,2)} = 0.1 \pm \epsilon, \tag{22}$$

$$\beta_1^{(1,2)} = 0.05 \pm \epsilon, \quad \beta_2^{(1,2)} = 0.125 \pm \epsilon, \tag{23}$$

where ϵ is a small mismatch parameter.

We set

$$\delta = 0.25 \tag{24}$$

for the low-threshold switching value.

The coupling strengths σ_x and σ_y are varied to explore their effect on the degree of synchronization.

Description:

- Simulations examine how varying the mismatch ϵ and coupling strengths σ_x, σ_y affects synchronization in the herd.
- We begin with a two-cow system and can later extend to larger herds if desired.