APMA 2822B Final Project Proposal

Distributed Preconditioned Conjugate Gradient (PCG) Solver with GPU Offload

Joshua Dover

1 Project Overview

I propose to implement a distributed, GPU-accelerated Preconditioned Conjugate Gradient (PCG) solver for large, sparse symmetric positive-definite (SPD) linear systems arising from finite-difference discretizations of a partial differential equation. The solver will combine a matrix-free stencil operator, MPI-based domain decomposition with halo exchange, and GPU offload using HIP. A Jacobi preconditioner will be used to improve convergence. The primary focus will be on a two-dimensional problem, with a three-dimensional extension as a stretch goal.

2 Mathematical Model

The target PDE is the Poisson equation on the unit square,

$$-\Delta u(x, y) = f(x, y), \qquad (x, y) \in (0, 1)^2,$$

with Dirichlet boundary conditions. I will use the analytic solution

$$u(x, y) = \sin(\pi x)\sin(\pi y),$$

which gives the exact right-hand side

$$f(x, y) = 2\pi^2 \sin(\pi x)\sin(\pi y).$$

After discretization on a uniform grid, this results in a linear system

$$Ax = b,$$

where A corresponds to the standard five-point finite-difference Laplacian. Dirichlet boundary values will be incorporated by modifying b. If time permits, I will extend the solver to the three-dimensional manufactured solution

$$u(x, y, z) = \sin(\pi x)\sin(\pi y)\sin(\pi z),$$

leading to a seven-point stencil.

3 Parallel and GPU Implementation

MPI Decomposition

The solver will employ a one-dimensional row-wise domain decomposition, where each MPI rank owns a contiguous block of grid rows and the associated ghost layers. Halo values will be updated using non-blocking `MPI_Isend` and `MPI_Irecv` calls, and global quantities such as inner products and residual norms will be evaluated via `MPI_Allreduce`.

GPU Kernels

The GPU will perform all compute-intensive operations, including the matrix-free stencil application, vector updates, and the Jacobi preconditioner. Local reductions will be carried out on the GPU before participating in MPI reductions, and I will measure memory-bandwidth utilization to characterize stencil performance.

4 Evaluation and Analysis

The solver will be evaluated in terms of:

- **Convergence:** Relative residual norms $\|r_k\|_2/\|r_0\|_2$ and, using the manufactured solution, discrete error norms.
- **Scaling:** Strong and weak scaling studies on distributed GPU hardware, including iteration times and total runtime.
- **Roofline and Profiling:** Arithmetic intensity and achieved bandwidth of the stencil kernel, along with GPU profiling to assess compute, communication, and reduction costs.

If time allows, I will repeat selected experiments for the three-dimensional extension using the seven-point stencil and analytic solution.